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ON THE FOURIER COEFFICIENTS OF POWERS OF A BLASCHKE FACTOR AND STRONGLY ANNULAR FONCTIONS

ALEXANDER BORICHEV, KARINE FOUCHET, AND RACHID ZAROUF

ABSTRACT. We compute asymptotic formulas for the k^{th} Fourier coefficients of b_λ^n, where $b_\lambda(z) = \frac{z-\lambda}{1-\bar{\lambda}z}$ is the Blaschke factor associated to $\lambda \in \mathbb{D}$, $k \in [0, \infty)$ and n is a large integer. We distinguish several regions of different asymptotic behavior of those coefficients in terms of k and n. Given $\beta \in ((1-\lambda)/(1+\lambda), (1+\lambda)/(1-\lambda))$ their decay is oscillatory for $k \in [\beta n, n/\beta]$. Given $\alpha \in (0, (1-\lambda)/(1+\lambda))$ their decay is exponential for $k \in [0, n\alpha] \cup [n/\alpha, \infty)$. Airy-type behavior is happening near the k-transition points $n(1-\lambda)/(1+\lambda)$ and $n(1+\lambda)/(1-\lambda)$. The asymptotic formulas for the k^{th} Fourier coefficients of b_λ^n are derived using standard tools of asymptotic analysis of Laplace-type integrals. More precisely, the integral defining the k^{th} Fourier coefficient of b_λ^n is perfectly suited for an application of the method of stationary phase when $k \in (n(1-\lambda)/(1+\lambda), n(1+\lambda)/(1-\lambda))$ and requires the use of the method of the steepest descent when $k \notin [n(1-\lambda)/(1+\lambda), n(1+\lambda)/(1-\lambda)]$. Uniform versions of those standard methods are required when k approaches one of the boundaries $n(1-\lambda)/(1+\lambda)$, $n(1+\lambda)/(1-\lambda)$. As an application, we construct strongly annular functions with Taylor coefficients satisfying sharp summation properties.

1. Introduction

1.1. **Notation.** Let $\mathbb{D} = \{z : \ |z| < 1\}$ be the open unit disk and $\partial\mathbb{D}$ its boundary. For a given $\lambda \in \mathbb{D}$ we denote by

$$b_\lambda(z) = \frac{z-\lambda}{1-\bar{\lambda}z},$$

the Blaschke factor corresponding to $\lambda \in \mathbb{D}$. It is well-known that the function b_λ is an automorphism of $\mathbb{D}$ and that $|b_\lambda(z)| = 1 \iff z \in \partial\mathbb{D}$. Given a nonnegative integer n we recall the definition of the $k^{\text{th}}-$Taylor/Fourier coefficient of the $n^{\text{th}}-$power of b_λ:

$$\frac{(b_\lambda^n)^{(k)}(0)}{k!} = \widehat{b_\lambda^n}(k) = \frac{1}{2i\pi}\int_{\partial\mathbb{D}} b_\lambda^n(z) z^{-k}\frac{\mathrm{d}z}{z},$$

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whose asymptotic behavior we wish to determine as $n \to \infty$. Let $b_\lambda^n(z) = \sum_{k\geq 0} \widehat{b_\lambda^n}(k) z^k$ be the Taylor expansion of b_λ^n. Then we have

$$\left(\frac{z-\lambda}{1-\bar{\lambda}z}\right)^n = e^{in\theta}\left(\frac{ze^{-i\theta}-|\lambda|}{1-|\lambda|ze^{-i\theta}}\right)^n = \sum_{k\geq 0} \widehat{b_{|\lambda|}^n}(k) e^{i(n-k)\theta} z^k,$$

with $\theta = \arg\lambda$, which shows that $\widehat{b_\lambda^n}(k) = \widehat{b_{|\lambda|}^n}(k) e^{i(n-k)\theta}$. Therefore, without loss of generality we assume from now on that $\lambda \in (0,1)$. In this article we compute asymptotic formulas for $\widehat{b_\lambda^n}(k)$ as $n \to \infty$, when $k \in [0,\infty)$. Furthermore, we apply these asymptotic formulas to construct strongly annular functions with small Taylor coefficients.

1.2. **Motivations.** Various motivations have led to study the asymptotic behavior of $\widehat{b_\lambda^n}(k)$ in the limit of large n. We begin by mentioning a line of research in which the question of estimating l^p norms of the sequence $\left(\widehat{b_\lambda^n}(k)\right)_{k\geq 0}$ plays a central role, see Subsection 1.2.1 below. Another motivation, described in Subsection 1.2.2 is the construction of so called strongly annular functions with specific decay of the Taylor coefficients.

1.2.1. *l^p norms of $\widehat{b_\lambda^n}$ for $p \in [1,\infty]$ and related topics.*

We use standard notation from asymptotic analysis: From now on, for two positive functions $f, g : \mathbb{C} \to \mathbb{R}^+$ we say that f is dominated by g, denoted by $f \lesssim g$, if there is a constant $c > 0$ such that $f \leq cg$. We say that f and g are comparable, denoted by $f \asymp g$, if both $f \lesssim g$ and $g \lesssim f$.

(1) The study of the l^p norms of $\widehat{b_\lambda^n}$ was probably initiated by J.-P. Kahane [27] who was interested in the case $p = 1$. He applied van der Corput type estimates on $\widehat{b_\lambda^n}(k)$ [27, p. 253] to get information on the asymptotic behavior of the l^1 norm of $\widehat{b_\lambda^n}$

$$\|\widehat{b_\lambda^n}\|_1 := \sum_{k\geq 0} |\widehat{b_\lambda^n}(k)|.$$

Kahane's motivation [27, Theorem 1] was to generalize a theorem by Z. K. Leibenson [31], which is a special case of a theorem [38, Theorem 4.1.3] about homomorphisms of group algebras due to P. T. Cohen. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous, non-constant and 2π-periodic function. A. Beurling and H. Helson [6] proved that if $\|\widehat{e^{in\varphi}}\|_1 = \mathcal{O}(1)$, $n \in \mathbb{Z}$, then φ is affine. Kahane proved that:
(a) If φ is piecewise linear, then $\|\widehat{e^{in\varphi}}\|_1 = \mathcal{O}(\log(|n|))$, [27, Theorem III] and
(b) if φ is analytic, then $\|\widehat{e^{in\varphi}}\|_1 \asymp \sqrt{|n|}$, [27, Theorem V].
Writing $b_\lambda(e^{it})$ as $e^{i\varphi(t)}$ for $t \in (-\pi, \pi]$, we deduce from (b) that

$$\|\widehat{b_\lambda^n}\|_1 \sim c_1\sqrt{n}, \qquad n \to \infty.$$

The precise value c_1 of the limit

$$\lim_{n\to\infty} n^{-1/2}\|\widehat{b_\lambda^n}\|_1$$

was computed in [24]. A discussion on l^p norms for $p \in [1,\infty]$ occurred in [9], where the asymptotic behavior

$$\|\widehat{b_\lambda^n}\|_p \asymp n^{\frac{2-p}{2p}} \text{ for } p \in [1,2] \tag{1.1}$$

is derived. The discussion in [9] is more general and motivated by investigating the boundedness of the composition operator C_b, $C_b(f) = f \circ b$, where $b = b_\lambda$. To assess whether C_b is a bounded linear operator from one Banach space X of analytic functions into another, say Y, it is often enough to know the asymptotic behavior of $\|b_\lambda^n\|_Y$.

Let us mention that the asymptotic formulas for $\widehat{b_\lambda^n}(k)$ obtained in the present paper could be used to compute the exact values of c_p defined as follows:

(a) If $p \in (0,4)$, then

$$\lim_{n\to\infty} n^{-\frac{2-p}{2p}} \|\widehat{b_\lambda^n}\|_p = c_p,$$

(b) if $p = 4$, then

$$\lim_{n\to\infty} \left(\frac{n}{\log n}\right)^{1/4} \|\widehat{b_\lambda^n}\|_4 = c_4,$$

(c) and if $p \in (4,\infty]$, then

$$\lim_{n\to\infty} n^{-\frac{1-p}{3p}} \|\widehat{b_\lambda^n}\|_p = c_p,$$

which generalizes Girard's result [24] and strengthens [42, Theorem 1]. The constants c_p are not studied in this article; their computations are part of a forthcoming work.

(2) O. Szehr and R. Zarouf proved upper and lower bounds on $|\widehat{b_\lambda^n}(k)|$ [42] to complete the result of M. Blyudze and S. Shimorin (1.1) on l^p norms of the sequence $\widehat{b_\lambda^n}$, extending (1.1) to the range $p \in [1,4)$ and providing sharp estimates on $\|\widehat{b_\lambda^n}\|_p$ for the remaining range $p \in [4,\infty]$. Later on, Szehr and Zarouf [42, Proposition 2] applied those results to estimate analytic capacities in Beurling–Sobolev spaces. Finally, the same authors [41, 43] proved upper bounds on $|\widehat{(1-z^2)b_\lambda^n}(k)|$ to construct a class of counterexamples to Schäffer's conjecture on optimal estimates for norms of inverses of matrices [39, 25, 36, 35].

Namely, in 1970 J.J. Schäffer [39, Theorem 3.8] proved that for any invertible $n \times n$ matrix T and for any operator norm $\|\cdot\|$ the inequality

$$|\det T| \left\|T^{-1}\right\| \leq \mathcal{S} \|T\|^{n-1}$$

holds with $\mathcal{S} = \mathcal{S}(n) \leq \sqrt{en}$. He conjectured that in fact this inequality holds with an $\mathcal{S}$ independent of n. This conjecture was refuted in the early 1990-s by E. Gluskin, M. Meyer and A. Pajor [25] who have shown that for certain $T = T(n)$ the inequality can only hold when $\mathcal{S}$ is growing with n. Subsequent contributions of J. Bourgain [25] and H. Queffélec [36, 35] provided increasing lower estimates on $\mathcal{S}$. The currently best known lower estimate on $\mathcal{S}$ is due to H. Queffélec [35] :

$$\mathcal{S} \geq \sqrt{n}(1 - \mathcal{O}(1/n)).$$

Those results rely on probabilistic and number theoretic arguments. The common point in the mentioned lower bounds is that they rely on an inequality of Bourgain [43, Inequality (2.2)] that relates Schäffer's problem to a geometric property of the spectrum of T: For $\mathcal{S}$ to grow the eigenvalues of T should satisfy a Turán-type power sum inequality. The construction of explicit solutions to such inequalities appears to be a well-studied but open problem in number theory [45, 32, 21, 4, 3]. More precisely, Bourgain's inequality relates Schäffer's question to Turán's tenth problem [4, 45]. The latter has no constructive solution and relies on deep number-theoretic existence arguments [4, 32, 35]. In [25] as well as in [36, question 5] the construction of explicit matrices with growing $\mathcal{S}$ is formulated as an open problem. Constructive counterexamples to Schäffer's conjecture are proposed in [43] where the authors present an explicit sequence of *Toeplitz matrices* $T_\lambda \in \mathcal{M}_n$ with *singleton spectrum* $\{\lambda\} \in \mathbb{D} \setminus \{0\}$ such that

$$\mathcal{S} \geq |\lambda|^n \left\|T_\lambda^{-1}\right\| \geq c(\lambda)\sqrt{n}\,\|T_\lambda\|^{\,n-1},$$

$c(\lambda) > 0$. The authors use a duality method to prove an analog of Bourgain's inequality and thereby estimate $\left\|T_\lambda^{-1}\right\|$ from below. Their lower bound on $\left\|T_\lambda^{-1}\right\|$ involves the l^∞ norm of the sequence

$$\widehat{(1-z^2)}b_\lambda^n(k) = \widehat{b_\lambda^n}(k) - \widehat{b_\lambda^n}(k-2), \qquad k \geq 2.$$

Better numerical estimates on $|\det T_\lambda| \left\|T_\lambda^{-1}\right\|$ can be obtained by considering more elaborate test functions than the simplest one they chose [43, Remark 10 and Remark 15]. Exact asymptotic expansions for $\widehat{b_\lambda^n}(k)$ are therefore of interest to derive numerical lower estimates on $\mathcal{S}/\sqrt{n}$ as $n \to \infty$.

1.2.2. *Strongly annular fonctions.* A function f analytic in the unit disc is said to be annular if there exists an embedded sequence of open domains Ω_n, $\Omega_n \subset \Omega_{n+1}$, $\overline{\Omega_n} \subset \mathbb{D}$, $n \geq 1$ such that $\cup_{n\geq 1}\Omega_n = \mathbb{D}$ and

$$\lim_{n\to\infty} \min_{\partial\Omega_n} |f| = \infty.$$

Such a function f does not belong to the Nevanlinna class, and, in particular, it does not belong to the Hardy space H^2, that is $\widehat{f} \notin \ell^2$. The function f is said to be strongly annular if it is annular with $\Omega_n = \mathcal{D}(0, r_n)$, $r_n \to 1$. The short book of Bonar [11] dedicated to this subject contains several constructions of such functions coming back, in particular, to Lusin–Privalov, 1925, Paley, 1930, and Littlewood, 1944.

Let us also mention here some more recent results on strongly annular functions. In 1997 Daquila [18] studied strongly annular solutions of Mahler's functional equation and in 2010 he studied [19] the density of such solutions in the space $\mathcal{H}(\mathbb{D})$ of the functions holomorphic in the unit disc. In 2007 Redett [37] constructed strongly annular functions in standard Bergman spaces. In 2013 Bernal–González–Bonilla [7] proved that the set of the strongly annular functions is algebraically large (maximal dense–lineable and algebrable in $\mathcal{H}(\mathbb{D})$). For random strongly annular functions see, for example, [28, Chapter 13, Theorem 7] and [26].

1.3. **Known results.**

1.3.1. *Estimates on* $\widehat{b^n_\lambda}(k)$.

Below we recall the known upper/lower bounds on $\widehat{b^n_\lambda}(k)$ as well as the known asymptotic formulas for these coefficients.

(1) D. D. Bonar, F. Carroll, and G. Piranian [13, Theorem 1] proved that there exist positive numbers A_1 and A_2 such that for all k and n the coefficients $\widehat{b^n_{1/2}}(k)$ satisfy the inequality

$$|\widehat{b^n_{1/2}}(k)| \le A_1 n^{-1/3}$$

and such that, for every nonnegative integer j,

$$\liminf_{n\to\infty} n^{1/3}|\widehat{b^n_{1/2}}(3k+j)| > A_2.$$

(2) It is also shown in [13, Theorem 2] that if $k < n/3$, then

$$|\widehat{b^n_{1/2}}(k)| \le \frac{6}{\pi}\frac{1}{n-3k},$$

and that if $k > 3n$, then

$$|\widehat{b^n_{1/2}}(k)| \le \frac{2}{\pi}\frac{1}{k-3n}.$$

(3) Szehr–Zarouf [42, Proposition 2] proved that if $\alpha \in (0, \alpha_0)$, $\alpha_0 := \frac{1-\lambda}{1+\lambda}$, then the following assertions hold for large enough n.

(a) If $k/n \le \alpha$, then $|\widehat{b^n_\lambda}(k)|$ decays exponentially as n tends to ∞, i.e. there exists $q \in (0,1)$ depending on α and λ only such that

$$|\widehat{b^n_\lambda}(k)| \le q^n.$$

Similarly, if $k/n \ge \alpha^{-1}$ then $|\widehat{b^n_\lambda}(k)|$ decays exponentially as n tends to ∞.

(b) If $k/n \in (\alpha, \alpha_0 - n^{-2/3}) \cup (\alpha_0^{-1} + n^{-2/3}, \alpha^{-1})$ then

$$|\widehat{b^n_\lambda}(k)| \lesssim \max\left\{\frac{1}{|\alpha_0 n - k|}, \frac{1}{|\alpha_0^{-1} n - k|}\right\}.$$

(c) If $k/n \in [\alpha_0 - n^{-2/3}, \alpha_0 + n^{-2/3}) \cup (\alpha_0^{-1} - n^{-2/3}, \alpha_0^{-1} + n^{-2/3}]$ then

$$|\widehat{b^n_\lambda}(k)| \lesssim \frac{1}{n^{1/3}}.$$

(d) If $k/n \in (\alpha_0 + n^{-2/3}, \alpha_0^{-1} - n^{-2/3})$ then

$$|\widehat{b^n_\lambda}(k)| \lesssim \max\left\{\frac{1}{n^{1/2}|\alpha_0 - \frac{k}{n}|^{1/4}}, \frac{1}{n^{1/2}|\alpha_0^{-1} - \frac{k}{n}|^{1/4}}\right\}.$$

(4) An asymptotic expansion of $\widehat{b^n_\lambda}(k)$ as k and n tend simultaneously to ∞ and k approaches the right boundary of $[\alpha_0 n, \alpha_0^{-1} n]$ from inside, i.e. $\lim_{n\to\infty}\left(\alpha_0^{-1} - k/n\right) = 0^+$, is computed in [42, Proposition 6]. In this region the asymptotic behavior of

$\widehat{b^n_\lambda}(k)$ can be written in terms of the Airy function $Ai(x)$. For real arguments the latter can be defined as an improper Riemann integral

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) \mathrm{d}t.$$

The authors in [42] were interested in the oscillatory behavior of Ai for large negative arguments for which we have the asymptotic approximation:

$$Ai(-x) \sim \frac{1}{x^{1/4}\sqrt{\pi}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right),\ x \to +\infty. \tag{1.2}$$

More precisely it is shown in [42, Proposition 6] (making use of a uniform version of the method of stationary phase, see, for example, [14, Section 2.3]) that for sequences $k = k(n)$ with $k \in [\alpha_0 n, \alpha_0^{-1} n]$ such that $\lim_{n\to\infty} \frac{k}{n} = \alpha_0^{-1}$, the following asymptotic formula holds as $n \to \infty$

$$\widehat{b^n_\lambda}(k) \sim \frac{(1-\lambda)^{1/4}}{(\lambda(1+\lambda))^{1/12}} \frac{\sqrt{2}}{\sqrt{k/n}\,(k/n - \alpha_0)^{1/4}} \frac{Ai(n^{2/3}\gamma^2)}{n^{1/3}},$$

where

$$\gamma^2 = \gamma^2_{\alpha_0^{-1}} \sim \frac{1-\lambda}{(\lambda(1+\lambda))^{1/3}} \left(k/n - \alpha_0^{-1}\right).$$

We will see that the above asymptotic formula for $\widehat{b^n_\lambda}(k)$ remains valid also when k/n approaches α_0^{-1} from outside of the compact interval $[\alpha_0, \alpha_0^{-1}]$, see below Theorem 1 (4). When $k/n > \alpha_0^{-1}$ and $\lim_{n\to\infty} \frac{k}{n} = \alpha_0^{-1}$ we will use the fact that the Airy function has exponential asymptotics for large positive arguments

$$Ai(x) \sim \frac{1}{2x^{1/4}\sqrt{\pi}} \exp\left(-\frac{2}{3}x^{3/2}\right),\ x \to +\infty. \tag{1.3}$$

Let us finally mention that in what follows, Theorem 1 (4) and Theorem 2 (1), (2), show a similar asymptotic formula for $\widehat{b^n_\lambda}(k)$ as k/n approaches the left boundary α_0 (both from the left and the right).

1.3.2. *Strongly annular fonctions.* Most of the known examples of strongly annular functions involve lacunary series. Frequently, the Taylor coefficients of the functions in such examples are unbounded. That is why Bonar asked in [11, Question 6.9] whether every strongly annular function is a sum of a bounded function and the sum of a lacunary Taylor series. In 1977 Bonar, Carroll, and Piranian [13] constructed a strongly annular function f such that $\lim_{n\to\infty} \widehat{f}(n) = 0$ and

$$\sum_{n\geq 0} \min(|\widehat{f}(2n)|^2, |\widehat{f}(2n+1)|^2) = \infty.$$

In other words, if s_k are positive integers, $s_{k+1} > s_k + 1$, then

$$\sum_{n\geq 0,\, n\notin(s_k)} \widehat{f}(n) z^n \notin H^2.$$

This construction was based on the above mentioned estimates of the asymptotics of the Taylor coefficients of b_λ^n.

Another construction of strongly annular functions whose Taylor coefficients tend to 0 was given by Bonar, Carroll, and Erdös in [12].

1.4. **Goals of the paper.**

1.4.1. *Asymptotic analysis of $\widehat{b_\lambda^n}(k)$ as $n \to \infty$.*

The first goal of this paper is to state all asymptotic formulas for $\widehat{b_\lambda^n}(k)$ as $n \to \infty$ depending on the region to which $k = k(n) \in [0, \infty)$ belongs. The above mentioned upper bounds on $\widehat{b_\lambda^n}(k)$ are usually based on van der Corput type estimates, and the standard *Laplace-type methods* which we describe below, will be used to derive exact asymptotic formulas for $\widehat{b_\lambda^n}(k)$ as $n \to \infty$. We write the integral defining $\widehat{b_\lambda^n}(k)$ in a way that is convenient for asymptotic analysis:

$$\widehat{b_\lambda^n}(k) = \frac{1}{2i\pi} \int_{\partial\mathbb{D}} e^{n\Phi(z)} \frac{\mathrm{d}z}{z} \tag{1.4}$$

(the so-called complex Laplace-type integral) where

$$\Phi(z) = \Phi_{k/n}(z) = \log\left(z^{-\frac{k}{n}} b_\lambda(z)\right), \tag{1.5}$$

and log denotes a branch of the complex logarithm chosen in the following way: if $k/n \leq c < \alpha_0^{-1}$, then we can take the branch cut $[0, \infty)$ and fix $\log(-1) = i\pi$, and if $k/n \geq c > \alpha_0$, then we can take the principal branch of the complex logarithm. In particular, if $\alpha_0 < c_1 \leq k/n \leq c_2 < \alpha_0^{-1}$, then we could take either of these two definitions. The asymptotic behavior of this integral is studied using standard tools of asymptotic analysis: the *method of stationary phase* [20, 23, 22, 14] or the *method of the steepest descent* [8, 15, 17, 44], depending on the location of the critical points of Φ. To apply the method of stationary phase we need to introduce the real function

$$h(\varphi) = h_{k/n}(\varphi) := -i\Phi_{k/n}(e^{i\varphi}) = \arg\left(\left(z^{-k/n} b_\lambda(z)\right)_{|z=e^{i\varphi}}\right) \qquad \varphi \in [0, \pi], \tag{1.6}$$

observing that $\left|z^{-k/n} b_\lambda(z)\right| = 1$ for $z \in \partial\mathbb{D}$, so that

$$\widehat{b_\lambda^n}(k) = \frac{1}{\pi} \Re\left\{\int_0^\pi e^{inh_{k/n}(\varphi)} \mathrm{d}\varphi\right\}. \tag{1.7}$$

As usual, the dominant contribution to integrals of the form (1.4) (respectively (1.7)) comes from a small neighborhood around the stationary points of Φ (respectively h). We refer to Lemma 5 below for an identification of the critical points of Φ which we denote by $z_\pm$, see also [41, Section 6]. It turns out that when $a = k/n \in [\alpha_0, \alpha_0^{-1}]$ we have $z_\pm \in \partial\mathbb{D}$ and the integral (1.4) is especially suited for an application of the method of stationary phase [20, 23, 22], whereas if $a = k/n \notin [\alpha_0, \alpha_0^{-1}]$, then $z_\pm \in \mathbb{R}$ and this method fails. In this case, a deformation of the contour $\partial\mathbb{D}$ will be required in order to apply the method of the steepest descent. As k/n approaches one of the boundaries α_0 or α_0^{-1}, uniform versions of these methods [14, Section 2.3] [8, Section 9.2] [46, p. 366–372] (all of them being based on [16]) will be required, see Proposition 3 below. A summary of the asymptotics of $\widehat{b_\lambda^n}(k)$

is provided in Figure 2.1 below, depending on k. The asymptotic formulas for $\widehat{b^n_\lambda}(k)$ are discussed in full detail in Section 2, see Theorem 1 and Theorem 2.

1.4.2. *Strongly annular functions.* Using the ideas from [13] and [12] and estimates on the asymptotics of $\widehat{b^n_\lambda}(k)$ obtained in our paper, we construct strongly annular functions f such that (a) $\widehat{f}$ belongs to $\ell^q \setminus \ell^p$ for any given $2 \leq p < q$ or (b) $\widehat{f}$ belongs to $\ell^2_\varphi \setminus \ell^2$ where ℓ^2_φ is the set of sequences (a_n) such that

$$\sum_{n\geq 0} |a_n|^2/\varphi(1/|a_n|) < \infty,$$

and φ is such that $\lim_{t\to\infty} \varphi(t) = \infty$. Furthermore, the functions f we construct are not lacunary in the sense that if (s_k) is a sequence of positive integers such that $s_{k+1} > s_k + 1$, then $\widehat{f} \cdot \chi_{\mathbb{Z}_+\setminus(s_k)} \notin \ell^p$ and $\widehat{f} \cdot \chi_{\mathbb{Z}_+\setminus(s_k)} \notin \ell^2$, correspondingly, in the cases (a) and (b).

1.5. **Outline of the paper.** In Section 2 below, we state asymptotic formulas for $\widehat{b^n_\lambda}(k)$ as $n \to \infty$. We distinguish seven regions of k where the asymptotic behavior of $\widehat{b^n_\lambda}(k)$ differs. Given $\alpha \in [\epsilon, \alpha_0)$ we compute an asymptotic formula for $\widehat{b^n_\lambda}(k)$ when $k \in [0, \alpha n] \cup [n/\alpha, \infty]$ and thereby sharpen the known fact asserting that $\widehat{b^n_\lambda}(k)$ decays exponentially for k in those regions, see Theorem 1 below. Given $\beta \in (\alpha_0, \alpha_0^{-1})$ we find that for $k \in [\beta n, n/\beta]$ the asymptotic of $\widehat{b^n_\lambda}(k)$ is oscillatory and witnesses a decay of order $\mathcal{O}(n^{-1/2})$, see Theorem 2 (2) below. We also compute an asymptotic formula for $\widehat{b^n_\lambda}(k)$ as k and n tend simultaneously to ∞ and k approaches the boundaries $\alpha_0 n$, $\alpha_0^{-1} n$. In these regions the asymptotic behavior of $\widehat{b^n_\lambda}(k)$ is described in terms of the Airy function $Ai(x)$, see Theorem 1 (3), (4), Theorem 2 and Proposition 3 for more details. We end Section 2 summing up $\widehat{b^n_\lambda}(k)$'s asymptotics depending on the region where k belongs, see Figure 2.1 below. The proofs of Theorem 1, Theorem 2 and Proposition 3 are collected in Section 3. In Section 4 we give two constructions of strongly annular functions with small Taylor coefficients in Theorems 6 and 7. These constructions are based on auxiliary Lemmas 8 and 9 concerning, correspondingly, properties of $b^n_{1/2}$ and flat polynomials.

2. Asymptotic formulas for $\widehat{b^n_\lambda}(k)$

It is known [42, Proposition 2], [40, Lemma 7] that given $\alpha < \alpha_0$, $\widehat{b^n_\lambda}(k)$ decays exponentially for $k \in [0, \alpha n] \cup [\alpha^{-1} n, +\infty)$ as n tends to $+\infty$. Theorem 1 below sharpens the previous results in [13, Theorem 2], [42, Proposition 2], [40, Lemma 7] by stating asymptotic formulas for $\widehat{b^n_\lambda}(k)$ as n tends to $+\infty$ when k belongs to those regions:

(1) If k is fixed (Region I), then the proof of the asymptotic formula for $\widehat{b^n_\lambda}(k)$ follows by induction on k.
(2) If $k = k(n) \to \infty$ as $n \to \infty$ with $k \leq \alpha n$ (Region II) or $k \geq \alpha^{-1} n$ (Region VIII), then the integral defining $\widehat{b^n_\lambda}(k)$ is treated by a direct application of the method of the steepest descent [8, Chapter 7], [17, Chapters 7-8], [15, Chapter 5], [44, Chapter 4], which we will use intensively in our proof.

(3) If $k \in [\alpha n, \alpha_0 n - n^{1/3})$ and in addition $n^{2/3}(\alpha_0 - k/n) \to +\infty$ (Region III) or if $k \in [\alpha_0^{-1} n + n^{1/3}, \alpha^{-1} n]$ and in addition $n^{2/3}(k/n - \alpha_0^{-1}) \to +\infty$ (Region VII), then a uniform version of the steepest descent method based on [16], see [8, Section 9.2], [46, p. 366–372], is required to obtain the asymptotic formula for $\widehat{b_\lambda^n}(k)$. More precisely, the proof of our asymptotic formulas for k in Regions III and VII will follow from an application of Proposition 3 (stated below) together with the approximation (1.3) of the Airy function for large positive arguments.

Our asymptotic formulas witnessing exponential decay of $\widehat{b_\lambda^n}(k)$ for k in Regions I-II-III-VII-VIII are sharp, new and agree on the intersections of Regions I-II, II-III and VII-VIII: We refer to the comments below just afer the statement of Theorem 1 for a detailed discussion, where we also compare our results to the previous upper estimates from [42, Proposition 2]. We recall that the value of α_0 is given by $\alpha_0 = \frac{1-\lambda}{1+\lambda}$ and that Φ is defined according to (1.5) by

$$\Phi(z) = \Phi_{k/n}(z) = \log\left(z^{-\frac{k}{n}} b_\lambda(z)\right).$$

Theorem 1. *Let $\alpha \in (0, \alpha_0)$. Consider a sequence $\omega(n^{1/3})$ such that $\omega(n^{1/3})/n^{1/3} \to \infty$ as $n \to \infty$ and assume additionally that $\omega(n^{1/3}) = o(n)$ as $n \to \infty$. The following asymptotic formulas for the $k^{\text{th}}-$Fourier coefficients of b_λ^n hold as n tends to $+\infty$.*

(1) *If k is fixed (Region I), then*

$$\widehat{b_\lambda^n}(k) \sim \frac{(-\lambda)^{n-k}\left(n(1-\lambda^2)\right)^k}{k!}.$$

(2) *If $k = k(n) \to \infty$ as $n \to \infty$ with $k \leq \alpha n$ (Region II) or $k \geq \alpha^{-1} n$ (Region VIII), then*

$$\widehat{b_\lambda^n}(k) \sim \frac{1}{\sqrt{2k\pi}} \frac{1}{\left[(\alpha_0 - k/n)(\alpha_0^{-1} - k/n)\right]^{1/4}} \left(\frac{b_\lambda(z_+)}{z_+^{k/n}}\right)^n,$$

where z_+ is defined by

$$z_+ = z_+(k/n) = \frac{\frac{k}{n}(1+\lambda^2) - (1-\lambda^2)}{2\lambda\frac{k}{n}} + \sqrt{\left(\frac{\frac{k}{n}(1+\lambda^2) - (1-\lambda^2)}{2\lambda\frac{k}{n}}\right)^2 - 1}. \tag{2.1}$$

(3) *If $k \in [\alpha n, \alpha_0 n - \omega(n^{1/3})]$ (Region III), then*

$$\widehat{b_\lambda^n}(k) \sim \frac{(-1)^{n-k}}{\sqrt{2n\pi}} \frac{1}{\sqrt{k/n}\left[(\alpha_0^{-1} - k/n)(\alpha_0 - k/n)\right]^{1/4}} \exp\left(-\frac{2}{3} n |\gamma_{\alpha_0}|^3\right),$$

where $\gamma_{\alpha_0}^3$ is given by

$$\gamma_{\alpha_0}^3 = \frac{3}{2}\left[\Phi(z_+) - i\pi\left(1 - \frac{k}{n}\right)\right], \tag{2.2}$$

and in particular

$$\gamma_{\alpha_0}^3 \sim -\frac{(\alpha_0 - k/n)^{3/2}(1+\lambda)^{3/2}}{(\lambda(1-\lambda))^{1/2}}, \qquad k/n \to \alpha_0, \qquad k/n < \alpha_0. \tag{2.3}$$

(4) *If* $k \in [\alpha_0^{-1}n + \omega(n^{1/3}), \alpha^{-1}n]$ *(Region VII), then*

$$\widehat{b_\lambda^n}(k) \sim \frac{1}{\sqrt{2n\pi}} \frac{1}{\sqrt{k/n}\left[(k/n - \alpha_0^{-1})(k/n - \alpha_0)\right]^{1/4}} \exp\left(-\frac{2}{3}n|\gamma_{\alpha_0^{-1}}|^3\right),$$

where $\gamma^3_{\alpha_0^{-1}}$ *is given by*

$$\gamma^3_{\alpha_0^{-1}} = \frac{3}{2}\Phi(z_+), \tag{2.4}$$

and in particular

$$\gamma^3_{\alpha_0^{-1}} \sim -\frac{(k/n - \alpha_0^{-1})^{3/2}(1-\lambda)^{3/2}}{(\lambda(1+\lambda))^{1/2}}, \qquad k/n \to \alpha_0^{-1}, \qquad k/n > \alpha_0^{-1}. \tag{2.5}$$

We proceed with a series of remarks and observations highlighting the coincidence of our formulas for k on the intersections of Regions I-II, II-III and VII-VIII, and comparing our results to [42, Proposition 3 (1)] and [42, Proposition 3(2)].

(1) The asymptotic formula stated for k in Region II agrees with the one for k in Region I. Indeed, a direct computation shows that if k is fixed, then

$$z_+^{-k} \sim (-1)^k \frac{(n(1-\lambda^2))^k}{k^k\lambda^k}, \qquad n \to \infty,$$

and

$$(b_\lambda(z_+))^n \sim (-1)^n e^k \lambda^n, \qquad n \to \infty.$$

The coincidence of the asymptotics follows from an application of Stirling's formula: if $k = o(n)$, then we have

$$\widehat{b_\lambda^n}(k) \sim \frac{1}{\sqrt{2k\pi}} \frac{b_\lambda(z_+)^n}{z_+^k} \sim \frac{(-\lambda)^{n-k}}{\sqrt{2k\pi}} \frac{(n(1-\lambda^2))^k}{k^k e^{-k}} \sim (-\lambda)^{n-k} \frac{(n(1-\lambda^2))^k}{k!}.$$

(2) Theorem 1 (1), (2) sharpens the result from [42, Proposition 3 (1)] for $k \in [0, \alpha n] \cup [\alpha^{-1}n, \infty)$. The latter asserts that $\widehat{b_\lambda^n}(k)$ decays exponentially and uniformly for $k \le \alpha n$ respectively $k \ge \alpha^{-1}n$. We observe that since the function $a \mapsto \frac{|b_\lambda(z_+(a))|}{|z_+(a)|^a}$ is increasing on the interval $[0, \alpha_0]$, we have

$$\frac{|b_\lambda(z_+(k/n))|}{|z_+(k/n)|^{k/n}} \le \frac{|b_\lambda(z_+(\alpha))|}{|z_+(\alpha)|^\alpha} < 1,$$

and therefore Theorem 1 gives that

$$|\widehat{b_\lambda^n(k)}| \le \frac{C}{\sqrt{k}} \left(\frac{|b_\lambda(z_+(\alpha))|}{|z_+(\alpha)|^\alpha}\right)^n$$

uniformly for $k \in [0, \alpha n]$, where $C = C(\lambda, \alpha) > 0$. A similar argument for $k \in [\alpha^{-1}n, \infty)$ (Region VIII) leads to the same conclusion.

(3) The formulation of Theorem 1 (3), (4) includes the number $\gamma^3 \in \left\{\gamma^3_{\alpha_0}, \gamma^3_{\alpha_0^{-1}}\right\}$ whose value is given by [8, formula (9.2.9), p. 370]:

$$\gamma^3 = \frac{3}{4}\left[\Phi(z_+) - \Phi(z_-)\right] \tag{2.6}$$

where z_+ is defined by (2.1) and

$$z_- = z_-(k/n) = \frac{\frac{k}{n}(1+\lambda^2)-(1-\lambda^2)}{2\lambda\frac{k}{n}} - \sqrt{\left(\frac{\frac{k}{n}(1+\lambda^2)-(1-\lambda^2)}{2\lambda\frac{k}{n}}\right)^2 - 1}.$$

Formulas (2.2), (2.3), (2.4), and (2.5) all follow from the above definition (2.6) of γ^3.

(4) The formulas stated for k belonging to Regions II–III–VII–VIII coincide. Assuming $k/n < \alpha_0$ (Region II) – respectively $k/n > \alpha_0^{-1}$ (Region VI) – we have $\gamma^3_{\alpha_0} < 0$ – respectively $\gamma^3_{\alpha_0^{-1}} < 0$. Therefore

$$\begin{aligned}\exp\left(-\frac{2}{3}n|\gamma_{\alpha_0}|^3\right) &= \exp\left(n\left(\Phi(z_+) - i\pi\left(1-\frac{k}{n}\right)\right)\right)\\ &= (-1)^{k-n}\left(\frac{b_\lambda(z_+)}{z_+^{k/n}}\right)^n\end{aligned}$$

and

$$\begin{aligned}\exp\left(-\frac{2}{3}n|\gamma_{\alpha_0^{-1}}|^3\right) &= \exp\left(n(\Phi(z_+))\right)\\ &= \left(\frac{b_\lambda(z_+)}{z_+^{k/n}}\right)^n,\end{aligned}$$

which shows that our asymptotic formulas in these four regions are actually the same.

(5) The asymptotic formulas stated for k in Regions III and VII, see Theorem 1 (3), (4), significantly improve the estimate from [42, Proposition 2, (2)] where the decay of $\widehat{b^n_\lambda}(k)$ is only shown to be

$$\mathcal{O}\left(\max\left\{\frac{1}{|\alpha_0 n - k|}, \frac{1}{|\alpha_0^{-1}n - k|}\right\}\right).$$

The following result, Theorem 2 below, establishes asymptotic formulas for $\widehat{b^n_\lambda}(k)$ as n tends to $+\infty$ when k belongs to the remaining regions where it turns out that the decay of $\widehat{b^n_\lambda}(k)$ is no longer exponential but either of order $\mathcal{O}(n^{-1/3})$ or oscillatory and of order $\mathcal{O}(n^{-1/2})$. More precisely:

(1) Airy-type behaviour for $\widehat{b^n_\lambda}(k)$ near the k-transition points $n\alpha_0$ and $n\alpha_0^{-1}$ is established, which extends the formula from [42, Proposition 6] to the case $k > \alpha_0^{-1}n$ and generalizes it to the left boundary $\alpha_0 n$ (for k both from the left and from the right of $\alpha_0 n$). Our asymptotic formulas are respectively given below for k near $n\alpha_0$ (Region IV) see Theorem 2 (1), and for k near $n\alpha_0^{-1}$ (Region VI), see Theorem 2(3), asserting that the decay of $\widehat{b^n_\lambda}(k)$ for k in these regions is of order $\mathcal{O}(n^{-1/3})$ at least when k lies in neighbourhoods of the boundaries $\alpha_0 n$, $\alpha_0^{-1}n$ of length proportional to $n^{1/3}$. For k in those neighborhoods, the quantity $n^{2/3}\gamma^2$ is always bounded in n. The main tool to prove Theorem 2 (1), (3) is the uniform version of the steepest

descent method based on [16] already mentioned above, which we apply following [8, Section 9.2] to prove Proposition 3 as an intermediate step.

(2) If k lies in the remaining central region, $(\alpha_0 n+n^{1/3}, \alpha_0^{-1}n-n^{1/3})$, and if in addition $n^{2/3}(k/n-\alpha_0)\to+\infty$ or $n^{2/3}(\alpha_0^{-1}-k/n)\to+\infty$, we find that the decay of $\widehat{b^n_\lambda}(k)$ is oscillatory and of order $\mathcal{O}(n^{-1/2})$. The corresponding asymptotic formula is stated below, see Theorem 2 (2). To prove the latter for $k/n\to\alpha_0$ or $k/n\to\alpha_0^{-1}$ we choose $\beta\in(\alpha_0,1)$ close enough to α_0, and combine a uniform version of the method of stationary phase [14, Section 2.3] (again based on [16], see the proof of Proposition 3 below) with the approximation (1.2) of the Airy function for large negative arguments. The proof of Theorem 2 (2) for k in the remaining interval $[\beta n,\beta^{-1}n]$ follows from an application of the standard version of the stationary phase method [20, Theorem 4]. The proof is however rather long and technical, and we will use a more elaborate version of this method due to M.V. Fedoryuk [22, Theorem 2.4 p. 80], [23, Theorem 1.6 p.107], which will make the argument much shorter, see Section 3.3.2 for more details.

The asymptotic approximations (1.3) – respectively (1.2) – for large positive – respectively negative – arguments of the Airy function, show that our asymptotic formulas coincide for k at the intersection of Regions III-IV, IV-V, V-VI and VI-VII.

Theorem 2. *Let $\omega(n^{1/3})$ be a sequence such that $\omega(n^{1/3})/n^{1/3}\to\infty$ as $n\to\infty$. We assume in addition that $\omega(n^{1/3})=o(n)$ as $n\to\infty$. The following asymptotic formulas for the k^{th} Fourier coefficients of b^n_λ hold as n tends to $+\infty$.*

(1) *If $k\in[\alpha_0 n-\omega(n^{1/3}),\alpha_0 n+\omega(n^{1/3})]$ (Region IV), then*

$$\widehat{b^n_\lambda}(k)\sim\frac{(-1)^{n-k}\sqrt{2}}{n^{1/3}}\frac{(1+\lambda)^{1/4}}{(\lambda(1-\lambda))^{1/12}}\frac{1}{\sqrt{k/n}(\alpha_0^{-1}-k/n)^{1/4}}Ai\left(n^{2/3}\gamma^2_{\alpha_0}\right),$$

where $\gamma^2_{\alpha_0}$ is asymptotically given by

$$\gamma^2_{\alpha_0}\sim\frac{(\alpha_0-k/n)\,(1+\lambda)}{(\lambda(1-\lambda))^{1/3}},\qquad k/n\to\alpha_0. \tag{2.7}$$

(2) *If $k\in[\alpha_0 n+\omega(n^{1/3}),\alpha_0^{-1}n-\omega(n^{1/3})]$ (Region V), then*

$$\widehat{b^n_\lambda}(k)\sim\sqrt{\frac{2}{n\pi}}\frac{\cos\left(nh(\varphi_+)-\pi/4\right)}{\sqrt{k/n}\left[(\alpha_0^{-1}-k/n)(k/n-\alpha_0)\right]^{1/4}},$$

where $h=h_{k/n}$ is defined in (1.6) and the parameter $\varphi_+\in[0,\pi]$ is defined by

$$e^{i\varphi_+}=z_+=\frac{\frac{k}{n}(1+\lambda^2)-(1-\lambda^2)}{2\lambda\frac{k}{n}}+i\sqrt{1-\left(\frac{\frac{k}{n}(1+\lambda^2)-(1-\lambda^2)}{2\lambda\frac{k}{n}}\right)^2}.$$

(3) *If $k\in[\alpha_0^{-1}n-\omega(n^{1/3}),\alpha_0^{-1}+\omega(n^{1/3})]$ (Region VI), then*

$$\widehat{b^n_\lambda}(k)\sim\frac{\sqrt{2}}{n^{1/3}}\frac{(1-\lambda)^{1/4}}{(\lambda(1+\lambda))^{1/12}}\frac{1}{\sqrt{k/n}(k/n-\alpha_0)^{1/4}}Ai\left(n^{2/3}\gamma^2_{\alpha_0^{-1}}\right),$$

where

$$\gamma^2_{\alpha_0^{-1}} \sim \frac{(k/n-\alpha_0^{-1})(1-\lambda)}{(\lambda(1+\lambda))^{1/3}}, \qquad k/n \to \alpha_0^{-1}. \tag{2.8}$$

The formulas given in Theorem 2 (1) respectively (3) are actually valid for k/n in a *fixed* neighbourhood of α_0 respectively α_0^{-1}. In fact, they hold more generally if $k \in [\alpha n, \beta n]$ respectively $k \in [\beta^{-1}n, \alpha^{-1}n]$ as long as $\alpha \in (0, \alpha_0)$ and $\beta \in (\alpha_0, 1)$ are chosen close enough to α_0. This is the content of Proposition 3 below, which entirely describes the Airy-type behaviour of $\widehat{b^n_\lambda}(k)$ near the k-transition points $n\alpha_0$ and $n\alpha_0^{-1}$. It is a modified version of [43, Proposition 17] where only upper bounds were stated and where the factor $(1-z^2)$ has been replaced by 1. The main tool to prove Proposition 3 is a result from [16], which we apply following [8, Section 9.2].

Proposition 3. *Fix $\alpha \in (0, \alpha_0)$ and $\beta \in (\alpha_0, 1)$. Suppose that α and β are close enough to α_0. If $k/n \in [\alpha, \beta]$, then*

$$\widehat{b^n_\lambda}(k) \sim_{n\to\infty} (-1)^{n-k} \sqrt{\frac{2|\gamma|}{k/n}} \frac{1}{|\Delta|^{1/4}} \frac{Ai(n^{2/3}\gamma^2)}{n^{1/3}},$$

where $\Delta = (k/n - \alpha_0)(\alpha_0^{-1} - k/n)$ and $\gamma^2 = \gamma^2_{\alpha_0}$ is asymptotically given by (2.7) as $k/n \to \alpha_0$. If $k/n \in [\beta^{-1}, \alpha^{-1}]$, then

$$\widehat{b^n_\lambda}(k) \sim_{n\to\infty} \sqrt{\frac{2|\gamma|}{k/n}} \frac{1}{|\Delta|^{1/4}} \frac{Ai(n^{2/3}\gamma^2)}{n^{1/3}},$$

where $\gamma^2 = \gamma^2_{\alpha_0^{-1}}$ is asymptotically given by (2.8) as $k/n \to \alpha_0^{-1}$.

Remark. The factor $(-1)^{n-k}$ in the first formula of Proposition 3 corresponds to that in Theorem 1 for k in Regions I and II. Indeed, the Airy function is positive in a neighborhood of 0, and for $k \in [0, \alpha n]$ (Region I) the sign of the factor $\left(\frac{b_\lambda(z_+)}{z_+^{k/n}}\right)^n$ is $(-1)^{n-k}$ because $z_+ = z_+(k/n)$ is negative.

Proposition 3 shows in particular that:

(1) For $k \in [\alpha n, \alpha_0 n - n^{1/3})$ respectively $k \in [\alpha_0^{-1}n + n^{1/3}, \alpha^{-1}n]$ such that $n^{2/3}(\alpha_0 - k/n) \to +\infty$ respectively $n^{2/3}(k/n - \alpha_0^{-1}) \to +\infty$, we use (2.7) respectively (2.8) to observe that $n^{2/3}\gamma^2 \to +\infty$ and then use the asymptotic approximation (1.3) for large positive arguments of the Airy function, to deduce the precise nature of $\widehat{b^n_\lambda}(k)$'s exponential decay in these regions, see Theorem 1 (3), (4) above.

(2) For $k \in (\alpha_0 n + n^{1/3}, \beta n] \cup [\beta^{-1}n, \alpha_0^{-1}n - n^{1/3})$ such that either $n^{2/3}(k/n - \alpha_0) \to +\infty$ or $n^{2/3}(\alpha_0^{-1} - k/n) \to +\infty$, we use (2.7) and (2.8) to observe that $n^{2/3}\gamma^2 \to -\infty$ and then apply the asymptotic approximation (1.2) for large negative arguments of the Airy function, which shows that the decay of $\widehat{b^n_\lambda}(k)$ is oscillatory in these regions, see Theorem 2 (2) above.

2.1. **Summary of $\widehat{b^n_\lambda}(k)$'s asymptotics.** The table below, see Figure 2.1, shows values of $A(n,k)$ such that

$$\widehat{b^n_\lambda}(k) \asymp A(n,k)$$

depending on the region to which k belongs. Again we use Landau standard notation and denote by $\omega(n^{1/3})$ a sequence such that $\omega(n^{1/3})/n^{1/3} \to \infty$ as $n \to \infty$. We assume in addition that $\omega(n^{1/3}) = o(n)$ as $n \to \infty$. The numbers γ_{α_0} and $\gamma_{\alpha_0{}^{-1}}$ are asymptotically given by

$$\gamma^2_{\alpha_0} \asymp \alpha_0 - k/n \qquad \text{and} \qquad \gamma^2_{1/\alpha_0} \asymp k/n - 1/\alpha_0$$

respectively as $k/n \to \alpha_0$ and α_0^{-1}. The table shows that the asymptotic behavior of $\widehat{b^n_\lambda}(k)$ is symmetric with respect to Region V. A possible explanation for that symmetry is due to the following observation, which is a consequence of a simple change of variable.

Proposition 4. *Given $\lambda \in (0,1)$, $k \geq 1$, and $n \geq 1$, the following identity holds*

$$\widehat{b^n_\lambda}(k) = \frac{(-1)^{n-k}}{2i\pi} \int_{\partial\mathbb{D}} \tilde{\varphi}(z) \exp\left(k\tilde{\Phi}(z)\right) \mathrm{d}z$$

where

$$\tilde{\varphi}(z) = \frac{1}{z} \frac{1-\lambda^2}{|1-\lambda z|^2} \qquad \text{and} \qquad \tilde{\Phi}(z) = \log\left(\frac{b_\lambda(z)}{z^{\frac{n}{k}}}\right) = \Phi_{n/k}(z),$$

$\Phi_{n/k}$ *being defined by* (1.5).

Proof. We first write

$$\begin{aligned}
\widehat{b^n_\lambda}(k) &= \frac{1}{2i\pi} \int_{\partial\mathbb{D}} (b_\lambda(z))^n \, z^{-k-1} \mathrm{d}z \\
&= \frac{(-1)^n}{2i\pi} \int_{\partial\mathbb{D}} \left(\widetilde{b_\lambda}(z)\right)^n z^{-k-1} \mathrm{d}z
\end{aligned}$$

where $\widetilde{b_\lambda}(z) = -b_\lambda(z) = \frac{\lambda - z}{1-\lambda z}$ satisfies $\widetilde{b_\lambda} \circ \widetilde{b_\lambda} = \mathrm{id}$. Changing the variable z by $u = \widetilde{b_\lambda}(z)$ we get $z = \widetilde{b_\lambda}(u)$, $\mathrm{d}z = -\frac{1-\lambda^2}{(1-\lambda u)^2}\mathrm{d}u$ and therefore

$$\begin{aligned}
\widehat{b^n_\lambda}(k) &= -\frac{(-1)^n}{2i\pi} \int_{\partial\mathbb{D}} u^n \left(\widetilde{b_\lambda}(u)\right)^{-k-1} \frac{1-\lambda^2}{(1-\lambda u)^2} \mathrm{d}u \\
&= \frac{(-1)^{n-k}}{2i\pi} \int_{\partial\mathbb{D}} u^n \left(b_\lambda(u)\right)^{-k-1} \frac{1-\lambda^2}{(1-\lambda u)^2} \mathrm{d}u. \\
&= \frac{(-1)^{n-k}}{2\pi} \int_{\partial\mathbb{D}} u^{n+1} \left(b_\lambda(u)\right)^{-k-1} \frac{1-\lambda^2}{(1-\lambda u)^2}|_{u=e^{it}} \mathrm{d}t
\end{aligned}$$

Taking into account the fact that $\widehat{b_\lambda^n}(k)$ is real and using complex conjugation we find

$$\begin{aligned}
\widehat{b_\lambda^n}(k) &= \frac{(-1)^{n-k}}{2\pi} \int_{\partial\mathbb{D}} \left(b_\lambda(u)\right)^{k+1} u^{-n-1} u^2 \frac{1-\lambda^2}{(u-\lambda)^2}|_{u=e^{it}} \mathrm{d}t \\
&= \frac{(-1)^{n-k}}{2\pi} \int_{\partial\mathbb{D}} \left(b_\lambda(u)\right)^{k} u^{-n+1} \frac{1-\lambda^2}{(u-\lambda)(1-\lambda u)}|_{u=e^{it}} \mathrm{d}t \\
&= \frac{(-1)^{n-k}}{2i\pi} \int_{\partial\mathbb{D}} \frac{1-\lambda^2}{|1-\lambda u|^2} \left(b_\lambda(u)\right)^{k} u^{-n} \frac{\mathrm{d}u}{u},
\end{aligned}$$

which completes the proof. □

Values of $k(n)$ in interval	Asymptotics of $\widehat{b^n_\lambda}(k)$	Region
$[0,\, \alpha n]$	$\frac{1}{\sqrt{k/n}\big[(\alpha_0 - k/n)(\alpha_0^{-1} - k/n)\big]^{1/4}} \frac{1}{\sqrt{n}} \left(\frac{b_\lambda(z_+)}{z_+^{k/n}} \right)^n$	I-II
$(\alpha n,\, \alpha_0 n - \omega(n^{1/3})]$	$\frac{1}{\sqrt{k/n}\big[(\alpha_0^{-1} - k/n)(\alpha_0 - k/n)\big]^{1/4}} \frac{1}{\sqrt{n}} e^{-\frac{2}{3} n \lvert \gamma_{\alpha_0{}^{-1}} \rvert^3}$	III
$[\alpha_0 n - \omega(n^{1/3}),\, \alpha_0 n + \omega(n^{1/3})]$	$\frac{1}{\sqrt{k/n}(\alpha_0^{-1} - k/n)^{1/4}} \frac{Ai(n^{2/3}\gamma^2_{\alpha_0})}{n^{1/3}}$	IV
$[\alpha_0 n + \omega(n^{1/3}),\, \alpha_0^{-1} n - \omega(n^{1/3})]$	$\frac{1}{\sqrt{n}} \frac{\cos(nh(\varphi_+) - \pi/4)}{\sqrt{k/n}\big[(\alpha_0^{-1} - k/n)(k/n - \alpha_0)\big]^{1/4}}$	V
$[\alpha_0^{-1} n - \omega(n^{1/3}),\, \alpha_0^{-1} n + \omega(n^{1/3})]$	$\frac{1}{\sqrt{k/n}(k/n - \alpha_0)^{1/4}} \frac{Ai(n^{2/3}\gamma^2_{\alpha_0{}^{-1}})}{n^{1/3}}$	VI
$[\alpha_0^{-1} n + \omega(n^{1/3}),\, \alpha^{-1} n)$	$\frac{1}{\sqrt{k/n}\big[(k/n - \alpha_0^{-1})(k/n - \alpha_0)\big]^{1/4}} \frac{1}{\sqrt{n}} e^{-\frac{2}{3} n \lvert \gamma_{\alpha_0{}^{-1}} \rvert^3}$	VII
$[\alpha^{-1} n,\, \infty)$	$\frac{1}{\sqrt{k/n}\big[(\alpha_0 - k/n)(\alpha_0^{-1} - k/n)\big]^{1/4}} \frac{1}{\sqrt{n}} \left(\frac{b_\lambda(z_+)}{z_+^{k/n}} \right)^n$	VIII

FIGURE 2.1. Asymptotic formulas for $\widehat{b^n_\lambda}(k)$ as $n \to \infty$, up to numerical factors. For k in Regions I–II and VIII, we have $|z_+^{-k/n} b_\lambda(z_+)| < 1$ and the decay of $\widehat{b^n_\lambda}(k)$ is exponential. The values γ_{α_0} and $\gamma_{\alpha_0{}^{-1}}$ are asymptotically given by $\gamma^2_{\alpha_0} \asymp \alpha_0 - k/n$ and $\gamma^2_{\alpha_0^{-1}} \asymp k/n - \alpha_0^{-1}$ respectively as $k/n \to \alpha_0$ and α_0^{-1}. The formulas for k in Regions III and VII ensure the transition between the exponential decay (Regions I–II and VIII) and the $\mathcal{O}(n^{-1/3})$ decay, which occurs in Regions IV and VI when the distance between k and $\alpha_0 n$ respectively $\alpha_0^{-1} n$ does not exceed $n^{1/3}$. Finally, the formula for k in Region V ensures the transition to an oscillatory decay of order $\mathcal{O}(n^{-1/2})$ when k is away from the boundaries $\alpha_0 n$ and $\alpha_0^{-1} n$ (we refer to Theorem 2 (2) for the definition of $h(\varphi_+)$).

3. Proofs of the asymptotic formulas for $\widehat{b^n_\lambda}(k)$

As usual, the dominant contribution to integrals of the form (1.4) comes from a small neighborhood around the stationary points of Φ_a. Therefore we start by recalling the critical points of Φ_a. The following lemma is a more complete version of [41, Lemma 11]. We prove it below for completeness.

Lemma 5. *Let $a = k/n$ and let $\Phi(z) = \Phi_a(z)$ be defined as above. We have the following assertions.*

(1) *If $a \in (\alpha_0, \alpha_0^{-1})$, then $\Phi_a(\cdot)$ has two distinct stationary points $z_\pm \in \partial\mathbb{D}$ of order one, i.e. $\frac{\partial\Phi_a}{\partial z}(z_\pm) = 0$ but $\frac{\partial^2\Phi_a}{\partial z^2}(z_\pm) \neq 0$, satisfying $z_- = \overline{z_+}$.*

(2) *If $a \in \{\alpha_0, \alpha_0^{-1}\}$, then $\Phi_a(\cdot)$ has one stationary point $z_0 \in \{-1, 1\}$ of order two, i.e. $\frac{\partial\Phi_a}{\partial z}(z_0) = \frac{\partial^2\Phi_a}{\partial z^2}(z_0) = 0$, but $\frac{\partial^3\Phi_a}{\partial z^3}(z_0) \neq 0$. More precisely, if $a = \alpha_0$ then $z_0 = -1$ and*

$$\frac{\partial^3\Phi_{\alpha_0}}{\partial z^3}(z_0) = \frac{2\lambda(1-\lambda)}{(1+\lambda)^3}.$$

If $a = \alpha_0^{-1}$ then $z_0 = 1$ and

$$\frac{\partial^3\Phi_{\alpha_0^{-1}}}{\partial z^3}(z_0) = \frac{2\lambda(1+\lambda)}{(1-\lambda)^3}.$$

(3) *If $a \notin [\alpha_0, \alpha_0^{-1}]$, then $\Phi_a(\cdot)$ has two stationary points $z_\pm \in \mathbb{R}$ of order one, i.e. $\frac{\partial\Phi_a}{\partial z}(z_\pm) = 0$ but $\frac{\partial^2\Phi_a}{\partial z^2}(z_\pm) \neq 0$, satisfying $z_- = z_+^{-1}$.*

The stationary points z_+ and z_- are given by the formula

$$z_\pm = z_\pm(a) = \frac{a(1+\lambda^2) - (1-\lambda^2)}{2\lambda a} \pm \sqrt{\left(\frac{a(1+\lambda^2) - (1-\lambda^2)}{2\lambda a}\right)^2 - 1} \tag{3.1}$$

and if $a \notin \{\alpha_0, \alpha_0^{-1}\}$ then

$$\left.\frac{\partial^2\Phi_a}{\partial z^2}\right|_{z=z_\pm} = \frac{(1-\lambda^2)(z_\pm - z_\mp)\lambda}{(z_\pm - \lambda)^2(1-\lambda z_\pm)^2}. \tag{3.2}$$

Proof. Computing derivatives we obtain

$$\begin{aligned}
\frac{\partial\Phi_a}{\partial z} &= \frac{1}{z-\lambda} - \frac{a}{z} + \frac{\lambda}{1-\lambda z},\\
\frac{\partial^2\Phi_a}{\partial z^2} &= -\frac{1}{(z-\lambda)^2} + \frac{a}{z^2} + \frac{\lambda^2}{(1-\lambda z)^2},\\
\frac{\partial^3\Phi_a}{\partial z^3} &= \frac{2}{(z-\lambda)^3} - \frac{2a}{z^3} + \frac{2\lambda^3}{(1-\lambda z)^3}.
\end{aligned}$$

The function $\Phi_a(z)$ has a stationary point if and only if $\partial\Phi_a/\partial z = 0$, i.e. if and only if

$$a = 1 + \frac{\lambda}{z-\lambda} + \frac{\lambda z}{1-\lambda z}.$$

Solving the latter for z yields the representation (3.1) for the roots $z_\pm$ of $\frac{\partial \Phi_a}{\partial z}$. If $a \notin \left\{\alpha_0,\, \alpha_0^{-1}\right\}$, then z_+ and z_- are distinct. If $a \in (\alpha_0,\, \alpha_0^{-1})$, then $z_\pm \in \partial\mathbb{D} \setminus \{-1, 1\}$ and if $a \notin [\alpha_0,\, \alpha_0^{-1}]$, then $z_\pm \in \mathbb{R} \setminus \{-1, 1\}$. Plugging in the values of $z_\pm$ we obtain formula (3.2) for the value of $\frac{\partial^2 \Phi_a}{\partial z^2}\Big|_{z=z_\pm}$ when $a \notin \left\{\alpha_0,\, \alpha_0^{-1}\right\}$. If $a \in \left\{\alpha_0,\, \alpha_0^{-1}\right\}$, then $\frac{\partial \Phi_a}{\partial z}$ has a unique zero. If $a = \alpha_0^{-1}$, then $z_+ = z_- = 1 = z_0$ and

$$\Phi_{\alpha_0^{-1}}(1) = \frac{\partial \Phi_{\alpha_0^{-1}}}{\partial z}(1) = \frac{\partial^2 \Phi_{\alpha_0^{-1}}}{\partial z^2}(1) = 0,$$

with

$$\frac{\partial^3 \Phi_{\alpha_0^{-1}}}{\partial z^3}(1) = \frac{2\lambda(1+\lambda)}{(1-\lambda)^3} \neq 0.$$

If $a = \alpha_0$, then $z_+ = z_- = -1 = z_0$ and

$$\frac{\partial \Phi_{\alpha_0}}{\partial z}(-1) = \frac{\partial^2 \Phi_{\alpha_0}}{\partial z^2}(-1) = 0, \qquad \frac{\partial^3 \Phi_{\alpha_0}}{\partial z^3}(-1) = \frac{2\lambda(1-\lambda)}{(1+\lambda)^3} \neq 0.$$

□

3.1. **Proof of Theorem 1 (1), (2).**

Proof. We start with part (1). We establish by induction on k that

$$(b_\lambda^n)^{(k)}(0) \sim (-\lambda)^{n-k} \left(n(1-\lambda^2)\right)^k, \qquad k \geq 0,\, n \to \infty.$$

This asymptotic formula clearly holds for $k = 0$. We assume that the above induction hypothesis holds for all $0 \leq j \leq k$. We first observe that

$$(b_\lambda^n)^{(k+1)}(z) = \left((b_\lambda^n)'\right)^{(k)}(z) = n(1-\lambda^2)\left((1-\lambda z)^{-2} \cdot b_\lambda^{n-1}\right)^{(k)}(z),$$

and then apply Leibniz formula to the product $z \mapsto (1-\lambda z)^{-2} \cdot b_\lambda^{n-1}(z)$, at $z = 0$. Computation shows that

$$\left((1-\lambda z)^{-2}\right)^{(j)}(0) = (j+1)!\lambda^j$$

and therefore

$$\left((1-\lambda z)^{-2} \cdot b_\lambda^{n-1}\right)^{(k)}(0) = \sum_{j=0}^{k} \binom{k}{j} (j+1)!\lambda^j \left(b_\lambda^{n-1}\right)^{(k-j)}(0).$$

Applying our induction hypothesis to the factor $\left(b_\lambda^{n-1}\right)^{(k-j)}(0)$, it turns out that the main contribution to the above sum is due to its first term (whose index is $j = 0$):

$$\binom{k}{0}(0+1)!\lambda^0 \left(b_\lambda^{n-1}\right)^{(k)}(0) \sim (-\lambda)^{n-1-k}\left((n-1)(1-\lambda^2)\right)^k \sim (-\lambda)^{n-k-1}\left(n(1-\lambda^2)\right)^k.$$

We conclude that

$$(b_\lambda^n)^{(k+1)}(0) = n(1-\lambda^2)\left((1-\lambda z)^{-2} \cdot b_\lambda^{n-1}\right)^{(k)}(0) \sim (-\lambda)^{n-k-1}\left(n(1-\lambda^2)\right)^{k+1},$$

which completes the proof of part (1).

Proof of part (2). The integral defining $\widehat{b^n_\lambda}(k)$ is of the form:

$$\widehat{b^n_\lambda}(k) = \frac{1}{2i\pi}\int_{\partial\mathbb{D}} \varphi(z)e^{n\Phi(z)}\mathrm{d}z \tag{3.3}$$

where $\varphi(z) = z^{-1}$ and $\Phi = \Phi_a$ with $a = k/n$.

Case 1: $a \in [\epsilon, \alpha]$. We first assume that $a \in [\epsilon, \alpha]$ for a given $\epsilon \in (0, \alpha)$ and apply the saddle point/steepest descent method [8, Chapter 7], [15, Chapters 5-6], [17, Chapters 7-8] to determine an asymptotic formula for the integral (3.3). This method essentially consists in deforming the original contour of integration (here $\partial\mathbb{D}$) into a suitable one, say C, so that C remains inside the domain U where our integrand is holomorphic (here $U = \mathbb{C} \setminus \{1/\lambda\}$) and the classical conditions – which we recall below and which relate to geometrical considerations specific to our situation – are satisfied. We refer to Figure 3.1 and Figure 3.2 for an illustration.

1) First of all C *must pass through the relevant saddle point(s) of* Φ i.e. the solutions $z_\pm$ of the equation $\Phi'(z) = 0$. In our case $a \leq \alpha < \alpha_0$ it can be checked that only z_+ is relevant: on the interval $[z_-,\, z_+]$ the continuous function

$$\psi : z \mapsto e^{\Re\Phi(z)}$$

achieves its minimum at $z = z_+$, its maximum at $z = z_-$ and

$$\psi(z_+) < 1 < \psi(z_-).$$

We also observe that the function $a \mapsto z_+(a)$ is negative and monotonically decreasing for $a \in (0, \alpha_0)$; moreover $\lim_{a\to 0} z_+(a) = 0$ and $\lim_{a\to\alpha_0} z_+(a) = -1$. In particular for $a \in [\epsilon, \alpha]$ we have $-1 < z_+(a) < 0$ and $z_+(a)$ is separated from 0.

2) The level curve

$$L(z_+) = \{z \in U :\ \Re\Phi(z) = \Re\Phi(z_+)\}$$

passes two times through z_+ making angle of $\pi/2$ and divides U into two domains $V(z_+)$ and $H(z_+)$ respectively named valleys and hills separating the neighborhood of the saddle point z_+:

$$V(z_+) = \{z \in U :\ \Re\Phi(z) < \Re\Phi(z_+)\},$$

$$H(z_+) = \{z \in U :\ \Re\Phi(z) > \Re\Phi(z_+)\},$$

and the new contour of integration C *must be contained in* $V(z_+)$. Here we observe that $L(z_+)$ is symmetric with respect to the real axis, which is the bisector in $H(z_+)$ of the angle between the two tangents to the curve $L(z_+)$ at z_+. We have $\psi(z) = 1$ for $z \in \partial\mathbb{D}$ and therefore $\partial\mathbb{D} \subset H(z_+)$. Furthermore we observe that $H(z_+)$ contains both a neighborhood of $1/\lambda$ because $\lim_{z\to 1/\lambda} \psi(z) = \infty$, and a neighborhood of 0 since $\lim_{z\to 0} \psi(z) = \infty$. The fact that $\lim_{z\to\infty} \psi(z) = 0$ shows that $V(z_+)$ contains a neighborhood of ∞ and that the distance from any point of $L(z_+)$ to z_+ is finite. $V(z_+)$ also contains a neighborhood of λ since $\psi(\lambda) = 0$. Let us finally mention that $L(z_+)$ is actually composed of two curves: A closed curve contained in $\mathbb{D}$ passing two times through z_+ and another one surrounding $\partial\mathbb{D}$, which is not of interest for us. We refer to Figure 3.1 for a depiction of the behavior of $L(z_+)$, $H(z_+)$ and $V(z_+)$ near the unit disc.

3) We recall that the curves of steepest descent respectively steepest ascent from z_+, respectively named S_d and S_a, see Figure 3.2, are the curves defined by the equation

$$\Im\Phi(z) = \Im\Phi(z_+)$$

and contained in $V(z_+)$ – respectively in $H(z_+)$ – and in a neighborhood of z_+. If $T(z_+)$ denotes the tangent at z_+ to the curve of steepest descent from z_+, then $T(z_+)$ *must also be tangent to the new contour of integration* C *at* z_+ and it is more convenient to choose C such that it coincides with $T(z_+)$ on a fixed neighborhood of z_+. Here $T(z_+)$ is the vertical line passing through z_+. It is usually obtained as the bisector in $V(z_+)$ of the angle formed by the two perpendicular tangents to the level curve $L(z_+)$ at z_+. The other bisector of this angle is part of the real axis, and necessarily lies in $H(z_+)$: $z \mapsto \psi(z)$ achieves its minimum at z_+ on $[z_-, 0)$ whereas $z \mapsto \psi(z)$ attains its maximum at z_+ on $T(z_+)$, which is required to apply the method of the steepest descent.

If such a choice of C is possible – which is the case here, see Figure 3.2 – then [8, formula (7.2.10)], [17, formula (36.7)], [15, formula (5.7.2)] we have

$$\begin{aligned}\int_{\partial\mathbb{D}} \varphi(z)e^{n\Phi(z)}\mathrm{d}z &= \int_C \varphi(z)e^{n\Phi(z)}\mathrm{d}z \\ &\sim \varphi(z_+)e^{n\Phi(z_+)+i\theta}\sqrt{\frac{2\pi}{n|\Phi''(z_+)|}}, \qquad n\to\infty,\end{aligned} \tag{3.4}$$

where θ is the angle between $T(z_+)$ and the real axis. It follows from Lemma 5, formula (3.2), that

$$\Phi''(z_+) = \frac{\lambda(z_+ - z_-)(1-\lambda^2)}{(z_+-\lambda)^2(1-\lambda z_+)^2}, \tag{3.5}$$

which is strictly positive, and taking into account the fact that $\theta = 3\pi/2$ we find

$$\begin{aligned}\int_{\partial\mathbb{D}} \varphi(z)e^{n\Phi(z)}\mathrm{d}z &\sim i\sqrt{\frac{2\pi}{n}}\frac{1}{z_+}\left(\frac{b_\lambda(z_+)}{z_+^{k/n}}\right)^n \frac{(z_+-\lambda)(1-\lambda z_+)}{\sqrt{\lambda(1-\lambda^2)(z_+-z_-)}} \\ &= i\sqrt{\frac{2\pi}{n}}\left(\frac{b_\lambda(z_+)}{z_+^{k/n}}\right)^n \frac{(z_+-\lambda)(z_- -\lambda)}{\sqrt{\lambda(1-\lambda^2)(z_+-z_-)}},\end{aligned}$$

where we used the identity $z_+z_- = 1$ (see Lemma 5). It follows from (3.1) that

$$\begin{aligned}z_+ - z_- &= \frac{\sqrt{(\lambda^2-1)(a(\lambda-1)+1+\lambda)(a(1+\lambda)+\lambda-1)}}{a\lambda} \\ &= \frac{1-\lambda^2}{a\lambda}\sqrt{(a-\alpha_0^{-1})(a-\alpha_0)},\end{aligned} \tag{3.6}$$

and that

$$(z_+-\lambda)(z_- -\lambda) = \frac{1-\lambda^2}{a}, \tag{3.7}$$

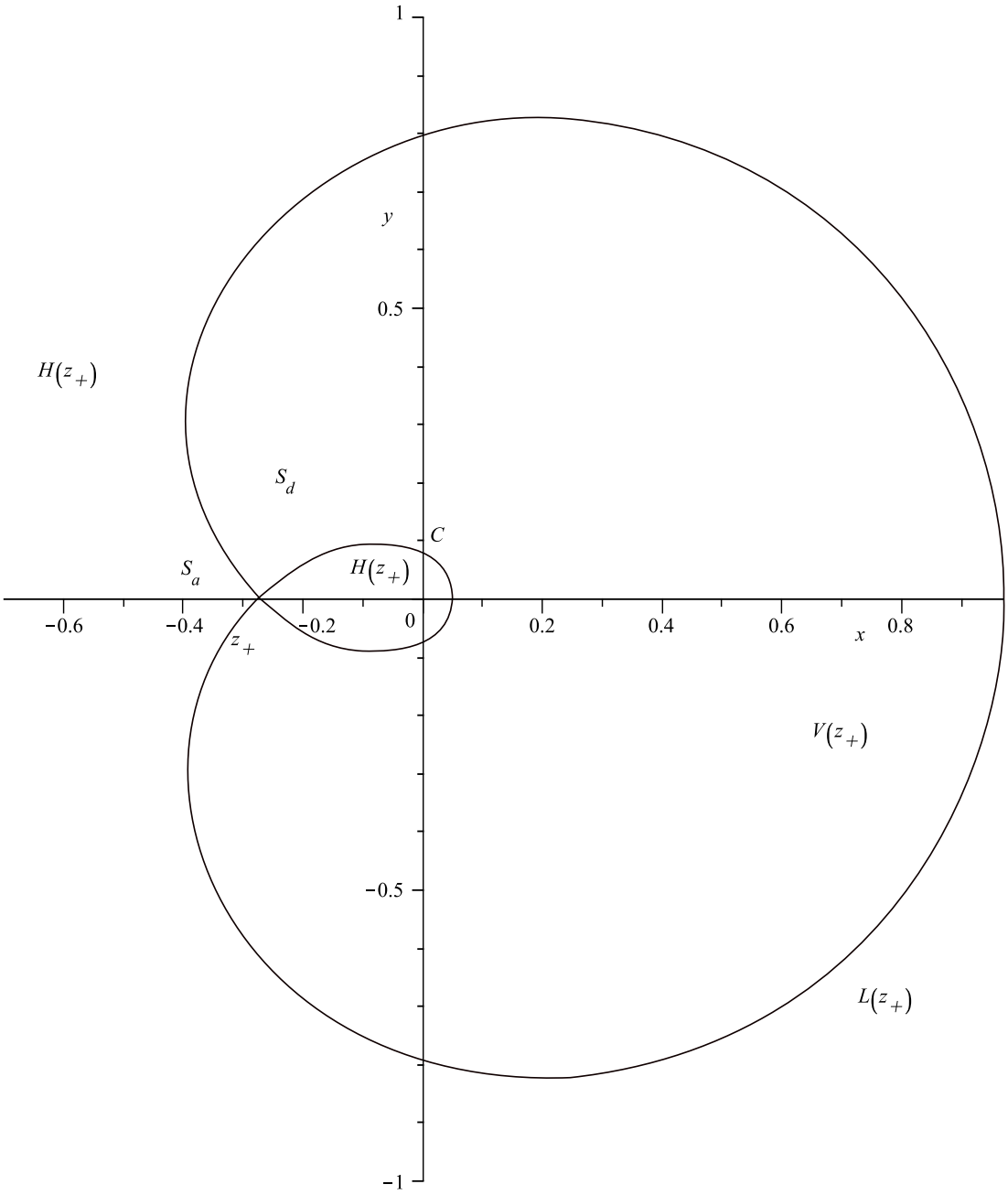

FIGURE 3.1. This figure depicts $L(z_+)$ $H(z_+)$ and $V(z_+)$ where $\lambda = 0.5$ and $k/n = 0.32$.

where $a = k/n$. Therefore

$$\frac{(z_+ - \lambda)(z_- - \lambda)}{\sqrt{\lambda(1-\lambda^2)(z_+ - z_-)}} = \frac{1-\lambda^2}{a}\frac{1}{\sqrt{\lambda(1-\lambda^2)}}\sqrt{\frac{a\lambda}{1-\lambda^2}}\frac{1}{\left[(a-\alpha_0^{-1})(a-\alpha_0)\right]^{1/4}}$$
$$= \frac{1}{\sqrt{a}\left[(a-\alpha_0^{-1})(a-\alpha_0)\right]^{1/4}}.$$

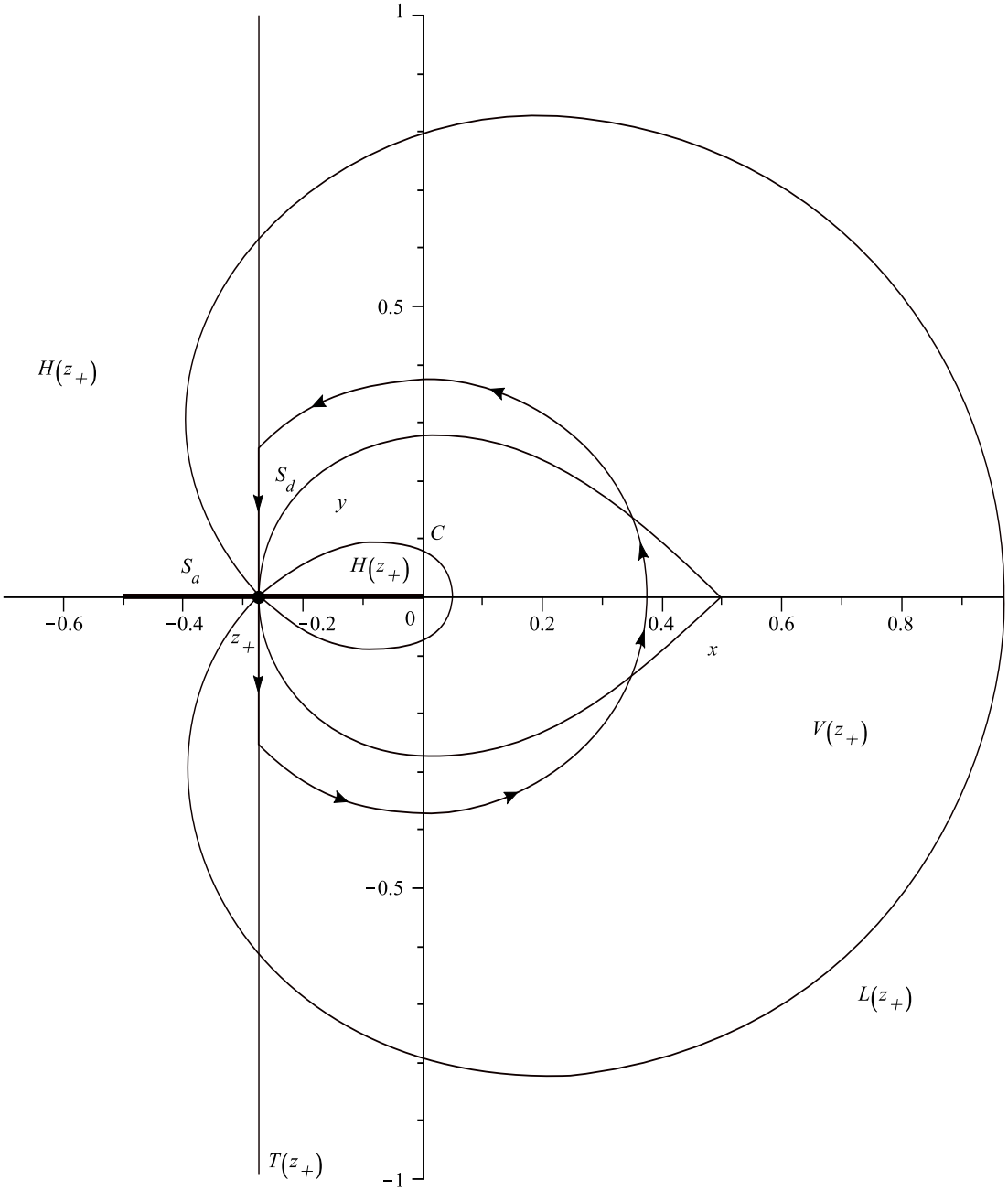

FIGURE 3.2. This figure depicts the new contour of integration C, the level curve $L(z_+)$, the curve S_d of steepest descent from z_+, the curve S_a of steepest ascent from z_+, the tangent $T(z_+)$ to S_d at z_+, the domain $V(z_+)$ and the domain $H(z_+)$, when $k/n \in [\epsilon n, \alpha n]$. Here we chose $\lambda = 0.5$ and $k/n = 0.32$.

Dividing the above asymptotic formula for $\int_{\partial\mathbb{D}} \varphi(z) e^{n\Phi(z)} \mathrm{d}z$ by $2i\pi$ we conclude that

$$\widehat{b_\lambda^n}(k) \sim \frac{1}{\sqrt{2n\pi}} \frac{1}{\sqrt{k/n}\left[(\alpha_0 - k/n)(\alpha_0^{-1} - k/n)\right]^{1/4}} \left(\frac{b_\lambda(z_+)}{z_+^{k/n}}\right)^n .$$

Case 2: $a = k/n \to 0$ and $k \to \infty$. Now we assume that $k = k(n)$ is such that $k(n) \to \infty$ and $k(n)/n \to 0$ as $n \to \infty$. The situation is essentially the same as before in the sense that again $z_+ = z_+(k/n)$ is the only relevant saddle point of Φ, but it is slightly more delicate because this time z_+ approaches the origin as $n \to \infty$. The new contour of integration C is chosen in $V(z_+)$ the same way as previously but the straight steepest descent line $C \cap T(z_+)$ – along which $\Phi''(z_+)(z - z_+)^2$ is negative – must lie in a neighborhood of z_+ where Φ can be expanded as a convergent power series

$$\Phi(z) = \Phi(z_+) + \sum_{j \geq 2} \frac{\Phi^{(j)}(z_+)}{j!}(z - z_+)^j.$$

A computation shows that

$$z_+ = -a\frac{\lambda}{1 - \lambda^2} + \mathcal{O}(a^2)$$

as $a = k/n$ tends to 0, and for $j \geq 2$

$$\begin{aligned} \frac{\Phi^{(j)}(z_+)}{j!} &= \frac{(-1)^j a}{j z_+^j} + \frac{\lambda^j}{j}\frac{1}{(1 - \lambda z_+)^j} - \frac{(-1)^j}{j(z_+ - \lambda)^j} \\ &\sim \frac{(-1)^j a}{j z_+^j} \sim \frac{1}{j a^{j-1}} \left(\frac{1 - \lambda^2}{\lambda}\right)^j. \end{aligned} \tag{3.8}$$

In particular, for large enough n the radius of convergence R of the power series of Φ near z_+ is proportional to a. We put

$$G(z) = \sum_{j \geq 3} \frac{\Phi^{(j)}(z_+)}{j!}(z - z_+)^j.$$

We follow and adapt the approach from [17, p. 92-93] to our situation. Let $x = 2/5$ and $u_k = k^{-x}$ so that $\lim_{k\to\infty} u_k = 0$, $\lim_{k\to\infty} k u_k^3 = 0$ and $\lim_{k\to\infty} k u_k^2 = \infty$. We choose C such that $C \cap T(z_+)$ lies in the disc $|z - z_+| \leq \rho$ where $\rho = a u_k = \frac{k}{n} u_k$. For z in the disc $|z - z_+| \leq \rho$ we have

$$\begin{aligned} |G(z)| &\leq \sum_{j \geq 3} \frac{|\Phi^{(j)}(z_+)|}{j!}|z - z_+|^j \\ &\lesssim a \sum_{j \geq 3} \frac{1}{j}\left(\frac{1 - \lambda^2}{\lambda} u_k\right)^j \lesssim a u_k^3. \end{aligned}$$

It follows that for $z \in C \cap T(z_+)$ we have

$$\exp\left(n\Phi(z)\right) = \exp\left(n\Phi(z_+) + n\frac{\Phi''(z_+)}{2}(z - z_+)^2\right) \cdot \left(1 + \mathcal{O}\left(k u_k^3\right)\right).$$

Observing that $\left|\frac{z_+-z}{z_+}\right| \lesssim u_k$, we obtain

$$\varphi(z) = \frac{1}{z_+ + z - z_+} = \frac{1}{z_+\left(1+\frac{z-z_+}{z_+}\right)} = \frac{1}{z_+}\left(1+\sum_{j\geq 1}\frac{1}{z_+}\left(\frac{z_+-z}{z_+}\right)^j\right) = \frac{1}{z_+}+\mathcal{O}\left(u_k\right).$$

Taking into account the fact that $x < 2^{-1}$ we find that for $z \in C \cap T(z_+)$

$$\varphi(z)\exp\left(n\Phi(z)\right) = z_+^{-1}\exp\left(n\Phi(z_+)+n\frac{\Phi''(z_+)}{2}(z-z_+)^2\right)\cdot\left(1+\mathcal{O}\left(ku_k^3\right)\right)$$

The contribution of the neighbourhood $|z-z_+| \leq \rho$ of the saddle point z_+ is therefore

$$\begin{aligned}(3.9)\quad &\int_{C\cap T(z_+)}\varphi(z)\exp(n\Phi(z))\mathrm{d}z\\ &\qquad = z_+^{-1}\exp\left(n\Phi(z_+)\right)\int_{C\cap T(z_+)}\exp\left(n\frac{\Phi''(z_+)}{2}(z-z_+)^2\right)\mathrm{d}z\cdot\left(1+\mathcal{O}\left(ku_k^3\right)\right)\end{aligned}$$

It follows from (3.5), (3.6), and (3.7) that

$$(3.10)\qquad \Phi''(z_+) = \frac{k}{n}z_+^{-2}\left(\alpha_0-\frac{k}{n}\right)^{1/2}\left(\alpha_0^{-1}-\frac{k}{n}\right)^{1/2}.$$

We let r vary from $-\rho$ to ρ and put $z = z_+ - ir$. Then (3.9) gives

$$\begin{aligned}&\int_{C\cap T(z_+)}\varphi(z)\exp(n\Phi(z))\mathrm{d}z\\ &= -iz_+^{-1}\left(1+\mathcal{O}\left(ku_k^3\right)\right)\exp\left(n\Phi(z_+)\right)\int_{-\rho}^{\rho}\exp\left(-kz_+^{-2}\frac{\left(\alpha_0-\frac{k}{n}\right)^{1/2}\left(\alpha_0^{-1}-\frac{k}{n}\right)^{1/2}}{2}r^2\right)\mathrm{d}r.\end{aligned}$$

Changing the variable r by

$$v = \sqrt{k}z_+^{-1}\frac{\left(\alpha_0-\frac{k}{n}\right)^{1/4}\left(\alpha_0^{-1}-\frac{k}{n}\right)^{1/4}}{\sqrt{2}}r,$$

we get

$$\begin{aligned}(3.11)\quad &\int_{C\cap T(z_+)}\varphi(z)\exp(n\Phi(z))\mathrm{d}z\\ &\qquad = i\left(1+o(1)\right)\exp\left(n\Phi(z_+)\right)\sqrt{\frac{2}{k}}\int_{-\omega}^{\omega}\exp\left(-v^2\right)\mathrm{d}v,\end{aligned}$$

where

$$\omega = \sqrt{\frac{k}{2}}|z_+|^{-1}\left(\alpha_0-\frac{k}{n}\right)^{1/4}\left(\alpha_0^{-1}-\frac{k}{n}\right)^{1/4}\rho \sim \sqrt{\frac{k}{2}}\frac{1-\lambda^2}{\lambda}u_k \asymp \sqrt{k}u_k,$$

and, in particular, ω tends to ∞ with k. Moreover, as $k\to\infty$, we have

$$\int_{\omega}^{\infty}\exp(-v^2)\mathrm{d}v = \mathcal{O}\left(\frac{\exp(-\omega^2)}{\omega}\right) = \mathcal{O}\left(\frac{\exp(-Cku_k^2)}{\sqrt{k}u_k}\right),$$

for some absolute constant $C > 0$. Therefore,

$$\int_{C\cap T(z_+)} \varphi(z)\exp(n\Phi(z))\mathrm{d}z = i\exp\left(n\Phi(z_+)\right)\sqrt{\frac{2\pi}{k}}\cdot(1+o(1))\,,$$

and, hence,

$$\frac{1}{2i\pi}\int_{C\cap T(z_+)} \varphi(z)\exp(n\Phi(z))\mathrm{d}z \sim \frac{1}{\sqrt{2k\pi}}\left(\frac{b_\lambda(z_+)}{z_+^{k/n}}\right)^n.$$

To complete the proof we choose C so that $C\setminus T(z_+)$ coincides with the circle centered at 0 of radius $|z_+| = -z_+$ intersected with the half-plane $\{\Re z > z_+\}$, and show that

$$\frac{1}{2i\pi}\int_{C\setminus T(z_+)} \varphi(z)\exp(n\Phi(z))\mathrm{d}z = o\left(\frac{1}{2i\pi}\int_{C\cap T(z_+)} \varphi(z)\exp(n\Phi(z))\mathrm{d}z\right)$$

as $k\to\infty$. The endpoints of $C\setminus T(z_+)$ are denoted by $|z_+|e^{i(\pi-\eta)}$ and $|z_+|e^{i(-\pi+\eta)}$ where $\eta > 0$ is such that

$$\eta \asymp \sin\eta \asymp \frac{\rho}{|z_+|} \asymp u_k.$$

We write

$$\frac{1}{2i\pi}\int_{C\setminus T(z_+)} \varphi(z)\exp(n\Phi(z))\mathrm{d}z$$
$$= \varphi(z_+)\exp(n\Phi(z_+))\frac{1}{2i\pi}\int_{C\setminus T(z_+)}\frac{\varphi(z)\exp(n\Phi(z))}{\varphi(z_+)\exp(n\Phi(z_+))}\mathrm{d}z,$$

put $z = |z_+|e^{it} = -z_+e^{it}$, and observe that

$$\left|\frac{\varphi(z)\exp(n\Phi(z))}{\varphi(z_+)\exp(n\Phi(z_+))}\right| = \left|\frac{b_\lambda\left(|z_+|e^{it}\right)}{b_\lambda(z_+)}\right|^n, \qquad |z| = |z_+|.$$

A direct computation shows that

$$\left|b_\lambda\left(|z_+|e^{it}\right)\right|^2 = 1 - \frac{(1-\lambda^2)(1-|z_+|^2)}{1+\lambda^2|z_+|^2 - 2\lambda|z_+|\cos t}. \tag{3.12}$$

This function is increasing on $[0,\pi]$ and decreasing on $[-\pi,0]$. Therefore,

$$\left|\frac{\varphi(z)\exp(n\Phi(z))}{\varphi(z_+)\exp(n\Phi(z_+))}\right| \le \left|\frac{b_\lambda\left(|z_+|e^{i(\pi-\eta)}\right)}{b_\lambda(z_+)}\right|^n.$$

By (3.12) we obtain that there exists $C > 0$ such that

$$\left|\frac{b_\lambda\left(|z_+|e^{i(\pi-\eta)}\right)}{b_\lambda(z_+)}\right| \le 1 - Ca\eta^2,$$

which proves that

$$\frac{1}{2i\pi}\int_{C\setminus T(z_+)}\frac{\varphi(z)\exp(n\Phi(z))}{\varphi(z_+)\exp(n\Phi(z_+))}\mathrm{d}z = \mathcal{O}\left(\exp(-Cku_k^2)\right). \tag{3.13}$$

This completes the proof in case 2.

Case 3: $a \in [\alpha^{-1}, \epsilon^{-1}]$. A discussion similar to that for case 1 leads to the same formula for $a = k/n \in [\alpha^{-1}, \epsilon^{-1}]$ where $\epsilon \in (0, \alpha)$ is fixed. We first reproduce the three steps from the first case required to deform the original contour of integration $\partial\mathbb{D}$ into the suitable one C, which remains inside the domain U where our integrand is holomorphic. The geometrical considerations corresponding to conditions (1)–(3) are sometimes slightly different in this case. We detail them below for completeness and refer to Figure 3.3 for an illustration.

1) As in case 1, C should pass through the relevant saddle point of Φ. Again, it can be checked that only the critical point z_+ is relevant: For z on the interval $[\lambda, \lambda^{-1})$ the continuous function $z \mapsto \psi(z)$ achieves its minimum at $z = z_+$, its maximum at $z = z_-$ and

$$\psi(z_+) < 1 < \psi(z_-).$$

We also observe that the function $a \mapsto z_+(a)$ is nonnegative and monotonically increasing for $a \in (\alpha_0^{-1}, \epsilon^{-1})$; moreover $\lim_{a \to \alpha_0^{-1}} z_+(a) = 1$ and $\lim_{a \to +\infty} z_+(a) = 1/\lambda$. In particular for $a \in [\epsilon, \alpha]$ we have $1 < z_+(a) < 1/\lambda$.

2) Again, the level curve $L(z_+)$ passes two times through z_+ making angle of $\pi/2$ and divides U into $V(z_+)$ (valleys) and $H(z_+)$ (hills). The new contour of integration C will be contained in $V(z_+)$ as required. $L(z_+)$ is symmetric with respect to the real axis and it consists again of two parts. The first one is not of interest for us: it is a closed curve contained in $\mathbb{D}$ surrounding λ. The second one, which is the one we are interested in, is a closed curve that surrounds $\partial\mathbb{D}$ to the left of z_+ and a neighborhood of $1/\lambda$ to the right of z_+. As in case 1, the real axis is the bisector in $H(z_+)$ of the angle between the two tangents to this part of $L(z_+)$ at z_+. Finally $H(z_+)$ still contains $\partial\mathbb{D}$ since $\psi(z) = 1$ for $z \in \partial\mathbb{D}$, and it also contains a neighborhood of $1/\lambda$ because $\lim_{z \mapsto 1/\lambda} \psi(z) = \infty$. $V(z_+)$ contains a neighborhood of ∞ because $\lim_{z \mapsto \infty} \psi(z) = 0$ and also contains a neighborhood of λ because $\lim_{z \mapsto \lambda} \psi(z) = 0$.

3) We do not reproduce the discussion on the curves of steepest descent/ascent S_d and S_a from z_+, since it is identical to the previous one (case 1). (This time $z \mapsto \psi(z)$ attains its minimum on $(\lambda, 1/\lambda)$ at z_+ whereas $z \mapsto \psi(z)$ attains its maximum at z_+ on $T(z_+)$.)

Since such a choice of C is possible – see Figure 3.3 for an illustration – the asymptotic formula (3.4) used in case 1 applies also here and we get

$$\widehat{b_\lambda^n}(k) = \frac{1}{2i\pi} \int_C \varphi(z) e^{n\Phi(z)} \mathrm{d}z \sim \frac{1}{2i\pi} \varphi(z_+) e^{n\Phi(z_+) + i\theta} \sqrt{\frac{2\pi}{n|\Phi''(z_+)|}},$$

as $n \to \infty$, where $\theta = \pi/2$ is the angle between $T(z_+)$ and the real axis. The rest of the proof is identical to the one we have detailed in case 1.

Case 4: $a = k/n \in [\alpha^{-1} n, \infty)$ and $k/n \to \infty$. This case is analogous to case 2. As in case 3, $z_+ = z_+(k/n)$ is the only relevant saddle point of Φ, but this time z_+ approaches $1/\lambda$ as $n \to \infty$. The new contour of integration C is chosen in $V(z_+)$ the same way as in case 3 but the straight steepest descent line $C \cap T(z_+)$ – along which $\Phi''(z_+)(z - z_+)^2$ is negative – must lie (as in case 2) in a neighborhood of z_+ where Φ can be expanded as a

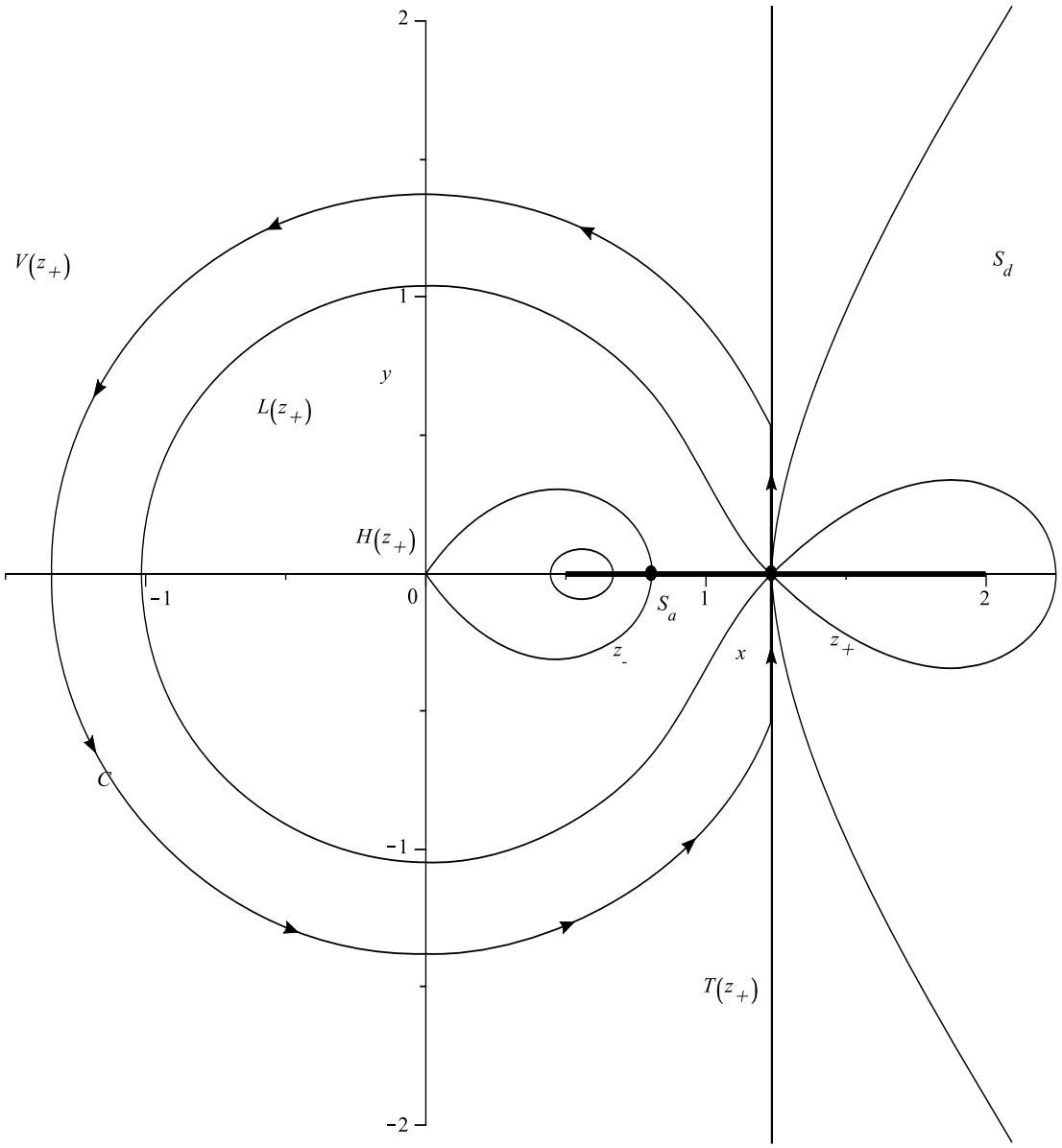

FIGURE 3.3. This figure depicts the new contour of integration C, the level curve $L(z_+)$, the curve S_d of steepest descent from z_+, the curve S_a of steepest ascent from z_+, the tangent $T(z_+)$ to S_d at z_+, the domain $V(z_+)$ and the domain $H(z_+)$, when $k/n \in [\alpha^{-1}n, \epsilon^{-1}n]$. Here we chose $\lambda = 0.5$ and $k/n = 3.3$.

convergent power series

$$\Phi(z) = \Phi(z_+) + \sum_{j \geq 2} \frac{\Phi^{(j)}(z_+)}{j!}(z - z_+)^j$$

whose radius of convergence – which can be computed using (3.8) – is this time proportional to $1/a$, whereas it was proportional to a when $k/n \to 0$ (see case 2). We omit the rest of the proof, which is identical to the one we have detailed in case 2. □

3.2. **Proof of Proposition 3.**

Proof of Proposition 3. We omit the proof of the second asymptotic formula (i.e. when k/n is in a neighborhood of α_0^{-1}) because it follows from an almost word-for-word adaptation of the one of [41, Proposition 10] (the part corresponding to (2)–(4), replacing the factor $(1 - z^{-2})$ by 1). We choose to sketch the proof of the asymptotic formulas for k/n in a neighborhood of α_0 , which is similar to those in [41, 43], but where computations are slighlty different. We refer to the proof of [43, Proposition 17] for more technical details. Again, we recall that for any k and n:

$$\widehat{b_\lambda^n}(k) = \frac{1}{2i\pi} \int_{\partial \mathbb{D}} \frac{e^{n\Phi(z)}}{z} \mathrm{d}z$$

where $\Phi = \Phi_a$ and $a = k/n$. It is explained in [41, 42] that the standard method of stationary phase cannot be applied when k/n approaches α_0^{-1} because in this case the saddle points z_+ and z_- which are of order 1, are coalescing to the saddle point $z_0 = 1$, which is of order 2. If k/n approaches α_0, then the same phenomena occurs and $z_\pm$ are coalescing this time to $z_0 = -1$. As the main contribution of the above integral is due to the critical points $z_\pm = z_\pm(a)$ of Φ_a, if $a < \alpha_0$ it is required to locally deform the unit circle to a new contour that passes through z_+, z_- (which are real and negative) and -1. If $a > \alpha_0$, then the critical points $z_\pm \in \partial\mathbb{D}$ (are complex conjugates) and there is no need to deform the contour as the unit circle already passes through z_+, z_- and -1: In this case the proof below is actually reduced to an application of the uniform version of the method of stationary phase [14, Section 2.3]. Let $\mathcal{D}(-1, \varepsilon)$ be the closed disk centered at -1 of radius $\varepsilon > 0$ chosen in such a way that $z_\pm \in \mathcal{D}(-1, \varepsilon)$. We denote by $\mathcal{C}_\epsilon \subset D(-1, \varepsilon)$ a corresponding local deformation of the unit circle $\partial\mathbb{D}$ and illustrate it below.

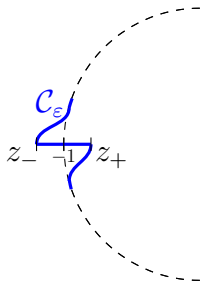

FIGURE 3.4. The contour $\mathcal{C}_\epsilon$ for $\frac{k}{n} < \alpha_0$.

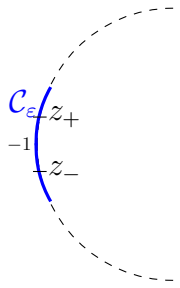

FIGURE 3.5. The contour $\mathcal{C}_\epsilon$ for $\frac{k}{n} > \alpha_0$.

We shall use a uniform version of the steepest descent method [16] as described in [8, p. 369–376], where the case of two nearby saddle points is considered and the first step is to observe that:

$$\widehat{b_\lambda^n}(k) \sim \frac{1}{2i\pi} \int_{\mathcal{C}_\varepsilon} \frac{e^{n\Phi(z)}}{z} \mathrm{d}z, \qquad n \to \infty,$$

the contribution to the integral (1.4) from the part of the contour outside $\mathcal{D}(-1,\varepsilon)$ being asymptotically smaller than the integral itself [16, Subsection 5]. This can usually be proved by the familiar arguments of the ordinary method of steepest descents, similar to those we previously used to prove (3.13). Following [8, (9.2.6)], to simplify the dependence of $z_\pm$ on k/n we change the variable of integration via a locally one-to-one transformation, implicitly given by $s = s_a(z)$ solving the equation

$$\Phi(z) = -\left(\frac{1}{3}s^3 - \gamma_{\alpha_0}^2 s\right) + \eta, \tag{3.14}$$

where the parameters $\gamma = \gamma_{\alpha_0}$ and η are determined in such a way that $s = 0$ is mapped to $z = -1$ and the saddle points $z_\pm$ are mapped symmetrically to $s = \pm\gamma$. For $z = z(s)$ to define a conformal map of $\mathcal{D}(-1,\varepsilon)$ it is necessary that $\gamma_{\alpha_0}^3$ and η be respectively defined by (2.6) and

$$\eta = \frac{\Phi(z_+) + \Phi(z_-)}{2},$$

so that

$$\gamma^2 = \gamma_{\alpha_0}^2 = \frac{(1+\lambda)\,(\alpha_0 - k/n)}{(\lambda(1-\lambda))^{1/3}} + o(\alpha_0 - k/n),$$

and

$$\eta = i\pi\left(1 - \frac{k}{n}\right).$$

For each value of z, (3.14) defines three possible values of s, that is, there are three branches of the inverse transformation. It is shown in [16] that there is one branch of the transformation (3.14) that defines, for each a in a neighborhood of α_0, a conformal map of $\mathcal{D}(-1,\varepsilon)$. More precisely, the transformation (3.14) has exactly one branch $s = s(z,a)$ that can be expanded into a power series in z with coefficients that are continuous in a. On this branch the points $z = z_\pm$ correspond to $s = \pm\gamma_{\alpha_0}$, and the mapping of z to s is one-to-one on $\mathcal{D}(-1,\varepsilon)$. This is an analog of [41, Proposition 12] and of [42, Proposition 9] for k/n in a neighborhood of α_0 instead of k/n close to α_0^{-1}. Following [8, Section 9.2] we get

$$\frac{1}{2i\pi}\int_{\mathcal{C}_\epsilon} \exp\left(n\Phi_a(z)\right)\frac{\mathrm{d}z}{z} = \frac{1}{2i\pi}\int_{\hat{\mathcal{C}}_\epsilon} G_0(s)\exp\left(n\left(-\frac{s^3}{3} + \gamma^2 s + \eta\right)\right)\mathrm{d}s$$

where we made the notation less cluttered writing briefly γ^2 for $\gamma_{\alpha_0}^2$, and where

$$G_0(s) = \frac{1}{z(s)}\frac{\mathrm{d}z}{\mathrm{d}s}$$

is regular on the image $\hat{D}(-1,\epsilon)$ of $D(-1,\epsilon)$ under the transformation $z \mapsto s(z)$. We exploit the fact that if the integrand vanishes near a critical point then its contribution to the asymptotic expansion is diminished. Thus we expand

$$G_0(s) = A_0 + A_1 s + (s^2 - \gamma^2)H_0(s),$$

with A_0, A_1, and H_0 to be determined. As long as H_0 is regular in $\hat{D}(-1,\epsilon)$ the last term of the above identity vanishes at the two saddle points $s = \pm\gamma$. We can then determine

A_0, A_1 by setting $s = \pm\gamma$ in the above equality to get

$$A_0 = \frac{G_0(\gamma) + G_0(-\gamma)}{2}, \quad A_1 = \frac{G_0(\gamma) - G_0(-\gamma)}{2\gamma}. \tag{3.15}$$

With A_0, A_1 defined by these formulas, it is shown in [8, p. 373] that $H_0 = \frac{G_0(s) - A_0 - A_1 s}{s^2 - \gamma^2}$ is regular in $\hat{D}(-1, \epsilon)$ as desired. We conclude that

$$\int_{\mathcal{C}_\epsilon} \exp\left(n\Phi_a(z)\right) \frac{\mathrm{d}z}{z} \sim e^{i\pi(n-k)} \int_{\hat{\mathcal{C}}_\epsilon} (A_0 + A_1 s) \exp\left(n\left(-\frac{s^3}{3} + \gamma^2 s\right)\right) \mathrm{d}s.$$

Following the procedure described in [8, p. 371–375] we consider a contour C_1 which is asymptotically equivalent to $\hat{\mathcal{C}}_\epsilon$. This means that the contribution of C_1 near the critical points coincides with that of $\hat{\mathcal{C}}_\epsilon$, but C_1 continues to ∞ as a contour of steepest descent. C_1 starts at infinity with points of argument $-2\pi/3$ and ends at infinity with points of argument $2\pi/3$. See Figure 3.6, Figure 3.7 and Figure 3.8 below, for a description of C_1 and $\hat{\mathcal{C}}_\epsilon$. We refer to [8, Section 7.2] for a detailed description of such contours.

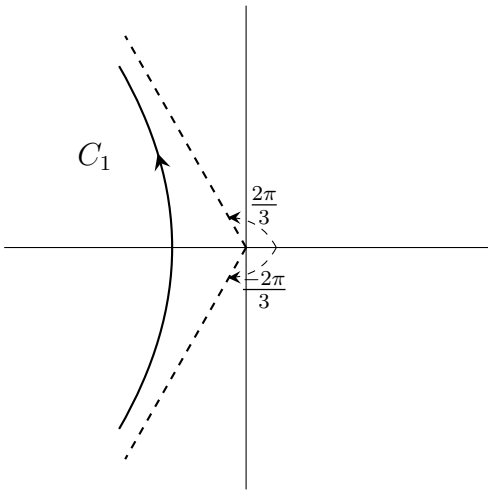

FIGURE 3.6. Introduction of the asymptotically equivalent contour C_1.

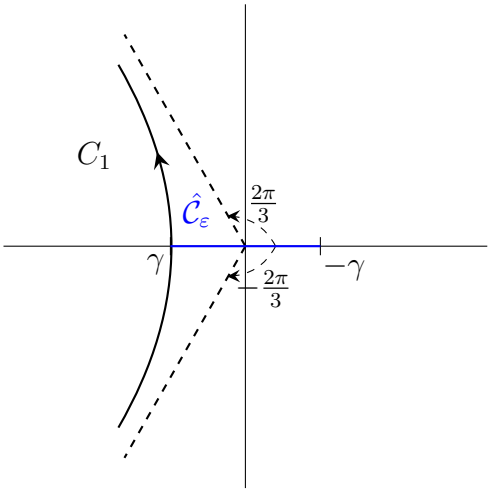

FIGURE 3.7. The contour $\hat{\mathcal{C}}_\epsilon$ for $\frac{k}{n} < \alpha_0$ and the asymptotically equivalent contour C_1.

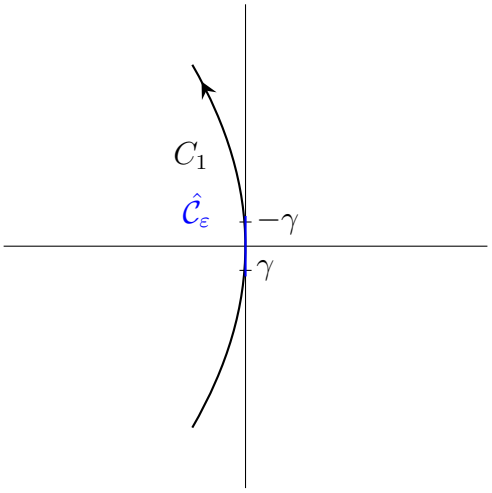

FIGURE 3.8. The contour $\hat{\mathcal{C}}_\epsilon$ for $\frac{k}{n} > \alpha_0$ and the asymptotically equivalent contour C_1.

When we replace $\hat{\mathcal{C}}_\epsilon$ by C_1 in (3.2), the introduced error is negligible, since the integral of $(A_0 + A_1 t)\exp\left(n\left(-\frac{s^3}{3} + \gamma^2 s\right)\right)$ over $C_1 \setminus \hat{D}(1,\epsilon)$, is asymptotically smaller than the integral over $\hat{\mathcal{C}}_\epsilon$, see [8, p. 372] for details. The Airy function can be represented as an integral over C_1. By a change of variable $\tau \mapsto i\tau$ and a deformation of the contour of integration one obtains

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \cos\left(\frac{\tau^3}{3} + \tau x\right) \mathrm{d}\tau = \frac{1}{2i\pi}\int_{C_1} \exp\left(-\frac{u^3}{3} + ux\right) \mathrm{d}u$$

and therefore

(3.16)
$$\frac{1}{2i\pi}\int_{\mathcal{C}_\epsilon} \exp\left(n\Phi_a(z)\right)\frac{\mathrm{d}z}{z} \sim (-1)^{n-k}\left(\frac{A_0}{n^{1/3}}Ai(n^{2/3}\gamma^2) + \frac{A_1}{n^{2/3}}Ai'(n^{2/3}\gamma^2)\right), \qquad n \to \infty,$$

where A_0, A_1 are defined in (3.15). To compute A_0, A_1 we write

$$G_0(\pm\gamma) = G_0(s_\pm) = \frac{1}{z_\pm} z'(s_\pm).$$

A computation (see the proof of [43, Proposition 17] for more details) shows that:

$$z'(t_\pm) = z_\pm \sqrt{\frac{2|\gamma|}{a}}\frac{1}{|\Delta|^{1/4}}, \quad \text{where } a = k/n, \quad \text{and } \Delta = (a - \alpha_0)\left(\alpha_0^{-1} - a\right).$$

Therefore

$$G_0(\gamma) = G_0(-\gamma) = \sqrt{\frac{2|\gamma|}{a}}\frac{1}{|\Delta|^{1/4}}$$

and

$$A_0 = \frac{G_0(\gamma) + G_0(-\gamma)}{2} = G_0(\gamma) = \sqrt{\frac{2|\gamma|}{a}} \frac{1}{|\Delta|^{1/4}},$$

$$A_1 = \frac{G_0(\gamma) - G_0(-\gamma)}{2\gamma} = 0.$$

□

3.3. **Proofs of Theorem 1 (3), (4) and of Theorem 2(2).**

3.3.1. *The case where $a = k/n$ is close to the boundaries α_0, α_0^{-1}.* We first discuss the situation where $a = k/n$ approaches the boundaries α_0, α_0^{-1} and start by proving Theorem 1 (3), (4). Here we apply Proposition 3 together with (1.3).

Proof of Theorem 1 (3), (4). First we prove part (3). If $k \in [\alpha n, \alpha_0 n - n^{1/3})$ and if, in addition, $n^{2/3}(\alpha_0 - k/n) \to +\infty$ (Region III) then $n^{2/3}\gamma^2 \to +\infty$ as n tends to ∞. Since $Ai(x) \sim \frac{1}{2x^{1/4}\sqrt{\pi}} \exp\left(-\frac{2}{3}x^{3/2}\right)$ as $x \to +\infty$, we have

$$\sqrt{\frac{2|\gamma|}{k/n}} \frac{(-1)^{n-k}}{|\Delta|^{1/4}} Ai(n^{2/3}\gamma^2) \sim \sqrt{\frac{2|\gamma|}{k/n}} \frac{(-1)^{n-k}}{|\Delta|^{1/4}} \frac{1}{2\sqrt{\pi} n^{1/6} |\gamma|^{1/2}} \exp\left(-\frac{2}{3} n|\gamma|^3\right)$$

$$\sim \frac{1}{\sqrt{2\pi}} \frac{(-1)^{n-k}}{\sqrt{k/n}\left[(\alpha_0 - k/n)(\alpha_0^{-1} - k/n)\right]^{1/4}} \frac{\exp\left(-\frac{2}{3}n|\gamma|^3\right)}{n^{1/6}}.$$

It remains to use (3.16) and to divide both parts by $n^{1/3}$. We omit the proof of part (4) which is almost identical. □

Next we apply Proposition 3 together with (1.2) to prove Theorem 2 (2) for $k \le \beta n$ or $k \ge \beta^{-1} n$.

Proof of Theorem 2 (2) *for $k \le \beta n$ or $k \ge \beta^{-1} n$.* Let $k \in (\alpha_0 n + n^{1/3}, \beta n] \cup [\beta^{-1} n, \alpha_0^{-1} n - n^{1/3})$. We assume in addition that either $n^{2/3}(k/n - \alpha_0) \to +\infty$ or $n^{2/3}(\alpha_0^{-1} - k/n) \to +\infty$ (i.e. k lies in Region IV$\backslash(\beta n, \beta^{-1} n)$):

i) If $n^{2/3}(k/n - \alpha_0) \to +\infty$ then $\gamma^2 = \gamma_{\alpha_0}^2$ and $n^{2/3}\gamma^2 \to -\infty$. Recalling that $Ai(-x) \sim \frac{1}{x^{1/4}\sqrt{\pi}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right)$ as $x \to +\infty$ we obtain

$$\sqrt{\frac{2|\gamma|}{k/n}} \frac{(-1)^{n-k}}{|\Delta|^{1/4}} Ai(n^{2/3}\gamma^2) \sim \sqrt{\frac{2|\gamma|}{k/n}} \frac{(-1)^{n-k}}{|\Delta|^{1/4}} \frac{1}{\sqrt{\pi} n^{1/6} |\gamma|^{1/2}} \cos\left(\frac{2}{3} n|\gamma|^3 - \frac{\pi}{4}\right),$$

and therefore

$$\sqrt{\frac{2|\gamma|}{k/n}} \frac{(-1)^{n-k}}{|\Delta|^{1/4}} Ai(n^{2/3}\gamma^2)$$

$$\sim \sqrt{\frac{2}{\pi}} \frac{(-1)^{n-k}}{\sqrt{k/n}\left[(\alpha_0 - k/n)(\alpha_0^{-1} - k/n)\right]^{1/4}} \frac{\cos\left(n|ih(\varphi_+) - i\pi(1 - k/n)| - \frac{\pi}{4}\right)}{n^{1/6}},$$

where we use the definitions of γ^3 (see (2.2)) and h (see (1.6)). Using the fact that $\gamma^3 \in i\mathbb{R}_+$ we obtain

$$\cos\left(n|ih(\varphi_+) - i\pi(1-k/n)| - \frac{\pi}{4}\right) = \cos\left(nh(\varphi_+) - \pi(n-k) - \frac{\pi}{4}\right)$$
$$= (-1)^{n-k}\cos\left(nh(\varphi_+) - \frac{\pi}{4}\right).$$

It remains to use Proposition 3 and to divide by $n^{1/3}$.

ii) If $n^{2/3}(\alpha_0^{-1} - k/n) \to +\infty$, then our argument is similar. We use the second formula in Proposition 3 and the fact that this time $\gamma^3 = \gamma^3_{\alpha_0^{-1}} = -\frac{3}{2}ih(\varphi_+)$, see (2.4). □

3.3.2. *The case where $a = k/n$ is separate from the boundaries α_0, α_0^{-1}.* Lemma 5 shows that the location of stationary points of Φ_a in $\mathbb{C}$ is determined by the location of a relative to the critical interval $[\alpha_0, \alpha_0^{-1}]$. The situation where a approaches the boundaries α_0, α_0^{-1} was discussed in the previous subsection. In this case, the stationary points $z_\pm$ degenerate and uniform methods are required. The situation where a is separate from α_0, α_0^{-1}, that is there exists $\beta \in (\alpha_0, 1)$ that separates a from the boundary, $a \in [\beta, \beta^{-1}]$, is different and even simpler. In this case the stationary points $z_\pm = e^{i\varphi_\pm}$ of Φ_a belong to the contour of integration $\partial\mathbb{D}$ and remain separate from ± 1, see below. Since $\left|z^{-k/n}\frac{z-\lambda}{1-\lambda z}\right| = 1$ for any $z \in \partial\mathbb{D}$ we can introduce the real function

$$\tilde{h}(\varphi) = \widetilde{h_a}(\varphi) = -h_a(\varphi) = i\Phi_a(e^{i\varphi}), \qquad \varphi \in [0, \pi],$$

to write the integral as a generalized Fourier integral (the Fourier/Taylor coefficients of b_λ^n are real because $\lambda \in (0,1)$),

$$\overline{\widehat{b_\lambda^n}(k)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-n\Phi_a(e^{i\varphi})}\mathrm{d}\varphi = \frac{1}{\pi}\Re\left\{\int_0^\pi e^{in\widetilde{h_a}(\varphi)}\mathrm{d}\varphi\right\}.$$

The asymptotic behavior of this integral can be determined using A. Erdélyi's standard *method of stationary phase* [20] and the approach from [40, Section 3.1], which will be done at the end of this section. Before that let us mention that a more elaborate version of the classical method of stationary phase, due to M.V. Fedoryuk [22, Theorem 2.4 p. 80] (see [23, Theorem 1.6 p.107] for a simple version in one dimension), will make our proof much shorter. Moreover, Fedoryuk's method immediately provides us with a sharp error term for the first order approximation of $\widehat{b_\lambda^n}(k)$, which holds uniformly for $k \in [\beta n, \beta^{-1}n]$. We first provide this simple proof making use of Fedoryuk's result, and then write in full detail a classical (but longer and more technical) proof of the same formula, using A.Erdélyi's standard method of stationary phase.

Proof of Theorem 2 (2) for $\beta n \le k \le \beta^{-1}n$ using Fedoryuk's method. Suppose that $a = k/n \in [\beta, \beta^{-1}]$. The stationary points of $\tilde{h} = \widetilde{h_a}$ are given by

$$z_\pm = \frac{a(1+\lambda^2) - (1-\lambda^2)}{2\lambda a} \pm i\sqrt{1 - \left(\frac{a(1+\lambda^2) - (1-\lambda^2)}{2\lambda a}\right)^2} \in \partial\mathbb{D}$$

and we write $z_\pm = e^{i\varphi_\pm}$ with $\varphi_+ \in [0, \pi]$ and $\varphi_- \in (-\pi, 0]$. Only z_+ is relevant since we integrate over $[0, \pi]$ and the unique critical point $\varphi_+ = \varphi_+(k/n)$ of $\widetilde{h_a}$ in $(0, \pi)$ satisfies

$x \leq \varphi_+ \leq \pi - x$ for some $x = x(\beta, \lambda) > 0$ because

$$\left|e^{i\varphi_+} - 1\right| \geq (1-\lambda)\sqrt{\frac{\beta}{\lambda}}\sqrt{\alpha_0^{-1} - \beta^{-1}}, \qquad \left|e^{i\varphi_+} + 1\right| \geq (1+\lambda)\sqrt{\frac{\beta}{\lambda}}\sqrt{\beta - \alpha_0}.$$

These inequalities follow from the identities

$$\left|e^{i\varphi_+} - 1\right|^2 = \frac{(1-\lambda)^2\left(\alpha_0^{-1} - a\right)}{a\lambda}, \qquad \left|e^{i\varphi_+} + 1\right|^2 = \frac{(1+\lambda)^2\left(a - \alpha_0\right)}{a\lambda}.$$

For the second derivative we have

$$-i\tilde{h}''(\varphi) = \frac{\partial}{\partial\varphi}\left(\frac{\partial\Phi}{\partial z}\frac{dz}{d\varphi}\right) = \frac{\partial^2\Phi}{\partial z^2}\left(\frac{dz}{d\varphi}\right)^2 + \frac{\partial\Phi}{\partial z}\frac{d^2z}{(d\varphi)^2}.$$

It follows from (3.5) that

$$i\frac{\partial^2\tilde{h}}{\partial\varphi^2}\bigg|_{\varphi=\varphi_+} \quad = \quad z_+^2\Phi''(z_+),$$

which gives, by (3.10), that

$$\tilde{h}''(\varphi_+) = \frac{k}{n}\sqrt{\left(\frac{k}{n} - \alpha_0\right)\left(\alpha_0^{-1} - \frac{k}{n}\right)} \geq \min_{a\in[\beta,\beta^{-1}]} a\sqrt{(a-\alpha_0)(\alpha_0^{-1} - a)} =: C(\beta,\lambda) > 0.$$

We are now ready to apply Fedoryuk's result [22, Theorem 2.4 p. 80] with $d = 1$ and $\Omega = (0, \pi)$ to

$$I(n,a) = \int_0^\pi \nu(\varphi)e^{in\tilde{h}(\varphi)}\mathrm{d}\varphi,$$

where $\tilde{h} = \widetilde{h_a}$, $a = k/n \in [\beta, \beta^{-1}] =: M$, $x = x(\beta,\lambda) > 0$ and $\nu : [0,\pi] \longrightarrow \mathbb{R}$ is a *neutralizer* satisfying $\nu = 1$ on $[x/2, \pi - x/2]$, $\nu = 0$ on $[0, x/4] \cup [\pi - x/4, \pi]$ and $0 \leqslant \nu \leqslant 1$. The compact $K := \left[\frac{x}{4}, \pi - \frac{x}{4}\right]$ satisfies Assumption 2 in [22]. The function $\nu : \varphi \mapsto \nu(\varphi)$ does not depend neither on a nor on $\xi = n$ and Assumption 3 in [22] is satisfied with $m = 0$. Finally for $a = \frac{k}{n} \in M$ the unique critical point $\varphi_+ = \varphi_+(a)$ of $\varphi \mapsto \widetilde{h_a}(\varphi)$ satisfies

$$\tilde{h}''(\varphi_+) \geq C(\beta,\lambda) > 0$$

and Assumptions 4 and 5 in [22] are also satisfied. Applying Fedoryuk's asymptotic formula with $l = 1$, $\alpha_1 = \frac{3}{2}$, $b_1 = \sqrt{2\pi}\left(\tilde{h}''(\varphi_+)\right)^{-1/2}\exp\left(\frac{i\pi}{4}\right)$, we obtain that

$$I(n,a) = \sqrt{2\pi}\left(\tilde{h}''(\varphi_+)\right)^{-1/2}\exp\left(\frac{i\pi}{4}\right)n^{-1/2}e^{in\tilde{h}(\varphi_+)} + \mathcal{O}\left(n^{-3/2}\right)$$

where $\mathcal{O}\left(n^{-3/2}\right)$ is uniform over $k/n \in [\beta, \beta^{-1}]$. It remains to observe that $\int_0^\pi e^{in\widetilde{h_a}(\varphi)}\mathrm{d}\varphi - I(n,a) = \mathcal{O}(n^{-2})$ uniformly for $k/n \in [\beta, \beta^{-1}]$ to conclude that

$$\widehat{b_\lambda^n}(k) = \sqrt{\frac{2}{n\pi}}\frac{\cos\left(n\tilde{h}(\varphi_+) + \pi/4\right)}{\sqrt{k/n}\left[(\alpha_0^{-1} - k/n)(k/n - \alpha_0)\right]^{1/4}} + \mathcal{O}\left(n^{-3/2}\right), \tag{3.17}$$

where $\mathcal{O}\left(n^{-3/2}\right)$ is uniform over $k/n \in [\beta, \beta^{-1}]$. □

For the sake of completeness we end this section by proving the above asymptotic expansion (3.17) using the standard *method of stationary phase* [20].

Proof of Theorem 2 (2) *for* $\beta n \leq k \leq \beta^{-1} n$ *using Erdélyi's method.* To determine the asymptotic behavior we apply a standard result of A. Erdélyi [20, Theorem 4] (see also [1, Theorem 1.3] for a detailed discussion of this result and the involved error estimates), which however requires that the stationary point is an endpoint of the interval of integration. Hence we begin by splitting our generalized Fourier integral:

$$\int_0^\pi e^{in\tilde{h}(\varphi)} \mathrm{d}\varphi = \int_0^{\varphi_+} e^{in\tilde{h}(\varphi)} \mathrm{d}\varphi + \int_{\varphi_+}^\pi e^{in\tilde{h}(\varphi)} \mathrm{d}\varphi.$$

For the second integral, Theorem 4 of [20] yields

$$\begin{aligned}\int_{\varphi_+}^\pi e^{in\tilde{h}(\varphi)} \mathrm{d}\varphi = &\frac{1}{2}\Gamma(1/2)\kappa_1(0) e^{i\frac{\pi}{4}} n^{-1/2} e^{in\tilde{h}(\varphi_+)} + \frac{1}{2}\Gamma(1)\kappa_1'(0) e^{i\frac{\pi}{2}} n^{-1} e^{in\tilde{h}(\varphi_+)} \\ &+ \frac{1}{2}\Gamma(3/2)\kappa_1''(0) e^{i\frac{3\pi}{4}} n^{-3/2} e^{in\tilde{h}(\varphi_+)} + e^{inh(\pi)} \frac{i}{n} \frac{1}{\tilde{h}'(\pi)} \\ &+ R_3^{(1)}(n) + R_3^{(2)}(n),\end{aligned}$$

where

$$\kappa_1(0) = 2^{1/2} \left(\tilde{h}''(\varphi_+)\right)^{-1/2},$$

$$\kappa_1'(0) = -\frac{2}{h''(\varphi_+)} \frac{\tilde{h}^{(3)}(\varphi_+)}{3\tilde{h}''(\varphi_+)},$$

$$\kappa_1''(0) = \frac{2^{5/2}}{\left(\tilde{h}''(\varphi_+)\right)^{3/2}} \left(\frac{5}{36} \left(\tilde{h}^{(3)}(\varphi_+)\right)^2 - \frac{\tilde{h}''(\varphi_+)\tilde{h}^{(4)}(\varphi_+)}{12} \right) \frac{3}{4\tilde{h}''(\varphi_+)^2},$$

the error terms $R_3^{(1)}(n)$, $R_3^{(2)}(n)$ will be explicitly estimated from above in what follows, according to [1, Theorem 1.3], and the function κ_1 will be explicitly defined later on. First of all we observe that for $k \in [\beta n,\, \beta^{-1} n]$ we have

$$\tilde{h}''(\varphi_+) \geq C(\beta, \lambda) > 0$$

which shows in particular that

$$\frac{1}{2}\Gamma(3/2)\kappa_1''(0) e^{i\frac{3\pi}{4}} n^{-3/2} e^{in\tilde{h}(\varphi_+)} = \mathcal{O}\left(n^{-3/2}\right)$$

and that the $\mathcal{O}\left(n^{-3/2}\right)$-term is uniform over $a = \frac{k}{n} \in [\beta, \beta^{-1}]$. Observing that $\tilde{h}'(\pi) = \frac{k}{n} - \alpha_0$ we find that the third term in the expansion of $\int_{\varphi_+}^\pi e^{in\tilde{h}(\varphi)} \mathrm{d}\varphi$ is purely imaginary. The second one will cancel out when we will add to $\int_{\varphi_+}^\pi e^{in\tilde{h}(\varphi)} \mathrm{d}\varphi$, the integral $\int_0^{\varphi_+} e^{in\tilde{h}(\varphi)} \mathrm{d}\varphi$ whose asymptotic expansion is computed below, see (3.20). Now, we show that the error terms $R_3^{(1)}(n)$, $R_3^{(2)}(n)$ both satisfy

$$R_3^{(j)}(n) = \mathcal{O}\left(n^{-3/2}\right), \qquad j = 1, 2$$

again uniformly for $a = \frac{k}{n} \in [\beta, \beta^{-1}]$. To this aim we first recall that for $k \in [\beta n,\, \beta^{-1} n]$ the unique critical point φ_+ of $\tilde{h}$ satisfies $x \leq \varphi_+ \leq \pi - x$ for some $x = x(\beta, \lambda) > 0$. We use the notation from [1, Section 1] and choose $\eta = \frac{x}{4} \in \left(0, \frac{\pi - \varphi_+}{2}\right)$. For $j = 1, 2$, let $\psi_j = I_j \to \mathbb{R}$ be the functions defined by

$$\psi_1(\varphi) = \left(\tilde{h}(\varphi) - \tilde{h}\left(\varphi_+\right)\right)^{\frac{1}{2}}, \quad \psi_2(\varphi) = \tilde{h}\left(\pi\right) - \tilde{h}(\varphi)$$

with $I_1 := [\varphi_+, \pi - \eta]\,, I_2 = [\varphi_+ + \eta, \pi]$ and $s_1 := \psi_1\left(\pi - \eta\right), s_2 := \psi_2\left(\varphi_+ + \eta\right)$. ψ_j is shown to be a diffeomorphism between I_j and $[0, s_j]$, see [1, Proposition 3.2]. For $j = 1, 2$, let $\kappa_j : (0, s_j] \to \mathbb{C}$ be the functions defined by

$$\kappa_j(s) := \left(\psi_j^{-1}\right)'(s).$$

It is shown in [1, Proposition 3.3] that κ_j can be continuously extended to $[0, s_j]$ and that $\kappa_j \in \mathcal{C}^3([0, s_j])$. Let $\nu : [\varphi_+, \pi] \longrightarrow \mathbb{R}$ be a neutralizer such that $\nu = 1$ on $[\varphi_+, \varphi_+ + \eta]$, $\nu = 0$ on $[\pi - \eta, \pi]$ and $0 \leqslant \nu \leqslant 1$, where η is defined above. For $j = 1, 2$, let $\nu_j = [0, s_j] \to \mathbb{R}$ be the functions defined by

$$\nu_1(s) = \nu \circ \psi_1^{-1}(s) \quad , \quad v_2(s) = (1 - \nu) \circ \psi_2^{-1}(s).$$

It is shown in [1, Theorem 1.3] that

$$\left|R_3^{(j)}(n)\right| \leqslant \frac{1}{4}\Gamma\left(\frac{3}{2}\right) n^{-\frac{3}{2}} \int_0^{s_j} \left|\frac{d^3}{ds^3}\left[\nu_j \kappa_j\right](s)\right| \mathrm{d}s, \qquad j = 1, 2.$$

To prove that $R_3^{(j)}(n) = \mathcal{O}\left(n^{-3/2}\right)$ uniformly for $\frac{k}{n} \in [\beta, \beta^{-1}]$ we write

$$\int_0^{s_j} \left|\frac{d^3}{ds^3}\left[\nu_j \kappa_j\right](s)\right| \mathrm{d}s \leq s_j \max_{s \in [0, s_j]} \left|\frac{d^3}{ds^3}\left[\nu_j \kappa_j\right](s)\right|, \qquad j = 1, 2.$$

First, we treat in details the case $j = 2$. We need to show that

$$\max_{s \in [0, s_2]} \left|\kappa_2^{(l)}(s)\right| = \mathcal{O}\left(1\right), \qquad l = 0, \ldots, 3, \tag{3.18}$$

uniformly for $\frac{k}{n} \in [\beta, \beta^{-1}]$. We have

$$\kappa_2(s) = -\frac{1}{\tilde{h}'(\psi_2^{-1}(s))}.$$

Computing the derivatives of κ_2 and taking into account that $\tilde{h}$ and each of its derivatives are uniformly bounded on $[0,\, \pi] \supset [\varphi_+ + \eta,\, \pi]$, we observe that the proof of (3.18) follows from the fact that $\min_{s \in [0, s_2]} |\tilde{h}'(\psi_2^{-1}(s))|$ is uniformly separated from 0. More precisely, for $s \in [0, s_2]$, $\psi_2^{-1}(s) \in [\varphi_+ + \eta,\, \pi]$ and for $\varphi \in (0, \pi)$, we have

$$\tilde{h}''(\varphi) = \frac{2\lambda(1 - \lambda^2)\sin(\varphi)}{(1 + \lambda^2 - 2\lambda\cos(\varphi))^2} > 0$$

which implies that $\tilde{h}'$ is increasing on $[\varphi_+ + \eta,\, \pi]$. We know that $h'(\varphi_+) = 0$, h' does not vanish on $(\varphi_+, \pi]$ and $\tilde{h}'(\pi) = \frac{k}{n} - \alpha_0 > 0$. Therefore, $\min_{s \in [0, s_2]} |\tilde{h}'(\psi_2^{-1}(s))| = \tilde{h}'(\varphi_+ + \eta)$.

By the mean-value theorem there is $\theta \in (\varphi_+,\, \varphi_+ + \eta)$ such that

$$\begin{aligned}\tilde{h}'(\varphi_+ + \eta) &= \tilde{h}'(\varphi_+ + \eta) - \tilde{h}'(\varphi_+) \\ &= \eta \tilde{h}''(\theta) \geq \eta \frac{2\lambda(1-\lambda^2)\sin\theta}{(1+\lambda^2-2\lambda\cos\theta)^2} \\ &\geq \frac{x\lambda(1+\lambda)}{2(1-\lambda)^3} \min_{t\in[x,\pi-3x/4]} \sin(t)\end{aligned}$$

because $x \leq \varphi_+ \leq \theta \leq \varphi_+ + \eta \leq \pi - \frac{3x}{4}$. The same type of argument yields

$$\max_{s\in[0,s_2]} \left|\nu_2^{(l)}(s)\right| = \mathcal{O}\left(1\right), \qquad 0 \leq l \leq 3,$$

uniformly for $\frac{k}{n} \in [\beta, \beta^{-1}]$. Indeed, a direct computation shows that $\nu_2'(s) = -\frac{(1-\nu)'(\psi_2^{-1}(s))}{\tilde{h}'(\psi_2^{-1}(s))}$ and ν_2' is of the same nature as κ_2. We conclude that $R_3^{(2)}(n) = \mathcal{O}\left(n^{-3/2}\right)$ uniformly for $\frac{k}{n} \in [\beta, \beta^{-1}]$.

Now, we deal with the case $j = 1$. We apply the same type of reasoning to show that $R_3^{(1)}(n) = \mathcal{O}\left(n^{-3/2}\right)$ uniformly for $\frac{k}{n} \in [\beta, \beta^{-1}]$. First of all, since $\nu_1(s) = 1$ in some neighborhood of $s = 0$ and $\nu_1'(s) = -\frac{\nu'(\psi_1^{-1}(s))}{\tilde{h}'(\psi_1^{-1}(s))}$, we have

$$\max_{s\in[0,s_1]} \left|\nu_1^{(l)}(s)\right| = \mathcal{O}\left(1\right), \qquad 0 \leq l \leq 3,$$

uniformly for $\frac{k}{n} \in [\beta, \beta^{-1}]$. Indeed, if s is separated from 0, a direct computation shows that $\nu_1^{(l)}$ are expressed as quotients whose numerators are uniformly bounded from above and whose denominators are powers of $\tilde{h}'(\psi_1^{-1})$ which are therefore uniformly separated from 0 (this can be seen, for example, by an application of the mean-value theorem as above). For $s \in [0, s_1]$, we have $\psi_1^{-1}(s) \in [\varphi_+, \pi - \eta]$ and

$$\kappa_1(s) = \frac{2s}{\tilde{h}'(\psi_1^{-1}(s))}.$$

To show that

$$\max_{s\in[0,s_1]} \left|\kappa_1^{(l)}(s)\right| = \mathcal{O}\left(1\right), \qquad 0 \leq l \leq 3, \tag{3.19}$$

uniformly for $\frac{k}{n} \in [\beta, \beta^{-1}]$, we begin with a series of preliminary observations. First, for $0 \leq l \leq 3$, the functions $\kappa_1^{(l)}$ are continuous on the compact $[0, s_1]$, see [1, Proposition 3.3]. Therefore, the function $s \mapsto |\kappa_1^{(l)}(s)|$ attains its maximum on this interval. Second, we recall that the three explicit formulas we have previously written for $\kappa_1^{(l)}(0)$, $0 \leq l \leq 2$, show that these quantities are expressed as quotients whose numerators are uniformly bounded (because $\tilde{h}$ and its derivatives are bounded) and whose denominators are expressed as powers of $\tilde{h}'(\varphi_+) \geq C(\beta, \lambda) > 0$. Therefore for $0 \leq l \leq 3$, we have $|\kappa_1^{(l)}(0)| = \mathcal{O}\left(1\right)$ uniformly for $\frac{k}{n} \in [\beta, \beta^{-1}]$. Third, if s is separated from 0, a direct computation shows again that $\kappa_1^{(l)}$ are expressed as quotients whose numerators are uniformly bounded from above and whose denominators are powers of $h'(\psi_1^{-1})$ which are therefore uniformly separated from 0. We use these observations to prove that for any $0 \leq l \leq 3$, (3.19) holds uniformly

for $\frac{k}{n} \in [\beta, \beta^{-1}]$. We only provide a proof of (3.19) for the case $l = 0$, the other cases $1 \leq l \leq 3$, can be proved similarly. Let $t = t(n) \in [0, s_1]$ be such that

$$\max_{s\in[0,s_1]} |\kappa_1(s)| = |\kappa_1(t(n))| = |\kappa_1(\psi_1(\varphi(n)))|,$$

where $\varphi(n) \in [x, \pi - \eta]$. If $|\kappa_1(t(n))|$ is not uniformly bounded for $\frac{k}{n} = \frac{k(n)}{n} \in [\beta, \beta^{-1}]$ as n tends to ∞, then $|\kappa_1(t(n_l))| \to_l \infty$ for some subsequence $(n_l)_l$ and $\frac{k(n_l)}{n_l} \in [\beta, \beta^{-1}]$. A direct computation shows that

$$\tilde{h}'(\psi_1^{-1}(t(n_l))) = \frac{k(n_l)}{n_l} - \frac{1-\lambda^2}{1+\lambda^2-2\lambda\cos(\varphi(n_l))}.$$

By compactness, we can construct a new subsequence (n_q) (actually extracted from (n_l)) such that both $\frac{k(n_q)}{n_q}$ converges to some $\tilde{\beta} \in [\beta, \beta^{-1}]$ and $\varphi(n_q)$ converges to some $\tilde{\varphi} \in [x, \pi - \eta]$. Passing to the limit as q tends to ∞ we find that

$$\lim_q \tilde{h}'(\psi_1^{-1}(t(n_q))) = \tilde{h}'_{\tilde{\beta}}(\tilde{\varphi}) = \tilde{\beta} - \frac{1-\lambda^2}{1+\lambda^2-2\lambda\cos(\tilde{\varphi})}.$$

Therefore,

$$\lim_q \kappa_1(t(n_q)) = \tilde{\kappa_1}(\tilde{\psi_1}(\tilde{\varphi})),$$

where $\tilde{\psi_1}(\varphi) = \sqrt{\tilde{h}_{\tilde{\beta}}(\varphi) - \tilde{h}_{\tilde{\beta}}(\varphi_+)}$, $\tilde{h}'(\varphi_+) = 0$ and $\tilde{\kappa_1}(s) = \frac{2s}{\tilde{h}'_{\tilde{\beta}}(\tilde{\psi_1}^{-1}(s))}$. This contradicts the assumption $\lim_{q\to\infty} |\kappa_1(t(n_q))| = \infty$.

The analysis of the first integral $\int_0^{\varphi_+} e^{in\tilde{h}(\varphi)}\mathrm{d}\varphi$ is essentially the same but we change the variable of integration $\varphi \mapsto -\varphi$ as suggested in [20, p. 23]. We get

$$\int_0^{\varphi_+} e^{in\tilde{h}(\varphi)}\mathrm{d}\varphi = \int_{-\varphi_+}^{0} e^{in\tilde{h}(-\varphi)}\mathrm{d}\varphi.$$

Applying Theorem 4 of [20] (together with [1, Theorem 1.3] to estimate the $\mathcal{O}-$term), we obtain that

$$\int_{-\varphi_+}^{0} e^{in\tilde{h}(-\varphi)}\mathrm{d}\varphi = \frac{1}{2}\Gamma(1/2)\kappa_3(0)e^{i\frac{\pi}{4}}n^{-1/2}e^{in\tilde{h}(\varphi_+)} + \frac{1}{2}\Gamma(1)\kappa_3(0)e^{i\frac{\pi}{2}}n^{-1}e^{in\tilde{h}(\varphi_+)} - \frac{i}{n}e^{in\tilde{h}(0)}\frac{1}{\tilde{h}'(0)} + \mathcal{O}\left(n^{-3/2}\right) \tag{3.20}$$

with

$$\kappa_3(0) = 2^{1/2}\left(\tilde{h}''(\varphi_+)\right)^{-1/2},$$

$$\kappa_3'(0) = \frac{2}{\tilde{h}''(\varphi_+)}\frac{\tilde{h}^{(3)}(\varphi_+)}{3\tilde{h}''(\varphi_+)},$$

where, as for the above asymptotic expansion of $\int_{\varphi_+}^{\pi} e^{in\tilde{h}(\varphi)}\mathrm{d}\varphi$, the $\mathcal{O}\left(n^{-3/2}\right)$-term is again uniform over $a = \frac{k}{n} \in [\beta, \beta^{-1}]$. Observing that $\tilde{h}(0) = 0$, $\tilde{h}(\pi) = (a-1)\pi$, $\tilde{h}'(0) =$

$\frac{(a-1)(1-\lambda)-2\lambda}{1-\lambda}$ and $\tilde{h}'(\pi) = -\frac{(a-1)(1+\lambda)+2\lambda}{1+\lambda}$ we compute

$$\begin{aligned}\int_0^\pi e^{in\tilde{h}(\varphi)}\mathrm{d}\varphi &= \Gamma(1/2)\left(2^{1/2}\left(\tilde{h}''(\varphi_+)\right)^{-1/2}\right)e^{i\frac{\pi}{4}}n^{-1/2}e^{in\tilde{h}(\varphi_+)} + \mathcal{O}\left(n^{-3/2}\right)\\ &= \frac{\sqrt{2}\Gamma(1/2)e^{in\tilde{h}(\varphi_+)+i\frac{\pi}{4}}}{\sqrt{k/n}\left[(k/n-\alpha_0)\left(\alpha_0^{-1}-k/n\right)\right]^{1/4}} + \mathcal{O}\left(n^{-3/2}\right)\\ &= \frac{\sqrt{2\pi}}{\sqrt{k/n}\left[(k/n-\alpha_0)\left(\alpha_0^{-1}-k/n\right)\right]^{1/4}}e^{in\tilde{h}(\varphi_+)+i\frac{\pi}{4}} + \mathcal{O}\left(n^{-3/2}\right).\end{aligned}$$

We conclude that

$$\begin{aligned}&\frac{1}{\pi}\Re\left\{\int_0^\pi e^{in\tilde{h}(\varphi)}\mathrm{d}\varphi\right\}\\ &= \sqrt{\frac{2}{\pi n}}\frac{\cos\left(n\tilde{h}(\varphi_+)+\frac{\pi}{4}\right)}{\sqrt{k/n}\left[(k/n-\alpha_0)\left(\alpha_0^{-1}-k/n\right)\right]^{1/4}}\big(1+O(n^{-1})\big),\end{aligned}$$

where $\mathcal{O}\left(n^{-3/2}\right)$ is uniform over $k/n \in [\beta, \beta^{-1}]$. □

4. Strongly annular functions with small Taylor coefficients

Let us recall that a function f analytic in the unit disc is said to be strongly annular (we use the notation $f \in \mathcal{SA}$) if

$$\limsup_{r\to 1}\min_{\partial\mathcal{D}(0,r)}|f| = \infty.$$

The question we are interested in here is how small and how (non)-lacunar could be the Taylor coefficients $\widehat{f}(n)$ of f:

$$f(z) = \sum_{n\geq 0}\widehat{f}(n)z^n, \qquad z\in\mathbb{D}.$$

In 1977, Bonar, Carroll, and Piranian [13] produced $f \in \mathcal{SA}$ such that $\widehat{f} \in c_0$. It is clear that if $f \in \mathcal{SA}$, then $\widehat{f} \notin \ell^2$. Furthermore, the function constructed in [13] is far from being lacunary. Given $0 < p < \infty$, set

$$\widetilde{\ell}^p = \Big\{\{a_n\}_{n\geq 0} : \sum_{k\geq 0}\min(|a_{2k}|^p, |a_{2k+1}|^p) < \infty\Big\}.$$

Then, the function f constructed in [13] is such that $\widehat{f} \in c_0 \setminus \widetilde{\ell}^2$.

In this section we are going to get new results in this direction.

Theorem 6. *Let $2 \leq p < q$. There exists $f \in \mathcal{SA}$ such that $\widehat{f} \in \ell^q \setminus \widetilde{\ell}^p$.*

Given a positive function φ on $\mathbb{R}_+$, we set

$$\ell^2_\varphi = \Big\{\{a_n\}_{n\geq 0} : \sum_{n\geq 0}\frac{|a_n|^2}{\varphi(1/|a_n|)} < \infty\Big\}.$$

Theorem 7. *Let φ be an increasing positive function on $\mathbb{R}_+$ such that $\lim_{x\to\infty}\varphi(x)=\infty$. There exists $f\in\mathcal{SA}$ such that $\widehat{f}\in\ell^2_\varphi\setminus\widetilde{\ell^2}$.*

Given $N\geq 1$ we denote

$$g_N(z)=b^N_{1/2}(z)=\left(\frac{z-\frac{1}{2}}{1-\frac{z}{2}}\right)^N.$$

Set

$$u_p(N)=\begin{cases} N^{\frac{1}{p}-\frac{1}{2}}, & 2\leq p<4,\\ (\log N)^{\frac{1}{4}}N^{-\frac{1}{4}}, & p=4,\\ N^{\frac{1}{3p}-\frac{1}{3}}, & p>4,\end{cases}$$

and

$$v_p=\begin{cases}\frac{1}{2}-\frac{1}{p}, & 2\leq p<4,\\ \frac{1}{3}-\frac{1}{3p}, & p>4.\end{cases}$$

We use the following corollary of Theorem 1 and Theorem 2.

Lemma 8. *Given $N\geq 10$, for some $\delta>0$ we have*

(i) $\|g_N\|_{H^\infty(\mathbb{D})}=1$,
(ii) $\min_{\partial\mathcal{D}(0,1-N^{-1})}|g_N|\geq e^{-4}$,
(iii) $|\widehat{g_N}(k)|\leq e^{-\delta k}$, $k\geq 4N$,
(iv) $\|\widehat{g_N}\|_\infty\lesssim N^{-1/2}$,
(v) $\left(\sum_{k\geq 0}\min(|\widehat{g_N}(2k)|^p,|\widehat{g_N}(2k+1)|^p)\right)^{1/p}\asymp\|\widehat{g_N}\|_p\asymp u_p(N)$, $p\geq 2$.

Proof. The properties (i) and (ii) follow immediately from the definition of g_N. Furthermore, we use that by Theorem 1 and Theorem 2, we have several upper estimates on $|\widehat{g_N}(k)|$ for different values of k.

$$|\widehat{g_N}(k)|\lesssim e^{-cN},\qquad 0\leq k<\frac{N}{4}, \tag{4.1}$$

$$|\widehat{g_N}(k)|\lesssim\frac{\exp(-cN(\frac{1}{3}-\frac{k}{N})^{3/2})}{N^{1/2}(\frac{1}{3}-\frac{k}{N}+N^{-2/3})^{1/4}},\qquad \frac{N}{4}\leq k<\frac{N}{3}, \tag{4.2}$$

$$|\widehat{g_N}(k)|\lesssim\frac{1}{N^{1/2}(\frac{k}{N}-\frac{1}{3}+N^{-2/3})^{1/4}},\qquad \frac{N}{3}\leq k<N, \tag{4.3}$$

$$|\widehat{g_N}(k)|\lesssim\frac{1}{N^{1/2}(3-\frac{k}{N}+N^{-2/3})^{1/4}},\qquad N\leq k<3N, \tag{4.4}$$

$$|\widehat{g_N}(k)|\lesssim\frac{\exp(-cN(\frac{k}{N}-3)^{3/2})}{N^{1/2}(\frac{k}{N}-3+N^{-2/3})^{1/4}},\qquad 3N\leq k<4N, \tag{4.5}$$

$$|\widehat{g_N}(k)|\lesssim e^{-ck},\qquad k\geq 4N. \tag{4.6}$$

Next, by Theorem 2 we have two lower estimates on $|\widehat{g_N}|$ for some intervals of values of k:

$$|\widehat{g_N}(k)|\gtrsim N^{-1/3},\qquad \frac{N}{3}\leq k<\frac{N}{3}+N^{1/3}, \tag{4.7}$$

and

$$|\widehat{g_N}(k)| \asymp N^{-1/2} \cos A_N(k), \qquad N \le k \le \frac{6N}{5},$$

where

$$\begin{aligned} A_N(t) &= NH_N(t) - \frac{\pi}{4}, \\ H_N(t) &= -\frac{t\varphi_N(t)}{N} + \psi(\varphi_N(t)), \\ \psi'(s) &= \frac{3}{5 - 4\cos s}, \\ \varphi_N(t) &\in (0, \pi), \\ \cos \varphi_N(t) &= \frac{5}{4} - \frac{3N}{4t}. \end{aligned}$$

Furthermore,

$$\begin{aligned} A_N'(t) &= -\varphi_N(t), \\ A_N''(t) &= \frac{3N}{4t^2 \sin \varphi_N(t)}. \end{aligned}$$

For $t \in [N, 6N/5]$ we have

$$\begin{aligned} \cos \varphi_N(t) &\in [1/2, 5/8], \\ A_N''(t) &\asymp 1/t, \\ -\pi/3 \le A_N'(t) &\le -\pi/4, \end{aligned}$$

and, hence,

$$-\frac{2\pi}{5} \le A_N(k+1) - A_N(k) \le -\frac{\pi}{5}, \qquad N \le k \le \frac{6N}{5} - 1.$$

Thus, for every $k \in [N, \frac{6N}{5} - 1]$,

$$\min(|\widehat{g_N}(k)|, |\widehat{g_N}(k+1)|) \gtrsim N^{-1/2}. \tag{4.8}$$

Finally, (iii) is (4.6), (iv) follows from (4.1)–(4.6), and (v) follows from (4.1)–(4.8). □

Another proof of the second asymptotic relation in Lemma 8 (v) is given in [42].

Proof of Theorem 6. Choose $r \in (p, q) \setminus \{4\}$. Given an integer $A > 1$, set

$$f(z) = \sum_{k \ge 1} A^{kv_r} g_{A^k}(z) z^{A^k}.$$

First of all, the function f is analytic in the unit disc. Furthermore,

$$\begin{aligned} \min_{\partial \mathcal{D}(0, 1-A^{-k})} |f| &\ge \min_{\partial \mathcal{D}(0, 1-A^{-k})} |A^{kv_r} g_{A^k}(z) z^{A^k}| - \sum_{s \ge 1,\, s \ne k} \max_{\partial \mathcal{D}(0, 1-A^{-k})} |A^{sv_r} g_{A^s}(z) z^{A^s}| \\ &\ge e^{-6} A^{kv_r} - \sum_{1 \le s < k} A^{sv_r} - \sum_{s > k} A^{sv_r} \exp(-A^{s-k}) \gtrsim A^{kv_r} \to \infty, \qquad k \to \infty, \end{aligned}$$

if $A^{v_r} \ge A_0$. Thus, $f \in \mathcal{SA}$.

Next, given $\eta > 2$ we have

$$\sum_{n\ge 0} |\widehat{f}(n)|^\eta = \sum_{\ell\ge 1} \sum_{A^\ell \le n < A^{\ell+1}} |\widehat{f}(n)|^\eta$$
$$= \sum_{\ell\ge 1} \Big(\sum_{A^\ell \le n < A^{\ell+1}} A^{\eta\ell v_r} |\widehat{g_{A^\ell}}(n - A^\ell)|^\eta + O(A^{\ell+1}(\ell A^{\eta\ell v_r} e^{-\delta A^\ell})^\eta) \Big)$$
$$= O(1) + \sum_{\ell\ge 1} A^{\eta\ell v_r} \sum_{n\ge 0} |\widehat{g_{A^\ell}}(n)|^\eta,$$

if $A \ge A_1(\delta)$. Thus, $\widehat{f} \in \ell^q$ and $\widehat{f} \notin \widetilde{\ell}^p$. □

Proof of Theorem 7. Given $A > 1$, choose integer N_k such that

$$N_{k+1} \ge A N_k, \qquad \min_{[N_k^{1/4},\infty)} \varphi \ge A^{3k}, \qquad k \ge 1.$$

Now set

$$f(z) = \sum_{k\ge 1} A^k g_{N_k}(z) z^{N_k}.$$

As in the proof of Theorem 6, f is analytic in the unit disc and for $A \ge A_0$ we have

$$\min_{\partial\mathcal{D}(0,1-N_k^{-1})} |f| \gtrsim A^k.$$

Thus, $f \in \mathcal{SA}$.

Next,

$$\sum_{n\ge 0} \frac{|\widehat{f}(n)|^2}{\varphi(1/|\widehat{f}(n)|)} \lesssim O(1) + \sum_{k\ge 1} A^{2k} \frac{\|g_{N_k}\|_2^2}{\varphi(cN_k^{1/3})} < \infty,$$

and, again by Lemma 8, we conclude that $\widehat{f} \in \ell^2_\varphi$ and $\widehat{f} \notin \widetilde{\ell}^2$. □

4.1. **Flat polynomials.** Here we discuss an alternative approach to Theorems 6 and 7 in such a way that they use different constructions of flat polynomials.

Lemma 9. *Given a large N, there exists a polynomial g_N of degree N such that*

(i) $\|g_N\|_{H^\infty(\mathbb{D})} = 1$,
(ii) $\min_{\partial\mathcal{D}(0,1-N^{-2})} |g_N| \gtrsim 1$,
(iii) $\|\widehat{g_N}\|_\infty \lesssim N^{-1/2}$,
(iv) $\big(\sum_{0\le k\le N} \min(|\widehat{g_N}(2k)|^p, |\widehat{g_N}(2k+1)|^p)\big)^{1/p} \asymp \|\widehat{g_N}\|_p \asymp N^{\frac{1}{p}-\frac{1}{2}}$.

One can easily modify the proofs of Theorems 6 and 7 in such a way that they use Lemma 9 instead of Lemma 8.

Furthermore, Lemma 9 follows from a 1978 result of Körner. Solving a Littlewood problem he established in [29, Theorem 6] the existence of polynomials of degree N with unimodular coefficients equivalent to $\sqrt{N}$ on the unit circle. This gives Lemma 9 immediately. This result of Körner is non-constructive. For further progress in this direction including some explicit constructions see [10] and the recent paper [5].

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