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# Boundary-Layer Meteorology

## Nondimensionalization of A Turbulent Boundary--Layer System: Obukhov Length and Monin-Obukhov Similarity Theory --Manuscript Draft--

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<b>Abstract:</b>	<p>The Obukhov length, although often adopted as a characteristic scale of atmospheric boundary layers, has been introduced purely based on a dimensional argument without a deductive derivation from a governing equation system. Here, its derivation is pursued by the nondimensionalization method in the same manner as for the Rossby deformation radius and the Ekman-layer depth. Physical implications of the Obukhov length is inferred by nondimensionalizing the turbulent kinetic energy (TKE) equation for the horizontally-homogeneous boundary layer.</p> <p>A nondimensionalization scale for a full set of equations for the boundary-layer eddies formally reduces to the Obukhov length by multiplying a re-scaling factor to the former. This re-scaling factor increases with increasing stable stratifications of the boundary layer, in which flows tend to be more horizontal and gentler, thus the Obukhov length increasingly loses its relevance. A heuristic, but deductive derivation of the Monin-Obukhov similarity theory is also outlined based on the obtained nondimensionalization results.</p>	
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1 **Nondimensionalization of A Turbulent**  
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4 **Jun–Ichi Yano · Marta Waclawczyk**

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8 **Abstract** The Obukhov length, although often adopted as a characteris-  
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22 nondimensionalization results.

23 **Keywords** Nondimensionalization scale · Obukhov length · Stably–stratified  
24 turbulence · Vertical scale · TKE equation ·

25 **1 Introduction**

26 Theories of atmospheric boundary–layer turbulence have been developed by  
27 heavily relying on the so–called dimensional analyses (Barenblatt 1996). This

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methodology is alternatively called *scaling* in atmospheric boundary-layer studies, as reviewed by *e.g.*, Holtslag and Nieuwstadt (1986), Foken (2006). Some key variables controlling a given regime of turbulence are first identified, then various characteristic scales of the system (*e.g.*, length, velocity, temperature) are determined from these key controlling variables by a dimensional consistency. For example, the Obukhov length (Obukhov 1948) is defined, by dimensional analysis, from the frictional velocity and the buoyancy flux. Resulting theories from these analyses are called “similarity theories”, because they remain similar regardless of specific cases by re-scaling the variables in concern by characteristic scales.

In performing a dimensional analysis correctly, a certain ingenuity is required for choosing proper controlling variables of a given system. No systematic methodology exists for choosing them, but the choice is solely based on physical intuitions. A wrong choice of controlling variables can lead to totally meaningless results (*cf.*, Batchelor 1954). With the absence of an analytical solution to turbulent flows as well as difficulties in observations and numerical modelling, usefulness of those proposed scales is often hard to judge. As a result, the Obukhov length is hardly a unique choice. There are various efforts to introduce alternative scales, as further discussed in Sects. 2.4 and 6.1, but the theory based on gradient-based scales (Sorbjan 2006, 2010, 2016) is probably the most notable.

However, the dimensional analysis is not a sole possibility of defining characteristic scales of a system. In atmospheric large-scale dynamics, these scales are typically derived by nondimensionalizations. No doubt, this procedure is more straightforward and formal: characteristic scales are introduced for all the variables of a system for nondimensionalizing them. These scales cannot be arbitrary, because we expect that terms in an equation balance each other, thus their orders of magnitudes must match each other. These conditions, in turn, constrain these characteristic scales in a natural manner. Advantage of the nondimensionalization analysis is that these scales are defined not only by dimensional consistencies, but also by requirements of balance by order of magnitude between the terms in a system. The Rossby radius of deformation is a classical example of a characteristic scale identified by a nondimensionalization. This scale characterizes the quasi-geostrophic system (Sect. 3.12, Pedlosky 1987). The depth of the Ekman layer is another such example (Sect. 4.3, Pedlosky 1987). Yano and Tsujimua (1987), Yano and Bonazzola (2009) systematically apply this methodology for the scale analysis.

In a certain sense, nondimensionalization is a *brute* approach without relying on any physical intuitions, even observation or modeling. The procedure is totally formal and abstract. However, it can lead us to identify physically meaningful scales, because it is applied to equations that govern a given system physically: see *e.g.*, discussions concerning Eqs. (5a, b) in Sect. 2.3 below.

Note a subtle difference of the concept of the scale in nondimensionalization from that in similarity theories: in similarity theories, precise values are used to scale the variables in universal functions that define a solution of a system. On the other hand, under the nondimensionalization analysis, the scales are

74 typically defined by orders of magnitudes, as just stated. This difference must  
75 clearly be kept in mind in the following analyses: see further discussions in  
76 Sects. 5. However, in principle, the characteristic scales adopted in boundary–  
77 layer similarity theories should be linked to the nondimensionalization scales  
78 identified by nondimensionalization analysis, if the former have physical basis  
79 to be adopted. The most basic motivation of the present study is to investigate  
80 whether this is the case.

81 As a first step of such a systematic investigation, the present study focuses  
82 on the Obukhov length. This scale is a core of the celebrated Monin–Obukhov  
83 similarity theory (Monin and Obukhov 1954). In spite of its importance in de-  
84 scribing boundary-layer turbulence, the basis of this Obukhov length is often  
85 questioned. In particular, previous studies suggest that this similarity theory  
86 breaks down in a strongly-stratified limit (*cf.*, King 1990, Howell and Sun  
87 1999, Mahrt 1999). Various efforts for generalization of the Monin–Obukhov  
88 theory already exist (*e.g.*, Zilitinkevich and Calanca 2000, Zilitinkevich 2002,  
89 Zilitinkevich and Esau 2007). However, as far as the authors are aware of, all  
90 these efforts are under frameworks of the dimensional analysis. The present pa-  
91 per is going to suggest a procedure beyond those efforts by analysing governing  
92 equations of turbulence more directly.

93 The nondimensionalization procedure itself is hardly new in boundary–  
94 layer meteorology. For example, Mahrt (1982) adopts it for analyzing gravity-  
95 wave currents in the boundary layer. A full nondimensionalization performed  
96 by Nieuwstadt (1984) is closer to the spirit of the present study by focusing  
97 on the stably–stratified boundary layer. However, unlike the present study,  
98 applicability of his nondimensionalization result is rather limited by adopting  
99 a model already including a closure, thus a final nondimensionalization result  
100 also depends on this closure. In the present study, we consider boundary–layer  
101 governing equation systems without closure for this reason.

102 The present analysis is close to the spirit of George *et al.* (2000) in seek-  
103 ing to identify characteristic scales of a given system by directly examining a  
104 balance condition in a governing equation, although their study does not go  
105 through a path of nondimensionalization. This link further suggests possibil-  
106 ities of applying various methodologies of multiscale asymptotic expansions  
107 (*cf.*, Yano and Tsujimua 1987) to the atmospheric boundary-layer problems,  
108 though attempts are left for future studies.

109 The paper proceeds as follows. The basics of nondimensionalization method  
110 are introduced in Sect. 2. First (Sect. 2.1), by taking a linear stability problem  
111 of a vertical–shear flow with stratification as a simple pedagogical example, it  
112 is demonstrated in explicit manner how characteristic scales can be identified  
113 by a nondimensionalization analysis. The basic premises of the method are  
114 then discussed (Sect. 2.3), because they become crucial in its application to  
115 the atmospheric boundary layer in subsequent sections. The adopted example  
116 is a standard stability problem in fluid mechanics (*cf.*, Miles 1961, Howard  
117 1961).

118 This example also serves for a purpose of introducing the Richardson num-  
119 ber, which plays a key role in the Monin–Obukhov theory (*cf.*, Łobocki 2013),

120 as a nondimensional parameter of the problem. In standard analyses, the  
 121 Richardson number is introduced in retrospect only after solving the stabil-  
 122 ity problem in dimensional form, with an exception of Sect. 8.1 of Townsend  
 123 (1976), which outlines a nondimensionalization analysis of this problem.

124 The main analyses are presented in Sects. 3–4. The turbulent-kinetic energy  
 125 (TKE) equation is considered in Sect. 3 as a preliminary analysis. However,  
 126 the TKE equation for the boundary-layer turbulence is hardly self-contained,  
 127 thus its generalization is much desirable. For this reason, the linear analysis of  
 128 Sect. 2 is generalized into a fully nonlinear case in Sect. 4. The analysis identi-  
 129 fies a vertical scale for the nondimensionalization with an explicit dependence  
 130 of its definition on the Richardson number. The identified nondimensional-  
 131 ization scale can directly be linked to the Obukhov length with a re-scaling  
 132 factor, which is defined in terms of the nondimensional eddy amplitude and the  
 133 aspect ratio of dominant turbulent eddies. Furthermore, the similarity theory  
 134 is derived in Sect. 5 in a heuristic manner from a derived nondimensionalized  
 135 system in Sect. 4. The paper is concluded by final remarks in Sect. 6.

136 Throughout the paper, Boussinesq approximation is adopted for the analy-  
 137 sis for simplicity. Though an extension of the analysis under anelastic approx-  
 138 imation is straightforward, only with little advantage by making the analysis  
 139 more involved. For the same reason, a two-dimensional system is considered in  
 140 Sect. 2 with  $x$  and  $z$  taken as horizontal and vertical coordinates, respectively.  
 141 Also keep in mind that throughout the paper, the vertical nondimensional-  
 142 ization scale is always that of the dominant eddy of a system. Thus, a cer-  
 143 tain caution is required to link it to the Obukhov length, because the latter is  
 144 adopted to be in many studies a vertical characteristic scale for vertical profiles  
 145 of vertical heat and momentum fluxes. Obukhov (1948) originally introduced  
 146 this scale to measure the height of the surface boundary layer (“sub-layer of  
 147 dynamic turbulence”).

148 The bar,  $\bar{\phantom{x}}$ , and the prime,  $\prime$ , signs are used throughout for designating the  
 149 mean and the deviation from the former. The former is assumed to depend  
 150 only on height,  $z$ . The latter corresponds to the turbulence fluctuation when  
 151 a fully nonlinear problem is considered, as in Sects. 3–4. On the other hand,  
 152 under a linear stability analysis in Sect. 2, the primed quantities designate the  
 153 perturbation variables. Due to the linearization, the perturbation variables  
 154 may grow to infinity under an unstable situation, whereas the turbulent fluc-  
 155 tuations are bounded by full nonlinearity. Nevertheless, the formal definition  
 156 of the prime sign itself does not change over these sections.

## 2 A Linear Perturbation Problem: Basic Premises of Nondimensionalization

### 2.1 Analysis

The purpose of this section is to introduce the basics of the nondimensionalization method, that is systematically exploited in the following sections. As a simple concrete example for a demonstration, we consider a linear perturbation equation for a standard shear instability. Viscosity is neglected by following the standard formulation (*cf.*, Miles 1961, Howard 1961). Thus, the governing equations of the problem are:

$$\frac{\partial u'}{\partial t} + w' \frac{d\bar{u}}{dz} + \bar{u} \frac{\partial u'}{\partial x} = -\frac{\partial \phi'}{\partial x}, \quad (1a)$$

$$\frac{\partial w'}{\partial t} + \bar{u} \frac{\partial w'}{\partial x} = -\frac{\partial \phi'}{\partial z} + b', \quad (1b)$$

$$\frac{\partial b'}{\partial t} + \bar{u} \frac{\partial b'}{\partial x} + w' \frac{d\bar{b}}{dz} = 0, \quad (1c)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0. \quad (1d)$$

Here,  $u$  and  $w$  are the horizontal and the vertical components of the velocity,  $\phi$  a perturbation pressure divided by a reference density,  $b'$  the perturbation buoyancy, which may be evaluated from the perturbation potential temperature,  $\theta'$ , as  $b' = g\theta'/\theta_0$ , where  $g$  is the acceleration of the gravity,  $\theta_0$  is a reference value of the potential temperature. The goal of a perturbation problem in this section is to identify the unstable eddy modes.

Throughout the paper, nondimensionalizations are performed on the variables by designating the nondimensional variables by the dagger,  $\dagger$ . The nondimensionalization scales for given variables are designated by superscript  $*$ . Note that the nondimensionalization scales are always defined to be positive definite, for example, regardless of whether the atmosphere is stably stratified or not, unlike the standard convention of the boundary-layer meteorology. This definition is consistent with the basic nature of the nondimensionalization that only the balances by the order of magnitude are in concern, thus the signs of the terms are not issues.

Consequently,

$$u' = u^* u^{\dagger}, \quad w' = u^* w^{\dagger}, \quad \phi' = \phi^* \phi^{\dagger}, \quad b' = b^* b^{\dagger},$$

$$\frac{\partial}{\partial t} = \frac{1}{t^*} \frac{\partial}{\partial t^{\dagger}}, \quad \frac{\partial}{\partial x} = \frac{1}{z^*} \frac{\partial}{\partial x^{\dagger}}, \quad \frac{\partial}{\partial z} = \frac{1}{z^*} \frac{\partial}{\partial z^{\dagger}},$$

and also

$$\bar{u} = \bar{u}^* \bar{u}^{\dagger}, \quad \frac{d\bar{u}}{dz} = \left(\frac{d\bar{u}}{dz}\right)^* \frac{d\bar{u}^{\dagger}}{dz^{\dagger}}, \quad \frac{d\bar{b}}{dz} = \left(\frac{d\bar{b}}{dz}\right)^* \frac{d\bar{b}^{\dagger}}{dz^{\dagger}}.$$

Here, the velocity and the spatial scales are not distinguished in horizontal and vertical directions for simplicity, thus  $x^* = z^*$  and  $w^* = u^*$ . Also note

186 that the background-state gradients,  $d\bar{u}/dz$  and  $d\bar{b}/dz$ , are nondimensionalized  
 187 directly by using the scales for these gradients, rather than by setting  $\bar{u}/z^*$   
 188 and  $\bar{b}/z^*$ , respectively, for the purpose of deriving a standard definition of the  
 189 Richardson number in the following. Furthermore, to simplify the notions, the  
 190 prime sign,  $\prime$ , has been omitted from the nondimensionalization scales for the  
 191 perturbations.

192 By substituting these expressions into Eqs. (1a, b, c, d), we obtain

$$\frac{u^*}{t^*} \frac{\partial u^\dagger}{\partial t^\dagger} + u^* \left( \frac{d\bar{u}}{dz} \right)^* w^\dagger \left( \frac{d\bar{u}}{dz} \right)^\dagger + \frac{\bar{u}^* u^*}{z^*} \bar{u}^\dagger \frac{\partial u^\dagger}{\partial x^\dagger} = -\frac{\phi^*}{z^*} \frac{\partial \phi^\dagger}{\partial x^\dagger}, \quad (2a)$$

$$\frac{u^*}{t^*} \frac{\partial w^\dagger}{\partial t^\dagger} + \frac{\bar{u}^* u^*}{z^*} \bar{u}^\dagger \frac{\partial w^\dagger}{\partial x^\dagger} = -\frac{\phi^*}{z^*} \frac{\partial \phi^\dagger}{\partial z^\dagger} + b^* b^\dagger, \quad (2b)$$

$$\frac{b^*}{t^*} \frac{\partial b^\dagger}{\partial t^\dagger} + \frac{b^* \bar{u}^*}{z^*} \bar{u}^\dagger \frac{\partial b^\dagger}{\partial x^\dagger} + u^* \left( \frac{d\bar{b}}{dz} \right)^* w^\dagger \left( \frac{d\bar{b}}{dz} \right)^\dagger = 0, \quad (2c)$$

$$\frac{u^*}{z^*} \frac{\partial u^\dagger}{\partial x^\dagger} + \frac{u^*}{z^*} \frac{\partial w^\dagger}{\partial z^\dagger} = 0. \quad (2d)$$

193 The basic procedure of defining the scales,  $u^*$ ,  $\phi^*$ , etc. under the nondimen-  
 194 sionalization is to set the prefactors in front of all the terms in the equations  
 195 to be unity as much as possible. The continuity (Eq. 2d) already satisfies this  
 196 condition, because two prefactors are identical. A final nondimensional equation  
 197 is obtained simply by dividing both sides by the common prefactor,  $u^*/z^*$ ,  
 198 thus a final prefactor in front of all the terms becomes unity. In the other equa-  
 199 tions, the same goal is achieved by setting those prefactors to be equal each  
 200 other as much as possible. These conditions define the characteristic scales of  
 201 the system. For example, by setting the prefactors in front of the first and the  
 202 third terms (temporal and advection tendencies) in Eq. (2a) equal, we obtain

$$\frac{u^*}{t^*} = \frac{\bar{u}^* u^*}{z^*},$$

203 which leads to a definition of the time scale,  $t^*$ .

204 By proceeding in this manner, we identify the nondimensionalization scales  
 205 as:

$$t^* = \frac{z^*}{\bar{u}^*}, \quad \phi^* = \bar{u}^* u^*, \quad b^* = \frac{\bar{u}^* u^*}{z^*}. \quad (3a, b, c)$$

206 The resulting nondimensionalized set of equations is

$$\frac{\partial u^\dagger}{\partial t^\dagger} + \left[ \frac{z^*}{\bar{u}^*} \left( \frac{d\bar{u}}{dz} \right)^* \right] \frac{d\bar{u}}{dz^\dagger} w^\dagger + \bar{u}^\dagger \frac{\partial u^\dagger}{\partial x^\dagger} = -\frac{\partial \phi^\dagger}{\partial x^\dagger}, \quad (4a)$$

$$\frac{\partial w^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial w^\dagger}{\partial x^\dagger} = -\frac{\partial \phi^\dagger}{\partial z^\dagger} + b^\dagger, \quad (4b)$$

$$\frac{\partial b^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial b^\dagger}{\partial x^\dagger} + \left( \frac{z^*}{\bar{u}^*} \right)^2 \left( \frac{d\bar{b}}{dz} \right)^* \frac{d\bar{b}}{dz^\dagger} w^\dagger = 0, \quad (4c)$$

$$\frac{\partial u^\dagger}{\partial x^\dagger} + \frac{\partial w^\dagger}{\partial z^\dagger} = 0. \quad (4d)$$

207 As intended, there is no constant prefactor in front of almost all the terms in  
 208 the equations, except for the two in Eqs. (4a) and (4c). Here, by choosing a  
 209 length scale,  $z^*$ , in an appropriate manner, we can remove a constant prefactor  
 210 from one of these two terms, but not from both of them.

211 As a result, in turn, there are only two options for defining the length  
 212 scale,  $z^*$ , consistently under the nondimensionalization analysis: set either of  
 213 those two undefined prefactors to be unity. This is a fundamental difference  
 214 from the dimensional analysis: with the latter, we can find variety of different  
 215 ways of defining a length scale solely based on a dimensional consistency by  
 216 taking various available dimensional parameters. However, under the nondi-  
 217 mensionalization analysis, we do not have such a liberty: the given equation  
 218 set dictates us in a more specific manner, what the options are, and we have  
 219 to follow them.

220 Thus, these two options for setting the length scale,  $z^*$ , are: (i) by setting a  
 221 prefactor in front of the second term of Eq. (4a) to be unity, or (ii) by setting  
 222 a prefactor in front of the second term in Eq. (4c) to be unity. The option  
 223 (i) amounts to set the spatial scale (shear scale) to be that of the background  
 224 wind shear, and we obtain

$$z^* = z^{*u} \equiv \bar{u}^*/(d\bar{u}/dz)^*. \quad (5a)$$

225 The option (ii) leads to

$$z^* = z^{*b} \equiv \bar{u}^*/(d\bar{b}/dz)^{1/2}. \quad (5b)$$

226 The latter may be called the buoyancy–gradient scale. Recall that all the  
 227 nondimensionalization scales are defined to be positive definite, thus the def-  
 228 inition of the buoyancy scale above is applicable to both the stable and the  
 229 unstable situations.

230 Consequently, the set of equations also reduces with the option (i) to:

$$\frac{\partial u^\dagger}{\partial t^\dagger} + \frac{d\bar{u}^\dagger}{dz^\dagger} w^\dagger + \bar{u}^\dagger \frac{\partial u^\dagger}{\partial x^\dagger} = -\frac{\partial \phi^\dagger}{\partial x^\dagger}, \quad (6a)$$

$$\frac{\partial w^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial w^\dagger}{\partial x^\dagger} = -\frac{\partial \phi^\dagger}{\partial z^\dagger} + b^\dagger, \quad (6b)$$

$$\frac{\partial b^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial b^\dagger}{\partial x^\dagger} + Ri \left( \frac{d\bar{b}^\dagger}{dz^\dagger} \right) w^\dagger = 0, \quad (6c)$$

$$\frac{\partial u^\dagger}{\partial x^\dagger} + \frac{\partial w^\dagger}{\partial z^\dagger} = 0, \quad (6d)$$

231 and with the option (ii) to:

$$\frac{\partial u^\dagger}{\partial t^\dagger} + Ri^{-1/2} \frac{d\bar{u}^\dagger}{dz^\dagger} w^\dagger + \bar{u}^\dagger \frac{\partial u^\dagger}{\partial x^\dagger} = -\frac{\partial \phi^\dagger}{\partial x^\dagger}, \quad (7a)$$

$$\frac{\partial w^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial w^\dagger}{\partial x^\dagger} = -\frac{\partial \phi^\dagger}{\partial z^\dagger} + b^\dagger, \quad (7b)$$

$$\frac{\partial b^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial b^\dagger}{\partial x^\dagger} + \frac{d\bar{b}^\dagger}{dz^\dagger} w^\dagger = 0, \quad (7c)$$

$$\frac{\partial u^\dagger}{\partial x^\dagger} + \frac{\partial w^\dagger}{\partial z^\dagger} = 0. \quad (7d)$$

232 Here,  $Ri$  is the Richardson number defined by a ratio of the two characteristic  
233 scales:

$$Ri = \left( \frac{z^{*u}}{z^{*b}} \right)^2 = \frac{(d\bar{b}/dz)^*}{(d\bar{u}/dz)^{*2}}. \quad (8)$$

234 Note again that the Richardson number is positive definite due to our defini-  
235 tions of the nondimensional scales to be positive definite.

236 We find that when the shear is more dominant than the buoyancy gradient  
237 (stratification), *i.e.*,  $Ri < 1$ , the nondimensionalization based on the shear  
238 scale,  $z^{*u}$ , (Eq. 5a) is relevant, and when the stratification is more dominant  
239 than the shear, *i.e.*,  $Ri > 1$ , the nondimensionalization based on the buoyancy-  
240 gradient scale,  $z^{*b}$ , (Eq. 5b) becomes relevant.

## 241 2.2 Asymptotic Limit of $Ri \rightarrow 0$

242 The system under asymptotic limit<sup>1</sup> of  $Ri \rightarrow 0$ , *i.e.*, when the Richardson  
243 number is very small, warrants further discussions, because in this limit, the  
244 leading-order of the buoyancy equation (6c) reduces to a homogeneous solu-  
245 tion:

$$\frac{\partial b^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial b^\dagger}{\partial x^\dagger} = 0.$$

246 The evolution of the buoyancy is simply described by advection by the back-  
247 ground flow,  $\bar{u}^\dagger$ . The time-evolving buoyancy, in turn, acts as forcing in  
248 Eq. (6b) to generate a perturbation flow as a consequence.

249 However, this leading-order solution of the buoyancy never becomes un-  
250 stable, being purely advective. Thus, if we decide to focus only on the unstable  
251 modes, the leading-order buoyancy can be neglected, and we must re-scale it  
252 into:

$$\tilde{b}^\dagger = Ri^{-1} b^\dagger \quad (9)$$

253 so that the buoyancy equation is also re-scaled into:

$$\frac{\partial \tilde{b}^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial \tilde{b}^\dagger}{\partial x^\dagger} + \frac{d\bar{b}^\dagger}{dz^\dagger} w^\dagger = 0.$$

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<sup>1</sup> Refer to *e.g.*, Olver (1974), Bender and Orszag (1978) on the concept of the asymptotic limit; see Sect. 5 of Yano (2015) for a pedagogical introduction. Not be confused with the *analytical* limit.

254 After the re-scaling, the third term with the background stratification,  $d\bar{b}^\dagger/dz^\dagger$ ,  
 255 contributes to a weak generation of the perturbation buoyancy. In turn, how-  
 256 ever, the generated weak buoyancy no longer feeds back to the momentum  
 257 equation (7b) to the leading order.

## 258 2.3 Discussions

259 We have demonstrated in Sect. 2.1 how characteristic scales (not only the  
 260 length scale) of a system can be determined by nondimensionalization nat-  
 261 urally. A question that we are going to address in the next two sections is  
 262 whether the Obukhov length can be derived in a similar manner for turbulent  
 263 boundary-layer flows. In addressing this question, we will also proceed with  
 264 the identical basic strategy as in the present section: to constrain a given par-  
 265 tial differential equation solely based on the nondimensionalization analysis.  
 266 No observational measurement will be taken into account.

267 From this demonstration, the following points may be noted:

- 268 1) For maintaining the generality, it is *imperative* to retain all the terms in  
 269 the equation system as much as possible, by keeping prefactors unity. In the  
 270 example here, only a single term remains with a prefactor nonunity ( $Ri$  or  
 271  $Ri^{-1/2}$  depending on the size of  $Ri$ ). There is no arbitrariness in the nondi-  
 272 mensionalization: the result is unique.
- 273 2) The nondimensional set of equations obtained can solve the given shear-  
 274 instability problem in general manner, without any approximations, once a  
 275 profile of the background flow,  $\bar{u}$ , is fixed, regardless of the actual scales (*e.g.*,  
 276 in a tank, over a planet) as well as magnitude of a given flow, but only by  
 277 specifying a value of a single nondimensional number,  $Ri$ . An asymptotic limit  
 278 of  $Ri \rightarrow 0$  or  $Ri \rightarrow \infty$  may be taken to consider a limiting background state.  
 279 In that case, one of the terms in the system can be dropped, but still a full  
 280 set of solutions under these limiting states can be obtained.
- 281 3) However, if we drop any extra terms from this system, a resulting stability  
 282 analysis loses its generality, by limiting a possibility of solutions by further  
 283 simplifications. That is why it is crucial to keep all the terms to be of the same  
 284 order of magnitude under nondimensionalization. Exactly the same principle  
 285 will be applied to boundary-layer problems in Sects. 3 and 4 for achieving the  
 286 same goal.
- 287 4) Not all the terms may contribute equally, in practice, with some spe-  
 288 cific solutions arising from this equation system. For example, as discussed  
 289 in Sect. 2.2, under the asymptotic limit of  $Ri \rightarrow 0$ , the buoyancy can be re-  
 290 scaled by Eq. (9). More generally, when steady solutions are sought, terms  
 291 with time derivatives drop out. One can also consider a perturbation mode  
 292 only slowly varying with time (*e.g.*, a weakly-unstable mode). We can focus  
 293 on those situations by re-scaling the time by  $\partial/\partial t^\dagger = \hat{\epsilon}\partial/\partial\tau^\dagger$  with a small  
 294 nondimensional parameter,  $\hat{\epsilon}$ , then  $\tau^\dagger$  is a resulting nondimensional slow time.  
 295 However, importantly, these cases are only subcategories of the general case  
 296 retained by Eqs. (4a, b, c, d).

Having said those, definition of the time scale,  $t^*$ , by advection (Eq. 3a) may appear to be rather restrictive. However, this appearance is rather superfluous: by substituting the two possible spatial scales, given by Eqs. (5a, b), we obtain  $t^* = (d\bar{u}/dz)^{-1}$  and  $t^* = (d\bar{b}/dz)^{-1/2}$ , corresponding to the shear- and the buoyancy-driven situations, respectively.

Keep in mind that in applying the same methodology to the atmospheric boundary layer in Sects. 3 and 4, the *same level* of generality as in the present section is also maintained. Due to this generality, no particular boundary-layer regime is specified in most part of the analyses, simply because the analyses are performed in a general manner. Identification of the regimes only follow from there based on nondimensional parameters characterizing a given system.

## 2.4 Implications to the boundary-layer problems

The result obtained in this section already has implications in the boundary-layer problems, because the identified characteristic scales,  $z^{u^*}$  and  $z^{b^*}$ , (Eqs. 5a, b) are expected to characterize the typical size of eddies of given regimes, and thus, also characterize the resulting mixing lengths,  $l^*$ . In this respect, it may be worthwhile to note that, for example, Grisogono (2010) proposes to use two different mixing lengths,

$$l^* = \frac{u^*}{(d\bar{u}/dz)^*}, \quad (10a)$$

$$l^* = \frac{u^*}{(d\bar{b}/dz)^{*/2}}, \quad (10b)$$

depending on the Richardson number,  $Ri$ . A similar scale (buoyancy scale),

$$l^* = \frac{w^*}{(d\bar{b}/dz)^{*/2}}, \quad (10c)$$

is introduced by Stull (1973), Zeman and Tennekes (1977), Brost and Wyngaard (1978). Hunt *et al.* (1985), in turn, suggest by field data analysis that the buoyancy scale (Eq. 10c) characterizes both the vertical heat transport and the temperature-variance production in the stably-stratified boundary layer. These definitions reduces to  $z^{*u}$  and  $z^{*b}$ , respectively, with small and large Richardson numbers by re-setting as  $u^* = \bar{u}^*$  and  $w^* = \bar{w}^*$ . This condition is expected to be satisfied when a system is fully turbulent.

## 3 Boundary-Layer System: Turbulent Kinetic Energy Equation

### 3.1 Obukhov Length

Now, we turn to the nondimensionalization of boundary-layer systems. The goal is to derive the Obukhov length as a natural consequence of nondimensionalization in a similar manner as in Sect. 2, and more specifically as discussed in Sect. 2.3.

329 The Obukhov length is defined by a ratio of fractional powers of the scales  
 330 for the vertical momentum stress and the vertical buoyancy flux,  $\overline{u'w'^*}$  and  
 331  $\overline{w'b'^*}$ :

$$L = \frac{\overline{u'w'^*}^{3/2}}{\overline{w'b'^*}}. \quad (11)$$

332 The definition of the Obukhov length is often alternatively presented as

$$L = \frac{u_{*r}^3}{(\overline{w'b'})^*}$$

333 by introducing a friction velocity,  $u_{*r}$ , defined by

$$u_{*r}^2 = (\overline{u'w'})^*.$$

334 Note that here the Obukhov length (11) is introduced as one of the nondi-  
 335 mensionalization scales, thus it differs from the standard definition in two  
 336 major respects: i) scales of fluxes are used rather than flux values themselves.  
 337 ii) these scales only represent orders of magnitudes rather than the actual  
 338 measured values at the surface. Also keep in mind that the Obukhov length  
 339 defined by Eq. (11) is positive definite with the flux scales chosen positive  
 340 definite.

### 341 3.2 Turbulent kinetic energy equation

342 Can we derive the Obukhov length by following a principle of the nondimen-  
 343 sionalization? That is a question considered in the following two sections. As  
 344 seen in Eq. (11), the Obukhov length is defined from the vertical buoyancy  
 345 flux,  $\overline{w'b'}$ , and the vertical momentum stress,  $\overline{u'w'}$ . Thus, for the purpose of  
 346 deriving this scale as directly as possible, we would need to take an equa-  
 347 tion system that contains these two variables for nondimensionalization. We  
 348 consider the turbulent kinetic energy (TKE) equation for this reason in this  
 349 section.

350 By following a standard formulation of the atmospheric boundary-layer  
 351 turbulence, only the vertical-flux terms are retained assuming horizontal ho-  
 352 mogeneity. This is solely for simplifying the analysis focusing on the goal of  
 353 deriving the Obukhov length. Horizontal heterogeneity is expected to be im-  
 354 portant for some stably-stratified atmospheric boundary-layer flows, but this  
 355 extension is left for a future study. Note that as in Sect. 2, the following anal-  
 356 yses are performed in general manner without discriminating the sign of the  
 357 stratification. However, as just suggested, our focus in applications will be in  
 358 stably stratified cases.

359 The TKE equation (*e.g.*, Deardorff 1983) is given by:

$$\frac{\partial}{\partial t} \frac{\overline{\mathbf{v}'^2}}{2} = \overline{w'b'} - \overline{u'w'} \frac{d\bar{u}}{dz} - \frac{\partial}{\partial z} \overline{w' \left( \frac{\mathbf{v}'^2}{2} + \phi' \right)} - D. \quad (12)$$

360 Here, the overbar designates a horizontal average,  $\mathbf{v}$  is a velocity vector, and  
 361  $D$  the dissipation rate. We refer to Wyngaard and Coté (1971) for the basics  
 362 of the boundary-layer TKE budget.

363 Here, in the present paper, we retain the temporal tendency in the equa-  
 364 tions for the generality of the analyses. Although the original Monin–Obukhov  
 365 similarity theory assumes quasi-stationary, the question here is an extent that  
 366 this theory may also be applicable to a transient system.

### 367 3.3 Nondimensionalization

368 In boundary-layer similarity theories, flux values are directly used as dimen-  
 369 sional variables, corresponding to the nondimensionalization scales in the ter-  
 370 minology here. By following this approach, fluxes are now nondimensionalized  
 371 by their own nondimensionalization scales, thus:

$$\begin{aligned}\overline{w'b'} &= (\overline{w'b'})^* \overline{w'b'}^\dagger, \\ \overline{u'w'} &= (\overline{u'w'})^* \overline{u'w'}^\dagger, \\ \overline{w'\mathbf{v}'^2} &= (\overline{w'\mathbf{v}'^2})^* \overline{w'\mathbf{v}'^2}^\dagger, \\ \overline{w'\phi'} &= (\overline{w'\phi'})^* \overline{w'\phi'}^\dagger, \\ \overline{w'^2} &= w^{*2} \overline{w'^2}^\dagger, \\ \overline{b'^2} &= b^{*2} \overline{b'^2}^\dagger.\end{aligned}$$

372 As in Sect. 2.1, the background-state gradients are also nondimensionalized  
 373 directly by using the scales for these gradients. Also keep in mind that the  
 374 nondimensionalization scales are assumed to be positive definite.

375 By nondimensionalizing Eq. (12) in this manner, we obtain

$$\begin{aligned}\frac{u^{*2}}{t^*} \frac{\partial}{\partial t^\dagger} \frac{\overline{\mathbf{v}'^2}}{2} &= (\overline{w'b'})^* \overline{w'b'}^\dagger - (\overline{u'w'})^* \left( \frac{d\bar{u}}{dz} \right)^* \overline{u'w'}^\dagger \left( \frac{d\bar{u}}{dz} \right)^\dagger \\ &\quad - \frac{(\overline{w'\mathbf{v}'^2})^*}{z^*} \left[ \frac{\partial}{\partial z^\dagger} \overline{w' \left( \frac{\mathbf{v}'^2}{2} + \phi' \right)}^\dagger \right] - D^* D^\dagger \quad (13)\end{aligned}$$

376 with  $(\overline{w'\phi'})^* = (\overline{w'\mathbf{v}'^2})^*$ .

377 The goal of the nondimensionalization analysis is to set the constant pref-  
 378 actors in front of the terms equal as much as possible, by following the prin-  
 379 ciples outlined in Sect. 2.3. However, keep in mind that setting two prefactors  
 380 equal does not mean *at all* that these two terms are well balanced with any  
 381 boundary-layer regimes: this *balance* is only by an order of magnitude, and  
 382 it does not say anything directly about the TKE budget. In the following  
 383 two subsections, we examine possible *balances*, in the sense just stated, in the  
 384 equation with the goal of identifying the Obukhov Length in mind.

385 3.4 The Balance:  $\overline{w'b'} \sim (\overline{u'w'})d\bar{u}/dz$ :

386 The first balance to be considered by order of magnitude is between two terms  
 387 that involve the vertical buoyancy flux,  $\overline{w'b'}$  (buoyancy production), and the  
 388 vertical momentum stress,  $\overline{u'w'}$  (shear production), respectively. This leads to  
 389 a constraint:

$$(\overline{w'b'})^* = (\overline{u'w'})^* \left( \frac{d\bar{u}}{dz} \right)^*. \quad (14)$$

390 It can lead to the Obukhov length in the following manner: introduce a vertical  
 391 scale,  $z^*$ , by setting

$$\left( \frac{d\bar{u}}{dz} \right)^* = \frac{\bar{u}^*}{z^*}; \quad (15)$$

392 substituting of Eq. (15) into Eq. (14) leads to:

$$z^* = \frac{\bar{u}^* (\overline{u'w'})^*}{(\overline{w'b'})^*}.$$

393 This length scale reduces to the Obukhov length by further assuming:

$$\bar{u}^* \overline{u'w'}^* \simeq \overline{u'w'}^{*3/2}. \quad (16)$$

394 3.5 The Balance:  $\overline{w'b'} \sim \partial(\overline{w'\mathbf{v}'^2})/\partial z$ :

395 Alternative possibility to be considered is the order-of-magnitude balance  
 396 between the buoyancy production (first) and the turbulent transport (third):

$$\overline{w'b'}^* = \frac{(\overline{w'u'^2})^*}{z^*},$$

397 which leads to a length scale:

$$z^* = \frac{\overline{w'u'^2}^*}{(\overline{w'b'})^*}.$$

398 If one can further set

$$\overline{w'u'^2}^* \simeq \overline{w'u'}^{*3/2}, \quad (17)$$

399 it reduces to the Obukhov length.

### 400 3.6 Discussions

401 This section have attempted to derive the Obukhov length directly by nondimensionalizing the TKE equation. The Obukhov length has been derived by  
 402 considering two possible balances by order of magnitude in this equation,  
 403 however, only with additional assumptions (Eqs. 16, 17). Making any further  
 404 progress is difficult due to the fact that the TKE equation is not self-contained.  
 405 Although adding more equations for turbulence statistics may help, they never  
 406 close the system. Based on these considerations, in Sect. 4, we generalize a set  
 407 of equations considered in Sect. 2 into a fully nonlinear three-dimensional version  
 408 so that the TKE equation can be nondimensionalized in a self-consistent  
 409 manner. However, as a major drawback of this alternative approach, the length  
 410 scale is not defined by ratio of fractional powers of flux scales directly any more.

## 412 4 Full Eddy Equations

### 413 4.1 Formulation

414 As decided at the end of Sect. 3, we now turn to a fully-nonlinear set of  
 415 equations, including diffusivity:

$$\begin{aligned} \frac{\partial \mathbf{u}'}{\partial t} + w' \frac{d\bar{\mathbf{u}}}{dz} + \bar{\mathbf{u}} \cdot \nabla_H \mathbf{u}' + \nabla_H \cdot \mathbf{u}' \mathbf{u}' + \frac{\partial}{\partial z} [\mathbf{u}' w' - \overline{\mathbf{u}' w'}] \\ = -\nabla_H \phi' + \nu \nabla^2 \mathbf{u}', \end{aligned} \quad (18a)$$

$$\begin{aligned} \frac{\partial w'}{\partial t} + \bar{\mathbf{u}} \cdot \nabla_H w' + \nabla_H \cdot \mathbf{u}' w' + \frac{\partial}{\partial z} [w'^2 - \overline{w'^2}] \\ = -\frac{\partial \phi'}{\partial z} + b' + \nu \nabla^2 w', \end{aligned} \quad (18b)$$

$$\begin{aligned} \frac{\partial b'}{\partial t} + \bar{\mathbf{u}} \cdot \nabla_H b' + w' \frac{d\bar{b}}{dz} + \nabla_H \cdot \mathbf{u}' b' + \frac{\partial}{\partial z} [w' b' - \overline{w' b'}] \\ = \kappa \nabla^2 b', \end{aligned} \quad (18c)$$

$$\nabla_H \cdot \mathbf{u}' + \frac{\partial w'}{\partial z} = 0. \quad (18d)$$

416 Here,  $\mathbf{u}$  and  $w$  are the horizontal and the vertical components of the velocity,  
 417  $\nu$  and  $\kappa$  are coefficients for diffusions of momentum and heat.

### 418 4.2 Nondimensionalization Scales

419 The above system can be nondimensionalized by setting the variables as:

$$\mathbf{u}' = u^* \mathbf{u}^\dagger, \quad w' = w^* w^\dagger, \quad \phi' = \phi^* \phi^\dagger, \quad b' = b^* b^\dagger,$$

420

$$\frac{\partial}{\partial t} = \frac{1}{t^*} \frac{\partial}{\partial t^\dagger}, \quad \nabla_H = \frac{1}{x^*} \nabla_H^\dagger, \quad \frac{\partial}{\partial z} = \frac{1}{z^*} \frac{\partial}{\partial z^\dagger},$$

421 and also

$$\bar{\mathbf{u}} = \bar{u}^* \bar{\mathbf{u}}^\dagger, \quad \frac{d\bar{\mathbf{u}}}{dz} = \left(\frac{d\bar{u}}{dz}\right)^* \frac{d\bar{\mathbf{u}}^\dagger}{dz^\dagger}, \quad \frac{d\bar{b}}{dz} = \left(\frac{d\bar{b}}{dz}\right)^* \frac{d\bar{b}^\dagger}{dz^\dagger}.$$

422 By substituting these expressions into Eqs. (18a, b, c, d), we obtain

$$\begin{aligned} \frac{u^*}{t^*} \frac{\partial \mathbf{u}^\dagger}{\partial t^\dagger} + \frac{\bar{u}^* u^*}{x^*} \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger \mathbf{u}^\dagger + w^* \left(\frac{d\bar{u}}{dz}\right)^* w^\dagger \frac{d\bar{\mathbf{u}}^\dagger}{dz^\dagger} + \frac{u^{*2}}{x^*} \nabla_H^\dagger \mathbf{u}^{\prime 2} + \frac{u^* w^*}{z^*} \frac{\partial}{\partial z} [\mathbf{u}^\dagger w^\dagger - \overline{\mathbf{u}^\dagger w^\dagger}] \\ = -\frac{\phi^*}{x^*} \nabla_H^\dagger \phi^\dagger + \nu u^* \left(\frac{1}{x^{*2}} \nabla_H^{\prime 2} + \frac{1}{z^{*2}} \frac{\partial^2}{\partial z^{\prime 2}}\right) \mathbf{u}^\dagger, \end{aligned} \quad (19a)$$

$$\begin{aligned} \frac{w^*}{t^*} \frac{\partial w^\dagger}{\partial t^\dagger} + \frac{\bar{u}^* w^*}{x^*} \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger w^\dagger + \frac{u^* w^*}{x^*} \nabla_H^\dagger \cdot \mathbf{u}^\dagger w^\dagger + \frac{w^{*2}}{z^*} \frac{\partial}{\partial z} [w^{\prime 2} - \overline{w^{\prime 2}}] \\ = -\frac{\phi^*}{z^*} \frac{\partial \phi^\dagger}{\partial z^\dagger} + b^* b^\dagger + \nu w^* \left(\frac{1}{x^{*2}} \nabla_H^{\prime 2} + \frac{1}{z^{*2}} \frac{\partial^2}{\partial z^{\prime 2}}\right) w^\dagger, \end{aligned} \quad (19b)$$

$$\begin{aligned} \frac{b^*}{t^*} \frac{\partial b^\dagger}{\partial t^\dagger} + \frac{\bar{u}^* b^*}{x^*} \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger b^\dagger + w^* \left(\frac{d\bar{b}}{dz}\right)^* w^\dagger \left(\frac{d\bar{b}}{dz}\right)^\dagger + \frac{u^* b^*}{x^*} \nabla_H^\dagger \cdot \mathbf{u}^\dagger b^\dagger \\ + \frac{w^* b^*}{z^*} \frac{\partial}{\partial z} [w^\dagger b^\dagger - \overline{w^\dagger b^\dagger}] = \kappa b^* \left(\frac{1}{x^{*2}} \nabla_H^{\prime 2} + \frac{1}{z^{*2}} \frac{\partial^2}{\partial z^{\prime 2}}\right) b^\dagger, \end{aligned} \quad (19c)$$

$$\frac{u^*}{x^*} \nabla_H^\dagger \cdot \mathbf{u}^\dagger + \frac{w^*}{z^*} \frac{\partial w^\dagger}{\partial z^\dagger} = 0. \quad (19d)$$

423 First, we note from the continuity equation (19d) that

$$\frac{u^*}{x^*} = \frac{w^*}{z^*}.$$

424 To take into account this condition, we introduce the aspect ratio,  $\alpha$ , defined  
425 by

$$\alpha = \frac{w^*}{u^*} = \frac{z^*}{x^*}. \quad (20a)$$

426 Second, we introduce an amplitude,  $\hat{\epsilon}$ , for the eddies defined by

$$\hat{\epsilon} = \frac{u^*}{\bar{u}^*}. \quad (20b)$$

427 These two unspecified parameters,  $\alpha$  and  $\hat{\epsilon}$ , become necessary for closing the  
428 nondimensionalized system in the following.

429 By setting the factors for the first two terms in Eqs. (19a, b, c) equal, *i.e.*,  
430 the temporal and advection tendencies, we define the time scale as:

$$t^* = \frac{x^*}{\bar{u}^*} = \left(\frac{1}{\alpha}\right) \frac{z^*}{\bar{u}^*}, \quad (20c)$$

431 *i.e.*, the advective time scale. Furthermore, by equating the factors for the sec-  
432 ond terms in the left-hand and right-hand sides, *i.e.*, the advection tendency  
433 and the pressure-gradient force, respectively, in Eq. (19a), the pressure scale  
434 is defined by:

$$\phi^* = \hat{\epsilon} \bar{u}^{*2}, \quad (20d)$$

435 that means that the magnitude of the pressure force is constrained by the  
 436 advection term. By equating the factors for the first two terms in right-hand  
 437 side of Eq. (19b), *i.e.*, the pressure-gradient and buoyancy forces,

$$b^* = \hat{\epsilon} \frac{\bar{u}^{*2}}{z^*}. \quad (20e)$$

438 Thus, the magnitude of buoyancy is constrained by the vertical pressure gra-  
 439 dient, as expected from the hydrostatic balance.

440 The resulting nondimensionalized set of equations is:

$$\begin{aligned} \frac{\partial u^\dagger}{\partial t^\dagger} + \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger \mathbf{u}^\dagger + \left[ \frac{z^*}{\bar{u}^*} \left( \frac{d\bar{u}}{dz} \right)^* \right] w^\dagger \frac{d\bar{\mathbf{u}}^\dagger}{dz^\dagger} + \hat{\epsilon} [\nabla_H^\dagger \mathbf{u}^{\dagger 2} + \frac{\partial(\mathbf{u}^\dagger w^\dagger - \overline{\mathbf{u}^\dagger w^\dagger})}{\partial z^\dagger}] \\ = -\nabla_H^\dagger \phi^\dagger + \frac{1}{Re} \left( \frac{\partial^2}{\partial z^{\dagger 2}} + \alpha^2 \nabla_H^{\dagger 2} \right) \mathbf{u}^\dagger, \end{aligned} \quad (21a)$$

$$\begin{aligned} \frac{\partial w^\dagger}{\partial t^\dagger} + \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger w^\dagger + \hat{\epsilon} [\nabla_H^\dagger \cdot \mathbf{u}^\dagger w^\dagger + \frac{\partial(w^{\dagger 2} - \overline{w^{\dagger 2}})}{\partial z^\dagger}] \\ = \frac{1}{\alpha^2} \left[ -\frac{\partial \phi^\dagger}{\partial z^\dagger} + b^\dagger \right] + \frac{1}{Re} \left( \frac{\partial^2}{\partial z^{\dagger 2}} + \alpha^2 \nabla_H^{\dagger 2} \right) w^\dagger, \end{aligned} \quad (21b)$$

$$\begin{aligned} \frac{\partial b^\dagger}{\partial t^\dagger} + \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger b^\dagger + \left( \frac{z^*}{\bar{u}^*} \right)^2 \left( \frac{d\bar{b}}{dz} \right)^* w^\dagger \frac{d\bar{b}^\dagger}{dz^\dagger} + \hat{\epsilon} [\nabla_H^\dagger \cdot \mathbf{u}^\dagger b^\dagger + \frac{\partial(w^\dagger b^\dagger - \overline{w^\dagger b^\dagger})}{\partial z^\dagger}] \\ = \frac{1}{Pr Re} \left( \frac{\partial^2}{\partial z^{\dagger 2}} + \alpha^2 \nabla_H^{\dagger 2} \right) b^\dagger, \end{aligned} \quad (21c)$$

$$\nabla_H^\dagger \cdot \mathbf{u}^\dagger + \frac{\partial w^\dagger}{\partial z^\dagger} = 0, \quad (21d)$$

441 where

$$Re = \alpha \frac{z^* \bar{u}^*}{\nu}, \quad (22a)$$

$$Pr = \frac{\nu}{\kappa}, \quad (22b)$$

442 are the Reynolds number and the Prandtl number, respectively.

443 The nondimensionalized version of the TKE equation (Eq. 12) can be ob-  
 444 tained directly from Eqs. (21a, b), and we find:

$$\frac{\partial}{\partial t^\dagger} \frac{\overline{\mathbf{v}^{\dagger 2}}}{2} = \overline{w^\dagger b^\dagger} - \left[ \frac{z^*}{\bar{u}^*} \left( \frac{d\bar{u}}{dz} \right)^* \right] \overline{u^\dagger w^\dagger} \frac{d\bar{\mathbf{u}}^\dagger}{dz^\dagger} - \hat{\epsilon} \frac{\partial}{\partial z^\dagger} \overline{w^\dagger \left( \frac{\mathbf{v}^{\dagger 2}}{2} + \phi^\dagger \right)} - \frac{1}{Re} D^\dagger, \quad (23a)$$

445 where

$$D^\dagger = \overline{\left( \frac{\partial \mathbf{u}}{\partial z} \right)^2} + \alpha^2 \left[ \overline{\left( \frac{\partial \mathbf{u}}{\partial x} \right)^2} + \overline{\left( \frac{\partial \mathbf{u}}{\partial y} \right)^2} + \overline{\left( \frac{\partial w}{\partial z} \right)^2} \right] + \alpha^4 \overline{(\nabla_H w)^2}. \quad (23b)$$

## 446 4.3 Reynolds Number and the Dissipation Term

447 Note that the viscosity coefficient of the air is  $\nu \sim 10^{-5} \text{ m}^2/\text{s}$ . Also assuming  
 448  $z^* \sim 10^2 \text{ m}$  and  $\bar{u}^* \sim 1 \text{ m/s}$ , we obtain  $Re \sim 10^7$  as an estimate of the Reynolds  
 449 number in atmospheric boundary layers. Further note that the Prandtl number  
 450 of the air is  $Pr \sim 1$ . These values suggest that both kinetic and thermal  
 451 diffusions are negligible in a typical situation of the atmospheric boundary  
 452 layer: the orders of magnitudes of those terms, *i.e.*,  $10^{-7}$ , are simply too small  
 453 to treat properly even numerically. It is known that those molecular dissipation  
 454 terms play a leading role over a very thin molecular-dissipative layer of only  
 455 few centimetres above the surface (*cf.*, Eq. 29 below). For this reason, in  
 456 standard large-scale eddy simulations of atmospheric boundary layers, these  
 457 molecular diffusions are *effectively* replaced by the so-called eddy diffusions.

458 However, this conclusion immediately contradicts with another known fact  
 459 that the dissipation term,  $D$ , in the TKE equation balances well with the  
 460 shear-production term (*cf.*, Wyngaard and Coté 1971, Lenschow *et al.* 1988),  
 461 *i.e.*,

$$\overline{\mathbf{u}'w'} \cdot \frac{d\bar{\mathbf{u}}}{dz} \sim D \quad (24)$$

462 in typical stably-stratified boundary layers.

463 More generally, under a standard “cascade” picture of turbulence (*cf.*,  
 464 Pope 2000), the energy cascade rate in the wavenumber space balances with  
 465 the molecular energy dissipation rate. In the TKE equation, this picture is  
 466 translated as the *local* turbulent-transport term balances with the *local* energy  
 467 dissipation:

$$D \sim \nabla \cdot \mathbf{v}'^2 \mathbf{v}'.$$

468 Note that in this balance statement, no spatial average is applied: a different  
 469 conclusion follows after a horizontal average. Thus, we expect the magnitude  
 470 of the energy dissipation is constrained by

$$D^* = \frac{u^{*3}}{x^*}. \quad (25)$$

471 This relation is often referred as the Taylor’s dissipation law (Taylor 1935).  
 472 However this phenomenological conclusion obviously contradicts with the or-  
 473 der of magnitude of the dissipation term which is estimated to be  $Re^{-1} \sim 10^{-7}$   
 474 by nondimensionalization in Eq. (23a). The tendency that the dissipation term  
 475 does not asymptotically vanish in the high Reynolds number limit is sometimes  
 476 called *anomalous dissipation* (*e.g.*, Salewski *et al.* 2012).

477 To resolve this ostensive contradiction, we have to first note that in deriving  
 478 Eq. (23a), we have assumed that the nondimensionalization scale for the energy  
 479 dissipation is:

$$D^* = \hat{\epsilon}^2 \frac{\nu \bar{u}^{*2}}{z^{*2}} = \left( \frac{1}{\hat{\epsilon} \alpha Re} \right) \frac{u^{*3}}{x^*}. \quad (26)$$

480 An obvious conclusion is that for the nondimensionalization scale to be con-  
 481 sistent with the phenomenologically-identified scale (25), the nondimension-  
 482 alization scale must be re-scaled into:

$$\tilde{D}^* = \hat{\epsilon}\alpha ReD^*. \quad (27)$$

483 so that the new nondimensionalization scale,  $\tilde{D}^*$  becomes identical to the  
 484 phenomenologically-identified scale (25).

485 Thus, the question reduces to: why such a re-scaling is necessary, although  
 486 it appears to be not necessary for the momentum equation? Realize that in  
 487 performing the nondimensionalization, we have assumed that the system is  
 488 dominated by a single pair of spatial scales,  $x^*$  and  $z^*$ . This reasoning works  
 489 to a good extent when a simulation of a flow is of concern. However, when the  
 490 focus is to close the energy budget with a horizontal average, different issues  
 491 arise.

492 The problem may be better understood in the wavenumber space with  
 493  $k_0 \sim x^{*-1}$ . The higher wavenumbers,  $k_1 (\gg k_0)$ , do not contribute to the motions  
 494 of the scale,  $k_0$ , in any significant manner, because any possible nonlinear  
 495 interactions arising with those higher wavenumbers,  $k_1 \pm k_1$ ,  $k_1 \pm k_0$ , do not  
 496 project to the scale,  $k_0$ , of the interest significantly. On the other hand, when  
 497 the horizontally-averaged energy budget is considered, the higher wavenum-  
 498 bers can project to the horizontal average by a nonlinear interaction,  $k_1 - k_1$ ,  
 499 in association with the second-order spatial derivative in the dissipation term,  
 500 which amplifies its contribution by the factor of  $(k_1/k_0)^2$ .

501 As a result, the nondimensionalization scale of the dissipation must be re-  
 502 scaled by taking into account of the contributions from those higher wavenum-  
 503 bers:

$$D^\dagger \sim \left(\frac{z^*}{\delta^*}\right)^2, \quad (28)$$

504 where  $\delta^*$  is a dissipation scale. It constitutes the re-scaling factor for  $D^*$ , thus  
 505 by comparing it with the factor in Eq. (27), we find that the dissipation scale,  
 506  $\delta^*$  is defined by:

$$\delta^*/x^* = (\hat{\epsilon}\alpha Re)^{-1/2} \sim 10^{-7/2}, \quad (29)$$

507 which is much smaller than a characteristic scale,  $x^*$ , of dominant boundary-  
 508 layer flows. A similar conclusion also follows from a typical boundary-layer  
 509 balance (24).

#### 510 4.4 Two Regimes

511 As in Sect. 2, the two regimes are identified depending on the magnitude of  
 512 the Richardson number,  $Ri$ . In respective cases, the equation set reduces to:

513 (i)  $Ri \ll 1$ ,  $z^* = \bar{u}^* \left(\frac{d\bar{u}}{dz}\right)^{-1}$  :

$$\begin{aligned} \frac{\partial \mathbf{u}^\dagger}{\partial t^\dagger} + \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger \mathbf{u}^\dagger + \frac{d\bar{\mathbf{u}}^\dagger}{dz^\dagger} w^\dagger + \hat{\epsilon} [\nabla_H^\dagger \mathbf{u}^{\dagger 2} + \frac{\partial(\mathbf{u}^\dagger w^\dagger - \overline{\mathbf{u}^\dagger w^\dagger})}{\partial z^\dagger}] \\ = -\nabla_H^\dagger \phi^\dagger + \frac{1}{Re} \left( \frac{\partial^2}{\partial z^{\dagger 2}} + \alpha^2 \nabla_H^{\dagger 2} \right) \mathbf{u}^\dagger, \end{aligned} \quad (30a)$$

$$\begin{aligned} \frac{\partial b^\dagger}{\partial t^\dagger} + \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger b^\dagger + Ri w^\dagger \frac{d\bar{b}^\dagger}{dz^\dagger} + \hat{\epsilon} [\nabla_H^\dagger \cdot \mathbf{u}^\dagger b^\dagger + \frac{\partial(w^\dagger b^\dagger - \overline{w^\dagger b^\dagger})}{\partial z^\dagger}] \\ = \frac{1}{PrRe} \left( \frac{\partial^2}{\partial z^{\dagger 2}} + \alpha^2 \nabla_H^{\dagger 2} \right) b^\dagger, \end{aligned} \quad (30b)$$

$$\frac{\partial \overline{\mathbf{v}^{\dagger 2}}}{\partial t^\dagger} \frac{1}{2} = \overline{w^\dagger b^\dagger} - \overline{u^\dagger w^\dagger} \frac{d\bar{u}^\dagger}{dz^\dagger} - \hat{\epsilon} \frac{\partial}{\partial z^\dagger} \overline{w^\dagger (\frac{\mathbf{v}^{\dagger 2}}{2} + \phi^\dagger)} - \frac{1}{Re} D^\dagger. \quad (30c)$$

514 (ii)  $Ri \gg 1$ ,  $z^* = \bar{u}^* \left( \frac{d\bar{b}}{dz} \right)^{*-\frac{1}{2}}$  :

$$\begin{aligned} \frac{\partial \mathbf{u}^\dagger}{\partial t^\dagger} + \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger \mathbf{u}^\dagger + Ri^{-\frac{1}{2}} w^\dagger \frac{d\bar{\mathbf{u}}^\dagger}{dz^\dagger} + \hat{\epsilon} [\nabla_H^\dagger \mathbf{u}^{\dagger 2} + \frac{\partial(\mathbf{u}^\dagger w^\dagger - \overline{\mathbf{u}^\dagger w^\dagger})}{\partial z^\dagger}] \\ = -\nabla_H^\dagger \phi^\dagger + \frac{1}{Re} \left( \frac{\partial^2}{\partial z^{\dagger 2}} + \alpha^2 \nabla_H^{\dagger 2} \right) \mathbf{u}^\dagger, \end{aligned} \quad (31a)$$

$$\begin{aligned} \frac{\partial b^\dagger}{\partial t^\dagger} + \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger b^\dagger + w^\dagger \frac{d\bar{b}^\dagger}{dz^\dagger} + \hat{\epsilon} [\nabla_H^\dagger \cdot \mathbf{u}^\dagger b^\dagger + \frac{\partial(w^\dagger b^\dagger - \overline{w^\dagger b^\dagger})}{\partial z^\dagger}] \\ = \frac{1}{PrRe} \left( \frac{\partial^2}{\partial z^{\dagger 2}} + \alpha^2 \nabla_H^{\dagger 2} \right) b^\dagger, \end{aligned} \quad (31b)$$

$$\frac{\partial \overline{\mathbf{v}^{\dagger 2}}}{\partial t^\dagger} \frac{1}{2} = \overline{w^\dagger b^\dagger} - Ri^{-\frac{1}{2}} \overline{u^\dagger w^\dagger} \frac{d\bar{u}^\dagger}{dz^\dagger} - \hat{\epsilon} \frac{\partial}{\partial z^\dagger} \overline{w^\dagger (\frac{\mathbf{v}^{\dagger 2}}{2} + \phi^\dagger)} - \frac{1}{Re} D^\dagger. \quad (31c)$$

515 As already found in Sect. 2, the stratification term with  $d\bar{b}^\dagger/dz^\dagger$  in the heat  
516 equation (30b) is scaled by  $Ri$  in the regime with weak stratifications,  $Ri \ll 1$ ,  
517 whereas the shear-driven terms with  $d\bar{\mathbf{u}}^\dagger/dz^\dagger$  in the momentum equation (31a)  
518 and the TKE budget (31c) are scaled by  $Ri^{-1/2}$  in the regimes with strong  
519 stratifications,  $Ri \gg 1$ . Thus, those respective terms become less significant in  
520 these respective limits. Note that the nonlinear terms solely due to the eddies  
521 are scaled by the nondimensional eddy amplitude,  $\hat{\epsilon}$ . As already discussed in  
522 Sect. 4.3, due to a very large value of  $Re$ , diffusion terms proportional to  $1/Re$   
523 both in momentum and heat equations practically drop off. On other hand,  
524 in the TKE budget, the energy dissipation term may be re-scaled by re-setting  
525  $D^\dagger/Re = \tilde{D}^\dagger$ , as also suggested in Sect. 4.3.

#### 526 4.5 Asymptotic Limit of $Ri \rightarrow 0$

527 The system under the asymptotic limit of  $Ri \rightarrow 0$  warrants further discussions,  
528 as in the linear case (Sect. 2.2). Under this limit, the leading-order of the fully-

529 nonlinear buoyancy equation (30b) reduces to:

$$\frac{\partial b^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial b^\dagger}{\partial x^\dagger} + \hat{\epsilon} [\nabla_H^\dagger \cdot \mathbf{u}^\dagger b^\dagger + \frac{\partial(w^\dagger b^\dagger - \overline{w^\dagger b^\dagger})}{\partial z^\dagger}] = 0.$$

530 With the absence of background stratification, the buoyancy is simply ad-  
531 vective. The time-evolving buoyancy, in turn, drives the flow in the vertical  
532 momentum equation (21b).

533 However, in this regime, the buoyancy is never generated to the leading  
534 order. Thus, when a system is initialized with no buoyancy, only a weak buoy-  
535 ancy is generated from the term with the background buoyancy stratification,  
536  $\overline{db}/dz$ , which remains  $O(Ri)$ . Thus, in this case, the buoyancy must be re-  
537 scaled into:

$$\tilde{b}^\dagger = Ri^{-1} b^\dagger \quad (32)$$

538 so that the buoyancy equation is also re-scaled into:

$$\frac{\partial \tilde{b}^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial \tilde{b}^\dagger}{\partial x^\dagger} + w^\dagger \frac{d\bar{b}^\dagger}{dz^\dagger} + \hat{\epsilon} [\nabla_H^\dagger \cdot \mathbf{u}^\dagger \tilde{b}^\dagger + \frac{\partial(w^\dagger \tilde{b}^\dagger - \overline{w^\dagger \tilde{b}^\dagger})}{\partial z^\dagger}] = 0$$

539 to the leading order.

540 After this re-scaling, in turn, the buoyancy no longer contributes to the  
541 vertical momentum equation (21b) to the leading order. In the other words,  
542 the flows are no longer driven by buoyancy. Here, we face a minor difficulty  
543 with this regime, because the pressure gradient  $\partial\phi^\dagger/\partial z$ , has no longer any  
544 term to balance with, to the leading order, when  $\alpha \ll 1$ . It simply suggests  
545 that the only types of flows consistent with weak buoyancy,  $b^\dagger = O(Ri)$ , are  
546 isotropic with  $\alpha = 1$ .

547 When the buoyancy is re-scaled by Eq. (32), the buoyancy flux,  $\overline{w^\dagger b^\dagger}$ , in  
548 the TKE equation must also be re-scaled accordingly, and we obtain

$$\frac{\partial}{\partial t^\dagger} \frac{\overline{\mathbf{v}^{\dagger 2}}}{2} = Ri \overline{w^\dagger \tilde{b}^\dagger} - \overline{u^\dagger w^\dagger} \frac{d\bar{u}^\dagger}{dz^\dagger} - \hat{\epsilon} \frac{\partial}{\partial z^\dagger} w^\dagger \left( \frac{\mathbf{v}^{\dagger 2}}{2} + \phi^\dagger \right) - \frac{1}{Re} D^\dagger. \quad (33)$$

549 This result would be consistent with typically observed stably-stratified boundary-  
550 layer states (*cf.*, Wyngaard and Coté 1971, Lenschow *et al.* 1988), if the con-  
551 ditions  $1 \gg Ri \gg \hat{\epsilon}$  are satisfied in asymptotic sense.

552 We may further speculate that the buoyancy flux must be re-scaled with  
553  $Ri \ll 1$  even when buoyancy is found to be of the leading order, because the  
554 buoyancy would simply act like a random force in the momentum equation  
555 (21b) to which the vertical velocity responds off phase, without spatial co-  
556 herency. Thus, vertical motions generated by a finite buoyancy perturbation  
557 does not correlate with the buoyancy. In this case, we need to more directly  
558 set:

$$\overline{w^\dagger \tilde{b}^\dagger} = Ri^{-1} \overline{w^\dagger b^\dagger}. \quad (34)$$

559 Eq. (34) also leads to the re-scaled TKE equation (33).

## 560 4.6 Obukhov Length: Derivation

561 The buoyancy scale defined by Eq. (20e) can alternatively be interpreted as a  
562 definition of the vertical scale:

$$z^* = \hat{\epsilon} \frac{\bar{u}^{*2}}{b^*}. \quad (35)$$

563 This scale can also be interpreted, effectively, to be equivalent to the Obukhov  
564 length (Eq. 11) as seen by noting that

$$\begin{aligned} \overline{u'w'^*} &= u^*w^* = \alpha u^{*2} = \hat{\epsilon}^2 \alpha \bar{u}^{*2}, \\ \overline{w'b'^*} &= w^*b^* = \hat{\epsilon} \alpha b^* \bar{u}^*, \end{aligned}$$

565 in which Eqs. (20a, b) are invoked in deriving final expressions. Substituting  
566 these two expressions into (11), we obtain

$$L = \hat{\epsilon}^2 \alpha^{1/2} z^*. \quad (37)$$

567 Furthermore, when  $Ri \ll 1$ , there is a possibility that the buoyancy must be  
568 further re-scaled by Eq. (32), and as a result, the buoyancy-flux scale must  
569 be revised into:

$$\overline{w'b'^*} = Ri w^* b^* = \hat{\epsilon} Ri b^* \bar{u}^*,$$

570 also using the constraint,  $\alpha = 1$ , under the re-scaling. As a result, the Obukhov  
571 length becomes.

$$L = \frac{\hat{\epsilon}^2}{Ri} z^*. \quad (38)$$

## 572 4.7 Obukhov Length: Discussions

573 The obtained main result is Eq. (37), which shows that the Obukhov length,  
574  $L$ , underestimates the characteristic vertical scale,  $z^*$ , of the system by the  
575 factors determined by the nondimensional eddy amplitude,  $\hat{\epsilon}$ , and the aspect  
576 ratio,  $\alpha$ , of the flow: the discrepancy between the Obukhov length,  $L$ , and the  
577 characteristic scale,  $z^*$  increases for the smaller  $\hat{\epsilon}$  and  $\alpha$ . Thus, the observation-  
578 ally known increasing discrepancy in the limit of strong stable stratification  
579 can easily be explained by dramatic decrease of both factors,  $\hat{\epsilon}$  and  $\alpha$ . On the  
580 other hand, with a strongly unstably-stratified regime (*i.e.*,  $Ri \gg 1$  with a  
581 negative stratification), we rather expect to be  $\alpha \sim 1$  and  $\hat{\epsilon}$  to increase with  
582 the increasing  $Ri$ . In contrast, an overall relevance of the Obukhov length,  $L$ ,  
583 identified in observations with weak stratification may partially be explained  
584 by a re-scaling by the Richardson number,  $Ri$ , in Eq. (38), which compensates  
585 the tendency of making the Obukhov length smaller than the actual length  
586 scale by the factor,  $\hat{\epsilon}^2$ .

587 Keep in mind that  $z^*$  measures a typical vertical scale of turbulent eddies  
588 in a given boundary layer, and not necessarily the actual vertical scale of the  
589 boundary layer itself. Although it may be intuitive enough to expect that the  
590 most dominant sale of the eddies in a boundary layer is comparable to the  
591 boundary-layer depth, this claim is still to be substantiated.

## 592 5 Heuristic Derivation of the Monin–Obukhov Similarity Theory

593 From the nondimensionalization considered so far, essence of the Monin–  
 594 Obukhov similarity theory can be derived in a heuristic manner. If the nondi-  
 595 mensionalization has been properly accomplished for a given equation system  
 596 describing a boundary layer, in principle, steady solutions for all the dependent  
 597 variables in the equation can be expressed as functions of  $z^\dagger = z/z^*$ , where  $z^*$   
 598 is the vertical scale defined by Eq. (35), after nondimensionalization. In other  
 599 words, steady solutions for these nondimensional variables are given in terms  
 600 of nondimensional functions, say,  $f^\dagger(z^\dagger)$ ,  $g^\dagger(z^\dagger)$ ,  $q^\dagger(z^\dagger)$ , by

$$\overline{w'b'}^\dagger = f^\dagger(z^\dagger), \quad (39a)$$

$$\overline{u'w'}^\dagger = g^\dagger(z^\dagger), \quad (39b)$$

$$\frac{d\bar{u}^\dagger}{dz^\dagger} = q^\dagger(z^\dagger), \quad (39c)$$

601 etc. We can consider these functions to be universal in the sense that we obtain  
 602 the same solutions whenever the same boundary conditions are imposed on the  
 603 system, regardless of its dimension, because the system reduces to identical  
 604 form by re-scaling by  $z^*$ . After dimensionalizations, the above equations read  
 605 as:

$$\overline{w'b'} = w^*b^* f^\dagger\left(\frac{z}{z^*}\right), \quad (40a)$$

$$\overline{u'w'} = u^*w^* g^\dagger\left(\frac{z}{z^*}\right), \quad (40b)$$

$$\frac{d\bar{u}}{dz} = \left(\frac{d\bar{u}}{dz}\right)^* q^\dagger\left(\frac{z}{z^*}\right). \quad (40c)$$

606 Furthermore, let us suppose that the only necessary boundary conditions  
 607 required for determining these variables are their surface values. In that case,  
 608 we may set  $w^*b^* = (\overline{w'b'})_0$ ,  $u^*w^* = (\overline{u'w'})_0$ , and  $(d\bar{u}/dz)^* = (d\bar{u}/dz)_0$  with  
 609 the subscript, 0, designating the surface values. Here, the surface values, more  
 610 precisely, refer to those at the top of the viscous–boundary layers for the fluxes.  
 611 Defining the wind–shear close to the surface is even trickier, because over the  
 612 surface layer, the wind speed typically increases logarithmically with height.  
 613 For this reason, the Monin–Obukhov theory further replaces  $(d\bar{u}/dz)_0$  by  $\bar{u}^*/z$ .

614 As a result, general solutions to the system are given by

$$\overline{w'b'} = (\overline{w'b'})_0 f^\dagger\left(\frac{z}{z^*}\right), \quad (41a)$$

$$\overline{u'w'} = (\overline{u'w'})_0 g^\dagger\left(\frac{z}{z^*}\right), \quad (41b)$$

$$\frac{d\bar{u}}{dz} = \frac{\bar{u}^*}{z} q^\dagger\left(\frac{z}{z^*}\right). \quad (41c)$$

615 Based on the arguments so far, we may conclude that the nondimensional  
 616 functions,  $f^\dagger(z/z^*)$ ,  $g^\dagger(z/z^*)$ ,  $q^\dagger(z/z^*)$ , are universal only depending on the

617 nondimensionalization scale,  $z^*$ . In this manner, Monin–Obukhov similarity  
 618 theory (*cf.*, Sorbjan 1989) is essentially derived simply as steady solutions of  
 619 the system under nondimensionalization.

620 Realize, however, that there are various caveats in this derivation. The first  
 621 is replacement of the nondimensionalization scales in the right hand side of  
 622 Eqs. (40a, b, c) by the surface values in Eqs. (41a, b, c). As already emphasized  
 623 in the introduction, the nondimensionalization scales only refer to the typical  
 624 physical values by orders of magnitudes. Conversely, the surface values may  
 625 not necessarily represent typical values of the whole boundary–layer depth.  
 626 However, this merely reduces to a matter of re–scaling the nondimensional  
 627 functions,  $f^\dagger$ ,  $g^\dagger$ ,  $q^\dagger$  with no further consequence.

628 A more serious problem is in replacing the nondimensionalization length  
 629 scale,  $z^*$ , defined by Eq. (35) with the Obukhov length. The nondimensional-  
 630 ization length scale,  $z^*$ , is related to the Obukhov length by Eq. (37). Even  
 631 after these re–scalings, these functions remain universal so long as these re–  
 632 scaling factors are constants for a given dynamical regime as defined, say,  
 633 by the Richardson number, thus a rescaling of Eq. (37) into Eq. (38) is not  
 634 an issue by adding an extra Richardson–number dependence. The most seri-  
 635 ous constraint of the above derivation of the Monin–Obukhov similarity the-  
 636 ory from a nondimensionalized equation is an assumption that the system is  
 637 solely controlled by surface values. It may not be the case with all types of  
 638 boundary–layer turbulence.

## 639 6 Summary and Further Remarks

### 640 6.1 Summary and Link to Previous Studies

641 An important aim behind the present study has been to demonstrate how  
 642 characteristic scales of turbulent boundary–layer systems can be identified  
 643 directly by nondimensionalization of partial differential–equations describing  
 644 the system. For a demonstrative purpose, the analysis has begun with a linear–  
 645 stability problem (Sect. 2), then we have turned to the TKE equation for  
 646 turbulent boundary layers (Sect. 3). Finally, a full set of nonlinear equations  
 647 for the boundary layer has been examined (Sect. 4). All the analyses have  
 648 been performed under the Boussinesq approximation, and also assuming a  
 649 horizontal homogeneity for simplicity.

650 With the general aim in mind, the present study has focused on identifying  
 651 the Obukhov length. Nondimensionalization of the full nonlinear equation set  
 652 in Sect. 4 shows that the characteristic vertical scale is related to the Obukhov  
 653 length by Eq. (37), or alternatively by Eq. (38) in the limit of weak stratifica-  
 654 tion (*i.e.*,  $Ri \rightarrow 0$ ). Note that equivalence of these two scales can be established  
 655 only by including a re–scaling factor, which increases with the increasing rel-  
 656 ative amplitude of the eddies with respect to the background flow, as well as  
 657 the increasing aspect ratio of the flow. The value of these factors can be sub-  
 658 stantial, and as a result, the Obukhov length substantially underestimates the

659 characteristic vertical scale of the boundary layer, especially with increasing  
660 stratifications.

661 Under weak stratifications, the shear-production term significantly con-  
662 tributes in defining the characteristic vertical scale (*cf.*, Eq. 5a). This is con-  
663 sistent with a preliminary stand-alone analysis with the TKE equation in  
664 Sect. 3.4, which suggests that the Obukhov length may essentially be derived  
665 from a condition of an order-of-magnitude balance between buoyancy- and  
666 shear-productions.

667 On the other hand, with increasing stratifications with the Richardson  
668 number much larger than unity (*i.e.*,  $Ri \rightarrow \infty$ ), the shear production is ex-  
669 pected to become insignificant, being scaled by  $Ri^{-1/2}$ . In this limit, a more  
670 relevant vertical scale becomes the buoyancy-gradient scale,  $z^{*b}$ , defined by  
671 Eq. (5b). This scale (5b) is somehow akin to the external static-stability scale

$$L_N = \frac{\overline{(u'w')^{*1/2}}}{(\overline{db/dz})^{*1/2}}$$

672 introduced by Kitaigorodskii (1988), and considered, especially, by Zilitinke-  
673 vich and Esau (2005). Moreover, Zilitinkevich and Calanca (2000) introduces  
674 a nondimensional parameter,  $L/L_N$ , to define a transition from a regime dom-  
675 inated by the Obukhov length,  $L$ , to  $L_N$ . Zilitinkevich and Esau (2005) argue  
676 that the scale,  $L_N$ , becomes relevant when the vertical eddy heat flux is small.  
677 Note that the static-stability scale,  $L_N$ , is further linked to the buoyancy  
678 scale (Eqs. 10b, c) introduced by Stull (1973), Zeman and Tennekes (1977),  
679 Brost and Wyngaard (1978). Sorbjan (2006, 2010, 2016), in turn, develops his  
680 gradient-based similarity theory based on the buoyancy scale.

681 By performing nondimensionalization of a full set of boundary-layer equa-  
682 tions, we have also arrived at the order of magnitude estimate of each term  
683 in the TKE equation separately for two regimes with  $Ri \ll 1$  and  $Ri \gg 1$  in  
684 Eqs. (31c) and (32c), respectively. In the case with  $Ri \ll 1$ , Eq. (31c) may  
685 further be re-scaled into Eq. (34), which appears to be consistent with the  
686 observed budget of weakly-stratified regimes (*cf.*, Wyngaard and Coté 1971,  
687 Lenschow *et al.* 1988).

688 It would be important to keep in mind that those order-of-magnitude es-  
689 timates of the TKE budget terms have been classified solely in terms of the  
690 absolute value of the Richardson number, without discriminating between the  
691 stable and unstable stratifications. Thus, rather unintuitively, if this analysis  
692 is self-consistent, the characteristics of the TKE budget, say, under strong  
693 stratification must somehow be akin to that of the convective boundary layer  
694 with the absence of the wind shear. Certainly, the signs of the budget terms  
695 would change with the change in sign of the stratification. However, impor-  
696 tance of the buoyancy generation term compared to the shear generation must  
697 be valid in both regimes, although buoyancy generation *may not* be positive  
698 under strong stratifications. Also keep in mind that two additional nondimen-  
699 sional parameters,  $\alpha$  and  $\hat{\epsilon}$ , are expected to depend on  $Ri$  in different manners  
700 depending on the sign of the stratification.

701 Finally, a heuristic derivation of the Monin-Obukhov similarity theory from  
702 a steady nondimensionalized solution has been outlined in Sect. 5.

## 703 6.2 Further Remarks

704 The present study should be considered only a first step for more extensive  
705 nondimensionalization analysis of the atmospheric boundary-layer system, es-  
706 pecially for the stably-stratified regimes. Nevertheless, the present preliminary  
707 analysis already suggests a fruitfulness of such an investigation. The present  
708 study has suggested that different turbulent boundary-layer regimes can be  
709 identified by changing orders of magnitudes of the Richardson number. Such  
710 an analysis is expected to provide a more solid theoretical basis for interpret-  
711 ing the various different regimes phenomenologically identified for the stably-  
712 stratified boundary layer (*cf.*, Holtslag and Nieuwstadt 1986, Mahrt 1999).  
713 Yano and Tsujimura (1987) demonstrate how such a systematic analysis is  
714 possible by the nondimensionalization method for a different system.

715 Various further generalizations are equally feasible. The present analy-  
716 sis has been focused on quasi-steady states. However, some of the turbu-  
717 lence regimes under stable stratification may be fundamentally transient (*cf.*,  
718 Caughey *et al.* 1979). A role of horizontal heterogeneity is another aspect to  
719 be investigated, especially in a context of stably-stratified boundary layers.  
720 For example, under certain situations, the horizontal heat transport, a term  
721 that is often neglected in theoretical studies, becomes a key process in heat  
722 budget (*e.g.*, Wittch 1991). A role of anisotropy of the flow with  $\alpha \ll 1$  is still  
723 to be fully examined, as well.

724 Another aspect, that is not discussed herein, is a role of boundary condi-  
725 tions in solving the turbulence problems. When our focus is on a layer close  
726 enough to the surface (*e.g.*, surface layer), a contribution from a top of the plan-  
727 etary layer may be neglected, as a basic premise of the Monin-Obukhov theory  
728 as well as in subsequent generalizations. However, when a problem concerns  
729 a whole depth of the boundary layer, this depth becomes another parameter  
730 to be considered. As pointed out by *e.g.*, Holtslag and Nieuwstadt (1986), the  
731 problem must be solved by explicitly taking into account the condition at the  
732 top of the boundary layer.

733 In all those respects, the major weakness of the present study is to take the  
734 Richardson number as a sole parameter for characterizing the boundary-layer  
735 regimes. Although the Richardson number can be interpreted as a measure  
736 of the stratification of the system, its correspondence to the Obukhov length,  
737 which is traditionally adopted for this measure, is not quite obvious. It is most  
738 likely that we still need to identify another controlling parameter of the system.  
739 However, the identification of this parameter is left for our future study.

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[Click here to view linked References](#)

## Cover Letter:

28 June 2021

Ref: BOUN-D-21-00049

Dear Dr. Fedorovich,

Thank you for your editorial message dated 24 June 2021.

The final manuscript has been prepared by following the comments by the Reviewer #2. Our point-by-point response to the Reviewer is appended below. Furthermore by following your suggestion, a final pain-taking proof reading has been applied to the text to minimize the technical issues as well as language flaws. Please also note that the manuscript has been prepared by following the latest guideline.

Thank you for your very patient assistances during the long editorial process.

Sincerely,

Jun-Ichi Yano

Marta Waclawczyk

## Response to the Reviewer #2:

As final comments, the Reviewer raises five issues, to which we respond as follows.

*Difference between the ND and the traditional scale analysis:* The nondimensionalization is a more systematic manner for performing the so-called scale analysis. For this reason, Pedlosky's textbook systematically applies the nondimensionalization to every system introduced in the text. On the other hand, Holton's textbook on the Dynamic Meteorology performs the scale analysis in the beginning of every chapter, but without taking a step of nondimensionalization explicitly. To make this point clearer, the last sentence (L63–64) of the third paragraph in the introduction has been modified as follows: Yano and Tsujimua (1987), Yano and Bonazzola (2009) systematically apply this methodology for the scale analysis.

*Section 2:* The Reviewer added final remarks on a need of section 2. However, the Reviewer also leaves the final fate of this section with us. In turn, we decide to maintain it, although with the Reviewer's remarks taken into account.

*Lack of correspondence between the regimes identified by the study and those traditionally established boundary-layer meteorology:* Due to the fact that the Richardson number is an only free parameter considered in the present study, correspondence between those Richardson-number dependent regimes and the various boundary-layer regimes discussed in the literature is not quite clear, as the Reviewer remarks. We believe that another controlling parameters must still be identified to make these correspondences clearer. However, this must be left for our future study, as remarked at the very end of the concluding section in revision (L733–739).

*Comments on prefactors in Eqs. (30)-(31):* By following the suggestion of the Reviewer, the following remarks have been added in revision after Eqs. (30) and (31):

»As already found in Sect. 2, the stratification term with  $d\bar{b}^\dagger/dz^\dagger$  in the heat equation (30b) is scaled by  $Ri$  in the regime with weak stratifications,  $Ri \ll 1$ , whereas the shear-driven terms with  $d\bar{u}^\dagger/dz^\dagger$  in the momentum equation (31a) and the TKE budget (31c) are scaled by  $Ri^{-1/2}$  in the regimes with strong stratifications,  $Ri \gg 1$ . Thus, those respective terms become less significant in these respective limits. Note that the nonlinear terms solely due to the eddies are scaled by the nondimensional eddy amplitude,  $\hat{\epsilon}$ . As already discussed in Sect. 4.3, due to a very large value of  $Re$ , diffusion terms proportional to  $1/Re$  both in momentum and heat equations practically drop off. On other hand, in the TKE budget, the energy dissipation term may be re-scaled by re-setting  $D^\dagger/Re = \tilde{D}^\dagger$ , as also

suggested in Sect. 4.3.«

*Diffusion and Diffusivity:* We much appreciate the efforts of the Reviewer for checking the meaning of the two terms in the AMS Glossary. However, we would rather stick to more basic etymologies: English nouns often constitute of pairs. Diffusion–diffusivity is a one, action–activity, human–humanity are the others. In all those cases, the first refers to a thing itself, and the latter to its nature. We rather slipped our tongue to call it also a process in our previous response. It would be more fair to say that diffusion itself is a process rather than its nature. Apart from those details, in both occasions, we are referring to the diffusion terms rather than to the diffusivity coefficients, thus these usages stand as these are.

*Spelling Errors:* We much appreciate the present Reviewer for pointing us out quite few spelling errors. In finalizing the manuscript, we have performed rather pain–taking proof reading to make sure that no further spelling errors will be left.