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Change-Point Detection in Dynamic Networks with Missing Links

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Abstract

Structural changes occur in dynamic networks quite frequently and its detection is an important question in many situations such as fraud detection or cybersecurity. Real-life networks are often incompletely observed due to individual non-response or network size. In the present paper we consider the problem of change-point detection at a temporal sequence of partially observed networks. The goal is to test whether there is a change in the network parameters. Our approach is based on the Matrix CUSUM test statistic and allows growing size of networks. We show that the proposed test is minimax optimal and robust to missing links. We also demonstrate the good behavior of our approach in practice through simulation study and a real-data application.

1 Introduction

Most of the real-life networks, such as social networks or biological networks of neurons connected by their synapses, evolve over the time. Detecting possible changes in a temporal sequence of networks is an important task with applications in such areas as intrusion detection or health care monitoring. In this problem we observe a sequence of graphs each of which is usually sparse with large dimension and heterogeneous degrees. The underlying distribution of this sequence of graphs may change at some unknown time moment called change-point. The goal of this paper is to design a testing procedure that allows the detection of the presence of a change-point.

Many of the real-life networks are only partially observed ([Handcock and Gile 2010](#), [Guimerà and Sales-Pardo 2009](#)). The exhaustive exploration of all interactions in a network requires significant efforts and can be expensive and time consuming. For example, graphs constructed from survey data are likely to be incomplete, due to non-response or drop-out of participants. Another example is online social network data. The gigantic size of these networks requires working with a sub-sample of the network ([Catanese et al. 2011](#)). In all these situations, being able to infer the properties of the networks from their partial observations is of particular interest. To the best of our knowledge, change-point detection for networks with missing links has not been considered in the literature. In the present paper we focus on the case when we only have access to partial observations of the network and we propose an efficient procedure for detection and estimation of a change-point that is adaptive to the missing data.

In real-life networks both the entities and relationships in a network can vary over time. In the literature we can often find approaches that detect vertex-based changes in a time series of graphs assuming a fixed set of nodes. Nevertheless, changes in the set of nodes are also quite frequent in applications. For example, the vertices of a citation network are scientific papers and an edge connects two papers if one of them cites the other one. New vertices are constantly added to such networks. Likewise, the set of nodes of the Web, social networks or, more generally, communication networks are constantly evolving and changing. To account for this possibility we generalize our approach to the case when nodes of the network may come and go. Considering graphs with varying number of nodes calls for a more general non-parametric model. In the present paper we build on a popular graphon model and show that our detection procedure can be adapted to this more general setting.

1.1 Contributions and comparison with previous results

We consider the problem of hypothesis testing for the presence of a change-point, commonly known as *change-point detection*. The related problem of estimating the unknown change-points and their number is usually called the *change-point localization* problem. Our primary motivation is to address the following fundamental question: *What is the detection boundary in the problem of change-point detection in dynamic networks?* That is, we are interested in the smallest amount of change in the underlying distribution of the network such that successful detection of a change-point is still possible.

The change-point detection in dynamic networks and the related problem of the change-point localization have attracted considerable attention in the past few years. Among others, the problem of change-point detection has been considered in (Peel and Clauset 2015) where the authors introduce a generalized hierarchical random graph model with a Bayesian hypothesis test. Wang et al. (2017) consider hierarchical aspects of the problem. A model based on non-homogeneous Poisson point processes with cluster dependent piecewise constant intensity functions and common discontinuity points is considered by Corneli et al. (2018). The authors of (Corneli et al. 2018) propose a variational expectation maximization algorithm for the change-point localization in this setting. More recently, in (Hewapathirana et al. 2020) the spectral embedding approach is applied to the problem of detection of vertex-based changes in dynamic networks. In (Zhang et al. 2020) the focus is on the online detection of a change in the community structure using a subspace projection procedure based on Gaussian model setting. An eigenspace based statistics is applied in (Cribben and Yu 2017) to the problem of localization of changes in the community structure for stochastic block model. These works focus mainly on the computational aspects of the problem without theoretical justifications.

In (Wang et al. 2014) the authors introduce two types of scan statistics for change-point detection for time-varying stochastic block model. Although the authors of (Wang et al. 2014) provide an asymptotic analysis of the test power for a simple situation of change in a single community connection probability, the testing procedure lacks a threshold allowing for testing at a given significance level. The problem of multiple change-point localization is studied by Zhao et al. (2019). The change-point algorithm proposed in this paper is built upon a refined network estimation procedure. The authors of (Zhao et al. 2019) show consistency of this procedure. A very complete comparison of different change point

localization algorithms (including the one introduced in the present paper) is given in [Sulem et al. \(2022\)](#).

A different line of work describes a general nonparametric approach to change-point detection for a general data sequence ([Chu and Chen 2019](#), [Chen and Zhang 2015](#), [Wang and Samworth 2018](#), [Pilliat et al. 2023](#)). For example, a graph-based non-parametric testing procedure is proposed in ([Chen and Zhang 2015](#)) and is shown to attain a pre-specified level of type I error. Another related problem is anomaly detection in dynamic networks. Here the task is to detect abrupt deviation of the network from its normal behavior. A comprehensive survey on this topic is given in ([Akoglu et al. 2015](#)).

To the best of our knowledge, none of the existing works provides the minimax separation rate for the change-point detection problem in dynamic network. We can only find some partial answers. In particular, the results of ([Ghoshdastidar et al. 2020](#)) can be applied to the case of a known location of the change-point. [Ghoshdastidar et al.](#) obtain the minimax separation rate for the two sample test and consider separation in spectral and Frobenius norms. In the present paper we are interested in a more general setting where no prior information about the existence or location of change-point is given, which is more realistic and applicable to real data. The problem of two sample test is also considered in ([Chen et al. 2020](#)) where the authors propose a test statistic similar to the one introduced in ([Ghoshdastidar et al. 2020](#)) which is based on the singular value of a generalized Wigner matrix. They apply this approach to the problem of change-point localization and derive consistency results.

We provide the minimax separation rate for the spectral norm separation (in [Section 3.1](#) we explain why we think that the spectral norm is an appropriate choice for this problem) in the case of an unknown change-point location with missing links. Besides the lower bound on the minimax separation rate, we also provide a test procedure based on the spectral norm of the matrix CUSUM statistics that is nearly minimax optimal. Our focus is on the challenging case when the networks are only partially observed. Missing values are a very common problem in the real life data. Usual imputation methods require observations from a homogeneous distribution. In the presence of unknown change-points such methods are not expected to perform well. This, in turn, will impact the performances of the change-point detection and estimation methods. These methods, in their large majority, are designed for the case of complete observations and we fall into a vicious circle. The only works considering missing values in the context of change-point detection that we are aware of are ([Londschien et al. 2021](#)), ([Xie et al. 2013](#)), and ([Follain et al. 2022](#)). In ([Londschien et al. 2021](#)) the authors consider the problem of multiple change-point localization for graphical models. [Xie et al.](#) propose a fast method for online tracking of a dynamic submanifold underlying very high-dimensional noisy data. ([Follain et al. 2022](#)) considers change-point estimation in partially-observed, high-dimensional time series.

We also propose a new procedure for *change-point estimation*. The problem of a single change-point estimation in a network generated by a dynamic stochastic block model has been considered, among others, in ([Bhattacharjee et al. 2020](#), [Wang et al. 2021](#), [Yu et al. 2021](#)). [Bhattacharjee et al.](#) establish the rate of convergence for the least squares estimate of the change-point and of the parameters of the model. They also derive the asymptotic distribution of the change-point estimator. The problem of multiple change-points localization has been studied by [Wang et al. \(2021\)](#). The authors of ([Wang et al. 2021](#)) provide optimal localization rate in the case when the magnitudes of the changes in the data gener-

ating distribution is measured using Frobenius norm. More recently the methods of (Wang et al. 2021) have been extended in (Yu et al. 2021) to the problem of online change-point localization. The algorithms proposed in (Yu et al. 2021) and (Wang et al. 2021) require two independent samplings. Unlike these methods, the algorithm that we introduce only requires one independent sample which is a more realistic scenario in many applications.

Usually in works on change-point detection and localization in dynamic networks the set of nodes is assumed to be fixed, e.g. (Ghoshdastidar et al. 2020, Wang et al. 2021) or, in asymptotic setting, the number of nodes is assumed to go to infinity, e.g. (Chen et al. 2020). Both settings may be limiting in some practical situations where the set of network’s nodes may change but not forced to go to infinity. Another common assumption in the existing literature is to suppose that the observed networks are mutually independent, notable exception being (Padilla et al. 2022, Xu et al. 2023). Using a popular graphon model, we propose a different paradigm when the underlying distribution of the network is independent of the number of nodes and networks may be time-dependent. We provide an upper bound condition on the separation rate that guarantees the detection of a change for two common classes of graphons, K -step graphons and Hölder continuous graphons. We prove that our procedure is minimax optimal for K -step graphons (up to a logarithmic factor and dependency on K).

To summarize, the key contributions of the paper are: 1) we obtain sharp (up to numerical constants) minimax detection boundary in spectral norm for the problem of the change-point detection in dynamic networks; 2) we introduce a new and more flexible notion of sparsity (see Section 2.1); 3) we generalize our setting to graphon model which allows us to consider time-dependent networks with a set of nodes that can change; 4) we obtain upper and lower bounds on the minimax separation rate for change-point detection in the step graphon model; 5) we introduce a new procedure for change-point estimation and show its consistency.

The rest of the paper is organized as follows. We start by summarizing the main notation used throughout the paper in Section 1.2. We introduce our model in the case of a fixed set of nodes in Section 2 and, in Section 3, we provide our main results for this case. We consider a more general setting which allows changes in the set of nodes and dependent networks in Section 4. The numerical performance of our method is illustrated in Section 5.

1.2 Notation

We start with some basic notation used in this paper. For any matrix M , we denote by M_{ij} its entry in the i th row and j th column. The notation $\text{diag}(M)$ stands for the diagonal of a square matrix M . The column vector of dimension n with unit entries is denoted by $\mathbf{1}_n = (1, \dots, 1)^T$ and the column vector of dimension n with zero entries is denoted by $\mathbf{0}_n = (0, \dots, 0)^T$. The identity matrix of dimension n is denoted by id_n . For a set A , we denote by $\mathbf{1}_A$ its indicator function.

For two matrices M and N of the same size, their Hadamard (elementwise) product is denoted by $M \odot N$. For any matrix $M \in \mathbb{R}^{n \times n}$, $\|M\|_F$ is its Frobenius norm, $\|M\|_{2 \rightarrow 2}$ is its operator norm. and $\|M\|_\infty = \max_{ij} |M_{ij}|$ is the largest absolute value of its entries. The column-wise $1, \infty$ -norm of M is denoted by $\|M\|_{1, \infty} = \max_j \sum_i |M_{ij}|$. For two sequences a_n and b_n we write $b_n = \Omega(a_n)$ if there exist $C > 0$ and $n_0 \in \mathbb{N}$ such that $b_n \geq Ca_n$ for any $n > n_0$. We define the function $q(t) = \sqrt{t(1-t)}$ for $t \in [0, 1]$ that will control the

impact of the change-point location on the rate. We denote by \mathcal{C}_n the set of all symmetric connection probability matrices:

$$\mathcal{C}_n = \left\{ \Theta \in [0, 1]^{n \times n} : \Theta = \Theta^\top \right\}$$

and by \mathcal{C}_n^0 the set of all symmetric connection probability matrices with zero diagonal:

$$\mathcal{C}_n^0 = \left\{ \Theta \in \mathcal{C}_n : \text{diag}(\Theta) = 0 \right\}.$$

2 Modeling dynamic networks with missing links

Assume that we have T consecutive independent observations of a network modeled by the adjacency matrix $A^t \in \{0, 1\}^{n \times n}$ ($1 \leq t \leq T$) of a simple undirected graph G^t with n vertices. The assumption of independent A^t is plausible in several applications, for example, for transportation networks or Internet of Things (IoT) networks. In Section 5.3 we apply our procedure to Transport for London (TfL) Open Data for which we may assume independent A^t . We will relax this assumption in Section 4 considering a more general graphon model.

We assume that at time t the network follows *inhomogeneous random graph model*: for $i < j$, the elements A_{ij}^t of the matrix A^t are independent Bernoulli random variables with the success probability $\Theta_{ij}^t \in [0, 1]$. It means that the edge (i, j) is present in the graph G^t with the probability Θ_{ij}^t . We consider undirected graphs with no loops. Then, the matrix A^t is symmetric and has zero diagonal. The corresponding matrix of connection probabilities is denoted by $\Theta^t \in [0, 1]^{n \times n}$; it is also symmetric with zero diagonal.

Often in practice the dynamic network is only partially observed. In this case, instead of observing $A = \{A^t, 1 \leq t \leq T\}$, we observe a sequence of matrices $Y = \{Y^t, 1 \leq t \leq T\}$ where each matrix Y^t contains the entries of the adjacency matrix A^t that are available at time t . We say that we sample the pair (i, j) at the time moment t , if we observe the presence or absence of the corresponding edge. We denote by Ω^t the sampling matrix such that $\Omega_{ij}^t = 1$ if the pair (i, j) is sampled at the time t , $\Omega_{ij}^t = 0$ otherwise. As the graph is undirected, the sampling matrix is symmetric. Importantly, our methods for change-point detection and estimation do not require the knowledge of the sample matrices Ω^t .

We assume that for each t , the entries Ω_{ij}^t ($1 \leq i < j \leq n$) are independent random variables and that Ω^t and A^t are also independent. We denote the expectation of Ω^t by $\Pi \in \mathbb{R}^{n \times n}$. Then, for any pair (i, j) and any t , $\Omega_{ij}^t \sim \text{Bernoulli}(\Pi_{ij})$ and, for any $i = 1, \dots, n$, we set $\Pi_{ii} = 1$. Note that we can attribute any value to the diagonal $\text{diag}(\Omega^t)$, since observing or not the diagonal elements does not carry any information about the diagonal of A^t that vanishes by definition. We assume that, for any pair (i, j) , we have non-zero probability to observe A_{ij} , that is, $\Pi_{ij} > 0$. For simplicity, we also assume that Π does not depend on t . This means that the probability of observation for each vertex is not changing over the time. This assumption is realistic in many situations, for example, when analysts sub-sample a very large network. Our proofs may be extended to the situation of t -dependent Π at the price of additional technicalities and we choose to avoid it. We will use notation $Y^t \sim \text{IRGML}(\Theta^t, \Pi)$ that stands for a realization of an inhomogeneous random graph with missing links according to connection probability matrix Θ^t and sampling matrix Π .

We can write our observations using following “signal-plus-noise” model:

$$Y^t = \Pi \odot \Theta^t + W^t, \quad 1 \leq t \leq T, \quad (1)$$

where $W^t \in [-1, 1]^{n \times n}$ is the matrix of centered independent Bernoulli random variables W_{ij}^t with the success probability $\Pi_{ij}\Theta_{ij}^t$ and $Y^t \in \{0, 1\}^{n \times n}$ is the matrix with the elements $Y_{ij}^t = \Omega_{ij}^t A_{ij}^t$.

2.1 Modeling sparse networks

Real-life networks are usually sparse with a number of connections that is much smaller than the maximum possible one, which is proportional to n^2 . This implies that Θ_{ij}^t may change with n and, in particular, $\Theta_{ij}^t \rightarrow 0$ as $n \rightarrow \infty$ for some (or all) (i, j) . Let $\rho_n = \max_t |\Theta^t|_\infty$. In the literature on the sparse network estimation ρ_n is usually called sparsity parameter and it is assumed that $\rho_n \rightarrow 0$ as $n \rightarrow \infty$. In the present paper we consider a more general notion of sparsity based on the column-wise $1, \infty$ -norm of the connection probability matrix.

Let $\kappa_n = \max_t \|\Theta^t\|_{1, \infty}$. We will say that the network is *sparse* if, on average, the degree of each node is much smaller than the maximal possible number of connections in the network, that is, we assume that $\kappa_n/n \rightarrow 0$ as $n \rightarrow \infty$. In particular, we have $\kappa_n \leq \rho_n n$. We will work with the set of all symmetric, zero diagonal connection probability matrices of sparsity at most κ_n :

$$\mathcal{M}_n(\kappa_n) = \left\{ \Theta \in \mathcal{C}_n^0 : \|\Theta\|_{1, \infty} \leq \kappa_n \right\}. \quad (2)$$

This new notion of sparsity is more general than the one based on the sup norm and has multiple advantages. First of all, this notion of sparsity allows each Θ_{ij}^t to decay at its own rate (or to be constant for some of them) which is a much more realistic scenario. Moreover, the methods for sparse network estimation often require the knowledge of the sparsity parameter ρ_n (see, e.g., (Klopp et al. 2017, Gao et al. 2016)) whose estimation is a tricky problem. In contrast to ρ_n , the parameter κ_n can be estimated using the observed degrees.

In the case of model with missing links, we will need an additional parameter ω_n , an upper bound on the mean of the observed degree of each node, $\max_{1 \leq t \leq T} \|\Pi \odot \Theta^t\|_{1, \infty} \leq \omega_n$. In the particular case of uniform sampling with probability p , we have $\omega_n = p\kappa_n$ and $\omega_n \leq np\rho_n$. In the case of missing links, we will work with the pairs of matrices (Θ^t, Π) with the sparsity level bounded by ω_n and consider the following set:

$$\mathcal{S}_n(\omega_n) = \{(\Theta, \Pi) \in \mathcal{C}_n^0 \times \mathcal{C}_n : \|\Pi \odot \Theta\|_{1, \infty} \leq \omega_n\}. \quad (3)$$

3 Change-point detection problem

We suppose that the connection probability matrix Θ^t might change at some location $\tau \in \{1, \dots, T-1\}$:

$$\Theta^t = \Theta^0 \mathbf{1}_{\{1 \leq t \leq \tau\}} + (\Theta^0 + \Delta\Theta^\tau) \mathbf{1}_{\{\tau+1 \leq t \leq T\}}, \quad t = 1, \dots, T. \quad (4)$$

Here Θ^0 is the connection probability matrix before the change and $\Delta\Theta^\tau \in [-1, 1]^{n \times n}$ is a symmetric jump matrix of a change that occurs at time τ . If Θ^t does not change, then $\Delta\Theta^\tau = 0$ for all $1 \leq \tau \leq T - 1$.

We consider the problem of testing whether there is a change in Θ^t at some (possibly unknown) point $\tau \in \mathcal{D}_T \subset \{1, \dots, T - 1\}$. Depending on the set of possible change-point positions \mathcal{D}_T , we can formulate two different testing problems: problem (P1) of testing the presence of a change at a given point τ with $\mathcal{D}_T = \{\tau\}$ and problem (P2) of testing the change at an unknown location within the set of all possible change-point locations $\mathcal{D}_T = \{1, \dots, T - 1\}$.

The difficulty of assessing the existence of a change-point can be quantified by what is called the change-point energy. It is defined as the product of the operator norm of the jump in the parameter matrix and the function $q(t) = \sqrt{t(1-t)}$ for $t \in [0, 1]$. The function $q(t)$ quantifies the impact of change-point location on the difficulty of detecting the change. Thus, we write the detection problem in terms of $q(\tau/T)\|\Pi \odot \Delta\Theta^\tau\|_{2 \rightarrow 2}$, *the change-point energy*. In the case of full observations the sampling matrix is given by $\Pi = \mathbf{1}_n \mathbf{1}_n^T$ and the energy equal to $q(\tau/T)\|\Delta\Theta^\tau\|_{2 \rightarrow 2}$.

To formulate the hypothesis testing problem we define the set of pairs of matrices before and after the change at some location τ with the jump energy at least $r > 0$:

$$\mathcal{W}_{n,T}^\tau(\omega_n, r) = \left\{ ((\Theta^a, \Pi), (\Theta^b, \Pi)) \in \mathcal{S}_n^{\otimes 2}(\omega_n) : q(\tau/T)\|\Pi \odot (\Theta^a - \Theta^b)\|_{2 \rightarrow 2} \geq r \right\}. \quad (5)$$

Let $\mathcal{W}_n(\omega_n, 0)$ denote the set without a jump:

$$\mathcal{W}_n(\omega_n, 0) = \left\{ \left\{ (\Theta^a, \Pi), (\Theta^b, \Pi) \right\} \in \mathcal{S}_n^{\otimes 2}(\omega_n) : \Theta^a = \Theta^b \right\}.$$

Then, the detection problem can be written as testing whether the jump $\Delta\Theta^\tau$ is zero under the null,

$$H_0 : \left\{ (\Theta^0 + \Delta\Theta^\tau, \Pi), (\Theta^0, \Pi) \right\} \in \mathcal{W}_n(\omega_n, 0), \text{ for all } \tau \in \mathcal{D}_T \quad (6)$$

against the alternative hypothesis of a change in Θ^t

$$H_1 : \left\{ (\Theta^0 + \Delta\Theta^\tau, \Pi), (\Theta^0, \Pi) \right\} \in \mathcal{W}_{n,T}^\tau(\omega_n, \mathcal{R}_{n,\mathcal{D}_T}) \text{ for some } \tau \in \mathcal{D}_T. \quad (7)$$

It is well known (see, e.g., (Ingster and Suslina 2003)) that the performance of a test depends on how close the sets of measures under the null and under the alternative hypotheses are. As a consequence, to determine the smallest possible distance between the null and the alternative hypothesis is a crucial question in minimax hypothesis testing. This question is formulated in terms of the radius $\mathcal{R}_{n,\mathcal{D}_T}$, the minimal amount of energy that guarantees the change-point detection. In this paper, we are interested in conditions on the minimal energy that separated a detectable change from an undetectable one.

3.1 Matrix CUSUM statistic

We start by introducing our test procedure. We observe the dynamic network $Y = \{Y^t, 1 \leq t \leq T\}$ defined in (1). We call *a test* (or decision rule) any measurable binary function $\psi_{n,T} : Y \rightarrow \{0, 1\}$ of the data. If its value is equal to 1, we reject the null hypothesis and say that there is a change in Θ^t . Otherwise we say that the dynamic network has no change in the connection probability matrix.

Recall that $Y^t \in \{0, 1\}^{n \times n}$ with the elements $Y_{ij}^t = \Omega_{ij}^t A_{ij}^t$, where Ω^t is the sampling matrix with mean Π (if there is no missing data, then $\Pi = \mathbf{1}_n \mathbf{1}_n^T$ is the matrix of unit entries). Define the following matrix process

$$Z_T(t) = \sqrt{\frac{t(T-t)}{T}} \left(\frac{1}{t} \sum_{s=1}^t Y^s - \frac{1}{T-t} \sum_{s=t+1}^T Y^s \right), \quad t = 1, \dots, T-1. \quad (8)$$

This process measures the difference between the average number of connections before and after the point t . Intuitively, if there is a change in some entries of the parameter matrix Θ^t at time τ , then, with high probability, the value of the process Z_T at these entries will be maximal in the neighborhood of τ . We call the process given by (8) Matrix CUSUM (Matrix Cumulative Sum) process since it is related to the cumulative sums of Y as

$$Z_T(t) = \sqrt{\frac{T}{t(T-t)}} \left[\sum_{s=1}^t Y^s - \frac{t}{T} \sum_{s=1}^T Y^s \right].$$

We can write our model (1) in the equivalent form

$$Z_T(t) = -\mu_T^\tau(t) \Pi \odot \Delta \Theta^\tau + \xi(t), \quad t = 1, \dots, T-1, \quad (9)$$

where

$$\mu_T^\tau(t) = \sqrt{\frac{t(T-t)}{T}} \left(\frac{\tau}{t} \mathbf{1}_{\{\tau+1 \leq t \leq T\}} + \frac{T-\tau}{T-t} \mathbf{1}_{\{1 \leq t \leq \tau\}} \right) \quad (10)$$

and the random matrices

$$\xi(t) = \sqrt{\frac{t(T-t)}{T}} \left(\frac{1}{t} \sum_{s=1}^t W^s - \frac{1}{T-t} \sum_{s=t+1}^T W^s \right) \quad (11)$$

are centered. The function $\mu_T^\tau(t)$ attains its maximum equal to $\sqrt{T}q(\tau/T)$ at the true change-point $t = \tau$. Thus, for any matrix norm, we have

$$\max_{1 \leq t \leq T-1} \|\mathbb{E}(Z_T(t))\| = \sqrt{T}q(\tau/T) \|\Pi \odot \Delta \Theta^\tau\|.$$

We need to ensure that the norm of the jump in the parameter matrix is larger than the norm of the noise term $\xi(t)$. We will use the test statistics based on the operator norm of $Z_T(t)$ since we can control the operator norm of the noise term using the matrix Bernstein inequality. On the other hand, we can easily see that the Frobenius norm is not a suitable choice here. Indeed, assuming that for any t , $\Theta_{ij}^t \approx \rho_n \rightarrow 0$, we get

$$\frac{\mathbb{E}\|\xi(t)\|_F^2}{n^2} \approx \rho_n \gg \frac{\|\Delta \Theta^\tau\|_F^2}{n^2} \approx \rho_n^2.$$

Thus our detection procedure is the following one: if the operator norm of the Matrix CUSUM statistic is sufficiently large at some point $t \in \mathcal{D}_T$, we conclude that there is a change in the connection matrix Θ^t of the network.

3.2 Definitions from the minimax testing theory

In this section, we recall some basic definitions from the minimax testing theory. Let $Y = \{Y_t, 1 \leq t \leq T\}$. Denote by $P_{(\Theta^0, \Pi)}$ the measure of observations Y under H_0 with $Y^t \sim \text{IRGML}(\Theta^0, \Pi)$ and by $P_{(\Theta^0 + \Delta\Theta^\tau, \Pi), (\Theta^0, \Pi)}$ the measure of Y under H_1 with $Y^t \sim \text{IRGML}(\Theta^t, \Pi)$ for $\Theta^t = \Theta^0 \mathbf{1}_{\{1 \leq t \leq \tau\}} + (\Theta^0 + \Delta\Theta^\tau) \mathbf{1}_{\{\tau < t \leq T\}}$, $\tau \in \mathcal{D}_T$. Let $\psi_{n,T} : Y \rightarrow \{0, 1\}$ be a test for one of the problems (P1) or (P2).

Definition 1. *The type I error of $\psi_{n,T}$ is given by*

$$\alpha(\psi_{n,T}) = \sup_{(\Theta^0, \Pi) \in \mathcal{S}_n(\omega_n)} P_{(\Theta^0, \Pi)} \left\{ \psi_{n,T} = 1 \right\}$$

and the type II error of $\psi_{n,T}$ is defined as

$$\beta(\psi_{n,T}, R_{n, \mathcal{D}_T}) = \sup_{\tau \in \mathcal{D}_T} \left\{ \sup_{\{(\Theta^0 + \Delta\Theta^\tau, \Pi), (\Theta^0, \Pi)\} \in \mathcal{W}_n^\tau(\omega_n, \mathcal{R}_{n, \mathcal{D}_T})} P_{(\Theta^0 + \Delta\Theta^\tau, \Pi), (\Theta^0, \Pi)} \left\{ \psi_{n,T} = 0 \right\} \right\}.$$

Let $\alpha \in (0, 1)$ be a given significance level. Denote by Ψ_α the set of all tests of level at most α :

$$\Psi_\alpha = \{ \psi_{n,T} : \alpha(\psi_{n,T}) \leq \alpha \}.$$

Definition 2. *A test $\psi_{n,T}^* \in \Psi_\alpha$ is called minimax if*

$$\beta(\psi_{n,T}^*, R_{n, \mathcal{D}_T}) = \inf_{\psi_{n,T} \in \Psi_\alpha} \beta(\psi_{n,T}, R_{n, \mathcal{D}_T}).$$

It is known (see (Ingster and Suslina 2003), Theorem 2.1, p. 55) that, for any $R_{n, \mathcal{D}_T} > 0$, the minimax test $\psi_{n,T}^*$ exists and

$$\beta(\psi_{n,T}^*, R_{n, \mathcal{D}_T}) = \inf_{\psi_{n,T} \in \Psi_\alpha} \beta(\psi_{n,T}, R_{n, \mathcal{D}_T}) \geq 1 - \alpha - \frac{1}{2} \inf_{\mathbb{P}_1 \in [\mathcal{P}_1]} \|\mathbb{P}_0 - \mathbb{P}_1\|_1, \quad (12)$$

where \mathbb{P}_0 is the measure of observations Y corresponding to the null hypothesis H_0 and $[\mathcal{P}_1]$ is the convex hull of set of measures $\mathcal{P}_1 = \mathcal{P}_1(\mathcal{R}_{n, \mathcal{D}_T})$ corresponding to the alternatives H_1 , and $\|\cdot\|_1$ is the L_1 -distance. It might happen that the minimax test $\psi_{n,T}^*$ is trivial, i.e. $\beta(\psi_{n,T}^*, R_{n, \mathcal{D}_T}) = 1 - \alpha$, $\forall \alpha \in (0, 1)$. In this case the global risk of testing defined as the sum of two testing errors is equal to 1 and the hypotheses H_0 and H_1 are not distinguishable. This happens if some points of the set of alternatives are too close to the null hypothesis set. To avoid this problem, we remove a ball of radius $\mathcal{R}_{n, \mathcal{D}_T}$ from the set of alternatives H_1 . Therefore, it is of crucial interest to know what are the conditions on the minimum radius $\mathcal{R}_{n, \mathcal{D}_T}$ that guarantee the existence of a non-trivial minimax test. These conditions are formulated in terms of the minimax separation rate.

Definition 3. *Let $\alpha, \beta \in (0, 1)$ be given. Let Ψ_α be the set of all tests ψ of level at most α . We say that the radius $\mathcal{R}_{n, \mathcal{D}_T}^*$ is (α, β) -minimax detection boundary in problems (P1)–(P2) of testing against the alternative $\mathcal{V}_n(\kappa_n, \mathcal{R}_{n, \mathcal{D}_T})$ if*

$$\mathcal{R}_{n, \mathcal{D}_T}^* = \inf_{\psi \in \Psi_\alpha} \mathcal{R}_{n, \mathcal{D}_T}(\alpha, \psi),$$

where

$$\mathcal{R}_{n, \mathcal{D}_T}(\alpha, \psi) = \inf \left\{ \mathcal{R} > 0 : \beta(\psi, \mathcal{R}) \leq \beta \right\}.$$

The minimax detection boundary is often written as the product $\mathcal{R}_{n,\mathcal{D}_T}^* = C\varphi_{n,\mathcal{D}_T}$, where $\varphi_{n,\mathcal{D}_T}$ is called *minimax separation rate* and C is a constant independent of n and T . We say that the radius $\mathcal{R}_{n,\mathcal{D}_T}$ satisfies the upper bound condition if there exists a constant $C^* > 0$ and a test $\psi_{n,T}^* \in \Psi_\alpha$ such that $\forall C > C^* \beta(\psi_{n,T}^*, \mathcal{R}_{n,\mathcal{D}_T}) \leq \beta$. We say that $\mathcal{R}_{n,\mathcal{D}_T}$ satisfies the lower bound condition if for any $0 < C \leq C_*$ there is no test of level α with type II error smaller than β . Our goal is to find the minimax separation rate $\varphi_{n,\mathcal{D}_T}$ and two constants C_* and C^* such that

$$C_*\varphi_{n,\mathcal{D}_T} \leq \mathcal{R}_{n,\mathcal{D}_T}^* \leq C^*\varphi_{n,\mathcal{D}_T}.$$

3.3 Change-point detection at a given location

We first consider problem (P1) of testing the presence of a change at some given point τ . Using the Matrix CUSUM statistic, we define the following decision rule

$$\psi_{n,T}^\tau(Y) = \mathbf{1}_{\{\|Z_T(\tau)\|_{2 \rightarrow 2} > H_{\alpha,n}\}}, \quad (13)$$

where, given the significance level $\alpha \in (0, 1)$, the threshold is

$$H_{\alpha,n} = 2\sqrt{2}(1 + \varepsilon)\sqrt{\omega_n} + C_\varepsilon \log\left(\frac{2n}{\alpha}\right). \quad (14)$$

Here $\varepsilon \in (0, 1/2]$ and C_ε is an absolute constant depending on ε , see Lemma 12. The result below provides the upper detection boundary.

Theorem 1. *Let $\alpha, \beta \in (0, 1)$ be given significance levels. Assume that for some universal constant C depending only on $\varepsilon \in (0, 1/2]$,*

$$\mathcal{R}_{n,\tau} \geq 4\sqrt{2}(1 + \varepsilon) \left(\frac{\omega_n}{T}\right)^{1/2} + \frac{C}{T^{1/2}} \log\left(\frac{n}{\alpha \wedge \beta}\right) \quad (15)$$

Then, for the test $\psi_{n,T}^\tau$ defined in (13) with threshold (14) we have

$$\alpha(\psi_{n,T}^\tau) \leq \alpha \quad \text{and} \quad \beta(\psi_{n,T}^\tau, R_{n,\tau}) \leq \beta.$$

Proof. The proof of this theorem is based on Lemmas 3 and 4. Lemma 3 implies that $\alpha(\psi_{n,T}) \leq \alpha$. By Lemma 4, the type II error smaller than β is guaranteed if $R_{n,\tau}$ satisfies (32). Condition (32) follows from (15) for a sufficiently large constant C . \square

The key point of this proof is to find a bound on the spectral norm of the noise term (11) that allows to control the Type I and II errors. This is done using the matrix concentration inequality from (Bandeira and van Handel 2016), see Lemma 12. In the next theorem, we provide the lower bound condition on the energy under which hypotheses H_0 and H_1 are indistinguishable. The result is written in terms of the global testing risk $\eta = \alpha + \beta$ and involves the constant $C_\eta = \log(1 + 4(1 - \eta)^2)$ that will be used throughout the paper.

Theorem 2. *Let $\alpha \in (0, 1)$, $\beta \in (0, 1 - \alpha]$ be given significance levels and $\eta = \alpha + \beta$. Assume that $\omega_n > \sqrt{2C_\eta}$. Then the (α, β) -minimax detection boundary in problem (P1) satisfies the following lower bound*

$$\mathcal{R}_{n,\tau}^* \geq (2C_\eta)^{1/4} \left(\frac{\omega_n}{T}\right)^{1/2}. \quad (16)$$

The proof of Theorem 2 uses the fact that the minimax testing error in deterministic setting is always greater than the one in the Bayesian setting. We reduce our testing problem to the Bayesian hypothesis testing by imposing appropriate prior distributions on the parameters Θ^0 and $\Theta^0 + \Delta\Theta$ under H_0 and H_1 respectively. The prior is built in the following way: under H_0 we have an Erdős–Rényi model with the connection probability ρ_n and, under H_1 , we observe networks with the probability matrix that deviates from the Erdős–Rényi model by a matrix with i.i.d. Rademacher entries. In particular, the choice of priors guarantees that, Θ^0 and $\Theta^0 + \Delta\Theta$ belong to the sets defined in (3) and (5) with high probability. We have to find a condition on the change-point energy that guarantees a non-trivial testing with the type II error at least β . We use (12), the lower bound on the type II error, which is written in terms of the total variation distance between the corresponding random measures that we need to bound from above. Usually, it is hard to bound the total variation distance, and we use the following relationship between the TV-distance and the χ^2 -divergence: $\|P - Q\|_{\text{TV}} \leq \sqrt{\chi^2(P, Q)}$. The key point in the proof is to find an upper bound on the χ^2 -divergence. This upper bound relies on a new technically demanding bound on the Laplace transform of the Rademacher chaos proven in Cortinovis and Kressner (2021). For a Rademacher vector ζ we show that

$$1 + \chi^2(P, Q) \lesssim \mathbb{E} \exp\left(\frac{T\mathcal{R}_{n,T}^2}{n\omega_n} \cdot \frac{1}{2}\zeta^T(\mathbf{1}_n\mathbf{1}_n^T - \text{id}_n)\zeta\right) \lesssim \exp\left(\left(\frac{T\mathcal{R}_{n,T}^2}{\omega_n}\right)^2 \left(1 - \frac{T\mathcal{R}_{n,T}^2}{n\omega_n}\right)^{-1}\right)$$

which implies that the χ^2 -divergence is small if the rate satisfies $\mathcal{R}_{n,T} \asymp (\omega_n/T)^{1/2}$.

Remark 1. Under a mild assumption that the sparsity parameter ω_n is of polylogarithmic order of n , that is, $\omega_n = \Omega(\log^{2+\delta} n)$ for some $\delta > 0$, the minimax detection boundary satisfies

$$(2C_\eta)^{1/4} \left(\frac{\omega_n}{T}\right)^{1/2} \leq \mathcal{R}_{n,\tau}^* \leq 4\sqrt{2} \left(\frac{\omega_n}{T}\right)^{1/2}.$$

Note that the lower and upper bounds match up to a constant with the detection rate $(\omega_n/T)^{1/2}$. Under the polylogarithmic condition the upper bound constant does not depend on the given global testing risk η . On the other hand, the lower bound constant is bounded from above by $(2\log 5)^{1/4} \approx 1.34$.

Remark 2. If we restrict our model to the trivial case of Erdős–Rényi graph model with connection probability ρ_n , it can be shown that the minimax detection boundary is given by

$$C_\eta^{1/2} \left(\frac{\rho_n}{T}\right)^{1/2} \leq \mathcal{R}_{n,\tau}^* \leq \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^{1/2} \left(\frac{\rho_n}{T}\right)^{1/2}$$

and the test

$$\psi_{\alpha,n}^\tau = \mathbf{1}_{\{(n-1)|S_{n,T}^\tau| > \sqrt{\rho_n/\alpha}\}}$$

based on the test statistic estimating the change in the connection probability

$$S_{n,T}^\tau = \sqrt{\frac{\tau(T-\tau)}{T}} \frac{1}{n(n-1)} \left(\frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{i \neq j} Y_{ij}^t - \frac{1}{T-\tau} \sum_{t=\tau+1}^T \sum_{i \neq j} Y_{ij}^t \right)$$

is rate optimal. Note that in this case test (13) is suboptimal.

3.4 Testing for change-point at an unknown location with missing links

In this section, we study problem (P2) of testing the presence of a change at an unknown location. To build a test in the case of unknown change-point location a natural idea is to use a decision rule based on the maximum of the matrix CUSUM statistic over a subset \mathcal{T} of \mathcal{D}_T : $L_{n,T}(Y) = \max_{t \in \mathcal{T}} \|Z_T(t)\|_{2 \rightarrow 2}$. For example, we can take the whole set $\mathcal{T} = \mathcal{D}_T$. However, the choice of the set \mathcal{T} may be optimized by taking a dyadic grid of $\{1, \dots, T-1\}$ (see, for example, (Liu et al. 2021)). Define the dyadic grid on the set $\mathcal{D}_T = \{1, \dots, T-1\}$ as $\mathcal{T} = \mathcal{T}^L \cup \mathcal{T}^R$, where

$$\mathcal{T}^L = \{2^k, k = 0, \dots, \lfloor \log_2(T/2) \rfloor\}, \quad \mathcal{T}^R = \{T - 2^k, k = 0, \dots, \lfloor \log_2(T/2) \rfloor\}. \quad (17)$$

Note that $|\mathcal{T}| = 2 + 2\lfloor \log_2(T/2) \rfloor \leq 2\log_2(T)$.

Our decision rule is based on the maximum of the matrix CUSUM statistic over the dyadic grid:

$$\psi_{n,T}(Y) = \mathbf{1}_{\left\{ \max_{t \in \mathcal{T}} \|Z_T^Y(t)\|_{2 \rightarrow 2} > H_{\alpha,n,T} \right\}} \quad (18)$$

where for some $\varepsilon \in (0, 1/2]$, the threshold is given by

$$H_{\alpha,n,T} = 2\sqrt{2}(1 + \varepsilon)\sqrt{\omega_n} + C_\varepsilon \log\left(\frac{4n \log_2(T)}{\alpha}\right). \quad (19)$$

The parameter ω_n in (19) is an upper bound on the mean of the observed degree of each node. Note that, using Kolmogorov's law of large numbers, it can be estimated by the maximum observed degree $\hat{\omega}_n = \max_{1 \leq t \leq T} \|Y^t\|_{1,\infty}$. The following theorem provides an upper detection condition for our test.

Theorem 3. *Let $\alpha, \beta \in (0, 1)$ be given significance levels. Suppose that for some $\varepsilon \in (0, 1/2]$ and for a constant C depending only on ε ,*

$$\mathcal{R}_{n,\mathcal{D}_T} \geq 4(1 + \varepsilon)\left(\frac{6\omega_n}{T}\right)^{1/2} + \frac{C}{\sqrt{T}} \log\left(\frac{4n \log_2(T)}{\alpha \wedge \beta}\right) \quad (20)$$

Then, for the test defined in (18)–(19), we have $\alpha(\psi_{n,T}) \leq \alpha$ and $\beta(\psi_{n,T}, \mathcal{R}_{n,\mathcal{D}_T}) \leq \beta$.

Proof. The proof follows from Lemmas 5 and 6. □

Remark 3. *As in the case of testing at a given change-point location, assume that $\omega_n = \Omega(\log^{2+\delta} n)$ for some $\delta > 0$. Then, if $n \geq \log T$, the minimax detection boundary satisfies*

$$(2C_\eta)^{1/4} \left(\frac{\omega_n}{T}\right)^{1/2} \leq \mathcal{R}_{n,\mathcal{D}_T}^* \leq 4\sqrt{6} \left(\frac{\omega_n}{T}\right)^{1/2}$$

and our test is minimax rate-optimal in the case of unknown change-point as well. Note that Theorems 1 and 3 imply that the unknown location of the change point may have an impact on the detection boundary only if $\omega_n \lesssim (\log n + \log \log T)^2$, that is, in the case of a very sparse network or a very large number of observations T .

Theorems 3 and 2 provide the minimax separation rate in terms of the operator norm of the Hadamard product of $\Delta\Theta^\tau$ and Π . Let us first consider the case of the uniform sampling with $\Pi_{ij} = p_n$ for all (i, j) such that $i \neq j$. In this case we have $\|\Pi \odot \Delta\Theta^\tau\|_{2 \rightarrow 2} = p_n \|\Delta\Theta^\tau\|_{2 \rightarrow 2}$ and we get the following result that provides the minimax separation rate for the problem of testing for a change at an unknown location in a dynamic sparse network with links missing uniformly at random.

Corollary 1 (Minimax separation rate for spectral norm separation). *Consider problem (6)–(7) with $\omega_n = \Omega(\log^{2+\delta} n)$ for some $\delta > 0$. Assume that links are missing uniformly at random with probability p_n and that conditions of Theorems 2 and 3 are satisfied. Then, for $n \geq \log T$, the minimax separation rate $\varphi_{n, \mathcal{D}_T}$ for spectral norm separation satisfies:*

$$\frac{(2C_\eta)^{1/4}}{p_n} \left(\frac{\kappa_n}{T}\right)^{1/2} \leq \varphi_{n, \mathcal{D}_T} \leq \frac{4\sqrt{6}}{p_n} \left(\frac{\kappa_n}{T}\right)^{1/2}. \quad (21)$$

For more general sampling schemes, we can quantify how the missing observations affect our detection problem by the following "distortion" parameter:

$$\delta_n := \delta_n(\Pi, \Theta) = \frac{\min_{ij} \Pi_{ij}}{\sqrt{r \vee 1}} \quad \text{where } r = \text{rank}(\Pi \odot \Theta). \quad (22)$$

Using Lemma 13, we have that

$$\|\Pi \odot \Delta\Theta\|_{2 \rightarrow 2} \geq \delta_n \|\Delta\Theta\|_{2 \rightarrow 2}.$$

Now, Theorem 3 implies that, for any fixed $\alpha, \beta \in (0, 1)$, $\omega_n = \Omega(\log^{2+\delta} n)$ and $n \geq \log T$, the risk of our test is bounded by $\eta = \alpha + \beta$ if the spectral norm of the jump matrix is sufficiently large:

$$\|\Delta\Theta^\tau\|_{2 \rightarrow 2} \geq \frac{4}{\delta_n} \sqrt{\frac{6\omega_n}{Tq(\tau/T)}}.$$

Remark 4. *Note that using Lemma 13 we can improve (22) and replace $r = \text{rank}(\Pi \odot \Theta)$ by*

$$r^* = \min_{M=(M_{ij}): M_{ij}=(\Pi \odot \Theta)_{ij} \text{ for } i \neq j} \text{rank}(M).$$

That is, we can modify the diagonal of $\Pi \odot \Theta$ to get a matrix of a smaller rank as, for example, in the case when both Π and Θ have a community structure.

Example. Let us consider a network with two communities and missing communication. We have two communities with k and $n - k$ nodes such that the links between the members of each community are fully observed. However, the links between the members of two different communities can be missing and are observed with the rate p_n . Then, after a suitable permutation, Π has the following block structure: $\Pi_{ij} = 1$ if $1 \leq i, j \leq k$ and $k + 1 \leq i, j \leq n$; otherwise $\Pi_{ij} = p_n$.

If the whole community changes its connection pattern, then $\Delta\Theta$ also follows the Stochastic Block Model with the same community structure as Π and we have

$$\|\Pi \odot \Delta\Theta\|_{2 \rightarrow 2} \geq \frac{p_n}{\sqrt{2}} \|\Delta\Theta\|_{2 \rightarrow 2}.$$

In this case, under the conditions of Corollary 1, the minimax separation rate $\varphi_{n, \mathcal{D}_T}$ satisfies

$$\frac{(2C_\eta)^{1/4}}{p_n} \left(\frac{\kappa_n}{T} \right)^{1/2} \leq \varphi_{n, \mathcal{D}_T} \leq \frac{8}{p_n} \left(\frac{3\kappa_n}{T} \right)^{1/2}.$$

3.5 Change-point localization

In this section, we turn to the twin problem of change-point localization which amounts to estimating the position of the change-point τ . That is, we seek for an estimator of τ , $\hat{\tau}_n$, such that $|\hat{\tau}_n - \tau| \leq \varepsilon$ with high probability. The ratio ε/T is usually called *localization rate* of an estimator and the estimator is deemed consistent if its localization rate vanishes as $T \rightarrow \infty$. As in the case of testing, our estimator is based on the operator norm of the Matrix CUSUM statistic and it is defined as follows:

$$\hat{\tau}_n \in \arg \max_{1 \leq t \leq T-1} \|Z_T(t)\|_{2 \rightarrow 2}. \quad (23)$$

Let $\Delta = \|\Pi \odot \Delta\Theta\|_{2 \rightarrow 2}$ denote the magnitude of the change in the matrix of connection probabilities. For presenting the results, we scale all the time points to the $[0, 1]$ interval, by dividing them by T . Let $x^* = \tau/T$ and $\hat{x} = \hat{\tau}_n/T$. The next proposition shows that our estimator is consistent.

Proposition 2. *Let $\gamma \in (0, 1)$. For any $\varepsilon \in (0, 1/2]$, with probability larger than $1 - \gamma$, the estimated change-point $\hat{x} = \hat{\tau}_n/T$ returned by (23) satisfies*

$$|\hat{x} - x^*| \leq \frac{6\sqrt{2}(1 + \varepsilon)}{\Delta q(x^*)} \left(\frac{\omega_n}{T} \right)^{1/2} + \frac{C}{\Delta q(x^*)} \frac{\log(2nT/\gamma)}{\sqrt{T}},$$

where C is a universal constant depending only on ε .

Remark 5. *Note that the minimax lower bound on the localization error obtained in (Wang et al. 2021) (without missing links, that is, when $\omega_n = \kappa_n$) implies that if $\Delta \leq \frac{\sqrt{\kappa_n}}{\sqrt{33T}q(x^*)}$, then no consistent change-point estimator can exist.*

4 Testing the change in the sparse graphon model

In this section, we generalize our model to the case when the set of network's nodes may change and to the case of dependent networks. Considering graphs with varying number of nodes call for a more general non-parametric model. The idea is to introduce a well-defined "limiting object" independent of the networks size n and such that the observed stochastic networks can be viewed as partial observations of this limiting object. Such objects, called *graphons*, play a central role in the theory of graph limits introduced by Lovász and Szegedy (2006) (see, for example, Lovász (2012), Borgs et al. (2008)). Graphons are symmetric measurable functions $W : [0, 1]^2 \rightarrow [0, 1]$. In the sequel, the space of graphons is denoted by \mathcal{W} .

Assume that we observe a dynamic network A^t ($1 \leq t \leq T$) where each adjacency matrix $A_t = (A_{ij}^t)$ is of size n_t that may change with time. We allow nodes to appear and disappear from our networks and denote by \mathcal{V} the set of all nodes of all networks, that is

$\vartheta \in \mathcal{V}$ if there exists a time moment $1 \leq t \leq T$ such that ϑ is a node of A^t . Let $N = |\mathcal{V}|$ be the cardinality of \mathcal{V} .

We consider the sparse graphon model and assume that each vertex $\vartheta \in \mathcal{V}$ is assigned to an independent random variable ε_ϑ uniformly distributed over $[0, 1]$. Now, the edge (i, j) is independently included in the network with probability $\Theta_{ij}^t = \rho_n W^t(\varepsilon_i, \varepsilon_j)$, where W^t is the graphon which defines the probability distribution of the network at time instant t and ρ_n is the sparsity parameter. Conditionally on $\varepsilon = (\varepsilon_\vartheta)_{\vartheta \in \mathcal{V}}$, we assume that A_{ij}^t are independent Bernoulli random variables with parameter Θ_{ij}^t , $A_{ij}^t \sim \text{Bernoulli}(\Theta_{ij}^t)$. Note that the networks $A^t \in \{0, 1\}^{n \times n}$ ($1 \leq t \leq T$) are dependent through the latent variables ε_i . This is similar to the time dependent network models recently introduced in (Padilla et al. 2022) and (Xu et al. 2023).

We suppose that the function W^t might change at some unknown time moment τ , that is, we assume that

$$W^t(x, y) = W_1(x, y)\mathbf{1}_{\{1 \leq t \leq \tau\}} + W_2(x, y)\mathbf{1}_{\{\tau+1 \leq t \leq T\}}. \quad (24)$$

Additionally, we can assume that the latent variables ε_ϑ may also change at time τ . We will denote the new latent variables by ε'_ϑ (for each ϑ , ε'_ϑ may be depended on ε_ϑ). Given the observations A_t ($1 \leq t \leq T$) of networks following the graphon model, we would like to test whether there is a change in the function W^t at some unknown time moment τ .

Different graphons can be at the origin of the same distribution on the space of graphs. More precisely, two graphons U_1 and U_2 define the same probability distribution on graphs if and only if there exists two measure-preserving maps ϕ_1, ϕ_2 such that, for all $(x, y) \in [0, 1]^2$, we have $U_1(\phi_1(x), \phi_1(y)) = U_2(\phi_2(x), \phi_2(y))$. Thus, we need to consider the quotient space of graphons that defines the same probability distribution on graphs equipped with the following distance:

$$\delta_2^2(U_1, U_2) = \inf_{\iota \in \mathcal{M}} \iint_{[0, 1]^2} |U_1(\iota(x), \iota(y)) - U_2(x, y)|^2 dx dy, \quad (25)$$

where \mathcal{M} is the set of all measure-preserving bijections $\iota : [0, 1] \rightarrow [0, 1]$. Here $\iota \in \mathcal{M}$ accounts for possible labels switching.

Let $W(x, y) : [0, 1]^2 \rightarrow [0, 1]$ be an integrable function. We can define the following operator $W : L_2[0, 1] \rightarrow L_2[0, 1]$:

$$Wf = \int_0^1 W(x, y)f(y) dy$$

and denote by $\|W\|_{2 \rightarrow 2}$ the corresponding operator norm. Now we can define the following operator-norm based distance in the quotient space of graphons:

$$\delta(W^\tau, W^{\tau+1}) = \inf_{\iota \in \mathcal{M}} \|W^\tau(x, y) - W^{\tau+1}(\iota(x), \iota(y))\|_{2 \rightarrow 2}. \quad (26)$$

The problem of detecting a possible change in the graphon function can be written as the following hypotheses testing problem:

$$H_0 : \delta(W^\tau, W^{\tau+1}) = 0 \quad \text{for all } \tau \in \mathcal{D}_T \quad \text{against } H_1 : \delta(W^\tau, W^{\tau+1}) \geq \delta_{n, T} \quad \text{for some } \tau \in \mathcal{D}_T.$$

As in the case of the inhomogeneous random graph model that we considered in Section 3, our test is based on the matrix CUSUM statistics. Let \tilde{A}^s denote the restriction of A^s on the set of nodes that is common to all the networks. Denote by n the size of this set. As we allow for a change in the latent variables ε_{ϑ} we should account for a possible mismatch of the labels. Let π denote a permutation of $\{1, \dots, n\}$. For any matrix $A \in \mathbb{R}^{n \times n}$ denote by $A \circ \pi$ a matrix obtained from A permuting simultaneously its rows and columns. Then, define

$$Z_T^\pi(t) = \frac{1}{t} \sum_{s=1}^t \tilde{A}^s - \frac{1}{T-t} \sum_{s=t+1}^T \tilde{A}^s \circ \pi.$$

Let $\pi^* \in \arg \min_{\pi} \|Z_T^\pi(t)\|_2$ and $Z_T(t) = Z_T^{\pi^*}(t)$. Now, the test is defined as follows:

$$\psi_{n,T}^*(A) = \mathbf{1} \left\{ \max_{t \in \mathcal{T}} \frac{\|Z_T(t)\|_{2 \rightarrow 2}}{H_{\alpha,n,t}^*} > 1 \right\} \quad (27)$$

where \mathcal{T} is the dyadic grid (17) and $H_{\alpha,n,t}^*$ is a threshold which will depend on the class of graphons under consideration. Note that we consider a tricky situation when we allow for a change in the features of nodes. If we consider a simpler setting, assuming that only the graphon function can change but features stay fixed, we can consider a simpler test, without taking a minimum over all possible permutations with the threshold given by (19).

For $i \in \{1, 2\}$, let $\mathcal{W}_i \subset \mathcal{W}$ be two classes of graphons, as, for example, step-function graphons or Hölder-smooth graphons. Define the following set of pairs of graphons separated by the distance at least $\delta_{n,T} > 0$:

$$\mathcal{W}(\delta_{n,T}) = \left\{ \{W_1, W_2\} \in \mathcal{W}_1 \times \mathcal{W}_2 : \delta(W_1, W_2) \geq \delta_{n,T} \right\}.$$

Denote by P_W the measure of observations of the networks A^1, \dots, A^T with the connection probabilities given by $\Theta_{ij}^t = \rho_n W(\varepsilon_i, \varepsilon_j)$. Similarly, given a pair of graphons W_1, W_2 denote by P_{W_1, W_2}^τ the measure of observations A^1, \dots, A^T with a change at τ . Then, the type I and II errors of testing for the graphon change-point detection are defined as follows,

$$\begin{aligned} \alpha(\psi_{n,T}) &= \sup_{W \in \mathcal{W}_0} P_W \left\{ \psi_{n,T} = 1 \right\} \\ \beta(\psi_{n,T}, \delta_{n,T}) &= \sup_{\tau \in \mathcal{D}_T} \sup_{W_1, W_2 \in \mathcal{W}(\delta_{n,T})} P_{W_1, W_2}^\tau \left\{ \psi_{n,T} = 0 \right\}. \end{aligned}$$

Let $\delta_{n,T}^*$ denotes the minimax separation rate for this problem. We consider two particular classes of graphons that have been considered in the literature on sparse graphon estimation: the class of step function graphons and the class of smooth graphons. We start by defining the class of step function graphons with K steps:

Definition 4. \mathcal{W}_K , the collection of K -step graphons, is the subset of graphons $W \in \mathcal{W}$ such that, for some symmetric matrix $Q \in \mathbb{R}^{K \times K}$ and some $\phi : [0, 1] \rightarrow \{1, \dots, K\}$

$$W(x, y) = Q_{\phi(x), \phi(y)} \quad \text{for all } x, y \in [0, 1].$$

Assuming that the underlying graphons belong to \mathcal{W}_{K_i} for some $K_i \leq K$ (we allow for a change in the number of steps) we define the following threshold:

$$H_{\alpha,n,t}^* = 2(1 + \varepsilon)\sqrt{2n\rho_n} + 4n^{3/4}\rho_n\sqrt{2T}q\left(\frac{t}{T}\right)\left((K_1 + K_2)\log n\right)^{1/4} + C_\varepsilon \log\left(\frac{8n\log_2(T)}{\alpha}\right), \quad (28)$$

where $\varepsilon \in (0, 1/2]$ and C_ε is a universal constant depending only on ε .

We have the following upper detection bound in the case of step graphons:

Theorem 4. *For $i = 1, 2$ let $W_i \in \mathcal{W}_{K_i}$ with $K_1, K_2 \leq \frac{n}{\log n}$ and let $\alpha, \beta \in (0, 1)$ be given significance levels such that $\alpha \wedge \beta \geq 8/n$. Let $K_1, K_2 \leq K$. Assume that our observations A^1, \dots, A^T follow graphon model (24) and that, for some $\varepsilon \in (0, 1/2]$*

$$q\left(\frac{\tau}{T}\right)\delta(W_1, W_2) \geq \frac{4\sqrt{6}(1 + \varepsilon)}{\sqrt{n\rho_n T}} + 18q\left(\frac{\tau}{T}\right)\left(\frac{K\log n}{n}\right)^{1/4} + \frac{4C_\varepsilon}{n\rho_n\sqrt{T}} \log\left(\frac{8n\log_2(T)}{\alpha \wedge \beta}\right). \quad (29)$$

Then, for the test defined in (27) with threshold (28), we have that $\alpha(\psi_{n,T}^*) \leq \alpha$ and $\beta(\psi_{n,T}^*, \delta_{n,T}) \leq \beta$.

The detection boundary in (29) has three parts. The one of the order $(K/n)^{1/4}$ is due to the ‘agnostic error’ coming from the variability of latent variables in the graphon model. The second one, $(n\rho_n T)^{-1/2}$, is related to the sampling of the observed dynamic network. The third part comes from the Bonferroni device in order to control the errors at the levels α and β . The next theorem provides a lower bound on the minimax separation rate:

Theorem 5. *Let $\alpha \in (0, 1)$ and $\beta \in (0, 1 - \alpha]$ be given significance levels. Let $\eta = \alpha + \beta$ and $C_\eta = \log(1 + 4(1 - \eta)^2)$. Then the (α, β) -minimax detection boundary for the change-point detection in step graphons model satisfies the following lower bound:*

$$q(\tau/T)\delta_{n,T}^* \geq \left[\left(\frac{8}{3}(1 - \eta)^2\right)^{1/4} \frac{q(\tau/T)}{n^{1/4}} \right] \vee \frac{\sqrt{2}(1 - \eta)e^{-C_\eta/2}}{\sqrt{n\rho_n T}}.$$

The proof of the lower bound consists of two parts. The first one is obtained by bounding the Kullback–Leibler divergence between measures in the case when the node assignment vector changes but the connection probability matrix remains unchanged. We derive a bound similar to the data processing inequality from the information theory which is then reduced to the divergence between two multinomial distributions. The second part is similar to the one considered in Theorem 2 and considers the case of a change in the transition matrix with fixed node assignment. We control the chi-squared divergence between measures under the null and alternative hypotheses. Combined with Theorem 4, this result provides the minimax separation rate for the problem of testing a change in step-function graphon model:

Remark 6. *Assume that the network satisfies the polylogarithmic assumption on the sparsity, $n\rho_n = \Omega(\log^{2+\delta} n)$ for some small $\delta > 0$ and that $n \geq \log T$. Then, Theorems 4 and 5 imply that the minimax separation rate for step graphons satisfies the following upper and lower bounds which are given up to a constant,*

$$\frac{q(\tau/T)}{n^{1/4}} + \frac{1}{\sqrt{n\rho_n T}} \lesssim q(\tau/T)\delta_{n,T}^* \lesssim q(\tau/T)\left(\frac{K\log n}{n}\right)^{1/4} + \frac{1}{\sqrt{n\rho_n T}}.$$

The network sampling rate $(n\rho_n T)^{-1/2}$ is exact (up to a constant) and the agnostic error rate $q(\tau/T)n^{-1/4}$ is optimal up to a log n factor and the number of blocks.

Next, we consider the class of smooth graphons.

Definition 5. For any $\gamma > 0$, $L > 0$, the class of γ -Hölder continuous functions $\Sigma(\gamma, L)$ is the set of all functions $W : [0, 1]^2 \rightarrow [0, 1]$ such that for all $(x', y'), (x, y) \in [0, 1]^2$,

$$|W(x', y') - \mathcal{P}_{\lfloor \gamma \rfloor}((x, y), (x' - x, y' - y))| \leq L \left(|x' - x|^{\gamma - \lfloor \gamma \rfloor} + |y' - y|^{\gamma - \lfloor \gamma \rfloor} \right).$$

where $\lfloor \gamma \rfloor$ is the maximal integer less than γ and the function $(x', y') \mapsto \mathcal{P}_{\lfloor \gamma \rfloor}((x, y), (x' - x, y' - y))$ is the Taylor polynomial of degree $\lfloor \gamma \rfloor$ at point (x, y) .

Assuming that the underlying graphons belong to $\Sigma(\gamma_i, L_i)$ for some (γ_i, L_i) with $\gamma_i \geq \gamma > 0$, we define the following threshold:

$$H_{\alpha, n, t}^* = 4(1 + \varepsilon)n\rho_n\sqrt{2T} + 2n\rho_n\sqrt{T}q\left(\frac{t}{T}\right) \left(\frac{\log n}{n}\right)^{\frac{\gamma \wedge 1}{2}} + C_\varepsilon \log\left(\frac{8n \log_2(T)}{\alpha}\right), \quad (30)$$

where $\varepsilon \in (0, 1/2]$ and C_ε is a universal constant depending only on ε .

The following theorem provides the upper detection condition for Hölder continuous graphons:

Theorem 6. For $i = 1, 2$ let W_i be γ_i -Hölder continuous functions and let $\alpha, \beta \in (0, 1)$ be given significance levels such that $\alpha \wedge \beta \geq 1/n$. Assume that, for some $\varepsilon \in (0, 1/2]$

$$q\left(\frac{\tau}{T}\right)\delta_{n, T}^* \geq \frac{4\sqrt{6}(1 + \varepsilon)}{\sqrt{n\rho_n T}} + 2q\left(\frac{\tau}{T}\right) \left(\frac{\log n}{n}\right)^{\frac{\gamma \wedge 1}{2}} + \frac{4C_\varepsilon}{\rho_n n\sqrt{T}} \log\left(\frac{8n \log_2(T)}{\alpha \wedge \beta}\right). \quad (31)$$

where $\gamma = \min(\gamma_1, \gamma_2)$. Then, for the test defined in (27) with threshold (30), we have $\alpha(\psi_{n, T}^*) \leq \alpha$ and $\beta(\psi_{n, T}^*, \delta_{n, T}^*) \leq \beta$.

5 Numerical experiments

In this section, we provide the study of numerical performance of our method. For each setting we applied three tests: the test $\psi_{n, T}^\tau$ at the given change-point τ defined in (13), the test $\psi_{n, T}$ over the dyadic grid \mathcal{T}^d defined in (18) (we add the point $\lfloor T/2 \rfloor$ to the grid in our simulations) and the test $\psi_{n, T}^{full}(Y)$ based on the maximum over the whole set $\mathcal{D}_T = \{1, \dots, T - 1\}$. Each test is calibrated to the significance level $\alpha = 0.05$.

The test defined in (13) is based on the threshold (14) depending on some given value $\varepsilon \in (0, 1/2]$ and an unknown universal constant C_ε . To overcome this difficulty, we use a slightly different threshold obtained from the matrix Bernstein inequality (see, for example, Theorem 1.4 in (Tropp 2011)). The threshold for the test $\psi_{n, T}^\tau$ is given by

$$\tilde{H}_{\alpha, n, T}^\tau = \frac{1}{3} \frac{\log(n/\alpha)}{\sqrt{T}q(\tau/T)} + \left(\frac{1}{9} \frac{\log^2(n/\alpha)}{Tq^2(\tau/T)} + 2\kappa_n \log(n/\alpha) \right)^{1/2}.$$

For test over the dyadic grid $\psi_{n,T}(Y)$ and the test $\psi_{n,T}^{full}(Y)$ we use the same threshold

$$\tilde{H}_{\alpha,n,T}(t) = \frac{1}{3} \frac{\log(n|\mathcal{T}|/\alpha)}{\sqrt{T}q(t/T)} + \left(\frac{1}{9} \frac{\log^2(n|\mathcal{T}|/\alpha)}{Tq^2(t/T)} + 2\kappa_n \log(n/\alpha) \right)^{1/2}, \quad t \in \mathcal{T}.$$

In order to compare the performance of the tests under different regimes (n, T, τ) , we introduce “energy-to-noise ratio” defined by

$$\text{ENR} := \text{ENR}_{n,T}(\tau/T, \Delta\Theta^\tau) = \frac{q(\tau/T)\|\Delta\Theta^\tau\|_{2 \rightarrow 2}}{\sqrt{\kappa_n/T}}.$$

This ratio provides a numerical upper bound on the minimax testing constant (see Theorem 1). We denote by ENR^τ , ENR^d , ENR^f the minimal detectable ENR for the tests $\psi_{n,T}^\tau$, $\psi_{n,T}$, and $\psi_{n,T}^{full}$, respectively. Here “detectable” means that the average power of the corresponding test is equal to 1 over 100 simulations. Note that the lower bound constant for any test of level α with $\beta = 0$ is equal to $c^* = \log^{1/4}(1 + (1 - \alpha)^2)/(4\sqrt{2})$ (see Theorem 2).

5.1 Results for Stochastic Block Models

In this section, we apply our method to four different scenarios of Stochastic Block Models (SBM). Recall that for an SBM model with K communities and connection probability matrix Q between the communities the matrix of connection probabilities Θ is defined as $\Theta = Z^T Q Z$, where $Z \in \{0, 1\}^{K \times n}$ is the membership matrix. Each row i of the matrix Z contains only zeros except one entry Z_{ij} that is equal to 1 if the node i belongs to the community j . We suppose in these simulations that the membership matrix Z does not change.

Scenario 1: SBM with 2 communities, change in connection probability between communities. We suppose that the network follows the Stochastic Block Model with two balanced communities (block sizes are $\lfloor n/2 \rfloor$ and $n - \lfloor n/2 \rfloor$.) The probabilities of connection between the communities change at some point and are given by the following matrices Q_1 (before the change) and Q_2 (after the change point):

$$Q_1 = \rho_n \begin{pmatrix} 0.6 & 1 \\ 1 & 0.6 \end{pmatrix}, \quad Q_2 = \rho_n \begin{pmatrix} 0.6 & \delta \\ \delta & 0.6 \end{pmatrix}, \quad \delta \in [0, 1].$$

Scenario 2: SBM with 2 communities, change in connection probability within one community. As in the previous scenario, we assume that the network follow the stochastic block model with two balanced communities. In this scenario, the matrices Q_1 (before the change) and Q_2 (after the change point) are defined by

$$Q_1 = \rho_n \begin{pmatrix} 1 & 0.5 \\ 0.5 & 0.6 \end{pmatrix}, \quad Q_2 = \rho_n \begin{pmatrix} \delta & 0.5 \\ 0.5 & 0.6 \end{pmatrix}, \quad \delta \in [0, 1].$$

Scenario 3: SBM with 2 communities, change in connection probability within two communities. Same setting as before, but now connection probabilities inside of both communities change:

$$Q_1 = \rho_n \begin{pmatrix} 1 & 0.2 \\ 0.2 & 1 \end{pmatrix}, \quad Q_2 = \rho_n \begin{pmatrix} \delta & 0.2 \\ 0.2 & \delta \end{pmatrix}, \quad \delta \in [0, 1].$$

Scenario 4: SBM with 3 communities and change in connection probability between communities. We suppose that the network follows the stochastic block model with three balanced communities (the block sizes are $k_1 = k_2 = \lfloor n/3 \rfloor$ and $k_3 = n - k_1 - k_2$). The probabilities of connection between the communities change at some point and are given by the following matrices Q_1 (before the change) and Q_2 (after the change point)

$$Q_1 = \rho_n \begin{pmatrix} 0.6 & 1 & 0.6 \\ 1 & 0.6 & 0.5 \\ 0.6 & 0.5 & 0.6 \end{pmatrix}, \quad Q_2 = \rho_n \begin{pmatrix} 0.6 & 1 - \delta & 0.6 \\ 1 - \delta & 0.6 & 0.5 + \delta \\ 0.6 & 0.5 + \delta & 0.6 \end{pmatrix}, \quad \delta \in [0, 0.5].$$

This scenario was considered in (Yu et al. 2021) with the constant parameter $\delta = 0.5$. In our study we vary δ and report the test power in terms of the change in the energy.

Scenario 5: SBM with 2 communities, change in connection probability within communities. We suppose that the network follows the Stochastic Block Model with two balanced communities (block sizes are $\lfloor n/2 \rfloor$ and $n - \lfloor n/2 \rfloor$.) The probabilities of connection within the communities change at some point and are given by the following matrices Q_1 (before the change) and Q_2 (after the change point):

$$Q_1 = \rho_n \begin{pmatrix} 0.6 & 1 \\ 1 & 0.6 \end{pmatrix}, \quad Q_2 = \rho_n \begin{pmatrix} \delta & 1 \\ 1 & \delta \end{pmatrix}, \quad \delta \in [0, 1].$$

The sparsity ρ_n is set to $n^{-1/2}$. The sparsity parameter κ_n is set to $\kappa_n = \frac{1}{2}n\rho_n(\|Q_1\|_{1,\infty} \vee \|Q_2\|_{1,\infty})$ for Scenarios 1–3, 5 and to $\kappa_n = \frac{1}{3}n\rho_n(\|Q_1\|_{1,\infty} \vee \|Q_2\|_{1,\infty})$ for Scenario 4. In all the scenarios we have $\kappa_n \leq \sqrt{n}$.

5.1.1 Varying n and T

In this part we study the dependency of the energy-to-noise ratio ENR on n and T . We report the results of simulations for five scenarios in Table 1. We see that globally the ENR decreases when the number of observations T increases. Some changes cannot be detected by our tests. For example, for Scenarios 1, 2 and 4 the change-point is undetectable for $T = 20$ and $n = 100$ by any test. It can be explained by the small number of observations T implying the threshold that is systematically greater than the value of the test statistic. Scenario 4 seems to be more difficult than the other ones, it might be, in particular, due to the fact that the allowed changes are within the interval $[0, 0.5]$ that is smaller than in Scenarios 1–3.

The ENR of the test $\psi_{n,T}^\tau$ is always smaller than the ENR of two other tests. Concerning the tests over the dyadic grid and over the whole set of observations, the test $\psi_{n,T}$ outperforms the test $\psi_{n,T}^{full}$ in the majority of parameter settings and scenarios.

5.1.2 Estimating the sparsity

In this section, we study the performance of our tests with the thresholds based on the estimated sparsity parameter $\hat{\kappa}_n$. Taking the maximum of $1, \infty$ -norms $\max_t \max_j A_{.j}^t$ as an estimator will systematically overestimate κ_n . Indeed, since $\mathbb{E} \max_j \xi_j = \sqrt{2 \log n}(1 + o(1))$, for $\xi_j \sim \mathcal{N}(0, 1)$ i.i.d., using the Gaussian approximation for binomial variables we can show

	n	T	ENR $^\tau$	ENR d	ENR f		n	T	ENR $^\tau$	ENR d	ENR f
Scenario 1 $\tau/T = 0.5$	100	20	NA	NA	NA	Scenario 2 $\tau/T = 0.5$	100	20	NA	NA	NA
	100	50	2.1546	2.4397	2.6298		100	50	2.1550	2.4766	2.7018
	100	100	2.0612	2.4197	2.6885		100	100	2.1834	2.5018	2.7292
	100	250	2.0546	2.4089	2.6923		100	250	2.0857	2.5172	2.8049
	150	20	2.1928	NA	NA		150	20	2.2389	NA	NA
	150	50	2.1363	2.4515	2.6266		150	50	2.1812	2.5387	2.7175
	150	100	2.0801	2.4763	2.6745		50	100	2.1239	2.5284	2.7812
	150	250	2.0360	2.4276	2.7408		150	250	2.1588	2.5586	2.7984
Scenario 3 $\tau/T = 0.5$	100	20	2.2098	NA	NA	Scenario 4 $\tau/T = 0.5$	100	50	NA	NA	NA
	100	50	2.1253	2.4495	2.6296		100	100	NA	NA	NA
	100	100	2.0377	2.4452	2.6999		100	150	2.0235	NA	NA
	100	250	2.0137	2.4146	2.7386		100	250	2.0483	2.3748	2.7013
	150	20	2.2528	NA	NA		150	50	NA	NA	NA
	150	50	2.1212	2.4814	2.6815		150	100	NA	NA	NA
	150	100	2.1508	2.4904	2.7168		150	150	2.0664	2.4236	NA
	150	250	2.0583	2.4163	2.7743		150	250	2.0420	2.3713	2.2007

Table 1: The results of simulations for five proposed scenarios for $n = 100$ and 150 .

that $E(\max_j A_j^t) \asymp \kappa_n + \sqrt{2\kappa_n \log n}/n$. To estimate κ_n , we first calculate $A_j^t = \sum_{i=1}^n A_{ij}^t$ for each $t = 1, \dots, T$. Then, we obtain a robust estimator of the sparsity taking the 0.9-level empirical quantile of A_j^t . The final estimator of κ_n maximizes the obtained estimated sparsities $\hat{\kappa}_n^t$ over t : $\hat{\kappa}_n = \max_t Q\left(\left\{\sum_{i=1}^n A_{ij}^t, j = 1, \dots, n\right\}, 0.9\right)$. Here $Q(Z, \alpha)$ denotes the α -level empirical quantile of the sample Z . The relative risk of estimation of κ_n is shown for different values of n and sparsity ρ_n in Table 2. We can see that the choice of the quantile $Q = 0.9$ guarantees $\hat{\kappa}_n \geq \kappa_n$ which is important to maintain the test significance at level smaller than α and it also has a good estimation precision for a large range of graph sparsity regimes.

Sparsity	Q	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 1200$
$\rho_n = n^{-0.1}$	0.7	-0.0334	-0.0352	-0.0407	-0.0294	-0.0212
	0.8	-0.0018	0.0070	-0.0082	-0.0200	-0.0067
	0.9	0.0586	0.0545	0.0230	0.0029	0.0154
$\rho_n = n^{-0.25}$	0.6	0.0239	0.0090	-0.0008	-0.0066	-0.0114
	0.8	0.0350	0.0823	0.0460	0.0310	0.0269
	0.9	0.1342	0.1631	0.1029	0.0746	0.0677
$\rho_n = n^{-0.5}$	0.7	0.1299	0.1412	0.0837	0.0806	0.0850
	0.8	0.2712	0.2189	0.1800	0.1489	0.1486
	0.9	0.4892	0.3719	0.3396	0.2623	0.2577

Table 2: Approximation of the relative risk of estimation $R(\hat{\kappa}_n) = E(\hat{\kappa}_n - \kappa_n)/\kappa_n$ over 100 simulations for $T = 10$, $\tau = T/2$, $n \in \{100, 200, 500, 1000, 1200\}$, three different levels of sparsity ρ_n and the quantiles $Q \in \{0.7, 0.8, 0.9\}$.

In Fig. 1 we compare the performance of the test adaptive to the unknown sparsity level with the test where we use the true value κ_n . We consider Scenario 1 with $n = 100$, $T = 100$ and Scenario 4 with $n = 100$, $T = 250$. For Scenario 1, $\kappa_n \approx 0.8n\rho_n$ and, for Scenario 4, $\kappa_n \approx 0.77n\rho_n$. In our simulations the change-point is located in the middle. For

Scenario 1 with $T = 100$, our estimator $\hat{\kappa}_n$ slightly overestimates the true value $\kappa_n = 7.94$ with the average value $\hat{\kappa}_n = 12.9129$ calculated over 100 simulations and over all values of the parameter δ . For Scenario 4 and $T = 250$ we obtain a better estimation that is equal to $\hat{\kappa}_n = 11.6858$ while the true value $\kappa_n = 7.26$. For Scenario 1, the tests adaptive to the unknown sparsity level behave quite well with a reasonable power and with the ENR that is about 1.25 times greater than the ENR of the corresponding test with known κ_n . For Scenario 4 the proportion between the ENRs is about 1.07 times. Better performances of the adaptive test for Scenario 4 is expected as in this case the change in the sparsity level of the network is less important than in the Scenario 1. Our test construction is based on the upper bound for the sparsity level for all t and naturally gives better results in the case of the change which is more homogeneous in terms of sparsity.

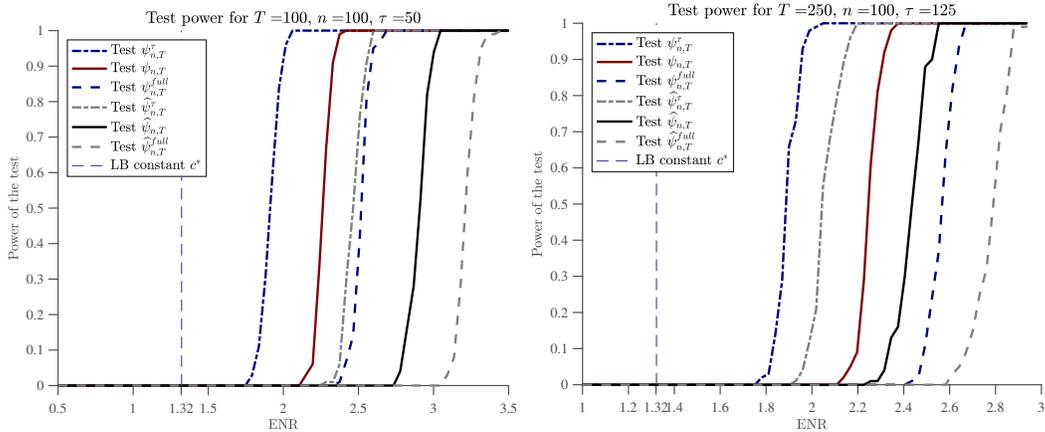


Figure 1: The test powers with known and estimated sparsity parameters for $n = 100$, $\tau/T = 0.5$ and $T = 100$ for Scenario 1 (on the left) and $T = 250$ for Scenario 4 (on the right).

5.1.3 Change-point localization in the case of uniform sampling

We have simulated the networks of size $n = 100$ from Scenario 1 with $T = 100$ and a change-point located at $\tau \in \{5, 25, 50\}$. The networks have missing links generated according to the uniform sampling matrix $\Pi = p_n(\mathbf{1}_n \mathbf{1}_n^t - \text{id}_n)$ with the sampling rate $p_n \in (0, 1]$. We compute the average absolute error of our estimator $\hat{\tau}$ defined in (23) over $N = 100$ simulations normalized by the number of observations T : $R_N(\hat{\tau}, \tau) = (NT)^{-1} \sum_{i=1}^N |\hat{\tau}_i - \tau|$. We present the dependence of this risk on the sampling rate p_n and on the norm of the jump $\Delta\Theta^\tau$.

In Fig. 2 we observe the dependence of the risk on the location τ : the closer τ is to the middle of the interval, the easier the estimation is. The dependence of the rate of convergence of $\hat{\tau}$ on the norm $\|\Pi \odot \Delta\Theta^\tau\|_{2 \rightarrow 2} = p_n \|\Delta\Theta^\tau\|_{2 \rightarrow 2}$ is represented by the level curve $p_n \|\Delta\Theta^\tau\|_{2 \rightarrow 2} \approx \text{const}$ separating the black area corresponding to a low change-point localization error from the light one with the higher error.

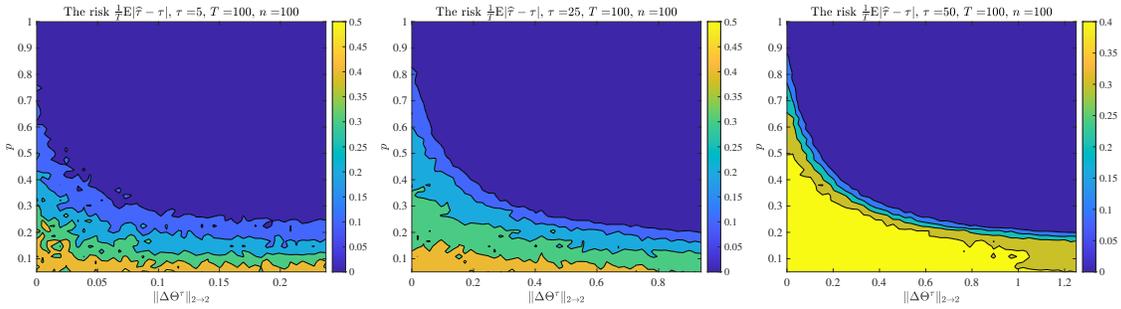


Figure 2: The risk of the change-point estimator under Scenario 1 for $T = 100$, $n = 100$ and the change-points $\tau \in \{5, 25, 50\}$ (left to right). The links are observed at the constant sampling rate $p_n \in (0, 1]$. The estimation is easier when the change-point is located in the middle (the graph to the right).

5.1.4 Change-point localization for non-uniform sampling patterns

We have simulated the networks of size $n = 100$ with the change point in the middle of $T = 100$ observations following three different sampling patterns:

Setting A. Change-point in missing communication. The networks before and after the change follow Scenario 1. The sampling matrix $\Pi = Z^T \tilde{\Pi} Z$ has the same community structure as the networks before and after the change and follows “missing in communication” pattern: $\tilde{\Pi} = \begin{pmatrix} 1 & p \\ p & 1 \end{pmatrix}$. This example was considered in Section 3.4.

Setting B. Change-point in communication, within groups missing values The networks before and after the change follow Scenario 1. The sampling matrix $\Pi = Z^T \tilde{\Pi} Z$ has the same community structure as the networks before and after the change and follows “missing within groups” pattern: $\tilde{\Pi} = \begin{pmatrix} p & 1 \\ 1 & p \end{pmatrix}$.

Setting C. Change-point and missing links withing communities. The networks before and after the change follow Scenario 5. The sampling matrix $\Pi = Z^T \tilde{\Pi} Z$ has the same community structure as the networks before and after the change and follows “missing in communication” pattern: $\tilde{\Pi} = \begin{pmatrix} p & 1 \\ 1 & 1 - p \end{pmatrix}$.

In Fig. 3 we present the results of the simulations that show the dependence of the risk of estimation on the sampling probability p and the jump norm $\|\Delta\Theta\|_{2 \rightarrow 2}$. Under Setting A, the distortion parameter is $\delta_n(\Pi, \Theta) = p/\sqrt{2}$ and we see that the level curve is given by $p\|\Delta\Theta^\tau\|_{2 \rightarrow 2} \approx \text{const}$ since the links between the communities with a change-point are sampled at the uniform rate p . Under Setting B, the missing links do not affect the change-point estimation since $\|\Pi \odot \Delta\Theta\|_{2 \rightarrow 2} = \|\Delta\Theta\|_{2 \rightarrow 2}$, the level curve is constant and the change-point estimation risk is independent of the link sampling. Finally, under Setting C, the distortion parameter is equal to $\delta_n(\Pi, \Theta) = (p \wedge 1 - p)/\sqrt{2}$ and we see a different level curve $(p \wedge 1 - p)\|\Delta\Theta^\tau\|_{2 \rightarrow 2} \approx \text{const}$ with the change in connection probabilities within blocks and links observed with different sampling probabilities p and $1 - p$.

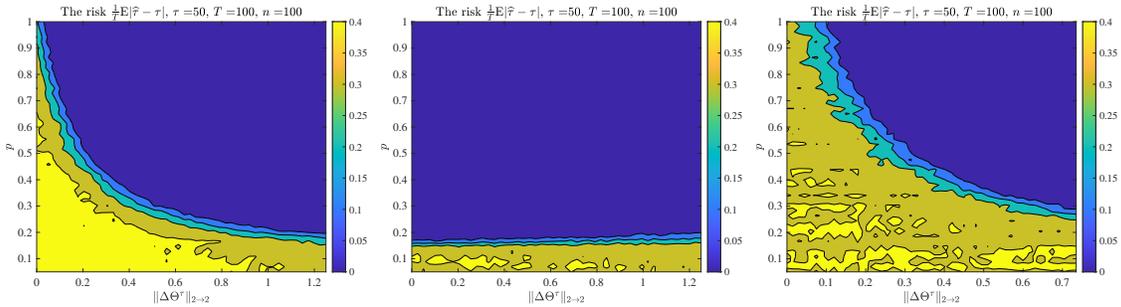


Figure 3: The risk of the change-point estimator under three different patterns of missing links for $T = 100$, $n = 100$ and $\tau = 50$ (Setting A–C, from left to right).

5.2 Results for graphon model

In this section, we simulate a dynamic network from graphon model with the graphon function in Hölder classes with

$$W_1(x, y) = xy \quad \text{before the change and} \quad W_2^\gamma(x, y) = (xy)^\gamma \quad \text{after the change.}$$

Here $\gamma \geq 1$ is the smoothness parameter that defines the impact of the change. We suppose that the assignment vector ε does not change. The sparsity parameter is set to $\rho_n = 1/\sqrt{n}$. The smoothness parameter γ varies from 1 to 5, the change-point is at the middle of the interval, that is $\tau = T/2$, and the number of observations is $T = 100$. The matrix size varies from 50 to 400.

On the left hand side of Fig. 4 we see the dependence of the power of three tests $\psi_{n,T}^\tau$, $\psi_{n,T}$, $\psi_{n,T}^{full}$ on the smoothness parameter γ . The results are similar to those obtained for the SBMs, the test over the dyadic grid outperforms the test over the whole grid and both are less powerful than testing at a given change-point τ . The change in graphons with smoothness $\gamma > 3$ can be detected with power close to 1. The graph on the right hand side of Fig. 4 shows the power of the test $\psi_{n,T}^\tau$ for different sizes of networks $n \in \{50, 100, 200, 300, 400\}$ and for different values of the smoothness γ . We can see that the detection power grows with n that confirms the detection rate $1/\sqrt{n\rho_n T}$. On the other hand, the smaller is the smoothness γ , the harder the detection will be. For example, if $\gamma = 4$, the detection power is 1 starting from $n = 100$ and for $\gamma = 2$, the detection power becomes close to 1 only for $n = 400$.

5.3 Transport for London (TfL) Open Data

In this section, we apply our test to the real data coming from the Transport for London (TfL) Open Data API¹. The data contains information about London Bicycle Sharing Network collected since 2012. The dataset contains the following information: the ID of each bicycle, the ID and name of the origin and the destination trip stations, the journey (rental) starting and ending time and date, and the unique ID and the duration of each trip.

¹Access to the data via <https://api.tfl.gov.uk>

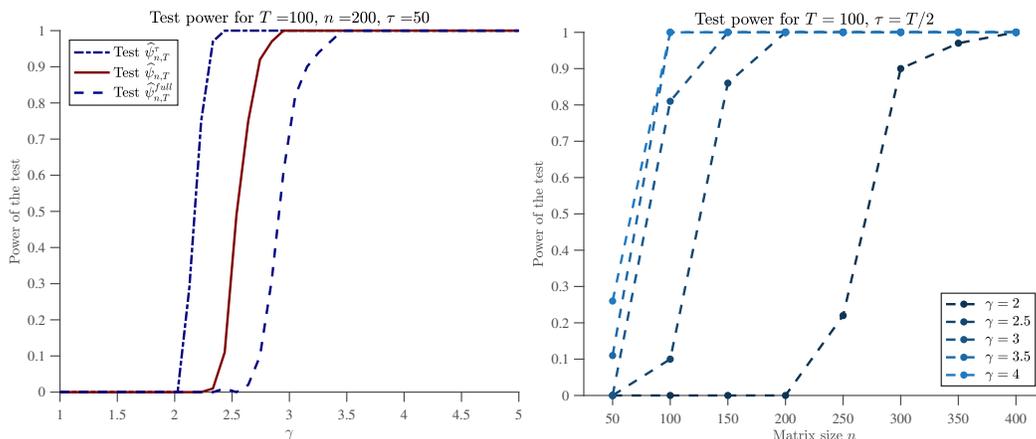


Figure 4: Power of the testing the change in the Hölder class graphons at $\tau = T/2, T = 100$. The graph to the left displays the power for three different test for $n = 200$. The graph to the right shows the power of the test $\hat{\psi}_{n,T}^\tau$ for different values of γ depending on the matrix size n .

We have analyzed the data during the two-month period from June 24, 2012 to August 31, 2012. The summer of 2012 is a remarkable period because of the Games of the XXX Olympiad that was held from July 27 to August 12, 2012 in London. The dynamic network is a sequence of $T = 69$ daily observations. Each observation is a graph with $n = 595$ vertices corresponding to the bike rental stations. We say that two vertices are connected if the minimal trip duration between the corresponding stations is not less than 3 minutes and the number of trips is greater than a predefined threshold. For each day, the threshold on the number of trips is equal to the 0.9975-level empirical quantile of the distribution of the total number of trips between every couple of stations excluding disconnected stations (zero trips during the day). The obtained network has the average sparsity $\bar{\kappa}_n = 43.2319$ (over $T = 69$ observations). The corresponding value of $\rho_n = \kappa_n/n = 0.0727 \asymp n^{-0.4}$.

Fig. 5 on the left shows the graph of the matrix CUSUM statistic calculated over the whole period from June 24, 2012 to August 31, 2012. We can see that the maximum of the statistic is attained at the position corresponding to July 22, 2012. This date corresponds to the day of the arrival of the Olympic Torch to London.² Our test detects this change-point at the significance level $\alpha = 0.05$ and our estimator correctly estimates it. The value of the test statistic is 53.3311, the corresponding threshold is equal to 40.5244.

We see several peaks on the graph of the matrix CUSUM statistics which may imply that actually this data exhibits several change points. One of them corresponds to the end of the Olympics on August 12, 2012. It is possible to combine our test with the segmentation methods for multiple change-point localization (see, for example, SMUCE (Frick et al. 2014), WBS (Fryzlewicz 2014) or the method proposed recently in (Verzelen et al. 2023)).

For example, if we take the data covering the period from July 23 to August, 22 (see

²The details about the traffic perturbation in London on July 22, 2012 can be found at the TfL website: <https://tfl.gov.uk/info-for/media/press-releases/2012/july/olympic-torch-relay-has-arrived-in-london--plan-your-travel-and-get-ahead-of-the-games-tomorrow---sunday-22-july-2012>.

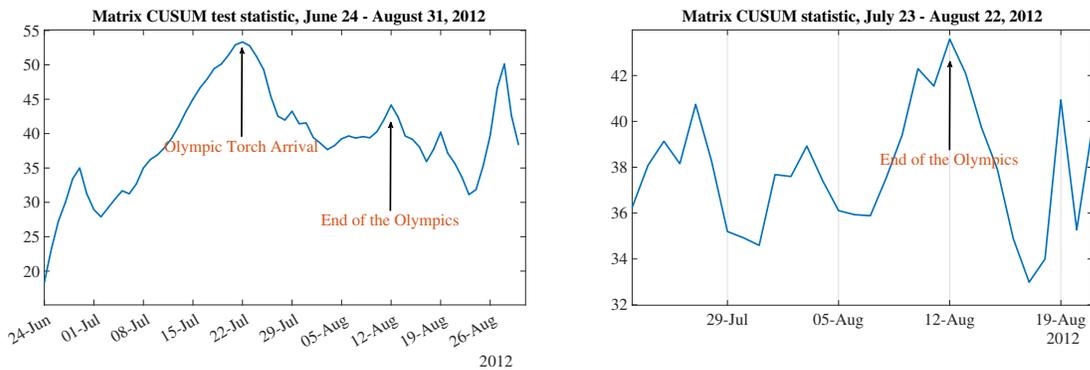


Figure 5: On the left: values of the matrix CUSUM statistic calculated during the whole period of observations. On the right: values of the matrix CUSUM statistic calculated during 31 days from July, 23 to August, 22.

Fig. 5), our test detects the change-point corresponding to the date of the London Olympics closing ceremony on August 12, 2012. Here we observe the network during $T = 31$ day and the values of the test statistic and the corresponding threshold at the level $\alpha = 0.05$ are, respectively, 43.5854 and 41.7997. The estimator estimates correctly the change-point.

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A Upper bound results

In this section, we provide the Lemmas that control the type I and II errors of the tests for problems (P1) and (P2).

A.1 Testing at a given point

Lemma 3. *For any $\alpha \in (0, 1)$ the type I error of test (13) is bounded by α .*

Proof. Recall that $Z_T(t) = -\mu_T(t)\Pi \odot \Delta\Theta^\tau + \xi(t)$, where

$$\xi(t) = \sqrt{\frac{t(T-t)}{T}} \left(\frac{1}{t} \sum_{s=1}^t W^s - \frac{1}{T-t} \sum_{s=t+1}^T W^s \right)$$

and that under the null hypothesis $W^s = (W_{ij}^s) \in [-1, 1]^{n \times n}$ are independent centered Bernoulli matrices with independent entries taking values in $\{1 - \Pi_{ij}\Theta_{ij}^0, -\Pi_{ij}\Theta_{ij}^0\}$ with the success probability $\Pi_{ij}\Theta_{ij}^0$. We have

$$\begin{aligned} \alpha(\psi_{n,T}^\tau) &= \sup_{(\Theta_0, \Pi) \in \mathcal{S}_n(\omega_n)} \mathbb{P}_{(\Theta_0, \Pi)} \left\{ \|Z_T(\tau)\|_{2 \rightarrow 2} > H_{\alpha, n} \right\} \\ &= \sup_{(\Theta_0, \Pi) \in \mathcal{S}_n(\omega_n)} \mathbb{P}_{(\Theta_0, \Pi)} \left\{ \|\xi(\tau)\|_{2 \rightarrow 2} > 2\sqrt{2}(1 + \varepsilon)\sqrt{\omega_n} + C_\varepsilon \log\left(\frac{2n}{\alpha}\right) \right\}. \end{aligned}$$

Using the bound from Lemma 12 with $\|\Pi \odot \Theta^0\|_{1, \infty} \leq \omega_n$ we get $\alpha(\psi_{n,T}) \leq \alpha$. \square

Lemma 4. Let $\alpha, \beta \in (0, 1)$ and $H_{\alpha, n}$ be given by (14). Suppose that

$$R_{n, \tau} \geq 4\sqrt{2}(1 + \varepsilon) \left(\frac{\kappa_n}{T}\right)^{1/2} + \frac{C_\varepsilon}{\sqrt{T}} \left(\log\left(\frac{2n}{\alpha}\right) + \log\left(\frac{2n}{\beta}\right) \right) \quad (32)$$

Then the type II error of test (13) is bounded by β .

Proof. For ease of notation we denote

$$(\Theta, \Delta\Theta^\tau) = \{(\Theta + \Delta\Theta^\tau, \Pi), (\Theta, \Pi)\}.$$

By definition of $\beta(\psi_{n, T}, R_{n, \mathcal{D}_T})$ we have

$$\beta(\psi_{n, T}^\tau, \mathcal{R}_{n, \tau}) = \sup_{(\Theta, \Delta\Theta^\tau) \in \mathcal{W}_{n, T}^\tau(\omega_n, R_n, \mathcal{D}_T)} \mathbb{P}_{(\Theta, \Delta\Theta^\tau)} \{ \|Z_T(\tau)\|_{2 \rightarrow 2} \leq H_{\alpha, n} \}.$$

Using the triangle inequality, (32), the choice of $H_{\alpha, n}$ and the fact that $\mu_T(\tau) = \sqrt{T}q(\tau/T)$ we compute

$$\begin{aligned} \beta(\psi_{n, T}^\tau, R_{n, \tau}) &\leq \sup_{(\Theta, \Delta\Theta^\tau) \in \mathcal{W}_{n, T}^\tau(\omega_n, R_n, \mathcal{D}_T)} \mathbb{P}_{(\Theta, \Delta\Theta^\tau)} \left\{ \|\xi(\tau)\|_{2 \rightarrow 2} > \mu_T(\tau) \|\Delta\Theta^\tau\|_{2 \rightarrow 2} - H_{\alpha, n} \right\} \\ &\leq \sup_{(\Theta, \Delta\Theta^\tau) \in \mathcal{W}_{n, T}^\tau(\omega_n, R_n, \mathcal{D}_T)} \mathbb{P}_{(\Theta, \Delta\Theta^\tau)} \left\{ \|\xi(\tau)\|_{2 \rightarrow 2} > \sqrt{T}R_{n, \tau} - 2\sqrt{2}(1 + \varepsilon)\omega_n^{1/2} - C_\varepsilon \log\left(\frac{2n}{\alpha}\right) \right\} \\ &\leq \sup_{(\Theta, \Delta\Theta^\tau) \in \mathcal{W}_{n, T}^\tau(\omega_n, R_n, \mathcal{D}_T)} \mathbb{P}_{(\Theta, \Delta\Theta^\tau)} \left\{ \|\xi(\tau)\|_{2 \rightarrow 2} > 2\sqrt{2}(1 + \varepsilon)\omega_n^{1/2} + C_\varepsilon \log\left(\frac{2n}{\beta}\right) \right\}. \end{aligned}$$

Applying Lemma 12 and using $\|\Pi \odot \Theta^0\|_{1, \infty} \leq \omega_n$, $\|\Pi \odot (\Theta^0 + \Delta\Theta^\tau)\|_{1, \infty} \leq \omega_n$ we obtain the statement of the lemma. \square

A.2 Testing at an unknown change-point

Lemma 5. For any $\alpha \in (0, 1)$ the type I error of test (18) is less than α .

Proof. Using the union bound we can bound the type I error as follows

$$\begin{aligned} \alpha(\psi_{n, T}) &= \mathbb{P}_{\mathbb{H}_0} \left\{ \max_{t \in \mathcal{T}} \|Z_T^Y(t)\|_{2 \rightarrow 2} > H_{\alpha, n, T} \right\} \\ &\leq \sum_{t \in \mathcal{T}} \sup_{(\Theta^0, \Pi) \in \mathcal{S}_n(\omega_n)} \mathbb{P}_{(\Theta^0, \Pi)} \left\{ \|Z_T^Y(t)\|_{2 \rightarrow 2} > H_{\alpha, n, T} \right\} \\ &= \sum_{t \in \mathcal{T}} \sup_{(\Theta^0, \Pi) \in \mathcal{S}_n(\omega_n)} \mathbb{P}_{(\Theta^0, \Pi)} \left\{ \|\xi(t)\|_{2 \rightarrow 2} > 2\sqrt{2}(1 + \varepsilon)\omega_n^{1/2} + C_\varepsilon \log\left(\frac{4n \log_2(T)}{\alpha}\right) \right\}. \end{aligned}$$

Using Lemma 12 with $\|\Pi_n \odot \Theta^0\|_{1, \infty} \leq \omega_n$, we immediately get for every $t \in \mathcal{T}$

$$\sup_{(\Theta^0, \Pi) \in \mathcal{S}_n(\omega_n)} \mathbb{P}_{(\Theta^0, \Pi)} \left\{ \|\xi(t)\|_{2 \rightarrow 2} > 2\sqrt{2}(1 + \varepsilon)\omega_n^{1/2} + C_\varepsilon \log\left(\frac{4n \log_2(T)}{\alpha}\right) \right\} \leq \frac{\alpha}{2 \log_2(T)}.$$

Using the fact that $|\mathcal{T}| \leq 2 \log_2(T)$, we obtain $\alpha(\psi_{n, T}) \leq \alpha$. \square

Lemma 6. Let $\alpha, \beta \in (0, 1)$ and $H_{\alpha, n, T}$ be given by (19). Suppose that for some $\varepsilon \in (0, 1/2]$

$$R_{n, \mathcal{D}_T} \geq 4\sqrt{6}(1 + \varepsilon) \left(\frac{\omega_n}{T}\right)^{1/2} + \frac{\sqrt{3}C_\varepsilon}{\sqrt{T}} \left(\log \frac{4n \log_2(T)}{\alpha} + \log \frac{2n}{\beta}\right).$$

Then, the type II error of the test $\psi_{n, T}$ is bounded by β .

Proof. For ease of notation we denote

$$(\Theta, \Delta\Theta^\tau) = \{(\Theta + \Delta\Theta^\tau, \Pi), (\Theta, \Pi)\}.$$

By definition of $\beta(\psi_{n, T}, R_{n, \mathcal{D}_T})$ we have

$$\begin{aligned} \beta(\psi_{n, T}, R_{n, \mathcal{D}_T}) &= \sup_{\tau \in \mathcal{D}_T} \sup_{(\Theta, \Delta\Theta^\tau) \in \mathcal{W}_{n, T}^\tau(\omega_n, R_{n, \mathcal{D}_T})} \mathbb{P}_{(\Theta, \Delta\Theta^\tau)} \left\{ \max_{t \in \mathcal{T}} \|Z_T^Y(t)\|_{2 \rightarrow 2} \leq H_{\alpha, n, T} \right\} \\ &\leq \inf_{t \in \mathcal{T}} \sup_{\tau \in \mathcal{D}_T} \sup_{(\Theta, \Delta\Theta^\tau) \in \mathcal{W}_{n, T}^\tau(\omega_n, R_{n, \mathcal{D}_T})} \mathbb{P}_{(\Theta, \Delta\Theta^\tau)} \left\{ \|Z_T^Y(t)\|_{2 \rightarrow 2} \leq H_{\alpha, n, T} \right\} \\ &\leq \inf_{t \in \mathcal{T}} \sup_{\tau \in \mathcal{D}_T} \sup_{(\Theta, \Delta\Theta^\tau) \in \mathcal{W}_{n, T}^\tau(\omega_n, R_{n, \mathcal{D}_T})} \mathbb{P}_{(\Theta, \Delta\Theta^\tau)} \left\{ \|\xi(t)\|_{2 \rightarrow 2} > \mu_T^\tau(t) \|\Pi \odot \Delta\Theta^\tau\|_{2 \rightarrow 2} - H_{\alpha, n, T} \right\}. \end{aligned}$$

If $\tau \leq T/2$, there exists a $t^* \in \mathcal{T}^L$ such that $\tau/2 \leq t^* < \tau$. It is easy to see that

$$\mu_T^\tau(t^*) = \sqrt{\frac{t^*(T - \tau)}{(T - t^*)\tau}} \sqrt{\frac{\tau(T - \tau)}{T}} \geq \sqrt{\frac{\tau/2(T - \tau)}{(T - \tau/2)\tau}} \sqrt{\frac{\tau(T - \tau)}{T}} \geq \frac{1}{\sqrt{3}} \sqrt{T} q(\tau/T),$$

since $\tau < T/2$ iff $(T - \tau)/(2T - \tau) > 1/3$. If $\tau \geq T/2$, noting that $\mu_T^\tau(t) = \mu_T^{T-\tau}(T - t)$, we can reduce the estimation of $\mu_T^\tau(t)$ to the previous case: there exists $T - t' \in \mathcal{T}^R$ such that $(T - \tau)/2 < T - t' < T - \tau$ and $\mu_T^\tau(t') = \mu_T^{T-\tau}(T - t') \geq \frac{1}{\sqrt{3}} \sqrt{T} q(\tau/T)$.

Thus, for any $\tau \in \mathcal{D}_T$ we have the following bound of the type II error

$$\begin{aligned} &\sup_{(\Theta, \Delta\Theta^\tau) \in \mathcal{W}_{n, T}^\tau(\omega_n, R_{n, \mathcal{D}_T})} \mathbb{P}_{(\Theta, \Delta\Theta^\tau)} \left\{ \|\xi(t^*)\|_{2 \rightarrow 2} > \mu_T^\tau(t) \|\Pi \odot \Delta\Theta^\tau\|_{2 \rightarrow 2} - H_{\alpha, n, T} \right\} \\ &\leq \sup_{(\Theta, \Delta\Theta^\tau) \in \mathcal{W}_{n, T}^\tau(\omega_n, R_{n, \mathcal{D}_T})} \mathbb{P}_{(\Theta, \Delta\Theta^\tau)} \left\{ \|\xi(t^*)\|_{2 \rightarrow 2} > \frac{1}{\sqrt{3}} \sqrt{T} q(\tau/T) \|\Pi \odot \Delta\Theta^\tau\|_{2 \rightarrow 2} - H_{\alpha, n, T} \right\} \\ &\leq \sup_{(\Theta, \Delta\Theta^\tau) \in \mathcal{W}_{n, T}^\tau(\omega_n, R_{n, \mathcal{D}_T})} \mathbb{P}_{(\Theta, \Delta\Theta^\tau)} \left\{ \|\xi(t^*)\|_{2 \rightarrow 2} > \frac{1}{\sqrt{3}} \sqrt{T} R_{n, \mathcal{D}_T} - H_{\alpha, n, T} \right\} \\ &\leq \sup_{(\Theta, \Delta\Theta^\tau) \in \mathcal{W}_{n, T}^\tau(\omega_n, R_{n, \mathcal{D}_T})} \mathbb{P}_{(\Theta, \Delta\Theta^\tau)} \left\{ \|\xi(t^*)\|_{2 \rightarrow 2} > 2\sqrt{2}(1 + \varepsilon) \omega_n^{1/2} + C_\varepsilon \log \frac{2n}{\beta} \right\} \leq \beta. \end{aligned}$$

The first inequality follows from the fact that $\mu_T^\tau(t) \geq \sqrt{(T/3)} q(\tau/T)$, the second inequality is a general fact, the third one follows from the definition of $\mathcal{R}_{n, \mathcal{D}_T}$ and in the last one we use Lemma 12 together with $\|\Pi \odot \Theta\|_{1, \infty} \leq \omega_n$ and $\|\Pi \odot (\Theta + \Delta\Theta)\|_{1, \infty} \leq \omega_n$. \square

B Lower bound results

B.1 General idea of the lower bound construction

Let $Y = (Y_1, \dots, Y_T)$ be the observations of the dynamic network following the inhomogeneous random graph model defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Using (12), we can see that a lower bound on the type II error can be obtained by bounding from above the total variation distance between the measures of Y under the null and the alternative hypotheses. The total variation distance is usually hard to bound and we can use instead the chi-squared or the Kullback–Leibler divergences as $\|\mathbb{P}_1 - \mathbb{P}_0\|_{\text{TV}} \leq \sqrt{\chi^2(\mathbb{P}_1, \mathbb{P}_0)}$ and $\|\mathbb{P}_1 - \mathbb{P}_0\|_{\text{TV}} \leq \sqrt{2\text{KL}(\mathbb{P}_1, \mathbb{P}_0)}$. Thus, the problem of bounding the TV-distance is reduced to the problem of bounding one of these two divergences. This can be done using *the second moment method* or *the fuzzy hypotheses method* as follows.

Let $\pi_{n,0}$ and $\pi_{n,1}$ be some prior distributions on the set of parameters (Θ, Π) and $(\Theta + \Delta\Theta^\tau, \Pi)$ of the network under \mathbb{H}_0 and \mathbb{H}_1 , respectively. Define the mixture distributions $\mathbb{p}_{n,0}^T(Y) = \mathbb{E}_{\pi_{n,0}^T} \mathbb{P}(Y)$ and $\mathbb{p}_{n,1}^T(Y) = \mathbb{E}_{\pi_{n,1}^T} \mathbb{P}(Y)$, where \mathbb{P} is the probability measure of Y . The expectations w.r.t. to the measures $\mathbb{p}_{n,i}^T$ are denoted by $\mathbb{E}_{n,i}^T$, $i = 0, 1$. The following bounds hold true (see, for example, (Ingster and Suslina 2003)):

$$\begin{aligned} \inf_{\psi_{n,T}^\tau \in \Psi_\alpha} \beta(\psi_{n,T}^\tau, \mathcal{R}_{n,\tau}) &= \inf_{\psi_{n,T}^\tau \in \Psi_\alpha} \sup_{(\Theta^\tau, \Theta^{\tau+1}) \in \mathcal{W}_{n,T}^\tau(\omega_n, \mathcal{R}_{n,\tau})} \mathbb{P}_{(\Theta^\tau, \Theta^{\tau+1})} \left\{ \psi_{n,T}^\tau = 0 \right\} \\ &\geq 1 - \frac{1}{2} \|\mathbb{p}_{n,1}^T - \mathbb{p}_{n,0}^T\|_{\text{TV}} - \alpha \\ &\geq 1 - \frac{1}{2} \sqrt{\chi^2(\mathbb{p}_{n,1}, \mathbb{p}_{n,0})} - \alpha \\ &= 1 - \frac{1}{2} \left(\mathbb{E}_{n,0}^T \left[\frac{d\mathbb{p}_{n,1}^T}{d\mathbb{p}_{n,0}^T}(Y) \right]^2 - 1 \right)^{1/2} - \alpha. \end{aligned} \quad (33)$$

$$= 1 - \frac{1}{2} \left(\mathbb{E}_{n,0}^T \left[\frac{d\mathbb{p}_{n,1}^T}{d\mathbb{p}_{n,0}^T}(Y) \right]^2 - 1 \right)^{1/2} - \alpha. \quad (34)$$

Let $\alpha \in (0, 1)$ and $\beta \in (0, 1 - \alpha]$. Set $\eta = \alpha + \beta$. To establish a non-asymptotic lower bound $\inf_{\psi_{n,T}^\tau \in \Psi_\alpha} \beta(\psi_{n,T}, \mathcal{R}_{n,\tau}) \geq \beta$ and the corresponding (α, β) -minimax detection rate, we need to find the conditions on $\mathcal{R}_{n,\tau}$ such that

$$\mathbb{E}_{n,0}^T \left[\frac{d\mathbb{p}_{n,1}^T}{d\mathbb{p}_{n,0}^T}(Y) \right]^2 \leq 1 + 4(1 - \alpha - \beta)^2 = 1 + 4(1 - \eta)^2.$$

In the case of problem (P2) of unknown change-point $\tau \in \mathcal{D}_T$, bounding the type II error can be reduced to the case of a given change-point location provided in (33):

$$\begin{aligned} &\inf_{\psi_{n,T} \in \Psi_\alpha} \beta(\psi_{n,T}, \mathcal{R}_{n,\mathcal{D}_T}) \\ &= \inf_{\psi_{n,T} \in \Psi_\alpha} \sup_{\tau \in \mathcal{D}_T} \sup_{(\Theta^0 + \Delta\Theta^\tau, \Pi), (\Theta^0, \Pi) \in \mathcal{W}_{n,T}^\tau(\omega_n, \mathcal{R}_{n,\mathcal{D}_T})} \mathbb{P}_{(\Theta^0 + \Delta\Theta^\tau, \Pi), (\Theta^0, \Pi)} \left\{ \psi_{n,T} = 0 \right\} \\ &\geq \inf_{\psi_{n,T} \in \Psi_\alpha} \sup_{(\Theta^0 + \Delta\Theta^{\tau^*}, \Pi), (\Theta^0, \Pi) \in \mathcal{W}_{n,T}^{\tau^*}(\omega_n, \mathcal{R}_{n,\mathcal{D}_T})} \mathbb{P}_{(\Theta^0 + \Delta\Theta^{\tau^*}, \Theta^0)} \left\{ \psi_{n,T} = 0 \right\}, \end{aligned} \quad (35)$$

$$\geq \inf_{\psi_{n,T} \in \Psi_\alpha} \sup_{(\Theta^0 + \Delta\Theta^{\tau^*}, \Pi), (\Theta^0, \Pi) \in \mathcal{W}_{n,T}^{\tau^*}(\omega_n, \mathcal{R}_{n,\mathcal{D}_T})} \mathbb{P}_{(\Theta^0 + \Delta\Theta^{\tau^*}, \Theta^0)} \left\{ \psi_{n,T} = 0 \right\}, \quad (36)$$

where $\tau^* \in \mathcal{D}_T$ is any possible change-point from the set of alternatives. Thus, we can reduce the construction of the lower bound for the case of an unknown change-point to the case of a given change-point τ^* .

B.2 Auxiliary lemma

Let $\rho_n \in (0, 1/2]$ and $q \in [-1, 1]$. Denote by p_0 and p_q the Bernoulli measures with the parameters ρ_n and $\rho_n(1+q)$ with the corresponding densities dp_0 and dp_q with respect to some dominating measure λ . The following simple formulas will be useful in the proof of the lower bound.

Lemma 7. *Let $\rho_n \in (0, 1/2]$, $q, q_1, q_2 \in [-1, 1]$. The following relations hold true for a Bernoulli variable $X \sim p_0$:*

$$\begin{aligned} \mathbb{E}_0 \left[\frac{dp_q}{dp_0} \right]^2 (X) &= 1 + \frac{\rho_n}{1 - \rho_n} q^2, \\ \mathbb{E}_0 \left[\frac{dp_{q_1}}{dp_0} \frac{dp_{q_2}}{dp_0} \right] (X) &= 1 + \frac{\rho_n}{1 - \rho_n} q_1 q_2. \end{aligned}$$

B.3 Lower bound for Inhomogeneous Random Graph Model

We will establish the lower bound for the case of the known change-point location τ . Let Π_n be the sampling matrix with the unit entries on the diagonal, $\text{diag}(\Pi_n) = \mathbf{1}_n$ and with non-zero entries, $\min_{ij} \Pi_{ij} > 0$. In case of $\min_{ij} \Pi_{ij} = 1$ there is no missing links. Recall that $\alpha \in (0, 1)$, $\beta \in (0, 1 - \alpha]$ and $\eta = \alpha + \beta$.

Proof of Theorem 2. In what follows we denote by δ_x the Dirac measure concentrated at x , where x can be a real or a matrix value. Denote by $P(Y) = \prod_{t=1}^T P(Y^t)$ the measure of the observations $Y = (Y^1, \dots, Y^T)$ from (1).

Denote by $\tilde{\Theta}^t = \Pi_n \odot \Theta^t$ the parameter of the observed adjacency matrix. We will impose the following priors on the matrix parameters $\tilde{\Theta}^t$ of the dynamic network $Y = (Y^1, \dots, Y^T)$.

Step 1. Priors on the transition matrices.

Set $\tilde{\rho}_n = (1 - \varepsilon_n) \frac{\omega_n}{n - 1}$ for some $\varepsilon_n \in (0, 1)$ that will be chosen later. Assume that under the null hypothesis H_0 all the observed connections occur independently with the same probability $\tilde{\rho}_n$ for all $1 \leq t \leq T$. Set $V_0 = \tilde{\rho}_n (\mathbf{1}_n \mathbf{1}_n^T - \text{id}_n)$ and define the prior under H_0 on the sequence of the sampled connection probability matrices $\tilde{\Theta}^t$ ($1 \leq t \leq T$):

$$\pi_{n,0}^T(\tilde{\Theta}^1, \dots, \tilde{\Theta}^T) = \prod_{t=1}^T \delta_{V_0}(\tilde{\Theta}^t) = \prod_{t=1}^T \prod_{i \neq j} \delta_{\tilde{\rho}_n}(\tilde{\Theta}_{ij}^t). \quad (37)$$

Here δ_{V_0} stands for the Dirac measure concentrated at V_0 and defined on the set of matrices $\Pi_n \odot \Theta$ such that $(\Theta, \Pi_n) \in \mathcal{S}_n(\omega_n)$. The prior is indeed concentrated on $\mathcal{S}_n(\omega_n)$, since $\|V_0\|_{1,\infty} = (n - 1)\tilde{\rho}_n = (1 - \varepsilon)\omega_n < \omega_n$.

Let us define the prior under H_1 . Let $\zeta = (\zeta_1, \dots, \zeta_n)$ be a vector of i.i.d. Rademacher random variables taking values in $\{-1, 1\}$ with probability 1/2. Assume that the

sampled connection probability matrices before and after the change are defined by

$$V_{1,\zeta} = V_0 - \left(1 - \frac{\tau}{T}\right) \Lambda_{n,\zeta}, \quad V_{2,\zeta} = V_0 + \frac{\tau}{T} \Lambda_{n,\zeta}, \quad (38)$$

where $\Lambda_{n,\zeta} = \frac{r_{n,\tau}}{n-1}(\zeta\zeta^\top - \text{id}_n)$ is the change matrix with $r_{n,\tau} = \frac{\mathcal{R}_{n,\tau}}{q(\tau/T)}$. Note that the operator norm of $\Lambda_{n,\zeta}$ is equal to $r_{n,\tau}$ and the energy of the change-point is $q(\tau/T)\|\Lambda_{n,\zeta}\|_{2 \rightarrow 2} = \mathcal{R}_{n,\tau}$.

To define the prior concentrated on $\mathcal{W}(\omega_n, \mathcal{R}_{n,\tau})$, we need to show that $\|V_{i,\zeta}\|_{1,\infty} \leq \omega_n$, $i = 1, 2$ for sufficiently large n . We have that for all $n \geq 2$ and for $i = 1, 2$,

$$\begin{aligned} \|V_{i,\zeta}\|_{1,\infty} &\leq \|V_0\|_{1,\infty} + \frac{\tau \vee (T - \tau)}{T} \|\Lambda_{n,\zeta}\|_{1,\infty} \\ &= (1 - \varepsilon_n)\omega_n + r_{n,\tau} \frac{\tau \vee (T - \tau)}{T} \\ &= (1 - \varepsilon_n)\omega_n + q\left(\frac{\tau}{T}\right)r_{n,\tau}\sqrt{T-1} \\ &\leq (1 - \varepsilon_n)\omega_n + R_{n,\tau}\sqrt{T}. \end{aligned}$$

Let $\varepsilon_n = (2C_\eta)^{1/4}\omega_n^{-1/2}$, then for all $\omega_n > \sqrt{2C_\eta}$ we have $\varepsilon_n \in (0, 1)$. It will be shown later in (43) that $R_{n,\tau}\sqrt{T} \leq \varepsilon_n\omega_n$ and, consequently, $\|V_{i,\zeta}\|_{1,\infty} \leq \omega_n$. Thus, the prior under H_1 is well defined and is given by

$$\pi_{n,1}^\tau(\tilde{\Theta}^1, \dots, \tilde{\Theta}^T) = \prod_{t=1}^{\tau} \delta_{V_{1,\zeta}}(\tilde{\Theta}^t) \prod_{t=\tau+1}^T \delta_{V_{2,\zeta}}(\tilde{\Theta}^t). \quad (39)$$

Step 2. Likelihood ratio of mixtures.

To shorten the notation, denote

$$q_{1,\tau} := -\left(1 - \frac{\tau}{T}\right) \frac{r_{n,\tau}}{\tilde{\rho}_n(n-1)}, \quad q_{2,\tau} := \frac{\tau}{T} \frac{r_{n,\tau}}{\tilde{\rho}_n(n-1)}.$$

We can now calculate the mixtures under H_0 that are given by

$$p_{n,0}(Y) = E_{\pi_{n,0}^\tau} P(Y) = \prod_{i>j} \prod_{t=1}^T \tilde{\rho}_n^{Y_{ij}^t} (1 - \tilde{\rho}_n)^{1-Y_{ij}^t}$$

and under H_1 that are given by

$$\begin{aligned} p_{n,1}^\tau(Y) = E_{\pi_{n,1}^\tau} P(Y) = E_\zeta &\left[\prod_{i>j} \left(\prod_{t=1}^{\tau} (\tilde{\rho}_n + \tilde{\rho}_n q_{1,\tau} \zeta_i \zeta_j)^{Y_{ij}^t} (1 - \tilde{\rho}_n - \tilde{\rho}_n q_{1,\tau} \zeta_i \zeta_j)^{1-Y_{ij}^t} \right. \right. \\ &\left. \left. \times \prod_{t=\tau+1}^T (\tilde{\rho}_n + \tilde{\rho}_n q_{2,\tau} \zeta_i \zeta_j)^{Y_{ij}^t} (1 - \tilde{\rho}_n - \tilde{\rho}_n q_{2,\tau} \zeta_i \zeta_j)^{1-Y_{ij}^t} \right) \right], \end{aligned}$$

where E_ζ stands for the expectation w.r.t. to the distribution of ζ .

Denote by $\mathcal{Z} = \{-1, +1\}^n$ the set of all sequences $\zeta = (\zeta_1, \dots, \zeta_n)$ taking values in $\{-1, +1\}$. We can write the likelihood ratio of mixtures,

$$\frac{d\mathbf{p}_{n,1}^\tau}{d\mathbf{p}_{n,0}}(Y) = \frac{1}{2^n} \sum_{\zeta \in \mathcal{Z}} \prod_{i>j} \left(\prod_{t=1}^{\tau} \frac{d\mathbf{p}_{q_1, \tau \zeta_i \zeta_j}}{d\mathbf{p}_0}(Y_{ij}^t) \prod_{t=\tau+1}^T \frac{d\mathbf{p}_{q_2, \tau \zeta_i \zeta_j}}{d\mathbf{p}_0}(Y_{ij}^t) \right),$$

where \mathbf{p}_0 stands for the Bernoulli measure with the parameter $\tilde{\rho}_n$ and \mathbf{p}_q denotes the Bernoulli measure with the parameter $\tilde{\rho}_n + q$, as in Lemma 7, Section B.2.

Step 3. Second moment of the likelihood ratio.

Let $\tilde{\zeta}$ be an independent copy of the vector ζ . Using Lemma 7, we can calculate the second moment of the likelihood ratio,

$$\begin{aligned} \mathbb{E}_0 \left[\frac{d\mathbf{p}_{n,1}^\tau}{d\mathbf{p}_{n,0}} \right]^2(Y) &= \frac{1}{2^{2n}} \sum_{\zeta, \tilde{\zeta} \in \mathcal{Z}} \mathbb{E}_0 \left[\prod_{i>j} \prod_{t=1}^{\tau} \frac{d\mathbf{p}_{q_1, \tau \zeta_i \zeta_j}}{d\mathbf{p}_0} \frac{d\mathbf{p}_{q_1, \tau \tilde{\zeta}_i \tilde{\zeta}_j}}{d\mathbf{p}_0}(Y_{ij}^t) \prod_{t=\tau+1}^T \frac{d\mathbf{p}_{q_2, \tau \zeta_i \zeta_j}}{d\mathbf{p}_0} \frac{d\mathbf{p}_{q_2, \tau \tilde{\zeta}_i \tilde{\zeta}_j}}{d\mathbf{p}_0}(Y_{ij}^t) \right] \\ &= \frac{1}{2^{2n}} \sum_{\zeta, \tilde{\zeta} \in \mathcal{Z}} \prod_{i>j} \left(1 + \frac{\tilde{\rho}_n}{1 - \tilde{\rho}_n} q_{1, \tau}^2 \zeta_i \zeta_j \tilde{\zeta}_i \tilde{\zeta}_j \right)^\tau \left(1 + \frac{\tilde{\rho}_n}{1 - \tilde{\rho}_n} q_{2, \tau}^2 \zeta_i \zeta_j \tilde{\zeta}_i \tilde{\zeta}_j \right)^{T-\tau} \\ &= \frac{1}{2^{2n}} \sum_{\zeta, \tilde{\zeta} \in \mathcal{Z}} \exp \left(\sum_{i>j} \tau \log \left(1 + \frac{\tilde{\rho}_n}{1 - \tilde{\rho}_n} q_{1, \tau}^2 \zeta_i \zeta_j \tilde{\zeta}_i \tilde{\zeta}_j \right) \right. \\ &\quad \left. + (T - \tau) \log \left(1 + \frac{\tilde{\rho}_n}{1 - \tilde{\rho}_n} q_{2, \tau}^2 \zeta_i \zeta_j \tilde{\zeta}_i \tilde{\zeta}_j \right) \right). \end{aligned}$$

Note that

$$\frac{\tilde{\rho}_n}{1 - \tilde{\rho}_n} (\tau q_{1, \tau}^2 + (T - \tau) q_{2, \tau}^2) = \frac{T q^2 (\tau/T) r_{n, \tau}^2}{(1 - \tilde{\rho}_n) \tilde{\rho}_n (n-1)^2} = \frac{T R_{n, \tau}^2}{\omega_n} \frac{1}{(1 - \tilde{\rho}_n)(n-1)(1 - \varepsilon_n)}$$

and denote the last quantity by $\mu_n = \frac{T R_{n, \tau}^2}{\omega_n} \frac{1}{(1 - \tilde{\rho}_n)(n-1)(1 - \varepsilon_n)}$. Applying the inequality $\log(1+x) \leq x$ and using the fact that the distribution of $\sum_{i \neq j} \zeta_i \zeta_j \tilde{\zeta}_i \tilde{\zeta}_j$ is the same as the one of $\sum_{i \neq j} \zeta_i \zeta_j$, we obtain the upper bound

$$\begin{aligned} \mathbb{E}_0 \left[\frac{d\mathbf{p}_{n,1}^\tau}{d\mathbf{p}_{n,0}} \right]^2(Y) &\leq \frac{1}{2^{2n}} \sum_{\zeta, \tilde{\zeta} \in \mathcal{Z}} \exp \left(\frac{1}{2} \mu_n \sum_{i \neq j} \zeta_i \zeta_j \tilde{\zeta}_i \tilde{\zeta}_j \right) \\ &= \mathbb{E}_\zeta \exp \left(\frac{1}{2} \mu_n \sum_{i \neq j} \zeta_i \zeta_j \right) = \mathbb{E}_\zeta \exp \left(\frac{1}{2} \mu_n \zeta^\top (\mathbf{1}_n^\top \mathbf{1}_n - \text{id}_n) \zeta \right). \quad (40) \end{aligned}$$

Step 4. Upper bound on the second moment.

Using Theorem 2 in (Cortinovis and Kressner 2021), we can bound the Laplace transform of the Rademacher chaos $\sum_{i \neq j} \zeta_i \zeta_j$ as follows. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix with zero diagonal. Then, $\forall 0 < \mu < 1/4$,

$$\log \mathbb{E} \left(e^{\mu \zeta^\top A \zeta} \right) \leq \mu \|A\|_F^2 \log \frac{1 - 2\mu}{1 - 4\mu} \leq \frac{2\mu^2 \|A\|_F^2}{1 - 4\mu}. \quad (41)$$

Using this inequality, from (40) we obtain that for any $\mu_n < 1/2$

$$\mathbb{E}_0 \left[\frac{dp_{n,1}^\tau}{dp_{n,0}} \right]^2 (Y) \leq \mathbb{E}_\zeta \exp \left\{ \frac{1}{2} \mu_n \zeta^\top (\mathbf{1}_n^\top \mathbf{1}_n - \text{id}_n) \zeta \right\} \leq \exp \left(\frac{\frac{1}{2} \mu_n^2 n(n-1)}{1-2\mu_n} \right).$$

This bound implies that the second moment of the likelihood ratio is less than $1 + 4(1-\eta)^2$ if

$$\mu_n \leq \frac{2C_\eta}{n^2} \left(\sqrt{1 + \frac{n^2}{2C_\eta}} - 1 \right), \quad (42)$$

where $C_\eta = \log(1 + 4(1-\eta)^2)$. Note that this condition will imply $\mu_n < 1/2$. Now, (42) is satisfied if

$$\frac{T\mathcal{R}_{n,\tau}^2}{\omega_n} \leq (1 - \tilde{\rho}_n)(1 - \varepsilon_n) \left(1 - \frac{1}{n}\right) \sqrt{2C_\eta} \left(\sqrt{1 + \frac{2C_\eta}{n^2}} - \sqrt{\frac{2C_\eta}{n^2}} \right).$$

Noting that $\sqrt{x+1} - \sqrt{x} \geq (1+2\sqrt{x})^{-1}$, we get that this inequality is satisfied if

$$\frac{T\mathcal{R}_{n,\tau}^2}{\omega_n} \leq \sqrt{2C_\eta} (1 - \tilde{\rho}_n)(1 - \varepsilon_n) \left(1 - \frac{1}{n}\right) \left(1 + \frac{2\sqrt{2C_\eta}}{n}\right)^{-1}. \quad (43)$$

It means that all the signals with energy $\mathcal{R}_{n,\tau}$ satisfying (43) are not detectable by any α -level test with the type II error smaller than β . Therefore the lower bound on the minimal detectable energy for an α -level test with type II errors bounded β is given by

$$\mathcal{R}_{n,\tau}^* \geq \left(2 \log(1 + 4(1-\eta)^2)\right)^{1/4} \sqrt{\frac{\omega_n}{T}}$$

and the theorem follows. \square

C Proof of result on the change-point localization

Proof of Proposition 2. Lemma 12 implies that for any $s \in [T]$, with probability at least $1 - \frac{\gamma}{T}$

$$\|\xi(s)\|_{2 \rightarrow 2} \leq 2\sqrt{2}(1 + \epsilon)\sqrt{\omega_n} + C_\epsilon \log(2nT/\gamma). \quad (44)$$

By the definition of $\hat{\tau}_n$ we have $\|Z_T(\hat{\tau}_n)\|_{2 \rightarrow 2} \geq \|Z_T(\tau)\|_{2 \rightarrow 2}$ which implies that

$$\mu_T^\tau(\tau)\Delta - \|\xi(\tau)\|_{2 \rightarrow 2} \leq \mu_T^\tau(\hat{\tau}_n)\Delta + \|\xi(\hat{\tau}_n)\|_{2 \rightarrow 2}.$$

Using (44) and the union bound we get that with probability at least $1 - \gamma$

$$(\mu_T^\tau(\tau) - \mu_T^\tau(\hat{\tau}_n))\Delta \leq 4\sqrt{2}(1 + \epsilon)\sqrt{\omega_n} + C_\epsilon \log(2nT/\gamma). \quad (45)$$

First, consider the case $\hat{\tau}_n \leq \tau$. Using the definition of $\mu_T^\tau(t)$ (10), we compute

$$\begin{aligned}
\mu_T^\tau(\tau) - \mu_T^\tau(\hat{\tau}_n) &= \sqrt{T} \left(q(x^*) - q(\hat{x}) \frac{1-x^*}{1-\hat{x}} \right) \\
&= \sqrt{T}(1-x^*) \left(\frac{q(x^*)}{1-x^*} - \frac{q(\hat{x})}{1-\hat{x}} \right) \\
&= \sqrt{T}(1-x^*) \left(\sqrt{\frac{x^*}{1-x^*}} - \sqrt{\frac{\hat{x}}{1-\hat{x}}} \right) \\
&= \sqrt{\frac{T(1-x^*)}{1-\hat{x}}} \frac{x^* - \hat{x}}{\sqrt{\hat{x}(1-x^*)} + \sqrt{x^*(1-\hat{x})}} \\
&\geq \sqrt{T(1-x^*)} \frac{x^* - \hat{x}}{1.5}
\end{aligned}$$

where we use that for any $x \in (0, 1)$, $x(1-x) \leq 1/4$. Plugging this calculation into (45) we get

$$(x^* - \hat{x}) \Delta \leq 6(1 + \epsilon) \sqrt{\frac{2\omega_n}{T(1-x^*)}} + \frac{1.5C_\epsilon \log(2nT/\gamma)}{\sqrt{T(1-x^*)}}. \quad (46)$$

Now assume that $\hat{\tau}_n \geq \tau$. Then, using the definition of $\mu_T^\tau(t)$, (10), we compute

$$\begin{aligned}
\mu_T^\tau(\tau) - \mu_T^\tau(\hat{\tau}_n) &= \sqrt{T} \left(q(x^*) - q(\hat{x}) \frac{x^*}{\hat{x}} \right) \\
&= \sqrt{T}x^* \left(\frac{q(x^*)}{x^*} - \frac{q(\hat{x})}{\hat{x}} \right) \\
&= \sqrt{T}x^* \left(\sqrt{\frac{1-x^*}{x^*}} - \sqrt{\frac{1-\hat{x}}{\hat{x}}} \right) \\
&= \sqrt{\frac{Tx^*}{\hat{x}}} \frac{\hat{x} - x^*}{\sqrt{\hat{x}(1-x^*)} + \sqrt{x^*(1-\hat{x})}} \\
&\geq \sqrt{T(1-x^*)} \frac{\hat{x} - x^*}{1.5}
\end{aligned}$$

which implies

$$(\hat{x} - x^*) \Delta \leq 6(1 + \epsilon) \sqrt{\frac{2\omega_n}{Tx^*}} + \frac{1.5C_\epsilon \log(2nT/\gamma)}{\sqrt{Tx^*}}. \quad (47)$$

Combining (46) and (47) and using $q^2(x^*) \leq x^* \wedge (1-x^*)$ we get the statement of the Proposition 2. \square

D Proofs of results for the sparse graphon model

We start by summarizing some notation that we use in the proofs:

Given a matrix $\Theta \in [-1, 1]^{n \times n}$, we define the empirical graphon associated with Θ as follows:

$$\tilde{f}_\Theta(x, y) = \Theta_{[nx], [ny]}, \quad (x, y) \in [0, 1]^2. \quad (48)$$

In the same spirit, given a vector $v = (v_1, \dots, v_n)$, for any $x \in [0, 1]$, we define the following piecewise constant function

$$\psi_v(x) = \sqrt{nv} \mathbf{1}_{[nx]}, \quad x \in [0, 1] \quad (49)$$

and set

$$\mathcal{F} = \left\{ \psi_v : \|v\|_{\ell_2} \leq 1 \right\}. \quad (50)$$

We have that $\|v\|_{\ell_2} \leq 1$ implies

$$\|\psi_v\|_{L_2[0,1]} = \frac{1}{n} \sum_{i=1}^n nv_i^2 \leq 1.$$

We will need to work with a difference of two graphons, so we extend the definition of graphon space. In what follows \mathcal{W} refers to the collection of bounded symmetric measurable functions $W : [0, 1]^2 \rightarrow [-1, 1]$.

D.1 Proofs of upper bounds

Proof of Theorem 4. We have to find a threshold $H_{\alpha, n, T}^*$ such that

$$\alpha(\psi_{n, T}) = \sup_{W \in \mathcal{W}_0} \mathbb{P}_{W, \varepsilon} \left\{ \psi_{n, T} = 1 \right\} \leq \alpha$$

and show that

$$\beta(\psi_{n, T}, \delta_{n, T}) = \sup_{\tau \in \mathcal{D}_T} \left(\sup_{W^\tau, W^{\tau+1} \in \mathcal{W}_0(\delta_{n, T})} \mathbb{P}_{W^\tau, W^{\tau+1}, \varepsilon, \varepsilon'} \left\{ \psi_{n, T} = 0 \right\} \right) \leq \beta.$$

Under H_0 there is no change in the graphon function W but it can be a change in the features. Let $\Theta_1 = (\rho_n W(\varepsilon_i, \varepsilon_j))_{(i, j) \in [n] \times [n]}$ denotes the matrix of connection probabilities before the time point τ and $\Theta_2 = (\rho_n W(\varepsilon'_i, \varepsilon'_j))_{(i, j) \in [n] \times [n]}$ the matrix of connection probabilities after τ . We have that

$$Z_T(t) = -\mu_T^\tau(t) \Delta \Theta^\tau + \xi^*(t), \quad t = 1, \dots, T-1,$$

where

$$\mu_T^\tau(t) = \sqrt{\frac{t(T-t)}{T}} \left(\frac{\tau}{t} \mathbf{1}_{\{\tau+1 \leq t \leq T\}} + \frac{T-\tau}{T-t} \mathbf{1}_{\{1 \leq t \leq \tau\}} \right), \quad \Delta \Theta^\tau = \Theta_1 - \Theta_2 \circ \pi^*$$

and the random matrices

$$\xi^*(t) = \sqrt{\frac{t(T-t)}{T}} \left(\frac{1}{t} \sum_{s=1}^t W^s - \frac{1}{T-t} \sum_{s=t+1}^T W^s \circ \pi^* \right)$$

are centered. Let π' be a permutation of $\{1, \dots, n\}$ such that

$$\pi' \in \arg \min_{\pi} \|\Theta_1 - \Theta_2 \circ \pi\|_{2 \rightarrow 2}.$$

Let

$$\xi(t) = \sqrt{\frac{t(T-t)}{T}} \left(\frac{1}{t} \sum_{s=1}^t W^s - \frac{1}{T-t} \sum_{s=t+1}^T W^s \circ \pi' \right)$$

be the corresponding noise term. Using the definition of π^* and the triangle inequality, we have that

$$\begin{aligned} \|Z_T(t)\|_{2 \rightarrow 2} &\leq \mu_T^\tau(t) \|\Theta_1 - \Theta_2 \circ \pi'\|_{2 \rightarrow 2} + \|\xi(t)\|_{2 \rightarrow 2} \\ &\leq \sqrt{T}q(t/T) \|\Theta_1 - \Theta_2 \circ \pi'\|_{2 \rightarrow 2} + \|\xi(t)\|_{2 \rightarrow 2}. \end{aligned}$$

Now, using Lemma 9, we have that with probability at least $1 - 4/n$

$$\begin{aligned} \|\Theta_1 - \Theta_2 \circ \pi'\|_{2 \rightarrow 2}^2 &\leq n^2 \delta_2^2(\tilde{f}_{\Theta_1}, \tilde{f}_{\Theta_2}) \leq 2n^2 \left(\delta_2^2(\tilde{f}_{\Theta_1}, \rho_n W) + \delta_2^2(\tilde{f}_{\Theta_2}, \rho_n W) \right) \\ &\leq 32n^2 \rho_n^2 \sqrt{\frac{K}{n}} \log(n) \end{aligned}$$

where \tilde{f}_{Θ_i} is the empirical graphon associated with Θ_i . Note that, for any $x > 0$,

$$\mathbb{P}(\|\xi(t)\|_{2 \rightarrow 2} > x) = \mathbb{E}_{\{\varepsilon_\vartheta, \varepsilon'_\vartheta\}_{\vartheta \in \mathcal{V}}} \left[\mathbb{P}(\|\xi(t)\|_{2 \rightarrow 2} > x \mid \{\varepsilon_\vartheta, \varepsilon'_\vartheta\}_{\vartheta \in \mathcal{V}}) \right].$$

Now, we can bound $\mathbb{P}(\|\xi(t)\|_{2 \rightarrow 2} > x \mid \{\varepsilon_\vartheta, \varepsilon'_\vartheta\}_{\vartheta \in \mathcal{V}})$ using Lemma 12: for every $\delta \in (0, 1)$, conditionally on $\{\varepsilon_\vartheta, \varepsilon'_\vartheta\}_{\vartheta \in \mathcal{V}}$, we have

$$\|\xi(t)\|_{2 \rightarrow 2} \leq 2\sqrt{2}(1+\varepsilon) \sqrt{\frac{t(T-t)}{T}} \left[\frac{1}{t^2} \sum_{s=1}^t \|\Theta^s\|_{1, \infty} + \frac{1}{(T-t)^2} \sum_{s=t+1}^T \|\Theta^s\|_{1, \infty} \right]^{1/2} + C_\varepsilon \log\left(\frac{2n}{\delta}\right)$$

with the probability larger than $1 - \delta$ where $\Theta^t = \rho_n (W^t(\varepsilon_i, \varepsilon_j))_{(i,j) \in [n] \times [n]}$. Note that $\|\Theta^t\|_{1, \infty} \leq n\rho_n$ and we get

$$\|\xi(t)\|_{2 \rightarrow 2} \leq 2(1+\varepsilon) \sqrt{2n\rho_n} + C_\varepsilon \log\left(\frac{2n}{\delta}\right) \quad (51)$$

with the probability larger than $1 - \delta$. Taking $\delta = \alpha/(2|\mathcal{T}|)$, using the union bound and $\alpha \geq 8/n$, we obtain

$$\alpha(\psi_{n,T}) = \sup_{W \in \mathcal{W}_0} \mathbb{P}_{W, \varepsilon} \left\{ \max_{t \in \mathcal{T}} \|Z_T(t)\|_{2 \rightarrow 2} \geq H_{\alpha, n, T}^* \right\} \leq \alpha.$$

Let us turn to the type II error. Following the proof of Lemma 6, we can show that, if $\tau \leq T/2$, there exists $t^* \in \mathcal{T}^L$ such that $\tau/2 \leq t^* < \tau$ and

$$\begin{aligned} &\beta(\psi_{n,T}, \delta_{n,T}) \\ &\leq \sup_{W^\tau, W^{\tau+1} \in \mathcal{W}_0(\delta_{n,T})} \mathbb{P}_{W^\tau, W^{\tau+1}} \left\{ \|\xi^*(t^*)\|_{2 \rightarrow 2} > \frac{\sqrt{T}q(\tau/T)}{\sqrt{3}} \|\Delta\Theta^\tau\|_{2 \rightarrow 2} - H_{\alpha, n, t^*}^* \right\}. \end{aligned}$$

Using the triangle inequality, Lemma 8 and the definition of the empirical graphon (48) we get

$$\begin{aligned}
\|\xi^*(t^*)\|_{2 \rightarrow 2} &> \frac{\sqrt{T}q(\tau/T)}{\sqrt{3}} \|\Theta_1 - \Theta_2 \circ \pi^*\|_{2 \rightarrow 2} - H_{\alpha,n,t^*}^* \\
&\geq \frac{n\sqrt{T}q(\tau/T)}{\sqrt{3}} \|\tilde{f}_{\Theta_1 - \Theta_2 \circ \pi^*}\|_{2 \rightarrow 2} - H_{\alpha,n,t^*}^* \\
&= \frac{n\sqrt{T}q(\tau/T)}{\sqrt{3}} \|\tilde{f}_{\Theta_1} - \tilde{f}_{\Theta_2 \circ \pi^*}\|_{2 \rightarrow 2} - H_{\alpha,n,t^*}^* \\
&\geq \frac{n\sqrt{T}q(\tau/T)}{\sqrt{3}} \delta(\tilde{f}_{\Theta_1}, \tilde{f}_{\Theta_2}) - H_{\alpha,n,t^*}^*.
\end{aligned}$$

Now, using twice the triangle inequality, we get

$$\|\xi^*(t^*)\|_{2 \rightarrow 2} \geq \frac{n\sqrt{T}q(\tau/T)}{\sqrt{3}} \left(\delta(\rho_n W_1, \rho_n W_2) - \delta(\tilde{f}_{\Theta_1}, \rho_n W_1) - \delta(\rho_n W_2, \tilde{f}_{\Theta_2}) \right) - H_{\alpha,n,t^*}^*. \quad (52)$$

Note that Lemma 9 and $\delta(\tilde{f}_{\Theta_i}, \rho_n W_i) \leq \delta_2(\tilde{f}_{\Theta_i}, \rho_n W_i)$ imply

$$\delta(\tilde{f}_{\Theta_i}, \rho_n W_i) \leq 4\rho_n \left(\frac{K_i \log n}{n} \right)^{1/4}$$

with probability at least $1 - 2/n$. Taking $\beta \geq 8/n$ and using (51) with $\delta = \beta/2$ we get that (52) implies $\beta(\psi_{n,T}, \delta_{n,T}) \leq \beta$ if

$$\frac{n\sqrt{T}q(\tau/T)}{\sqrt{3}} \left(\rho_n \delta(W_1, W_2) - 4\rho_n \left(\frac{(K_1 + K_2) \log n}{n} \right)^{1/4} \right) \geq H_{\alpha,n,t^*}^* + 2(1+\varepsilon)\sqrt{2n\rho_n} + C_\varepsilon \log \left(\frac{4n}{\beta} \right)$$

Combining this condition with the threshold $H_{\alpha,n,t}^*$ defined in (28) and the facts that $q(t^*/T) \leq q(\tau/T)$ for $t^* < \tau \leq T/2$ and $K_1, K_2 \leq K$, we obtain the detection condition (29). The case of $\tau > T/2$ is analogous. \square

Proof of Theorem 6. Theorem 6 follows from combining the bounds obtained in Lemma 8, Lemma 10 and Theorem 3. \square

Lemma 8. *Let $\Theta = (\Theta_{ij}) \in [-1, 1]^{n \times n}$ be symmetric matrix. Then*

$$\|\Theta\|_{2 \rightarrow 2} \geq n \|\tilde{f}_\Theta\|_{2 \rightarrow 2}.$$

Proof. By the definition of the operator norm, we have that

$$\begin{aligned}
\|\tilde{f}_\Theta\|_{2 \rightarrow 2} &= \sup_{\psi \in L_2[0,1], \|\psi\|_{L_2} \leq 1} \left| \iint_{[0,1]^2} \tilde{f}_\Theta(x,y) \psi(x) \psi(y) dx dy \right| \\
&= \sup_{\psi \in L_2[0,1], \|\psi\|_{L_2} \leq 1} \left| \sum_{ij} \Theta_{ij} \int_{i/n}^{(i+1)/n} \psi(x) dx \int_{j/n}^{(j+1)/n} \psi(y) dy \right| \\
&= \sup_{\psi \in L_2[0,1], \|\psi\|_{L_2} \leq 1} \left| \frac{1}{n} \sum_{ij} \Theta_{ij} v_i^\psi v_j^\psi \right|
\end{aligned}$$

where $v_i^\psi = \sqrt{n} \int_{i/n}^{(i+1)/n} \psi(x) dx$ and $v^\psi = (v_i^\psi)_{i=1}^n$. Note that the Cauchy-Schwartz inequality implies

$$\left(\int_{i/n}^{(i+1)/n} \psi(x) dx \right)^2 \leq \frac{1}{n} \int_{i/n}^{(i+1)/n} \psi^2(x) dx$$

and we get that $\|v^\psi\|_{\ell_2} \leq 1$ for ψ such that $\|\psi\|_{L_2} \leq 1$. Now we can write

$$n \|\tilde{f}_\Theta\|_{2 \rightarrow 2} \leq \sup_{\|v\|_{\ell_2} \leq 1} \left| \sum_{ij} \Theta_{ij} v_i v_j \right| = \|\Theta\|_{2 \rightarrow 2}$$

and the statement of Lemma 8 follows. \square

Lemma 9. *For any $K \leq \frac{n}{\log(n)}$ assume that $W \in \mathcal{W}_K$. Let $\Theta = (\Theta_{ij}) \in [0, 1]^{n \times n}$ be symmetric matrix with entries $\Theta_{ij} = W(\xi_i, \xi_j)$ for $i < j$, where ξ_i are i.i.d. uniform random variables on $[0, 1]$. We have that, with probability large than $1 - 2/n$,*

$$\delta(\tilde{f}_\Theta, W) \leq 4 \left(\frac{K}{n} \log(n) \right)^{1/4}.$$

Proof. Following the proof of Proposition 3.2 in (Klopp et al. 2017), we get

$$\delta_2(\tilde{f}_{\Delta\Theta^\tau}, \Delta W^\tau) \leq \frac{1}{n} + \sum_{a=1}^K |\lambda_a - \hat{\lambda}_a|,$$

with

$$\hat{\lambda}_a = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{\varepsilon_i \in \phi^{-1}(a)\}}$$

and $\lambda_a = \lambda(\phi^{-1}(a))$, where λ stands for the Lebesgue measure. Since $\varepsilon_1, \dots, \varepsilon_n$ are i.i.d. uniform random variables, $n\hat{\lambda}_a$ has a binomial distribution with parameters (n, λ_a) . We have $n\hat{\lambda}_a - n\lambda_a = \sum_{i=1}^n (Y_i - \lambda_a)$, where $Y_i \sim \text{Bernoulli}(\lambda_a)$. Applying the Bernstein inequality we obtain that for any $t > 0$

$$|n\hat{\lambda}_a - n\lambda_a| \leq \left(2t \sum_{a=1}^K \lambda_a (1 - \lambda_a) \right)^{1/2} + 2t/3$$

with probability $1 - 2e^{-t}$. Taking $t = \log(nK)$ implies that with probability $1 - 2/(nK)$

$$|n\hat{\lambda}_a - n\lambda_a| \leq (2n\lambda_a \log(nK))^{1/2} + \frac{2}{3} \log(nK).$$

Using $K \leq n$ and the union bound we obtain that, with probability $1 - 2/(n)$,

$$\delta^2(\tilde{f}_\Theta, W) \leq \frac{1}{n} + \frac{2}{n} \sum_{a=1}^K (n\lambda_a \log n)^{1/2} + \frac{4K \log(n)}{3n} \leq \frac{1}{n} + 2 \left(\frac{K \log n}{n} \right)^{1/2} + \frac{4K \log(n)}{3n}$$

where we use $\sum_{a=1}^K \lambda_a = 1$ and the Cauchy-Schwarz inequality. Using $\frac{K \log(n)}{n} \leq 1$ we complete the proof of Lemma 9. \square

Lemma 10. Assume that $W \in \Sigma(\gamma, L)$. Let $\Theta = (\Theta_{ij}) \in [0, 1]^{n \times n}$ be symmetric matrix with entries $\Theta_{ij} = W(\varepsilon_i, \varepsilon_j)$ for $i < j$, where ε_i are i.i.d. uniform random variables on $[0, 1]$. We have that, with probability at least $1 - 2/n$,

$$\delta(\tilde{f}_\Theta, W) \leq 2 \left(\frac{\log n}{n} \right)^{\frac{\gamma \wedge 1}{2}}$$

Proof. Following the proof of Proposition 3.6 in (Klopp et al. 2017), we get

$$\delta^2(\tilde{f}_\Theta, W) \leq \frac{2}{n} + \frac{1}{n} \sum_{m=1}^n \left| \frac{m}{n+1} - \varepsilon_{(m)} \right|^{2\gamma'}$$

where $\gamma' = \gamma \wedge 1$ and $\varepsilon_{(m)}$ stands for the m -th largest element of the set $\{\varepsilon_1, \dots, \varepsilon_n\}$. Note that, the random variable $\varepsilon_{(m)}$ follows β -distribution with parameters $(m, n+1-m)$, $\varepsilon_{(m)} \sim \text{Beta}(m, n+1-m)$. The β -distribution is sub-Gaussian and the proxy variance σ^2 for $\text{Beta}(m, n+1-m)$ is bounded by $\frac{1}{4(n+2)}$ (see, for example, (Marchal and Arbel 2017)). By the exponential Markov inequality (see, for example, (Vershynin 2018), Lemma 5.5) we get

$$\mathbb{P} \left\{ \left| \varepsilon_{(m)} - \frac{m}{n+1} \right| > t \right\} \leq 2e^{-t^2/(4\sigma^2)}.$$

Taking $t = (\log n / (n+2))^{1/2}$ implies that, with probability at least $1 - 2/n^2$,

$$\left| \varepsilon_{(m)} - \frac{m}{n+1} \right| < \left(\frac{\log n}{n+2} \right)^{1/2}.$$

Now, applying the union bound we obtain

$$\delta^2(\tilde{f}_\Theta, W) \leq \frac{2}{n} + \left(\frac{\log n}{n+2} \right)^{\gamma'}$$

and Lemma 10 follows. \square

D.2 Proof of the lower bound for K -step graphons

We will start with the definition of a class of K -step graphons used throughout the proof.

Let $u = (u_1, \dots, u_K) \in (-\frac{1}{K}, \frac{1}{K})^K$ be a given vector satisfying $\sum_{k=1}^K u_k = 0$. Define the partition $\Pi = \bigcup_{1 \leq k, l \leq K} \Pi_{kl}$ of the set $[0, 1]^2$ into K^2 blocks:

$$\Pi_{kl}(u) = \left[\frac{k-1}{K} + \sum_{i=1}^{k-1} u_i, \frac{k}{K} + \sum_{i=1}^k u_i \right) \times \left[\frac{l-1}{K} + \sum_{i=1}^{l-1} u_i, \frac{l}{K} + \sum_{i=1}^l u_i \right) \quad \forall 1 \leq k, l \leq K.$$

Let $Q = (Q_{kl})_{1 \leq k, l \leq K} \in [0, 1]^{K \times K}$ be a matrix of connection probabilities. The K -step graphon W_u is a blockwise constant function defined by

$$W_u(x, y) = \sum_{k, l \in [K]^2} Q_{kl} \mathbf{1}\{(x, y) \in \Pi_{kl}(u)\}. \quad (53)$$

Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in [0, 1]^n$ be the vector of i.i.d. features uniformly distributed over $[0, 1]$. Note that the community assignment of each vertex ϑ is defined by the corresponding variable ε_ϑ :

$$\mathbb{P}\left\{\vartheta \text{ belongs to the block } k\right\} = \mathbb{P}\left\{\varepsilon_\vartheta \in \left[\frac{k-1}{K} + \sum_{i=1}^{k-1} u_i, \frac{k}{K} + \sum_{i=1}^k u_i\right)\right\} = \frac{1}{K} + u_k.$$

We can introduce a new random variable $\xi_\vartheta = \xi_\vartheta(u)$, $\vartheta \in [n]$ of the block assignment following multinomial distribution $\mathcal{M}(K, p_1, \dots, p_K)$ with parameters $p_k = 1/K + u_k$. Denote the corresponding vector of i.i.d. multinomial variables by $\xi_u = (\xi_1(u), \dots, \xi_n(u)) \in [K]^n$. Given ξ_u , the connection probabilities are given by

$$\Theta_{ij}(\xi_u) = \begin{cases} \rho_n Q_{\xi_i(u), \xi_j(u)}, & i \neq j \\ 0, & i = j. \end{cases}$$

Denote by $\mathbb{P}_{W_u, \xi_u}(A^t)$ the conditional distribution of the dynamic network at time t given the node assignment ξ_u :

$$\begin{aligned} \mathbb{P}_{W_u, \xi_u}(A^t) &= \prod_{i < j} \Theta_{ij}(\xi_u)^{A_{ij}^t} (1 - \Theta_{ij}(\xi_u))^{1 - A_{ij}^t} \\ &= \prod_{i < j} \sum_{k, l \in [K]^2} (\rho_n Q_{kl})^{A_{ij}^t} (1 - \rho_n Q_{kl})^{1 - A_{ij}^t} \mathbf{1}\{(\xi_i(u), \xi_j(u)) = (k, l)\}. \end{aligned}$$

In what follows we denote by $A = (A^1, \dots, A^T)$ the full set of observations and by $A^{\leq \tau} = (A^1, \dots, A^\tau)$ and $A^{> \tau} = (A^{\tau+1}, \dots, A^T)$ the realizations before and after the time τ . We denote by $\mathbb{P}^{\otimes \tau}(A^{\leq \tau})$ and $\mathbb{P}^{\otimes (T-\tau)}(A^{> \tau})$ the corresponding product measures.

A K -step graphon depends on two main ingredients: the partition Π_K of $[0, 1]^2$ and the connection probability matrix Q . We will see that choosing different prior distributions on Q and Π will lead to two different lower bounds. The first lower bound, that we call *agnostic error lower bound*, will be derived from the uncertainty of sampling vector of features ε . The second one, that we call *network sampling lower bound*, comes from the uncertainty of random realizations of the network.

Without loss of generality, we will prove the result for $K = 2$. Indeed, for any $K > 2$,

$$\begin{aligned} &\inf_{\psi \in \Psi_\alpha} \sup_{\varepsilon, \varepsilon' \in [0, 1]^n} \sup_{(W^\tau, \varepsilon), (W^{\tau+1}, \varepsilon') \in \mathcal{W}_K(\delta_{n, T})} \mathbb{P}_{(W^\tau, \varepsilon), (W^{\tau+1}, \varepsilon')} \left\{ \psi = 0 \right\} \\ &\geq \inf_{\psi \in \Psi_\alpha} \sup_{\varepsilon, \varepsilon' \in [0, 1]^n} \sup_{(W^\tau, \varepsilon), (W^{\tau+1}, \varepsilon') \in \mathcal{W}_2(\delta_{n, T})} \mathbb{P}_{(W^\tau, \varepsilon), (W^{\tau+1}, \varepsilon')} \left\{ \psi = 0 \right\} \end{aligned}$$

and the boundary for the case of two blocks will imply the one for K blocks.

Proof of Theorem 5.

I. Agnostic error lower bound. The first lower bound is related to the error coming from the sampling of ε . We start by choosing a prior distribution on the graphons and the assignment vectors. Based on the prior, we will bound the Kullback-Leibler divergence between measures under the null and the alternative hypotheses. Note that it follows from (33) and the inequality $\frac{1}{2} \|\mathbb{P}_1 - \mathbb{P}_0\|_{\text{TV}} \leq \sqrt{\frac{1}{2} \text{KL}(\mathbb{P}_0, \mathbb{P}_1)}$ that the type II error is

bounded from below by β if $\text{KL}(\mathbf{P}_0, \mathbf{P}_1) \leq 2(1-\alpha-\beta)^2 = 2(1-\eta)^2$. Thus, we need to provide an upper bound on the Kullback–Leibler divergence that will imply the corresponding lower bound on the minimax detectable distance between graphons.

Step 1. Choice of priors. We will use W_u graphons defined in (53). We suppose that the connection probability matrix Q is the same under \mathbf{H}_0 and under \mathbf{H}_1 and is defined as $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

1. *Prior under \mathbf{H}_0 :* we choose vector $u = 0$ and get a partition Π with blocks of a constant size $1/2$. The corresponding graphon is denoted by W_0 . Let ξ_0 and ξ'_0 be two independent blocks assignment vectors following the multinomial distribution with class probabilities $p_k = 1/2$, $k = 1, 2$. Since matrix Q does not change, the corresponding conditional distributions \mathbf{P}_{W_0, ξ_0} and \mathbf{P}_{W_0, ξ'_0} coincide. Then, the measure under \mathbf{H}_0 be given by

$$\mathbf{P}_0(A) = \left(\sum_{a \in \{1,2\}^n} \mathbf{P}(\xi_0 = a) \mathbf{P}_{W_0, \xi_0=a}^{\otimes \tau}(A^{\leq \tau}) \right) \left(\sum_{b \in \{1,2\}^n} \mathbf{P}(\xi'_0 = b) \mathbf{P}_{W_0, \xi'_0=b}^{\otimes (T-\tau)}(A^{> \tau}) \right).$$

2. *Prior under \mathbf{H}_1 :* fix some $0 < \varepsilon < 1/2$. Let $u = (\varepsilon, -\varepsilon)$, $v = (-\varepsilon, \varepsilon)$ and W_u, W_v be the corresponding graphons with the probabilities of classes $1/2 + \varepsilon$ and $1/2 - \varepsilon$. The only difference between these two graphons is a slight disequilibrium around $(x, y) \in [1/2 - \varepsilon, 1/2 + \varepsilon]^2$. It is not difficult to see that $\delta^2(W_u, W_v) \geq 2\varepsilon$.

Let $\xi_u \in \{1, 2\}^n$ and $\xi_v \in \{1, 2\}^n$ be two independent class assignment vectors before and after the change in the graphon such that $\mathbf{P}(\xi_i(u) = k) = \frac{1}{2} + u_k$, $\mathbf{P}(\xi_i(v) = k) = \frac{1}{2} + v_k \forall i = 1, \dots, n, k = 1, 2$. The measure under \mathbf{H}_1 is defined as

$$\mathbf{P}_1(A) = \left(\sum_{a \in \{1,2\}^n} \mathbf{P}(\xi_u = a) \mathbf{P}_{W_u, \xi_u=a}^{\otimes \tau}(A^{\leq \tau}) \right) \left(\sum_{b \in \{1,2\}^n} \mathbf{P}(\xi_v = b) \mathbf{P}_{W_v, \xi_v=b}^{\otimes (T-\tau)}(A^{> \tau}) \right).$$

Step 2. Bounding the divergence. Denote for brevity $\mathbf{P}(\xi = a)$ by $\mathbf{P}_\xi(a)$. Since the matrix Q is the same for all graphons, the conditional probabilities generating the networks under \mathbf{H}_0 and under \mathbf{H}_1 are the same. Denote

$$P_Q(A^{\leq \tau} | a) := \mathbf{P}_{W_0, \xi_0=a}^{\otimes \tau}(A^{\leq \tau}) = \mathbf{P}_{W_u, \xi_u=a}^{\otimes \tau}(A^{\leq \tau})$$

and

$$P_Q(A^{> \tau} | b) := \mathbf{P}_{W_0, \xi'_0=b}^{\otimes (T-\tau)}(A^{> \tau}) = \mathbf{P}_{W_v, \xi_v=b}^{\otimes (T-\tau)}(A^{> \tau}).$$

Then, we have

$$\begin{aligned}
\text{KL}(P_0, P_1) &= \sum_A \left(\sum_{a \in \{1,2\}^n} P_{\xi_0}(a) P_Q(A^{\leq \tau} | a) \right) \left(\sum_{b \in \{1,2\}^n} P_{\xi'_0}(b) P_Q(A^{> \tau} | b) \right) \\
&\times \log \left(\frac{\left(\sum_{a \in \{1,2\}^n} P_{\xi_0}(a) P_Q(A^{\leq \tau} | a) \right) \left(\sum_{b \in \{1,2\}^n} P_{\xi'_0}(b) P_Q(A^{> \tau} | b) \right)}{\left(\sum_{a \in \{1,2\}^n} P_{\xi_u}(a) P_Q(A^{\leq \tau} | a) \right) \left(\sum_{b \in \{1,2\}^n} P_{\xi_v}(b) P_Q(A^{> \tau} | b) \right)} \right) \\
&= \sum_{A^{\leq \tau}} \sum_{a \in \{1,2\}^n} P_{\xi_0}(a) P_Q(A^{\leq \tau} | a) \log \left(\frac{\sum_{a \in \{1,2\}^n} P_{\xi_0}(a) P_Q(A^{\leq \tau} | a)}{\sum_{a \in \{1,2\}^n} P_{\xi_u}(a) P_Q(A^{\leq \tau} | a)} \right) \\
&+ \sum_{A^{> \tau}} \sum_{b \in \{1,2\}^n} P_{\xi'_0}(b) P_Q(A^{> \tau} | b) \log \left(\frac{\sum_{b \in \{1,2\}^n} P_{\xi'_0}(b) P_Q(A^{> \tau} | b)}{\sum_{b \in \{1,2\}^n} P_{\xi_v}(b) P_Q(A^{> \tau} | b)} \right).
\end{aligned}$$

Thus, taking into account that the function $f(x, y) = x \log(x/y)$ is convex, we can apply the Jensen's inequality and obtain that

$$\begin{aligned}
\text{KL}(P_0, P_1) &\leq \sum_{a \in \{1,2\}^n} P_{\xi_0}(a) \log \frac{P_{\xi_0}(a)}{P_{\xi_u}(a)} \sum_{A^{\leq \tau}} P_Q(A^{\leq \tau} | a) \\
&+ \sum_{b \in \{1,2\}^n} P_{\xi'_0}(b) \log \frac{P_{\xi'_0}(b)}{P_{\xi_v}(b)} \sum_{A^{> \tau}} P_Q(A^{> \tau} | b) \\
&= n \left(\text{KL}(P_{\xi_0}, P_{\xi_u}) + \text{KL}(P_{\xi'_0}, P_{\xi_v}) \right).
\end{aligned}$$

The last inequality follows from the fact that $P_{\xi}(a)$ are product probabilities. Thus we have to bound the Kullback–Leibler divergence between two binomial distributions. Using the inequality $\log(1+x) \geq x/(1+x) \forall x > -1$, we obtain

$$\text{KL}(P_{\xi_0}, P_{\xi_u}) = \frac{1}{2} \log \left(\frac{1/4}{1/4 - \varepsilon^2} \right) \leq \frac{2\varepsilon^2}{1 - 4\varepsilon^2}$$

which implies $\text{KL}(P_0, P_1) \leq 4n\varepsilon^2/(1 - 4\varepsilon^2)$. Recall that $\delta^2(W_u, W_v) \geq 2\varepsilon$. Consequently, if $\delta = \delta(W_u, W_v)$, we can write

$$\text{KL}(P_0, P_1) \leq \frac{n}{2} \frac{\delta^4}{1 - \delta^4} \leq 2(1 - \eta)^2$$

if

$$\delta^4 \leq \frac{4n^{-1}(1 - \eta)^2}{1 + 4n^{-1}(1 - \eta)^2}.$$

The last inequality is true if $\delta^4 \leq \frac{8}{3}n^{-1}(1 - \eta)^2$ for all $n \geq 8$. It implies the lower bound condition on the distance between graphons:

$$\delta(W_u, W_v) \leq \left(\frac{8}{3} \right)^{1/4} (1 - \eta)^{1/2} n^{-1/4}.$$

II. Network Sampling lower bound. In this part we will suppose that the transition matrix Q changes but the partition Π does not change. In order to bound the type II error by β from below, we need to show that the chi-squared divergence between the mixtures under H_0 and H_1 is smaller than $4(1 - \eta)^2$.

Step 1. Choice of priors. It will be sufficient to show the result for the case of $K = 2$, since as we will see the lower bound on the separation rate is independent of K . We will work with 2-step graphons with fixed partition Π into 4 equal blocks $\Pi_{kl} = [k - 1/2, k/2) \times [l - 1/2, l/2)$, $1 \leq k, l \leq 2$.

Prior under H_0 . We suppose that under H_0 the connection probabilities are all equal to $1/2$, that is $Q = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ and $\Theta_{ij} = \rho_n/2$, $\forall i \neq j$. The corresponding graphon is denoted by W_0 . Denote $p_0 = \rho_n/2$. Then, independently of the feature vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$,

$$P_0(A) = \prod_{i < j} \prod_{i=1}^T p_0^{\sum_{t=1}^T A_{ij}^t} (1 - p_0)^{T - \sum_{t=1}^T A_{ij}^t}$$

Prior under H_1 . Denote by Q_1 and Q_2 the connection probability matrices before and after the change-point. Let $\varepsilon > 0$. We assume that

$$Q_1 = \rho_n^{-1} \begin{pmatrix} p_1 & p_2 \\ p_2 & p_1 \end{pmatrix} \quad \text{and} \quad Q_2 = \rho_n^{-1} \begin{pmatrix} p_3 & p_4 \\ p_4 & p_3 \end{pmatrix}$$

where

$$p_1 = \rho_n \left(\frac{1}{2} + \left(1 - \frac{\tau}{T}\right) \varepsilon \right), \quad p_2 = \rho_n \left(\frac{1}{2} - \left(1 - \frac{\tau}{T}\right) \varepsilon \right), \\ p_3 = \rho_n \left(\frac{1}{2} - \frac{\tau}{T} \varepsilon \right), \quad p_4 = \rho_n \left(\frac{1}{2} + \frac{\tau}{T} \varepsilon \right).$$

Denote the corresponding graphons by W_1 and W_2 and the corresponding matrices of connection probabilities by Θ_i , $i = 1, 2$. Let $\xi = (\xi_1, \dots, \xi_n)$ be the class assignment vector of i.i.d. variables taking values $\{1, 2\}$ with probability $1/2$. Then

$$P_1(A) = P_{W_1}^{\otimes \tau}(A^{\leq \tau}) P_{W_2}^{\otimes (T - \tau)}(A^{> \tau})$$

where

$$P_{W_1}(A^t) = \prod_{i < j} \left(p_1^{A_{ij}^t} (1 - p_1)^{1 - A_{ij}^t} \mathbf{1}\{\xi_i = \xi_j\} + p_2^{A_{ij}^t} (1 - p_2)^{1 - A_{ij}^t} \mathbf{1}\{\xi_i \neq \xi_j\} \right)$$

and

$$P_{W_2}(A^t) = \prod_{i < j} \left(p_3^{A_{ij}^t} (1 - p_3)^{1 - A_{ij}^t} \mathbf{1}\{\xi_i = \xi_j\} + p_4^{A_{ij}^t} (1 - p_4)^{1 - A_{ij}^t} \mathbf{1}\{\xi_i \neq \xi_j\} \right)$$

Step 2. Bounding the χ^2 -divergence. To prove the lower bound, we will need an upper bound on the chi-squared divergence

$$\chi^2(P_0, P_1) = E_{P_0} \left(\frac{dP_1}{dP_0} \right)^2 - 1 \leq 4(1 - \alpha - \beta)^2 = 4(1 - \eta)^2.$$

This bound will follow from the upper bound on the second moment of the likelihood ratio:

$$\mathbb{E}_{P_0} L^2(A) \leq 1 + 4(1 - \eta)^2$$

where $L(A) = \frac{dP_1}{dP_0}(A)$. Define the set $S = \{ \{a, b\} \in [n]^2 : a < b, \xi_a = \xi_b \}$ and its complement S^c . Denote by $N = n(n - 1)/2$ the cardinality of $S \cup S^c$ and by μ the distribution of S . Then, $L(A) = \int L_S(A) d\mu(S)$, where

$$\begin{aligned} L_S(A) &= \left(\frac{1 - p_1}{1 - p_0} \right)^{\tau|S|} \left(\frac{1 - p_3}{1 - p_0} \right)^{(T-\tau)|S|} \left(\frac{1 - p_2}{1 - p_0} \right)^{\tau|S^c|} \left(\frac{1 - p_4}{1 - p_0} \right)^{(T-\tau)|S^c|} \\ &\times \prod_{\{i,j\} \in S} \left(\frac{p_1(1 - p_0)}{p_0(1 - p_1)} \right)^{\sum_{t=1}^{\tau} A_{ij}^t} \left(\frac{p_3(1 - p_0)}{p_0(1 - p_3)} \right)^{\sum_{t=\tau+1}^T A_{ij}^t} \\ &\times \prod_{\{i,j\} \in S^c} \left(\frac{p_2(1 - p_0)}{p_0(1 - p_2)} \right)^{\sum_{t=1}^{\tau} A_{ij}^t} \left(\frac{p_4(1 - p_0)}{p_0(1 - p_4)} \right)^{\sum_{t=\tau+1}^T A_{ij}^t}. \end{aligned}$$

We need to find the second moment of $L(A)$. Let S_1 and S_2 be two independent copies of S , then $\mathbb{E}_{P_0}[L^2(A)] = \int \mathbb{E}_{P_0}[L_{S_1}(A)L_{S_2}(A)] d\mu(S_1)d\mu(S_2)$. We have that

$$\begin{aligned} \mathbb{E}_{P_0}[L_{S_1}(A)L_{S_2}(A)] &= \left(\frac{1 - p_1}{1 - p_0} \right)^{\tau(|S_1|+|S_2|)} \left(\frac{1 - p_3}{1 - p_0} \right)^{(T-\tau)(|S_1|+|S_2|)} \\ &\times \left(\frac{1 - p_2}{1 - p_0} \right)^{\tau(|S_1^c|+|S_2^c|)} \left(\frac{1 - p_4}{1 - p_0} \right)^{(T-\tau)(|S_1^c|+|S_2^c|)} \\ &\times \mathbb{E}_{P_0} \left[\prod_{\{i,j\} \in S_1 \cap S_2} \left(\frac{p_1^2(1 - p_0)^2}{p_0^2(1 - p_1)^2} \right)^{\sum_{t=1}^{\tau} A_{ij}^t} \left(\frac{p_3^2(1 - p_0)^2}{p_0^2(1 - p_3)^2} \right)^{\sum_{t=\tau+1}^T A_{ij}^t} \right. \\ &\times \prod_{\{i,j\} \in S_1^c \cap S_2^c} \left(\frac{p_2^2(1 - p_0)^2}{p_0^2(1 - p_2)^2} \right)^{\sum_{t=1}^{\tau} A_{ij}^t} \left(\frac{p_4^2(1 - p_0)^2}{p_0^2(1 - p_4)^2} \right)^{\sum_{t=\tau+1}^T A_{ij}^t} \\ &\left. \times \prod_{\{i,j\} \in S_1 \Delta S_2} \left(\frac{p_1 p_2(1 - p_0)^2}{p_0^2(1 - p_1)(1 - p_2)} \right)^{\sum_{t=1}^{\tau} A_{ij}^t} \left(\frac{p_3 p_4(1 - p_0)^2}{p_0^2(1 - p_3)(1 - p_4)} \right)^{\sum_{t=\tau+1}^T A_{ij}^t} \right]. \end{aligned}$$

Taking into account the relations $p_1 - p_0 = (1 - \tau/T)\rho_n \varepsilon$, $p_2 - p_0 = -(1 - \tau/T)\rho_n \varepsilon$,

$p_3 - p_0 = -(\tau/T)\rho_n\varepsilon$, $p_4 - p_0 = (\tau/T)\rho_n\varepsilon$ and Lemma B.2, we obtain

$$\begin{aligned} \mathbb{E}_{P_0}[L_{S_1}(A)L_{S_2}(A)] &= \left(1 + \left(1 - \frac{\tau}{T}\right)^2 \frac{\varepsilon^2 \rho_n^2}{p_0(1-p_0)}\right)^{\tau(|S_1 \cap S_2| + |S_1^c \cap S_2^c|)} \\ &\quad \times \left(1 + \left(\frac{\tau}{T}\right)^2 \frac{\varepsilon^2 \rho_n^2}{p_0(1-p_0)}\right)^{(T-\tau)(|S_1 \cap S_2| + |S_1^c \cap S_2^c|)} \\ &\quad \times \left(1 - \left(1 - \frac{\tau}{T}\right)^2 \frac{\varepsilon^2 \rho_n^2}{p_0(1-p_0)}\right)^{\tau|S_1 \Delta S_2|} \\ &\quad \times \left(1 - \left(\frac{\tau}{T}\right)^2 \frac{\varepsilon^2 \rho_n^2}{p_0(1-p_0)}\right)^{(T-\tau)|S_1 \Delta S_2|}. \end{aligned}$$

Next, using the fact that $|S_1 \Delta S_2| + |S_1 \cap S_2| + |S_1^c \cap S_2^c| = N$, we obtain

$$\begin{aligned} \mathbb{E}_{P_0}[L_{S_1}(A)L_{S_2}(A)] &\leq \\ &\exp \left[(|S_1 \cap S_2| + |S_1^c \cap S_2^c|) \left(\tau \left(1 - \frac{\tau}{T}\right)^2 + (T-\tau) \left(\frac{\tau}{T}\right)^2 \right) \frac{\varepsilon^2 \rho_n^2}{p_0(1-p_0)} \right. \\ &\quad \left. - |S_1 \Delta S_2| \left(\tau \left(1 - \frac{\tau}{T}\right)^2 + (T-\tau) \left(\frac{\tau}{T}\right)^2 \right) \frac{\varepsilon^2 \rho_n^2}{p_0(1-p_0)} \right] \\ &= \exp \left[q^2 \left(\frac{\tau}{T}\right) \frac{T\varepsilon^2 \rho_n^2}{p_0(1-p_0)} (2|S_1 \cap S_2| + 2|S_1^c \cap S_2^c| - N) \right]. \end{aligned}$$

Note that $p_0 = \rho_n/2 < 1/2$. Thus, in order to bound the second moment likelihood ratio, we need to control the exponential moment

$$\mathbb{E}_{P_0}[L^2(A)] \leq \mathbb{E}_{S_1, S_2} \left\{ \exp \left[\left(2|S_1 \cap S_2| + 2|S_1^c \cap S_2^c| - N\right) 4Tq^2 \left(\frac{\tau}{T}\right) \varepsilon^2 \rho_n \right] \right\}.$$

We need to control the exponential moment of the random variable $U = |S_1 \cap S_2| + |S_1^c \cap S_2^c|$, where S_1 and S_2 are independent random variables distributed according to μ . Following the last lines of Lemma 4.9 in (Klopp et al. 2017), denote by $\xi^{(1)} = (\xi_1, \dots, \xi_n)$ and $\xi^{(2)} = (\xi_1, \dots, \xi_n)$ the assignment vectors corresponding to the variables S_1 and S_2 , respectively. For any $(i, j) \in \{1, 2\}^2$ introduce the random variable that counts the number of nodes in the classes i and j according to the first and the second assignement:

$$N_{ij} = \left| \left\{ a \in [n] : \xi_a^{(1)} = i, \xi_a^{(2)} = j \right\} \right|, \quad (i, j) \in \{1, 2\}^2.$$

Then $2|S_1 \cap S_2| + n = N_{11}^2 + N_{12}^2 + N_{21}^2 + N_{22}^2$ and $2|S_1^c \cap S_2^c| = 2N_{11}N_{22} + 2N_{12}N_{21}$. Hence, $2U + n = (N_{11} + N_{22})^2 + (N_{12} + N_{21})^2$. Note that $N_{11} + N_{22} + N_{12} + N_{21} = n$. Let $Z := N_{11} + N_{22} - n/2$. It is a centered binomial random variable with parameters $(n, 1/2)$ and

$$2U - N = (n/2 + Z)^2 + (n/2 - Z)^2 - n - N = 2Z^2 - n/2.$$

Consequently, we need to control the exponential moment of Z^2 :

$$\mathbb{E}_{P_0}[L^2(A)] \leq \mathbb{E} \exp \left[8Tq^2 \left(\frac{\tau}{T}\right) \varepsilon^2 \rho_n Z^2 \right].$$

Using Hoeffding's inequality, we can show that $P(Z^2 > t) \leq 2e^{-2t/n}$, thus Z^2 is subexponential with the moments $E[Z^{2k}] \leq n^k k!$. Consequently, for any γ_n such that $0 < n\gamma_n < 1$ we have

$$Ee^{\gamma_n Z^2} \leq 1 + \sum_{k=1}^{+\infty} \frac{\gamma_n^k E[Z^{2k}]}{k!} \leq \sum_{k=0}^{+\infty} (n\gamma_n)^k = \frac{1}{1 - n\gamma_n}.$$

Set $\gamma_n = 8Tq^2 \left(\frac{\tau}{T}\right) \varepsilon^2 \rho_n$. We can see that if $n\gamma_n \leq 4(1-\eta)^2(1+4(1-\eta)^2)^{-1} < 1$, then $E_{P_0}[L^2(A)] \leq 1 + 4(1-\eta)^2$. Since $\delta^2(W_1, W_2) = 4\varepsilon^2$, we have $\gamma_n = 2Tn\rho_n q^2 \left(\frac{\tau}{T}\right) \delta^2$ and the above condition on $n\gamma_n$ implies the lower bound

$$q\left(\frac{\tau}{T}\right) \delta(W_1, W_2) \leq \frac{\sqrt{2}(1-\eta)}{(1+4(1-\eta)^2)^{1/2}} \frac{1}{\sqrt{n\rho_n T}}$$

and the second part of the theorem follows. □

E Auxiliary results

E.1 Concentration inequalities for matrix processes

The first result is the concentration inequality for the operator norm of a random matrix with independent entries (see [Bandeira and van Handel \(2016\)](#), Corollary 3.12 and Remark 3.13):

Proposition 11 (Bandeira and Van Handel, 2016). *Let W be an $m \times m$ symmetric matrix whose entries W_{ij} are independent centered random variables bounded (in absolute value) by some $\sigma_* > 0$. Then, for any $0 < \epsilon \leq 1/2$ there exists a universal constant c_ϵ such that, for every $x \geq 0$*

$$P \left\{ \|W\|_{2 \rightarrow 2} \geq 2\sqrt{2}(1+\epsilon)\sigma + x \right\} \leq m \exp \left(-\frac{x^2}{c_\epsilon \sigma_*^2} \right),$$

where $\sigma = \max_i \left[\sum_j \text{var}(W_{ij}) \right]^{1/2}$.

Next we apply this concentration inequality to the Matrix CUSUM statistics.

Let $X^t \in [-1, 1]^{n \times n}$ ($1 \leq t \leq T$) be a sequence of matrices with independent entries X_{ij}^t for any $1 \leq i, j \leq n$ and for any $t = 1, \dots, T$. Assume that X_{ij}^t are centered Bernoulli random variables taking values in $\{1 - B_{ij}^t, -B_{ij}^t\}$ with success probability B_{ij}^t . Consider the following centered matrix process defined in (11):

$$\xi(t) = \sqrt{\frac{t(T-t)}{T}} \left(\frac{1}{t} \sum_{s=1}^t X^s - \frac{1}{T-t} \sum_{s=t+1}^T X^s \right), \quad 1 \leq t \leq T-1.$$

Lemma 12. For any $\varepsilon \in (0, 1/2]$ there exists an absolute constant C_ε such that, for every $\delta \in (0, 1)$ we have

$$\|\xi(t)\|_{2 \rightarrow 2} \leq 2\sqrt{2}(1+\varepsilon)\sqrt{\frac{t(T-t)}{T}} \left[\frac{1}{t^2} \sum_{s=1}^t \|B^s\|_{1,\infty} + \frac{1}{(T-t)^2} \sum_{s=t+1}^T \|B^s\|_{1,\infty} \right]^{1/2} + C_\varepsilon \log \frac{2n}{\delta} \quad (54)$$

with the probability larger than $1 - \delta$.

Proof. The result follows from the direct application of Proposition 11. Since X^s are independent, we can easily estimate σ^2 from above:

$$\begin{aligned} \sigma^2 &= \max_i \sum_j \text{Var}[\xi_{ij}(t)] \\ &= \frac{t(T-t)}{T} \max_i \sum_j \left(\frac{1}{t^2} \sum_{s=1}^t B_{ij}^s (1 - B_{ij}^s) + \frac{1}{(T-t)^2} \sum_{s=t+1}^T B_{ij}^s (1 - B_{ij}^s) \right) \\ &\leq \frac{t(T-t)}{T} \max_i \left(\frac{1}{t^2} \sum_{s=1}^t \sum_j B_{ij}^s + \frac{1}{(T-t)^2} \sum_{s=t+1}^T \sum_j B_{ij}^s \right) \\ &= \frac{t(T-t)}{T} \left(\frac{1}{t^2} \sum_{s=1}^t \|B^s\|_{1,\infty} + \frac{1}{(T-t)^2} \sum_{s=t+1}^T \|B^s\|_{1,\infty} \right). \end{aligned}$$

We will show now that the norm $\|\xi(t)\|_\infty$ is bounded by some σ^* with high probability. Consider the entries of the matrix $\xi(t)$ defined by

$$\xi_{ij}(t) = \sqrt{\frac{t(T-t)}{T}} \left(\frac{1}{t} \sum_{s=1}^t X_{ij}^s - \frac{1}{T-t} \sum_{s=t+1}^T X_{ij}^s \right) = \sum_{s=1}^T V_{ij}^s$$

Since $-B_{ij}^s \leq X_{ij}^s \leq 1 - B_{ij}^s$ we have $a_s \leq V_{ij}^s \leq b_s$ with

$$b_s - a_s \leq \sqrt{\frac{t(T-t)}{T}} \left(\frac{1}{t} \mathbf{1}_{\{1 \leq s \leq t\}} + \frac{1}{T-t} \mathbf{1}_{\{t < s \leq T\}} \right).$$

Since X_{ij}^s are independent, applying the Hoeffding inequality, we obtain for any $x > 0$ that for any $1 \leq i, j \leq n$,

$$\mathbb{P}\left\{ |\xi_{ij}(t)| > x \right\} \leq 2 \exp\left\{ -\frac{2x^2}{\sum_s (b_s - a_s)^2} \right\} = 2e^{-2x^2}.$$

Using the union bound, we get that

$$\mathbb{P}\left\{ \|\xi(t)\|_\infty > x \right\} \leq n^2 \mathbb{P}\left\{ |\xi_{ij}(t)| > x \right\} = 2n^2 e^{-2x^2}.$$

Consequently, for any $\delta \in (0, 1)$ we have $\|\xi(t)\|_\infty \leq \log^{1/2}(2n/\sqrt{\delta}) \leq \log^{1/2}(2n/\delta)$ with probability larger than $1 - \delta/2$. Applying Proposition 11 given the event $\left\{ \|\xi(t)\|_\infty \leq \sigma_* \right\}$ with $\sigma_* = \log^{1/2}(2n/\delta)$ we get

$$\mathbb{P}\left\{ \|\xi(t)\|_{2 \rightarrow 2} \geq 2\sqrt{2}(1+\varepsilon)\sigma + x \right\} \leq n \exp\left\{ -\frac{x^2}{c_\varepsilon \log(2n/\delta)} \right\} + \frac{\delta}{2}.$$

Choosing $x = \sqrt{c_\varepsilon} \log(2n/\delta)$ and $C_\varepsilon = c_\varepsilon^{1/2}$ we obtain (54). \square

E.2 Result on the Hadamard product of two matrices

Lemma 13. Let $A = (A_{ij}) \in [0, \infty)^{n \times n}$ and $B = (B_{ij}) \in \mathbb{R}^{n \times n}$. Assume that $\text{diag}(B) = 0$. Then,

$$\|A \odot B\|_{2 \rightarrow 2} \geq \frac{\min_{(ij): i \neq j} A_{ij}}{\sqrt{r} \sqrt{1}} \|B\|_{2 \rightarrow 2}, \quad (55)$$

where $r = \text{rank}(A \odot B)$. Moreover, if A and B are symmetric, we have that

$$\|A \odot B\|_{2 \rightarrow 2} \geq \frac{\min_{ij} A_{ij}}{2\sqrt{r^*} \sqrt{1}} \|B\|_{2 \rightarrow 2}, \quad (56)$$

where

$$r^* = \min_{M=(M_{ij}): M_{ij}=(A \odot B)_{ij} \text{ for } i \neq j} \text{rank}(M).$$

Proof. If $\min_{(ij): i \neq j} A_{ij} = 0$, then the statement of the Lemma is trivially true. Now assume that $\min_{(ij): i \neq j} A_{ij} > 0$. We have that

$$\begin{aligned} \|B\|_{2 \rightarrow 2}^2 &\leq \|B\|_F^2 = \sum_{i \neq j} A_{ij}^2 (A_{ij}^{-1})^2 B_{ij}^2 \\ &\leq \max_{i \neq j} (A_{ij}^{-1})^2 \|A \odot B\|_F^2 \leq r \max_{i \neq j} (A_{ij}^{-1})^2 \|A \odot B\|_{2 \rightarrow 2}^2 \end{aligned}$$

which implies (55). On the other hand, let M be a solution to

$$M \in \arg \min_{M=(M_{ij}): M_{ij}=(A \odot B)_{ij} \text{ for } i \neq j} \text{rank}(M).$$

Let $\text{rank}(M) = r^* < r$. We have that

$$\begin{aligned} \|B\|_{2 \rightarrow 2}^2 &\leq \|B\|_F^2 = \sum_{i \neq j} A_{ij}^2 (A_{ij}^{-1})^2 B_{ij}^2 \leq \max_{i \neq j} (A_{ij}^{-1})^2 \left\{ \sum_{i \neq j} (A_{ij} B)_{ij}^2 + \sum_i M_{ii}^2 \right\} \\ &= \max_{i \neq j} (A_{ij}^{-1})^2 \|M\|_F^2 \leq r^* \max_{i \neq j} (A_{ij}^{-1})^2 \|M\|_{2 \rightarrow 2}^2 \end{aligned}$$

which implies

$$\|B\|_{2 \rightarrow 2} \leq \sqrt{r^*} \max_{i \neq j} (A_{ij}^{-1}) \|M\|_{2 \rightarrow 2} \leq 2\sqrt{r^*} \max_{i \neq j} (A_{ij}^{-1}) \|A \odot B\|_{2 \rightarrow 2} \quad (57)$$

where in the last inequality we use that $\|M\|_{2 \rightarrow 2} \leq 2\|A \odot B\|_{2 \rightarrow 2}$. To prove it, using the triangle inequality, it is enough to prove that $\|M - A \odot B\|_{2 \rightarrow 2} \leq \|A \odot B\|_{2 \rightarrow 2}$. Let denote by $\lambda_1(X) \leq \lambda_2(X) \leq \dots \leq \lambda_n(X)$ the eigenvalues of a symmetric matrix X . Then, using Weyl's inequality, we have that

$$\lambda_{j+k-n}(A \odot B) \leq \lambda_j(A \odot B - M) + \lambda_k(M) \leq \lambda_{j+k-1}(A \odot B). \quad (58)$$

Note that $r^* < r$ implies that there exist a k such that $\lambda_k(M) = 0$ but $\lambda_k(A \odot B) \neq 0$. Assume first that $\|M - A \odot B\|_{2 \rightarrow 2} = -\lambda_1(M - A \odot B)$. Then, taking in (58) $j = 1$ we get $-\lambda_1(M - A \odot B) \leq \lambda_k(A \odot B) \leq \|A \odot B\|_{2 \rightarrow 2}$. Now, if $\|M - A \odot B\|_{2 \rightarrow 2} = \lambda_n(M - A \odot B)$, taking $j = n$ we also get $\lambda_n(M - A \odot B) \leq -\lambda_k(A \odot B) \leq \|A \odot B\|_{2 \rightarrow 2}$.

To conclude the proof, note that (57) implies (56). \square