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Reduction Theorems For Stability of Compact Sets in Time-Varying Systems

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Abstract

Reduction theorems provide a framework for stability analysis that consists in breaking down a complex problem into a hierarchical list of subproblems that are simpler to address. This paper investigates the following reduction problem for time-varying ordinary differential equations on $\mathbb{R}^n$. Let Γ_1 be a compact set and Γ_2 be a closed set, both positively invariant and such that $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. Suppose that Γ_1 is uniformly asymptotically stable relative to Γ_2. Find conditions under which Γ_1 is uniformly asymptotically stable. We present a reduction theorem for uniform asymptotic stability that completely addresses the local and global version of this problem, as well two reduction theorems for either local or global uniform stability and uniform attractivity. These theorems generalize well-known equilibrium stability results for cascade-connected systems as well previous reduction theorems for time-invariant systems. We also present Lyapunov characterizations of the stability properties required in the reduction theorems that to date have not been investigated in the stability theory literature.

Key words: Time-varying systems, Cascades, Stability of sets

1 Introduction

The *reduction problem* was originally posed by P. Seibert in 1969 in the context of semidynamical systems [25,26]. In its most elementary formulation it concerns a differential equation with locally-Lipschitz right-hand side,

$$\dot{x} = f(x),\ x \in \mathbb{R}^n, \tag{1}$$

with no particular structure, and two nested subsets of the state space, $\Gamma_1 \subset \Gamma_2$, that are both positively invariant and have the property that Γ_1 is asymptotically stable *relative to* Γ_2. Loosely speaking, this means that solutions generated by (1) starting from initial states that are restricted to lie in Γ_2 converge, and remain close, to the set Γ_1. Then, the problem consists in finding conditions under which Γ_1 is asymptotically stable. Hence, in particular, attractive to solutions starting away from the set Γ_2. In addition, several refinements may be of interest; for instance, to admit arbitrarily large initial conditions, as well as versions addressing the properties of stability and attractivity, in place of asymptotic stability.

Such problems are far from being of pure academic interest. The solution leads to the *reduction theorems* on stability, which are technical statements that form a framework of analysis and design of dynamical systems, based on breaking down a complex problem into a prioritized sequence of simpler sub-problems —one step at a time. Instances of following such a natural methodology in popular control methods such as *backstepping* [12] and sliding-modes [33], as well as in stability theory for cascaded systems,

$$\dot{x}_1 = f_1(x_1, x_2) \tag{2a}$$
$$\dot{x}_2 = f_2(x_2). \tag{2b}$$

Cascaded systems illustrate well the essence of the reduction problem. The basic (stability analysis) problem is to find conditions under which asymptotic stability of $\{x_1 = 0\}$ for $f_1(x_1, 0)$ and of $\{x_2 = 0\}$ for $f_2(x_2)$ leads to conclude that $\{x = 0\}$ is asymptotically stable for (1) with $f := [f_1^\top\ f_2^\top]^\top$. The extensive literature on cascaded systems originates, for time-invariant systems, with work by Vidyasagar in [34] focusing on local asymptotic stability of the zero equilibrium, followed by research aimed at establishing global results, e.g., [28,24,19,15,3,9,17,31]. Now, the stability

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questions investigated in the literature on time-invariant cascaded systems are, as a matter of fact, reduction problems such as asking under which conditions $\Gamma_1 = \{(x_1, x_2) = (0, 0)\}$ is asymptotically stable provided that so is $\Gamma_2 = \{(x_1, x_2) : x_2 = 0\}$.

Stability analysis of cascaded systems is also important for control design; for instance, when one considers not only the control of a plant itself, but also of the actuators [22]. The central idea consists in constructing a controller that ensures that the systems trajectories converge *asymptotically* to an invariant manifold having the property that trajectories contained in it converge to the origin (or a set for that matter).

This rationale however, is not bound to cascaded systems. For instance, it is also reminiscent of the well-known result in [2] that a passive system is stabilizable via static output feedback if it is zero-state detectable (namely, if the state trajectories converge to the origin provided that so does the output). This connection was explored in [6]. Another clear example where the same rationale holds is the Slotine & Li Controller [29], one of the first tracking controllers for robot manipulators ensuring global asymptotic stability. The operation and stability properties of this controller can be naturally understood using the reduction viewpoint, and this is illustrated in Section 5.1.

In the previous discussion we have motivated the study of the reduction problem only for the case in which two sets, $\Gamma_1 \subset \Gamma_2$ are involved. Some control problems, however, may be conveniently broken down into a prioritized sequence of more than two elementary subproblems, which are then solved *separately*. That is, in general, the control specification of asymptotically stabilize a subset Γ of the state space may be solved by breaking it down in sub-tasks and defining a suitable collection of nested subsets $\Gamma_1 \subset \cdots \subset \Gamma_l \subset \Gamma_{l+1} := \mathbb{R}^n$ (a hierarchy of control specifications), with $\Gamma_1 = \Gamma$. Then, by asymptotically stabilizing Γ_i relative to Γ_{i+1}, for $i = 1, \ldots, l$. The reduction theorems allow to recursively deduce the asymptotic stability of Γ. Following such premise, in [7] was introduced the hierarchical control framework, which has direct implications on backstepping control.

Existing literature on the reduction problem is focused entirely on time-invariant systems. Reduction theorems for stability and asymptotic stability of compact sets were developed by Seibert and Florio in [27] in the context of time-invariant semidynamical systems. See also work by B.S. Kalitin [10] and co-workers [8]. The work in [7] presents reduction theorems for non-compact sets and a new reduction theorem for attractivity. Recently, reduction theorems for hybrid dynamical systems were presented in [16].

Contributions of this paper. The literature on the reduction problem is focused entirely on time-invariant systems. This paper presents three reduction theorems for time-varying systems, focusing on the properties of uniform stability, uniform attractivity, and uniform asymptotic stability. Both local and global versions of these properties are characterized. These theorems recover analogous results for time-invariant systems found in [27,7], as well as statements on uniform global asymptotic stability of cascaded time-varying systems. In addition to presenting reduction theorems, this paper presents Lyapunov characterizations of certain key qualitative properties invoked in the most general formulation of the reduction theorems.

Organization. In Section 2 we present definitions of relative stability properties and other stability notions. Section 3 provides a precise formulation of the reduction problem. In Section 4 we present our reduction theorems for uniform stability, uniform attractivity, and uniform asymptotic stability; then, some useful implications of these theorems; and finally, Lyapunov characterizations of the key stability properties used in the reduction theorems. In Section 5 we provide examples illustrating the use and rationale of reduction theory, and in Section 6 we prove the three reduction theorems. The paper is wrapped up with concluding remarks in Section 7, and completed with two technical appendices used in the proofs of our main statements. *Notation.* We denote by 0_k, $k \in \mathbb{N}$, the vector of zeros in $\mathbb{R}^k$, and if $x \in \mathbb{R}^k$, we denote by $\|x\| := (x^\top x)^{1/2}$, the Euclidean norm of x. We denote by $\mathbb{S}^1$ the set of real numbers modulo 2π. If $\Gamma \subset \mathbb{R}^n$ is a closed set and $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is a vector norm, we denote by $\|x\|_\Gamma := \inf_{y\in\Gamma} \|x - y\|$ the point-to-set distance of $x \in \mathbb{R}^n$ to Γ. If $A, B \subset \mathbb{R}^n$, we define $d(A, B) := \sup_{x\in A}\{\|x\|_B\}$. If $\delta > 0$, we let $B_\delta(\Gamma) := \{x \in \mathbb{R}^n : \|x\|_\Gamma < \delta\}$. For a set K, ∂K denotes the boundary of K, $\mathrm{int}(K)$ its interior, and $\overline{K}$ its closure. For $t_0 \in \mathbb{R}$, we denote $\mathbb{R}_{\geq t_0} := \{t \in \mathbb{R} : t \geq t_0\}$. A function $\alpha : [0, r) \to \mathbb{R}$, with $r > 0$, belongs to class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}$ belongs to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and $\alpha(s) \to \infty$ as $s \to \infty$.

Table 1 summarizes the notational conventions of this paper. Table 2 summarizes all stability-related acronyms used in this paper.

2 Preliminaries

In this paper we investigate the time-varying differential equation

$$\dot{x} = f(t, x), \tag{3}$$

with state space[4] $\mathbb{R}^n$. We denote by $x(t, t_0, x_0)$ the solution of (3) satisfying $x(t_0) = x_0$, where t_0 is the *initial time* and x_0 is the *initial state*. The pair (t_0, x_0) is called the *initial data* of the solution. We denote by $T^+_{t_0,x_0}$ the *right maximal interval of existence* of the solution with initial data (t_0, x_0), i.e., the maximal interval contained in $\mathbb{R}_{\geq t_0}$ on which the solution $x(t, t_0, x_0)$ is defined. If $I \subset \mathbb{R}$ and $U \subset \mathbb{R}^n$, we define $x(I, t_0, U) :=$

[4] The main results of this paper continue to hold if the state space is a smooth complete Riemannian manifold, see Remark 20.

Table 1
Mathematical notation used in the paper.

Symbol	Meaning	Where
0_k	The vector of zeros in $\mathbb{R}^k$	Section 1
$\mathbb{BS}$	Set of in. cond. giving t_0-uniformly bounded sol'ns	Defn. 8
$\|x\|_\Gamma$	Point-to-set distance of x to Γ	Section 1
$B_\delta(\Gamma)$	The set $\{x \in \mathbb{R}^n : \|x\|_\Gamma < \delta\}$	Section 1
$\mathbb{B}(\Gamma)$	Basin of t_0-uniform attraction of Γ	Defn. 5
$d(A, B)$	The maximum distance of set A to set B	Section 1
$\operatorname{int}(K)$	Interior of set K	Section 1
∂K	Boundary of set K	Section 1
$\overline{K}$	Closure of set K	Section 1
$\mathbb{R}_{\geq t_0}$	The closed half-line $[t_0, \infty)$	Section 1
$x(t, t_0, x_0)$	Maximal solution of (3) from (t_0, x_0)	Section 2
$T^+_{t_0,x_0}$	Right maximal int. of existence of the solution with initial data (t_0, x_0)	Section 2

Table 2
List of stability-related acronyms used in the paper.

Acronym	Meaning	Where
US	uniformly stable	Defn. 3
of attraction of set UGS	uniformly globally stable	Defn. 3
in. cond. giving UA	uniformly attractive	Defn. 3
Point-to-set distance of UGA	uniformly globally attractive	Defn. 3
UAS	uniformly asymptotically stable	Defn. 3
of UGAS	uniformly globally asymptotically stable	Defn. 3
t_0-US	t_0-uniformly stable	Defn. 5
half-line t_0-UA	t_0-uniformly attractive	Defn. 5
t_0-UGA	t_0-uniformly globally attractive	Defn. 5
t_0-UAS	t_0-uniformly asymptotically stable	Defn. 5
t_0-UGAS	t_0-uniformly globally asymptotically stable	Defn. 5
LUS-Γ	locally uniformly stable near Γ	Defn. 10

$\{x(t, t_0, x_0) \in \mathbb{R}^n : t \in I, x_0 \in U\}$. This set is well-defined as long as $I \subset T^+_{t_0,x_0}$ for all $(t_0, x_0) \in \mathbb{R} \times U$.

We require the time-varying vector field f in (3) to possess a basic continuity property, stated in the next assumption.

Basic Assumption. The function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is piecewise continuous with respect to its first argument and satisfies the following Lipschitz continuity property with respect to its second argument. For any compact set $K \subset \mathbb{R}^n$, there exists a constant $L > 0$ such that for each $x_1, x_2 \in K$ and for each $t \in \mathbb{R}$, $\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|$. We refer to L as a *Lipschitz constant of f on K*. △

Remark 1. The Basic Assumption holds, for instance, if $f(t, x)$ has the form $f(t, x) = f_1(x, f_2(t))$, where $f_1 : \mathbb{R}^n \times D \to \mathbb{R}^n$ is C^1, $D \subset \mathbb{R}^k$ is a bounded open set, and $f_2 : \mathbb{R} \to D$ is a piecewise continuous function whose image is contained in a compact subset of D. This is the setup used in [14] to prove a converse Lyapunov theorem for uniform global asymptotic stability of compact sets. When f does not depend on t, the Basic Assumption reduces to the familiar notion of local Lipschitz continuity. △

Definition 2 (positive invariance)**.** A set $\Gamma \subset \mathbb{R}^n$ is *positively invariant* for (3) if $x(T^+_{t_0,x_0}, t_0, x_0) \subset \Gamma$ for all $t_0 \in \mathbb{R}$ and all $x_0 \in \Gamma$. In other words, for any initial data $(t_0, x_0) \in \mathbb{R} \times \Gamma$, the solution remains in Γ for all $t \geq t_0$ for which the solution is defined. △

Next, we present some notions of uniform stability and uniform attractivity of compact sets.

Definition 3 (uniform stability and attractivity of compact sets)**.** Consider system (3) and let $\Gamma \subset \mathbb{R}^n$ be a compact, positively invariant, set.

- Γ is *uniformly stable* (US) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma)) \subset B_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$.
- Γ is *uniformly globally stable* (UGS) if Γ is US and for each $\delta > 0$ there exists $\varepsilon > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma)) \subset B_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$.
- Γ is *uniformly attractive* (UA) if there exists $r > 0$ such that for each $\varepsilon > 0$ there exists $T > 0$ such that $x(\mathbb{R}_{\geq t_0+T}, t_0, B_r(\Gamma)) \subset B_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$.
- Γ is *uniformly globally attractive* (UGA) if the UA property holds for all $r > 0$.

- Γ is *uniformly asymptotically stable* (UAS) if it is US and UA.
- Γ is *uniformly globally asymptotically stable* (UGAS) if it is UGS and UGA.

$\triangle$

Remark 4. All properties in Definition 3 are analogous to familiar definitions concerning equilibria found in [11, Section 4.5], and the definition that a compact set Γ is UGAS is equivalent to the one found, e.g., in [14]. $\triangle$

Next, we present some notions of stability and attractivity of closed, but not necessarily compact sets. The notion of t_0-UA used in this paper is taken from [23].

Definition 5 (t_0-uniform stability and t_0-uniform attractivity of closed sets)**.** Consider system (3), and let $\Gamma \subset \mathbb{R}^n$ be a closed, positively invariant set.

- Γ is *t_0-uniformly stable* (t_0-US) if for each $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}^n$ such that $\Gamma \subset U$, and for each $x_0 \in U$, for each $t_0 \in \mathbb{R}$, and each $t \in T^+_{t_0,x_0}$, it holds that $x(t, t_0, x_0) \in B_\varepsilon(\Gamma)$.
- The *basin of t_0-uniform attraction of* Γ is the set $\mathbb{B}(\Gamma)$ of initial states for which solutions converge to Γ uniformly with respect to t_0:
$$\begin{aligned}\mathbb{B}(\Gamma) := \{x_0 \in \mathbb{R}^n : (\forall \varepsilon > 0)(\exists T > 0)(\forall t_0 \in \mathbb{R}) \\ t_0 + T \in T^+_{t_0,x_0} \text{ and} \\ x(\mathbb{R}_{\geq t_0+T} \cap T^+_{t_0,x_0}, t_0, x_0) \subset B_\varepsilon(\Gamma)\}\,.\end{aligned}$$
- Γ is *t_0-uniformly attractive* (t_0-UA) if $\Gamma \subset \operatorname{int}(\mathbb{B}(\Gamma))$.
- Γ is *t_0-uniformly globally attractive* (t_0-UGA) if $\mathbb{B}(\Gamma) = \mathbb{R}^n$.
- Γ is *t_0-uniformly asymptotically stable* (t_0-UAS) if Γ is t_0-US and t_0-UA.
- Γ is *t_0-uniformly globally asymptotically stable* (t_0-UGAS) if Γ is t_0-US and t_0-UGA.

$\triangle$

Remark 6. US is defined for compact sets only (See Def. 3), but an identical definition may be formulated for closed and unbounded sets. In such case US implies t_0-US, but not vice versa; only for compact sets these properties are equivalent —see item (i) of Proposition 9 below. More precisely, for the US property, given $\varepsilon > 0$ one requires the existence of a neighborhood of initial states of the form $B_\delta(\Gamma)$ whose associated solutions remain in $B_\varepsilon(\Gamma)$ for arbitrary initial times. For the t_0-US property, the neighborhood of initial states is only required to be an open set U containing Γ. When Γ is compact, there is no loss of generality in assuming that U has the form $B_\delta(\Gamma)$, which is the reason why US and t_0-US are equivalent properties for compact sets. On the other hand, if Γ is unbounded then Γ may be t_0-US without being US. This is illustrated in Figure 1, in which it is showed that solutions starting close to U, or even to Γ but laying out of U, may leave the band $B_\varepsilon(\Gamma)$. Also, note that in the definition of the US property in Definition 3 it is tacitly assumed that $T^+_{t_0,x_0} = \mathbb{R}_{\geq t_0}$ for each $x_0 \in B_\delta(\Gamma)$ and each $t_0 \in \mathbb{R}$. This is because if a solution remains in the *bounded* set $B_\varepsilon(\Gamma)$, then its right-maximal interval of existence is $\mathbb{R}_{\geq t_0}$. On the other hand, if Γ is unbounded and t_0-US, then we can no longer assume that $T^+_{t_0,x_0} = \mathbb{R}_{\geq t_0}$, and indeed Definition 5 allows for finite escape times.

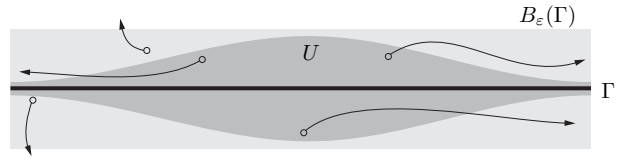

Fig. 1. The set Γ is t_0-US but not US.

The notions of UA and t_0-UA (and their global counterparts) are both uniform with respect to the initial time t_0, but differ in their requirements on initial states. For the UA property, all solutions with initial states in a neighborhood $B_r(\Gamma)$ get to an arbitrarily small neighborhood $B_\varepsilon(\Gamma)$ of Γ in some time $T > 0$ which depends on ε and is independent of t_0. For the t_0-UA property, the time T depends on x_0 and ε, and is independent of t_0. Even when Γ is compact, UA and t_0-UA are non-equivalent properties. In particular, UA implies t_0-UA, but not vice versa. $\triangle$

Remark 7. If system (3) is time-invariant, i.e., f does not depend on t, the t_0-UA property in Definition 5 coincides with the notion of semi-attractivity in [1], and in this case $\mathbb{B}(\Gamma)$ defined above coincides with the basin of attraction of Γ in [1]. $\triangle$

Definition 8 (uniform boundedness of solutions)**.** Let $x_0 \in \mathbb{R}^n$. The solutions with initial state x_0 are *t_0-uniformly bounded* if there exists a constant $c > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, x_0) \subset B_c(0)$ for all $t_0 \in \mathbb{R}$. The set of initial states giving rise to t_0-uniformly bounded solutions is defined as
$$\begin{aligned}\mathbb{BS} := \{x_0 \in \mathbb{R}^n : (\exists c > 0)(\forall t_0 \in \mathbb{R}) \\ x(\mathbb{R}_{\geq t_0}, t_0, x_0) \subset B_c(0)\}. \qquad (4)\end{aligned}$$
$\triangle$

The next result clarifies the relationships between the concepts of stability and attractivity in Definitions 3 and 5.

Proposition 9. *Consider the differential equation* (3)*, in which the vector field $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the Basic Assumption. Let $\Gamma \subset \mathbb{R}^n$ be a compact positively invariant set. Then:*

(i) Γ *is* US *if and only if* Γ *is* t_0-US*;*
(ii) Γ *is* UAS *if and only if* Γ *is* UA*;*
(iii) Γ *is* UGAS *if and only if* Γ *is* UGA *and all solutions are t_0-uniformly bounded, i.e., $\mathbb{BS} = \mathbb{R}^n$;*
(iv) Γ *is* UAS *if and only if* Γ *is* t_0-UAS*;*
(v) Γ *is* UGAS *if and only if* Γ *is* t_0-UGAS *and all solutions are t_0-uniformly bounded, i.e., $\mathbb{BS} = \mathbb{R}^n$.*

The proof is provided in Appendix A.

We conclude this section with definitions of local uniform stability, local t_0-uniform attractivity, and relative stability and attractivity. These are adaptations of notions found in [27,7,16] to the time-varying setting. Unlike the stability notions reviewed earlier, the notions in

the next definitions are not widespread in the stability theory literature, but they turn out to be important for the formulation and solution of the reduction problem investigated in this paper.

Definition 10 (local uniform stability)**.** Let $\Gamma_1 \subset \Gamma_2$ be two closed subsets of $\mathbb{R}^n$, with Γ_1 compact. The set Γ_2 is *locally uniformly stable near* Γ_1 (LUS-Γ_1) for (3) if there exists $r > 0$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $t_0 \in \mathbb{R}$ and any $x_0 \in B_\delta(\Gamma_1)$ the following implication holds:

$$(\forall t \in T^+_{t_0,x_0})\,(x([t_0,t],t_0,x_0) \subset B_r(\Gamma_1) \implies x([t_0,t],t_0,x_0) \subset B_\varepsilon(\Gamma_2))\,. \quad (5)$$

△

Definition 11 (t_0-uniform attractivity near a set)**.** Consider system (3). The closed set $\Gamma_2 \subset \mathbb{R}^n$ is t_0-*uniformly attractive near* Γ_1 (t_0-UA near Γ_1) if there exists $r > 0$ such that $B_r(\Gamma_1) \subset \mathbb{B}(\Gamma_2)$. △

Definition 12 (relative properties)**.** Consider system (3), and let $\Gamma_1 \subset \Gamma_2$ be two closed positively invariant subsets of $\mathbb{R}^n$, with Γ_1 compact.

- Γ_1 is US *relative to* Γ_2 for (3) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma_1) \cap \Gamma_2) \subset B_\varepsilon(\Gamma_1)$ for all $t_0 \in \mathbb{R}$.
- Γ_1 is UGS *relative to* Γ_2 if Γ_1 is US relative to Γ_2 and for each $\delta > 0$ there exists $\varepsilon > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma_1) \cap \Gamma_2) \subset B_\varepsilon(\Gamma_1)$.
- Γ_1 is UA *relative to* Γ_2 if there exists $r > 0$ such that for each $\varepsilon > 0$ there exists $T > 0$ such that $x(\mathbb{R}_{\geq t_0+T}, t_0, B_r(\Gamma_1) \cap \Gamma_2) \subset B_\varepsilon(\Gamma_1)$ for all $t_0 \in \mathbb{R}$.
- Γ_1 is UGA[5] *relative to* Γ_2 if $r > 0$ can be chosen arbitrarily large in the definition of UA relative to Γ_2.
- Γ_1 is, respectively, UAS *relative to* Γ_2 or UGAS *relative to* Γ_2, if Γ_1 is US (resp., UGS) and UA (resp., UGA) relative to Γ_2.

△

3 Problem formulation and motivation

Consider the system (3) under the Basic Assumption and let $\Gamma_1 \subset \Gamma_2$ be two closed, positively invariant sets, with Γ_1 compact. Suppose that Γ_1 is P *relative to* Γ_2, where P corresponds to any of the following properties: US, t_0-UA, t_0-UGA, UAS, or UGAS. In its general form, the reduction problem consists in finding conditions under which the property P holds in $\mathbb{R}^n$. As it turns out, however, this problem is meaningful only if it is assumed that Γ_1 is UAS or UGAS relative to Γ_2. The reason is that the properties of uniform stability of Γ_1 relative to Γ_2 and t_0-uniform attractivity of Γ_1 relative to Γ_2 are fragile, in the sense that, in general, they may fail to hold in the whole $\mathbb{R}^n$, even if Γ_2 possesses strong stability properties. This was first pointed out in [27,4] in the time-invariant setting and, for the purpose of motivation, it is illustrated below with two examples.

[5] Similarly, one may define the notion that Γ_1 is t_0-UA or t_0-UGA relative to Γ_2, but it is not used in this paper.

Example 1. (Uniform stability of Γ_1 relative to Γ_2 is a fragile property). Consider the cascade-connected system with state $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$,

$$\dot{x}_1 = x_2 f(t)$$
$$\dot{x}_2 = -x_2^3,$$

where $f(t)$ is a continuous bounded function such that $f(t) \geq 1$. Let $\Gamma_1 = \{0_2\}$ and $\Gamma_2 = \{(x_1, x_2) : x_2 = 0\}$. The set Γ_2 is positively invariant because $x_2 = 0$ is an equilibrium of the subystem with state x_2. On Γ_2, the subsystem with state x_1 reduces to $\dot{x}_1 = 0$, and therefore Γ_1 is US relative to Γ_2.

Since the equilibrium $x_2 = 0$ is globally asymptotically stable for the differential equation $\dot{x}_2 = -x_2^3$, and since the system has no finite escape times, the set Γ_2 is t_0-UGAS (in fact, UGAS). Yet, Γ_1 is unstable. To see why this is the case, pick $\epsilon > 0$ and $t_0 \in \mathbb{R}$, and let $(x_1(t), x_2(t))$ be the solution with initial state $x(t_0) = [0 \;\; \epsilon]^\top$. Then $x_2(t) \to 0$ at a rate of $t^{-1/2}$, and using the fact that $x_2(t) > 0$, we deduce that

$$x_1(t) = \int_{t_0}^{t} x_2(\tau) f(\tau) d\tau \geq \int_{t_0}^{t} x_2(\tau) d\tau \to \infty \text{ as } t \to \infty.$$

Since $\epsilon > 0$ is arbitrary, the origin is unstable.

In conclusion, Γ_1 is US relative to Γ_2 and Γ_2 is t_0-UGAS, but Γ_1 is not US in $\mathbb{R}^2$ because $t \mapsto x_2(t)$ is not integrable. In [20, Theorem 1, condition (10)] it is shown that the integrability of $t \mapsto x_2(t)$ plays a crucial role in the UGS property. △

Example 2. (t_0-Uniform attractivity of Γ_1 relative to Γ_2 is a fragile property). This example is adapted from [6]. Consider the time-varying system

$$\dot{x}_1 = x_2(x_1 - 1) - x_1(x_1^2 + x_2^2 - 1) - x_2 x_3 \sin(t)^2 \quad (6a)$$
$$\dot{x}_2 = -x_1(x_1 - 1) - x_2(x_1^2 + x_2^2 - 1) + x_1 x_3 \sin(t)^2 \quad (6b)$$
$$\dot{x}_3 = -x_3^3, \quad (6c)$$

and let $\Gamma_1 = \{(x_1, x_2, x_3) = (1, 0, 0)\}$ and $\Gamma_2 = \{(x_1, x_2, x_3) : x_3 = 0\}$. As in the previous example, Γ_2 is positively invariant and t_0-UGAS. We claim that Γ_1 is t_0-UA relative to Γ_2. To see why this is the case, let $(r, \theta) \in \mathbb{R}_{>0} \times \mathbb{S}^1$ be polar coordinates for the (x_1, x_2) plane, excluding the origin, so that $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. In (r, θ, x_3) coordinates, the above time-varying system reads as

$$\dot{r} = -r(r^2 - 1) \quad (7a)$$
$$\dot{\theta} = 1 - r\cos(\theta) + x_3 \sin(t)^2 \quad (7b)$$
$$\dot{x}_3 = -x_3^3. \quad (7c)$$

In (r, θ, x_3) coordinates, the sets Γ_1, Γ_2 are given by, respectively, $\tilde{\Gamma}_1 = \{(r, \theta, x_3) = (1, 0, 0)\}$ and $\tilde{\Gamma}_2 = \{(r, \theta, x_3) : x_3 = 0\}$. The dynamics on $\tilde{\Gamma}_2$ are described by the time-invariant system

$$\dot{r} = -r(r^2 - 1) \quad (8a)$$
$$\dot{\theta} = 1 - r\cos(\theta). \quad (8b)$$

For each $t_0 \in \mathbb{R}$, if $r(t_0) \neq 0$ then the solution $r(t) \to 1$ uniformly with respect to t_0, and if $\theta(t_0) \neq \pi$, then

$\theta(t) \to (0 \bmod 2\pi)$. This proves that Γ_1 is t_0-UA relative to Γ_2. On the other hand, Γ_1 is *not* US relative to Γ_2 because the unit circle is a homoclinic orbit of system (8) (see the left-hand side of Figure 2) which implies that there are initial states in Γ_2 arbitrarily close to Γ_1 leading to solutions following the whole circle before converging to Γ_1.

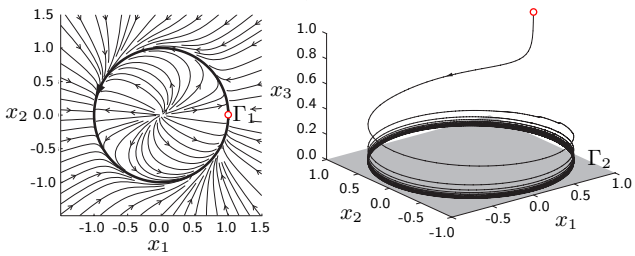

Fig. 2. On the left-hand side, the phase portrait of system (8) in $(x_1, x_2) = (r\cos(\theta), r\sin(\theta))$ coordinates, representing the dynamics on Γ_2. The set Γ_1, an equilibrium, is t_0-UA relative to Γ_2, but unstable. On the right-hand side, an orbit of the time-varying system (6) converging to Γ_2, but not to Γ_1.

Consider initial data (t_0, x_0) where $t_0 \in \mathbb{R}$ is arbitrary and $x_0 \in \mathbb{R}^3$ is a vector whose third component is positive and whose first two components lie on the unit circle, i.e., in (r, θ, x_3) coordinates, $r(t_0) = 1$, $x_3(t_0) > 0$. The corresponding solution $(r(t), \theta(t), x_3(t))$ has the property that $r(t) \equiv 1$, and $x_3(t)$ tends to zero with rate $t^{-1/2}$. Thus, Equation (7b) may be rewritten as

$$\dot{\theta} = 1 - \cos(\theta) + \mu(t),$$

where $\mu(t) \geq 0$ converges to 0 with rate $t^{-1/2}$. The solution $\theta(t)$ satisfies

$$\begin{aligned}\theta(t) &= \theta(t_0) + \int_{t_0}^{t} 1 - \cos(\theta(\tau))d\tau + \int_{t_0}^{t} \mu(\tau)d\tau \\ &\geq \theta(t_0) + \int_{t_0}^{t} \mu(\tau)d\tau \to \infty \text{ as } t \to \infty.\end{aligned}$$

Thus, in (x_1, x_2, x_3) coordinates, the solution does not converge to Γ_1, and in fact it converges to the unit circle on Γ_2, see the right-hand side of Figure 2. This proves that Γ_1 is not t_0-UA.

In conclusion, Γ_1 is t_0-UA relative to Γ_2 and Γ_2 is t_0-UGAS, but Γ_1 is not t_0-UA in $\mathbb{R}^3$. △

We are now ready to precisely state the reduction problem.

Reduction Problem. Suppose that Γ_1 is UAS or UGAS relative to Γ_2. Find conditions under which a property $P \in \{$US, t_0-UA, t_0-UGA, UAS, UGAS$\}$ holds in $\mathbb{R}^n$. △

Remark 13. Note that in the list of properties P of interest, we did not include uniform attractivity (UA). The reason is that, by Proposition 9, uniform attractivity of compact sets is equivalent to uniform asymptotic stability, therefore there is no need to state a separate reduction problem for uniform attractivity. The t_0-uniform attractivity property (t_0-UA), on the other hand, is complementary to uniform stability (US) in that, together, these two properties are equivalent to uniform asymptotic stability (see Proposition 9, parts (i) and (iv)). An analogous remark holds for the global version of these properties. △

4 The reduction theorems

In this section we first present three reduction theorems for the properties of uniform stability, t_0-uniform attractivity, uniform attractivity, and their global counterparts for time-varying systems. Then, we present useful consequences of these theorems. For clarity of exposition, the proofs of the main statements are provided in Section 6.

4.1 Main statements for time-varying systems

Theorem 14 (Reduction theorem for uniform stability)**.** *Consider the time-varying system* (3) *under the Basic Assumption. Let Γ_1 be a compact set and Γ_2 be a closed set, both positively invariant and such that $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. Then Γ_1 is* US *if*

(i) Γ_1 is UAS *relative to Γ_2, and*
(ii) Γ_2 is LUS-Γ_1.

Theorem 14 is proved in Section 6.1. Assumption (ii) is a necessary condition for Γ_1 to be US.

Theorem 15 (Reduction theorem for t_0-uniform (global) attractivity)**.** *Consider the time-varying system* (3) *under the Basic Assumption. Let Γ_1 be a compact set and Γ_2 be a closed set, both positively invariant and such that $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. Assume that*

(i) Γ_1 is UAS *relative to Γ_2,*
*(ii) Γ_2 is t_0-*UA *near Γ_1, and*
(iii) there exists $\delta > 0$ such that the set

$$K_\delta := \overline{\bigcup_{t_0 \in \mathbb{R}} x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma_1))}$$

is compact and such that $K_\delta \cap \Gamma_2 \subset \mathbb{B}(\Gamma_1)$.

*Then, Γ_1 is t_0-*UA *and $B_\delta(\Gamma_1) \subset \mathbb{B}(\Gamma_1)$.*

Moreover, if

(i)' Γ_1 is UGAS *relative to Γ_2, and*
*(ii)' Γ_2 is t_0-*UGA*,*

then all initial states giving rise to t_0-uniformly bounded solutions are contained in the basin of t_0-uniform attraction of Γ, i.e., $\mathbb{BS} \subset \mathbb{B}(\Gamma_1)$. In particular, if all solutions of (3) *are t_0-uniformly bounded, i.e., $\mathbb{BS} = \mathbb{R}^n$, then Γ_1 is t_0-*UGA.

Theorem 15 is proved in Section 6.2.

Remark 16. Assumption (ii) is a necessary condition for Γ_1 to be t_0-UA, and assumption (ii)' is necessary for Γ_1 to be t_0-UGA. Assumption (iii) is hard to check in general, but the first part of Theorem 15, which asserts that Γ_1 is t_0-UA, is useful to establish other statements, such as the first one in Theorem 17 below.

Theorem 17 (Reduction theorem for uniform (global) asymptotic stability)**.** *Consider the time-varying system* (3) *under the Basic Assumption. Let Γ_1 be a compact set and Γ_2 be a closed set, both positively invariant and such that $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. Then Γ_1 is* UAS *if and only if*

(i) Γ_1 *is* UAS *relative to* Γ_2,
(ii) Γ_2 *is* LUS-Γ_1, *and*
(iii) Γ_2 *is* t_0-UA *near* Γ_1.

Moreover, Γ_1 *is* UGAS *if and only if*

(i)' Γ_1 *is* UGAS *relative to* Γ_2,
(ii) Γ_2 *is* LUS-Γ_1,
(iii)' Γ_2 *is* t_0-UGA, *and*
(iv) *all solutions are* t_0*-uniformly bounded, i.e.,* $\mathbb{BS} = \mathbb{R}^n$.

Finally, if assumptions (i)', (ii), and (iii)' hold, then Γ_1 *is* UAS *and all initial states giving rise to* t_0*-uniformly bounded solutions are contained in the basin of* t_0*-uniform attraction of* Γ, *i.e.,* $\mathbb{BS} \subset \mathbb{B}(\Gamma_1)$.

Theorem 17 is proved in Section 6.3.

Remark 18. Theorems 14, 15, and 17 may be used recursively to analyze the stability of chains of nested closed positively invariant sets $\Gamma_1 \subset \cdots \subset \Gamma_k \subset \mathbb{R}^n$ in which Γ_1 is compact. This was done in the context of the hierarchical control problem in [7, Proposition 14] and then applied to backstepping. See also [16, Theorem 4.9]. Furthermore, the results of this paper can be directly used to extend the method proposed in [7] to the context of time-varying systems with minimal modifications. △

Remark 19. Theorems 14, 15, and 17 establish uniform stability and attractivity properties in $\mathbb{R}^n$. If $\mathcal{X} \subset \mathbb{R}^n$ is a positively invariant set such that $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, and if the assumptions of these theorems hold for initial states restricted to lie in $\mathcal{X}$, then the results of the theorems hold relative to $\mathcal{X}$. In Section 5.2, we illustrate this fact with an example. △

Remark 20. The observation made in Remark 19 is important as it implies that Theorems 14, 15, and 17 apply to time-varying systems whose state spaces are smooth complete Riemannian manifolds $\mathcal{X}$, not necessarily diffeomorphic to $\mathbb{R}^n$. For instance, in Section 5.3 we present an example concerning a kinematic unicycle, the state space of which is non-Euclidean. The reason that our results can be applied to such state spaces is this. The Nash isometric embedding theorem [18, Theorem 3] guarantees the existence of a C^1 embedding of $\mathcal{X}$ in a Euclidean space $\mathbb{R}^n$ of suitable dimension, in such a way that the Riemannian metric of $\mathcal{X}$ is the restriction to $\mathcal{X}$ of the Euclidean metric of $\mathbb{R}^n$. By smoothly extending the vector field $f(t,x)$ from $\mathcal{X}$ to $\mathbb{R}^n$ (which can be done globally in virtue of [13, Lemma 8.6]), one obtains a time-varying differential equation on $\mathbb{R}^n$ for which $\mathcal{X} \subset \mathbb{R}^n$ is a positively invariant set. Thus, Theorems 14, 15, and 17 apply and all the statements in this section continue to hold if the state space of the differential equation is a smooth complete Riemannian manifold. △

4.2 Consequences of the reduction theorems

We present now some useful consequences of the reduction theorems above. The first statement, which is a straightforward consequence of the reduction theorem for UGAS (Theorem 17), replaces Assumption (ii) in that theorem (Γ_2 is LUS-Γ_1) with the assumption that Γ_2 is t_0-US. Even though the latter is more conservative, it is generally easier to verify than Assumption (ii) in Theorem 17.

Proposition 21. *Consider the time-varying system* (3) *under the Basic Assumption. Let* Γ_1 *be a compact set and* Γ_2 *be a closed set, both positively invariant and such that* $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. *If*

(i) Γ_1 *is* UAS *relative to* Γ_2, *and*
(ii) Γ_2 *is* t_0-UAS,

then Γ_1 *is* UAS. *Moreover,* Γ_1 *is* UGAS *if*

(iii) Γ_1 *is* UGAS *relative to* Γ_2,
(iv) Γ_2 *is* t_0-UGAS,
(v) *all solutions are* t_0*-uniformly bounded, i.e.,* $\mathbb{BS} = \mathbb{R}^n$.

Finally, if assumptions (iii) and (iv) hold, then Γ_1 *is* UAS *and all initial states giving rise to* t_0*-uniformly bounded solutions are contained in the basin of* t_0*-uniform attraction of* Γ, *i.e.,* $\mathbb{BS} \subset \mathbb{B}(\Gamma_1)$.

PROOF. Assumption (ii) implies that Γ_2 is t_0-US and t_0-UA, while if Γ_2 is t_0-UA then it is also t_0-UA near Γ_1. Therefore, conditions (i) and (iii) of Theorem 17 hold. That is, in order to prove the first statement, that Γ_1 is UAS, it suffices to establish the implication

$$(\Gamma_2 \text{ is } t_0\text{-US}) \implies (\Gamma_2 \text{ is LUS-}\Gamma_1). \tag{9}$$

Similarly, Assumption (iv) implies that Γ_2 is t_0-US and t_0-UGA. Therefore, the remaining statements in the proposition also follow directly from Theorem 17, provided that the implication (9) holds. Thus, to show that this is the case, assume Γ_2 is t_0-US. Then for each $\varepsilon > 0$, there exists an open set $U \subset \mathbb{R}^n$ such that $\Gamma_2 \subset U$ and for each $(t_0, x_0) \in \mathbb{R} \times U$ and each $t \in T^+_{t_0,x_0}$, $x(t, t_0, x_0) \in B_\varepsilon(\Gamma_2)$. By the compactness of Γ_1 and the fact that $\Gamma_1 \subset \Gamma_2$, there exists $\delta > 0$ such that $B_\delta(\Gamma_1) \subset U$. Then we have

$$(\forall x_0 \in B_\delta(\Gamma_1))(\forall t_0 \in \mathbb{R})(\forall t \in T^+_{t_0,x_0}) \quad x([t_0,t], t_0, x_0) \subset B_\varepsilon(\Gamma_2). \tag{10}$$

Comparing with (5) in the definition of local uniform stability, we see that (10) implies (5) for arbitrary $r > 0$, and thus Γ_2 is LUS-Γ_1. □

From the proposition above we recover a well-known result concerning the stability of equilibria for cascade-connected systems (see [28, Theorem 1.1] for the time-invariant case, and Lemma 2 in [21] for the time-varying case).

Corollary 22 (Cascade-connected systems)**.** *Consider the cascade-connected system*

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, x_2) \\ \dot{x}_2 &= f_2(t, x_2) \end{aligned} \tag{11}$$

where $f_1 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *and* $f_2 : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ *satisfy the Basic Assumption and* $f_1(\cdot, 0_n, 0_m) \equiv 0_n$,

$f_2(\cdot, 0_m) \equiv 0_m$. *Then the equilibrium* $(x_1, x_2) = (0_n, 0_m)$ *is* UGAS *for* (11) *if and only if*

(i) the equilibrium $x_1 = 0_n$ *is UGAS for* $\dot{x}_1 = f_1(t, x_1, 0_m)$,

(ii) the equilibrium $x_2 = 0_m$ *is UGAS for* $\dot{x}_2 = f_2(t, x_2)$, *and*

(iii) all solutions of (11) *are* t_0*-uniformly bounded, i.e.,* $\mathbb{BS} = \mathbb{R}^n \times \mathbb{R}^m$.

On the other hand, if only Assumptions (i) and (ii) hold and the set $\mathbb{BS}$ *of* t_0*-uniformly bounded solutions is only a subset of* $\mathbb{R}^n \times \mathbb{R}^m$, *then the equilibrium* $(x_1, x_2) = (0_n, 0_m)$ *is* UAS *and the set* $\mathbb{BS}$ *is contained in the basin of* t_0*-uniform attraction of the equilibrium* $(0_n, 0_m)$, *i.e.,* $\mathbb{BS} \subset \mathbb{B}((0_n, 0_m))$.

PROOF. The sufficiency part follows directly from Proposition 21 by setting $\Gamma_1 := \{(0_n, 0_m) \in \mathbb{R}^n \times \mathbb{R}^m\}$ and $\Gamma_2 := \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m : x_2 = 0_m\}$. Then assumption (i) implies that Γ_1 is UGAS relative to Γ_2, while assumption (ii) implies that Γ_2 is t_0-UGAS.

For the necessity part, assuming $(x_1, x_2) = (0_n, 0_m)$ is UGAS for the cascaded system (11), we need to show that properties (i)-(iii) hold. Property (iii) follows from Theorem 17. As for property (i) and (ii), we will only prove their uniform stability component. The proofs for the remaining components use analogous arguments. Denote by $B_\delta(0_n)$, $B_\delta(0_m)$, and $B_\delta(0_n \times 0_m)$ the open δ balls centred at the origin in $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^n \times \mathbb{R}^m$, respectively, and partition the solution map as $x(t, t_0, x_0) = (x_1(t, t_0, x_0), x_2(t, t_0, x_0)) \in \mathbb{R}^n \times \mathbb{R}^m$. Then the following set inclusions hold: $0_n \times B_\delta(0_m) \subset B_\delta(0_n, 0_m)$; $B_\delta(0_n) \times 0_m \subset B_\delta(0_n, 0_m)$; and, $B_\delta(0_n \times 0_m) \subset B_\delta(0_n) \times B_\delta(0_m)$.

For any $\varepsilon > 0$, there exists $\delta > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(0_n \times 0_m)) \subset B_\varepsilon(0_n \times 0_m)$ for all $t_0 \in \mathbb{R}$. Then,

$$x(\mathbb{R}_{\geq t_0}, t_0, 0_n \times B_\delta(0_m)) \subset B_\varepsilon(0_n \times 0_m) \,\forall\, t_0 \in \mathbb{R}, \quad (12)$$

$$x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(0_n) \times 0_m) \subset B_\varepsilon(0_n \times 0_m) \,\forall\, t_0 \in \mathbb{R}. \quad (13)$$

Since the x_2 subsystem is decoupled from the x_1 dynamics, from (12) we deduce that $x_2(\mathbb{R}_{\geq t_0}, t_0, 0_n \times B_\delta(0_m)) \subset B_\varepsilon(0_m)$ for all $t_0 \in \mathbb{R}$, and thus the equilibrium 0_m is US for $\dot{x}_2 = f_2(t, x_2)$. Since $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(0_n) \times 0_m) = x_1(\mathbb{R}_{\geq t_0}, t_0, B_\delta(0_n) \times 0_m) \times 0_n$, from (13) we deduce that

$$x_1(\mathbb{R}_{\geq t_0}, t_0, B_\delta(0_n) \times 0_m) \subset B_\varepsilon(0_n)$$

for all $t_0 \in \mathbb{R}$, and therefore the equilibrium $x_1 = 0_n$ is US for $\dot{x}_1 = f_1(t, x_1, 0)$. □

4.3 Lyapunov characterizations

The reduction theorems in Section 4 and Proposition 21 rely on assumptions that are somewhat unusual in the stability theory literature:

- Γ_2 is LUS-Γ_1. Used in Theorems 14 and 17.
- Γ_2 is either t_0-UA near Γ_1 or t_0-UGA. Used in Theorems 15 and 17.
- Γ_2 is either t_0-UAS or t_0-UGAS. Used in Proposition 21.

In this section we give Lyapunov characterizations of the properties listed above. Even though these characterizations are more conservative in general, they may result easier to verify in concrete cases. An example that illustrates this assertion is given in Section 5.1.

Proposition 23 (Lyapunov characterization of LUS-Γ_1 property)**.** *Consider the time-varying system* (3) *under the Basic Assumption. Let* Γ_1 *be a compact set and* Γ_2 *be a closed set, both positively invariant and such that* $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. *Suppose there exist* $r, s > 0$ *and a* C^1 *nonnegative function* $V : \mathbb{R} \times B_r(\Gamma_1) \to \mathbb{R}$ *such that*

$$\alpha(\|x\|_{\Gamma_2}) \leq V(t, x) \leq \beta(\|x\|_{\Gamma_1}) \quad (14)$$

$$\partial_t V(t, x) + \partial_x V(t, x) f(t, x) \leq 0, \quad (15)$$

for all $(t, x) \in \mathbb{R} \times B_r(\Gamma_1)$, *where* $\alpha : [0, s) \to \mathbb{R}$ *and* $\beta : [0, r) \to \mathbb{R}$ *are two class* $\mathcal{K}$ *functions. Then,* Γ_2 *is* LUS-Γ_1.

PROOF. Let $V : \mathbb{R} \times B_r(\Gamma_1) \to \mathbb{R}$ be a function satisfying the hypotheses of the proposition. Since α and β are class-$\mathcal{K}$ functions, so is the function $\beta^{-1} \circ \alpha : [0, c) \to \mathbb{R}$, for suitable $c > 0$. For each $\varepsilon > 0$, pick $\delta > 0$ such that $\delta < \beta^{-1} \circ \alpha(\min\{c, \varepsilon\})$. By property (15), for each $(t_0, x_0) \in \mathbb{R} \times B_\delta(\Gamma_1)$ and each $\bar{t} \in T^+_{t_0, x_0}$, if $x([t_0, \bar{t}], t_0, x_0) \subset B_r(\Gamma_1)$, then the function $t \mapsto V(t, x(t, t_0, x_0))$ is nonincreasing for $t \in [t_0, \bar{t}]$. By property (14) we have

$$\alpha(\|x(t, t_0, x_0)\|_{\Gamma_2}) \leq V(t, x(t, t_0, x_0)) \leq V(t_0, x_0) \leq \beta(\|x_0\|_{\Gamma_1}) \leq \beta(\delta),$$

for all $t \in [t_0, \bar{t}]$. From the above we deduce that

$$\|x(t, t_0, x_0)\|_{\Gamma_2} \leq \alpha^{-1} \circ \beta(\delta) < \varepsilon$$

for all $t \in [t_0, \bar{t}]$. We have thus shown that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $(t_0, x_0) \in \mathbb{R} \times B_\delta(\Gamma_1)$, if $x([t_0, \bar{t}], t_0, x_0) \subset B_r(\Gamma_1)$ then $x([t_0, \bar{t}], t_0, x_0) \subset B_\varepsilon(\Gamma_2)$. This proves that Γ_2 is LUS-Γ_1. □

Next, we provide a Lyapunov characterization of the t_0-UA, t_0-UGA, and t_0-UGAS properties for closed, but not necessarily compact sets.

Proposition 24 (Lyapunov characterization of t_0-UA, t_0-UGA, and t_0-UGAS properties)**.** *Consider the time-varying system* (3) *under the Basic Assumption. Let* $\Gamma \subset \mathbb{R}^n$ *be a closed, positively-invariant set, and* $U \subset \mathbb{R}^n$ *be an open set such that* $\Gamma \cap U \neq \emptyset$. *Let* $V : \mathbb{R} \times U \to \mathbb{R}$ *be a* C^1 *nonnegative function such that*

$$W_1(x) \leq V(t, x) \leq W_2(x) \quad (16)$$

$$\partial_t V(t, x) + \partial_x V(t, x) f(t, x) \leq -W_3(x), \quad (17)$$

for all $(t, x) \in \mathbb{R} \times U$, *where* $W_1, W_2, W_3 : U \to \mathbb{R}$ *are continuous nonnegative functions such that* $W_1^{-1}(0) = W_2^{-1}(0) = W_3^{-1}(0) = \Gamma \cap U$. *Let* $U^\star \subset U$ *be defined as*[6] $U^\star := \{x_0 \in U : (\forall t_0 \in \mathbb{R})\, x(T^+_{t_0, x_0}, t_0, x_0) \subset U\}$. *Then, the following implications hold:*

[6] The set $U^\star$ may be empty. If $U = \mathbb{R}^n$, then $U^\star = \mathbb{R}^n$.

(a) *All initial states in $U^\star$ giving rise to solutions that are t_0-uniformly bounded are contained in the basin of t_0-uniform attraction of Γ, i.e.,*
$$\mathbb{BS} \cap U^\star \subset \mathbb{B}(\Gamma).$$
(b) *If $U = \mathbb{R}^n$ and $\Gamma \subset \operatorname{int}(\mathbb{BS})$, then Γ is t_0-UA.*
(c) *If $U = \mathbb{R}^n$ and all solutions are t_0-uniformly bounded, i.e., $U = \mathbb{BS} = \mathbb{R}^n$, then Γ is t_0-UGA.*
(d) *If $U = \mathbb{BS} = \mathbb{R}^n$, and there exist $r > 0$ and a class $\mathcal{K}$ function $\alpha_1 : [0, r) \to \mathbb{R}$ such that $\alpha_1(\|x\|_\Gamma) \leq W_1(x)$ for all $x \in B_r(\Gamma)$, then Γ is t_0-UGAS.*

PROOF. Part (a). Letting $x_0 \in \mathbb{BS} \cap U^\star$ be arbitrarily fixed, we want to show that $x_0 \in \mathbb{B}(\Gamma)$, that is
$$(\forall \varepsilon > 0)(\exists T > 0)(\forall t_0 \in \mathbb{R})$$
$$t_0 + T \in T^+_{t_0,x_0} \text{ and } x(\mathbb{R}_{\geq t_0+T} \cap T^+_{t_0,x_0}, t_0, x_0) \subset B_\varepsilon(\Gamma). \tag{18}$$
If $x_0 \in \Gamma$, then $x_0 \in \mathbb{B}(\Gamma)$ because Γ is positively invariant. Suppose $x_0 \in U^\star \setminus \Gamma$. Since $x_0 \in \mathbb{BS}$, by definition there exists $c > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, x_0) \subset B_c(0)$ for all $t_0 \in \mathbb{R}$, which implies that $T^+_{t_0,x_0} = \mathbb{R}_{\geq t_0}$ for all $t_0 \in \mathbb{R}$. Letting $K := \overline{B_c(0)}$, a compact set and using the fact that $x_0 \in \mathbb{BS} \cap U^\star$, we have
$$(\forall t_0 \in \mathbb{R})\ x(\mathbb{R}_{\geq t_0}, t_0, x_0) \subset K \cap U. \tag{19}$$
Let $\varepsilon > 0$ be arbitrarily fixed and define
$$\delta_1 := \min_{x \in K, \|x\|_\Gamma \geq \varepsilon/2} W_1(x), \quad \hat{\delta}_2 := \min_{x \in K, W_2(x) \geq \delta_1} W_1(x),$$
$$\delta_2 := \min\{\hat{\delta}_2, W_2(x_0)\}.$$
Since W_1 and W_2 are continuous nonnegative functions and the set K is compact, the constants $\delta_1, \hat{\delta}_2$ exist and are nonnegative. Moreover, since $W_1 \neq 0$ on the set $\{x \in K : \|x\|_\Gamma \geq \varepsilon/2\}$, we have that $\delta_1 > 0$. Further, since $W_2(x) \geq \delta_1$ implies that $x \notin \Gamma$, and since $W_1(x) \neq 0$ for all $x \notin \Gamma$, it follows that $\hat{\delta}_2 > 0$. Finally, since $x_0 \notin \Gamma$, $W_2(x_0) > 0$, so $\delta_2 > 0$ as well. Next, let
$$k := \min_{x \in K, W_1(x) \geq \delta_2/2} W_3(x). \tag{20}$$
Following similar arguments as above, we conclude that $k > 0$. Furthermore, in view of the definition of δ_1 and δ_2, were defined we have
$$\big\{x \in K : W_1(x) \leq \delta_1\big\} \subset \overline{B_{\varepsilon/2}(\Gamma)} \subset B_\varepsilon(\Gamma), \tag{21}$$
$$\big\{x \in K : W_1(x) \leq \delta_2\big\} \subset \big\{x \in K : W_2(x) \leq \delta_1\big\}. \tag{22}$$
We claim that
$$(\exists T > 0)(\forall t_0 \in \mathbb{R})\ W_1(x(t_0 + T, t_0, x_0)) \leq \delta_2. \tag{23}$$
By way of contradiction, suppose that
$$(\forall T > 0)(\exists t_0 \in \mathbb{R})\ W_1(x(t_0 + T, t_0, x_0)) > \delta_2. \tag{24}$$
Using (19) and (24), we get $x([t_0, t_0 + T], t_0, x_0) \subset K \cap \{x \in \mathbb{R}^n : W_1(x) \geq \delta_2/2\}$. Let $T := (W_2(x_0) - \delta_2)/k \geq 0$, and let $t_0 \in \mathbb{R}$ be such that (24) holds. Using (17), (19), and the definition of k in (20), we have
$$\begin{aligned} V(t_0 + T, x(t_0 + T, t_0, x_0)) &\leq V(t_0, x_0) - kT \\ &\leq W_2(x_0) - kT = \delta_2. \end{aligned}$$
By the first inequality in (16), $W_1(x(t_0+T, t_0, x_0)) \leq \delta_2$, contradicting (24). Thus (23) holds. Henceforth, fix $T \geq 0$ such that (23) holds. Since $W_1(x(t_0 + T, t_0, x_0)) \leq \delta_2$, by (19) and (22) we have that $W_2(x(t_0 + T, t_0, x_0)) \leq \delta_1$ for any $t_0 \in \mathbb{R}$. Since for any $t_0 \in \mathbb{R}$ the function $t \mapsto V(t, x(t, t_0, x_0))$ is nonincreasing, and since $V(t_0 + T, x(t_0 + T, t_0, x_0)) \leq W_2(x(t_0 + T, t_0, x_0)) \leq \delta_1$, we have that $V(t, x(t, t_0, x_0)) \leq \delta_1$ for all $t_0 \in \mathbb{R}$ and all $t \geq t_0 + T$. Using the first inequality in (16), we deduce that $W_1(x(t, t_0, x_0)) \leq \delta_1$ for all $t_0 \in \mathbb{R}$ and all $t \geq t_0+T$. By (19) and (21), we conclude that for all $t_0 \in \mathbb{R}$, $x(\mathbb{R}_{\geq t_0+T}, t_0, x_0) \subset B_\varepsilon(\Gamma)$, and thus (18) holds. We have thus shown that for each $x_0 \in \mathbb{BS} \cap U^\star$, $x_0 \in \mathbb{B}(\Gamma)$. This concludes the proof of part (a).

Part (b). If $U = \mathbb{R}^n$ and $\Gamma \subset \operatorname{int}(\mathbb{BS})$, then $U^\star = \mathbb{R}^n$, and by part (a), $\Gamma \subset \operatorname{int}(\mathbb{BS}) \subset \operatorname{int}(\mathbb{B}(\Gamma))$, which implies that Γ is t_0-UA.

Part (c). If $U = \mathbb{BS} = \mathbb{R}^n$ then $U^\star = \mathbb{R}^n$, and by part (a), $\mathbb{B}(\Gamma) = \mathbb{R}^n$, which implies that Γ is t_0-UGA.

Part (d). Now suppose that $U = \mathbb{BS} = \mathbb{R}^n$ so that, by part (c), Γ is t_0-UGA, and there exist $r > 0$ and a class $\mathcal{K}$ function $\alpha_1 : [0, r) \to \mathbb{R}$ such that $\alpha_1(\|x\|_\Gamma) \leq W_1(x)$ for all $x \in B_r(\Gamma)$. We need to show that Γ is t_0-US. Let $\varepsilon > 0$ be arbitrary, without loss of generality $\varepsilon \in (0, r)$. Define the open set $U := \{x \in \mathbb{R}^n : W_2(x) < \alpha_1(\varepsilon)\}$. For any initial data $(t_0, x_0) \in \mathbb{R} \times U$, we have $W_2(x_0) < \alpha_1(\varepsilon)$, and by (16) and (17) we have that $W_1(x(t, t_0, x_0)) < \alpha_1(\varepsilon)$ for all $t \in T^+_{t_0,x_0}$. Since $W_1(x) \geq \alpha_1(\|x\|_\Gamma)$, $\|x(t, t_0, x_0))\|_\Gamma < \varepsilon$ for all $t \in T^+_{t_0,x_0}$. This proves that Γ is t_0-US. In conclusion, we have shown that Γ is both t_0-UGA and t_0-US, which implies that Γ is t_0-UGAS. This concludes the proof of the proposition. □

Part (a) of Proposition 24 yields the next Lyapunov characterization of the property that a set Γ_2 is t_0-UA near Γ_1, used in Theorems 15 and 17.

Corollary 25 (Lyapunov characterization of the property that Γ_2 is t_0-UA near Γ_1)**.** *Consider the time-varying system* (3) *under the Basic Assumption. Let Γ_1 be a compact set and Γ_2 be a closed set, both positively invariant and such that $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. Suppose that Γ_1 is* US*, and for some open set $U \subset \mathbb{R}^n$ such that $\Gamma_1 \subset U$, there exists a C^1 nonnegative function $V : \mathbb{R} \times U \to \mathbb{R}$ satisfying* (16) *and* (17)*, where $W_1, W_2, W_3 : U \to \mathbb{R}$ are continuous nonnegative functions such that $W_1^{-1}(0) = W_2^{-1}(0) = W_3^{-1}(0) = \Gamma \cap U$. Then Γ_2 is t_0-UA near Γ_1.*

PROOF. Since Γ_1 is compact and contained in the open set U, there exists $\varepsilon > 0$ such that $B_\varepsilon(\Gamma_1) \subset U$. Since Γ_1 is US, there exists $\delta > 0$ such that
$$(\forall t_0 \in \mathbb{R})\ x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma_1)) \subset B_\varepsilon(\Gamma_1) \subset U, \tag{25}$$
which implies that $B_\delta(\Gamma_1) \subset U^\star$, with $U^\star$ defined in the statement of Proposition 24. Moreover, since Γ_1 is compact the set $B_\varepsilon(\Gamma_1)$ is bounded, and thus property (25) implies that $B_\delta(\Gamma_1) \subset \mathbb{BS}$. We have thus established that $B_\delta(\Gamma_1) \subset \mathbb{BS} \cap U^\star$. By part (a) of Proposition 24,

$B_\delta(\Gamma_1) \subset \mathbb{BS} \cap U^\star \subset \mathbb{B}(\Gamma_2)$, and thus Γ_2 is t_0-UA near Γ_1. □

5 Examples

In this section we present three examples demonstrating the utility of the theoretical results in Section 4. In the first example we revisit the Slotine & Li controller mentioned in the introduction, considering (for simplicity) the special case of one degree-of-freedom mechanical systems, and we propose a reduction viewpoint to understand its operation. We show, in particular, that its uniform global tracking properties can be derived using Propositions 23, 24, and Theorem 17. The second example illustrates the reduction theorem for t_0-uniform attractivity (Theorem 15). Finally, in the third example we use Proposition 21 to derive a global path following controller for a kinematic unicycle meeting a position tracking requirement on the path.

5.1 *The Slotine & Li controller*

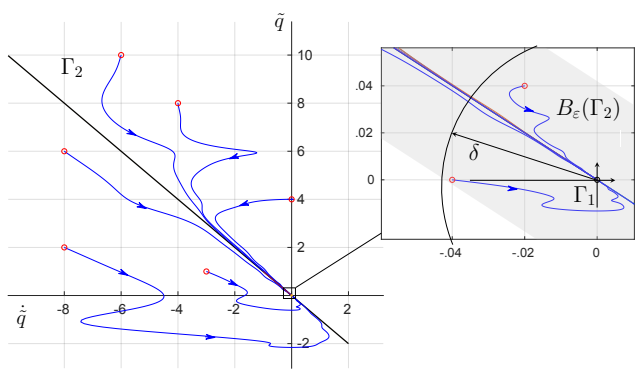

Fig. 3. Trajectories generated by the Slotine-and-Li controller represented on the plane

Consider the following Lagrangian control system

$$d(q)\ddot{q} + c(q)\dot{q}^2 + g(g) = u,$$

where, for simplicity of exposition, we assume that $q \in \mathbb{R}$. The function $q \mapsto d(q)$ denotes the system's inertia and it is bounded, smooth and bounded away from zero uniformly for all $q \in \mathbb{R}$, i.e., $0 < d_m \leq d(\cdot) \leq d_M$; the function $q \mapsto c(q)$ is uniformly bounded and satisfies $2c(q) := \dot{d}(q)$; the function $q \mapsto g(q)$ denotes forces stemming from potential energy and it is also uniformly bounded. Consider the problem of making the generalized positions and velocities q and $\dot{q}$ follow some given desired smooth bounded reference trajectories $q_d(t)$ and $\dot{q}_d(t)$. This problem was solved (for systems with $q \in \mathbb{R}^n$, $n \geq 1$) in [29], where the now well-known *Slotine & Li controller* was proposed. This is defined as follows. Let $\lambda, k_d > 0$ be two design parameters and let

$$u = d(q)\ddot{q}_r + c(q)\dot{q}\dot{q}_r + g(g) - k_d s \tag{26a}$$

$$s := \dot{q} - \dot{q}_r \tag{26b}$$

$$\dot{q}_r := \dot{q}_d - \lambda\tilde{q}, \quad \tilde{q} := q - q_d. \tag{26c}$$

Then, the closed-loop nonlinear time-varying system is given by

$$d\left(\tilde{q} + q_d(t)\right)\dot{s} + c(\tilde{q} + q_d(t))\left(s + \dot{q}_d + \lambda\tilde{q}\right)s + k_d s = 0 \tag{27a}$$

$$\dot{\tilde{q}} = -\lambda\tilde{q} + s. \tag{27b}$$

It is well known that for the system (27) the origin, $\{(\tilde{q}, s) = (0,0)\}$, is uniformly globally asymptotically stable; this may be established via various methods, including Lyapunov's first [32]. We revisit the analysis of this system because the rationale that leads to the design of this controller in [30] captures well the essence of the reduction theorems. Indeed, the controller is designed in a manner to steer the trajectory $\dot{q}(t)$ to the artificially-defined reference $\dot{q}_r$ generated by (26c). Given that $\dot{q} = \dot{q}_r$ is equivalent to $s = 0$, the controller is designed to steer the trajectories towards the set $\Gamma_2 := \{(\tilde{q}, s) : s = 0\}$ (see Figure 3 for an illustration with $\lambda = 1$ and $k_d = 3$) on which the dynamics is *reduced* to $\dot{\tilde{q}} = -\lambda\tilde{q}$. More precisely, the function

$$V(t, \tilde{q}, s) := \frac{1}{2}d\left(\tilde{q} + q_d(t)\right)s^2, \tag{28}$$

satisfies

$$\frac{1}{2}d_m s^2 \leq V(t, \tilde{q}, s) \leq d_M s^2 + \tilde{q}^2 \tag{29a}$$

$$\dot{V}(t, \tilde{q}, s) \leq -k_d s^2 \tag{29b}$$

in view of the assumption that $0 < d_m \leq d(\cdot) \leq d_M$ and $\dot{d}(q) = 2c(q)\dot{q}$. From these inequalities, it follows that $s \to 0$ for any $k_d > 0$. That is, the trajectories tend to the set Γ_2 on which they satisfy $\dot{\tilde{q}} = -\lambda\tilde{q}$, so $\tilde{q} \to 0$ for any $\lambda > 0$. It is important to stress that, although intuitive, this argument tacitly relies on the set Γ_2 being reached in finite time, which is not the case for this controller; the trajectories only *tend asymptotically* to Γ_2. A formal argument may be made using Theorem 17 even if the convergence to Γ_2 is only asymptotic. To this end, letting $\Gamma_1 := \{(\tilde{q}, s) = (0,0)\}$ and $\Gamma_2 := \{(\tilde{q}, s) : s = 0\}$, the following remarks are in order:

- All solutions of (27) are t_0-uniformly bounded. This follows from (29) and from the fact that (27b) constitutes an exponentially stable linear time-invariant system with uniformly bounded input $s(t)$.
- The set Γ_2 is LUS-Γ_1. This follows from Proposition 23. Clearly, $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^2$. Also, Γ_2 is positively invariant since $s = 0$ is a solution of (27a). Finally, (14) and (15) hold in view of (29) with $\alpha(\|x\|_{\Gamma_2}) = (1/2)d_m s^2$, $\beta(\|x\|_{\Gamma_1}) = d_M s^2 + \tilde{q}^2$. The property is also illustrated in the zoomed plot in Figure 3: solutions that start in a neighbourhood of the origin [7], $B_\delta(\Gamma_1)$, remain in a neighbourhood of Γ_2, $B_\varepsilon(\Gamma_2)$, the gray band.
- The set Γ_2 is t_0-UA near $\Gamma_1 = \{(\tilde{q}, s) = (0,0)\}$. This follows from part (b) of Proposition 24, in view of the t_0-uniformly boundedness of the solutions, with V as

[7] Strictly speaking, in Figure 3 Γ_1 is represented as the point $\{(\tilde{q}, \dot{\tilde{q}}) = (0,0)\}$ which is equivalent to $\{(\tilde{q}, s) = (0,0)\}$

in (28), $U = \mathbb{R}$, and $\Gamma = \Gamma_2$. Also, the property is illustrated in the zoomed plot in Figure 3: solutions starting in a neighbourhood of the origin Γ_1, are attracted to Γ_2.

- The set Γ_1 is UGAS relative to Γ_2. This follows from the fact that, for the system (27b) with $s = 0$, $\{\tilde{q} = 0\}$ is UGAS.
- By Theorem 17, we conclude that Γ_1 is UGAS.
- One can also use part (d) of Proposition 24 with V as in (28), $\Gamma = \Gamma_2$, $U = \mathbb{R}^2$, and $\alpha_1(\|x\|_{\Gamma_2}) = (1/2)d_m s^2$ to arrive at the conclusion that Γ_2 is t_0-UGAS, then use Proposition 21 to conclude that Γ_1 is UGAS. From the left plot in Figure 3 it is appreciated that solutions with larger initial conditions, tend *asymptotically* to Γ_2, the line on the plane $\{\dot{\tilde{q}} = -\tilde{q}\}$.

Even though the Slotine-Li controller does not make trajectories $s(t)$ converge to zero in finite time, the reduction argument presented above captures the intuition behind the operation of the controller presented at the beginning of this discussion, namely the idea that the controller makes solutions approach the line $s = 0$, that on this line solutions converge exponentially to the origin, and that these two properties imply that solutions converge to the origin.

5.2 *Illustration of reduction theorem for t_0-uniform attractivity*

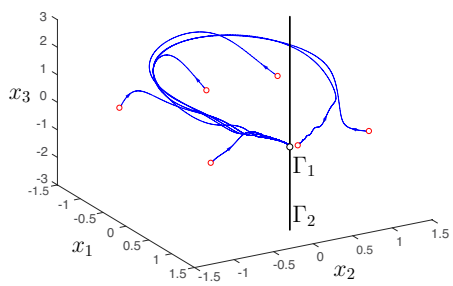

Fig. 4. A few solutions for the example in Section 5.2. The equilibrium Γ_1 is almost globally t_0-UA but unstable.

Consider the time-varying system

$$\dot{x}_1 = x_2(x_1 - 1) - x_1(x_1^2 + x_2^2 - 1) \tag{30a}$$
$$\dot{x}_2 = -x_1(x_1 - 1) - x_2(x_1^2 + x_2^2 - 1) \tag{30b}$$
$$\dot{x}_3 = -x_3^3 + (x_1 - 1)^2 + x_2^2 f(t), \tag{30c}$$

where $f(t)$ is a continuous bounded function. This system satisfies the Basic Assumption. Letting $\Gamma_2 = \{(x_1, x_2, x_3) : x_1 = 1, x_2 = 0\}$ and $\Gamma_1 = \{(x_1, x_2, x_3) = (1, 0, 0)\}$, we claim that Γ_1 is t_0-UA with basin of attraction given by the whole state space minus a set of measure zero (i.e., it is *almost globally* t_0-UA). On Γ_2 the dynamics are described by the differential equation $\dot{x}_3 = -x_3^3$, whose origin represents the set Γ_1, and therefore Γ_1 is UGAS relative to Γ_2. In Example 2 we showed that the equilibrium $(x_1, x_2) = (1, 0)$ of the subsystem

$$\dot{x}_1 = x_2(x_1 - 1) - x_1(x_1^2 + x_2^2 - 1) \tag{31a}$$
$$\dot{x}_2 = -x_1(x_1 - 1) - x_2(x_1^2 + x_2^2 - 1) \tag{31b}$$

is t_0-UA with basin of attraction given by $\mathbb{R}^2 \setminus \{(0, 0)\}$. In particular, all its solutions are bounded, and in fact t_0-uniformly bounded because this system is time-invariant. Since the control system $\dot{x}_3 = -x_3^3 + u$ is input-to-state stable, all solutions of the subsystem

$$\dot{x}_3 = -x_3^3 + (x_1 - 1)^2 + x_2^2 f(t),$$

are also t_0-uniformly bounded because the pair $(x_1(t), x_2(t))$ and the function $f(t)$ are bounded. The considerations above show that all solutions of system (30) are t_0-uniformly bounded, i.e., $\mathbb{BS} = \mathbb{R}^3$. Letting $\mathcal{X} := \mathbb{R}^3 \setminus \{(x_1, x_2, x_3) : x_1 = x_2 = 0\}$, $\mathcal{X}$ has full measure in $\mathbb{R}^3$, and it positively invariant because its complement, the set $\{(x_1, x_2, x_3) : x_3 = 0\}$, is invariant. Since, for system (31), $\mathbb{B}(\{(1, 0)\}) = \mathbb{R}^2 \setminus \{(0, 0)\}$, we have that, for system (30), $\mathbb{B}(\Gamma_2) = \mathcal{X}$, i.e., Γ_2 is t_0-UGA relative to $\mathcal{X}$. To summarize, we have determined that (a) Γ_1 is UGAS relative to Γ_2, (b) Γ_2 is t_0-UGA relative to $\mathcal{X}$, and (c) $\mathbb{BS} = \mathcal{X}$. By Theorem 15, Γ_1 is t_0-UGA relative to $\mathcal{X}$ or, what is the same, Γ_1 is almost globally t_0-UA, as claimed. A few solutions of the system with $f(t) = \sin(t)^2$ and $t_0 = 0$ are depicted in Figure 4. Note that Γ_1 is unstable, and indeed Figure 4 shows an initial state very close to Γ_1 giving rise to an orbit with a large excursion away from Γ_1.

5.3 *Circular path following for a kinematic unicycle*

This example illustrates the use of reduction theorems in the context of hierarchies of control specifications that were mentioned in the introduction. Consider the kinematic unicycle

$$\dot{x}_1 = u_1 \cos(\theta) \tag{32a}$$
$$\dot{x}_2 = u_1 \sin(\theta) \tag{32b}$$
$$\dot{\theta} = u_2, \tag{32c}$$

where $x \in \mathbb{R}^2$ are the Cartesian coordinates of the unicycle in the plane, $\theta \in \mathbb{S}^1$ is the unicycle heading, and $(u_1, u_2) \in \mathbb{R} \times \mathbb{R}$, the linear and angular speeds of the unicycle, are the control inputs. We denote by $\chi := (x, \theta)$ the state of the unicycle, and by $\mathcal{X} := \mathbb{R}^2 \times \mathbb{S}^1$ its state space. For a vector $x \in \mathbb{R}^2$, we denote by angle(x) the angle that the vector makes with the positive x_1 axis. Let $C_r := \{x \in \mathbb{R}^2 : x^\top x = r^2\}$ denote the circle of radius $r > 0$ centred at the origin, and consider the following list of control specifications:

(a) For each initial position $x(0) \in C_r$ and initial heading $\theta(0) = \text{angle}(x(0)) + \pi/2$ (i.e., heading tangent to C_r with counterclockwise orientation), $x(t)$ must remain on C_r for all $t \geq 0$.

(b) For all other initial states, the unicycle position, $x(t)$, must asymptotically converge to C_r.

(c) For each initial state in some neighborhood of the reference signal

$$\chi_d(t) = \big(r\cos(\alpha_d(t)), r\sin(\alpha_d(t)), \alpha_d(t) + \pi/2\big),$$

where $\alpha_d : \mathbb{R} \to \mathbb{S}^1$ is a given C^1 function, the unicycle state must asymptotically converge to $\chi_d(t)$.

In essence, for any initial state we want the unicycle to approach and follow the circle C_r counterclockwise,

rendering the circle invariant for the position dynamics. Moreover, we want to ensure that, on C_r, the motion of the unicycle matches a prescribed reference signal. This latter specification is only required to be met locally.

A controller meeting specifications (a) and (b) was presented in [5]. Using Proposition 21, we now enhance the controller in [5] to meet also specification (c). Define the set

$$\begin{aligned}\Gamma &= \{\chi = (x_1, x_2, \theta) \in \mathcal{X} : x^\top x = r^2, \\ &\qquad\qquad \theta = \text{angle}(x) + \pi/2\} \\ &= \{\chi = (x_1, x_2, \theta) \in \mathcal{X} : x_1 = r\sin(\theta), \\ &\qquad\qquad x_2 = -r\cos(\theta)\}.\end{aligned}$$

The set Γ corresponds to the situation when the unicycle position is on C_r and its heading is tangent to C_r with counterclockwise orientation. It is clear that in order to meet specifications (a) and (b), we need to render Γ UGAS. The controller in [5, Proposition III.1] does just that. For any $v \in \mathbb{R}$, the smooth feedback

$$u_1 = v \tag{33a}$$

$$u_2 = \frac{u_1}{r} + r\left(x_1\cos(\theta) + x_2\sin(\theta)\right), \tag{33b}$$

renders Γ UGAS. One can replace $v \in \mathbb{R}$ by any smooth real-valued function without affecting the result. In order to meet specification (c), we assign v in the feedback (33a) so as to incorporate an additional stabilization mechanism. Define $\theta_d(t) := \alpha_d(t) + \pi/2$, and note that, having met specifications (a) and (b), specification (c) corresponds to making $\theta \to \theta_d(t)$ for suitable initial states. This control objective can be attained without affecting the UGAS property of Γ, as follows. On Γ, the feedback (33) reduces to $(u_1, u_2) = (v, v/r)$, and therefore the evolution of $\theta(t)$ is governed by $\dot\theta = v/r$. Letting $v = r\left[\dot\theta_d(t) - \sin(\theta - \theta_d(t))\right]$, i.e., letting

$$u_1 = r\left[\dot\theta_d - \sin(\theta - \theta_d(t))\right] \tag{34a}$$

$$u_2 = \frac{u_1}{r} + r\left[x_1\cos(\theta) + x_2\sin(\theta)\right], \tag{34b}$$

makes $\theta(t) \to \theta_d(t)$ for almost all initial states on Γ. Of course, this does not yet imply that specification (c) is met, since we need to allow initial states outside of Γ, but Proposition 21 will allow us to obtain the desired result. In order to formulate a reduction problem, we define the error state $\tilde\chi \in \mathcal{X}$ as $\tilde\chi := (\tilde x, \tilde\theta)$, with $\tilde x := (x_1 - r\sin(\theta), x_2 + r\cos(\theta))$ and $\tilde\theta = \theta - \theta_d(t)$. The closed-loop system in error coordinates reads as

$$\begin{aligned}\dot{\tilde x}_1 = -r^2\Big[&\tilde x_1\cos(\tilde\theta + \theta_d(t)) \\ &+ \tilde x_2\sin(\tilde\theta + \theta_d(t))\cos(\tilde\theta + \theta_d(t))\Big]\end{aligned} \tag{35a}$$

$$\begin{aligned}\dot{\tilde x}_2 = -r^2\Big[&\tilde x_2\sin(\tilde\theta + \theta_d(t)) \\ &+ \tilde x_1\sin(\tilde\theta + \theta_d(t))\cos(\tilde\theta + \theta_d(t))\Big]\end{aligned} \tag{35b}$$

$$\dot{\tilde\theta} = -\sin(\tilde\theta) + r\left[\tilde x_1\cos(\tilde\theta + \theta_d(t)) + \tilde x_2\sin(\tilde\theta + \theta_d(t))\right]. \tag{35c}$$

The system above satisfies the Basic Assumption (see Remark 1), and in $\tilde\chi$-coordinates the set Γ becomes

$$\Gamma_2 = \{\tilde\chi \in \mathcal{X} : \tilde x = 0\},$$

and meeting specification (c) corresponds to stabilizing the equilibrium $\Gamma_1 = \{0 \in \mathcal{X}\}$. Clearly, Γ_1 is compact, Γ_2 is closed, and $\Gamma_1 \subset \Gamma_2$. By [5, Proposition III.1], the feedback (34) renders Γ_2 UGAS, and to meet specification (c), we need to show that Γ_1 is UAS. On Γ_2, the dynamics are described by the differential equation $\dot{\tilde\theta} = -\sin(\tilde\theta)$. The equilibrium $\tilde\theta = 0$ is asymptotically stable for the above differential equation, which means that Γ_1 is UAS relative to Γ_2. By Proposition 21, Γ_1 is UAS. In conclusion, the feedback (34) simultaneously renders Γ_2 UGAS and Γ_1 UAS, therefore meeting specifications (a)-(c). Figure 5 shows simulation results for

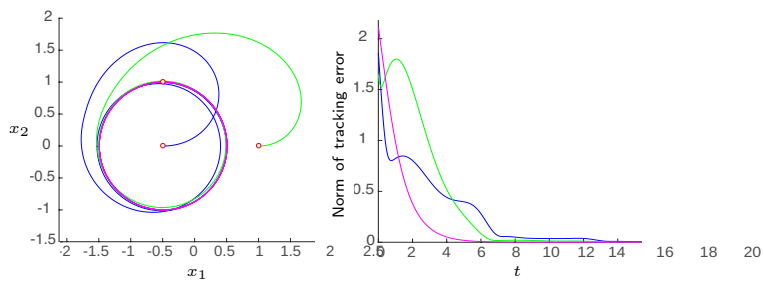

Fig. 5. Unicycle path following with simultaneous trajectory tracking. On the left-hand side, behaviour of the unicycle on the plane for three different initial conditions, including one of the circle. On the right-hand side, the corresponding norm of the tracking error for each solution.

$r = 1$ and $\alpha_d(t) = t + \sin(t)$. As expected, all solutions converge to the circle, and move counterclockwise around it. The corresponding tracking errors (specification (c)) converge to zero. One of the displayed solutions (the one in magenta) corresponds to the unicycle being initialized on the circle, with heading tangent to it. In accordance with specification (a), the unicycle remains on the circle even though its initial tracking error is not zero. In accordance with specification (c), the unicycle adjusts its linear speed to synchronize with the reference signal without leaving the circle.

As a final remark, the closed-loop system (35) does not have a cascade-connected structure, because the $\tilde x$ dynamics depend on $\tilde\theta$. Therefore, in this example one cannot use the cascade systems theory of [20,21] or Corollary 22 of this paper.

6 Proofs of Theorems 14, 15, and 17

6.1 Proof of Theorem 14

Before proving Theorem 14, we need the following result on continuous dependence of solutions on initial data. It is not an original statement of this paper but, for completeness, we present the proof in Appendix B.

Lemma 26. *Assume the differential equation* (3) *satisfies the Basic Assumption. Then for each compact set $K \subset \mathbb{R}^n$, each $\varepsilon > 0$, and each $T > 0$, there exists $\delta > 0$ such that for any initial data $(t_0, x_0) \in \mathbb{R} \times K$ such that*

$x([t_0, t_0+T], t_0, x_0) \subset K$, *the property* $\|x(t, t_0, x_0) - x(t, t_0, x_1)\| < \varepsilon$ *holds for all* $x_1 \in B_\delta(x_0)$ *and for all* $t \in [t_0, t_0+T]$.

The proof of Theorem 14 follows similar lines of an analogous statement for time-invariant systems in [27], and is divided into two parts. First, we prove that uniform attractivity of Γ_1 relative to Γ_2 implies the so-called threshold property, in the terminology of [27]. Then, we prove that the threshold property and the assumption that Γ_2 is LUS-Γ_1 together imply that Γ_1 is US.

Lemma 27. *Consider the time-varying system* (3) *under the Basic Assumption. Let* Γ_1 *be a compact set and* Γ_2 *be a closed set, both positively invariant and such that* $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. *If* Γ_1 *is* UA *relative to* Γ_2, *then the threshold property holds:*

$$(\forall \varepsilon > 0)(\exists \delta, \eta > 0)(\forall x_0 \in B_\delta(\Gamma_1)(\forall t_0 \in \mathbb{R})(\forall t \geq t_0)$$
$$x([t_0, t], t_0, x_0) \subset B_\eta(\Gamma_2) \implies x([t_0, t], t_0, x_0) \subset B_\varepsilon(\Gamma_1). \tag{36}$$

PROOF. Suppose, by way of contradiction, that property (36) does not hold, so that

$$(\exists \varepsilon > 0)(\forall \delta, \eta > 0)(\exists x_0 \in B_\delta(\Gamma_1)(\exists t_0 \in \mathbb{R})(\exists t \geq t_0)$$
$$x([t_0, t], t_0, x_0) \subset B_\eta(\Gamma_2) \text{ and}$$
$$(\exists s \in [t_0, t]) x(s, t_0, x_0) \notin B_\varepsilon(\Gamma_1). \tag{37}$$

Fix $\varepsilon > 0$ as in (37), and pick sequences $\{\delta_n\}_{n\in\mathbb{N}} \subset (0, \varepsilon)$, $\{\eta_n\}_{n\in\mathbb{N}}$ such that $\delta_n, \eta_n \to 0$. Then by (37) there exist sequences $\{x_{0n}\}_{n\in\mathbb{N}}$, $\{t_{0n}\}_{n\in\mathbb{N}}$, and $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n > t_{0n}$ for all $n \in \mathbb{N}$, and the following properties hold:

$$\|x_{0n}\|_{\Gamma_1} \to 0 \text{ as } n \to \infty, \tag{38}$$
$$x([t_{0n}, t_n), t_{0n}, x_{0n}) \subset B_\varepsilon(\Gamma_1), \tag{39}$$
$$x(t_n, t_{0n}, x_{0n}) \notin B_\varepsilon(\Gamma_1), \tag{40}$$
$$d(x([t_{0n}, t_n], t_{0n}, x_{0n}), \Gamma_2) \to 0 \text{ as } n \to \infty. \tag{41}$$

Since Γ_1 is UA relative to Γ_2, we have that

$$(\exists \sigma > 0)(\exists T > 0)(\forall t_0 \in \mathbb{R})$$
$$x(\mathbb{R}_{\geq t_0+T}, B_\sigma(\Gamma_1) \cap \Gamma_2) \subset B_{\varepsilon/2}(\Gamma_1). \tag{42}$$

Let σ and T be as above. Since (37) continues to hold if ε is replaced by any smaller number, and since σ in (42) is independent of ε, we may assume that $\varepsilon < \sigma$. Let $\beta \in \mathbb{R}$ be such that

$$\beta > \sigma > \varepsilon. \tag{43}$$

By the Basic Assumption, and using Lemma 26 with $K := \overline{B_\beta(\Gamma_1)}$ (a compact set), we have:

$$(\forall T > 0)(\exists \delta' > 0)(\forall t_0 \in \mathbb{R})(\forall x_0 \in \overline{B_\beta(\Gamma_1)})$$
$$\text{if } x([t_0, t_0+T], t_0, x_0) \subset \overline{B_\beta(\Gamma_1)} \text{ then}$$
$$(\forall x_1 \in B_{\delta'}(x_0))(\forall t \in [t_0, t_0+T])$$
$$\|x(t, t_0, x_0) - x(t, t_0, x_1)\| < \varepsilon/2. \tag{44}$$

Let the UA property of Γ_1 generate, through (42), a constant $T > 0$ that, in turn, generates $\delta' > 0$ such that (44) holds. If necessary, reduce δ' such that

$$\delta' < \sigma - \varepsilon. \tag{45}$$

By (38), $\|x_{0n}\|_{\Gamma_1} \to 0$. Since Γ_1 is positively invariant, (40) and (44) imply that $t_n - t_{0n} \to \infty$ as $n \to \infty$, so we may assume that

$$(\forall n \in \mathbb{N})\ t_n - t_{0n} > T. \tag{46}$$

Define $x_n := x(t_n - T, t_{0n}, x_{0n})$, and note that, by (39) and (46), $x_n \in B_\varepsilon(\Gamma_1)$ for all $n \in \mathbb{N}$. By (41), $\|x_n\|_{\Gamma_2} \to 0$ as $n \to \infty$, so we may assume that $\|x_n\|_{\Gamma_2} < \delta'$ for all $n \in \mathbb{N}$. Then there exists a sequence $\{x'_n\}_{n\in\mathbb{N}} \subset \Gamma_2$ such that $\|x_n - x'_n\| < \delta'$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we have that $x'_n \in B_{\delta'}(x_n) \cap \Gamma_2$. By (45), $x'_n \in B_{\sigma-\varepsilon}(x_n) \cap \Gamma_2$, and since $x_n \in B_\varepsilon(\Gamma_1)$, we conclude that

$$x'_n \in B_\sigma(\Gamma_1) \cap \Gamma_2. \tag{47}$$

This fact and (42) imply that

$$(\forall n \in \mathbb{N})\ x(t_n, t_n - T, x'_n) \subset B_{\varepsilon/2}(\Gamma_1). \tag{48}$$

By the definition of x_n we have that

$$x([t_n - T, t_n], t_n - T, x_n) =$$
$$x([t_n - T, t_n], t_{0n}, x_{0n}) \underset{(46)}{\subset} x([t_{0n}, t_n], t_{0n}, x_{0n})$$
$$\underset{(39)}{\subset} \overline{B_\varepsilon(\Gamma_1)} \underset{(43)}{\subset} B_\beta(\Gamma_1).$$

Since $\|x_n - x'_n\| < \delta'$, using (44) we deduce that

$$\|x(t_n, t_n - T, x_n) - x(t_n, t_n - T, x'_n)\| < \varepsilon/2. \tag{49}$$

By (48) and (49) we conclude that $x(t_n, t_n - T, x_n) \in B_\varepsilon(\Gamma_1)$. However, in view of (40), $x(t_n, t_n - T, x_n) = x(t_n, t_{0n}, x_{0n}) \notin B_\varepsilon(\Gamma_1)$, which leads to a contradiction. □

Lemma 28. *Consider the time-varying system* (3) *under the Basic Assumption. Let* Γ_1 *be a compact set and* Γ_2 *be a closed set, both positively invariant and such that* $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. *If* Γ_2 *is* LUS-Γ_1 *and the threshold property* (36) *holds, then* Γ_1 *is* US.

PROOF. Suppose, by way of contradiction, that Γ_1 is not US, so that

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x_0 \in B_\delta(\Gamma_1))(\exists t_0 \in \mathbb{R})(\exists t \geq t_0)$$
$$x(t, t_0, x_0) \notin B_\varepsilon(\Gamma_1). \tag{50}$$

Let $\varepsilon > 0$ be as in (50), and consider a sequence $\{\delta_n\}_{n\in\mathbb{N}} \subset (0, \varepsilon)$ such that $\delta_n \to 0$ as $n \to \infty$. Then there exist sequences $\{x_{0n}\}_{n\in\mathbb{N}} \subset \mathbb{R}^n$, $\{t_{0n}\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $t_n > t_{0n}$ and

$$x([t_{0n}, t_n), t_{0n}, x_{0n}) \subset B_\varepsilon(\Gamma_1), \tag{51}$$
$$x(t_n, t_{0n}, x_{0n}) \notin B_\varepsilon(\Gamma_1), \tag{52}$$
$$\|x_{0n}\|_{\Gamma_1} \to 0 \text{ as } n \to \infty. \tag{53}$$

By the threshold property (36) we have

$$(\exists \delta', \eta > 0)(\forall x_0 \in B_{\delta'}(\Gamma_1)(\forall t_0 \in \mathbb{R})(\forall t \geq t_0)$$
$$\text{if } x([t_0, t], t_0, x_0) \subset B_\eta(\Gamma_2) \text{ then } x([t_0, t], t_0, x_0) \subset B_\varepsilon(\Gamma_1). \tag{54}$$

By (53) we may assume that $\{x_{0n}\}_{n\in\mathbb{N}} \subset B_{\delta'}(\Gamma_1)$. From the definition that Γ_2 is LUS-Γ_1 (Definition 10), with ε replaced by η, we have

$$(\exists r > 0)(\exists \delta > 0)(\forall t_0 \in \mathbb{R})(\forall x_0 \in B_\delta(\Gamma_1))(\forall t \in T^+_{t_0,x_0})$$
$$\text{if } x([t_0, t], t_0, x_0) \subset B_r(\Gamma_1) \text{ then } x([t_0, t], t_0, x_0) \subset B_\eta(\Gamma_2). \tag{55}$$

Fix r, δ as in (55), and note that (50) continues to hold if ε is replaced by a smaller number, so we may assume that $\varepsilon < r$. By (53), we may assume that $\|x_{0n}\|_{\Gamma_1} <$

$\min\{\delta, \delta'\}$ for all $n \in \mathbb{N}$. Thus we have that for all $n \in \mathbb{N}$, $x_{0n} \in B_\delta(\Gamma_1)$ and

$$x([t_{0n}, t_n], t_{0n}, x_{0n}) \underset{(51)}{\subset} \overline{B_\varepsilon(\Gamma_1)} \subset B_r(\Gamma_1).$$

By (55), the above implies that $x([t_{0n}, t_n], t_{0n}, x_{0n}) \subset B_\eta(\Gamma_2)$. By (54) and the fact that $x_{0n} \in B_{\delta'}(\Gamma_1)$, we deduce that $x([t_{0n}, t_n], t_{0n}, x_{0n}) \subset B_\varepsilon(\Gamma_1)$, which contradicts (52). □

PROOF OF THEOREM 14. By assumption (i), Γ_1 is UAS relative to Γ_2, and therefore UA relative to Γ_2. By Lemma 27, the threshold property (36) holds. By Lemma 28, the threshold property and assumption (ii) that Γ_2 is LUS-Γ_1 imply that Γ_1 is US. □

6.2 Proof of Theorem 15

Suppose assumptions (i)-(iii) in Theorem 15 hold and let $x_0 \in B_\delta(\Gamma_1)$ be arbitrarily fixed. We need to show that $x_0 \in \mathbb{B}(\Gamma_1)$, or

$$(\forall \varepsilon > 0)(\exists T > 0)(\forall t_0 \in \mathbb{R})\ x(\mathbb{R}_{\geq t_0+T}, t_0, x_0) \subset B_\varepsilon(\Gamma_1). \tag{56}$$

By assumption (ii), the basin of attraction $\mathbb{B}(\Gamma_2)$ contains a neighbourhood of Γ_1. Therefore, without loss of generality we may assume that δ in assumption (iii) is small enough so that

$$B_\delta(\Gamma_1) \subset \mathbb{B}(\Gamma_2). \tag{57}$$

By the definition of K_δ in the theorem statement, we have that for each $t_0 \in \mathbb{R}$, $x(\mathbb{R}_{\geq t_0}, t_0, x_0) \subset K_\delta$. Since, by assumption, K_δ is compact, for each $t_0 \in \mathbb{R}$, $T^+_{t_0,x_0} = \mathbb{R}_{\geq t_0}$.

Let $\varepsilon > 0$ be arbitrary, and pick $\varepsilon' \in (0, \varepsilon)$. By assumption (i), Γ_1 is UA relative to Γ_2, so by Lemma 27 the threshold property (36) holds, i.e.,

$$\begin{aligned}&(\exists \delta', \eta_1 > 0)(\forall x_0 \in B_{\delta'}(\Gamma_1))(\forall t_0 \in \mathbb{R})(\forall t \geq t_0)\\ &\text{ if } x([t_0, t], t_0, x_0) \subset B_{\eta_1}(\Gamma_2) \text{ then } x([t_0, t], t_0, x_0) \subset B_{\varepsilon'}(\Gamma_1).\end{aligned} \tag{58}$$

By assumption (i), Γ_1 is UAS relative to Γ_2, and by assumption (iii), $K_\delta \cap \Gamma_2 \subset \mathbb{B}(\Gamma_1)$. Since K_δ is compact, by Corollary 30 in Appendix A the set $K_\delta \cap \Gamma_2$ enjoys the uniform attraction property (A.5) in the Appendix, and thus

$$(\exists T_2 > 0)(\forall t_0 \in \mathbb{R})\ x(\mathbb{R}_{\geq t_0+T_2}, t_0, K_\delta \cap \Gamma_2) \subset B_{\delta'/2}(\Gamma_1). \tag{59}$$

By the Basic Assumption, and using Lemma 26 with K replaced by K_δ and $T_2 > 0$ given as in (59), we have:

$$\begin{aligned}&(\exists \eta_2 > 0)(\forall t_0 \in \mathbb{R})(\forall z_0 \in B_{\eta_2}(x_0))(\forall t \in [t_0, t_0 + T_2])\\ &\qquad \|x(t, t_0, x_0) - x(t, t_0, z_0)\| < \delta'/2.\end{aligned} \tag{60}$$

Let $\eta := \min\{\eta_1, \eta_2\}$. From (57) we get

$$(\exists T_1 > 0)(\forall t_0 \in \mathbb{R})\ x(\mathbb{R}_{\geq t_0+T_1}, t_0, x_0) \subset B_\eta(\Gamma_2). \tag{61}$$

Let $t_0 \in \mathbb{R}$ be arbitrary. By (61), and since $x(t_0 + T_1, t_0, x_0) \in K_\delta$, there exists $z_0 \in K_\delta \cap \Gamma_2$ such that $\|x(t_0 + T_1, t_0, x_0) - z_0\| < \eta$. By (60),

$$\begin{aligned}&\|x(t_0 + T_1 + T_2, t_0 + T_1, x(t_0 + T_1, t_0, x_0))\\ &\qquad - x(t_0 + T_1 + T_2, t_0 + T_1, z_0)\| < \delta'/2\end{aligned} \tag{62}$$

and, since $z_0 \in K_\delta \cap \Gamma_2$, by (59) it follows that

$$x(t_0 + T_1 + T_2, t_0 + T_1, z_0) \in B_{\delta'/2}(\Gamma_1). \tag{63}$$

Next, combining (62) and (63) we get

$$\begin{aligned}&x(t_0 + T_1 + T_2, t_0, x_0) =\\ &\quad x(t_0 + T_1 + T_2, t_0 + T_1, x(t_0 + T_1, t_0, x_0)) \in B_{\delta'}(\Gamma_1)\end{aligned} \tag{64}$$

and from (61) we have

$$\begin{aligned}&x(\mathbb{R}_{\geq t_0+T_1+T_2}, t_0, x_0) =\\ &\quad x(\mathbb{R}_{\geq t_0+T_1+T_2}, t_0 + T_1 + T_2, x(t_0 + T_1 + T_2, t_0, x_0))\\ &\qquad \subset B_\eta(\Gamma_2).\end{aligned} \tag{65}$$

By the threshold property in (58), (64) and (65) imply that

$$\begin{aligned}&x(\mathbb{R}_{\geq t_0+T_1+T_2}, t_0, x_0) =\\ &\quad x(\mathbb{R}_{\geq t_0+T_1+T_2}, t_0 + T_1 + T_2, x(t_0 + T_1 + T_2, t_0, x_0))\\ &\qquad \subset \overline{B_{\varepsilon'}(\Gamma_1)} \subset B_\varepsilon(\Gamma_1).\end{aligned} \tag{66}$$

Setting $T := T_1 + T_2$, (66) implies that property (56) holds. Hence, $B_\delta(\Gamma_1) \subset \mathbb{B}(\Gamma_1)$ so we conclude that Γ_1 is t_0-UA.

Now suppose that conditions (i)'-(ii)' hold and let $x_0 \in \mathbb{BS}$ be arbitrary, so that the set $K := \overline{\bigcup_{t_0 \in \mathbb{R}} x(\mathbb{R}_{\geq t_0}, t_0, x_0)}$ is compact. By (i)', $\Gamma_2 \subset \mathbb{B}(\Gamma_1)$, and therefore $K \cap \Gamma_2 \subset \mathbb{B}(\Gamma_1)$. Now repeating the proof above with K_δ replaced by K we reach the conclusion that (56) holds, thereby implying that $\mathbb{BS} \subset \mathbb{B}(\Gamma_1)$. This concludes the proof of Theorem 15. □

6.3 Proof of Theorem 17

($\Longrightarrow$) Suppose that Γ_1 is UAS. Since $\Gamma_1 \subset \Gamma_2$, Γ_1 is UAS relative to Γ_2, hence condition (i) holds. Since Γ_1 is UA, there exists $r > 0$ such that $B_r(\Gamma_1) \subset \mathbb{B}(\Gamma_1)$. Since $\Gamma_1 \subset \Gamma_2$, $\mathbb{B}(\Gamma_1) \subset \mathbb{B}(\Gamma_2)$, and thus $B_r(\Gamma_1) \subset \mathbb{B}(\Gamma_2)$, implying that condition (iii) holds. Since Γ_1 is US, we have

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall t_0 \in \mathbb{R})\ x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma_1)) \subset B_\varepsilon(\Gamma_1).$$

Hence condition (5) in the definition of LUS-Γ_1 holds for arbitrary $r > 0$, so condition (ii) holds.

Next, suppose Γ_1 is UGAS. Then it is UGAS relative to Γ_2, so condition (i)' holds. Since $\mathbb{B}(\Gamma_1) = \mathbb{R}^n$ and since $\mathbb{B}(\Gamma_1) \subset \mathbb{B}(\Gamma_2)$, Γ_2 is t_0-UGA and hence condition (iii)' holds. As for condition (iv), let $x_0 \in \mathbb{R}^n$ be arbitrary and define $\delta := 2\|x_0\|_{\Gamma_1}$, so $x_0 \in B_\delta(\Gamma_1)$. Since Γ_1 is UGS, there exists $\varepsilon > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma_1)) \subset B_\varepsilon(\Gamma_1)$ for all $t_0 \in \mathbb{R}$. Since Γ_1 is compact, there exists $c > 0$ such that $B_\varepsilon(\Gamma_1) \subset B_c(0)$. Thus for each $t_0 \in \mathbb{R}$, $x(\mathbb{R}_{\geq t_0}, t_0, x_0) \subset B_c(0)$, implying that $x_0 \in \mathbb{BS}$. Since x_0 is arbitrary, $\mathbb{BS} = \mathbb{R}^n$ and condition (iv) holds.

($\Longleftarrow$) Suppose conditions (i)-(iii) hold. By Theorem 14, conditions (i) and (ii) imply that Γ_1 is US. To prove that Γ_1 is UAS, in view of item (iv) of Proposition 9 it suffices to show that Γ_1 is t_0-UA. To this end, we invoke Theorem 15. Conditions (i) and (ii) of Theorem 15 correspond to conditions (i) and (iii) of Theorem 17, which hold by assumption. It is only left

to show that there exists $\delta > 0$ such that the set

$$K_\delta := \overline{\bigcup_{t_0 \in \mathbb{R}} x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma_1))}$$

is compact and $K_\delta \cap \Gamma_2 \subset \mathbb{B}(\Gamma_1)$. By assumption (i), there exists $\varepsilon > 0$ such that $\overline{B_\varepsilon(\Gamma_1)} \cap \Gamma_2 \subset \mathbb{B}(\Gamma_1)$. Since Γ_1 is US, there exists $\delta > 0$ such that

$$(\forall t_0 \in \mathbb{R})\ x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma_1)) \subset B_\varepsilon(\Gamma_1).$$

The above implies that for the value of δ just discussed, $K_\delta \subset \overline{B_\varepsilon(\Gamma_1)}$, and therefore

$$K_\delta \cap \Gamma_2 \subset \overline{B_\varepsilon(\Gamma_1)} \cap \Gamma_2 \subset \mathbb{B}(\Gamma_1).$$

Moreover, since Γ_1 is compact and $K_\delta \subset \overline{B_\varepsilon(\Gamma_1)}$, K_δ is compact too. Thus assumption (iii) of Theorem 15 holds, and Γ_1 is t_0-UA. By part (iv) of Proposition 9, Γ_1 is UAS.

Now suppose that assumptions (i)', (ii), (iii)', and (iv) hold. By Theorem 14, Γ_1 is US, and by Theorem 15 it is t_0-UGA. Part (v) of Proposition 9 implies that Γ_1 is UGAS.

Finally, suppose that assumptions (i)', (ii), and (iii)' hold. By the first part of Theorem 17, Γ_1 is UAS. By Theorem 15, assumptions (i)' and (iii)' imply that all initial states giving rise to t_0-uniformly bounded solutions are contained in the basin of t_0-uniform attraction of Γ, i.e., $\mathbb{BS} \subset \mathbb{B}(\Gamma_1)$. This concludes the proof of the theorem.

□

7 Conclusion

In this paper we presented reduction theorems for uniform stability, t_0-uniform attractivity, and uniform asymptotic stability of compact sets, as well as a number of consequences. We also presented Lyapunov characterizations of the properties of local uniform stability near a set and t_0-uniform attractivity. Further research on Lyapunov characterizations might provide useful extensions and new stability results. In an example we illustrated how in certain simple cases, reduction theorems can be used to assess the property of almost global t_0-uniform attractivity. The development of general reduction theorems for almost global uniform asymptotic stability remains an open problem.

A Proof of Proposition 9

Part (i). ($\Longrightarrow$) If Γ is US then for each $\varepsilon > 0$ there exists $\delta > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma)) \subset B_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$. Letting $U = B_\delta(\Gamma)$, the above property implies that Γ is t_0-US.

($\Longleftarrow$) If Γ is t_0-US then for each $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}^n$ such that $\Gamma \subset U$, and for each $x_0 \in U$, for each $t_0 \in \mathbb{R}$, and each $t \in T^+_{t_0,x_0}$, it holds that $x(t, t_0, x_0) \subset B_\varepsilon(\Gamma)$. Since Γ is compact, $B_\varepsilon(\Gamma)$ is bounded, and hence $T^+_{t_0,x_0} = \mathbb{R}_{\geq t_0}$ for all $(t_0, x_0) \in \mathbb{R} \times U$. By the compactness of Γ and the fact that $\Gamma \subset U$ with U open, there exists $\delta > 0$ such that $B_\delta(\Gamma) \subset U$. It then follows that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma)) \subset B_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$, proving that Γ is US.

Part (ii). ($\Longrightarrow$) By definition, if Γ is UAS then it is UA.

($\Longleftarrow$) Assume Γ is UA. We will show that Γ is US. Let $\varepsilon > 0$ be arbitrary. By definition of UA, we have

$$(\exists r > 0)(\forall \varepsilon > 0)(\exists T > 0)(\forall t_0 \in \mathbb{R}) \quad x(\mathbb{R}_{\geq t_0+T}, t_0, B_r(\Gamma)) \subset B_\varepsilon(\Gamma). \quad \text{(A.1)}$$

Let $\varepsilon > 0$ be arbitrary, and let $T > 0$ be such that (A.1) holds.

By the positive invariance of Γ, for all $x_0 \in \Gamma$ we have that $x([t_0, t_0+T], t_0, x_0) \subset \Gamma$. By the Basic Assumption and Lemma 26, and since Γ is compact, there exists $\delta_1 > 0$ such that $\|x(t, t_0, x_0) - x(t, t_0, x_1)\| < \varepsilon$ for all $x_0 \in \Gamma$, all $x_1 \in B_{\delta_1}(x_0)$, all $t_0 \in \mathbb{R}$, and all $t \in [t_0, t_0 + T]$. Therefore, $x([t_0, t_0+T], t_0, B_{\delta_1}(x_0)) \subset B_\varepsilon(\Gamma)$ for all $x_0 \in \Gamma$ and all $t_0 \in \mathbb{R}$. Since Γ is compact, there exists $\delta_2 > 0$ such that $B_{\delta_2}(\Gamma) \subset \bigcup_{x_0 \in \Gamma} B_{\delta_1}(x_0)$, using which we obtain that

$$x([t_0, t_0 + T], t_0, B_{\delta_2}(\Gamma)) \subset B_\varepsilon(\Gamma), \quad \text{(A.2)}$$

for all $t_0 \in \mathbb{R}$. Picking $\delta = \min\{r, \delta_2\}$, (A.1) and (A.2) imply that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma)) \subset B_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$, so that Γ is US. This proves that UA implies UAS.

Part (iii). ($\Longrightarrow$) If Γ is UGAS then by definition it is UGA and UGS. This latter property implies that all solutions are t_0-uniformly bounded, so that $\mathbb{BS} = \mathbb{R}^n$.

($\Longleftarrow$) Now suppose that Γ is UGA and $\mathbb{BS} = \mathbb{R}^n$. We need to show that Γ is UGS. We have already shown in part (ii) that Γ is US, so we need to show that for each $\delta > 0$ there exists $\varepsilon > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma)) \subset B_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$. Let $\delta > 0$ be arbitrary, and pick $\varepsilon_1 > 0$. Since Γ is UGA, there exists $T > 0$ such that

$$x(\mathbb{R}_{\geq t_0+T}, t_0, B_\delta(\Gamma)) \subset B_{\varepsilon_1}(\Gamma) \quad \text{(A.3)}$$

for all $t_0 \in \mathbb{R}$. Since $\mathbb{BS} = \mathbb{R}^n$, for each $x_0 \in \overline{B_\delta(\Gamma)}$ there exists a constant $c(x_0)$ such that $x(\mathbb{R}_{\geq t_0}, t_0, x_0) \subset B_{c(x_0)}(0)$ for all $t_0 \in \mathbb{R}$. By continuous dependence on initial data, there exists a constant $\mu(x_0) > 0$ such that $x([t_0, t_0 + T], t_0, B_{\mu(x_0)}(x_0)) \subset B_{2c(x_0)}(0)$ for all $t_0 \in \mathbb{R}$. The collection of open balls $\{B_{\mu(x_0)}(x_0) : x_0 \in \overline{B_\delta(\Gamma)}\}$ is an open cover of $\overline{B_\delta(\Gamma)}$, and since this latter set is compact, it has a finite subcover, so that there exists a finite collection of points $x_i \in \overline{B_\delta(\Gamma)}$, $i \in \mathbf{k}$, such that $B_\delta(\Gamma) \subset \bigcup_{i \in \mathbf{k}} B_{\mu(x_i)}(x_i)$. Let $M = \max_{i \in \mathbf{k}} 2c(x_i)$. Then for each $t_0 \in \mathbb{R}$, $x([t_0, t_0+T], t_0, B_\delta(\Gamma)) \subset B_M(0)$. Letting $\varepsilon_2 > 0$ be such that $B_M(0) \subset B_{\varepsilon_2}(\Gamma)$, we get

$$x([t_0, t_0 + T], t_0, B_\delta(\Gamma)) \subset B_{\varepsilon_2}(\Gamma) \quad \text{(A.4)}$$

for all $t_0 \in \mathbb{R}$. Setting $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$, by (A.3) and (A.4) we conclude that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma)) \subset B_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$. Thus Γ is UGS, and therefore also UGAS.

Part (iv). ($\Longrightarrow$) If Γ is UAS then by definition it is UA and US. The former property implies that Γ is t_0-UA and, by part (i), the latter property implies that Γ is t_0-US. Being t_0-US and t_0-US, Γ is t_0-UAS.

($\Longleftarrow$) Assume Γ is t_0-UAS. Since Γ is t_0-UA, $\Gamma \subset \text{int}(\mathbb{B}(\Gamma))$, where $\mathbb{B}(\Gamma)$ is the basin of t_0-uniform attraction of Γ. This fact and the assumption that Γ is compact imply that there exists $r > 0$ such that $B_r(\Gamma) \subset \text{int}(\mathbb{B}(\Gamma))$. The set $K = \overline{B_r(\Gamma)} \subset \mathbb{B}(\Gamma)$ is compact, and

by Lemma 29 below it enjoys the uniform attraction property (A.5):

$$(\forall \varepsilon > 0)(\exists T > 0)(\forall t_0 \in \mathbb{R})\, x(\mathbb{R}_{\geq t_0+T}, t_0, \overline{B_r(\Gamma)}) \subset B_\varepsilon(\Gamma).$$

The above property implies that Γ is UA.

Part(v). ($\Longrightarrow$) If Γ is UGAS, then by definition it is UGA and UGS. By part (i), we deduce that Γ is t_0-US. Moreover, the UGS property implies that all solutions are t_0-uniformly bounded, so that $\mathbb{BS} = \mathbb{R}^n$. The UGA property implies that Γ is t_0-UGA. In conclusion, Γ is t_0-UGAS and $\mathbb{BS} = \mathbb{R}^n$.

($\Longleftarrow$) Assume Γ is t_0-US and t_0-UGA, and $\mathbb{BS} = \mathbb{R}^n$. Since Γ is compact and t_0-UGA, we have $\mathbb{B}(\Gamma) = \mathbb{R}^n$, so we may repeat the argument in the proof of part (iv) with arbitrary $r > 0$ to conclude that Γ is UGA. Then, in light of part (iii), Γ is UGAS. This concludes the proof of Proposition 9. □

Lemma 29. *Consider the differential equation* (3)*, in which the vector field $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the Basic Assumption. Let $\Gamma \subset \mathbb{R}^n$ be a compact positively invariant set that is t_0-UAS. Then, for each compact set $K \subset \mathbb{B}(\Gamma)$ the following* uniform attraction property *holds:*

$$(\forall \varepsilon > 0)(\exists T > 0)(\forall t_0 \in \mathbb{R})\; x(\mathbb{R}_{\geq t_0+T}, t_0, K) \subset B_\varepsilon(\Gamma). \tag{A.5}$$

Proof. Let $\varepsilon > 0$ be arbitrarily fixed. By part (i) of Proposition 9, since Γ is t_0-US it is also US, and we have

$$(\exists \delta > 0)(\forall t_0 \in \mathbb{R})\; x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma)) \subset B_\varepsilon(\Gamma). \tag{A.6}$$

By t_0-uniform attractivity of Γ and the fact that $K \subset \mathbb{B}(\Gamma)$, we have

$$(\forall x_0 \in K)(\exists T > 0)(\forall t_0 \in \mathbb{R})$$
$$t_0 + T \in T^+_{t_0,x_0} \text{ and } x(t_0 + T, t_0, x_0) \subset B_{\delta/2}(\Gamma). \tag{A.7}$$

Let $x_0 \in K$ be arbitrary, and let $T > 0$ be as in (A.7). Using Lemma 26 with K in the lemma given by the set $\{x([t_0, t_0 + T], t_0, x_0)\}$, we have that

$$(\exists \delta' > 0)(\forall z_0 \in B_{\delta'}(x_0))(\forall t \in [t_0, t_0 + T])$$
$$\|x(t, t_0, x_0) - x(t, t_0, z_0)\| < \delta/2. \tag{A.8}$$

By (A.7) and (A.8) we have that $x(t_0+T, t_0, B_{\delta'}(x_0)) \subset B_\delta(\Gamma)$, and by (A.6) we conclude that

$$x(\mathbb{R}_{\geq t_0+T}, t_0, B_{\delta'}(x_0)) \subset B_\varepsilon(\Gamma). \tag{A.9}$$

By property (A.9) and the fact that the set K is compact, there exists a finite cover of K by balls $B_{\delta_i}(x_i)$, $i \in \mathbf{k}$, where $x_i \in K$, and associated times $T_i > 0$, $i \in \mathbf{k}$, such that

$$(\forall t_0 \in \mathbb{R})\; x(\mathbb{R}_{\geq t_0+T_i}, t_0, B_{\delta_i}(x_i)) \subset B_\varepsilon(\Gamma). \tag{A.10}$$

Letting $T := \max\{T_1, \ldots, T_n\}$, we conclude that

$$(\forall t_0 \in \mathbb{R})\; x(\mathbb{R}_{\geq t_0+T}, t_0, K) \subset B_\varepsilon(\Gamma),$$

proving that (A.5) holds. □

Corollary 30. *In the setup of Lemma 29, let Γ_1 be a compact set and Γ_2 be a closed set, both positively invariant and such that $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. If Γ_1 is* UAS *relative to Γ_2, then for each compact set $K \subset \mathbb{B}(\Gamma_1) \cap \Gamma_2$ the uniform attraction property* (A.5) *holds.*

The proof of this corollary follows by repeating the argument of the proof of Lemma 29, replacing $B_\delta(\Gamma)$ in (A.6) by $B_\delta(\Gamma) \cap \Gamma_2$, and making analogous changes in (A.9) and (A.10).

B Proof of Lemma 26

Fix arbitrary $\varepsilon > 0$, $T > 0$, and a compact set $K \subset \mathbb{R}^n$. Define $K_\varepsilon := \overline{B_\varepsilon(K)}$. Let $L_\varepsilon > 0$ be the Lipschitz constant of f on K_ε in the Basic Assumption. Pick $\delta \in (0, \varepsilon \exp(-L_\varepsilon T))$, and let $(t_0, x_0) \in \mathbb{R} \times K$ be such that $x([t_0, t_0 + T], t_0, x_0) \subset K$. For any $x_1 \in B_\delta(x_0)$, we have that $x_1 \in \operatorname{int} K_\varepsilon$ because $x_0 \in K$ and $\|x_1 - x_0\| < \delta < \varepsilon$. We claim that $x([t_0, t_0 + T], t_0, x_1) \subset \operatorname{int} K_\varepsilon$. By way of contradiction, suppose there exists a time $t_1 \in (t_0, t_0+T]$ such that $x([t_0, t_1), t_0, x_1) \subset \operatorname{int}(K_\varepsilon)$ and $x(t_1, t_0, x_1) \in \partial K_\varepsilon$. Then for each $t \in [t_0, t_1]$ we have

$$\|x(t, t_0, x_0) - x(t, t_0, x_1)\| =$$
$$\left\| x_0 - x_1 + \int_{t_0}^{t} f(s, x(s, t_0, x_0)) - f(s, x(s, t_0, x_1)) ds \right\|$$
$$\leq \delta + L_\varepsilon \int_{t_0}^{t} \|x(s, t_0, x_0) - x(s, t_0, x_1)\| ds.$$

By the Bellman-Gronwall Lemma we deduce that for all $t \in [t_0, t_1]$,

$$\|x(t, t_0, x_0) - x(t, t_0, x_1)\| \leq \delta \exp(L_\varepsilon(t - t_0))$$
$$\leq \delta \exp(L_\varepsilon T) < \varepsilon. \tag{B.1}$$

Then, since $x(t_1, t_0, x_0) \in K$, $x(t_1, t_0, x_1) \in \operatorname{int}(K_\varepsilon)$ contradicting the assumption that $x(t_1, t_0, x_1) \in \partial K_\varepsilon$. Thus we have proved that $x([t_0, t_0 + T], t_0, x_1) \subset \operatorname{int} K_\varepsilon$. This implies that inequality (B.1) holds for all $t \in [t_0, t_0 + T]$, and therefore $\|x(t, t_0, x_0) - x(t, t_0, x_1)\| < \varepsilon$ for all $x_1 \in B_\delta(x_0)$ and all $t \in [t_0, t_0 + T]$. This concludes the proof of the lemma. □

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