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A 6-ring embedded strain sensor *

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June 30, 2021

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Abstract

The proposed embedded strain sensor consists of six elastic rings, placed in an icosahedral symmetry, whose measured elongations give access to the complete 3D strain tensor in the surrounding matrix. The linear relationship between the ring elongations and the strain tensor in the matrix is given. From this relation it is deduced that the tensor is isotropic, *i.e.* its sensitivity is independent of the rotation of the strain tensor and that the sensor is more sensitive to hydrostatic strains than to deviatoric strains. The additional perturbation of the strain field in the matrix, induced by the presence of the sensor, is studied by means of a numerical integration of the Kelvin solution and the use two dimensionless factors relating to geometry and elasticity. From this study we obtain, in a generic way, an evaluation of the intensity of this additional strain field and the value of the measurement bias due to the elasticity of the ring, which allows in practice to reduce this bias. A draft of the realisation is proposed, for which it is verified that the disturbance of the strain field in the matrix as well as the bias to be corrected are small.

keywords : strain sensor, 3D

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1 Introduction

Undergoing exponential industrial development [5], Structural Health Monitoring (SHM) aims both to improve reliability and to reduce structural inspection costs in many fields (Civil Engineering, Aeronautics, Marine Engineering...). Recent SHM methods, such as big-data methods, use information of different physical types. However strain measurements remain of primary interest due to their strong mechanical content. One of the main challenges of SHM is to measure the strain (or stress) in the body and compare it to a calculated value with respect to the shape of the structure, its present load, its history...

Most strain measurements are carried out on surfaces, using strain gauges, optical fibres or Digital Image Correlation. On the one hand, this is an advantage because the stresses are generally maximum on the surfaces but, on the other hand, surfaces are, especially in civil engineering, the site of great thermal variations and chemical aggression, which can lead to erroneous information. Furthermore, the data available for a surface measurement is reduced to the 3 components of the 2D strain tensors.

Embedded sensors have been developed since the 1940s for Civil Engineering, with halter-shaped devices embedded in concrete. These are nowadays available with various transducers such as vibrating wires, strain gauges, optical wires. Such a design has many flaws: it only provides axial strain, the rigidity of the spring body perturbs the strain field in the matrix, which induces a bias in the measurement and a stress concentration in the matrix. Embeddable strain gauges are also commercialised, but they still provide 1D information, their flat shape requires a good adhesion to the matrix and can initiate a crack. Embedded sensor (including the proposed one) are inherently limited to materials with cold casting processes such as concrete, polymers and composites. To our knowledge, the sole alternative method to measure a strain inside a body (of engineering material, this excludes optical methods) is the 3D Digital Image Correlation method [6] which requires both a heterogenous meso-structure (for the required 3D speckle) and a powerful X-ray source. These two points strongly restrict its use to laboratory testing.

The authors and colleagues have proposed a new sensor consisting of a spherical body instrumented with six regularly spaced, radial strain transducers [11] [10]. Thanks to the Eshelby's inclusion theory, the strain in the

sphere is homogenous and the strain in the matrix, which would exist in the absence of the sphere, depends linearly on the strain in the sphere (given the elasticity of the matrix). This concept has been successfully tested during concrete pouring and concrete compression [16]. A variant of this sensor, in which the six transducers are no longer radial but perimetric (ring shaped), has recently been proposed [12]. However, the use of a ball as a test body disturbs the nearby strain field in the matrix and the exact measurement of the strain requires, from Eshelby’s theory, the knowledge of the elasticity of both the sphere and the matrix. In this paper we study a variant of this concept in which the sphere is removed and only the six rings are kept. One can also consider that the sphere is made of the matrix material, the localization tensor being then degenerate to the identity if one neglects the stiffness of the thin rings.

The architecture of the sensor is presented in Section 2. In Section 3, the linear relationship between the six elongations of the rings and the six components of the matrix strain tensor is given. In Section 4 is detailed the internal sensitivity of the the sensor, *i.e.* the link between the ring strains and the matrix strain tensor, as well as its consequence on the accuracy of the measurement. Section 5 is devoted to the study of the perturbation strain field generated by the presence of the ring inside the strained matrix. In order to establish relations as general and as simple as possible, a semi-analytical method based on the Kelvin’s point-force solution is used, as well as dimensionless parameters. This gives access to the strain concentration around the sensor and to the induced bias in all cases. Moreover, the expression of the bias allows a correction of the measurement in order to recover the nominal matrix strain (the one that would exist in the absence of the sensor). The magnitude of the perturbation strain field and of the bias are specified in an example of a design with rings made of Constantan rings in a concrete matrix.

2 Architecture

Figure 3 shows a sketch of proposed design. The sensor is composed of six deformable rings, the normals of which are regularly spaced along the 6 (5th order) axis of an icosahedron. The sensor is supposed to be embedded inside the matrix during its casting. The six transducers, whose technology is not fixed, measure the mean elongation of the rings. This is facilitated if the rings are sliding in the matrix because then their deformation (and tension)

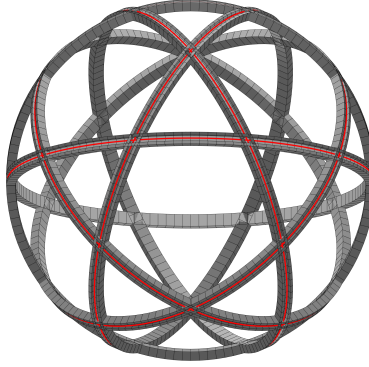


Figure 1: Sketch of the 6-ring 3D strain sensor. Red: transducers, gray: elastic rings.

is homogeneous. Depending upon the nature of the transducer, the elastic ring and the transducer can be a single part (*e.g* a coil of resistive wire) or two separate parts (*e.g* a strain gauge on a metallic ring).

Reduced to a minimum volume, this shape causes little disturbance to the strain (or stress) field in the matrix (this is studied in Section 5). Compared to the previous ball concept, it can be considered that the sphere is made of the same matter as the matrix, reducing the Eshelby problem to a trivial case. Wire or fibre transducers generally require a pre-tensioning in order to avoid micro-buckling in case of compressive states [1]. As an engineering formula indicates that the acceptable tension T on belt wound on a pulley evolves as $T = T_0 \exp^{\phi\beta}$ where T_0 is the tension on the slack strand (which can be weak and obtained by gluing), ϕ the Coulomb's friction and β the winding angle, such pre-tension of is easy to maintain on the rings.

3 Relationship between the strain tensor of the matrix and the elongation of the ring

The goal of this section is to find the most direct relationship between the measured strains of the rings and the strain tensor of the matrix. The latter is supposed to be homogeneous in the vicinity of the sensor (this hypothesis is common for any device, for example strain gauges). The ring is assimilated to its mean circle and is supposed (in this section) to have no volume

thus not to affect the matrix strain.

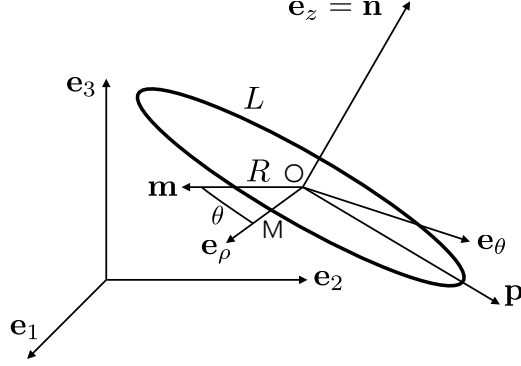


Figure 2: Geometrical parameterisation

The local basis of the circle is $(\mathbf{m}, \mathbf{p}, \mathbf{n})$, in which $\mathbf{n}$ is normal to the plane of the circle (see Figure 2). The radius of the ring is R , the global reference basis is $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $\boldsymbol{\varepsilon}$ is the (homogeneous) strain tensor. Relying in the plane defined by $\mathbf{m}$ and $\mathbf{p}$, the circular ring is subjected to the 2D projection $P(\boldsymbol{\varepsilon})$ of the strain tensor, of components in the local basis:

$$P(\boldsymbol{\varepsilon}) = \begin{bmatrix} \bar{\varepsilon}_I & 0 \\ 0 & \bar{\varepsilon}_{II} \end{bmatrix}_{(\mathbf{m}, \mathbf{p})} \quad (1)$$

in which, without any restriction, $(\mathbf{m}, \mathbf{p})$ are chosen to coincide with the eigenvectors of $P(\boldsymbol{\varepsilon})$. A portion of ring at point M such that $\mathbf{OM} = R\mathbf{e}_\rho$, where $(\mathbf{e}_\rho, \mathbf{e}_\theta)$ are the polar coordinates (see figure 2), has the local hoop strain:

$$\mathbf{e}_\theta \cdot \boldsymbol{\varepsilon} \cdot \mathbf{e}_\theta = \bar{\varepsilon}_I \sin^2 \theta + \bar{\varepsilon}_{II} \cos^2 \theta. \quad (2)$$

The global ring elongation ΔL is given by the integral of this local strain over one turn:

$$\Delta L = \pi R (\bar{\varepsilon}_I + \bar{\varepsilon}_{II}). \quad (3)$$

Because the initial length is $L = 2\pi R$, the mean strain of the ring is:

$$\frac{\Delta L}{L} = \frac{\bar{\varepsilon}_I + \bar{\varepsilon}_{II}}{2}. \quad (4)$$

This result can also be obtained by considering a compatible displacement field, which leads to deform the circle into an ellipse of semi-axis $R_I =$

$R(1 + \bar{\varepsilon}_I)$ and $R_{II} = R(1 + \bar{\varepsilon}_{II})$, and calculating the perimeter of the ellipse with the approximate formula $L + \Delta L = \pi \sqrt{2(R_I^2 + R_{II}^2)}$. Previous equation refers to the trace of the tensor $P(\varepsilon)$. Being an invariant, this expression:

$$2 \frac{\Delta L}{L} = \text{trace}(P(\varepsilon)) \quad (5)$$

is also true in the global basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$:

$$2 \frac{\Delta L}{L} = P_{pr} P_{ps} \varepsilon_{rs} \quad (6)$$

in which the components of the projector are:

$$P_{pq} = \delta_{pq} - n_p n_q \quad (7)$$

where δ is the Kronecker symbol. As a consequence:

$$2 \frac{\Delta L}{L} = P_{rs} \varepsilon_{rs}. \quad (8)$$

Above equation is available for each one of the six rings, leading to the system:

$$\frac{1}{2} \begin{bmatrix} P_{11}^1 & P_{22}^1 & P_{33}^1 & \sqrt{2}P_{23}^1 & \sqrt{2}P_{31}^1 & \sqrt{2}P_{12}^1 \\ P_{11}^2 & \dots & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ P_{11}^6 & \dots & & & & \end{bmatrix} \bullet \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \sqrt{2}\varepsilon_{23} \\ \sqrt{2}\varepsilon_{31} \\ \sqrt{2}\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \Delta L_1/L \\ \Delta L_2/L \\ \Delta L_3/L \\ \Delta L_4/L \\ \Delta L_5/L \\ \Delta L_6/L \end{bmatrix}, \quad (9)$$

where the extra indexes refer to the ring number. Such correspondence between the ε_I , for $I \in \{1, \dots, 6\}$, and the ε_{pq} was introduced at first by Bechterew [4] but are often called (in our opinion incorrectly) Kelvin's notation. Anyway, it is preferred to classical Voigt's notation for some of its later used properties. The normals $\mathbf{n}_I$ of the six ring correspond to the normals to the faces of an icosahedron, whose components are chosen as:

$$\begin{aligned} \mathbf{n}_1 &= (0, \varphi, 1)/\sqrt{2 + \varphi}, \\ \mathbf{n}_2 &= (0, \varphi, -1)/\sqrt{2 + \varphi}, \\ \mathbf{n}_3 &= (1, 0, \varphi)/\sqrt{2 + \varphi}, \\ \mathbf{n}_4 &= (1, 0, -\varphi)/\sqrt{2 + \varphi}, \\ \mathbf{n}_5 &= (\varphi, 1, 0)/\sqrt{2 + \varphi}, \\ \mathbf{n}_6 &= (\varphi, -1, 0)/\sqrt{2 + \varphi}, \end{aligned} \quad (10)$$

where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio. Gathering equations 9 and 10 gives a linear relation of the form:

$$M_{IJ}\varepsilon_J = \frac{\Delta L_I}{L}, \quad (11)$$

in which ε_J represents the Bechterew components of ε as defined by Equation 9. From equation 10 (and considering the property $\varphi^2 - 1 = \varphi$) the analytical expression of M is:

$$M = \frac{1}{2(2+\varphi)} \begin{bmatrix} 2+\varphi & 1 & 1+\varphi & -\sqrt{2}\varphi & 0 & 0 \\ 2+\varphi & 1 & 1+\varphi & \sqrt{2}\varphi & 0 & 0 \\ 1+\varphi & 2+\varphi & 1 & 0 & -\sqrt{2}\varphi & 0 \\ 1+\varphi & 2+\varphi & 1 & 0 & \sqrt{2}\varphi & 0 \\ 1 & 1+\varphi & 2+\varphi & 0 & 0 & -\sqrt{2}\varphi \\ 1 & 1+\varphi & 2+\varphi & 0 & 0 & \sqrt{2}\varphi \end{bmatrix} \quad (12)$$

whose inverse is:

$$M^{-1} = \frac{1}{2} \begin{bmatrix} 2 & 2 & \varphi-1 & \varphi-1 & -\varphi & -\varphi \\ -\varphi & -\varphi & 2 & 2 & \varphi-1 & \varphi-1 \\ \varphi-1 & \varphi-1 & -\varphi & -\varphi & 2 & 2 \\ \frac{1-2\varphi}{\sqrt{2}} & \frac{2\varphi-1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-2\varphi}{\sqrt{2}} & \frac{2\varphi-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\varphi}{\sqrt{2}} & \frac{2\varphi-1}{\sqrt{2}} \end{bmatrix}. \quad (13)$$

These leads to the formula of interest for use in engineering:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 2 & \varphi-1 & \varphi-1 & -\varphi & -\varphi \\ -\varphi & -\varphi & 2 & 2 & \varphi-1 & \varphi-1 \\ \varphi-1 & \varphi-1 & -\varphi & -\varphi & 2 & 2 \\ 1-2\varphi & 2\varphi-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-2\varphi & 2\varphi-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\varphi & 2\varphi-1 \end{bmatrix} \bullet \begin{bmatrix} \Delta L_1/L \\ \Delta L_2/L \\ \Delta L_3/L \\ \Delta L_4/L \\ \Delta L_5/L \\ \Delta L_6/L \end{bmatrix}. \quad (14)$$

4 Internal sensitivity and its influence on accuracy

Being linear, the above relationships lead to a proportionality of the ring mean strains $\Delta L_I/L$ to the Euclidean norm $\|\varepsilon\|$ of the strain tensor. Accordingly, we define internal sensitivity as :

$$s(\varepsilon) = \frac{\left| \frac{\Delta L_I}{L} \right|}{\|\varepsilon\|} \quad (15)$$

where the numerator is the Euclidean norm of the (pseudo-)vector of the strains of the rings, proportional to their root mean square. The Bechterew basis used in Equation 9 keeps the definition of the Euclidean norm: $\sum_{ij} \varepsilon_{ij}^2 = \sum_{I=1}^6 \varepsilon_I^2$. From Equation 11:

$$\begin{aligned} \|\varepsilon\|^2 s(\varepsilon)^2 &= \varepsilon_J C_{JK} \varepsilon_K \\ C_{JK} &= M_{JI}^T M_{IK} \end{aligned} \quad (16)$$

This matrix C_{JK} is representative of a fourth-rank tensor which has both the "major" index symmetry $C_{JK} = C_{KJ}$ or $C_{ijkl} = C_{klij}$ in the canonic basis and, the "minor" index symmetries $C_{ijkl} = C_{jikl} = C_{ijlk}$ (intrinsic to the Bechterew basis). As a consequence, it has the nature of an elasticity tensor. From Equation 12, its expression is:

$$C = \frac{1}{5} \begin{bmatrix} 4 & 3 & 3 & 0 & 0 & 0 \\ 3 & 4 & 3 & 0 & 0 & 0 \\ 3 & 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (17)$$

This one corresponds to an isotropic tensor [7] [9] [3] which can be decomposed in a sum of weighted projectors called the Kelvin's decomposition [14] [2] which, in this case, simply separates hydrostatic and deviatoric parts [18]:

$$C = 2P^H + 0.2P^D, \quad (18)$$

$$P^H = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (19)$$

$$P^D = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad (20)$$

where $\mathbf{P}^H$ is the projector on the hydrostatic (or spheric) subspace and $\mathbf{P}^D$ the projector on the deviatoric subspace. Finally, the internal sensitivity is:

$$s(\boldsymbol{\varepsilon}) = \sqrt{\frac{2\|\boldsymbol{\varepsilon}^H\|^2 + 0.2\|\boldsymbol{\varepsilon}^D\|^2}{\|\boldsymbol{\varepsilon}^H\|^2 + \|\boldsymbol{\varepsilon}^D\|^2}}, \quad (21)$$

where $\boldsymbol{\varepsilon}^H = \mathbf{P}^H : \boldsymbol{\varepsilon}$ is the spherical part of the strain and $\boldsymbol{\varepsilon}^D = \mathbf{P}^D : \boldsymbol{\varepsilon}$ its deviatoric complementary part. This equation shows that the sensor has an internal sensitivity of $s = \sqrt{2} \simeq 1.414$ for a spherical strain, greater than $s = \sqrt{0.2} \simeq 0.4472$ for a deviatoric strain. For a general strain tensor, the internal sensitivity is somewhere between the two bounds: $\sqrt{0.2} \leq s \leq \sqrt{2}$. In other words, a spherical strain tensor induce three times ($\simeq \sqrt{2}/\sqrt{0.2}$) more hoop strain in the rings than a deviatoric strain tensor of same norm. This result does not affect the measure of the device which, as given by Equation 14, is theoretically exact. However, transducers and measuring chain are generally more accurate with strong signals than with weak signals. Thus, the sensor should be more accurate for spherical strains than for deviatoric strains. The isotropy of $\mathbf{C}$ implies that s , therefore the root mean square of the ring strains, remains unchanged with respect to any rotation of $\boldsymbol{\varepsilon}$. In other words, the device has no preferred directions. This isotropy of the sensor response is clearly an advantage and is due to the regular tessellation chosen for the ring normals.

5 Influence of the sensor on the matrix strain field

Like any sensor, this one perturbs by its presence the quantity of interest, *i.e.* the nominal (locally homogeneous) matrix strain $\boldsymbol{\varepsilon}$ that exist in the absence of the sensor. Supposing a perfectly sliding ring, any strain puts the ring in homogenous tension or compression. This tension induces distributed radial forces from the ring to the matrix which generate a perturbation strain field in the matrix. The actual strain field in the matrix is the superposition of the nominal strain field and the perturbation strain field. In order not to generate too much stress concentration, the latter should be as evanescent as possible.

5.1 Estimation of the perturbation of the strain field in the matrix

The perturbation strain field is estimated below by a semi-analytical model in which we assimilate the ring by a circular beam whose tension T generates

linear forces on the matrix. Furthermore, we assume that the ring elongation ΔL , thus T , are that calculated in Section 3, *i.e.*, assuming that they depend only on the strain of the matrix. The matrix is supposed to be isotropic and to have a Young's modulus E and a Poisson's ratio ν .

Because the matter of the ring replaces that of the matrix, the tension of the ring is:

$$T = (E_r - E)S \frac{\Delta L}{L} \quad (22)$$

where E_r is the Young's modulus of the ring, S its cross section and $\Delta L/L$ its nominal strain. From the internal equilibrium of the ring, this tension induces an uniformly distributed radial force $f = T/R$. By reaction, the matrix is loaded along the circle by $-f$. Figure 3 shows the retained model.

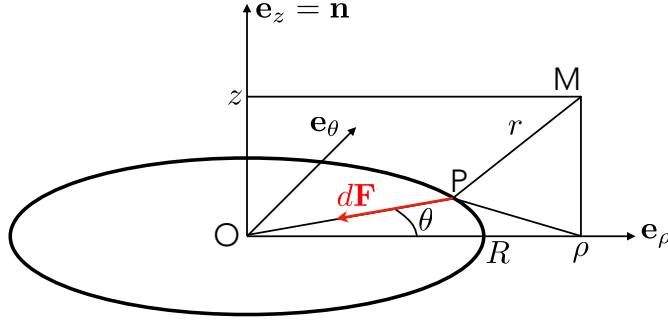


Figure 3: Line force ring model

Each differential element of the circle is subject to a radial differential force $dF = -Rf d\theta = -Td\theta$. For an unique point-force $d\mathbf{F}$, the Kelvin's solution in displacement [15], as expressed by [17] [8] [13] is:

$$d\mathbf{u}^p(M) = \frac{1}{16\pi\mu(1-\nu)} \left(\frac{3-4\nu}{r} \mathbf{I} + \frac{\mathbf{PM} \otimes \mathbf{PM}}{r^3} \right) \cdot d\mathbf{F}, \quad (23)$$

where p stands for perturbation, $\mu = E/(2(1+\nu))$ is the shear modulus of the matrix, M the point of interest, P the point where the force $d\mathbf{F}$ acts and $r = \|\mathbf{PM}\|$ is the distance from P to M. Considering the contribution of all forces of equal intensity, the problem is axisymmetric. It is therefore possible to consider only a point M belonging to the $(\mathbf{e}_\rho, \mathbf{e}_z)$ plane. The displacement due to the set of forces is obtained by integration along θ : $\mathbf{u}^p(M) = \int_0^{2\pi} d\mathbf{u}^p$. This integral does not have a simple analytical solution,

so is realised numerically, except for the points along the axis for which:

$$\mathbf{u}^p(O, z) = \frac{RTz}{8\mu(1-\nu)(R^2+z^2)^{3/2}} \mathbf{e}_z. \quad (24)$$

The strain field $\boldsymbol{\varepsilon}^p(\rho, z)$ is obtained by derivation of the displacement field. Being axisymmetric, it has only three independent components:

$$\boldsymbol{\varepsilon}^p = \begin{bmatrix} \int_{\theta} du_{\rho,\rho}^p & 0 & \frac{1}{2} \int_{\theta} du_{\rho,z}^p + du_{z,\rho}^p \\ 0 & \int_{\theta} du_{\rho}^p/\rho & 0 \\ \frac{1}{2} \int_{\theta} du_{\rho,z}^p + du_{z,\rho}^p & 0 & \int_{\theta} du_{z,z}^p \end{bmatrix}. \quad (25)$$

Annex 7.1 provides useful expressions of these derivatives. A dimensional analysis of the above equations shows that :

$$\frac{\mathbf{u}^p}{R} = \xi \bar{\mathbf{u}}^p \frac{\Delta L}{L}, \quad (26)$$

$$\boldsymbol{\varepsilon}^p = \xi \bar{\boldsymbol{\varepsilon}}^p \frac{\Delta L}{L}, \quad (27)$$

$$\xi = \frac{E_r - E}{E} \frac{S}{R^2}, \quad (28)$$

where ξ is a dimensionless parameter and where $\bar{\mathbf{u}}^p$ and $\bar{\boldsymbol{\varepsilon}}^p$ are respectively the unitary fields of displacement and strain which depend only on the Poisson's ratio ν . This unitary strain field $\bar{\boldsymbol{\varepsilon}}^p$ is shown by Figure 4 for a typical case $\nu = 0.25$. For this computation, the numerical integral of Kelvin's solution was realised using 2000 values of θ (with a relative difference of 10^{-4} from the result for 200 values) and the spatial resolution of the figure is $R/100$. Equation 27 shows that the perturbation strain field $\boldsymbol{\varepsilon}^p$ is proportional to these maps by the factor ξ (depending on the design) and the ring strain $\Delta L/L$.

However, in a design process, it is of interest to have a simple, dimensionless, index of magnitude of the perturbation field. On the boundary $z = 2R \cup \rho = 2R$, $\|\boldsymbol{\varepsilon}^p\|$ is maximum at the point $(\rho = 2R, z = 0)$ (bottom right on Figure 4) for any value of ν . For this reason, a perturbation index p is introduced as follows:

$$p(\nu) = \frac{\|\boldsymbol{\varepsilon}^p(2R, 0)\|}{\|\boldsymbol{\varepsilon}\|}, \quad (29)$$

whose general expression is given by Equations 27, 15 and 21:

$$p(\nu) = |\xi| \|\bar{\boldsymbol{\varepsilon}}^p(2R, 0)\|_s. \quad (30)$$

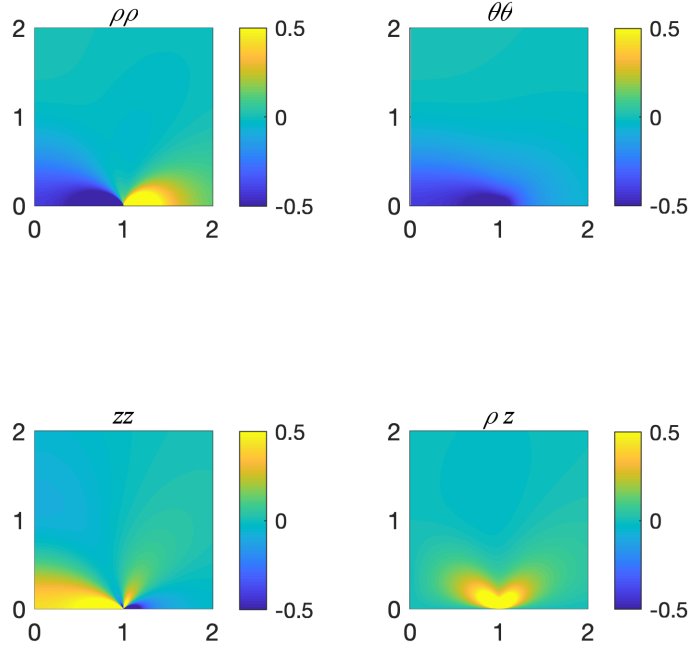


Figure 4: Unitary perturbation strain field $\bar{\epsilon}^p$ in the matrix around one ring in reduced coordinates $(\rho/R, z/R)$, for $\nu = 0.25$. Out-of-limit values close to the ring at $(1, 0)$ are clipped.

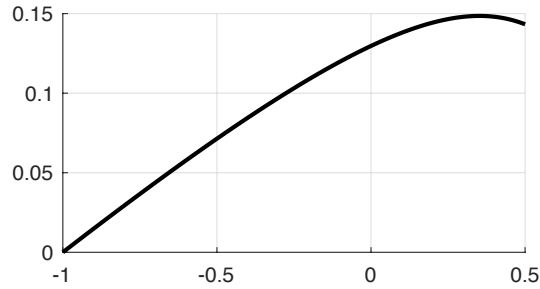


Figure 5: Dependence of $\|\bar{\epsilon}^p(2R, 0)\|$ to the Poisson's ratio

Figure 5 shows the value of $\|\bar{\epsilon}^p(2R, Z)\|$ in any cases. A simple bound on the intensity of the perturbation p is given by considering the last terms at

their maximum: $\|\bar{\varepsilon}^p(2R, 0)\| = 0.15$ and (from Equation 21) $s = \sqrt{2}$:

$$p \leq 0.2121 \frac{S}{R^2} \frac{|E_r - E|}{E}. \quad (31)$$

For the realisation example of Section 7.2, $\xi \simeq 0.0049$ and this relation gives $p < 10^{-3}$ thus the magnitude of the perturbation strain field is negligible with respect to the matrix strain field (of interest) from a distance to the center of the sensor of twice the size of the sensor.

This relation also indicates that, in general, the strain (or stress) field induced in the matrix by the presence of the ring will be minimised for a ring of small section S , of large radius R , or (unsurprisingly) for rings of the same elasticity E_r as the matrix E .

5.2 Estimation of the sensor bias due to the elasticity of the ring

The expression for the ring elongation ΔL as defined by Equation 5 was obtained by assuming that the ring strain was imposed by the matrix strain. However, the ring does not have the same elasticity as the matrix and this induces a discrepancy (a perturbation) ΔL^p between its real (measured) elongation and this theoretical elongation ΔL . The measurement bias b is:

$$b = \frac{\Delta L^p}{\Delta L}. \quad (32)$$

In order to obtain generic results, a discrete integration of the Kelvin solution 23 is again retained. In the same way, we will still assume that the tension T of the ring, and thus the radial forces which result from it, is due to the theoretical elongation ΔL : this amounts to assuming $\Delta L^p \ll \Delta L$. Previously, the field was searched far from the ring, so a line force model was sufficient. Now looking for the deformation of the ring itself, a more accurate surface force model is used. It consists of a surface force field $d^2\mathbf{F} = -(T/h)d\theta dz \mathbf{e}_\rho$ applied on a cylinder of radius R and height h , the height of the ring along z . This description will allow us to know the influence of the form factor:

$$\omega = \frac{h}{R}, \quad (33)$$

which conditions the thickness of the ring e , by $S = eh$ so $e = S/R\omega$. The radial displacement obtained with this model is not constant according to z (in the shape of a diabolo) so we retain in this calculation the displacement

of the point $(R, 0)$, at half height of the ring. The change in length of the ring is geometrically associated with the radial displacement:

$$\Delta L^p = 2\pi u_\rho^p(R, 0), \quad (34)$$

where u_ρ^p is the radial (and the only non-zero) component of $\mathbf{u}^p$.

The Kelvin solution gives an infinite displacement at the point of application of the force, regardless of its value. In order to overcome this, the numerical integration is done with integration points homogeneously distributed according to (θ, z) and such as the point of interest $(R, 0)$ is in the middle of the nearest integration points. Figure 6 shows that the displacement obtained stabilises as a function of the number of integration points. The result is similar for the R discretisation and thus the calculations were

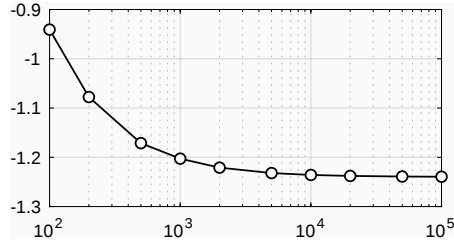


Figure 6: Unitary radial displacement of the ring $\bar{u}_\rho^p(R, 0)$ with respect to the number of computational points along θ , for $\nu = 0.25$ and $\omega = 1/30$

performed with 10^8 integration points. This calculation, purely analytical, takes only a few seconds on a current computer.

The dimensional analysis (Equation 26) remains true except that $\bar{\mathbf{u}}^p$, thus its component $\bar{u}_\rho^p(R, 0)$, depends upon both ω and ν . The result of this calculation is shown in Figure 7. From Equations 26, 34 the measurement bias is simply given by:

$$b = \xi \bar{u}_\rho^p(R, 0). \quad (35)$$

This analysis shows that the higher (thus thinner) the ring, the lower the bias. If the elasticity of the matrix is known, the bias b can be calculated once for all and used as a linear correction to recover the exact measurement of the strain tensor. If not, a calibration procedure can also be used.

For the example of the section 7.2, with $\omega = 0.0333$, is found $\bar{u}_\rho^p(R, 0) = -1.36$ thus a bias $b = -0.665\%$ which is acceptable in spite of the low ratio ω chosen (square section). As assumed, the perturbation displacement field

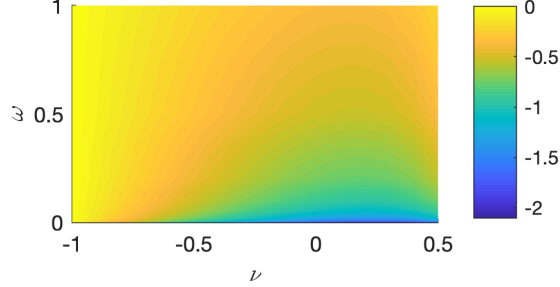


Figure 7: Unitary radial displacement at the middle of the ring $\bar{u}_\rho^p(R, 0)$ with respect to the Poisson's ratio ν and the relative width of the ring ω

$u_\rho^p = -0.01 \mu\text{m}$ is much smaller than the nominal displacement field $u_\rho = 15 \mu\text{m}$.

Other analytical models could have been used for this analysis, such as the two-dimensional fretting problem, leading to other simplifying assumptions. Anyway a better model, provided it remains in elasticity, will only modify the fields $\bar{\mathbf{u}}^p$ and $\bar{\boldsymbol{\varepsilon}}^p$, and leave others equations of this section, in particular the the role of the dimensionless parameters, unchanged. In an engineering approach, a finite element model, which requires the complete definition of the object, will of course be used.

6 Conclusion

The 6-ring sensor concept proposed in this article is capable of measuring the 6-components of a strain tensor within a matrix. Compared to the previous ball design proposed previously by the authors [11], its hollow shape makes it possible to do without the knowledge of the elasticity of the matrix in which the sensor is embedded. Moreover, the linear correspondence between the measured ring strains and the strain tensor (Equation 14) is as direct as for conventional rosette gauges. Its internal sensitivity has been shown to be isotropic but higher for spherical strains states. Again because of its hollow shape, its presence perturbs *a minima* the local strain field in the matrix thus minimises the stress concentration around the sensor. The proposed semi-analytical calculation, almost instantaneous on modern computers, allows engineers to both optimise the design and to correct the remaining bias induced by the presence of the sensor.

The concept can also be extended to more rings, in order to exploit

redundancy, or to fewer rings, if certain characteristics of the stress tensor are fixed. For example, a single ring is sufficient for a spherical stress (e.g. a pressure measurement), two rings are sufficient for a transverse isotropic, fixed-axis, strain state.

In this theoretical approach, the transducer type has not been fixed. Different measuring principles, such as strain gauges, ultrasonic measurements, optical fibre measurements, can be used to measure the elongation of rings. This study did not specify the presence or not of a spring body (for example metal rings on which the wires or fibres are folded). This one has the disadvantage of increasing the cross-section and rigidity of the ring (leading to an increase in both bias and disturbance field) but has the advantage, in the case of a technology with wires or fibres, to be more robust and to allow pre-tension in order to avoid micro-buckling in compression. Remaining technological difficulties are inherent to embedded transducers. For example, they may be sensitive to the lateral stress. Moreover, according to Equation 2, the hoop strain in the matrix is not constant. Thus, if it is necessary for the transducers to have a homogenous strain, some lubricant between the rings and the matrix should be envisaged.

This type of integrated physical sensor can only be used in cold-cast materials such as concrete or polymers. However, this concept could also be extended to lattice materials in which rings (or loops) could be added during the manufacturing process.

7 Annex

7.1 Derivatives of the displacement field

The derivatives of the displacement field (equation 23) are given by:

$$\begin{aligned}
d\mathbf{F} &= -Td\theta(c, s, 0), \\
\mathbf{PM} &= (\rho - Rc, -Rs, z), \\
r &= \sqrt{\rho^2 + R^2 + z^2 - 2\rho Rc}, \\
\frac{\partial(d\mathbf{u}^p)}{\partial\rho} &= K \left((3 - 4\nu) \frac{Rc - \rho}{r^3} \mathbf{I} + \frac{1}{r^3} \frac{\partial}{\partial\rho} (\mathbf{PM} \otimes \mathbf{PM}) - 3 \frac{\rho - Rc}{r^5} \mathbf{PM} \otimes \mathbf{PM} \right) \cdot d\mathbf{F}, \\
\frac{\partial(d\mathbf{u}^p)}{\partial z} &= K \left(-(3 - 4\nu) \frac{z}{r^3} \mathbf{I} + \frac{1}{r^3} \frac{\partial}{\partial z} (\mathbf{PM} \otimes \mathbf{PM}) - 3 \frac{z}{r^5} \mathbf{PM} \otimes \mathbf{PM} \right) \cdot d\mathbf{F}, \\
K &= \frac{1}{16\pi\mu(1 - \nu)},
\end{aligned} \tag{36}$$

where $c = \cos(\theta)$ and $s = \sin(\theta)$.

7.2 Example of possible realisation

For the sake of generality, the calculations are presented below in a dimensionless form. However, we shall apply them to the following exemple of design:

- the sensor size is defined by $R = 15$ mm, typical for strain gauges
- the ring is supposed to be made of Constantan for which $E_r = 162$ GPa
- the ring section is a square of side $h = 0.5$ mm thus of section $S = 0.25$ mm², strong enough to allow for hand manipulations
- the ring tension is $T = 33$ N corresponding, according to Equation 22, to a nominal ring strain $\Delta L/L = 10^{-3}$, this magnitude being representative of strain measurements
- the matrix is supposed to be made of concrete of Young's modulus $E = 30$ GPa and of Poisson's ratio $\nu = 0.25$

Conflict of interest

The authors declare that they have no conflict of interest.

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