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# A workshop on the epistemology and didactics of mathematical structuralism

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*This workshop was dedicated to a discussion of epistemological and didactic aspects of mathematical structuralism with a focus on Abstract Algebra, in particular Group Theory. The participants worked on a corpus of documents comprising excerpts of the Bourbaki Manifesto “the architecture of mathematics” and the transcript of a discussion thread from a mathematical forum online. To what extent does mathematical structuralism rely on didactic principles? How is this reflected in a teaching-learning environment? The key construct within the Anthropological Theory of the Didactic was the theoretical notion of “structuralist praxeology”, introduced by the author. The analyses of the corpus led to the discussion of the rationale of such praxeologies, their historical development, how they may be reconstructed by learners, and the role played by the “dialectic of objects and structures” that should be added to Chevallard’s model for study and research processes in order to account for the development of structuralist praxeologies.*

*Keywords: mathematical structuralism, Abstract Algebra, Group Theory, Anthropological Theory of the Didactic, structuralist praxeology, dialectic of objects and structures*

## Epistemology of mathematical structuralism

Mathematical structuralism takes its roots in the re-foundation of algebra operated by German algebraists at the beginning of the 20th Century. Workshop participants were thus given a few historical landmarks on the emergence of the group, ring and field structures, following Wussing (2007) and Corry (1996). Synthetic accounts may be found in (Hausberger, 2018b).

The idea of algebraic structures as a unifying principle is due to Noether: in the 1920s, she got Abstract Algebra away from thinking about operations on elements (such as addition or multiplication in groups or rings) but described structures in terms of selected subsets (such as normal subgroups of groups or ideals in Ring Theory) and homomorphisms. Noether and her school thus changed the way theorems were proved in algebra, focusing on general proofs that limit the calculations and put to the fore the “most general and fundamental” (hence simpler, according to structuralist views) concepts. This mathematical re-foundation of algebra also paved the way for unprecedented mathematical constructs, such as noetherian rings.

The structuralist method is well described in the Bourbaki (1950) Manifesto, entitled “the architecture of mathematics”, written by a group of French mathematicians who were the great promoters of structuralist thinking. Indeed, Bourbaki set out to apply the method developed by the German algebraists to all fields of mathematics. The first part of the workshop was dedicated to the discussion of excerpts of the Manifesto. Bourbaki describes the structuralist use of the axiomatic method both as a method of exposition of mathematical theories and a method of discovery of new results (thus a heuristic). He draws conclusions on the impact of structuralism on mathematical activity and the organization of mathematical theories.

The starting point of the workshop was a couple of remarks that guided the reading of the Manifesto. Firstly, Bourbaki is aiming at the “profound intelligibility” of mathematics through the axiomatic method which “teaches us to look for the deep-lying reasons for such a discovery, to find the common ideas of these theories, buried under the accumulation of details properly belonging to each of them, to bring these ideas forward and to put them in their proper light”. This suggests that *didactic principles* are governing Bourbaki’s reconstruction of mathematical knowledge. Secondly, Bourbaki sees structures as “tools for the mathematician”; he emphasizes the “standardization of mathematical techniques” by means of the axiomatic method which is “nothing but the Taylor system for mathematics”. This connects to the *praxeological point of view* of the Anthropological Theory of Didactic (ATD) which sees mathematical activity as the development of combinations of praxis and logos (praxeologies, Chevallard 2006). The following questions were thus discussed:

1. Which statements of the Bourbaki discourse express didactic concerns? To what extent does mathematical structuralism rely (or not) on didactic principles as a method of exposition and a method of discovery? How to phrase these principles in terms of didactics of mathematics?
2. Using the vocabulary of ATD, how to describe the impact of structuralism on mathematical praxeologies, in particular the praxis and logos blocks, the interrelations of both, on mathematical organizations in general?

Participants underlined that the writing of the Manifesto is itself a didactic gesture: in order to disseminate mathematics (Bourbaki wrote a treatise in several volumes: *Elements de mathématiques*), one must have a clear vision of what mathematics is. As pointed out in the Manifesto, “[it is] out of the question to give to the uninitiated an exact picture of that which the mathematicians themselves can not conceive in its totality”. This quote refers to a stage of development of mathematics in which diversity hindered the production, communication and dissemination of mathematics. This is where the structuralist method comes into play as a didactic method to promote understanding and sense-making through “separating out the principal mainsprings of the arguments; then taking each of them separately and formulating it in abstract forms, to develop the consequences which follow from it alone”. Bourbaki also refers to an “economy of thought”, which may be related to the economy of didactic memory. Nevertheless, setting out abstraction and generality as principles was also debated among workshop participants (as among mathematicians, for instance Mandelbrot who discussed relationships between explanation, generality and abstraction). At first sight, a few educators - perhaps those who least share the Bourbaki culture - exclaimed: “Don’t do that!” (the use of such principles in classrooms).

In fact, Bourbaki describes and justifies in the Manifesto techniques to solve problems and to write and communicate mathematical theories. In other words, all the elements of mathematical research praxeologies may be found. Readers may draw a list of structuralist techniques to solve problems: “to recognize among the elements relations which satisfy axioms of known types”, “to apply the arsenal of general theorems which belong to the structure of that type”, “to orient the intuitive course of one’s thought [according to the structural insights]”; and structuralist techniques for the exposition of mathematics: “to look for deep-lying reasons”, “to find common ideas of theories and bring them forward”, “to separate out the mainsprings of its arguments”, “to inquire how these different components influence each-other”, “to set up the axiomatic theory of a given structure”.

The phenomena of unification of punctual praxeologies into regional and global praxeologies was related by several participants to the Bourbaki discourse of unification of mathematics under the axiomatic method. The holistic vision of Bourbaki was questioned as an epistemological point of view that may be relativised (it applies to algebra but maybe not with the same pertinence to every mathematical domain) and as a socio-cultural norm. Participants pointed out that distinctions should be made between mathematical praxeologies in teaching institutions (including universities), in research institutions (also including universities), and in different fields or domains, considering also variations between societies and civilisations (in reference to the scale of levels of didactic codeterminacy). Nevertheless, it was argued that some common points (consistent with the Bourbaki discourse) may be found: privileging the “most general formulations” (of theorems, definitions, etc.); the role played by Theory to unify mathematical sectors and domains (regional praxeologies); a further Metatheory to unify the discipline in a global praxeology. Other participants underlined that the structuralist method induced changes at the discipline level, for instance the creation of new domains and sectors (general set-theoretic topology, algebraic topology,...). It was accompanied by important modifications in the praxis (and of course logos), as new questions appeared and new ways of proving results (new techniques). At this stage of the workshop, the need of new tools emerged to make such statements more precise.

### **Modeling mathematical structuralism within ATD**

The notion of *structuralist praxeology* (Hausberger, 2018a) aims at modeling within ATD the epistemological ideas developed above. Structuralist thinking is characterised by reasoning in terms of classes of objects, relationships between these classes and stability of properties under operations on structures. The application of the structuralist method relies on a *dialectic between the particular and the general*, or in other words between *objects and structures*. The questions and problems are raised to a higher level of generality in order to apply structuralist concepts (e.g., ideal, principal ideal domain, etc.) and tools (e.g., isomorphism theorems, structure theorems, combinatorial of structures, etc.) according to the motto “generalizing is simplifying”. In other words, the “structuralist methodology aims at replacing a praxeology  $[T, \tau, \theta, \Theta_{\text{particular}}]$  by a structuralist praxeology  $[T^g, \tau, \theta, \Theta_{\text{general}}]$ , where  $T^g$  is a generalization of  $T$  that allows the use of structuralist techniques” (loc. cit. p. 83). In order to illustrate the theoretical construct of structuralist praxeology, workshop participants were presented the example of the thread on decimal numbers from the online forum *mathematiques.net* (loc. cit. pp.83-87). To prove that the ring  $\mathbf{D}$  of decimals is a principal ideal domain, the forum students searched for a proof of the general statement that any subring of  $\mathbf{Q}$  is principal, then investigated whether principality was transferred from a ring to its subrings. The study of these dialogues emphasise the role of the dialectic of objects and structures in the development of structuralist praxeologies: structures are applied as a generalizing-simplifying viewpoint in order to demonstrate properties on mathematical objects and, conversely, a semantic control of the formal general statements on structures is exercised by putting them to the test of known examples ( $\mathbf{Q}[x]$  is principal but  $\mathbf{Z}[X]$  is not).

Inspired by Winsløw’s (2008) praxeological formalisation of the concrete to abstract transition in analysis (from calculus to more theoretical tasks involving continuity and differentiability of functions as well as the topology of real numbers), the author proposed a model for the

epistemological transition to Abstract Algebra, in two phases: the first phase is concerned with the transition from  $T$  to  $T^{\mathfrak{B}}$  described above and leads to the construction of a structuralist praxeology as a fertile strategy to prove properties of concrete objects; the second phase builds on structuralist praxeologies previously developed in order to introduce more abstract and theoretical types of tasks that only consider classes of objects with their structural properties (e.g.: show that a Noetherian integral domain such that every maximal ideal is principal is a principal ideal domain).

A new example related to arithmetic and abelian Group Theory (GT) was then introduced at the workshop. Indeed, Bourbaki cautiously explained that his account of structuralism is a schematic and idealized sketch, and so is the model described above. Its pertinence in the elucidation of teaching-learning phenomena related to GT needs be investigated through concrete case studies. The case of GT was also chosen to relate to other didactic studies. For instance, Bosch et al. (2018) focused “not only on the official *raison d’être* of GT within university teaching, but also on different possible alternative ones that could motivate or impel the use of GT to solve problematic questions”. In particular, they looked for external problems (that is, external to GT) that could lead to the reproduction of a substantial part of GT as a means to ascribe some rational to it. Precisely, they argued that a counting problem (such as that of symmetries of a square) may be a suitable candidate for a reconstruction of elementary GT. This choice is justified by the links between GT and the notions of symmetry and invariant in the historical development of GT and by the role played by Lagrange’s theorem as a tool to solve the problem. Nevertheless, it also raises epistemological issues: for instance, “is it substantial enough to motivate the study of the isomorphism theorems”?

The problem discussed at the workshop is classical, external to GT, and may be found in standard textbooks (Perrin, 1996): determine the set of primes  $p$  such that  $-1$  is a quadratic residue (congruent to a perfect square) modulo  $p$ . Such questions follow from the work of number theorists of the 17<sup>th</sup> and 18<sup>th</sup> centuries (Fermat, Euler, Lagrange, Legendre) and have been given a first systematic treatment in Gauss’s *Disquisitiones Arithmeticae* (1801). The answer to this question is called “first supplement to the law of quadratic reciprocity” ( $p = 2$  or  $p \equiv 1 \pmod{4}$ ). The ambition is not to cover a substantial part of GT but simply illustrate the structuralist methodology.

The problem may be considered inside arithmetic as a particular theory (in the sense of Bourbaki) in dialectical relationship with GT (through the group  $(\mathbf{Z}/p\mathbf{Z})^*$ ), Field Theory ( $\mathbf{F}_p$ ), or even Ring Theory ( $\mathbf{Z}[i]$ ,  $\mathbf{F}_p[X]$ ) as general structures. Students may stay in arithmetic (as a theory in the sense of ATD), use Fermat’s little theorem to prove that  $(-1)^{(p-1)/2} \equiv 1 \pmod{p}$  and deduce the necessity of the condition  $p \equiv 1 \pmod{4}$ . The converse implication is less straightforward (the reader may have a try or look up Gauss’s DA art 111), but gains much clarity when translated into structural terms. Indeed, the set  $Q$  of quadratic residues modulo  $p$  make up a subgroup of index 2 in  $\mathbf{F}_p^*$  (as the image of the homomorphism  $x \mapsto x^2$ ). It thus contains  $(p-1)/2$  elements (and we may recover the preceding result by means of Lagrange’s theorem). Moreover, the equation  $x^{(p-1)/2} = \bar{1}$  in  $\mathbf{F}_p$  admits at most  $(p-1)/2$  solutions since  $\mathbf{F}_p$  is a field. This proves that the set of solutions is exactly  $Q$  and the result follows. Another proof uses the argument that a group of even order always possesses an element of order 2 (a well-known result in elementary GT) to conclude that  $-1$  belongs to  $Q$  (under the hypothesis  $p \equiv 1 \pmod{4}$ ), since it is the unique element of order 2. The problem may thus be

related to standard praxeologies from GT (show that a subset is a group, determine its order and use known results on orders of elements) or mixed arguments using a property of polynomials defined over a field.

In this example, the type of task  $T$  (find conditions for the existence of a solution of a congruence) is handled with respect to the structural properties of the given congruence equation, and thus gives birth to the structuralist types of tasks (which are not direct generalizations of  $T$ ), once structures are identified. The first isomorphism theorem, Lagrange's theorem and other general results about groups play the role of technologies. Theories of structures (GT, abelian GT, FT, RT) orient the work and provide the (formalized part of the) theory in the sense of ATD.

### **The thread on $(\mathbb{Z}/32\mathbb{Z})^*$ and abelian Group Theory**

Although many mathematical problems from GT may be stated independently on GT as we have just seen, the concepts of GT often offer an adequate framework to state GT problems in a synthetic meaningful way, as well as inspiring new problems as further natural developments. Looking up at lists of problems collected and proposed by mathematicians to students, for instance <https://yutsumura.com/welcome-to-problems-in-mathematics/>, it is striking that the structuralist scope is put in the fore to give a title to the problems ([Prove a Group is Abelian if  \$\(ab\)^2=a^2b^2\$](#)  [Quotient Group of Abelian Group is Abelian](#) [A Group is Abelian if and only if Squaring is a Group Homomorphism](#)). Therefore, the success of students to solve these problems very much depend on their ability to connect the statements to the core structuralist praxeologies that they previously met, in other words their structuralist praxeological equipment and their understanding of the structuralist methodology.

For the last part of the workshop, participants focused on a problem that was submitted by a student (ianchenmu) to the Math Help Board (MHB) forum. MHB presents itself as an “online community that gives free mathematics help any time of the day about any problem, no matter what the level”. As stated by ianchenmu, “the question is to identify isomorphism type for each proper subgroup of  $(\mathbb{Z}/32\mathbb{Z})^*$ ”. The student wonders about the meaning of “isomorphism type”, guesses that isomorphisms need to be found between each such subgroup and “another group” and questions about the method to proceed. A math scholar and MHB moderator (Klaas\_van\_Aarsen) helps ianchenmu to advance through the task, and another student (jakncoke) joins in to share his views. The thread may be accessed via this link<sup>1</sup> and a transcript of the collective study process was provided at the workshop.

The study process may be modeled by a study and research path (SRP; Chevallard, 2006) as the author did for the thread on decimal numbers (Hausberger, 2018a; 2019). The chronogenesis is governed by a dialectic between questions and answers while the mesogenesis involves a dialectic of medias and milieus (Chevallard, 2009). Special attention should be paid to the mathematical praxeologies that are developed through the SRP, in particular structuralist praxeologies that, as we have seen, usually appear through the dialectic of objects and structures.

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<sup>1</sup><https://mathhelpboards.com/linear-abstract-algebra-14/identify-isomorphism-type-each-proper-subgroup-z-32z-3585.html>

The following questions were proposed for discussion at the workshop: 1. Do you have any idea about the historical origin of such a problem? To which sector and theme of the Abstract Algebra curriculum would you connect this problem to? 2. What are the main praxeologies developed during the study process online? 3. Which structuralist aspects can you identify? What are the objects and structures involved? Can you observe a dialectic between objects and structures? 4. What is your evaluation of the dialectic of medias and milieus, and the global vitality of the study process? 5. What are, according to you, the main conditions that hinder/foster the development of structuralist praxeologies (the ecological question)? If you were moderating the forum, what strategy would you adopt to “link problem solving and learning content” and thus address the “challenges of self-sustained study and research processes” (Bosch et Winsløw, 2016)? What would you retain from Bourbaki’s didactic principles? Answers or partial answers to these questions are proposed and discussed in the sequel.

Considering the formulation of the problem by ianchenmu, two types of tasks emerge: determine the lattice of subgroups of a given finite abelian group  $G$  ( $T_1$ ), determine the isomorphism type (or structure) of a given finite abelian group ( $T_2$ ). The type of tasks  $T_2$  must be understood in relation to the *fundamental theorem of finite abelian groups* (FTAG), which clarifies the notion of “type”: every finite abelian group can be expressed as the direct sum of cyclic subgroups of prime-power order. According to Wussing (2007), the FTAG, was proven by Kronecker in 1870, using a group-theoretic proof, though without stating it in group-theoretic terms. This generalised an earlier result of Gauss from *Disquisitiones Arithmeticae* (1801), which classified quadratic forms. The theorem was stated and proved in the language of groups by Frobenius and Stickelberger in 1878.

The type of tasks  $T_1$  and type of group  $G = (\mathbf{Z}/n\mathbf{Z})^\times$  both take roots in Galois Theory (GT). Indeed, such groups are the Galois groups of cyclotomic field extensions (generated over  $\mathbf{Q}$  by a primitive  $n$ -th root of unity), and GT states a correspondence between the lattice of field extensions and the lattice of subgroups (the latter being easier to determine). Finally, the isomorphism types of  $(\mathbf{Z}/n\mathbf{Z})^\times$  are known: by the Chinese remainder theorem, it is enough to determine the type in the case  $n$  is a primer power  $p^r$ ; moreover,  $(\mathbf{Z}/p^r\mathbf{Z})^\times$  is cyclic except in the case  $p = 2$  and  $r \geq 2$  in which  $(\mathbf{Z}/2^r\mathbf{Z})^\times \simeq \mathbf{Z}/2^{r-2}\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ . The parameters  $p$  and  $r$  ( $= 5$ ) are thus important didactic variables.

There are introductory as well as advanced textbooks on GT, covering GT or Abstract Algebra in general. The lattice of subgroups is often not a topic covered by elementary textbooks, the classification of groups according to isomorphism types being the culmination of the course. In Dummit & Foote (2003), a more advanced textbook, the lattice of subgroups appears as a topic of chapter 2 “subgroups” (the theme) within part 1 GT (sector). The FTAG (generalised to finitely generated abelian groups) and the classification problem are presented in chapter 5 on “direct and semidirect products and abelian groups”.

Several techniques are developed on the forum to solve the main types of tasks  $T_1$  and  $T_2$ . Regarding the former:  $\tau_{1,Klass}$ : determine cyclic subgroups; if  $g$  does not generate  $G$ , add other elements;  $\tau_{1,ianchenmu}$ : compute  $\langle g \rangle$  for all  $g$  in  $G$  [“it’s so much work”];  $\tau_{1a,Jakncoke}$ : if  $G = \mathbf{Z}/n\mathbf{Z}$ , use the fundamental theorem of cyclic groups;  $\tau_{1b,Jakncoke}$ : determine the possible orders of subgroups thanks to Lagrange’s theorem and the possible isomorphism types thanks to the FTAG. Then construct subgroups by looking at orders of elements and then *combining elements by means of direct*

*products* [erroneous part of the technique]. The technology includes the definition of a cyclic group, Lagrange's theorem and the fundamental theorems (used implicitly by Jakncoke). Regarding the latter:  $\tau_{2a,Klass}$ : if  $G$  is cyclic of order  $n$ , then the isomorphism type is that of  $\mathbf{Z}/n\mathbf{Z}$ ;  $\tau_{2b,Klass}$ : give all possible isomorphism types thanks to the FTAG and conclude by examining orders of elements;  $\tau_{2,ianchenmu}$ : if  $G = \langle a, b, c \rangle$  then the isomorphism type is that of the direct product of cyclic groups of orders given by those of the 3 generators [erroneous technique]. The technology here includes the definitions of the isomorphism type and cartesian product, and the FTAG. In fact, three related auxiliary types of tasks are discussed on the forum:  $T_3$ : determine the order of an element  $g$  in  $G$ ;  $T_4$ : determine the list of elements of an abelian group  $G = \langle a, b \rangle$ ;  $T_5$ : show that a given group of order 4 is isomorphic to the Klein group  $V_4$ .

Several structuralist aspects of the problem may be pointed out: 1. at the level of mathematical concepts, the problem involves: the mathematical structures of group and the partial order  $<$  on subgroups; the notion of isomorphism type (abstract structure of a group); the lattice of subgroups that "encodes" part of the group-theoretic information. 2. at the level of methods, the following structuralist principles are encountered: thinking in terms of generators (and relations); thinking up to isomorphism (isomorphisms preserve "structural information"); to decompose into simple components (structure theorems); to combine simple components (direct product).

Let us now focus on the dialectic of objects and structures. The main objects met during the study process are the groups  $\mathbf{Z}/32\mathbf{Z}$  and  $(\mathbf{Z}/32\mathbf{Z})^\times$ , and instances of products of cyclic groups (e.g.  $V_4$ ). The main structures are  $\langle a, b \rangle$  (abstract group on generators) and  $\prod \mathbf{Z}/p_i^{f_i}\mathbf{Z}$  (FTAG), which serve as abstract models. The dialectical aspects may be described as follows: the structures serve as tools to determine subgroups of  $(\mathbf{Z}/32\mathbf{Z})^\times$  (by means of generators) and their isomorphism types ( $\tau_{2,ianchenmu}$ ; the fundamental theorem as a means to predict possible models). Respectively, further elements of the logos on structures emerged (but only implicitly) to fulfill technical needs: the implicit erroneous formulas  $\langle a, b \rangle = \langle a \rangle \cup \langle b \rangle$  and  $\langle a, b, c \rangle \simeq \mathbf{Z}/o(a)\mathbf{Z} \times \mathbf{Z}/o(b)\mathbf{Z} \times \mathbf{Z}/o(c)\mathbf{Z}$ , which were refuted during the study process. Unfortunately, conditions for such an isomorphism to hold (in the simplest case of 2 elements) were not discussed, as well as more advanced structuralist questions such as the behavior of the lattice of subgroups under direct product of groups.

Let us now conclude on the ecology of structuralist praxeologies. Our analysis shows that the study process evolved towards more and more basic praxeologies ( $T_4$ ,  $T_5$ ) due to a lack in the praxeological equipment of ianchenmu that hindered the possibility to deal with the main tasks. Moreover, the moderator only focused on ianchenmu and didn't help jakncoke develop more advanced praxeologies. The dialectic of medias and milieus is very limited (very few elements of the medias were brought up in the milieu and no media on lattice of subgroups was identified): one hypothesis is that ianchenmu lacked skills to identify pertinent sources and therefore relied on Klaus who played the role of teacher. The dialectic of objects and structures is also limited: structures were applied as tools but techniques did not lead to further theoretical developments (new general results about structures to test on known examples). The two types of tasks  $T_1$  and  $T_2$  were not related as they could (role of information on the structure of  $G$  to construct its lattice of subgroups), and the structuralist methods were not supported by structuralist insights as a metadiscourse (a technology): e.g. the notion of isomorphism type was not clarified by Klaus.



Finally, if the FTAG illuminates the role of cyclic groups, it hinders the work on constructing isomorphisms “by hand” to link  $\langle a, b \rangle$  and  $\mathbf{Z}/o(a) \times \mathbf{Z}/o(b)$  (the issue of the economy of techniques). In Dummit and Foote (2003), the task  $T_1$  on lattice of subgroups is discussed three chapters before the FTAG (generalised) is taught. This suggests, by contrast with structuralist principles, that it would be preferable to separate the two types of tasks, at an early stage of the development of structuralist praxeologies in GT.

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