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On the general conditions of existence for linear MMSE filters: Wiener and Kalman

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ABSTRACT

State estimation is a fundamental task in a plethora of applications. Two important classes of linear minimum mean square error (MMSE) estimators are Wiener filters (WFs) and Kalman filters (KFs), the latter being the recursive form of the former for linear discrete state-space (LDSS) dynamic models. The minimum set of uncorrelation conditions regarding states and measurements which allows to formulate a general KF form has been recently introduced in the literature, but for both WF and KFs a standard key assumption is that the measurement noise covariance is invertible. In this contribution we provide i) an alternative derivation of the WF in the case of a non-invertible measurement covariance matrix, and ii) the proof that under the minimum set of uncorrelation conditions lately released, the KF can always be formulated irrespective of the measurement covariance matrix rank. Notice that these results also apply for Kalman predictors and smoothers, linearly constrained WFs and minimum variance distortionless response (MVDR) filters, therefore being a remarkable result of broad interest.

1. Introduction

State estimation is a fundamental task in a plethora of applications, i.e., robotics, tracking, guidance and navigation systems, to name a few [1–4]. Considering two complex random vectors \mathbf{x} and \mathbf{y} , the linear estimator of \mathbf{x} from noisy observations \mathbf{y} which minimizes the mean square error (MSE) is the so-called (linear or affine) Wiener filter (WF) [2]. The main assumption to derive the WF is that the covariance matrix of \mathbf{y} , \mathbf{C}_y , is invertible [1–3,5,6], which may not be the case in practice. Notice that the existence of the WF with a singular measurement noise covariance is mentioned in Kailath et al. [7, Theorem 3.2.3], but neither the solution nor the proof are provided, as well as in Anderson and Moore [8, Theorem 2.1], where a solution is given but no proof is provided.

In the case of linear discrete state-space (LDSS) models where we have a dynamic stochastic representation of both state and measurements, state estimation refers to the estimation of \mathbf{x}_k (at discrete time k) based on measurements up to discrete time l , typically denoted as $\hat{\mathbf{x}}_{k|l} \triangleq \hat{\mathbf{x}}_{k|l}(\mathbf{y}_1, \dots, \mathbf{y}_l) = \hat{\mathbf{x}}_{k|l}(\bar{\mathbf{y}}_l)$, with $\bar{\mathbf{y}}_l^T = (\mathbf{y}_1^T, \dots, \mathbf{y}_l^T)$. The term estimator includes filters ($l = k$), predictors ($l < k$) and smoothers ($l > k$). Indeed, the well-known Kalman filter (KF) is the recursive form of the WF for LDSS models [9].

Again, a key assumption to derive the KF as a recursive WF is that the covariance matrix $\mathbf{C}_{\bar{\mathbf{y}}_k}$ is invertible [1–3,5,6,10]. Notice that [3, p.214] states that “a singular covariance matrix does not present any theoretical problem”, but neither the solution nor the proof are provided. In addition, the following uncorrelation conditions for the LDSS model,

$$\begin{aligned} \mathbf{C}_{\mathbf{x}_0, \mathbf{w}_k} &= \mathbf{0}, \quad \mathbf{C}_{\mathbf{x}_0, \mathbf{v}_k} = \mathbf{0}, \quad \mathbf{C}_{\mathbf{w}_l, \mathbf{w}_k} = \mathbf{C}_{\mathbf{w}_k} \delta_k^l, \\ \mathbf{C}_{\mathbf{v}_l, \mathbf{v}_k} &= \mathbf{C}_{\mathbf{v}_k} \delta_k^l, \quad \mathbf{C}_{\mathbf{w}_l, \mathbf{v}_k} = \mathbf{C}_{\mathbf{w}_{k-1}, \mathbf{v}_k} \delta_k^{l+1}, \end{aligned} \quad (1)$$

with $\{\mathbf{w}_k\}$ and $\{\mathbf{v}_k\}$ the state and observation noise sequences, are typically assumed for the KF derivation in monographs [7, §9.1] [3, §7.1]. These conditions have long been regarded as leading to the general form of the KF including correlated process and measurement noise. This set of minimum uncorrelation conditions has been recently generalized in Chaumette et al. [9,11], leading to

$$\mathbf{C}_{\mathbf{w}_{k-1}, \bar{\mathbf{y}}_{k-1}} = \mathbf{0}, \quad \mathbf{C}_{\mathbf{v}_k, \bar{\mathbf{y}}_{k-1}} = \mathbf{0}, \quad \forall k \geq 2, \quad (2)$$

but such derivation also assumes that $\mathbf{C}_{\bar{\mathbf{y}}_k}$ is invertible.

In general, for the linear minimum MSE (MMSE) estimation problem, considering LDSS dynamic systems, a fundamental question is if the KF exists in the case where the measurement noise covariance matrix is non-invertible. This is the main concern of this article, where we generalize the results in Chaumette et al. [9] and Chaumette et al. [11] as follows:

- The first contribution is an alternative straightforward derivation of the general WF form, which can always be computed

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irrespective of the measurement covariance \mathbf{C}_y rank by resorting to the Moore–Penrose inverse.

- The second contribution is the application of the general WF result to the filtering problem in LDSS models, i.e., the KF. This allows to show that considering the general KF conditions of existence [9,11], the general KF solution always exist and can be formulated irrespective of the measurement covariance rank.

Notation:

Scalars, vectors and matrices are represented, respectively, by italic, bold lowercase and bold uppercase characters. $\mathcal{M}_{\mathbb{C}}(N, P)$ denotes the vector space of complex matrices with N rows and P columns. The transpose and transpose conjugate operators are indicated by $(\cdot)^T$ and $(\cdot)^H$. The identity matrix of size M is denoted by \mathbf{I}_M . $[\mathbf{A} \ \mathbf{B}]$ and $\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}$ denote the matrix resulting from the horizontal and the vertical concatenation of \mathbf{A} and \mathbf{B} , respectively. The vector resulting from the vertical concatenation of k vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$ is denoted as $\bar{\mathbf{a}}_k$. The matrix resulting from the vertical concatenation of k matrices $\mathbf{A}_1, \dots, \mathbf{A}_k$ of same column dimension is denoted as $\bar{\mathbf{A}}_k$. \mathbf{x} and \mathbf{y} denote P -dimensional and N -dimensional complex random state and observation vectors, belonging to the state space $\mathcal{X} \subset \mathbb{C}^P$ and observation space $\mathcal{Y} \subset \mathbb{C}^N$. $\mathbf{m}_x \triangleq E_{y,x}[\mathbf{x}] = E_x[\mathbf{x}]$ is the expectation of \mathbf{x} . If \mathbf{x} and \mathbf{y} are two complex random vectors: a) \mathbf{C}_x , \mathbf{C}_y and $\mathbf{C}_{x,y}$ are the covariance matrices of \mathbf{x} , \mathbf{y} , and their cross-covariance; b) If \mathbf{C}_y is invertible, then $\mathbf{C}_{x|y} \triangleq \mathbf{C}_x - \mathbf{C}_{x,y}\mathbf{C}_y^{-1}\mathbf{C}_{x,y}^H$. For the sake of legibility, $E_{y,x}[(f(\mathbf{y}, \mathbf{x}))(\cdot)^H] \triangleq E_{y,x}[(f(\mathbf{y}, \mathbf{x}))(f(\mathbf{y}, \mathbf{x}))^H]$.

2. Affine/linear MMSE estimator

The estimation error covariance matrix between a signal \mathbf{x} and the corresponding affine estimator, $\hat{\mathbf{x}} \triangleq \mathbf{K}\mathbf{y} + \mathbf{a}$, is

$$\mathbf{P}(\mathbf{K}, \mathbf{a}) = E_{y,x}[(\mathbf{K}\mathbf{y} + \mathbf{a} - \mathbf{x})(\cdot)^H]. \quad (3a)$$

We look for the affine linear minimum MSE (AMMSE) estimator of $\hat{\mathbf{x}}^1$:

$$\hat{\mathbf{x}}^b = \mathbf{K}^b\mathbf{y} + \mathbf{a}^b \mid (\mathbf{K}^b, \mathbf{a}^b) = \arg \min_{(\mathbf{K}, \mathbf{a})} \{\mathbf{P}(\mathbf{K}, \mathbf{a})\}. \quad (3b)$$

Since we can write that $\mathbf{K}\mathbf{y} + \mathbf{a} - \mathbf{x} = \mathbf{K}(\mathbf{y} - \mathbf{m}_y) - (\mathbf{x} - \mathbf{m}_x) + (\mathbf{a} - (\mathbf{m}_x - \mathbf{K}\mathbf{m}_y))$,

$$\begin{aligned} \mathbf{P}(\mathbf{K}, \mathbf{a}) &= E_{y,x}[(\mathbf{K}\mathbf{y} + \mathbf{a} - \mathbf{x})(\cdot)^H] = \mathbf{C}_x + \mathbf{K}\mathbf{C}_y\mathbf{K}^H - \mathbf{C}_{x,y}\mathbf{K}^H - \mathbf{K}\mathbf{C}_{x,y}^H \\ &+ (\mathbf{a} - (\mathbf{m}_x - \mathbf{K}\mathbf{m}_y))(\mathbf{a} - (\mathbf{m}_x - \mathbf{K}\mathbf{m}_y))^H. \end{aligned} \quad (4a)$$

This quadratic form in \mathbf{a} is positive semidefinite, so $\mathbf{P}(\mathbf{K}, \mathbf{a}) \geq \mathbf{C}_x + \mathbf{K}\mathbf{C}_y\mathbf{K}^H - \mathbf{C}_{x,y}\mathbf{K}^H - \mathbf{K}\mathbf{C}_{x,y}^H$ with equality for $\mathbf{a}_K^b = \arg \min_{\mathbf{a}} \{\mathbf{P}(\mathbf{K}, \mathbf{a})\} = \mathbf{m}_x - \mathbf{K}\mathbf{m}_y$, which leads to

$$\begin{aligned} \mathbf{P}(\mathbf{K}) &\triangleq \mathbf{P}(\mathbf{K}, \mathbf{a}_K^b) = E_{y,x}[(\mathbf{K}(\mathbf{y} - \mathbf{m}_y) - (\mathbf{x} - \mathbf{m}_x))(\cdot)^H] \\ &= \mathbf{C}_x + \mathbf{K}\mathbf{C}_y\mathbf{K}^H - \mathbf{C}_{x,y}\mathbf{K}^H - \mathbf{K}\mathbf{C}_{x,y}^H, \end{aligned} \quad (4b)$$

and therefore

$$\mathbf{K}^b = \arg \min_{\mathbf{K}} \{\mathbf{P}(\mathbf{K})\} = \arg \min_{\mathbf{K}} \{\mathbf{P}(\mathbf{K}, \mathbf{a}_K^b)\}, \quad (4c)$$

$$\hat{\mathbf{x}}^b - \mathbf{m}_x = \mathbf{K}^b(\mathbf{y} - \mathbf{m}_y). \quad (4d)$$

As \mathbf{K}^b (4c) is also the solution of the linear MMSE (LMMSE) estimator after centering \mathbf{x} and \mathbf{y} (4b), i.e., $\hat{\mathbf{x}} - \mathbf{m}_x = \mathbf{K}(\mathbf{y} - \mathbf{m}_y)$, we follow [7] hereinafter and only use the term linear to designate both linear and affine MMSE estimators. In the sequel two cases are considered: (i) \mathbf{C}_y has full rank, and (ii) \mathbf{C}_y has not full rank.

2.1. \mathbf{C}_y full rank

If \mathbf{C}_y has full rank, then it is an invertible positive definite matrix, and (4b) can be recast as Schreier and Scharf [5, p.121]

$$\mathbf{P}(\mathbf{K}) \triangleq \mathbf{P}(\mathbf{K}, \mathbf{a}_K^b) = \mathbf{C}_{x|y} + (\mathbf{K} - \mathbf{C}_{x,y}\mathbf{C}_y^{-1})\mathbf{C}_y(\mathbf{K} - \mathbf{C}_{x,y}\mathbf{C}_y^{-1})^H, \quad (5a)$$

which leads to the standard WF solution

$$\mathbf{K}^b = \arg \min_{\mathbf{K}} \{\mathbf{P}(\mathbf{K})\} = \mathbf{C}_{x,y}\mathbf{C}_y^{-1}; \quad \mathbf{P}(\mathbf{K}^b) \triangleq \mathbf{P}(\mathbf{K}^b, \mathbf{a}_{K^b}^b) = \mathbf{C}_{x|y}, \quad (5b)$$

$$\mathbf{a}_{K^b}^b = \mathbf{m}_x - \mathbf{K}^b\mathbf{m}_y; \quad \hat{\mathbf{x}}^b - \mathbf{m}_x = \mathbf{C}_{x,y}\mathbf{C}_y^{-1}(\mathbf{y} - \mathbf{m}_y). \quad (5c)$$

2.2. \mathbf{C}_y not full rank

If \mathbf{C}_y has not full rank, then $\dim\{\text{span}(\mathbf{C}_y)\} = Q < N$. Since it is an endomorphism whose matrix is hermitian positive, it admits an eigenvalue decomposition (EVD):

$$\mathbf{C}_y = [\mathbf{V} \ \mathbf{V}]\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}[\mathbf{V} \ \mathbf{V}]^H = \mathbf{V}\mathbf{D}\mathbf{V}^H, \quad (6)$$

$$\begin{cases} \mathbf{V} \text{ is a basis of } \ker(\mathbf{C}_y) \in \mathbb{C}^{N \times (N-Q)} \\ \mathbf{V} \text{ is a basis of } \text{span}(\mathbf{C}_y) \in \mathbb{C}^{N \times Q} \\ \mathbf{D} \in \mathbb{C}^{Q \times Q} \text{ is a diagonal positive definite matrix} \\ \mathbf{V}^H\mathbf{V} = \mathbf{I}_Q, \quad \mathbf{V}\mathbf{V}^H + \mathbf{V}\mathbf{V}^H = \mathbf{I} \\ \mathbf{V}^H\mathbf{C}_y\mathbf{V} = \mathbf{C}_{V^Hy} = \mathbf{D}, \quad \mathbf{V}^H\mathbf{C}_y\mathbf{V} = \mathbf{C}_{V^Hy} = \mathbf{0} \end{cases}$$

then in this case (4b) can be recast as

$$\begin{aligned} \mathbf{P}(\mathbf{K}) &= \mathbf{C}_x + \mathbf{K}\mathbf{V}\mathbf{D}\mathbf{V}^H\mathbf{K}^H - \mathbf{C}_{x,(\mathbf{V}\mathbf{V}^H + \mathbf{V}\mathbf{V}^H)}\mathbf{K}^H - \mathbf{K}\mathbf{C}_{x,(\mathbf{V}\mathbf{V}^H + \mathbf{V}\mathbf{V}^H)}^H \\ &= (\mathbf{C}_x + (\mathbf{K}\mathbf{V})\mathbf{D}(\mathbf{K}\mathbf{V})^H - \mathbf{C}_{x,\mathbf{V}^Hy}(\mathbf{K}\mathbf{V})^H - (\mathbf{K}\mathbf{V})\mathbf{C}_{x,\mathbf{V}^Hy}^H) \\ &\quad - (\mathbf{C}_{x,\mathbf{V}^Hy}(\mathbf{K}\mathbf{V})^H + (\mathbf{K}\mathbf{V})\mathbf{C}_{x,\mathbf{V}^Hy}^H). \end{aligned}$$

However $\forall \lambda \in \mathbb{C}^{P \times 1}$, $\forall \mu \in \mathbb{C}^{(N-Q) \times 1}$

$$\begin{aligned} 0 &\leq |\lambda^H \mathbf{C}_{x,\mathbf{V}^Hy} \mu|^2 = |C_{\lambda^H x, \mu^H \mathbf{V}^Hy}|^2 \leq C_{\lambda^H x} C_{\mu^H \mathbf{V}^Hy} \\ &= (\lambda^H \mathbf{C}_x \lambda)(\mu^H (\mathbf{V}^H \mathbf{C}_y \mathbf{V}) \mu) = 0 \Rightarrow \mathbf{C}_{x,\mathbf{V}^Hy} = \mathbf{0}, \end{aligned}$$

leading to

$$\begin{aligned} \mathbf{P}(\mathbf{K}) &= \mathbf{C}_x + (\mathbf{K}\mathbf{V})\mathbf{C}_{V^Hy}(\mathbf{K}\mathbf{V})^H - \mathbf{C}_{x,\mathbf{V}^Hy}(\mathbf{K}\mathbf{V})^H - (\mathbf{K}\mathbf{V})\mathbf{C}_{x,\mathbf{V}^Hy}^H \\ &= E_{y,x}[(\mathbf{K}\mathbf{V})(\mathbf{V}^H(\mathbf{y} - \mathbf{m}_y)) - (\mathbf{x} - \mathbf{m}_x)(\cdot)^H], \\ &= \mathbf{C}_{x|V^Hy} + (\mathbf{K}\mathbf{V} - \mathbf{C}_{x,\mathbf{V}^Hy}\mathbf{C}_{V^Hy}^{-1})\mathbf{C}_{V^Hy}(\mathbf{K}\mathbf{V} - \mathbf{C}_{x,\mathbf{V}^Hy}\mathbf{C}_{V^Hy}^{-1})^H \\ &= \mathbf{C}_x - \mathbf{C}_{x,y}\mathbf{C}_y^{\dagger}\mathbf{C}_{x,y} + (\mathbf{K} - \mathbf{C}_{x,y}\mathbf{C}_y^{\dagger})\mathbf{C}_y(\mathbf{K} - \mathbf{C}_{x,y}\mathbf{C}_y^{\dagger})^H \end{aligned} \quad (7a)$$

where \mathbf{C}_y^{\dagger} is the Moore–Penrose inverse of \mathbf{C}_y , i.e., considering $\mathbf{C}_y = \mathbf{V}\mathbf{D}\mathbf{V}^H = \mathbf{EVD}(\mathbf{C}_y)$, then

$$\text{if } \mathbf{C}_y \text{ is full rank : } \mathbf{C}_y^{\dagger} = \mathbf{C}_y^{-1}, \quad (7b)$$

$$\text{if } \mathbf{C}_y \text{ is not full rank : } \mathbf{C}_y^{\dagger} = \mathbf{V}\mathbf{D}^{-1}\mathbf{V}^H. \quad (7c)$$

Therefore

$$\begin{aligned} \underbrace{\mathbf{K}'}_{\mathbb{C}^{P \times N}} &= \underbrace{\mathbf{C}_{x,\mathbf{V}^Hy}}_{\mathbb{C}^{P \times (N-Q)}} \underbrace{\mathbf{C}_{V^Hy}^{-1}}_{\mathbb{C}^{(N-Q) \times N}} \underbrace{\mathbf{V}^H}_{\mathbb{C}^{N \times Q}} \Rightarrow \underbrace{\mathbf{K}'\mathbf{V}}_{\mathbb{C}^{P \times Q}} = \mathbf{C}_{x,\mathbf{V}^Hy} \mathbf{C}_{V^Hy}^{-1} \\ &\Rightarrow \mathbf{P}(\mathbf{K}') = \mathbf{C}_{x|V^Hy} = \min_{\mathbf{K}} \{\mathbf{P}(\mathbf{K})\}, \end{aligned} \quad (7d)$$

which means that there are infinite solutions when the observation space is not restricted ($\mathbf{y} \in \mathbb{C}^N$)². Since all these solutions lead

¹ The superscript ^b is used to remind that the value under consideration is the “best” one according to the MSE.

² Note that $E_{y,x}[\mathbf{V}^H(\mathbf{y} - \mathbf{m}_y)] = \mathbf{0} \Rightarrow \mathbf{V}^H(\mathbf{y} - \mathbf{m}_y) = \mathbf{0}, \forall \mathbf{y}$, i.e., $E_{y,x}[\mathbf{V}^H(\mathbf{y} - \mathbf{m}_y)] = \mathbf{0} \Rightarrow (\mathbf{y} - \mathbf{m}_y) \in \text{span}(\mathbf{V}), \forall \mathbf{y}$.

to the same minimum MSE matrix, one can choose a best \mathbf{K} , for instance the one with the minimum Frobenius norm,

$$\begin{cases} \mathbf{K}^b = \mathbf{C}_{\mathbf{x}, \mathbf{y}^H} \mathbf{C}_{\mathbf{y}^H}^{-1} \mathbf{V}^H \\ \mathbf{P}(\mathbf{K}^b) \triangleq \mathbf{P}(\mathbf{K}^b, \mathbf{a}_{\mathbf{K}^b}^b) = \mathbf{C}_{\mathbf{x}|\mathbf{y}^H} \end{cases} \quad (8a)$$

$$\begin{cases} \mathbf{K}^b = \mathbf{C}_{\mathbf{x}, \mathbf{y}} \mathbf{C}_{\mathbf{y}}^\dagger \\ \mathbf{P}(\mathbf{K}^b) \triangleq \mathbf{P}(\mathbf{K}^b, \mathbf{a}_{\mathbf{K}^b}^b) = \mathbf{C}_{\mathbf{x}} - \mathbf{C}_{\mathbf{x}, \mathbf{y}} \mathbf{C}_{\mathbf{y}}^\dagger \mathbf{C}_{\mathbf{x}, \mathbf{y}}^H \end{cases} \quad (8b)$$

and $\hat{\mathbf{x}}^b - \mathbf{m}_{\mathbf{x}} = \mathbf{C}_{\mathbf{x}, \mathbf{y}} \mathbf{C}_{\mathbf{y}}^\dagger (\mathbf{y} - \mathbf{m}_{\mathbf{y}})$. Another solution is to reformulate (3a) and (3b)

$$\mathbf{P}(\mathbf{K}, \mathbf{a}) = E_{\mathbf{y}, \mathbf{x}}[(\mathbf{K}(\mathbf{V}^H \mathbf{y}) + \mathbf{a} - \mathbf{x})(\cdot)^H], \quad (9a)$$

and to look for the AMMSE estimator of $\hat{\mathbf{x}}$ based on $\mathbf{V}^H \mathbf{y}$ (instead of \mathbf{y}),

$$\hat{\mathbf{x}}^b = \mathbf{K}^b(\mathbf{V}^H \mathbf{y}) + \mathbf{a}^b \mid (\mathbf{K}^b, \mathbf{a}^b) = \arg \min_{(\mathbf{K}, \mathbf{a})} \{\mathbf{P}(\mathbf{K}, \mathbf{a})\},$$

where $\mathbf{C}_{\mathbf{V}^H \mathbf{y}}$ is full rank. Then, according to (5a)–(5c),

$$\begin{aligned} \mathbf{K}^b &= \mathbf{C}_{\mathbf{x}, \mathbf{V}^H \mathbf{y}} \mathbf{C}_{\mathbf{V}^H \mathbf{y}}^{-1}; \quad \mathbf{P}(\mathbf{K}^b) \triangleq \mathbf{P}(\mathbf{K}^b, \mathbf{a}_{\mathbf{K}^b}^b) = \mathbf{C}_{\mathbf{x}|\mathbf{V}^H \mathbf{y}}, \\ \hat{\mathbf{x}}^b - \mathbf{m}_{\mathbf{x}} &= \mathbf{C}_{\mathbf{x}, \mathbf{V}^H \mathbf{y}} \mathbf{C}_{\mathbf{V}^H \mathbf{y}}^{-1} (\mathbf{V}^H (\mathbf{y} - \mathbf{m}_{\mathbf{y}})), \end{aligned} \quad (9b)$$

with $\mathbf{a}_{\mathbf{K}^b}^b = \mathbf{m}_{\mathbf{x}} - \mathbf{K}^b(\mathbf{V}^H \mathbf{m}_{\mathbf{y}})$.

2.3. General case

Finally, in any case, the solution of (3b)

$$\hat{\mathbf{x}}^b = \mathbf{K}^b \mathbf{y} + \mathbf{a}^b \mid (\mathbf{K}^b, \mathbf{a}^b) = \arg \min_{(\mathbf{K}, \mathbf{a})} \{\mathbf{P}(\mathbf{K}, \mathbf{a})\},$$

$$\begin{aligned} \mathbf{P}(\mathbf{K}, \mathbf{a}) &= E_{\mathbf{y}, \mathbf{x}}[(\mathbf{K} \mathbf{y} + \mathbf{a} - \mathbf{x})(\cdot)^H] = \mathbf{C}_{\mathbf{x}} - \mathbf{C}_{\mathbf{x}, \mathbf{y}} \mathbf{C}_{\mathbf{y}}^\dagger \mathbf{C}_{\mathbf{x}, \mathbf{y}}^H \\ &+ (\mathbf{K} - \mathbf{C}_{\mathbf{x}, \mathbf{y}} \mathbf{C}_{\mathbf{y}}^\dagger) \mathbf{C}_{\mathbf{y}} (\mathbf{K} - \mathbf{C}_{\mathbf{x}, \mathbf{y}} \mathbf{C}_{\mathbf{y}}^\dagger)^H \\ &+ (\mathbf{a} - (\mathbf{m}_{\mathbf{x}} - \mathbf{K} \mathbf{m}_{\mathbf{y}}))(\mathbf{a} - (\mathbf{m}_{\mathbf{x}} - \mathbf{K} \mathbf{m}_{\mathbf{y}}))^H, \end{aligned} \quad (10a)$$

is

$$\mathbf{K}^b = \mathbf{C}_{\mathbf{x}, \mathbf{y}} \mathbf{C}_{\mathbf{y}}^\dagger, \quad \mathbf{a}_{\mathbf{K}^b}^b = \mathbf{m}_{\mathbf{x}} - \mathbf{K}^b \mathbf{m}_{\mathbf{y}}, \quad (10b)$$

$$\mathbf{P}(\mathbf{K}^b) \triangleq \mathbf{P}(\mathbf{K}^b, \mathbf{a}_{\mathbf{K}^b}^b) = \mathbf{C}_{\mathbf{x}} - \mathbf{C}_{\mathbf{x}, \mathbf{y}} \mathbf{C}_{\mathbf{y}}^\dagger \mathbf{C}_{\mathbf{x}, \mathbf{y}}^H, \quad (10c)$$

$$\hat{\mathbf{x}}^b - \mathbf{m}_{\mathbf{x}} = \mathbf{C}_{\mathbf{x}, \mathbf{y}} \mathbf{C}_{\mathbf{y}}^\dagger (\mathbf{y} - \mathbf{m}_{\mathbf{y}}). \quad (10d)$$

Remark 1. The result summarized in (10b)–(10d) is the general WF form, which can always be computed irrespective of the rank of $\mathbf{C}_{\mathbf{y}}$ by resorting to the Moore–Penrose inverse, and provides the general affine/linear MMSE estimator.

3. LMMSE filter for LDSS models

3.1. LDSS models

We consider the general class of LDSS models represented with the pair state and measurement equation,

$$\mathbf{x}_k = \mathbf{F}_{k-1} \mathbf{x}_{k-1} + \mathbf{w}_{k-1}, \quad (11a)$$

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k, \quad (11b)$$

where the time index $k \geq 1$, $\mathbf{x}_k \in \mathbb{C}^{P_k}$ is the state vector, $\mathbf{y}_k \in \mathbb{C}^{N_k}$ is the measurement vector and the model matrices \mathbf{F}_k and \mathbf{H}_k are known. Unless otherwise stated, the process noise sequence $\{\mathbf{w}_k\}$ and the measurement noise sequence $\{\mathbf{v}_k\}$, as well as the initial state \mathbf{x}_0 , are random vectors with arbitrary distributions but at

least known mean values, covariance and cross-covariance matrices, containing elements with finite modulus. The objective is to estimate \mathbf{x}_k based on the measurements and our knowledge of the model dynamics. If the estimate of \mathbf{x}_k is based on measurements up to and including time l , we denote the estimator as $\hat{\mathbf{x}}_{k|l} \triangleq \hat{\mathbf{x}}_{k|l}(\mathbf{y}_1, \dots, \mathbf{y}_l)$, and we use the term estimator to refer to the class of algorithms that includes: (i) filtering, $\hat{\mathbf{x}}_{k|k}$, (ii) prediction $\hat{\mathbf{x}}_{k|l}$ ($l < k$), and (iii) smoothing, $\hat{\mathbf{x}}_{k|l}$ ($l > k$).

First, (11a) can be rewritten for $k \geq 2$ as

$$\begin{aligned} \mathbf{x}_k &= \mathbf{B}_{k,1} \mathbf{x}_1 + \mathbf{G}_k \bar{\mathbf{w}}_{k-1}, \quad \mathbf{G}_k \bar{\mathbf{w}}_{k-1} = \sum_{l=1}^{k-1} \mathbf{B}_{k,l+1} \mathbf{w}_l, \\ \mathbf{B}_{k,l} &= \begin{cases} \mathbf{F}_{k-1} \mathbf{F}_{k-2} \dots \mathbf{F}_l, & k > l \\ \mathbf{I} & , k = l \\ \mathbf{0} & , k < l \end{cases}, \quad \mathbf{G}_k \in \mathcal{M}_{\mathbb{C}}(P_k, P_{k-1}), \end{aligned}$$

where $P_k = \sum_{l=1}^k P_l$, then an equivalent form of (11b) is

$$\mathbf{y}_k = \mathbf{A}_k \mathbf{x}_1 + \mathbf{n}_k, \quad \mathbf{A}_k = \mathbf{H}_k \mathbf{B}_{k,1}, \quad \mathbf{n}_k = \mathbf{v}_k + \mathbf{H}_k \mathbf{G}_k \bar{\mathbf{w}}_{k-1}. \quad (12a)$$

Let $\mathbf{A}_1 = \mathbf{H}_1$ and $\mathbf{n}_1 = \mathbf{v}_1$, then (11b) can be extended on a horizon of k points from the first measurement as,

$$\bar{\mathbf{y}}_k = \bar{\mathbf{A}}_k \mathbf{x}_1 + \bar{\mathbf{n}}_k, \quad (12b)$$

where $\bar{\mathbf{y}}_k, \bar{\mathbf{n}}_k \in \mathcal{M}_{\mathbb{C}}(\mathcal{N}_k, 1)$, $\bar{\mathbf{A}}_k \in \mathcal{M}_{\mathbb{C}}(\mathcal{N}_k, P_k)$ and $\mathcal{N}_k = \sum_{l=1}^k N_l$.

3.2. LMMSEF for LDSS models with $\mathbf{C}_{\bar{\mathbf{y}}_k}$ full rank

The LMMSE filter (LMMSEF) of \mathbf{x}_k based on measurements up to time k , $k \geq 2$, is simply (5b) and (5c)

$$\hat{\mathbf{x}}_{k|k}^b = \mathbf{m}_{\mathbf{x}_k} + \mathbb{K}_{k|k}^b (\bar{\mathbf{y}}_k - \mathbf{m}_{\bar{\mathbf{y}}_k}) = \mathbf{m}_{\mathbf{x}_k} + [\mathbb{J}_{k-1}^b \mathbf{K}_k^b] (\bar{\mathbf{y}}_k - \mathbf{m}_{\bar{\mathbf{y}}_k}), \quad (13a)$$

with $[\mathbb{J}_{k-1}^b \mathbf{K}_k^b] \mathbf{C}_{\bar{\mathbf{y}}_k} = \mathbf{C}_{\mathbf{x}_k, \bar{\mathbf{y}}_k}$, provided that $\mathbf{C}_{\bar{\mathbf{y}}_k}$ has full rank, where $\mathbb{J}_{k-1}^b \in \mathcal{M}_{\mathbb{C}}(P_k, \mathcal{N}_{k-1})$ and $\mathbf{K}_k^b \in \mathcal{M}_{\mathbb{C}}(P_k, N_k)$. Thus, \mathbb{J}_{k-1}^b and \mathbf{K}_k^b are the solution of the system of linear equations

$$\begin{cases} \mathbb{J}_{k-1}^b \mathbf{C}_{\bar{\mathbf{y}}_{k-1}} + \mathbf{K}_k^b \mathbf{C}_{\mathbf{y}_k, \bar{\mathbf{y}}_{k-1}} = \mathbf{C}_{\mathbf{x}_k, \bar{\mathbf{y}}_{k-1}}, \\ \mathbb{J}_{k-1}^b \mathbf{C}_{\bar{\mathbf{y}}_{k-1}, \mathbf{y}_k} + \mathbf{K}_k^b \mathbf{C}_{\mathbf{y}_k} = \mathbf{C}_{\mathbf{x}_k, \mathbf{y}_k} \end{cases},$$

which yields to $\mathbb{J}_{k-1}^b = \mathbf{C}_{\mathbf{x}_k, \bar{\mathbf{y}}_{k-1}} \mathbf{C}_{\bar{\mathbf{y}}_{k-1}}^{-1} - \mathbf{K}_k^b \mathbf{C}_{\mathbf{y}_k, \bar{\mathbf{y}}_{k-1}} \mathbf{C}_{\bar{\mathbf{y}}_{k-1}}^{-1}$. Consequently, (13a) can be rewritten as,

$$\begin{aligned} \hat{\mathbf{x}}_{k|k}^b &= \mathbf{m}_{\mathbf{x}_k} + \mathbb{J}_{k-1}^b (\bar{\mathbf{y}}_{k-1} - \mathbf{m}_{\bar{\mathbf{y}}_{k-1}}) + \mathbf{K}_k^b (\mathbf{y}_k - \mathbf{m}_{\mathbf{y}_k}) \\ &= \hat{\mathbf{x}}_{k|k-1}^b + \mathbf{K}_k^b (\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}^b), \end{aligned} \quad (13b)$$

$$\hat{\mathbf{x}}_{k|k-1}^b = \mathbf{m}_{\mathbf{x}_k} + \mathbf{C}_{\mathbf{x}_k, \bar{\mathbf{y}}_{k-1}} \mathbf{C}_{\bar{\mathbf{y}}_{k-1}}^{-1} (\bar{\mathbf{y}}_{k-1} - \mathbf{m}_{\bar{\mathbf{y}}_{k-1}}),$$

$$\hat{\mathbf{y}}_{k|k-1}^b = \mathbf{m}_{\mathbf{y}_k} + \mathbf{C}_{\mathbf{y}_k, \bar{\mathbf{y}}_{k-1}} \mathbf{C}_{\bar{\mathbf{y}}_{k-1}}^{-1} (\bar{\mathbf{y}}_{k-1} - \mathbf{m}_{\bar{\mathbf{y}}_{k-1}}),$$

which is the general form of the so-called predictor/corrector format of the LMMSEF. Moreover, as $\mathbf{C}_{\mathbf{y}_k, \bar{\mathbf{y}}_{k-1}} = \mathbf{H}_k \mathbf{C}_{\mathbf{x}_k, \bar{\mathbf{y}}_{k-1}} + \mathbf{C}_{\mathbf{v}_k, \bar{\mathbf{y}}_{k-1}}$ and $\mathbf{C}_{\mathbf{x}_k, \bar{\mathbf{y}}_{k-1}} = \mathbf{F}_{k-1} \mathbf{C}_{\mathbf{x}_{k-1}, \bar{\mathbf{y}}_{k-1}} + \mathbf{C}_{\mathbf{w}_{k-1}, \bar{\mathbf{y}}_{k-1}}$ then according to (5b) and (5c),

$$\hat{\mathbf{y}}_{k|k-1}^b = \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}^b + \hat{\mathbf{v}}_{k|k-1}^b; \quad \hat{\mathbf{x}}_{k|k-1}^b = \mathbf{F}_{k-1} \hat{\mathbf{x}}_{k-1|k-1}^b + \hat{\mathbf{w}}_{k-1|k-1}^b,$$

$$\hat{\mathbf{v}}_{k|k-1}^b = \mathbf{m}_{\mathbf{v}_k} + \mathbf{C}_{\mathbf{v}_k, \bar{\mathbf{y}}_{k-1}} \mathbf{C}_{\bar{\mathbf{y}}_{k-1}}^{-1} (\bar{\mathbf{y}}_{k-1} - \mathbf{m}_{\bar{\mathbf{y}}_{k-1}}),$$

$$\hat{\mathbf{w}}_{k|k}^b = \mathbf{m}_{\mathbf{w}_k} + \mathbf{C}_{\mathbf{w}_k, \bar{\mathbf{y}}_k} \mathbf{C}_{\bar{\mathbf{y}}_k}^{-1} (\bar{\mathbf{y}}_k - \mathbf{m}_{\bar{\mathbf{y}}_k}),$$

and (13b) can be recast as

$$\begin{aligned} \hat{\mathbf{x}}_{k|k}^b &= \mathbf{F}_{k-1} \hat{\mathbf{x}}_{k-1|k-1}^b + \mathbf{m}_{\mathbf{w}_{k-1}} + \mathbf{K}_k^b (\mathbf{y}_k - (\mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}^b + \mathbf{m}_{\mathbf{v}_k})) \\ &+ (\mathbf{C}_{\mathbf{w}_{k-1}, \bar{\mathbf{y}}_{k-1}} - \mathbf{K}_k^b \mathbf{C}_{\mathbf{v}_k, \bar{\mathbf{y}}_{k-1}}) \mathbf{C}_{\bar{\mathbf{y}}_{k-1}}^{-1} (\bar{\mathbf{y}}_{k-1} - \mathbf{m}_{\bar{\mathbf{y}}_{k-1}}). \end{aligned} \quad (13c)$$

However, in general, no LDSS model for the noise sequences $\{\mathbf{w}_{k-1}, \mathbf{v}_k\}$ is known (i.e., otherwise it could be incorporated into (11a) and (11b)) and the computation of $\mathbf{C}_{\bar{\mathbf{y}}_{k-1}}^{-1}$ becomes computationally intractable as the number of observations k increases.

Therefore, the general assumptions required to obtain a computationally tractable form of $\hat{\mathbf{x}}_{k|k}^b$ (13c) are

$$\mathbf{C}_{\mathbf{w}_{k-1}, \bar{\mathbf{y}}_{k-1}} = \mathbf{0}, \quad \mathbf{C}_{\mathbf{v}_k, \bar{\mathbf{y}}_{k-1}} = \mathbf{0}, \quad k \geq 2. \quad (13d)$$

Indeed, under (13d), $\hat{\mathbf{x}}_{k|k}^b$ (13c) reduces to

$$\hat{\mathbf{x}}_{k|k}^b = \hat{\mathbf{x}}_{k|k-1}^b + \mathbf{K}_k^b (\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}^b), \quad (13e)$$

$$\hat{\mathbf{x}}_{k|k-1}^b = \mathbf{F}_{k-1} \hat{\mathbf{x}}_{k-1|k-1}^b + \mathbf{m}_{\mathbf{w}_{k-1}}; \quad \hat{\mathbf{y}}_{k|k-1}^b = \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}^b + \mathbf{m}_{\mathbf{v}_k}, \quad (13f)$$

which is the KF form of the LMMSEF for LDSS models.

3.3. Insightful breakdown of the MSE of linear filters

Another noteworthy point is that under the general assumptions (13d), the MSE of any linear filter $\hat{\mathbf{x}}_{k|k} = \mathbf{m}_{\mathbf{x}_k} + \mathbb{K}_k(\bar{\mathbf{y}}_k - \mathbf{m}_{\bar{\mathbf{y}}_k})$, $k \geq 2$, $\mathbb{K}_k = [\mathbb{J}_{k-1} \quad \mathbf{K}_k]$ where $\mathbb{J}_{k-1} \in \mathcal{M}_{\mathbb{C}}(P_k, \mathcal{N}_{k-1})$ and $\mathbf{K}_k \in \mathcal{M}_{\mathbb{C}}(P_k, N_k)$, that is,

$$\mathbf{P}_{k|k}(\mathbb{K}_k) = E_{\mathbf{y}, \mathbf{x}} \left[(\hat{\mathbf{x}}_{k|k} - \mathbf{x}_k)(\hat{\mathbf{x}}_{k|k} - \mathbf{x}_k)^H \right] = \mathbf{P}_{k|k}(\mathbb{J}_{k-1}, \mathbf{K}_k), \quad (14a)$$

breaks down into

$$\begin{aligned} \mathbf{P}_{k|k}(\mathbb{J}_{k-1}, \mathbf{K}_k) &= \mathbf{Q}_{k-1}(\mathbb{J}_{k-1}, \mathbf{K}_k) \\ &+ (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) (\mathbf{C}_{\mathbf{w}_{k-1}} + \mathbf{F}_{k-1} \mathbf{C}_{\mathbf{w}_{k-1}, \mathbf{x}_{k-1}}^H + \mathbf{C}_{\mathbf{w}_{k-1}, \mathbf{x}_{k-1}} \mathbf{F}_{k-1}^H) (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^H \\ &- (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{C}_{\mathbf{x}_k, \mathbf{v}_k} \mathbf{K}_k^H - \mathbf{K}_k \mathbf{C}_{\mathbf{x}_k, \mathbf{v}_k}^H (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^H + \mathbf{K}_k \mathbf{C}_{\mathbf{v}_k} \mathbf{K}_k^H, \end{aligned} \quad (14b)$$

$$\mathbf{Q}_{k-1}(\mathbb{J}_{k-1}, \mathbf{K}_k) = E_{\mathbf{y}, \mathbf{x}} [\hat{\mathbf{q}}_{k-1} \hat{\mathbf{q}}_{k-1}^H], \quad (14c)$$

$$\hat{\mathbf{q}}_{k-1} = \mathbb{J}_{k-1} (\bar{\mathbf{y}}_{k-1} - \mathbf{m}_{\bar{\mathbf{y}}_{k-1}}) - (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{F}_{k-1} (\mathbf{x}_{k-1} - \mathbf{m}_{\mathbf{x}_{k-1}}), \quad (14d)$$

which is a key result in order to derive straightforwardly the general form of the KF recursion (without extension of the state and measurement equations). (14b) is easily obtained using (13d) and the next breakdown of the error,

$$\hat{\mathbf{x}}_{k|k} - \mathbf{x}_k = \hat{\mathbf{q}}_{k-1} + (\mathbf{K}_k (\mathbf{v}_k - \mathbf{m}_{\mathbf{v}_k}) - (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) (\mathbf{w}_{k-1} - \mathbf{m}_{\mathbf{w}_{k-1}})). \quad (14e)$$

Note that (14b) holds irrespective of the rank of $\mathbf{C}_{\bar{\mathbf{y}}_k}$, provided that (13d) is satisfied.

3.4. General form of the KF recursion

From (14b), and irrespective of the rank of $\mathbf{C}_{\bar{\mathbf{y}}_{k-1}}$ (see (10c)), we have that

$$\mathbf{P}_{k|k}(\mathbb{J}_{k-1}, \mathbf{K}_k) \geq \mathbf{P}_{k|k}(\mathbb{J}_{k-1}^b(\mathbf{K}_k), \mathbf{K}_k), \quad (15a)$$

$$\mathbb{J}_{k-1}^b(\mathbf{K}_k) = \arg \min_{\mathbb{J}_{k-1}} \{ \mathbf{Q}_{k-1}(\mathbb{J}_{k-1}, \mathbf{K}_k) \} = (\mathbf{I}_p - \mathbf{K}_k \mathbf{H}_k) \mathbf{F}_{k-1} \mathbb{K}_{k-1}^b. \quad (15b)$$

Therefore,

$$\begin{aligned} \min_{\mathbb{K}_k} \{ \mathbf{P}_{k|k}(\mathbb{K}_k) \} &= \min_{\mathbb{J}_{k-1}, \mathbf{K}_k} \{ \mathbf{P}_{k|k}(\mathbb{J}_{k-1}, \mathbf{K}_k) \} \\ &= \min_{\mathbf{K}_k} \{ \mathbf{P}_{k|k}(\mathbb{J}_{k-1}^b(\mathbf{K}_k), \mathbf{K}_k) \}, \end{aligned} \quad (16a)$$

$$\begin{aligned} \mathbf{P}_{k|k}(\mathbb{J}_{k-1}^b(\mathbf{K}_k), \mathbf{K}_k) &= (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_{k|k-1}^b (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^H \\ &- (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{C}_{\mathbf{x}_k, \mathbf{v}_k} \mathbf{K}_k^H - \mathbf{K}_k \mathbf{C}_{\mathbf{x}_k, \mathbf{v}_k}^H (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^H + \mathbf{K}_k \mathbf{C}_{\mathbf{v}_k} \mathbf{K}_k^H, \end{aligned} \quad (16b)$$

which is the general form of the Joseph stabilized version of the covariance measurement update equation. Moreover, under (13d),

$$\hat{\mathbf{x}}_{k|k-1}^b = \mathbf{F}_{k-1} \hat{\mathbf{x}}_{k-1|k-1}^b + \hat{\mathbf{w}}_{k-1|k-1}^b = \mathbf{F}_{k-1} \hat{\mathbf{x}}_{k-1|k-1}^b + \mathbf{m}_{\mathbf{w}_{k-1}}, \quad (16c)$$

$$\begin{aligned} \mathbf{P}_{k|k-1}^b &= \mathbf{F}_{k-1} \mathbf{P}_{k-1|k-1}^b \mathbf{F}_{k-1}^H + \mathbf{C}_{\mathbf{w}_{k-1}} \\ &+ \mathbf{F}_{k-1} \mathbf{C}_{\mathbf{w}_{k-1}, \mathbf{x}_{k-1}}^H + \mathbf{C}_{\mathbf{w}_{k-1}, \mathbf{x}_{k-1}} \mathbf{F}_{k-1}^H. \end{aligned} \quad (16d)$$

Interestingly enough, $\mathbf{P}_{k|k}(\mathbb{J}_{k-1}^b(\mathbf{K}_k), \mathbf{K}_k)$ in (16b) can be written as

$$E_{\mathbf{y}, \mathbf{x}} [((\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^b) - \mathbf{K}_k \hat{\mathbf{e}}_{k|k-1}^b)(\cdot)^H], \quad (17a)$$

where $\hat{\mathbf{e}}_{k|k-1}^b = \mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}^b$ is the innovation vector, and under (13d) the corresponding covariance is

$$\mathbf{S}_{k|k-1}^b \triangleq \mathbf{C}_{\hat{\mathbf{e}}_{k|k-1}^b} = \mathbf{H}_k \mathbf{P}_{k|k-1}^b \mathbf{H}_k^H + \mathbf{C}_{\mathbf{v}_k} + \mathbf{H}_k \mathbf{C}_{\mathbf{v}_k, \mathbf{x}_k}^H + \mathbf{C}_{\mathbf{v}_k, \mathbf{x}_k} \mathbf{H}_k^H. \quad (17b)$$

Then finally we have that

$$\mathbf{K}_k^b = \arg \min_{\mathbf{K}_k} \{ \mathbf{P}_{k|k}(\mathbb{J}_{k-1}^b(\mathbf{K}_k), \mathbf{K}_k) \}, \quad (18a)$$

where \mathbf{K}_k^b is recursively computed according to (10c) as,

$$\mathbf{K}_k^b = (\mathbf{P}_{k|k-1}^b \mathbf{H}_k^H + \mathbf{C}_{\mathbf{v}_k, \mathbf{x}_k}^H) (\mathbf{S}_{k|k-1}^b)^{\dagger}, \quad (18b)$$

$$\mathbf{P}_{k|k}^b = (\mathbf{I} - \mathbf{K}_k^b \mathbf{H}_k) \mathbf{P}_{k|k-1}^b - \mathbf{K}_k^b \mathbf{C}_{\mathbf{v}_k, \mathbf{x}_k}. \quad (18c)$$

with $\mathbf{P}_{k|k-1}^b$ in (16d) and $\mathbf{S}_{k|k-1}^b$ in (17b). At time $k = 1$, the Kalman gain is given by

$$\mathbf{K}_1^b = \arg \min_{\mathbf{K}_1} \{ E_{\mathbf{y}, \mathbf{x}} [(\mathbf{K}_1 (\mathbf{y}_1 - \mathbf{m}_{\mathbf{y}_1}) - (\mathbf{x}_1 - \mathbf{m}_{\mathbf{x}_1}))(\cdot)^H] \},$$

where \mathbf{K}_1^b can be computed according to (18b) provided that one sets: $\hat{\mathbf{x}}_{0|0} = \mathbf{m}_{\mathbf{x}_0}$ and $\mathbf{P}_{0|0}^b = \mathbf{C}_{\mathbf{x}_0}$.

Remark 2. Since (14b)–(14e), (15a), (16a), (17a) and (18a) only depend on the general assumptions (13d), irrespective of the rank of $\mathbf{C}_{\bar{\mathbf{y}}_k}$, we can assert that under (13d) the Kalman form (13e) of the LMMSEF for LDSS models (13a) always exists and can be computed according to the recursion (16c)–(18c) provided that one sets $\hat{\mathbf{x}}_{0|0} = \mathbf{m}_{\mathbf{x}_0}$ and $\mathbf{P}_{0|0}^b = \mathbf{C}_{\mathbf{x}_0}$.

4. Conclusions

The goal of this contribution was to assess the conditions of existence of the KF form under more general settings w.r.t. the current literature. An alternative derivation of the general WF form was provided. Such general WF can always be computed irrespective of the measurement covariance $\mathbf{C}_{\mathbf{y}}$ rank by resorting to the Moore–Penrose inverse. Regarding state estimation for LDSS models, it was shown that considering the general KF conditions of existence the general KF solution always exist and can also be formulated irrespective of the measurement covariance rank. These results also apply for Kalman predictors and smoothers, linearly constrained WFs and MVDR filters, therefore being a remarkable finding of broad interest.

Declaration of Competing Interest

Authors declare that they have no conflict of interest.

CRediT authorship contribution statement

Eric Chaumette: Conceptualization, Methodology, Writing - original draft, Funding acquisition. **Jordi Vilà-Valls:** Methodology, Writing - original draft, Funding acquisition. **François Vincent:** Investigation, Supervision, Writing - review & editing, Project administration.

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