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Abstract. We define logarithmic tangent sheaves associated with complete intersections in connection with Jacobian syzygies and distributions. We analyse the notions of local freeness, freeness and stability of these sheaves.

We carry out a complete study of logarithmic sheaves associated with pencils of quadrics and compute their projective dimension from the classical invariants such as the Segre symbol and new invariants (splitting type and degree vector) designed for the classification of irregular pencils. This leads to a complete classification of free (equivalently, locally free) pencils of quadrics.

Finally we produce examples of locally free, non free pencils of surfaces in $\mathbb{P}^3$ of arbitrary degree $k \geqslant 3$, answering (in the negative) a question of Calvo-Andrade, Cerveau, Giraldo and Lins Neto about codimension foliations on $\mathbb{P}^3$.

Contents

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1. Introduction

Let κ be an algebraically closed field of characteristic zero and consider a regular sequence $\sigma = (f_1, \dots, f_k)$ of homogeneous polynomials $f_i \in R = \kappa[x_0, \dots, x_n]$ of degree $d_i + 1$, for some $0 \leqslant d_1 \leqslant \cdots \leqslant d_k$ and $k \leqslant n$. Let $I_\sigma := (f_1, \dots, f_k)$ denote the ideal generated by the sequence σ, and $V(\sigma)$ be the associated scheme-theoretic complete intersection in $\mathbb{P}^n$. Consider the Jacobian matrix of σ, namely:

$$\mathcal{J}_\sigma := \begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_k \end{pmatrix}.$$

This can be viewed as a morphism of sheaves:

$$\mathcal{J}_\sigma \colon \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \longrightarrow \oplus_{i=1}^k \mathcal{O}_{\mathbb{P}^n}(d_i).$$

The main focus of this paper concerns the sheaf:

$$\mathcal{T}_\sigma := \ker(\mathcal{J}_\sigma).$$

We call $\mathcal{T}_\sigma$ the *logarithmic tangent sheaf associated to* σ. This nomenclature for $\mathcal{T}_\sigma$ is motivated by the following observation. Set $X = V(\sigma)$ and recall that the Zariski tangent sheaf $\mathrm{T}X$ and the sheaf T'_X supported at $\mathrm{Sing}(X)$ fit into:

$$0 \to \mathrm{T}X \to \mathrm{T}\mathbb{P}^n|_X \to \bigoplus_{i=1}^k \mathcal{O}_X(d_i + 1) \to \mathrm{T}'_X \to 0.$$

The *sheaf* $\mathrm{T}_{\mathbb{P}^n}\langle X\rangle$ *of vector fields on* $\mathbb{P}^n$ *tangent to* X is the kernel of the natural morphism $\mathrm{T}\mathbb{P}^n \to \bigoplus_{i=1}^k \mathcal{O}_X(d_i + 1)$, see [10, Chapter 3]. It turns out that $\mathcal{T}_\sigma(1)$ is a subsheaf of rank $n - k + 1$ of $\mathrm{T}_{\mathbb{P}^n}\langle X\rangle$. More precisely (see Lemma 2.4), writing $\mathcal{V}_\sigma = \bigoplus_{i=1}^k \mathcal{I}_X(d_i + 1)/\mathcal{O}_{\mathbb{P}^n}$ and denoting by $\mathcal{Q}_\sigma$ the cokernel of $\mathcal{J}_\sigma$, we have:

$$0 \to \mathcal{T}_\sigma(1) \to \mathrm{T}_{\mathbb{P}^n}\langle X\rangle \to \mathcal{V}_\sigma \to \mathcal{Q}_\sigma(1) \to \mathrm{T}'_X \to 0.$$

When the sequence σ consists of a single polynomial f (so that $k = 1$), then $\mathcal{V}_\sigma = 0$ so $\mathcal{T}_\sigma(1) \simeq \mathrm{T}_{\mathbb{P}^n}\langle X\rangle$ is precisely the logarithmic tangent sheaf associated to the divisor $V(f)$, see for instance [11] or the celebrated [9].

Note that, for $k \geqslant 2$, the sheaf $\mathrm{T}_{\mathbb{P}^n}\langle X\rangle$ cannot be locally free. On the other hand, as we shall see, $\mathcal{T}_\sigma$ may be locally free or even completely decomposable. Hence we propose the following three definitions, whose goal is to generalize the usual concept of a *free divisor* introduced in [9].

Definition. *A regular sequence σ is said to be:*

(1) locally free *if the associated logarithmic tangent sheaf $\mathcal{T}_\sigma$ is locally free.*
(2) free *if the logarithmic tangent sheaf $\mathcal{T}_\sigma$ splits as a sum of line bundles.*
(3) strongly free *if every sequence σ' such that $I_{\sigma'} = I_\sigma$ is free.*

Clearly, every free regular sequence of length $k = 1$ is also strongly free. On the other extreme case, we observe that if σ is a regular sequence of length $k = n$, then $\mathcal{T}_\sigma = \mathcal{O}_{\mathbb{P}^n}(e)$ for some negative integer e, since every rank 1 reflexive sheaf on $\mathbb{P}^n$ is a line bundle. Therefore, every regular sequence σ in $\kappa[x_0, \dots, x_n]$ of length n is strongly free. We provide explicit examples of free and strongly free regular sequences of length 2 in $\kappa[x_0, x_1, x_2, x_3]$, see Example 2.7 and Example 2.8 below.

Regarding the middle range $1 < k < n$, recall that it is notoriously hard to construct indecomposable locally free sheaves of rank r on $\mathbb{P}^n$ when $2 \leqslant r \leqslant n-2$. In fact, only two examples are known, the Horrocks–Mumford rank 2 bundle on $\mathbb{P}^4$ and Horrocks' rank 3 parent bundle on $\mathbb{P}^5$. Furthermore, Hartshorne's conjecture predicts that every locally free sheaf of rank r on $\mathbb{P}^n$ with $3r < n$ splits as a sum of line bundles, which would imply that locally free regular sequences of length k in $\kappa[x_0, \dots, x_n]$ are free whenever $3k > 2n + 3$.

With these facts in mind, it seems natural to investigate regular sequences of length 2 in $R = \kappa[x_0, \dots, x_n]$. Following two directions usually pursued in the literature concerning logarithmic sheaves for hypersurfaces, our goal is to find criteria to determine when, on the one hand, a regular sequence $\sigma = (f_1, f_2)$ is (locally) free, and, on the other hand, when the associated logarithmic tangent sheaf $\mathcal{T}_\sigma$ is slope-stable.

We start in Section 2 by setting up basic definitions and examples. Most importantly, we provide criteria to determine when a regular sequence in R is locally free (see Lemma 2.2), as well as examples of free regular sequences that are not strongly free (Example 2.7), and of a strongly free regular sequence (Example 2.8). In Section 2.2 we show that regular sequences of length k induces codimension $k-1$ distributions on $\mathbb{P}^n$, see Lemma 2.3. Yet another important notion in this paper is the compressibility of a k-web σ (that is, a regular sequence of length k consisting only of forms of equal degree): σ is said to be *compressible* if there is a variable that occurs in none of the forms contained in σ; a regular sequence that is not compressible is said to be *incompressible* (see Definition 2.12).

Sections 3 through 6 are dedicated to a thorough study of regular sequences of consisting of 2 polynomials of degree 2, also referred to as *pencils of quadrics*.

First, recall that the *slope* of a torsion-free sheaf F of rank $p > 0$ on $\mathbb{P}^n$ of determinant $\left(\bigwedge^p F\right)^{\vee\vee} \simeq \mathcal{O}_{\mathbb{P}^n}(e)$ is defined as $\mu(F) = e/p$. The sheaf F is said to be *slope-(semi)stable* if any proper subsheaf K of E has slope $\mu(K) < (\leqslant)\ \mu(E)$; F is *slope-polystable* if it is the direct sum of slope-stable sheaves with the same slope, and F is *slope-unstable* if it is not slope-semistable. The following result is proved in Section 3.

Theorem A. *Let σ be a pencil of quadrics in $\mathbb{P}^n$ and let r_0 be the maximal corank of the Hessian matrix for each quadric of the pencil.*

1. *If σ contains two double hyperplanes, then $\mathcal{T}_\sigma = \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}$.*
2. *If σ contains only one double hyperplane, then $\mathcal{T}_\sigma$ is slope-stable if and only if σ is incompressible.*
3. *If σ is compressible and contains no double hyperplanes, then $\mathcal{T}_\sigma$ is slope-unstable.*
4. *If σ is incompressible and contains no double hyperplanes, then*
   - *i) $\mathcal{T}_\sigma$ is slope-stable when $2r_0 < n + 1$;*
   - *ii) $\mathcal{T}_\sigma$ is strictly slope-semistable when $2r_0 = n + 1$;*
   - *iii) $\mathcal{T}_\sigma$ is slope-unstable when $2r_0 > n + 1$.*

The upshot is that, for the most interesting case (namely that of incompressible pencils without double hyperplanes), stability depends only of the maximal corank r_0 of the quadrics in the pencil. By [1], semistability of a pencil of quadrics in the sense of geometric invariant theory is equivalent to the fact that the discriminant of the pencils is non-zero (i.e. the pencil is regular) and has no root of multiplicity greater than $(n+1)/2$. So there are many GIT-unstable pencils σ whose logarithmic sheaf $\mathcal{T}_\sigma$ is still slope-semistable or even slope-stable see Remark 3.14.

Next, we look at freeness and local freeness of pencils of quadrics and, more generally, at projective dimension of $\mathcal{T}_\sigma$, both in the local and in the graded senses. This turns out to depend on more subtle invariants of the pencil. To review them, note that the pencil of quadrics defined by σ gives a symmetric matrix ρ_σ of linear forms on $\mathbb{P}^1$, whose *generic corank* r_1 is the corank of the Hessian matrix of a generic quadric in the pencil. Note that $r_1 = 0$ if and only if σ contains smooth quadrics, we call σ *regular* in this case and *irregular* otherwise. When σ is irregular, there are integers $c_1 \leqslant \cdots \leqslant c_{r_1}$ such that the torsion free part of $\mathcal{C}_\sigma = \operatorname{coker}(\rho_\sigma)$ is $\bigoplus_{i=1}^{r_1} \mathcal{O}_{\mathbb{P}^1}(c_i)$. We call $\mathbf{c} = (c_1, \dots, c_{r_1})$ the *degree vector* of σ. If $\Lambda = \{\lambda_1, \dots, \lambda_\ell\} \subset \mathbb{P}^1$ is the support of the torsion part $\mathcal{C}_{\mathrm{t}}$ of $\mathcal{C}_\sigma$, then, for each $j \in \{1, \dots, \ell\}$, denoting by $\lambda_j^{(a)}$ the a-tuple structure over λ_j, the localization at λ_j of $\mathcal{C}_{\mathrm{t}}$ is $\bigoplus_{i=1}^{s_j} \mathcal{O}_{\lambda_j^{(a_{j,i})}}^{\oplus p_{j,i}}$, for some s_j and $(a_{j,i}, p_{j,i} \mid i \in \{1, \dots, s_j\})$. These data are arranged into the *Segre symbol* $\Sigma = [\Sigma_1, \dots, \Sigma_\ell]$, defined for all $j \in \{1, \dots, \ell\}$ by:

$$\Sigma_j = (\underbrace{a_{j,1}, \dots, a_{j,1}}_{p_{j,1}}, \dots, \underbrace{a_{j,s_j}, \dots, a_{j,s_j}}_{p_{j,s_j}}), \qquad \text{with } a_{j,1} > \cdots > a_{j,s_j}.$$

It turns out that the data (r_1, Λ, Σ) completely characterize an incompressible pencil of quadrics up to homography, thus generalizing a classical result attributed to Segre and Weierstrass for the case of regular pencils, see Section 4.1 for further details.

With these data, we describe the scheme-theoretic structure of the Jacobian scheme, when σ is regular, as a union of nilpotent structures on pairwise disjoint linear spaces whose dimension and degree of nilpotency depend on Σ and whose position depends on Λ. If σ is irregular, the Jacobian scheme contains an additional component which is a rational normal scroll of dimension r_1 and degree $c_1 + \cdots + c_{r_1}$ that connects all the linear spaces, with a prescribed intersection along each space.

The upshot is that these invariants also characterize the projective and global projective dimensions of $\mathcal{T}_\sigma$, as it is described in the following two results, proved in Section 4.2 and Section 4.3, respectively

Theorem B. *Let σ have Segre symbol Σ. For $q > 0$, $\mathcal{E}xt^q_{\mathbb{P}^n}(\mathcal{T}_\sigma, \mathcal{O}_{\mathbb{P}^n}) \neq 0$ if and only if there are $j \in \{1, \dots, \ell\}$ and $k \in \{1, \dots, s_j\}$ such that:*

$$q + p_{j,1} + \ldots + p_{j,k} = n - r_1 - 1,$$

or $r_1 > 0$ and $q + r_1 = n - 2$. In particular we have:

i) if σ is regular and $p = \min\{p_{j,1} \mid j \in \{1, \dots, \ell\}\}$, then $\operatorname{pdim}(\mathcal{T}_\sigma) = n - p - 1$;
ii) if σ is irregular, then $\operatorname{pdim}(\mathcal{T}_\sigma) = n - r_1 - 2$.

We have a rather different situation for the graded projective dimension $\operatorname{gpdim}(\mathcal{T}_\sigma)$ – namely, the projective dimension of the module of global sections of $\mathcal{T}_\sigma$. This is summarized in the following result.

Theorem C. *For a regular pencil of quadrics σ in $\mathbb{P}^n$ we have* $\operatorname{gpdim}(\mathcal{T}_\sigma) = n-2$ *except if σ has Segre symbol $[1^p, 1^q]$ for $p \geqslant q \geqslant 1$ or $[(2^q, 1^p)]$ with $q \geqslant 1$. In both these cases* $\operatorname{gpdim}(\mathcal{T}_\sigma) = n-q-1$.

For an irregular pencil of quadrics σ of generic corank r_1 we have $\operatorname{gpdim}(\mathcal{T}_\sigma) = n-1$ *except if σ has degree vector $(1, \ldots, 1)$, in which case* $\operatorname{gpdim}(\mathcal{T}_\sigma) = n-r_1-2$.

With this in mind, after a careful analysis of pencils of quadrics in $\mathbb{P}^3$, performed in Section 5, we come to the conclusion that freeness and local freeness are equivalent conditions for pencils of quadrics in $\mathbb{P}^n$ and we completely classify pencils satisfying such condition.

Theorem D. *A pencil of quadrics σ in $\mathbb{P}^n$, $n \geqslant 3$, is free if and only if $\mathcal{T}_\sigma$ is locally free. More precisely, the only free pencils of quadrics are displayed in Table 52.*

By contrast, we provide in Section 7 a series of examples of locally free pencils of degree $k \geqslant 3$ that are not free. This indicates that potentially interesting vector bundles may arise as logarithmic sheaves associated to regular sequences of higher degree having deep singularities. To understand our following result, recall that a *null correlation bundle* is defined as the cokernel of a non vanishing morphism $\mathcal{O}_{\mathbb{P}^3}(-1) \to \Omega^1_{\mathbb{P}^3}(1)$; every slope-stable rank 2 locally free sheaf N on $\mathbb{P}^3$ with $c_1(N) = 0$ and $c_2(N) = 1$ arises in this way.

Theorem E. *Fix $k \geqslant 0$, and consider the pencil $\sigma = (f, g)$ of degree $k+3$ with:*

$$f = x_0 x_1^{k+2} + x_2^{k+3} + x_2^{k+2} x_3 \quad \text{and} \quad g = x_2 x_3 (x_2^{k+1} - x_1^{k+1}).$$

Then $\mathcal{T}_\sigma \simeq N(-k-2)$, where N is a null correlation bundle.

We complete this paper with an application of our results to the study of rational codimension one foliations on $\mathbb{P}^n$, see Section 8. To be precise, recall that a *rational 1-form* is a twisted 1-form given by the expression

$$\omega = (d_1+1) f_1 \cdot \mathrm{d}f_2 - (d_2+1) f_2 \cdot \mathrm{d}f_1 \ \in H^0(\Omega^1_{\mathbb{P}^n}(d_1+d_2+2)),$$

where $f_i \in H^0(\mathcal{O}_{\mathbb{P}^n}(d_i+1))$ for $i = 1, 2$ and f_1, f_2 have no common factors. Regarding ω as a morphism $\mathrm{T}\mathbb{P}^n \to \mathcal{O}_{\mathbb{P}^n}(d_1+d_1+2)$, we consider the kernel sheaf $\mathcal{K}_\omega := \ker \omega$. We show in Section 8 that the natural 1-1 correspondence $(f_1, f_2) \leftrightarrow \omega$ between regular sequences of length 2 and rational 1-forms is such that $\mathcal{K}_\omega = \mathcal{T}_\sigma(1)$, see Lemma 8.1.

This fact has two important consequences. First, we can invoke a result from the general theory of codimension one distributions on $\mathbb{P}^3$, presented in [4], to obtain simple criteria to establish when $\mathcal{T}_\sigma$ is slope-(semi)stable, see Corollary 8.2.

Second, we provide a negative answer to a problem posed by Calvo-Andrade, Cerveau, Giraldo and Lins Neto, see [3, Problem 2]; namely, these authors asked whether the tangent sheaf of a codimension one foliation must split as a sum of line bundles whenever it is locally free. While Theorem D implies that this claim is true for rational foliations of type $(2, 2)$, Theorem E says that for each $k \geqslant 0$ there are rational foliations of type $(k+3, k+3)$ on $\mathbb{P}^3$ whose tangent sheaf is a slope-stable locally free sheaf.

All things considered, we believe that the results presented in this paper point to a rich, interesting general theory of (local/strong) freeness for complete intersection subschemes that in some sense parallels the widely known theory of freeness for divisors in $\mathbb{P}^n$.

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2. Basic definitions and examples

Let κ be a field of characteristic 0, $n \in \mathbb{N}$ and put $R = \kappa[x_0, \dots, x_n]$. In this section we will give some preliminary properties of logarithmic tangent sheaves associated with complete intersections. Many properties remain valid for κ of characteristic different from 2.

2.1. **General framework.** Let $\sigma := (f_1, \dots, f_k)$ be a regular sequence of R. Consider the associated complete intersection variety $X = V(\sigma)$ and set $\mathcal{J}_\sigma$ for the associated Jacobian matrix. Let us denote by ∂_i the partial derivative $\frac{\partial}{\partial x_i}$ and $\nabla f = (\partial_0 f, \dots, \partial_n f)$ the gradient of a homogeneous polynomial $f \in R$.

We put $\mathcal{T}_\sigma$ for the associated logarithmic tangent sheaf as defined in the introduction, namely $\mathcal{T}_\sigma$ is the kernel of $\mathcal{J}_\sigma$. In addition, we define the sheaves $\mathcal{M}_\sigma := \operatorname{im}(\mathcal{J}_\sigma)$ and $\mathcal{Q}_\sigma := \operatorname{coker}(\mathcal{J}_\sigma)$. The sheaf $\mathcal{M}_\sigma$ is torsion free, it can be thought of as the natural extension to $\mathbb{P}^n$ of the equisingular normal sheaf of $X = V(\sigma)$ and that $\mathcal{T}_\sigma$ is reflexive. We have the fundamental exact sequences:

$$0 \longrightarrow \mathcal{T}_\sigma \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \xrightarrow{\mathcal{J}_\sigma} \mathcal{M}_\sigma \longrightarrow 0 \tag{1}$$

$$0 \longrightarrow \mathcal{M}_\sigma \longrightarrow \oplus_{i=1}^k \mathcal{O}_{\mathbb{P}^n}(d_i) \longrightarrow \mathcal{Q}_\sigma \longrightarrow 0 \tag{2}$$

We define the *Jacobian scheme* Ξ_σ as the degeneracy locus of $\mathcal{J}_\sigma$,

$$\Xi_\sigma := V\left(\bigwedge^k \mathcal{J}_\sigma\right).$$

This is the subscheme of $\mathbb{P}^n$ defined by the common zeros of the $k \times k$ minors of $\mathcal{J}_\sigma$. The reduced structure $(\Xi_\sigma)_{\mathrm{red}}$ coincides with the support of the sheaf $\mathcal{Q}_\sigma$. Note that $(\Xi_\sigma)_{\mathrm{red}}$ may contain a hypersurface.

More precisely, the image of the exterior power morphism:

$$\bigwedge^k \mathcal{J}_\sigma : \mathcal{O}_{\mathbb{P}^n}^{\oplus \binom{n+1}{k}} \to \mathcal{O}_{\mathbb{P}^3}(d_1 + \cdots + d_k); \tag{3}$$

is of the form $\mathcal{I}_{W_\sigma}(d_1 + \cdots + d_k - l)$, where $l := c_1(\mathcal{Q}_\sigma)$ is the degree of the hypersurface contained in Ξ_σ, and $W_\sigma \subset \mathbb{P}^n$ is a subscheme of codimension at least 2, possibly not pure. Let us illustrate this discussion with an explicit example.

Example 2.1. Let $f = x_0x_1 + x_2x_3$ and $g = x_0x_1x_2x_3$, so that:

$$\mathcal{J}_\sigma := \begin{pmatrix} x_1 & x_0 & x_3 & x_2 \\ x_1x_2x_3 & x_0x_2x_3 & x_0x_1x_3 & x_0x_1x_2 \end{pmatrix}.$$

The curve $C = V(f,g)$ is the union of the four lines $V(x_i, x_j)$ with $i = 0, 1$ and $j = 2, 3$. Two of the 2×2 minors of $\mathcal{J}_\sigma$ vanish identically, and we have that

$$\bigwedge^2 \mathcal{J}_\sigma = (x_0x_1 - x_2x_3) \cdot (0 \ x_0x_3 \ x_1x_2 \ x_1x_3 \ x_0x_2 \ 0).$$

It follows that Ξ_σ consists of the quadric $V(x_0x_1 - x_2x_3)$, so that $l = c_1(\mathcal{Q}_\sigma) = 2$ in this case, plus two lines $V(x_0, x_1)$ and $V(x_2, x_3)$. So W_σ is the union of these two skew lines. Note that neither f and g have common factors, nor do ∇f and ∇g.

Lemma 2.2. *Let σ be a regular sequence. Then:*

i) σ is locally free if and only if $\mathcal{Q}_\sigma$ has no subsheaf of codimension $\geqslant 3$;
ii) if σ is locally free, Ξ_σ has no irreducible component of codimension $\geqslant 3$.

Note that $(\Xi_\sigma)_{\mathrm{red}}$ may have no irreducible component of codimension at least 3 even when $\mathcal{Q}_\sigma$ admits a subsheaf of codimension at least 3, see Section 7. This means that the converse of item *(ii)* above does not hold in general.

Proof. Taking duals of (1) and (2) we obtain $\mathcal{E}xt^{n-1}_{\mathbb{P}^n}(\mathcal{T}_\sigma, \mathcal{O}_{\mathbb{P}^n}) = 0$ and, for $j \leqslant n-1$:

$$\mathcal{E}xt^j_{\mathbb{P}^n}(\mathcal{T}_\sigma, \mathcal{O}_{\mathbb{P}^n}) \simeq \mathcal{E}xt^{j+1}_{\mathbb{P}^n}(\mathcal{M}_\sigma, \mathcal{O}_{\mathbb{P}^n}) \simeq \mathcal{E}xt^{j+2}_{\mathbb{P}^n}(\mathcal{Q}_\sigma, \mathcal{O}_{\mathbb{P}^n}).$$

The sheaf $\mathcal{T}_\sigma$ is locally free if and only if $\mathcal{E}xt^j_{\mathbb{P}^n}(\mathcal{T}_\sigma, \mathcal{O}_{\mathbb{P}^n}) = 0$ for $j \geqslant 1$, which is equivalent to requiring that $\mathcal{E}xt^j_{\mathbb{P}^n}(\mathcal{Q}_\sigma, \mathcal{O}_{\mathbb{P}^n}) = 0$ for $j \geqslant 3$. This gives the equivalence in the first claim.

If $(\Xi_\sigma)_{\mathrm{red}}$ has an irreducible component Y of codimension $j \geqslant 3$, then, since $(\Xi_\sigma)_{\mathrm{red}}$ is the support of $\mathcal{Q}_\sigma$, it follows that $\mathcal{Q}_\sigma$ has a non trivial subsheaf $\mathcal{V} \hookrightarrow \mathcal{Q}_\sigma$ supported on Y, hence $\operatorname{codim} \mathcal{V} = j$. The previous item then implies that $\mathcal{T}_\sigma$ is not locally free. □

2.2. **Regular sequences and distributions.** Recall that a *codimension r distribution on $\mathbb{P}^n$* is a short exact sequence of the form

$$\mathscr{D} : 0 \longrightarrow \mathcal{T}_{\mathscr{D}} \longrightarrow \mathrm{T}\mathbb{P}^n \longrightarrow \mathcal{N}_{\mathscr{D}} \longrightarrow 0$$

where $\mathcal{N}_{\mathscr{D}}$ is a torsion free sheaf of rank r and $\mathcal{T}_{\mathscr{D}}$ is a reflexive sheaf of rank $n - r$, respectively called the *normal* and *tangent* sheaves of $\mathscr{D}$. We refer to [4, Section 2.1] for further details on the general theory of distributions.

Let us point out how distributions are related to regular sequences. First, thinking of the Koszul complex attached to σ we consider $\tilde{\sigma} = ((d_1+1)f_1, \ldots, (d_k+1)f_k)^{\mathrm{t}}$ and the *Koszul syzygy* sheaf $\mathcal{S}_\sigma := \operatorname{coker}(\tilde{\sigma})$, so:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^n}(d_i) \to \mathcal{S}_\sigma \to 0. \tag{4}$$

Let $\eta : \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}^{\oplus n+1}_{\mathbb{P}^n}$ be the Euler morphism, namely $\eta = (x_0, \ldots, x_n)^{\mathrm{t}}$. The Euler relation gives $\eta \cdot \nabla f_i = (d_i + 1)f_i$ for all $i \in \{1, \ldots, k\}$. This allows us to

construct the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{\mathbb{P}^n}(-1) & = & \mathcal{O}_{\mathbb{P}^n}(-1) & & \\
 & & & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \tilde{\sigma}} & & \\
0 & \longrightarrow & \mathcal{T}_\sigma & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} & \xrightarrow{\jmath_\sigma} & \bigoplus_{i=1}^k \mathcal{O}_{\mathbb{P}^n}(d_i) & & \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}_\sigma & \longrightarrow & \mathrm{T}\mathbb{P}^n(-1) & \longrightarrow & \mathcal{S}_\sigma & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{5}$$

Here we used that κ is of characteristic zero, or rather that the characteristic of κ does not divide $d_i + 1$ for all $i \in \{1, \dots, k\}$.

Note that the image of $\tilde{\sigma}$ is contained in $\mathcal{M}_\sigma$ and set $\mathcal{N}_\sigma$ for the cokernel of $\tilde{\sigma}$, corestricted to $\mathcal{M}_\sigma$. The previous diagram gives:

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & \mathcal{O}_{\mathbb{P}^n}(-1) & = & \mathcal{O}_{\mathbb{P}^n}(-1) & & & & \\
 & & & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \tilde{\sigma}} & & & & \\
0 & \longrightarrow & \mathcal{T}_\sigma & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} & \xrightarrow{\jmath_\sigma} & \mathcal{M}_\sigma & \longrightarrow & 0 & & \\
 & & \| & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{T}_\sigma & \longrightarrow & \mathrm{T}\mathbb{P}^n(-1) & \longrightarrow & \mathcal{N}_\sigma & \longrightarrow & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array} \tag{6}$$

Furthermore, we have a second diagram featuring the cokernel sheaf $\mathcal{Q}_\sigma$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}^n}(-1) & = & \mathcal{O}_{\mathbb{P}^n}(-1) & & & & \\
 & & \downarrow & & \downarrow{\scriptstyle \tilde{\sigma}} & & & & \\
0 & \longrightarrow & \mathcal{M}_\sigma & \longrightarrow & \oplus_{i=1}^k \mathcal{O}_{\mathbb{P}^n}(d_i) & \longrightarrow & \mathcal{Q}_\sigma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{N}_\sigma & \longrightarrow & \mathcal{S}_\sigma & \longrightarrow & \mathcal{Q}_\sigma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{7}$$

It follows that the bottom line in diagram in display (6) defines, for $k \geqslant 2$, a *codimension* $k-1$ *distribution* $\mathscr{D}_\sigma$ *on* $\mathbb{P}^n$, given by the exact sequence

$$\mathscr{D}_\sigma : 0 \longrightarrow \mathcal{T}_\sigma(1) \longrightarrow \mathrm{T}\mathbb{P}^n \longrightarrow \mathcal{N}_\sigma(1) \longrightarrow 0. \tag{8}$$

Summing up, we have proved the following statement.

Lemma 2.3. *Every regular sequence σ of length k on $n+1$ variables induces a codimension $k-1$ distribution $\mathscr{D}_\sigma$ on $\mathbb{P}^n$ such that $\mathcal{T}_{\mathscr{D}_\sigma} = \mathcal{T}_\sigma(1)$.*

However, not every codimension $k-1$ distribution on $\mathbb{P}^n$ comes from a regular sequence via the construction above. For instance, given a codimension $k-1$ distribution $\mathscr{D}$ on $\mathbb{P}^n$, the monomorphism $\mathcal{T}_{\mathscr{D}} \hookrightarrow \mathrm{T}\mathbb{P}^n$ may not factor through $\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}$.

2.3. **Logarithmic tangent sheaf and deformations.** Let us point out the relationship between our sheaf and classical sheaves of tangent vector fields, in connection with locally trivial deformations of embeddings.

2.3.1. *Tangent vector fields along a complete intersection.* Given the regular sequence σ we have a complete intersection $X = V(\sigma) \subset \mathbb{P}^n$, whose ideal sheaf $\mathcal{I}_X$ is generated by $\tilde{\sigma}^\vee : \oplus_{i=1}^k \mathcal{O}_{\mathbb{P}^n}(-d_i) \to \mathcal{I}_X(1)$. In addition, we have the *equisingular normal sheaf* $N'_{X/\mathbb{P}^n}$, see [10, § 3.4.4], which is defined as the quotient sheaf $\mathrm{T}\mathbb{P}^n|_X/\mathrm{T}X$, and therefore satisfies the following exact sequence

$$0 \to \mathrm{T}X \to \mathrm{T}\mathbb{P}^n|_X \to N'_{X/\mathbb{P}^n} \to 0;$$

here, $\mathrm{T}X$ denotes the Zariski tangent sheaf of X. Note also that $N'_{X/\mathbb{P}^n}$ is a subsheaf of the normal bundle $N_{X/\mathbb{P}^n} \simeq \bigoplus_{i=1}^k \mathcal{O}_X(d_i+1)$, so:

$$N'_{X/\mathbb{P}^n} \hookrightarrow \bigoplus_{i=1}^k \mathcal{O}_X(d_i+1);$$

the quotient of this monomorphism is denoted by T'_X, see [10, § 1.1.3]; it is supported at the singular locus of X. For further reference, we write its defining exact sequence:

$$0 \to N'_{X/\mathbb{P}^n} \to \bigoplus_{i=1}^k \mathcal{O}_X(d_i+1) \to \mathrm{T}'_X \to 0. \tag{9}$$

The *sheaf of vector fields on $\mathbb{P}^n$ tangent to X*, denoted by $\mathrm{T}_{\mathbb{P}^n}\langle X\rangle$, is defined as the kernel of the composed epimorphism $\mathrm{T}\mathbb{P}^n \twoheadrightarrow \mathrm{T}\mathbb{P}^n|_X \twoheadrightarrow N'_{X/\mathbb{P}^n}$, yielding the exact sequence

$$0 \to \mathrm{T}_{\mathbb{P}^n}\langle X\rangle \to \mathrm{T}\mathbb{P}^n \to N'_{X/\mathbb{P}^n} \to 0. \tag{10}$$

The main motivation for introducing $\mathrm{T}_{\mathbb{P}^n}\langle X\rangle$ is given by [10, Proposition 3.4.17]; namely, $H^1(\mathrm{T}_{\mathbb{P}^n}\langle X\rangle)$ and $H^2(\mathrm{T}_{\mathbb{P}^n}\langle X\rangle)$ are the tangent space and the obstruction space of the semiuniversal space of locally trivial deformations of the embedding $X \hookrightarrow \mathbb{P}^n$. Here we show that $\mathcal{T}_\sigma(1)$ is a subsheaf of $\mathrm{T}_{\mathbb{P}^n}\langle X\rangle$ and, to a certain extent, describe the quotient $\mathrm{T}_{\mathbb{P}^n}\langle X\rangle/\mathcal{T}_\sigma(1)$.

Since the forms $f_1, \ldots, f_k$ generate the homogeneous ideal of X in $\mathbb{P}^n$, we may view $\tilde{\sigma} = ((d_1+1)f_1, \ldots, (d_k+1)f_k)^{\mathrm{t}}$ as a morphism $\mathcal{O}_{\mathbb{P}^n} \to \bigoplus_{i=1}^k \mathcal{I}_X(d_i+1)$. We define a torsion free sheaf $\mathcal{V}_\sigma = \operatorname{coker}(\tilde{\sigma})$ fitting into:

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \bigoplus_{i=1}^k \mathcal{I}_X(d_i+1) \to \mathcal{V}_\sigma \to 0.$$

Note that, when $k=1$, we have $\mathcal{V}_\sigma = 0$, as $\mathcal{I}_X(d_1+1) \simeq \mathcal{O}_{\mathbb{P}^n}$ in this case, so:

$$\mathcal{T}_\sigma(1) \simeq \mathrm{T}_{\mathbb{P}^n}\langle X\rangle, \qquad \text{for } k=1.$$

For $k \geqslant 2$ the relationship between the two sheaves $\mathcal{T}_\sigma(1)$ and $\mathrm{T}_{\mathbb{P}^n}\langle X\rangle$ is expressed by the following lemma.

Lemma 2.4. *We have an exact sequence:*

$$0 \to \mathcal{T}_\sigma(1) \to \mathrm{T}_{\mathbb{P}^n}\langle X\rangle \to \mathcal{V}_\sigma \to \mathcal{Q}_\sigma(1) \to \mathrm{T}'_X \to 0. \tag{11}$$

Proof. We use the Koszul syzygy sheaf $\mathcal{S}_\sigma$ of Subsection 2.2 to write the following exact sequence relating $\mathcal{S}_\sigma$ and $\mathcal{V}_\sigma$:

$$0 \to \mathcal{V}_\sigma \to \mathcal{S}_\sigma(1) \to \bigoplus_{i=1}^{k} \mathcal{O}_X(d_i+1) \to 0.$$

We get a commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^n} & \xrightarrow{\tilde{\sigma}} & \bigoplus_{i=1}^{k} \mathcal{I}_X(d_i+1) & & \\
 & & \downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}_\sigma(1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}(1) & \xrightarrow{\mathcal{J}_\sigma} & \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^n}(d_i+1) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{T}_{\mathbb{P}^n}\langle X\rangle & \longrightarrow & \mathrm{T}\mathbb{P}^n & \xrightarrow{\varpi} & \bigoplus_{i=1}^{k} \mathcal{O}_X(d_i+1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where ϖ is given by the composition $\mathrm{T}\mathbb{P}^n \twoheadrightarrow N'_{X/\mathbb{P}^n} \hookrightarrow \bigoplus_{i=1}^{k} \mathcal{O}_X(d_i+1)$. The exact sequence in display (11) is then obtained via the snake lemma, since $\mathcal{V}_\sigma := \operatorname{coker}(\tilde{\sigma})$, $\mathcal{Q}_\sigma := \operatorname{coker}(\mathcal{J}_\sigma)$ and $T'_X := \operatorname{coker}(\varpi)$. □

2.3.2. *Tangent vector fields along hypersurfaces.* We look at the relationship between $\mathcal{T}_\sigma$ and the tangent vector field to one of the hypersurfaces defining σ.

Lemma 2.5. *We have:*

$$\mathcal{T}_\sigma = \bigcap_{j=1}^{k} \mathcal{T}_{f_j}. \tag{12}$$

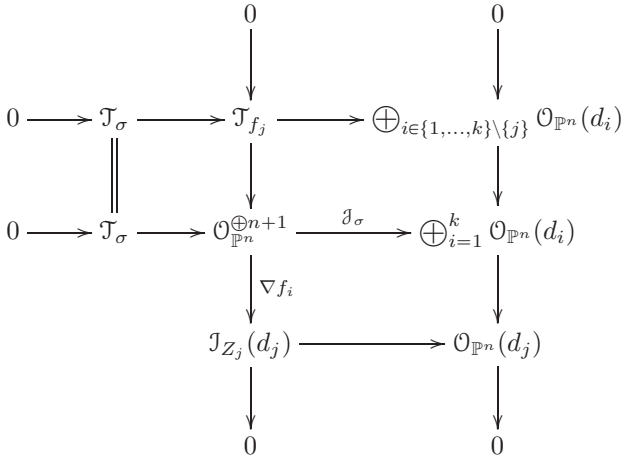
Further, for each $j \in \{1,\dots,k\}$, set $Z_j = \operatorname{Sing}\big(V(f_j)\big)$. Then there is an exact sequence:

$$0 \to \mathcal{T}_\sigma \to \mathcal{T}_{f_j} \to \bigoplus_{i\in\{1,\dots,k\}\setminus\{j\}} \mathcal{O}_{\mathbb{P}^n}(d_i) \to \mathcal{Q}_\sigma \to \mathcal{O}_{Z_j}(d_j) \to 0. \tag{13}$$

Proof. For any $j \in \{1,\dots,n\}$, we have:

$$\mathcal{T}_{f_j} = \ker\Big(\nabla f_j \colon \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n}(d_j)\Big).$$

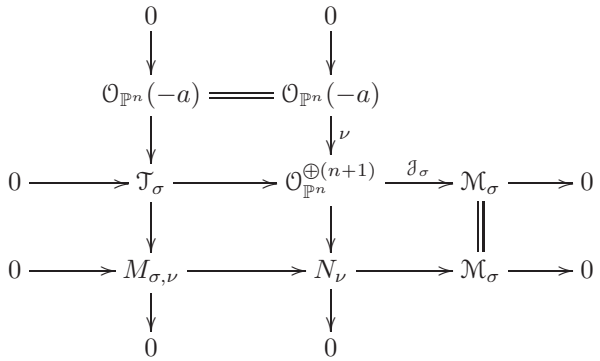
Therefore, since $\mathcal{T}_\sigma$ is defined as kernel of the matrix obtained by stacking $\nabla(f_1), \dots, \nabla(f_k)$, we get (12). Next, for any $j \in \{1, \dots, k\}$, we have the following commutative diagram:

(14)


Since Z_j is the Jacobian scheme of f_j, the completion of this diagram via the snake lemma leads to (13). □

These observations will play an important role in the proof of Theorem 7.1 below.

2.4. **Syzygies and global sections.** Let us point out the relationship between the Jacobian syzygies and the global sections of $\mathcal{T}_\sigma$. Let $\nu : \mathcal{O}_{\mathbb{P}^n}(-a) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1}$ be a sygyzy of degree a for the Jacobian matrix $\mathcal{J}_\sigma$ of a regular sequence σ, that is, $\mathcal{J}_\sigma \circ \nu = 0$; assume that the entries of ν have no common factors of positive degree, so that $N_\nu := \operatorname{coker}(\nu)$ is a torsion free sheaf. We have the commutative diagram:

(15)


It follows that every syzygy of degree a for $\mathcal{J}_\sigma$ induces a section in $H^0(\mathcal{T}_\sigma(a))$. Conversely, every non trivial section in $H^0(\mathcal{T}_\sigma(a))$ induces a syzygy of degree a for $\mathcal{J}_\sigma$, thus we obtain an isomorphism of vector spaces

$$H^0(\mathcal{T}_\sigma(a)) \simeq \operatorname{Syz}_a(\mathcal{J}_\sigma), \tag{16}$$

where $\operatorname{Syz}_a(\mathcal{J}_\sigma)$ is the vector space of syzygies of degree a for the matrix $\mathcal{J}_\sigma$.

Example 2.6. Let us consider the case $n = k = 1$ as a toy model for the theory we are proposing; Take $g \in H^0(\mathcal{O}_{\mathbb{P}^1}(d+1))$ and the associated morphism

$$\mathcal{J}_g = \nabla g : \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1}(d);$$

If $\partial_0 g$ and $\partial_1 g$ have no common factors, then $\mathcal{J}_g$ is surjective and $\mathcal{T}_g = \mathcal{O}_{\mathbb{P}^1}(-d)$. The observation (16) implies that ∇g has no syzygies of degree less than d.

More generally, if $\partial_0 g$ and $\partial_1 g$ have a common factor of degree e (for instance, if $g = x_0^2 x_1^2$, then $x_0 x_1$ is a common factor for $\partial_0 g$ and $\partial_1 g$), then $\mathcal{M}_g \simeq \mathcal{O}_{\mathbb{P}^1}(d-e)$, and $\mathcal{T}_g \simeq \mathcal{O}_{\mathbb{P}^1}(-d+e)$.

Example 2.7. Here is an example of a free regular sequence that is not strongly free. Consider the following regular sequences in $R = \kappa[x_0, x_1, x_2, x_3]$

$$\sigma := (x_0 x_1, g) \text{ and } \sigma' := (x_0 x_1, x_0^2 x_1 + g);$$

where g is a polynomial of degree 3 depending only on x_2 and x_3. Assume that $\partial_2 g$ and $\partial_3 g$ have no common factors, so that, as it was observed in Example 2.6 above, ∇g has no syzygies of degree < 2.

Clearly, $I_\sigma = I_{\sigma'}$. We argue that σ is free, while σ' is not. Indeed, their Jacobian matrices are given by:

$$\mathcal{J}_\sigma = \begin{pmatrix} x_1 & x_0 & 0 & 0 \\ 0 & 0 & \partial_2 g & \partial_3 g \end{pmatrix} \quad \text{and} \quad \mathcal{J}_{\sigma'} = \begin{pmatrix} x_1 & x_0 & 0 & 0 \\ 2x_0 x_1 & x_0^2 & \partial_2 g & \partial_3 g \end{pmatrix}.$$

Note that $\mathcal{J}_\sigma$ has two independent syzygies, given by

$$\nu_1 = (-x_0, x_1, 0, 0) \text{ and } \nu_2 = (0, 0, \partial_3 g, -\partial_2 g)$$

of degrees 1 and 2, respectively. Therefore, we have a monomorphism

$$\nu \colon \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \hookrightarrow \mathcal{T}_\sigma$$

whose cokernel, being a subsheaf of $\mathcal{I}_L(1) \oplus \mathcal{I}_C(2)$ with $L = V(x_0, x_1)$ and $C = V(\partial_2 g, \partial_3 g)$, must be torsion free. It follows that ν must be an isomorphism, thus $\mathcal{T}_\sigma \simeq \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2)$.

To see that $\mathcal{T}_{\sigma'}$ does not split as a sum of line bundles, note that ν_2 is also a syzygy for $\mathcal{J}_{\sigma'}$, thus $h^0(\mathcal{T}_{\sigma'}(2)) > 0$. On the other hand, since $\mathcal{J}_{\sigma'}$ has no syzygy of degree $\leqslant 1$, we have that $h^0(\mathcal{T}_{\sigma'}(1)) = 0$. In addition, the minors appearing in $\bigwedge^2 \mathcal{J}_{\sigma'}$ have no common factor, thus $c_1(\mathcal{Q}_{\sigma'}) = 0$ and $c_1(\mathcal{T}_{\sigma'}) = -c_1(\mathcal{M}_{\sigma'}) = -3$. Thus if $\mathcal{T}_{\sigma'} = \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b)$ with $a \leqslant b$, then $a + b = -3$, and $a, b \leqslant -2$, which is impossible.

In fact, note that $\Xi_{\sigma'}$ consists of the line $V(x_0, x_1)$ together with the following 0-dimensional schemes:

$$V(x_0, \partial_2 g, \partial_3 g) \qquad \text{and} \qquad V(x_1, \partial_2 g, \partial_3 g),$$

each of length equal to 4. Therefore, $(\Xi_{\sigma'})_{\mathrm{red}}$ contains at least two irreducible components of codimension 3; the second item of Lemma 2.2 implies that $\mathcal{T}_{\sigma'}$ is not locally free.

Example 2.8. We show that the regular sequence $\sigma = (x_0, x_3^2)$ in $R = \kappa[x_0, x_1, x_2, x_3]$ is a strongly free sequence consisting of polynomials of different degrees. Any regular sequence σ' such that $I_\sigma = I_{\sigma'}$ must be of the form $\sigma' = (\alpha x_0, x_0 l + \beta x_3^2)$ for some linear form $h \in H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ and $\alpha, \beta \in \kappa^*$. Setting $h = a x_0 + b x_1 + c x_2 + d x_3$, that Jacobian matrix for σ' is given by

$$\mathcal{J}_{\sigma'} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 2a x_0 & b x_0 & c x_0 & d x_0 + 2\beta x_3 \end{pmatrix}.$$

If $c \neq 0$, then

$$\nu_1 = (0, -c, b, 0) \qquad \text{and} \qquad \nu_2 = (0, 0, d x_0 + 2\beta x_3, -c x_0)$$

are independent syzygies of degree 0 and 1, respectively. Following the argument in Example 2.7, so we can conclude that $\mathcal{T}_{\sigma'} = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)$.

When $c = 0$ and $b \neq 0$ then

$$\nu_1 = (0, 0, 1, 0) \text{ and } \nu_2 = (0, dx_0 + 2\beta x_3, 0, -bx_0)$$

are independent syzygies of degree 0 and 1, respectively, so again we conclude that $\mathcal{T}_{\sigma'} = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)$.

Finally, if $b = c = 0$, then $\nu_1 = (0, 0, 1, 0)$ and $\nu_2 = (0, 1, 0, 0)$ are independent syzygies of degree 0, thus $\mathcal{T}_{\sigma'} = \mathcal{O}_{\mathbb{P}^3}^{\oplus 2}$.

It is worth noticing that, in general, $\det(\mathcal{T}_\sigma)$ is not fixed and may change with the choice of generators for I_σ.

2.5. **Webs.** Fix integers $d \geqslant 0$ and $k \geqslant 1$ and let $\sigma = (f_1, \ldots, f_k)$ be a regular sequence of forms of degree $d + 1$ in $R = \kappa[x_0, \ldots, x_n]$; we call σ a k-web in $\mathbb{P}^n$; a 2-web is usually called a *pencil*. In this section, we establish some basic properties of logarithmic tangent sheaves associated to k-webs, which will be useful later on.

2.5.1. *Freeness of webs.* Here is the first fundamental fact.

Lemma 2.9. *Let σ be a k-web. If σ is free, then it is strongly free.*

Proof. Let $\sigma' = (f'_1, \ldots, f'_k)$ be another regular sequence such that $I_{\sigma'} = I_\sigma$; one can check that there is a matrix $P \in \mathrm{GL}_k(\kappa)$ such that

$$\begin{pmatrix} f'_1 \\ \vdots \\ f'_k \end{pmatrix} = P \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix}.$$

It follows that $\mathcal{J}_{\sigma'} = P\mathcal{J}_\sigma$, thus in fact $\mathcal{T}_{\sigma'} \simeq \mathcal{T}_\sigma$, from which the desired statement follows immediately. □

A particular case of the previous result leads to the simplest example of a strongly free regular sequence.

Example 2.10. Take a regular sequence $\sigma = (f_1, \ldots, f_k)$ such that each f_i is a linear polynomial; note that $V(\sigma)$ is a linear subspace of codimension k. The Jacobian matrix is then a constant matrix of maximal rank, inducing a surjective morphism $\mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n}^{\oplus k}$. It follows that $\mathcal{T}_\sigma = \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1-k}$, $\mathcal{M}_\sigma = \mathcal{O}_{\mathbb{P}^n}^{\oplus k}$, and $\mathcal{Q}_\sigma = 0$.

2.5.2. *Webs versus regular sequence.* Let us point out how to associate a web to any regular sequence, keeping the logarithmic sheaf unchanged. Let $\sigma = (f_1, \ldots, f_k)$ be a regular sequence, with $\deg(f_i) = d_i + 1$, for some $d_1 + 1, \ldots, d_k + 1 \in \mathbb{N}$. Let e be the least common multiple of $d_1 + 1, \ldots, d_k + 1$. For $i \in \{1, \ldots, k\}$, put $\ell_i = e/(d_i + 1)$. Set:

$$\tau = (f_1^{\ell_1}, \ldots, f_k^{\ell_k}).$$

Note that τ is a web of degree e.

Lemma 2.11. *We have:*

$$\mathcal{T}_\tau = \mathcal{T}_\sigma.$$

Proof. For $i \in \{1, \dots, k\}$, set $g_i = f_i^{\ell_i}$, so that $\tau = (g_1, \dots, g_k)$. By the chain rule, for each $i \in \{1, \dots, k\}$ we have:

$$\nabla(g_i) = \ell_i f_i^{\ell_i - 1} \nabla(f_i).$$

In other words, considering the morphism defined by the diagonal matrix:

$$P = \operatorname{diag}(\ell_1 f_1^{\ell_1 - 1}, \dots, \ell_k f_k^{\ell_k - 1}) \colon \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^n}(d_i) \to \mathcal{O}_{\mathbb{P}^n}(e)^{\oplus k},$$

we get that:

$$\mathcal{J}_\tau = P \circ \mathcal{J}_\sigma.$$

Since κ is of characteristic zero, P is injective, so $\mathcal{T}_\tau = \ker(\mathcal{J}_\tau) = \ker(\mathcal{J}_\sigma) = \mathcal{T}_\sigma$. □

2.5.3. *Compressibility of webs.* Next, we introduce the following definitions.

Definition 2.12. We say that the k-web σ is:

(1) *regular* if there is $(z_1, \dots, z_k) \in \kappa^k$ such that the hypersurface $V(\sum_{i=1}^k z_i f_i)$ is non singular;
(2) *compressible* if, up to a linear coordinate change, there is a variable that occurs in none of the forms $f_1, \dots, f_k$.

A k-web that is not regular is called *irregular*; a k-web that is not compressible is called *incompressible*.

Note that regular k-webs are incompressible. Furthermore, as it was observed in the proof of Lemma 2.9 above, if σ is a k-web, then the logarithmic tangent sheaf $\mathcal{T}_\sigma$ is independent from the choice of generators of the ideal I_σ generated by σ.

Lemma 2.13. *A k-web σ is compressible if and only if $H^0(\mathcal{T}_\sigma) \neq 0$.*

Proof. The condition $H^0(\mathcal{T}_\sigma) \neq 0$ does not depend on the given choice of a system of coordinates, and if none of the forms $f_1, \dots, f_k$ depends on a given variable, then all partial derivatives of $f_1, \dots, f_k$ with respect to this variable are zero. This means that $\mathcal{J}_\sigma$ contains a column containing only 0 and thus the kernel sheaf $\mathcal{T}_\sigma$ contains a copy of $\mathcal{O}_{\mathbb{P}^n}$.

Conversely, assume $H^0(\mathcal{T}_\sigma) \neq 0$. For all $(i, j) \in \mathbb{N}^2$ with $1 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant n$, we set:

$$f_{i,j} = \frac{\partial f_i}{\partial x_j} \in R.$$

Since the sheaf $\mathcal{T}_\sigma$ satisfies $H^0(\mathcal{T}_\sigma) \neq 0$, there is a non-zero vector $(b_0, \dots, b_n) \in \kappa^{n+1}$ such that:

$$b_0 f_{i,0} + \cdots + b_n f_{i,n} = 0, \qquad \text{for all } 1 \leqslant i \leqslant k.$$

Then we define new coordinates $(x'_0, \dots, x'_n)$ by choosing an invertible matrix $(a_{i,j})$ of size $n+1$ with the condition $a_{j,0} = b_j$ for all $0 \leqslant j \leqslant n$ and putting :

$$x_j = \sum_{\ell=0}^{n} a_{j,\ell} x'_\ell, \qquad \text{for all } 0 \leqslant j \leqslant n.$$

Then, for all for all $1 \leqslant i \leqslant k$, we have:

$$\frac{\partial f_i}{\partial x'_0} = \sum_{j=0}^{n} \frac{\partial x_j}{\partial x'_0} f_{i,j} = \sum_{j=0}^{n} b_j f_{i,j} = 0.$$

Therefore, in the new coordinates $(x'_0, \dots, x'_n)$, none of the forms appearing in σ depends on x'_0. □

The *compressibility* of a k-web $\sigma = (f_1, \dots, f_k)$ is defined as the number of independent variables that can be removed from the polynomials f_i, in a suitable coordinate system. In other words, σ has compressibility m if and only if $h^0(\mathcal{T}_\sigma) = m$; note that $0 \leqslant m \leqslant n-1$. We set $\hat{n} := n - m$; it indicates the minimal number of variables where the web is defined.

Lemma 2.14. *If σ is a compressible k-web in R, then there is an incompressible k-web $\hat{\sigma}$ in $\kappa[x_0, \dots, x_{\hat{n}}]$ and a $\hat{n}$-dimensional linear space $L \subset \mathbb{P}^n$ such that $\mathcal{T}_\sigma = \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-\hat{n})} \oplus \mathcal{E}$ where $\mathcal{E}|_L \simeq \mathcal{T}_{\hat{\sigma}}$.*

Proof. Assume that $\sigma = (f_1, \dots, f_k)$ is compressible, so that $m := h^0(\mathcal{T}_\sigma) > 0$; set $\hat{n} := n - h^0(\mathcal{T}_\sigma)$. We get a monomorphism $\mathcal{O}_{\mathbb{P}^n}^{\oplus m} \hookrightarrow \mathcal{T}_\sigma$ so that the following composition

$$\mathcal{O}_{\mathbb{P}^n}^{\oplus m} \hookrightarrow \mathcal{T}_\sigma \hookrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus m}$$

is the identity morphism; it follows that $\mathcal{T}_\sigma = \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-\hat{n})} \oplus \mathcal{E}$, where the sheaf $\mathcal{E}$ fits in the exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^n}^{\oplus(\hat{n}+1)} \xrightarrow{\mu} \mathcal{O}_{\mathbb{P}^n}(d-1)^{\oplus k} \,. \tag{17}$$

As we have seen in the proof of Lemma (2.13), there are new coordinates $(x'_0 : \cdots : x'_n)$ such that the variables $x'_0, \dots, x'_{m-1}$ do not appear in the polynomials $f_i \in \sigma$.

This means that the first m columns of the Jacobian matrix consist only of zeros, and that the matrix μ in display (17) is precisely the submatrix of trivial columns of $\mathcal{J}_\sigma$.

In addition, the k-web σ can be regarded as a k-web in $\kappa[x'_{m+1}, \dots, x'_{\hat{n}}]$, which we rename $\hat{\sigma}$. Setting $L = V(x'_0, \cdots, x'_{m-1})$, we have that $\mu|_L = \mathcal{J}_{\hat{\sigma}}$, thus $\mathcal{E}|_L \simeq \mathcal{T}_{\hat{\sigma}}$. □

As an immediate consequence, we have:

Corollary 2.15. *The logarithmic tangent sheaf of a compressible k-web is never slope-stable.*

Recall that a coherent subsheaf $\mathcal{F}$ of a coherent sheaf $\mathcal{E}$ is *saturated* if $\mathcal{E}/\mathcal{F}$ is torsion free. The following technical observation will be useful later on.

Lemma 2.16. *If σ is an incompressible k-web, then every saturated subsheaf $\mathcal{K} \subset \mathcal{T}_\sigma$ satisfies $c_1(\mathcal{K}) < 0$.*

Proof. Any saturated subsheaf $\mathcal{K}$ of $\mathcal{T}_\sigma$ is also a saturated subsheaf of $\mathcal{O}_{\mathbb{P}^n}^{\oplus n+1}$, which is a slope polystable sheaf. It follows that $c_1(\mathcal{K}) \leqslant 0$, and if $c_1(\mathcal{K}) = 0$, then [8, Corollary 1.6.11] implies that $\mathcal{K} = \mathcal{O}_{\mathbb{P}^n}^{\oplus m}$, so σ is compressible. □

We complete this section with a characterization of the degeneracy locus of k-webs as the set of points which are singular for some hypersurface of the web.

Lemma 2.17. *Let σ be a k-web. The reduced degeneracy locus $(\Xi_\sigma)_{\mathrm{red}}$ of the Jacobian matrix $\mathcal{J}_\sigma$ coincides with the union of the singular loci of the singular hypersurfaces contained in the web.*

Proof. Set $\sigma = (f_1, \dots, f_k)$. A point $x \in \mathbb{P}^n$ belongs to $(\Xi_\sigma)_{\mathrm{red}}$ if and only if the gradients of f_i are linearly dependent, that is, there is $(z_1, \dots, z_k) \in \kappa^{n+1} \backslash \{0\}$ such that:

$$\nabla \left(\sum_{i=1}^{k} f_i \right)(x) = \sum_{i=1}^{k} z_i \nabla f_i(x) = 0.$$

But this is the same as saying that x lies in the singular locus of $V(\sum_{i=1}^k z_i f_i)$. □

3. Stability for pencils of quadrics

In this section κ is algebraically closed of characteristic different from 2. Given a quadric hypersurface Q in $\mathbb{P}^n$, we denote by $\mathrm{rk}(Q)$ the rank of the Hessian matrix of an equation of Q, that is, the rank of a quadratic form associated with Q. We set $\mathrm{cork}(Q) = n + 1 - \mathrm{rk}(Q)$. When non empty, the singular locus of Q is a linear subspace of $\mathbb{P}^n$ of dimension $\mathrm{cork}(Q) - 1$. The quadric Q is a double plane if and only if $\mathrm{rk}(Q) = 1$.

In this section, we focus on regular sequences $\sigma = (f_1, f_2)$ such that $\deg(f_1) = \deg(f_2) = 2$, to which we can associate the pencil of quadrics $Q_\lambda := V\big(z_1 f_1 + z_2 f_2\big)$, where $\lambda = [z_1 : z_2] \in \mathbb{P}^1$. Let

$$r_0 := \max \big\{ \mathrm{cork}(Q_\lambda) \mid \lambda \in \mathbb{P}^1 \big\}.$$

Our goal is to present the proof of Theorem A as follows. We will start by considering the easiest case, namely pencils that contain at least one double hyperplane, and prove the first two items of Theorem A in Section 3.1. The third item is an immediate consequence of Lemma 2.13; the most involved part of Theorem A is the last item, and its proof will take the bulk of Section 3.2.

3.1. **Stability of pencils with a double hyperplane.** If Q_λ contains two double hyperplanes, then we can take $f_1 = x_0^2$ and $f_2 = x_1^2$, so that

$$\mathcal{J}_\sigma = \begin{pmatrix} 2x_0 & 0 & 0 & \cdots & 0 \\ 0 & 2x_1 & 0 & \cdots & 0 \end{pmatrix}.$$

It is then easy to see that $\mathcal{T}_\sigma = \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}$, as desired.

Assume now that Q_λ contains only one double plane. We can take $f_1 = x_0^2$ and put $g = f_2$, so that

$$\mathcal{J}_\sigma = \begin{pmatrix} 2x_0 & 0 & \cdots & 0 \\ g_0 & g_1 & \cdots & g_n \end{pmatrix},$$

with $g_i = \partial g / \partial x_i$, for $i \in \{0, \dots, n\}$. The hypothesis that Q_λ contains only one double plane implies that at least two of the partial derivatives g_i are non trivial for $i \in \{1, \dots, n\}$. Since

$$\bigwedge^2 \mathcal{J}_\sigma = 2x_0 \cdot (g_1 \ \cdots \ g_n),$$

it follows that $\Xi_\sigma = V(x_0) \cup V(g_1, \dots, g_n)$, thus $c_1(\mathcal{T}_\sigma) = -1$.

Lemma 3.1. *Let σ be a pencil of quadrics containing only one double plane. Then $\mathcal{T}_\sigma$ is slope-stable if and only if σ is incompressible.*

Proof. If σ is compressible, then $h^0(\mathcal{T}_\sigma) \neq 0$ by Lemma 2.13, thus $\mathcal{T}_\sigma$ cannot be slope-stable since $\mu(\mathcal{T}_\sigma) = -1/(n-1)$.

Conversely, assume that $\mathcal{T}_\sigma$ is not slope-stable, and let $\mathcal{K} \hookrightarrow \mathcal{T}_\sigma$ be a destabilizing subsheaf; set $r = \mathrm{rk}(\mathcal{K})$. If $c_1(\mathcal{K}) \leqslant -1$, then

$$\frac{-1}{n-1} \leqslant \frac{c_1(\mathcal{K})}{r} \leqslant \frac{-1}{r} \iff r \geqslant n-1,$$

which is a contradiction. It follows that $c_1(\mathcal{K}) = 0$, and Lemma 2.16 implies that σ must be compressible. □

3.2. **The stability criterion for pencils of quadrics.** If a pencil of quadrics σ contains no double hyperplanes, then $c_1(\mathcal{T}_\sigma) = -2$. Lemma 2.13 then implies that compressible pencils of quadrics containing no double hyperplanes have slope-unstable logarithmic sheaves, thus proving the third item of Theorem A.

We can finally address incompressible pencils of quadrics $\sigma = (f_1, f_2)$ containing no double hyperplanes. The result depends on the maximal corank of the quadrics Q_λ, where for each $\lambda = (z_1 : z_2) \in \mathbb{P}^1$ we write $Q_\lambda = V(z_1 f_1 + z_2 f_2)$. To be precise, we prove the following result.

Theorem 3.2. *Let $n \geqslant 3$ and let σ be an incompressible pencil of quadrics in $\mathbb{P}^n$ containing no double hyperplane. Put $r_0 = \max(\mathrm{cork}(Q_\lambda) \mid \lambda \in \mathbb{P}^1)$.*

i) If $2r_0 < n+1$, then $\mathcal{T}_\sigma$ is slope-stable.
ii) If $2r_0 = n+1$, then $\mathcal{T}_\sigma$ is strictly slope-semistable.
iii) If $2r_0 > n+1$, then $\mathcal{T}_\sigma$ is slope-unstable.

Since σ contains no double plane, we have $\mathrm{codim}(\mathcal{Q}_\sigma) \geqslant 2$ so the slope of $\mathcal{T}_\sigma$ is:

$$\mu(\mathcal{T}_\sigma) = \frac{2}{1-n}.$$

The proof of Theorem 3.2 will be divided in three parts. We start by establishing items ii) and iii) in Section 3.3. For the proof of item i), we first consider regular pencils in Section 3.4, leaving the case of irregular pencils for Section 3.6. We start with the following observation.

Lemma 3.3. *Let σ be an incompressible pencil of quadrics and let $\lambda, \mu \in \mathbb{P}^1$ be distinct points such that Q_λ and Q_μ are singular. Then the singular loci of Q_λ and Q_μ are disjoint linear spaces of dimension* $\mathrm{cork}(Q_\lambda) - 1$ *and* $\mathrm{cork}(Q_\mu) - 1$.

Proof. The singular locy of $Q\lambda$ and Q_μ are defined by linear equations and the corank of $Q\lambda$ and Q_μ is precisely the number of independent equations. In addition, these two linear spaces are disjoint, as the coordinates of a point of $\mathbb{P}^n$ lying in the singular locus of two distinct quadrics f_1, f_2 of the pencil would annihilate the derivatives of f_1 and f_2, so such derivatives would fail to span $H^0(\mathcal{O}_{\mathbb{P}^n}(1))$. Thus we could choose coordinates so that one of the variables $x_0, \ldots, x_n$ occurs neither in f_1 nor in f_2. However, this is excluded by the hypothesis that σ is incompressible. □

3.3. **Koszul subsheaves.** As a preliminary step towards the proof of items ii) and iii), let $\mathcal{R}_M$ be the first Koszul syzygy sheaf of a linear subspace $M \subset \mathbb{P}^n$ of codimension $r > 1$, namely the sheaf fitting into the following short exact sequence:

$$0 \to \mathcal{R}_M \to \mathcal{O}_{\mathbb{P}^n}^{\oplus r}(-1) \to \mathcal{I}_M \to 0.$$

We claim that $\mathcal{R}_M$ is slope-stable. Indeed, note that $\mu(\mathcal{R}_M(1)) = -1/(r-1)$; moreover, any saturated subsheaf $\mathcal{F} \hookrightarrow \mathcal{R}_M(1)$ must, by the argument in the proof

of Lemma 2.16, have $c_1(\mathcal{F}) \leqslant -1$, since $h^0(\mathcal{R}_M(1)) = 0$. If $\mathcal{F}$ destabilizes $\mathcal{R}_M(1)$, then

$$\frac{c_1(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} > \frac{-1}{r-1} \implies c_1(\mathcal{F}) > -\frac{\mathrm{rk}(\mathcal{F})}{r-1} > -1,$$

providing a contradiction.

3.3.1. *Koszul subsheaves from singular quadrics.* A linear subspace of Ξ_σ is called *maximal* if it is not strictly contained in another linear subspace of Ξ_σ. The following technical lemma is quite useful.

Lemma 3.4. *Let σ be a incompressible pencil, $\varrho \geqslant 3$ be an integer and $L \subset \Xi_\sigma$ be a maximal linear subspace of dimension $\varrho - 1$. Then there is a linear subspace $M \subset \mathbb{P}^n$ of codimension ϱ and a subscheme $W \subset M$ such that $\mathcal{T}_\sigma$ fits into*

$$0 \to \mathcal{R}_M(1) \to \mathcal{T}_\sigma \to \mathcal{R}_L(1) \to \mathcal{I}_{W/M}(1) \to 0. \tag{18}$$

Proof. Since the pencil σ is incompressible, the linear forms appearing in the Jacobian matrix of σ span $H^0(\mathcal{O}_{\mathbb{P}^n}(1))$, hence the sheaf $\mathcal{Q}_\sigma$ has rank 1 at each point of its support, in particular this happens at each point of L, so $\mathcal{Q}_\sigma|_L$ is a line bundle on L, namely there is $e \in \mathbb{Z}$ such that $\mathcal{Q}_\sigma|_L \simeq \mathcal{O}_L(e)$. Since $\mathcal{O}_L(1)^{\oplus 2}$ surjects onto $\mathcal{O}_L(e)$ and $\varrho \geqslant 3$, we conclude that $e = 1$.

The surjection $\mathcal{Q}_\sigma \to \mathcal{O}_L(1)$ allows to write the following commutative exact diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}_\sigma & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2} & \longrightarrow & \mathcal{Q}_\sigma & \longrightarrow & 0 \\
 & & \vdots & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_L(1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(1) & \longrightarrow & \mathcal{O}_L(1) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Put $\mathcal{F}$ and $\mathcal{G}$ for the kernel and cokernel of the induced morphism $\mathcal{M}_\sigma \to \mathcal{I}_L(1)$, respectively; in addition, let $\mathcal{Q}'$ denote the kernel of the epimorphism $\mathcal{Q}_\sigma \twoheadrightarrow \mathcal{O}_L(1)$. The snake lemma provides the following exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_{\mathbb{P}^n}(1) \to \mathcal{Q}' \to \mathcal{G} \to 0,$$

thus there is a subscheme $W \subset \mathbb{P}^n$ such that $\mathcal{F} \simeq \mathcal{I}_W(1)$ and

$$\mathrm{Supp}(\mathcal{G}) \subset \mathrm{Supp}(\mathcal{Q}') \subset \mathrm{Supp}(\mathcal{Q}_\sigma) = (\Xi_\sigma)_{\mathrm{red}}.$$

Since $\mathcal{M}_\sigma$ is the image of the the Jacobian matrix $\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2}$, we get a morphism $\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \to \mathcal{I}_L(1)$, with cokernel $\mathcal{G}$. Therefore, either this morphism is surjective, or $\mathcal{G}$ is supported on a linear space strictly containing L. However, this second possibility is excluded because L is maximal.

Summing up, we obtain an epimorphism $\mathcal{M}_\sigma \twoheadrightarrow \mathcal{I}_L(1)$. Since L is cut by $n+1-\varrho$ equations, the induced epimorphism $\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \twoheadrightarrow \mathcal{I}_L(1)$ factors through $\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1-\varrho)} \to$

$\mathcal{I}_L(1)$, and we get a second diagram:

$$
\begin{array}{ccccccccc}
 & & & & \mathcal{O}_{\mathbb{P}^n}^{\oplus\varrho} & \longrightarrow & \mathcal{I}_W(1) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} & \longrightarrow & \mathcal{M}_\sigma & \longrightarrow & 0 \\
 & & \vdots & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{R}_L(1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1-\varrho)} & \longrightarrow & \mathcal{I}_L(1) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\tag{19}
$$

Note that the image of the morphism $\mathcal{O}_{\mathbb{P}^n}^{\oplus\varrho} \to \mathcal{I}_W(1)$ is the ideal sheaf of a linear subspace $M \subset \mathbb{P}^n$ of codimension ϱ containing W, twisted by $\mathcal{O}_{\mathbb{P}^3}(1)$; its cokernel is the ideal of W in M, also twisted by $\mathcal{O}_{\mathbb{P}^3}(1)$. The snake lemma then yields the exact sequence (18). □

3.3.2. *Destabilizing Kozsul subsheaves.* Now we can prove items ii) and iii). Indeed, we set $\varrho = r_0$ and consider a quadric Q_λ in the pencil σ having $\operatorname{cork}(Q_\lambda) = r_0$. The assumption $r_0 \geqslant (n+1)/2$ forces $r_0 \geqslant 3$ or $(n, r_0) = (3, 2)$. The latter case follows from the full classification of pencils of quadrics in $\mathbb{P}^3$ and their logarithmic tangent sheaves provided in Subsection 5. Hence we can assume $r_0 \geqslant 3$, so the linear space $L \subset \mathbb{P}^n$ of dimension $r_0 - 1$ appearing as the singular locus of Q_λ satisfies the hypotheses of Lemma 3.4, thus $\mathcal{T}_\sigma$ contains the Koszul subsheaf $\mathcal{R}_M(1)$ which has slope $1/(1-r_0)$. The condition $r_0 > (n+1)/2$ implies:

$$\mu(\mathcal{R}_M(1)) = \frac{1}{1-r_0} > \frac{2}{1-n} = \mu(\mathcal{T}).$$

Finally, for item ii), we use the exact sequence in display (18), which yields:

$$0 \to \mathcal{R}_M(1) \to \mathcal{T}_\sigma \to \mathcal{E} \to 0, \tag{20}$$

where $\mathcal{E}$ is the kernel of $\mathcal{R}_L(1) \twoheadrightarrow \mathcal{I}_{W/M}(1)$. Since M has codimension $r_0 \geqslant 2$, the sheaves $\mathcal{E}$ and $\mathcal{R}_L(1)$ share the same slope, namely $1/(1-r_0)$. This implies that any destabilizing subsheaf of $\mathcal{E}$ would also destabilize $\mathcal{R}_L(1)$, thus $\mathcal{E}$ is slope-stable. But if $r_0 = (n+1)/2$, then $\mathcal{R}_L(1)$ also has slope equal to $1/(1-r_0)$. Therefore, the exact sequence in display (20) shows that $\mathcal{T}$ is strictly slope-semistable; in addition, $\mathcal{R}_M(1)$ and $\mathcal{E}$ are the factors of the Jordan–Holder filtration of $\mathcal{T}_\sigma$.

3.4. **Proof of stability for regular pencils.** Let σ be a regular pencil of quadrics containing no double hyperplane, so that there are only finitely many points $\lambda \in \mathbb{P}^1$ such that Q_λ is singular and at each such point the singular locus of Q_λ is a linear space of dimension $\operatorname{cork}(Q_\lambda) - 1$. A regular pencil is incompressible so these spaces are disjoint by Lemma 3.3, so $(\Xi_\sigma)_{\mathrm{red}}$ is the union of finitely linear spaces of dimension at most $r_0 - 1$.

In order to prove i) we assume, by contradiction, that $\mathcal{T}$ has a saturated destabilizing subsheaf $\mathcal{K}$ of rank p, with $1 \leqslant p \leqslant n-2$ with $\left(\bigwedge^p \mathcal{K}\right)^{\vee\vee} \simeq \mathcal{O}_{\mathbb{P}^n}(-e)$. Since σ is incompressible, Lemma 2.16 implies that $e > 0$. The condition that $\mathcal{K}$ destabilizes $\mathcal{T}$ amounts to:

$$(n-1)e \leqslant 2p.$$

Since $p \leqslant n-2$, this gives $e = 1$. Also, we get $p \geqslant (n-1)/2$.

Choose a sufficiently general linear subspace $M \subset \mathbb{P}^n$ of dimension $n - r_0$. Since $\dim(M) + \dim(\Xi_\sigma) = n-1$, we may assume that M is disjoint from the degeneracy locus Ξ_σ and that M meets transversely the locus where $\mathcal{T}/\mathcal{K}$ is not locally free. The second assumption implies that $\mathcal{T}\mathrm{or}_1(\mathcal{T}/\mathcal{K}, \mathcal{O}_M) = 0$, so we get a subsheaf $\mathcal{K}|_M \hookrightarrow \mathcal{T}|_M$ which still destabilizes $\mathcal{T}|_M$. The first assumption yields $\mathcal{Q}_\sigma|_M = 0 = \mathcal{T}\mathrm{or}^1(\mathcal{M}_\sigma, \mathcal{O}_M)$, so the restricted Jacobian matrix gives an exact sequence:

$$0 \to \mathcal{T}|_M \to \mathcal{O}_M^{\oplus(n+1)} \to \mathcal{O}_M^{\oplus 2}(1) \to 0. \tag{21}$$

The sheaf $\mathcal{T}|_M$ is locally free and, setting $q = n - 1 - p$ we get:

$$\Big(\bigwedge^p \mathcal{T}|_M\Big)(1) \simeq \Big(\bigwedge^q \mathcal{T}^\vee|_M\Big)(-1).$$

Since $\mathcal{K}|_M$ is a subsheaf of $\mathcal{T}|_M$, we obtain a monomorphism

$$\mathcal{O}_M(-1) \simeq \big(\bigwedge^p \mathcal{K}|_M\big)^{\vee\vee} \hookrightarrow \big(\bigwedge^p \mathcal{T}|_M\big)^{\vee\vee} \simeq \big(\bigwedge^p \mathcal{T}|_M\big)$$

which in turn gives $H^0\Big(\Big(\bigwedge^q \mathcal{T}^\vee|_M\Big)(-1)\Big) \neq 0$. We need to prove that this is absurd.

In order to check this, we dualize the exact sequence in display (21) and take exterior powers to get a long exact sequence:

$$0 \to \mathcal{O}_M(-q-1)^{\oplus(q+1)} \to \mathcal{O}_M(-q)^{\oplus q(n+1)} \to \cdots \to \Big(\bigwedge^q \mathcal{T}^\vee|_M\Big)(-1) \to 0.$$

All of the terms in the sequence above, except for the rightmost one, are copies of $\mathcal{O}_M(-t)$ for some integer t with $1 \leqslant t \leqslant q+1$. In the range $p \geqslant (n-1)/2$, we have $q = n-1-p \leqslant (n-1)/2$ so $q+1 \leqslant (n+1)/2$. Now, the assumption $r_0 < (n+1)/2$ guarantees $(n+1)/2 < n - r_0 + 1 = \dim(M) + 1$, thus $q < \dim(M)$. Therefore $H^*(\mathcal{O}_M(-t)) = 0$ for all $1 \leqslant t \leqslant q+1$ and hence $H^0\Big(\Big(\bigwedge^q \mathcal{T}^\vee|_M\Big)(-1)\Big) = 0$. This is the contradiction we were looking for, thus proving i).

This finishes the proof of Theorem 3.2 for regular pencils.

3.5. **Irregular pencils of quadrics.** The goal of this section is to set up some basic analysis of irregular pencils that is necessary for the proof of item i) for irregular pencils of quadrics. In particular, we study in detail a special kind of pencil that we call *completely irregular*, which is actually the only kind of irregular pencils where the proof given in Subsection 3.4 fails.

3.5.1. *The regular part of an irregular pencil.* Considering the polarization (or Hessian) matrix of the quadrics in the pencil σ we obtain a *pencil of symmetric matrices* of size $n+1$:

$$\rho_\sigma : \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^1}^{\oplus(n+1)}.$$

Definition 3.5. The *splitting type* of a pencil of quadrics σ is the unique pair of integers (u, v), such that

$$\mathrm{im}(\rho_\sigma) \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus u} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus v}.$$

Note that σ is regular if and only if ρ_σ is injective, if and only the splitting type of σ is $(n+1, 0)$. In addition, if σ is incompressible, then $\mathrm{coker}(\rho_\sigma)$ contains no

trivial direct summand. Set $r_1 := \min\{\operatorname{cork}(Q_\lambda) \mid \lambda \in \mathbb{P}^1\}$. We call r_1 the *generic corank* of $\mathbb{P}^1$. We have:

$$r_1 = n + 1 - \operatorname{rk}(\rho_\sigma) = n + 1 - u - v.$$

Set $\mathcal{K} := \ker(\rho_\sigma)$, $\mathcal{F} := \operatorname{im}(\rho_\sigma)$, $\mathcal{C}_\sigma := \operatorname{coker}(\rho_\sigma)$ and write the long exact sequence:

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^1}^{\oplus(n+1)} \to \mathcal{C}_\sigma \to 0. \tag{22}$$

The cokernel sheaf $\mathcal{C}_\sigma$ decomposes as $\mathcal{C}_\sigma \simeq \mathcal{C}_{\mathrm{tf}} \oplus \mathcal{C}_{\mathrm{t}}$, where $\mathcal{C}_{\mathrm{t}}$ is its torsion part and $\mathcal{C}_{\mathrm{tf}}$ is its torsion free part.

Lemma 3.6. *If σ has splitting type (u, v) then $u \geqslant v$ and there is a regular pencil of symmetric matrices $\bar{\rho}_\sigma$ with size $u - v$ satisfying* $\operatorname{coker}(\bar{\rho}_\sigma) \simeq \mathcal{C}_{\mathrm{t}}$.

Proof. Using that ρ_σ is symmetric and that $\mathcal{C}_{\mathrm{t}}$ is zero-dimensional, we get:

$$\mathcal{C}_{\mathrm{tf}} \simeq \mathcal{C}_\sigma^{\vee\vee} \simeq \mathcal{K}^\vee(-1), \qquad \mathcal{C}_{\mathrm{t}} \simeq \mathcal{E}xt^1_{\mathbb{P}^1}(\mathcal{C}_\sigma, \mathcal{O}_{\mathbb{P}^1}(-1)).$$

Therefore, dualizing the above sequence we get:

$$\begin{aligned} &0 \to \mathcal{C}_\sigma^\vee(-1) \to \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(n+1)} \xrightarrow{\alpha} \mathcal{F}^\vee(-1) \to \mathcal{C}_{\mathrm{t}} \to 0, \\ &0 \to \mathcal{F}^\vee(-1) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^1}^{\oplus(n+1)} \to \mathcal{K}^\vee(-1) \to 0. \end{aligned} \tag{23}$$

Again by the symmetry of ρ_σ, the following composition of morphisms

$$\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(n+1)} \xrightarrow{\alpha} \mathcal{F}^\vee(-1) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^1}^{\oplus(n+1)}$$

coincides with ρ_σ, thus the image of the morphism α in display (23) is precisely $\mathcal{F}$. We get an injective twisted endomorphism $\overline{\rho}_\sigma$ of the vector bundle $\mathcal{F}$ and an exact sequence:

$$0 \to \mathcal{F} \xrightarrow{\overline{\rho}_\sigma} \mathcal{F}^\vee(-1) \to \mathcal{C}_{\mathrm{t}} \to 0.$$

This endomorphism is also symmetric. Let us rewrite this in terms of the splitting type:

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus u} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus v} \xrightarrow{\overline{\rho}_\sigma} \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus v} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus u} \to \mathcal{C}_{\mathrm{t}} \to 0.$$

The morphism $\mathcal{O}_{\mathbb{P}^1}^{\oplus v} \to \mathcal{O}_{\mathbb{P}^1}^{\oplus u}$ extracted from $\bar{\rho}_\sigma$ is thus injective. Therefore its transpose $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus u} \to \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus v}$ is surjective – in particular we must have $v \leqslant u$. These constant maps of maximal rank can be expunged from $\bar{\rho}_\sigma$ so that $\overline{\rho}_\sigma$ reduces to a symmetric morphism, still denoted by $\overline{\rho}_\sigma$, which takes the form:

$$\overline{\rho}_\sigma : \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(u-v)} \hookrightarrow \mathcal{O}_{\mathbb{P}^1}^{\oplus(u-v)}.$$

The torsion part $\mathcal{C}_{\mathrm{t}}$ is a sheaf of length $h^0(\mathcal{C}_{\mathrm{t}}) = u - v$ which is supported at the points $\lambda \in \mathbb{P}^1$ such that $\operatorname{cork}(Q_\lambda) > r_1$. We obtain the exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(u-v)} \xrightarrow{\overline{\rho}_\sigma} \mathcal{O}_{\mathbb{P}^1}^{\oplus(u-v)} \to \mathcal{C}_{\mathrm{t}} \to 0. \tag{24}$$

□

Definition 3.7. The pencil of quadrics associated with the pencil of matrices $\bar{\rho}_\sigma$ given by the previous lemma is called the *regular part of σ*.

Note that the regular part of a pencil of quadrics may be empty: this happens for $u = v$. We will shall call these pencils completely irregular and treat them in detail a bit further on. Note also that the regular part of an irregular pencil may fail to be a pencil, namely when $u = v + 1$. By convention, a pencil of symmetric matrices of size 0 (the empty pencil) and a non-zero pencil of symmetric matrices of size 1 are regular.

3.5.2. *Recovering the pencil.* The torsion-free part $\mathcal{C}_{\mathrm{tf}}$ of $\mathcal{C}_\sigma$ decomposes as a sum of ample line bundles. Namely, there are integers $c_1, \dots, c_{r_1}$ such that:

$$\mathcal{C}_{\mathrm{tf}} \simeq \bigoplus_{i=1}^{r_1} \mathcal{O}_{\mathbb{P}^1}(c_i), \qquad 1 \leqslant c_1 \leqslant \cdots \leqslant c_{r_1}, \qquad \sum_{i=1}^{r_1} c_i = v.$$

We call $\mathbf{c} = (c_1, \dots, c_{r_1})$ the *degree vector* of σ. The following result allows to recover a pencil from its regular part and the degree vector.

Proposition 3.8. *Let $n_1 \in \mathbb{N}$ and let ρ be a regular pencil of symmetric matrix of size n_1. Fix integers r_1 and $1 \leqslant c_1 \leqslant \cdots \leqslant c_{r_1}$. Then, up homography, there is a unique incompressible pencil σ such that $\mathcal{C}_\sigma \simeq \mathcal{C}_{\mathrm{t}} \oplus \mathcal{C}_{\mathrm{tf}}$ with:*

$$\mathcal{C}_{\mathrm{tf}} \simeq \bigoplus_{i=1}^{r_1} \mathcal{O}_{\mathbb{P}^1}(c_i), \qquad \mathcal{C}_{\mathrm{t}} \simeq \operatorname{coker}(\rho), \qquad \bar{\rho}_\sigma = \rho. \tag{25}$$

Proof. According to the previous subsection, we put:

$$v = \sum_{i=1}^{r_1} c_i, \qquad u = n_1 + v, \qquad n = r_1 + u + v - 1. \tag{26}$$

For all $m \in \mathbb{N}^*$ we consider the exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus m} \xrightarrow{\tau_m} \mathcal{O}_{\mathbb{P}^1}^{\oplus(m+1)} \to \mathcal{O}_{\mathbb{P}^1}(m) \to 0, \tag{27}$$

given by the mth symmetric power of the Euler sequence on $\mathbb{P}^1$. The matrix τ_m is unique up to a coordinate change on the source and target and up to homography of $\mathbb{P}^1$.

We consider the block-diagonal matrix formed by the morphisms τ_{c_i} for all $i \in \{1, \dots, r_1\}$. In view of the definition of v this gives:

$$\bigoplus_{i=1}^{r_1} \tau_{c_i} : \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus v} \to \mathcal{O}_{\mathbb{P}^1}^{\oplus(v+r_1)}.$$

Stacking this morphism together with its transpose and with ρ, in view of our definition of n we get a symmetric matrix pencil of the form:

$$\bigoplus_{i=1}^{r_1} (\tau_{c_i} \oplus \tau_{c_i}^{\mathrm{t}}) \oplus \rho : \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^1}^{\oplus(n+1)}. \tag{28}$$

We obtain a pencil of quadrics σ on $\mathbb{P}^n$. By construction σ is incompressible and satisfies the conditions in display (25).

About uniqueness of σ, we argue as follows. First note that, under the assumption of incompressibility, the dimension n and the splitting type (u, v) are determined by $r_1, c_1, \dots, c_{r_1}$ through the equalities in display (26). Next, we observe that for any pencil σ satisfying (25), the direct sum decomposition of $\mathcal{C}_\sigma$ gives:

$$0 \to \begin{array}{c} \mathcal{O}_{\mathbb{P}^1}^{\oplus(u-v)}(-1) \\ \oplus \\ \mathcal{O}_{\mathbb{P}^1}^{\oplus v}(-1) \end{array} \to \begin{array}{c} \mathcal{O}_{\mathbb{P}^1}^{\oplus(u-v)} \\ \oplus \\ \mathcal{O}_{\mathbb{P}^1}^{\oplus(v+r_1)} \end{array} \to \begin{array}{c} \mathcal{C}_{\mathrm{t}} \\ \oplus \\ \mathcal{C}_{\mathrm{tf}} \end{array} \to 0.$$

This is a block-diagonal matrix pencil having a block $\mathcal{O}_{\mathbb{P}^1}^{\oplus(u-v)}(-1) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus(u-v)}$ which is ρ and a block $\mathcal{O}_{\mathbb{P}^1}^{\oplus v}(-1) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus(v+r_1)}$ obtained by concatenating minimal presentation matrices of $\mathcal{O}_{\mathbb{P}^1}(c_1), \dots, \mathcal{O}_{\mathbb{P}^1}(c_{r_1})$. In other words, this second block is $\tau_{c_1} \oplus \cdots \oplus \tau_{c_{r_1}}$. Up to choosing coordinates of $\mathbb{P}^n$ adapted to this decomposition, the above pencil appears in a block decomposition of ρ_σ and by symmetry of ρ_σ the residual block must be $\tau_{c_1}^{\mathrm{t}} \oplus \cdots \oplus \tau_{c_{r_1}}^{\mathrm{t}}$. So σ is obtained as in (28). □

3.5.3. *The Jacobian scheme of an irregular pencil.* We continue with the assumption that σ is an incompressible irregular pencil and seek a set-theoretic description of degeneracy scheme Ξ_σ.

The torsion free part $\mathcal{C}_{\mathrm{tf}}$ defines a projective bundle $Y = \mathbb{P}(\mathcal{C}_{\mathrm{tf}})$ and, since the vector bundle $\mathcal{C}_{\mathrm{tf}}$ is very ample, $Y = \mathbb{P}(\mathcal{C}_{\mathrm{tf}})$ embeds via the linear system of the tautological relatively ample divisor h as a rational normal scroll of degree v, spanning a linear space $L \subset \mathbb{P}^n$ of dimension $n-u$.

Lemma 3.9. *Let σ be an incompressible pencil of quadrics. Then Ξ_σ satisfies:*

$$(\Xi_\sigma)_{\mathrm{red}} = Y \cup \bigcup_{\lambda \in \mathrm{Supp}(\mathcal{C}_{\mathrm{t}})} \mathbb{P}^{r_\lambda - 1}, \tag{29}$$

where the linear subspaces $\{\mathbb{P}^{r_\lambda-1} \mid \lambda \in \mathrm{Supp}(\mathcal{C}_{\mathrm{t}})\}$ are disjoint. In particular:

$$\dim(\Xi_\sigma) = \max(r_0 - 1, r_1), \qquad r_1 = n + 1 - u - v.$$

Proof. We look at the projectivization of the vector bundle $\mathcal{C}_{\mathrm{tf}}$ and of the coherent sheaves $\mathcal{C}_{\mathrm{t}}$, $\mathcal{C}_\sigma$ The epimorphisms $\mathcal{O}_{\mathbb{P}^1}^{\oplus(n+1)} \twoheadrightarrow \mathcal{C}_\sigma$, $\mathcal{O}_{\mathbb{P}^1}^{\oplus(n+1)} \twoheadrightarrow \mathcal{C}_{\mathrm{t}}$, and $\mathcal{O}_{\mathbb{P}^1}^{\oplus(n+1)} \twoheadrightarrow \mathcal{C}_{\mathrm{tf}}$ induce embeddings $\mathbb{P}(\mathcal{C}_\sigma) \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^n$, $\mathbb{P}(\mathcal{C}_{\mathrm{t}}) \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^n$ and $Y \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^n$. Similarly, the epimorphism $\mathcal{O}_{\mathbb{P}^n}^{\oplus 2} \twoheadrightarrow \mathcal{Q}(-1)$ induces an embedding $\mathbb{P}(\mathcal{Q}_\sigma(-1)) \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^n$. The two subschemes $\mathbb{P}(\mathcal{C}_\sigma)$ and $\mathbb{P}(\mathcal{Q}_\sigma(-1))$ of $\mathbb{P}^1 \times \mathbb{P}^n$ are defined by the same bihomogeneous equations. Indeed, denoting by $\lambda = (z_1 : z_2)$ and $x = (x_0 : \ldots : x_n)$ the points of $\mathbb{P}^1$ and $\mathbb{P}^n$ and recalling the notation $f_{i,j} = \frac{\partial f_i(x)}{\partial x_j}$, we have:

$$\mathbb{P}(\mathcal{Q}_\sigma(-1)) = \mathbb{P}(\mathcal{C}_\sigma) = \{((x, \lambda) \in \mathbb{P}^1 \times \mathbb{P}^n \mid f_{1,j} z_1 + f_{2,j} z_2 = 0, \forall j = 0, \dots, n\},$$

which in turn gives a Kozsul complex (in the obvious notation):

$$\cdots \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(-1,-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}(\mathcal{C}_\sigma)} \to 0. \tag{30}$$

We get thus a correspondence:

$$\begin{array}{ccc} & \mathbb{P}(\mathcal{C}_\sigma) & \\ {\scriptstyle\varphi}\swarrow & & \searrow{\scriptstyle\psi} \\ \mathbb{P}^1 & & \mathbb{P}^n \end{array} \tag{31}$$

where the map $\varphi : \mathbb{P}(\mathcal{C}_\sigma) \to \mathbb{P}^1$ is generically a $\mathbb{P}^{r_1-1}$-bundle and $\psi : \mathbb{P}(\mathcal{C}_\sigma) \to \Xi_\sigma \subset \mathbb{P}^n$ is an isomorphism at the points where $\mathcal{Q}_\sigma$ has rank 1.

At each point λ of the support of the torsion part $\mathcal{C}_{\mathrm{t}}$ we have a skyscraper sheaf supported at λ, whose rank we denote by r_λ. The surjection $\mathcal{O}_{\mathbb{P}^1}^{\oplus(n+1)} \to \mathcal{O}_\lambda^{\oplus r_\lambda}$ induces an embedding $\mathbb{P}^{r_\lambda-1} \subset \Xi_\sigma \subset \mathbb{P}^n$. We noticed in Lemma 3.3 that the linear spaces appearing as singular loci of distinct points of $\mathrm{Supp}(\mathcal{C}_{\mathrm{t}})$ are disjoint. This achieves the proof. □

In addition, we will also use the following statement later on.

Lemma 3.10. *An incompressible pencil of quadrics satisfies $r_1 \leqslant \frac{n+1}{3}$.*

Proof. We assume that σ is an incompressible pencil of quadrics. Recall that σ has a splitting type (u, v) with $u \geqslant v$. Since σ is incompressible, the integers $c_1, \dots, c_{r_1}$ appearing in the degree vector are strictly positive, hence:

$$v = \sum_{i=1}^{r_1} c_i \geqslant r_1.$$

Therefore we have $n + 1 = r_1 + u + v \geqslant r_1 + 2v \geqslant 3r_1$. □

3.5.4. *Completely irregular pencils.* We say that the incompressible pencil σ is *completely irregular* if has no regular part, which is to say, if $u = v$. This is equivalent to the condition $\mathcal{C}_{\mathrm{t}} = 0$, which in turn is tantamount to $r_0 = r_1$.

We take a closer look at the degeneration scheme in this case. Denote by F the divisor class of a fibre of $Y \to \mathbb{P}^1$, so $\mathcal{O}_Y(F) \simeq \varphi^*(\mathcal{O}_{\mathbb{P}^1}(1))$, in the notation of the diagram in display (31); write also $H = c_1(\psi^*(\mathcal{O}_{\mathbb{P}^n}(1)))$. Note that $\psi_*(\mathcal{O}_Y(F)) \simeq \mathcal{Q}_\sigma$ and that the Koszul complex (30) is exact at $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}$. Tensoring it with $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}(1, 0)$ and applying ψ_*, we get an exact sequence:

$$\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}(-1) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus 2} \to \mathcal{O}_Y(F) \to 0.$$

The rightmost morphism above agrees with the Jacobian matrix, so we have an exact sequence:

$$0 \to \mathcal{T}_\sigma \to \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \xrightarrow{\mathcal{J}_\sigma} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2} \to \mathcal{O}_Y(H + F) \to 0. \tag{32}$$

Example 3.11. Let us list all the possible splitting types (u, v) for $n = 5$ together with the data of the scroll, compressibility and so forth. Regular pencils give $\mathcal{C}_\sigma = \mathcal{C}_{\mathrm{t}}$, a finite-length scheme with $h^0(\mathcal{C}_\sigma) = 6$. For irregular pencils we have the following possibilities.

r_1	(u, v)	$h^0(\mathcal{C}_{\mathrm{t}})$	$\mathcal{C}_{\mathrm{tf}}$	m	Completely irregular
1	$(5, 0)$	5	$\mathcal{O}_{\mathbb{P}^1}$	yes	no
1	$(4, 1)$	3	$\mathcal{O}_{\mathbb{P}^1}(1)$	no	no
1	$(3, 2)$	1	$\mathcal{O}_{\mathbb{P}^1}(2)$	no	no
2	$(4, 0)$	4	$\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$	yes	no
2	$(3, 1)$	2	$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$	yes	no
2	$(2, 2)$	0	$\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$	no	yes
2	$(2, 2)$	0	$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$	yes	yes
3	$(3, 0)$	3	$\mathcal{O}_{\mathbb{P}^1}^{\oplus 3}$	yes	no
3	$(2, 1)$	1	$\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$	yes	no
4	$(2, 0)$	2	$\mathcal{O}_{\mathbb{P}^1}^{\oplus 4}$	yes	no
4	$(1, 1)$	0	$\mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$	yes	yes
5	$(1, 0)$	1	$\mathcal{O}_{\mathbb{P}^1}^{\oplus 5}$	yes	no

Observe that there is only one incompressible, completely irregular pencil of quadrics in $\mathbb{P}^5$; moreover, $n = 5$ is the lowest dimension in which such pencils can occur.

Lemma 3.12. *Let σ be an incompressible completely irregular pencil. Then there is an action of $\mathrm{SL}_2 = \mathrm{SL}_2(\kappa)$ on $\mathbb{P}^n$ for which the exact sequence in display* (32) *is equivariant.*

Proof. Let the group SL_2 act by homographies on $\mathbb{P}^1$ regarded as the base of the pencil. For any integer $m \in \mathbb{N}$ we let V_m be the irreducible representation SL_2 of

weight m, so V_0 is the trivial representation of rank 1, V_1 is the standard representation of rank 2, while $V_m = S^m V_1$ has rank $m+1$. By convention we set $V_{-1} = 0$. For all $m \in \mathbb{N}^*$ we rewrite (27) as in SL_2-equivariant form:

$$0 \to V_{m-1} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\tau_m} V_m \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(m) \to 0.$$

Given an incompressible, completely irregular pencil σ, the matrix pencil σ is formed by stacking the morphisms τ_{c_i} with their transpose for all $i \in \{1, \dots, n+1-2u\}$ whereby obtaining a symmetric block matrix. Thus we rewrite the exact sequence in display (22) in equivariant form:

$$0 \to \bigoplus_{i=1}^{n+1-2u} \mathcal{O}_{\mathbb{P}^1}(-1-c_i) \to \bigoplus_{i=1}^{n+1-2u} (V_{c_i-1} \oplus V_{c_i}) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\rho}$$
$$\xrightarrow{\rho} \bigoplus_{i=1}^{n+1-2u} (V_{c_i-1} \oplus V_{c_i}) \otimes \mathcal{O}_{\mathbb{P}^1} \to \bigoplus_{i=1}^{n+1-2u} \mathcal{O}_{\mathbb{P}^1}(c_i) \to 0.$$

Since the matrix pencil ρ is SL_2-equivariant, so is the pencil σ obtained by ρ. Hence the Jacobian matrix of σ is also SL_2-equivariant and this induces an SL_2-action on its kernel and cokernel sheaves. Since the pencil σ is determined by the degree vector $(c_1, \dots, c_{n+1-2u})$ by Proposition 3.8, we get that this construction holds for any incompressible completely irreducible pencil. □

Note that the SL_2-action on $\mathcal{O}_Y(H+f)$ is simply the action by homographies on the basis of the scroll Y. Also, the SL_2-action on $\mathbb{P}^n$ induces the isomorphism of SL_2-modules:

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = \bigoplus_{i=1}^{n+1-2u} (V_{c_i-1} \oplus V_{c_i}). \tag{33}$$

The following lemma says that, if σ is a completely irregular pencil of quadrics, then the logarithmic sheaf $\mathcal{T}_\sigma$ is simple.

Lemma 3.13. *Let σ be a completely irregular incompressible pencil of quadrics with $u \geqslant 2$. Then* $\mathrm{End}_{\mathbb{P}^n}(\mathcal{T}_\sigma) \simeq \kappa$.

Proof. We use the exact sequence in display (32). We first use its rightmost part, namely:

$$0 \to \mathcal{M}_\sigma \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2} \to \mathcal{O}_Y(H+F) \ \to 0. \tag{34}$$

Since $\mathcal{O}_Y$ is a line bundle on the smooth irreducible variety Y, we have $\mathrm{End}_X(\mathcal{O}_Y) \simeq \kappa$. Since the morphism $\mathcal{O}_{\mathbb{P}^n}^{\oplus 2} \to \mathcal{O}_Y(F)$ induces an isomorphism on global sections and $H^p(\mathcal{O}_Y(F)) = 0$ for $p > 0$, we get $H^*(\mathcal{M}_\sigma(-1)) = 0$. Also, $\mathrm{Ext}^p_{\mathbb{P}^n}(\mathcal{O}_Y(F+h), \mathcal{O}_{\mathbb{P}^n}(1)) = 0$ for $p = 0, 1$ by Serre duality since $\dim(Y) = n+1-2u < n-1$. Therefore applying $\mathrm{Hom}_{\mathbb{P}^n}(-, \mathcal{M}_\sigma)$ and $\mathrm{Hom}_{\mathbb{P}^n}(\mathcal{O}_Y(H+F), -)$ to the exact sequence in display (34) we get:

$$\mathrm{End}_{\mathbb{P}^n}(\mathcal{M}_\sigma) \simeq \mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{O}_Y(F+h), \mathcal{M}_\sigma) \simeq \mathrm{End}_{\mathbb{P}^n}(\mathcal{O}_Y) \simeq \kappa.$$

Also, note that Serre duality gives $\mathrm{Ext}^p_{\mathbb{P}^n}(\mathcal{O}_Y(F+h), \mathcal{O}_{\mathbb{P}^n}) = 0$ for $p = 1, 2$ as $\dim(Y) = n+1-2u < n-2$. We deduce that $\mathrm{Ext}^p_{\mathbb{P}^n}(\mathcal{M}_\sigma, \mathcal{O}_{\mathbb{P}^n}) = 0$ for $p = 0, 1$.

Next, we write the exact sequence:

$$0 \to \mathcal{T}_\sigma \to \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \to \mathcal{M}_\sigma \to 0. \tag{35}$$

Applying $\mathrm{Hom}_{\mathbb{P}^n}(\mathcal{M}_\sigma, -)$ to (35) and using that $\mathrm{Ext}^p_{\mathbb{P}^n}(\mathcal{M}_\sigma, \mathcal{O}_{\mathbb{P}^n}) = 0$ for $p = 0, 1$, we get:

$$\mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{M}_\sigma, \mathcal{T}_\sigma) \simeq \mathrm{End}_{\mathbb{P}^n}(\mathcal{M}_\sigma) \simeq \kappa.$$

Finally, we apply $\mathrm{Hom}_{\mathbb{P}^n}(-, \mathcal{T}_\sigma)$ to (35) and use that, since σ is incompressible, we have $H^0(\mathcal{T}_\sigma) = H^1(\mathcal{T}_\sigma) = 0$. This concludes the proof, since:

$$\mathrm{End}_{\mathbb{P}^n}(\mathcal{T}_\sigma) \simeq \mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{M}_\sigma, \mathcal{T}_\sigma) \simeq \kappa.$$

□

3.6. **Proof of stability for irregular pencils of quadrics.** It only remains for us to prove item i) for irregular incompressible pencils σ containing no double hyperplane.

By hypothesis, we have $2r_0 < n + 1$. As in Subsection 3.4, we assume by contradiction that $\mathcal{T}$ admits a saturated subsheaf $\mathcal{K}$ of rank p and again we get $p \geqslant (n-1)/2$ and $(\bigwedge^p \mathcal{K})^{\vee\vee} \simeq \mathcal{O}_{\mathbb{P}^n}(-1)$.

Next, observe that, if $r_1 < r_0$, then by Lemma 3.9 we have $\dim(\Xi_\sigma) \leqslant r_0 - 1$. In this case, the proof of i) given in Subsection 3.4 goes through as again a sufficiently general linear subspace $M \subset \mathbb{P}^n$ of dimension $n - r_0$ does not meet Ξ_σ, while the rest of the argument is still valid for irregular pencils.

Therefore, we may assume until the end of the subsection that $r_0 = r_1$, which is to say, that the pencil is completely irregular, so $u = v$, $\mathcal{C}_\sigma = \mathcal{C}_{\mathrm{tf}}$ and $\dim(\Xi_\sigma) = r_0$.

This time, with a slight difference with respect to the proof of item i) given in Subsection 3.4, we choose a general linear subspace $M \subset \mathbb{P}^n$ of dimension $n - r_0 - 1$. In particular, we may assume that M does not meet Ξ_σ and that the p^{th} exterior power of $\mathcal{K}|_M \hookrightarrow \mathcal{T}|_M$ gives a non-zero element of $H^0(\bigwedge^q \mathcal{T}^\vee|_M(-1))$, with $q = n - 1 - p$; this equality $q = n - 1 - p$ comes from duality of the sheaf $\mathcal{T}$, which is of rank $n - 1$.

If n is odd, we write $n = 2n_0 + 1$ and $2r_0 < n + 1$ gives $r_0 \leqslant n_0$, while $p \geqslant (n-1)/2 = n_0$ gives $q = 2n_0 - p \leqslant n_0$. Since $\dim(M) = 2n_0 - r_0 \geqslant n_0$ we get $q < \dim(M)$ unless $\dim(M) = n_0$. If $q < \dim(M)$, again the argument given in Subsection 3.4 remains valid, so we may assume, without loss of generality, that $\dim(M) = n_0$. It then follows that $n_0 = \dim(M) = 2n_0 - r_0$, thus $n_0 = r_0$; since $r_0 = n + 1 - 2u$, n_0 is even, say $n_0 = 2n_1$, and $u = n_1 + 1$. Summing up, if n is odd, then:

$$n = 4u - 3, \quad \dim(M) = p = q = r_0 = 2(u-1), \quad u \geqslant 2. \tag{36}$$

Similarly, if $n = 2n_0$, then we get that $r_0 \leqslant n_0 \leqslant p$ and thus $q \leqslant n_0 - 1 \leqslant \dim(M)$, so we may assume that $\dim(M) = n_0 - 1$. It follows that $n_0 = r_0 = n + 1 - 2u$ so n_0 is odd; setting $n_0 = 2n_1 + 1$, we obtain $u = n_1 + 1$. Summarizing, if n is even, then:

$$n = 4u - 2, \quad \dim(M) = q = 2(u-1), \quad r_0 = p = 2u - 1, \quad u \geqslant 2. \tag{37}$$

In any case, the sheaves $\mathcal{K}$ and $\mathcal{L} = \mathcal{T}/\mathcal{K}$ are slope-stable.

Having established these numerical constraints, we proceed with the next step of the proof, which requires looking at the exact sequence in display (32). Working

in the linear span $L = \mathbb{P}^{n+1-u} \subset \mathbb{P}^n$ of Y, we write an exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{I}_{Y/L}(1)^{\oplus 2} & = & \mathcal{I}_{Y/L}(1)^{\oplus 2} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{M}_L & \longrightarrow & \mathcal{O}_L(1)^{\oplus 2} & \longrightarrow & \mathcal{O}_Y(H+F) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_Y(h-F) & \longrightarrow & \mathcal{O}_Y(h)^{\oplus 2} & \longrightarrow & \mathcal{O}_Y(H+F) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Here, the sheaf $\mathcal{M}_L$, defined by the middle row, can be though of as the normal sheaf associated to the pencil of quadrics restricted to the smaller space L. Using the leftmost column of the previous diagram and the fact that the morphism $\mathcal{O}_L(1)^{\oplus 2} \to \mathcal{O}_Y(H+F)$ in the exact sequence in display (32) factors through $\mathcal{O}_L(1)^{\oplus 2} \to \mathcal{O}_Y(H+F)$, we get an exact sequence:

$$0 \to \mathcal{I}_{L/\mathbb{P}^n}(1)^{\oplus 2} \to \mathcal{M}_\sigma \to \mathcal{M}_L \to 0.$$

Using this exact sequence and the one in display (32) we get a second exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{R}_L(1)^{\oplus 2} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}^{\oplus 2u} & \longrightarrow & \mathcal{I}_{L/\mathbb{P}^n}(1)^{\oplus 2n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}^{\oplus (n+1)} & \longrightarrow & \mathcal{M}_\sigma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}^{\oplus (n+1-2u)} & \longrightarrow & \mathcal{M}_L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here the sheaf $\mathcal{G}$, defined with the bottom row, has $c_1(\mathcal{G}) = 0$ and $\mathrm{rk}(\mathcal{G}) = n+1-2u$, while $\mathcal{R}_L(1)$ is the Koszul syzygy of L which we already proved to be stable of slope $1/(1-u)$.

In view of the leftmost column of the previous diagram and of the slope-semistability of $\mathcal{R}_L(1)^{\oplus 2}$, the inclusion $\mathcal{K} \hookrightarrow \mathcal{T}$ must descend to an inclusion $\mathcal{K} \hookrightarrow \mathcal{G}$. We get thus two injections with isomorphic cokernel:

$$\mathcal{K} \hookrightarrow \mathcal{G}, \qquad \mathcal{R}_L(1)^{\oplus 2} \hookrightarrow \mathcal{L}. \tag{38}$$

Denote by $\mathcal{P}$ this common cokernel sheaf, so $\mathcal{P} \simeq \mathcal{G}/\mathcal{K} \simeq \mathcal{L}/\mathcal{R}_L(1)^{\oplus 2}$. Then, note that independently on whether n is even or odd, we get:

$$\mathrm{rk}(\mathcal{K}) = p = n + 1 - 2u = \mathrm{rk}(\mathcal{G}).$$

Therefore, since $(\bigwedge^p \mathcal{K})^{\vee\vee} \simeq \mathcal{O}_{\mathbb{P}^n}(-1)$, we get a hyperplane $H \subset \mathbb{P}^n$ as support $\mathcal{P}$. Note that the hyperplane H is determined uniquely by $\mathcal{T}$. Indeed, if $\mathcal{K}' \hookrightarrow \mathcal{T}$ is an embedding of any saturated destabilizing subsheaf of $\mathcal{T}$, then the induced morphism $\mathcal{K}' \to \mathcal{L}$ is either zero or an isomorphism, since $\mathcal{K}'$ and $\mathcal{L}$ are stable of

the same slope. If $\mathcal{K}' \to \mathcal{L}$ is zero then $\mathcal{K}' \to \mathcal{T}$ factors through $\mathcal{K}$ so it determines the same hyperplane H. If $\mathcal{K}' \to \mathcal{L}$ is an isomorphism, then $\mathcal{T}$ is decomposable, which is absurd by Lemma 3.13.

Now, recall from Lemma 3.12 that $\mathcal{T}_\sigma$ is equivariant for a natural SL_2-action on $\mathbb{P}^n$. So the hyperplane H must be fixed by this action, in other words, it must correspond to a trivial summand V_0 in the decomposition in display (33).

Set t for the number of indices $i \in \{1, \dots, n+1-2u\}$ such that $c_i = 1$, so:

$$u = t + \sum_{i=t+1}^{n+1-2u} c_i \geqslant 2(n-2u-t+1) + t. \tag{39}$$

If $t \geqslant 3$, then we can equip Y (and consequently $\mathcal{T}_\sigma$) with a further SL_2-action by letting SL_2 operate as $V_{t-1} \otimes \mathcal{O}_{\mathbb{P}^1}(1)$ on the summands of $\mathcal{C}_\sigma$ of the form $\mathcal{O}_{\mathbb{P}^1}(1)$. Again we obtain that $H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ contains no copy of V_0. In all these cases the SL_2-fixed hyperplane H cannot exist and we conclude that $\mathcal{T}_\sigma$ is stable.

Finally if $t \leqslant 2$ then using (39), depending on whether n is odd or even, we get from (36) or (37) that $u \leqslant 2$ or $u \leqslant 1$, which leaves the only case $n = 5$, $u = 2$, $c_1 = c_2 = 1$. This last case corresponds to the pencil of quadrics $\sigma = (x_1x_5 + x_3x_4, x_2x_4 + x_0x_5)$. For this explicit pencil, direct computation shows that $H^0\Big(\big(\bigwedge^2 \mathcal{T}_\sigma\big)^\vee(-1)\Big) = 0$ so that $\mathcal{T}_\sigma$ is stable.

Remark 3.14. One may check that GIT-semistability of σ implies slope-semistability of $\mathcal{T}_\sigma$. Indeed, [1, Theorem 3.1] says that GIT-semistability of σ amounts to σ being regular with $\mathcal{C}_\sigma$ supported at $\lambda_1, \dots, \lambda_\ell$ with the condition that for all $j \in \{1, \dots, \ell\}$, λ_j has multiplicity at most $(n+1)/2$ as a root of $\det(\rho_\sigma)$. In terms of the Segre symbol (see the introduction or the next paragraph), this means that $\sum_{i=1}^{s_j} a_{j,i}p_{j,i} \leqslant (n+1)/2$. This implies $\sum_{i=1}^{s_j} p_{j,i} \leqslant (n+1)/2$, which amounts to $r_{\lambda_j} \leqslant (n+1)/2$, so $\mathcal{T}_\sigma$ is slope-semistable, as $r_0 = \max\{\sum_{i=1}^{s_j} p_{j,i} \mid j \in \{1, \dots, \ell\}\}$. However, the converse implication fails as one can see reverting the argument or considering that there are irregular pencils σ having a slope-semistable sheaf $\mathcal{T}_\sigma$.

4. Projective dimension for pencils of quadrics

Also in this section, κ is an algebraically closed field of characteristic different from 2.

4.1. **Segre symbols.** Let σ be a pencil of quadrics. Following the notation introduced in Section 3.5.1, let (u, v) be the splitting type of σ, so that its generic corank r_1 satisfies $n+1 = u+v+r_1$. Let $\{\lambda_1, \dots, \lambda_\ell\} \subset \mathbb{P}^1$ be the support of the torsion sheaf $\mathcal{C}_\mathrm{t}$. Its localization at λ_j of $\mathcal{C}_\mathrm{t}$ can be written in the following way:

$$(\mathcal{C}_\mathrm{t})_{\lambda_j} \simeq \bigoplus_{i=1}^{s_j} \mathcal{O}_{\lambda_j^{(a_{j,i})}}^{\oplus p_{j,i}}.$$

Here, we denoted by $\lambda_j^{(a_{j,i})}$ the subscheme defined by the ideal $(g_j^{a_{j,i}})$ where g_j is a linear form vanishing at λ_j for each $j \in \{1, \dots, \ell\}$.

The integers $a_{j,i}$ can then be arranged into the so called *Segre symbol*. We write, for each $j \in \{1, \dots, \ell\}$:

$$\Sigma_j = (\underbrace{a_{j,1}, \dots, a_{j,1}}_{p_{j,1}}, \dots, \underbrace{a_{j,s_j}, \dots, a_{j,s_j}}_{p_{j,s_j}}), \qquad \text{with } a_{j,1} > \cdots > a_{j,s_j}.$$

The Segre symbol Σ for a pencil of quadrics σ is defined to be the multi-set $[\Sigma_1, \ldots, \Sigma_\ell]$. We will use exponential notation for repeated entries; for instance, the Segre symbol $[(1,1,1),(3,3,1),2]$ in exponential notation reads $[1^3,(3^2,1),2]$.

Note that, as we are dealing with potentially irregular pencils σ, we always refer to the Segre pencil of the regular part $\bar{\sigma}$ of σ. In case σ is completely irregular, its Segre symbol is $\varnothing$ by convention. Note that, in contrast to the behavior for regular pencils, the Segre symbol of an irregular pencil may be of the form $[1^p]$, for some integer p.

The Segre symbol is a the key invariant of regular pencils. Indeed, the content of the Segre-Weierstrass theorem is that the set of singular quadrics of a regular pencil together with its Segre symbol classifies the regular pencil up to homography of $\mathbb{P}^1$ and $\mathbb{P}^n$. For a reference of this theorem going back to Corrado Segre's thesis, the reader may look at the classical textbook [7, §XIII.10] or at the more recent paper [6]. This classical result is extended to irregular pencils as follows.

Proposition 4.1. *Given $\ell, r_1 \in \mathbb{N}$ and integers $1 \leqslant c_1 \leqslant \ldots \leqslant c_{r_1}$, fix distinct points $\{\lambda_1, \ldots, \lambda_\ell\} \subset \mathbb{P}^1$, and a multiset $\Sigma = [\Sigma_1, \ldots, \Sigma_\ell]$. Then there is an incompressible pencil of quadrics σ, unique up to homography, such that:*

$$\mathcal{C}_{\mathrm{tf}} \simeq \bigoplus_{i=1}^{r_1} \mathcal{O}_{\mathbb{P}^1}(c_i), \qquad \mathrm{Supp}(\mathcal{C}_{\mathrm{t}}) = \{\lambda_1, \ldots, \lambda_\ell\}, \qquad \Sigma(\bar{\rho}_\sigma) = \Sigma.$$

Proof. In view of the Segre-Weierstrass theorem, there is a regular pencil, and thus a regular symmetric pencil of matrices, uniquely defined by the datum of the set-theoretic support of $\mathcal{C}_{\mathrm{t}}$ together with the Segre symbol. Therefore the result follows from Proposition 3.8. □

Note that $p_{j,i} \leqslant n$ for all indices (i,j). The splitting type (u,v) satisfies:

$$\sum_{j=1}^{\ell} \sum_{i=1}^{s_j} a_{j,i} p_{j,i} = h^0(\mathcal{C}_{\mathrm{t}}) = u - v, \tag{40}$$

according to the exact sequence in display (24).

4.2. **Ext sheaves.** The main result of this section provides necessary and sufficient conditions in terms of Σ for the Ext sheaves $\mathcal{E}xt^q_{\mathbb{P}^n}(\mathcal{T}_\sigma, \mathcal{O}_{\mathbb{P}^n})$ to be non trivial.

Theorem 4.2. *For a pencil of quadrics σ with Segre symbol Σ and for $q > 0$, we have $\mathcal{E}xt^q_{\mathbb{P}^n}(\mathcal{T}_\sigma, \mathcal{O}_{\mathbb{P}^n}) \neq 0$ if and only if there are $j \in \{1, \ldots, \ell\}$ and $k \in \{1, \ldots, s_j\}$ such that:*

$$q + p_{j,1} + \ldots + p_{j,k} = n - r_1 - 1, \tag{41}$$

or $r_1 > 0$ and $q + r_1 = n - 2$.

Proof. We prove the theorem under the assumption that σ is incompressible, see the end of the proof for compressible pencils. Since the question is local and $\mathcal{T}_\sigma$ is free of rank $n-1$ locally around any point outside Ξ_σ, it is enough to prove the claim on charts containing a single primary component of Ξ_σ. In view of Lemma 3.9, these components are supported either at disjoint linear subspaces associated with distinct points in the support of $\mathcal{C}_{\mathrm{t}}$, or at the rational normal scroll Y. Note that the points of the support of $\mathcal{C}_{\mathrm{t}}$ correspond to the parenthesized pieces of the Segre symbol. Also, the proof at the points of Y is similar if the support of $\mathcal{C}_{\mathrm{t}}$

contains one point or many. So we may assume, without loss of generality, that $\ell = 1$ and simplify the notation to $\Sigma = (a_1^{p_1}, \dots, a_s^{p_s})$ with $a_1 > \cdots > a_s > 0$.

As we did in the proof of item i) of Theorem 3.2 (see Section 3.4), we observe that the sheaf $\mathcal{Q}_\sigma$ is a line bundle supported at Ξ_σ. Therefore, given $q > 0$, we have $\mathcal{E}xt^{q+2}_{\mathbb{P}^n}(\mathcal{Q}_\sigma, \mathcal{O}_{\mathbb{P}^n}) \neq 0$ if and only if $\mathcal{E}xt^{q+2}_{\mathbb{P}^n}(\mathcal{O}_{\Xi_\sigma}, \mathcal{O}_{\mathbb{P}^n}) \neq 0$, so this in turn is equivalent to $\mathcal{E}xt^{q}_{\mathbb{P}^n}(\mathcal{T}_\sigma, \mathcal{O}_{\mathbb{P}^n}) \neq 0$.

Let us analyze Ξ_σ more in detail and recall the notation of Subsection 3.5.3. Since we are working under the assumption $\ell = 1$ the sheaf $\mathcal{C}_{\mathrm{t}}$, if nonzero, is supported at a single point of $\mathbb{P}^1$ and we may fix coordinates so that this point is $\lambda = (0 : 1)$. In such coordinates and up to the action of $\mathrm{GL}_{n+1}(\kappa)$, the pencil of matrices $\rho = \bar{\rho}_\sigma$ of the regular part of σ is a block-diagonal matrix with blocks of sizes $a_1, \dots, a_s$, repeated $p_1, \dots, p_s$ times, where a block of size $a \in \mathbb{N}^*$ takes the form (below, (z_1, z_2) is a basis for $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$):

$$\rho_a = \begin{pmatrix} 0 & 0 & \cdots & 0 & z_2 & z_1 \\ 0 & \cdots & 0 & z_2 & z_1 & 0 \\ \vdots & \vdots & \iddots & \iddots & 0 & \vdots \\ 0 & z_2 & z_1 & 0 & \iddots & \iddots \\ z_2 & z_1 & 0 & \iddots & \iddots & \\ z_1 & 0 & \cdots & \iddots & & \end{pmatrix}.$$

The cokernel of this matrix is the structure sheaf of $\mathcal{O}_{\lambda^{(a)}}$, where $\lambda^{(a)}$ is the a-tuple point of $\mathbb{P}^1$ defined by the ideal (z_1^a). Therefore, we have:

$$\mathcal{C}_{\mathrm{t}} \simeq \bigoplus_{i=1}^{s} \mathcal{O}_{\lambda^{(a_i)}}^{\oplus p_i}.$$

For each $i \in \{1, \dots, s\}$, we consider the injection $\lambda^{(a_i)} \subset \lambda^{(a_s)}$. Concatenating the surjections $\mathcal{O}_{\lambda^{(a_s)}} \to \mathcal{O}_{\lambda^{(a_i)}}$ we get an epimorphism:

$$\mathcal{C}_{\mathrm{t}} \twoheadrightarrow \bigoplus_{i=1}^{s} \mathcal{O}_{\lambda^{(a_s)}}^{\oplus p_i}.$$

For $k \in \{1, \dots, s\}$, put $q_k = \sum_{i=1}^{k} p_i$. From the above epimorphism we get the exact sequence:

$$0 \to \bigoplus_{i=1}^{s-1} \mathcal{O}_{\lambda^{(a_i - a_s)}}^{\oplus p_i} \to \mathcal{C}_{\mathrm{t}} \to \mathcal{O}_{\lambda^{(a_s)}}^{\oplus q_s} \to 0.$$

Iterating this procedure we obtain a natural filtration:

$$0 = \mathcal{D}^{(0)} \subset \mathcal{D}^{(1)} \subset \cdots \subset \mathcal{D}^{(s)} = \mathcal{C}_{\mathrm{t}},$$

where, for all $k \in \{1, \dots, s\}$, we have (with the convention $a_{s+1} = 0$):

$$\mathcal{D}^{(k)} = \bigoplus_{i=1}^{k} \mathcal{O}_{\lambda^{(a_i - a_{k+1})}}^{\oplus p_i}, \qquad \mathcal{C}^{(k)} := \mathcal{D}^{(k)}/\mathcal{D}^{(k-1)} = \mathcal{O}_{\lambda^{(a_k - a_{k+1})}}^{\oplus q_k}.$$

To make the proof more transparent, we carry it out first under the assumption that σ is regular, hence $\rho = \rho_\sigma$, $r_1 = 0$, $(u, v) = (n + 1, 0)$, $\mathcal{C}_{\mathrm{tf}} = 0$ and $\mathcal{C}_{\mathrm{t}} = \mathcal{C}_\sigma$. For any $k \in \{1, \dots, s\}$, projectivizing the surjection $\mathcal{D}^{(k)} \twoheadrightarrow \mathcal{C}^{(k)}$ we get a closed embedding $\mathbb{P}(\mathcal{C}^{(k)}) \hookrightarrow \mathbb{P}(\mathcal{D}^{(k)})$ of schemes of the same dimension, with a residual subscheme of $\mathbb{P}(\mathcal{C}^{(k)})$ in $\mathbb{P}(\mathcal{D}^{(k)})$ which is isomorphic to $\mathbb{P}(\mathcal{D}^{(k-1)})$ and has thus

strictly smaller dimension – by convention, $\mathbb{P}(\mathcal{D}^{(0)}) = \varnothing$. Note that, for each $k \in \{1, \dots, s\}$, the exact sequence

$$0 \to \mathcal{I}_{\mathbb{P}(\mathcal{C}^{(k)})/\mathbb{P}(\mathcal{D}^{(k)})} \to \mathcal{O}_{\mathbb{P}(\mathcal{D}^{(k)})} \to \mathcal{O}_{\mathbb{P}(\mathcal{C}^{(k)})} \to 0$$

induces an exact sequence :

$$0 \to H^0(\mathcal{I}_{\mathbb{P}(\mathcal{C}^{(k)})/\mathbb{P}(\mathcal{D}^{(k)})}) \to H^0(\mathcal{O}_{\mathbb{P}(\mathcal{D}^{(k)})}) \to H^0(\mathcal{O}_{\mathbb{P}(\mathcal{C}^{(k)})}) \to 0.$$

Thus, since $\mathcal{I}_{\mathbb{P}(\mathcal{C}^{(k)})/\mathbb{P}(\mathcal{D}^{(k)})}$ is supported at $\mathbb{P}(\mathcal{D}^{(k-1)})$, we have an isomorphism $H^0(\mathcal{I}_{\mathbb{P}(\mathcal{C}^{(k)})/\mathbb{P}(\mathcal{D}^{(k)})}) \simeq H^0(\mathcal{O}_{\mathbb{P}(\mathcal{D}^{(k-1)})})$, which in turn implies that $\mathcal{I}_{\mathbb{P}(\mathcal{C}^{(k)})/\mathbb{P}(\mathcal{D}^{(k)})}$ is isomorphic to $\mathcal{O}_{\mathbb{P}(\mathcal{D}^{(k-1)})}$.

Recalling the correspondence (31), we send this filtration to $\mathbb{P}^n$ and define, for each $k \in \{1, \dots, s\}$ the subschemes $\Xi^{(k)} = \psi(\mathbb{P}(\mathcal{D}^{(k)}) \times_{\mathbb{P}^1} \mathbb{P}(\mathcal{C}_\sigma)) \subset \mathbb{P}^n$ and $\Upsilon^{(k)} = \psi(\mathbb{P}(\mathcal{C}^{(k)}) \times_{\mathbb{P}^1} \mathbb{P}(\mathcal{C}_\sigma)) \subset \mathbb{P}^n$. Since ψ is an embedding on the fibres of φ, this gives $\Upsilon^{(k)} \subset \Xi^{(k)}$ for each $k \in \{1, \dots, s\}$ and finally a stratification:

$$0 = \mathcal{O}_{\Xi^{(0)}} \subset \mathcal{O}_{\Xi^{(1)}} \subset \cdots \subset \mathcal{O}_{\Xi^{(s-1)}} \subset \mathcal{O}_{\Xi^{(s)}} = \mathcal{O}_{\Xi_\sigma}, \qquad \text{with :} \tag{42}$$

$$\mathcal{O}_{\Xi^{(k)}}/\mathcal{O}_{\Xi^{(k-1)}} \simeq \mathcal{O}_{\Upsilon^{(k)}}, \tag{43}$$

for all each $k \in \{1, \dots, s\}$, with $\Xi^{(1)} = \Upsilon^{(1)}$. Each component $\Upsilon^{(k)}$ is the projectivization of a trivial bundle of rank q_k over a subscheme of length a_k in $\mathbb{P}^1$. As such, it is equidimensional and Cohen–Macaulay of codimension $n - q_k + 1$ and therefore satisfies:

$$\mathcal{E}xt^{q+2}_{\mathbb{P}^n}(\mathcal{O}_{\Upsilon^{(k)}}, \mathcal{O}_{\mathbb{P}^n}) \simeq \begin{cases} 0, & \text{if } q+2 \neq n-q_k+1, \\ \omega_{\Upsilon^{(k)}}(n+1), & \text{if } q+2 = n-q_k+1. \end{cases} \tag{44}$$

So this sheaf is non zero if and only if $q = n - p_1 - \cdots - p_k - 1$, which in turn agrees with (41).

We apply $\mathcal{E}xt^*_{\mathbb{P}^n}(-, \mathcal{O}_{\mathbb{P}^n})$ to the filtration (42). To compute this we use (43) and induction on $k \in \{0, \dots, s\}$. Since for all $k \in \{1, \dots, s\}$ the sheaves (44) are line bundles supported on subschemes sharing no common component, the boundary morphisms induced by applying $\mathcal{E}xt^*_{\mathbb{P}^n}(-, \mathcal{O}_{\mathbb{P}^n})$ to (42) are all zero. We deduce that $\mathcal{E}xt^{q+2}_{\mathbb{P}^n}(\mathcal{O}_{\Xi_\sigma}, \mathcal{O}_{\mathbb{P}^n}) \neq 0$ if and only if there is $k \in \{1, \dots, s\}$ such that (41) is satisfied. This concludes the proof when σ is regular.

Now let us assume that σ is irregular. Restricting $\mathcal{C}_{\mathrm{tf}}$ to each of the subschemes $\lambda^{(a_1)}, \dots, \lambda^{(a_s)}$, we obtain the sheaves:

$$\hat{\mathcal{D}}^{(k)} = \mathcal{D}^{(k)} \oplus \mathcal{C}_{\mathrm{tf}}(-a_{k+1}), \qquad \hat{\mathcal{C}}^{(k)} = \hat{\mathcal{D}}^{(k)}/\hat{\mathcal{D}}^{(k-1)} = \mathcal{O}^{\oplus(q_k+r_1)}_{\lambda^{(a_k-a_{k+1})}}, \tag{45}$$

with the filtration:

$$\mathcal{C}_{\mathrm{tf}}(-a_1) = \hat{\mathcal{D}}^{(0)} \subset \hat{\mathcal{D}}^{(1)} \subset \cdots \subset \hat{\mathcal{D}}^{(s)} = \mathcal{C}_\sigma. \tag{46}$$

Again, for any $k \in \{1, \dots, s\}$, we get an embedding $\mathbb{P}(\hat{\mathcal{C}}^{(k)}) \hookrightarrow \mathbb{P}(\hat{\mathcal{D}}^{(k)})$ of schemes of the same dimension. The residual subscheme is $\mathbb{P}(\hat{\mathcal{D}}^{(k-1)})$ and has strictly smaller dimension – this time $\mathbb{P}(\hat{\mathcal{D}}^{(0)}) = Y$. The component Y of Ξ_σ is a rational normal scroll over $\mathbb{P}^1$. We denote by F the divisor class of a fibre of the scroll map $Y \to \mathbb{P}^1$.

Using the diagram (31) we define, for each $k \in \{1, \dots, s\}$ the subschemes $\hat{\Xi}^{(k)} = \psi(\mathbb{P}(\hat{\mathcal{D}}^{(k)}) \times_{\mathbb{P}^1} \mathbb{P}(\mathcal{C}_\sigma)) \subset \mathbb{P}^n$ and $\hat{\Upsilon}^{(k)} = \psi(\mathbb{P}(\hat{\mathcal{C}}^{(k)}) \times_{\mathbb{P}^1} \mathbb{P}(\mathcal{C}_\sigma)) \subset \mathbb{P}^n$. We get $\hat{\Upsilon}^{(k)} \subset \hat{\Xi}^{(k)}$. Note that $\psi_*(\varphi^*(\mathcal{O}_{\mathbb{P}^1}(1))) \simeq \mathcal{O}_Y(F)$ and that $F|_{\Upsilon^{(k)}} = 0$. Hence, in view of (45) we obtain for each $k \in \{1, \dots, s\}$ an exact sequence:

$$0 \to \mathcal{O}_{\hat{\Xi}^{(k-1)}}(-a_k F) \to \mathcal{O}_{\hat{\Xi}^{(k)}} \to \mathcal{O}_{\hat{\Upsilon}^{(k)}} \to 0. \tag{47}$$

With our convention, $Y = \hat{\Xi}^{(0)} = \hat{\Upsilon}^{(0)}$ so for $k = 1$ the leftmost term of the above sequence is $\mathcal{O}_{\hat{\Xi}^{(0)}}(-a_1F) \simeq \mathcal{O}_Y(-a_1F)$.

We have obtained a stratification of Ξ_σ that allows to compute the desired Ext sheaves. Indeed, to compute $\mathcal{E}xt^{q+2}_{\mathbb{P}^n}(\mathcal{O}_{\Xi_\sigma}, \mathcal{O}_{\mathbb{P}^n})$ for $q > 0$ we apply $\mathcal{E}xt^*_{\mathbb{P}^n}(-, \mathcal{O}_{\mathbb{P}^n})$ to (47) and use induction on $k \in \{0, \dots, s\}$ together with twists by $\mathcal{O}_Y(tF)$ for suitable $t \in \mathbb{Z}$. For $k = 0$ we observe that, since $Y \subset \mathbb{P}^n$ is smooth of codimension $n-r_1$, for any $t \in \mathbb{Z}$ we have $\mathcal{E}xt^{q+2}_{\mathbb{P}^n}(\mathcal{O}_Y(tF), \mathcal{O}_{\mathbb{P}^n}) \neq 0$ if and only if $q = n-r_1-2$. For $k \geqslant 1$, $\hat{\Upsilon}^{(k)}$ is the projectivization of a trivial bundle of rank $q_k + r_1$ over a subscheme of length a_k in $\mathbb{P}^1$, so:

$$\mathcal{E}xt^{q+2}_{\mathbb{P}^n}(\mathcal{O}_{\hat{\Upsilon}^{(k)}}, \mathcal{O}_{\mathbb{P}^n}) \neq 0 \qquad \text{if and only if } q = n - r_1 - q_k - 1, \tag{48}$$

and this sheaf is $\omega_{\hat{\Upsilon}^{(k)}}(n+1)$ if $q = n-r_1-q_k-1$. Again, since these sheaves are line bundles on $\hat{\Upsilon}^{(0)}, \dots, \hat{\Upsilon}^{(s)}$ and since these subschemes have no common component, we have the vanishing of all the boundary morphisms of the long exact sequence obtained by applying $\mathcal{E}xt^*_{\mathbb{P}^n}(-, \mathcal{O}_{\mathbb{P}^n})$ to (47). Therefore, $\mathcal{E}xt^{q+2}_{\mathbb{P}^n}(\mathcal{O}_{\Xi_\sigma}, \mathcal{O}_{\mathbb{P}^n}) \neq 0$ if and only if $q = n - r_1 - p_1 - \cdots - p_k - 1$ for some $k \in \{1, \dots, s\}$ or $q = n - r_1 - 2$. This concludes the proof if σ is incompressible.

Finally, if σ has compressibility m with $1 \leqslant m \leqslant n$, then we set $\hat{n} = n - m$ as in Lemma 2.14 and work with the incompressible pencil of quadrics $\hat{\sigma}$ in $\mathbb{P}^{\hat{n}}$ associated with σ. We obtained already a stratification of $\Xi_{\hat{\sigma}}$ Cohen–Macaulay closed subschemes of $\mathbb{P}^{\hat{n}}$ which are projective bundles over subschemes of $\mathbb{P}^1$, or the scroll Y. The equations of these subschemes, viewed in $\mathbb{P}^n$ define cones over such subschemes, which are still Cohen–Macaulay of the same codimension. Therefore, for all $q > 0$, we have $\mathcal{E}xt^q_{\mathbb{P}^n}(\mathcal{T}_\sigma, \mathcal{O}_{\mathbb{P}^n}) \neq 0$ if and only if $\mathcal{E}xt^q_{\mathbb{P}^{\hat{n}}}(\mathcal{T}_{\hat{\sigma}}, \mathcal{O}_{\mathbb{P}^{\hat{n}}}) \neq 0$. This concludes the proof. □

Let us give a couple of explicit examples to show the stratification appearing in the proof of the theorem.

Example 4.3. Consider a regular pencil with Segre symbol $[(6^3, 3^4, 2^3)]$, so that $\ell = 1$, $s = 3$, and $(a_1, a_2, a_3) = (6, 3, 2)$, $(p_1, p_2, p_3) = (3, 4, 3)$. We have a torsion sheaf $\mathcal{C}_\sigma = \operatorname{coker}(\rho)$ with $h^0(\mathcal{C}_\sigma) = 18 + 12 + 6 = 36 = n + 1$, so $n = 35$ and $\rho = \rho_6^{\oplus 3} \oplus \rho_4^{\oplus 4} \oplus \rho_2^{\oplus 3}$. We have:

$$\mathcal{C}_\sigma = \mathcal{D}^{(3)} = \mathcal{O}^{\oplus 3}_{\lambda^{(6)}} \oplus \mathcal{O}^{\oplus 4}_{\lambda^{(3)}} \oplus \mathcal{O}^{\oplus 3}_{\lambda^{(2)}}.$$

The Jacobian subscheme $\Xi_\sigma = \Xi^{(3)}$ is set-theoretically a linear subspace of $\mathbb{P}^{35}$ of dimension $3 + 4 + 3 - 1 = 9$. We have $\mathcal{C}^{(s)} = \mathcal{C}^{(3)} = \mathcal{O}^{\oplus 10}_{\lambda^{(2)}}$. The scheme Ξ_σ contains $\Upsilon^{(3)} = \psi_*(\varphi^*(\mathcal{C}^{(3)}))$ which is a double structure over $\mathbb{P}^9 \subset \mathbb{P}^{35}$. We have:

$$\mathcal{D}^{(2)} = \mathcal{O}^{\oplus 3}_{\lambda^{(4)}} \oplus \mathcal{O}^{\oplus 4}_{\lambda^{(1)}}, \qquad \mathcal{C}^{(2)} = \mathcal{O}^{\oplus 7}_{\lambda^{(1)}},$$

Note that $\Upsilon^{(2)} = \psi_*(\varphi^*(\mathbb{P}(\mathcal{C}^{(2)})))$ and $\Xi^{(2)} = \psi_*(\varphi^*(\mathbb{P}(\mathcal{D}^{(2)})))$ have dimension 6. The residual subscheme of $\Upsilon^{(3)}$ in $\Xi^{(3)} = \Xi_\sigma$ is $\Xi^{(2)}$. This is set-theoretically a $\mathbb{P}^6$ and contains $\Upsilon^{(2)}$ which is a reduced $\mathbb{P}^6$. We have:

$$\mathcal{D}^{(1)} = \mathcal{C}^{(1)} = \mathcal{O}^{\oplus 3}_{\lambda^{(3)}}.$$

The residual subscheme of $\Upsilon^{(2)}$ in $\Xi^{(2)}$ is $\Xi^{(1)} = \Upsilon^{(1)}$. This is a triple structure on $\mathbb{P}^2$. For $q > 0$, we have $\mathcal{E}xt^q_{\mathbb{P}^n}(\mathcal{T}_\sigma, \mathcal{O}_{\mathbb{P}^n}) \neq 0$ if and only $q \in \{24, 27, 31\}$.

Example 4.4. An incompressible pencil with $r_1 = 3$, $(c_1, c_2, c_3) = (1, 2, 2)$ and Segre symbol $\Sigma = [(3^2, 1^4), (4^5, 3^2, 2^3)]$, hence with $(u, v) = (47, 5)$, lives in $\mathbb{P}^{54}$. Its Jacobian scheme consists of a rational normal scroll Y of dimension 3 and degree 5 spanning a $\mathbb{P}^7$ and of two linear spaces $(\Xi_1)_{\mathrm{red}}$ and $(\Xi_2)_{\mathrm{red}}$ of dimension 5 and 9 meeting Y along two disjoint projective planes appearing as fibres of the scroll. The subscheme (Ξ_1) contains a simple $\mathbb{P}^5$ with a double line as residual subscheme. On the other hand the subscheme (Ξ_2) contains a double $\mathbb{P}^9$, whose residual subscheme still contains a simple $\mathbb{P}^6$ with a simple $\mathbb{P}^4$ as residual subscheme.

We have $\ell = 2$ and $\mathcal{E}xt^q_{\mathbb{P}^n}(\mathcal{T}_\sigma, \mathcal{O}_{\mathbb{P}^n}) \neq 0$ if and only $q = 54 - 2 - r_1$ or $q = 54 - 1 - r_1 - p_{1,1}$ or $q = 54 - 1 - r_1 - p_{1,1} - p_{1,2}$ or $q = 54 - 1 - r_1 - p_{2,1}$ or $q = 54 - 1 - r_1 - p_{2,1} - p_{2,2}$ or $q = 54 - 1 - r_1 - p_{2,1} - p_{2,2} - p_{2,3}$ which gives $q \in \{40, 43, 44, 45, 48, 49\}$.

From the proof of the previous theorem, we extract some precise information on the primary components of Ξ_σ. Assume σ is an incompressible pencil of quadrics having Segre symbol Σ and degree vector $\mathbf{c}$, with:

$$\Sigma = [\Sigma_1, \dots, \Sigma_\ell], \qquad \Sigma_j = (a_{j,1}^{p_{j,1}}, \dots, a_{j,s_j}^{p_{j,s_j}}),$$
$$\mathbf{c} = (c_1, \dots, c_{r_1}),$$

for some integers $r_1, \ell, s_1, \dots, s_\ell$, $\{(a_{j,i}, p_{j,i}) \mid j \in \{1, \dots, \ell\}, i \in \{1, \dots, s_j\}$ with $a_{j,1} > \cdots > a_{j,s_j}$ for all $j \in \{1, \dots, \ell\}$ and $c_1 \leqslant \cdots \leqslant c_{r_1}$. Recall the convention $a_{j,i} = 0$ for $i > s_j$ and for each $j \in \{1, \dots, \ell\}$ set $q_{j,k} - 1 = \sum_{i=1}^{k} p_{j,i} - 1$.

Corollary 4.5. *Let σ be an incompressible pencil of quadrics.*

i) If σ is regular, then the Jacobian scheme Ξ_σ has primary components:

$$\Upsilon_j^{(k)}, \qquad \text{for } j \in \{1, \dots, \ell\} \text{ and } k \in \{1, \dots, s_j\},$$

where the components $\Upsilon_j^{(k)}$ are projective spaces of dimension $q_{j,k} - 1$ over subschemes of length $a_{j,i} - a_{j,k+1}$ of $\mathbb{P}^1$. We have:

$$h^0(\mathcal{O}_{\Xi_\sigma}) = \sum_{j=1}^{\ell} a_{j,1}, \qquad \Upsilon_j^{(k)} \cap \Upsilon_{j'}^{(k')} = \varnothing, \text{ if } j \neq j'.$$

ii) If σ is irregular, then the Jacobian scheme Ξ_σ consists of a smooth scroll Y of dimension r_1 and degree $v = \sum_{i=1}^{r_1} c_i$ and of the primary components:

$$\hat{\Upsilon}_j^{(k)}, \qquad \text{for } j \in \{1, \dots, \ell\} \text{ and } k \in \{1, \dots, s_j\},$$

where the components $\hat{\Upsilon}_j^{(k)}$ are projective spaces of dimension $r_1 + q_{j,k} - 1$ over subschemes of length $a_{j,i} - a_{j,k+1}$ of $\mathbb{P}^1$. Also:

$$h^0(\mathcal{O}_{\Xi_\sigma}) = 1, \qquad \hat{\Upsilon}_j^{(k)} \cap \hat{\Upsilon}_{j'}^{(k')} = \varnothing, \text{ if } j \neq j'.$$

Finally, setting $\hat{\Xi}$ for the residual scheme of Y in Ξ_σ, we have:

$$h^0(\mathcal{O}_{\hat{\Xi}}) = \sum_{j=1}^{\ell} a_{j,1}.$$

Proof. We gave in Lemma 3.9 a set-theoretic description of the Jacobian scheme Ξ_σ which shows that Ξ_σ consists of ℓ pairwise disjoint linear spaces, together with the scroll Y in case σ is irregular, and in this case we also noticed that Y has dimension

r_1 and degree v. Also, in the proof of Theorem 4.2 we gave the structure of each primary component supported at any of the linear spaces mentioned above. Taking the union over all such spaces we get precisely the set $\{\Upsilon_j^{(k)} \mid j \in \{1,\dots,\ell\},\, k \in \{1,\dots,s_j\}\}$, or $\{\hat{\Upsilon}_j^{(k)} \mid j \in \{1,\dots,\ell\},\, k \in \{1,\dots,s_j\}\}$ depending on whether σ is regular or not. Note that in the proof of Theorem 4.2 we also had the component $\hat{\Upsilon}^{(0)}$, but this is just the scroll Y which is already accounted for.

To compute $h^0(\mathcal{O}_{\Xi_\sigma})$, note that taking h^0 of the structure sheaf is an additive operation on disjoint primary components, which is invariant under taking projective bundles and takes value a at $\lambda_j^{(a)} \subset \mathbb{P}^1$ for any $a \in \mathbb{N}^*$. So for regular pencils we get:

$$h^0(\mathcal{O}_{\Xi_\sigma}) = \sum_{j=1}^{\ell}\sum_{k=1}^{s_j} h^0(\mathcal{O}_{\Upsilon_j^k}) = \sum_{j=1}^{\ell}\sum_{k=1}^{s_j}(a_{j,i} - a_{j,k+1}) = \sum_{j=1}^{\ell} a_{j,1}.$$

For irregular pencils, the Jacobian scheme is connected as Y meets all the components $\{\hat{\Upsilon}_j^{(k)} \mid j \in \{1,\dots,\ell\},\, k \in \{1,\dots,s_j\}\}$, hence we have $h^0(\mathcal{O}_{\Xi_\sigma}) = 1$. Finally, the primary components of $\hat{\Xi}$ are precisely the $\{\hat{\Upsilon}_j^{(k)} \mid j \in \{1,\dots,\ell\},\, k \in \{1,\dots,s_j\}\}$, so the last formula follows as in the regular case. □

4.3. **Applications to projective dimension.** Theorem 4.2 allows to compute the projective dimension $\operatorname{pdim}(\mathcal{T}_\sigma)$ of the logarithmic tangent sheaf associated to a pencil o quadrics, namely, the minimal length of a locally free resolution of $\mathcal{T}_\sigma$.

Proposition 4.6. *Let σ be an incompressible pencil of quadrics.*

i) Assume σ is irregular. Then $\operatorname{pdim}(\mathcal{T}_\sigma) = n - r_1 - 2$.

ii) Assume σ is regular and put $p = \min\{p_{j,1} \mid j \in \{1,\dots,\ell\}\}$. Then:

$$\operatorname{pdim}(\mathcal{T}_\sigma) = n - p - 1.$$

Proof. By Theorem 4.2, we can compute for which values of $q \geqslant 1$ one has $\mathcal{E}xt^q_{\mathbb{P}^n}(\mathcal{T}_\sigma, \mathcal{O}_{\mathbb{P}^n}) \neq 0$. On the other hand, we have:

$$\operatorname{pdim}(\mathcal{T}_\sigma) = \max\{q \in \mathbb{N} \mid \mathcal{E}xt^q_{\mathbb{P}^n}(\mathcal{T}_\sigma, \mathcal{O}_{\mathbb{P}^n}) \neq 0\}.$$

Fixing $j \in \{1,\dots,\ell\}$ and letting k vary in $\{1,\dots,s_j\}$ the maximal value for $n - r_1 - p_{j,1} - \cdots - p_{j,k} - 1$ is attained by choosing $k = 1$. Such value is thus $n - r_1 - p_{j,1} - 1$. Letting j vary in $\{1,\dots,\ell\}$, the maximal value of $n - r_1 - p_{j,1} - 1$ is $n - r_1 - p - 1$. The maximum between $n - r_1 - p - 1$ and $n - r_1 - 2$ is $n - r_1 - 2$ because $p \geqslant 1$. This gives the result. □

Example 4.7. A pencil σ as in Example 4.3 has $\operatorname{pdim}(\mathcal{T}_\sigma) = 31$. For Example 4.4, we get $\operatorname{pdim}(\mathcal{T}_\sigma) = 49$.

Example 4.8. The completely irregular pencil $\sigma = (x_1x_5 + x_3x_4, x_2x_4 + x_0x_5)$ showing up at the end of the proof of Theorem 3.2 has $n = 5, u = v = r_1 = 2$. So for $q > 0$ we have $\mathcal{E}xt^q_{\mathbb{P}^n}(\mathcal{T}_\sigma, \mathcal{O}_{\mathbb{P}^n}) \neq 0$ if and only if $q = 1$. We get $\operatorname{pdim}(\mathcal{T}_\sigma) = 1$.

Corollary 4.9. *A regular pencil of quadrics is locally free if and only if $n \in \{1, 2\}$ or its Segre symbol is $[(1,1),(1,1)]$ or $[(2,2)]$.*

Proof. A regular pencil σ is locally free if and only if all the integers q satisfying (41) are non positive. This is always the case for $n \leqslant 2$ and holds true for the Segre symbols $[(1,1),(1,1)]$ or $[(2,2)]$ so one implication is proved.

Conversely, assume $n \geqslant 3$ and σ locally free. Set $a = \min\{a_{j,1} \mid j \in \{1, \dots, \ell\}\}$. For all $j \in \{1, \dots, \ell\}$, take $k = 1$ and define q by (41). The inequality $q \leqslant 0$ gives $p_{j,1} \geqslant n - 1$, which implies, in view of (40), that:

$$n + 1 \geqslant \sum_{j=1}^{\ell} \left(a_{j,1}(n-1) + \sum_{i=2}^{s_j} a_{j,i} p_{j,i} \right).$$

This gives $n(a\ell - 1) \leqslant a\ell + 1$ and therefore either $a = \ell = 1$, or $a\ell = 2$ and $n = 3$. In the former case $s_1 = 1$ so (40) gives $p_{1,1} = n + 1$, which is impossible. In the latter, either $(a, \ell) = (1, 2)$ and the Segre symbol is $[(1^2), (1^2)] = [(1, 1), (1, 1)]$ or $(a, \ell) = (2, 1)$ and the Segre symbol is $[(2^2)] = [(2, 2)]$. □

In the same spirit we have, more generally, the following.

Corollary 4.10. *Let σ be an incompressible pencil of quadrics. Then:*

i) If σ is regular, then $\operatorname{pdim}(\mathcal{T}_\sigma) \geqslant \frac{n-3}{2}$.

ii) If σ is irregular, then $\operatorname{pdim}(\mathcal{T}_\sigma) \geqslant \frac{2n-7}{3}$.

Proof. Assume σ is irregular. Then by Lemma 3.10 we get:

$$\operatorname{pdim}(\mathcal{T}_\sigma) = n - r_1 - 2 \geqslant n - 2 - \frac{n+1}{3} = \frac{2n-7}{3}.$$

Next, suppose σ is regular. Again put $a = \min\{a_{j,1} \mid j \in \{1, \dots, \ell\}\}$, $p = \min\{p_{j,1} \mid j \in \{1, \dots, \ell\}\}$ and use (40) to get $n + 1 \geqslant \ell a p$. By Corollary (4.6), we obtain:

$$n + 1 \geqslant \ell a (n - \operatorname{pdim}(\mathcal{T}_\sigma) - 1).$$

Rearranging the terms, this yields:

$$\operatorname{pdim}(\mathcal{T}_\sigma) \geqslant n - \frac{n+1}{\ell a} - 1.$$

We saw in the previous proof that $a\ell \geqslant 2$ so this gives $\operatorname{pdim}(\mathcal{T}_\sigma) \geqslant \frac{n-3}{2}$. □

Remark 4.11. The previous bounds are sharp. Indeed, if σ is a completely irregular incompressible pencil, then $3r_1 = n + 1$ so $\operatorname{pdim}(\mathcal{T}_\sigma) = \frac{n-3}{2}$.

Also, if σ is a regular pencil with $n \geqslant 3$ odd, say $n + 1 = 2m$, then we may take σ to have Segre symbol $[(1^m), (1^m)]$ or $[(2^m)]$ and we get $\operatorname{pdim}(\mathcal{T}_\sigma) = \frac{n-3}{2}$.

Example 4.12. Let us list all the possible cases for irregular pencils of quadrics in $\mathbb{P}^3$. We give the possible Segre symbols of the regular part.

r_1	(u, v)	$h^0(\mathcal{C}_{\mathrm{t}})$	$\mathcal{C}_{\mathrm{tf}}$	Compressible	Segre
1	$(3, 0)$	3	$\mathcal{O}_{\mathbb{P}^1}$	yes	$[1, 1, 1]$
1	$(3, 0)$	3	$\mathcal{O}_{\mathbb{P}^1}$	yes	$[2, 1]$
1	$(3, 0)$	3	$\mathcal{O}_{\mathbb{P}^1}$	yes	$[3]$
1	$(3, 0)$	3	$\mathcal{O}_{\mathbb{P}^1}$	yes	$[1^2, 1]$
1	$(3, 0)$	3	$\mathcal{O}_{\mathbb{P}^1}$	yes	$[(2, 1)]$
1	$(2, 1)$	1	$\mathcal{O}_{\mathbb{P}^1}(1)$	no	$[1]$
2	$(2, 0)$	2	$\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$	yes	$[1, 1]$
2	$(2, 0)$	2	$\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$	yes	$[2]$
2	$(1, 1)$	0	$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$	yes	$[\varnothing]$

The pencil with empty Segre symbol is completely irregular. We see that only one case gives an incompressible pencil. This one has $\operatorname{pdim}(\mathcal{T}_\sigma) = 1$.

4.4. **Graded projective dimension.** Let $n \geqslant 2$. We call *graded projective dimension* of a torsion free sheaf $\mathcal{E}$ on $\mathbb{P}^n$:

$$\operatorname{gpdim}(\mathcal{E}) = \max\{q \in \{0, \dots, n-1\} \mid \operatorname{Ext}^q_{\mathbb{P}^n}(\mathcal{E}, \mathcal{O}_{\mathbb{P}^n})_* \neq 0\}.$$

Here, $\operatorname{Ext}^q_{\mathbb{P}^n}(\mathcal{E}, \mathcal{O}_{\mathbb{P}^n})_*$ is a shortcut for $\oplus_{t\in\mathbb{Z}} \operatorname{Ext}^q_{\mathbb{P}^n}(\mathcal{E}, \mathcal{O}_{\mathbb{P}^n}(t))$. The graded projective dimension is the length of a sheafified minimal graded free resolution of the module of global sections of $\mathcal{E}$.

Theorem 4.13. *Let σ be a pencil of quadrics in $\mathbb{P}^n$.*

i) Assume σ is regular. Then:

$$\operatorname{gpdim}(\mathcal{T}_\sigma) = n - 2,$$

unless the Segre symbol Σ of σ $[1^p, 1^q]$ for some $p \geqslant q \geqslant 1$, or $[(2^q, 1^p)]$ for some $p \geqslant 0$ and $q \geqslant 1$, in which case:

$$\operatorname{gpdim}(\mathcal{T}_\sigma) = n - q - 1.$$

ii) Assume σ is irregular of generic corank r_1. Then:

$$\operatorname{gpdim}(\mathcal{T}_\sigma) = n - 1,$$

unless σ has degree vector $(1, \dots, 1)$, in which case:

$$\operatorname{gpdim}(\mathcal{T}_\sigma) = n - r_1 - 2.$$

We underline that the graded projective dimension of $\mathcal{T}_\sigma$ depends on the Segre symbol only if σ is regular; otherwise, $\operatorname{gpdim}(\mathcal{T}_\sigma)$ only depends on whether or not the degree vector $\mathbf{c} = (c_1, \dots, c_{r_1})$ contains a value strictly greater than 1.

Proof. First of all we observe that, without loss of generality, we can assume that the pencil σ is incompressible. Indeed, if σ has compressibility $m > 0$, then we may work in a projective space of dimension $\hat{n} = n - m$ whose coordinates do occur in the quadrics of σ. The minimal resolution obtained over the coordinate ring of this space is a minimal resolution of $\mathcal{T}_\sigma/\mathcal{O}_{\mathbb{P}^n}^{\oplus m}$ and thus computes $\operatorname{gpdim}(\mathcal{T}_\sigma)$.

Next we note that, according to the proof of Theorem 3.2, the sheaf $\mathcal{Q}_\sigma(-1)$ of an incompressible pencil σ is isomorphic to $\mathcal{O}_{\Xi_\sigma}(F)$, where F is the class of a fibre of the scroll map $Y \to \mathbb{P}^1$. The divisor F is trivial on the components $\{\hat{\Upsilon}_j^{(k)} \mid j \in \{1, \dots, \ell\}, k \in \{1, \dots, s_j\}\}$. Set $a = \sum_{j=1}^{\ell} a_{j,1}$. By Corollary 4.5, we get:

$$h^0(\mathcal{Q}(-1)) = a, \qquad \text{if } \sigma \text{ is regular.} \tag{49}$$

Also, if σ is irregular, denoting again by $\hat{\Xi}$ the residual scheme of Y in Ξ_σ, we get an exact sequence:

$$0 \to \mathcal{O}_Y(F) \to \mathcal{Q}(-1) \to \mathcal{O}_{\hat{\Xi}} \to 0. \tag{50}$$

In order to prove the result, we will need two more ingredients, namely two equivalent definitions of the graded projective dimension. For the first one, for any coherent sheaf $\mathcal{E}$ on $\mathbb{P}^n$ and $q \in \mathbb{N}$, put $H^q_*(\mathcal{E}) = \bigoplus_{t\in\mathbb{Z}} H^q(\mathcal{E}(t))$. Set:

$$q_0 = \min\{q \in \mathbb{N}^* \mid H^q_*(\mathcal{T}_\sigma) \neq 0\}.$$

We have, by Serre duality:

$$\operatorname{gpdim}(\mathcal{T}_\sigma) = n - q_0.$$

The second one is worked out in the framework of graded modules over the polynomial ring $R = \kappa[x_0, \dots, x_n]$. Consider the matrix $\mathcal{J}_\sigma$ as a map of graded

modules $R^{n+1} \to R(1)^2$ and define the R-modules Q_σ, M_σ and T_σ as the cokernel, image and kernel of this map, so that sheafifying the graded modules Q_σ, M_σ and T_σ we get back $\mathcal{Q}_\sigma$, $\mathcal{M}_\sigma$ and $\mathcal{T}_\sigma$. We write down the exact sequence of graded R-modules:

$$0 \to T_\sigma \to R^{n+1} \xrightarrow{\mathcal{J}_\sigma} R(1)^2 \to Q_\sigma \to 0.$$

The Auslander–Buchsbaum formula gives:

$$\operatorname{gpdim}(\mathcal{T}_\sigma) = n + 1 - \operatorname{depth}(T_\sigma).$$

We compute the depth of T_σ by the relation:

$$\operatorname{depth}(T_\sigma) = \min\{q \in \mathbb{N} \mid \operatorname{Ext}^q_R(\kappa, T_\sigma) \neq 0\},$$

where κ is the residual field, namely $\kappa = R/(x_0, \dots, x_1)$.

Having set up all this, we are in position to prove i). So assume σ is regular. First we compute $\operatorname{gpdim}(\mathcal{T}_\sigma)$ when $\Sigma = [1^p, 1^q]$ or $\Sigma = [(2^q, 1^p)]$. If $\Sigma = [1^p, 1^q]$ with $p \geqslant q \geqslant 1$ then $p + q = n + 1$ and the generators (f, g) of σ can be chosen to be $f = x_0^2 + \cdots + x_{p-1}^2$ and $g = x_p^2 + \cdots + x_n^2$. Set $L = V(x_0, \dots, x_{p-1})$ and $M = V(x_p, \dots, x_n)$. Looking at $\mathcal{J}_\sigma$, we see that:

$$\mathcal{T}_\sigma \simeq \mathcal{R}_L(1) \oplus \mathcal{R}_M(1),$$

where $\mathcal{R}_L$ and $\mathcal{R}_M$ are the Koszul syzygies of L and M, see §3.3.2. Now $\operatorname{gpdim}(\mathcal{R}_L(1)) = p - 2 \geqslant q - 2 = \operatorname{gpdim}(\mathcal{R}_M(1))$, so:

$$\operatorname{gpdim}(\mathcal{T}_\sigma) = p - 2 = n - q - 1.$$

Next, we deal with $\Sigma = [(2^q, 1^p)]$, for $q \geqslant 1$ and $p \geqslant 0$. Note that $p + 2q = n + 1$, so $n - q - 1 \geqslant q$. When $p = 0$ we have $\mathcal{Q}_\sigma(-1) \simeq \mathcal{O}_{\Upsilon^{(1)}}$, where $\Upsilon^{(1)}$ is a projective space $\mathbb{P}^{n-q}$ over a length-2 subscheme of $\mathbb{P}^1$, while for $p > 0$ we have a filtration:

$$0 \to \mathcal{O}_{\Upsilon^{(1)}} \to \mathcal{Q}_\sigma(-1) \to \mathcal{O}_{\Upsilon^{(2)}} \to 0,$$

where $\Upsilon^{(1)}$ is a reduced $\mathbb{P}^{n-q}$ and $\Upsilon^{(2)}$ is a reduced $\mathbb{P}^{q-1} \subset \mathbb{P}^{n-q}$. In both cases, since the coordinate rings of the subschemes $\Upsilon^{(k)}$ are graded Cohen–Macaulay rings, we have $\operatorname{gpdim}(\mathcal{O}_{\Upsilon^{(k)}}) = \operatorname{codim}(\Upsilon^{(k)})$. Therefore $\operatorname{gpdim}(\mathcal{T}_\sigma) + 2$ is the maximum of the $\operatorname{gpdim}(\mathcal{O}_{\Upsilon^{(k)}})$ for different values of k. Since $n - q - 1 \geqslant q$, in both cases we obtain the equality:

$$\operatorname{gpdim}(\mathcal{T}_\sigma) = n - q - 1.$$

Let us show that, unless $\Sigma = [1^p, 1^q]$ or $\Sigma = [(2^q, 1^p)]$, we have $H^2(\mathcal{T}_\sigma(-1)) \neq 0$, which implies $\operatorname{gpdim}(\mathcal{T}_\sigma) \geqslant n - 2$. It suffices to show $H^1(\mathcal{M}_\sigma(-1)) \neq 0$, which in turn holds true if $h^0(\mathcal{Q}_\sigma(-1)) > 2$. But by (49), we have $h^0(\mathcal{Q}_\sigma(-1)) > 2$ unless $\ell = 2$ and $s_1 = s_2 = a_{1,1} = a_{2,1} = 1$, or $\ell = 1$, $s_1 \in \{1, 2\}$, $a_{1,1} = 2$. Since these two cases correspond to the Segre symbols $\Sigma = [1^p, 1^q]$ for some $p, q \geqslant 1$ or $\Sigma = [(2^q, 1^p)]$ for some $q \geqslant 1, p \geqslant 0$, we get $\operatorname{gpdim}(\mathcal{T}_\sigma) \geqslant n - 2$ except in these cases.

Now we prove that, for regular pencils, we have $\operatorname{gpdim}(\mathcal{T}_\sigma) \leqslant n - 2$. It suffices to show that the module Q_σ contains no copy of the residual field κ. Indeed, otherwise there would be a non-zero element of Q_σ, represented by $(h, k) \in R(1)^2$, whose annihilator contains the maximal ideal $(x_0, \dots, x_n)$. Up to switching the factors, we may assume $h \neq 0$. Also, since the pencil σ is regular, we may choose the generators (f, g) of the pencil to be both associated to smooth quadrics. Also, we may select coordinates $x_0, \dots, x_n$ of $\mathbb{P}^n$ so that $2f = x_0^2 + \cdots + x_n^2$.

Then, for the pair $(h,k) \in R(1)^2$ with $h \neq 0$, there must be a matrix $(a_{i,j})_{0 \leqslant i,j \leqslant n}$, with $a_{i,j} \in R$ for all $0 \leqslant i,j \leqslant n$ such that:

$$\begin{pmatrix} h \\ k \end{pmatrix} \begin{pmatrix} x_0 & \cdots & x_n \end{pmatrix} = \begin{pmatrix} x_0 & \cdots & x_n \\ g_0 & \cdots & g_n \end{pmatrix} \begin{pmatrix} a_{0,0} & \cdots & a_{0,n} \\ \vdots & & \vdots \\ a_{n,0} & \cdots & a_{n,n} \end{pmatrix}, \tag{51}$$

where $\sigma = (f,g)$ and we wrote $g_i = \partial g/\partial x_i$, for all $i \in \{0,\dots,n\}$. We used here $2f = x_0^2 + \cdots + x_n^2$. Note that, by the symmetric role of f and g, we may assume $h \neq 0$. Hence, (51) implies that the matrix $A = (a_{i,j})_{0 \leqslant i,j \leqslant n}$ is $h\mathbb{I}_{n+1}$ and is thus invertible in $\kappa(x_0,\dots,x_n)$. Hence, we may rewrite (51) in $\kappa(x_0,\dots,x_n)$ as:

$$\begin{pmatrix} h \\ k \end{pmatrix} \begin{pmatrix} x_0 & \cdots & x_n \end{pmatrix} A^{-1} = \begin{pmatrix} x_0 & \cdots & x_n \\ g_0 & \cdots & g_n \end{pmatrix}.$$

Therefore $\mathcal{J}_\sigma$ should have generic rank 1, which is impossible by the Euler relation since f,g are not proportional.

Summing up, we have proved $\operatorname{Hom}_R(\kappa, Q_\sigma) = 0$. Therefore, applying $\operatorname{Ext}^*_R(\kappa,-)$ to the above sequence we get $\operatorname{Ext}^q_R(\kappa, T_\sigma) = 0$ for $q \leqslant 2$. Hence $\operatorname{depth}(T_\sigma) \geqslant 3$ and finally $\operatorname{gpdim}(\mathcal{T}_\sigma) \leqslant n-2$. This concludes the proof for regular pencils.

It remains to carry out the proof if σ is irregular. In view of the filtration (50) and since the coordinate rings of the primary components of $\hat{\Xi}$ are graded Cohen–Macaulay rings, we get:

$$\operatorname{gpdim}(\mathcal{O}_{\hat{\Xi}}) = \operatorname{codim}(\hat{\Xi}) \leqslant \operatorname{codim}(Y) = n - r_1.$$

Now, if $c_i = 1$ for all $i \in \{1,\dots,r_1\}$, the sheaf $\mathcal{O}_Y(F)$ has a minimal Buchsbaum–Rim resolution of length equal to $\operatorname{codim}(Y)$. This is obtained by the Buchsbaum–Rim resolution of $\mathcal{O}_Y(F)$ in the linear span $L = \mathbb{P}^{v-1} \subset \mathbb{P}^n$ of Y seen as cokernel of a matrix of linear forms of size $2 \times v$ over L, combined with the Koszul complex of L in $\mathbb{P}^n$; we refer to [5, Theorem A2.10, Exercice A2.19] for Buchsbaum–Rim complexes and matrices associated with scrolls.

We deduce $\operatorname{gpdim}(\mathcal{O}_Y(F)) = \operatorname{codim}(Y)$, which in turn yields:

$$\operatorname{gpdim}(\mathcal{T}_\sigma) = \max(\operatorname{gpdim}(\mathcal{O}_{\hat{\Xi}}), \operatorname{gpdim}(\mathcal{O}_Y(F)) - 2 = n - r_1 - 2.$$

This proves the last part of ii).

To conclude the proof, let us assume that there is $i \in \{1,\dots,r_1\}$ such that $c_i \geqslant 2$ and show that $\operatorname{gpdim}(\mathcal{T}_\sigma) = n-1$. It is enough to show that $H^1(\mathcal{T}_\sigma) \neq 0$. Note that $H^0(\mathcal{T}_\sigma) = 0$ by incompressibility of σ, hence:

$$h^1(\mathcal{T}_\sigma) \geqslant 2(n+1) - h^0(\mathcal{Q}_\sigma) - (n+1) \geqslant n + 1 - h^0(\mathcal{Q}_\sigma),$$

so it suffices to check $h^0(\mathcal{Q}_\sigma) < n+1$.

To show this inequality we recall the notation (u,v) for the splitting type of σ and take cohomology of (50) twisted by $\mathcal{O}_{\mathbb{P}^n}(1)$. Since the linear span of the residual subscheme $\hat{\Xi}$ of Y in Ξ_σ is $\mathbb{P}(H^0(\mathcal{C}_{\mathrm{t}}))$ and since $h^k(\mathcal{O}_Y(F+H)) = 0$ for $k > 0$ and $h^0(\mathcal{O}_Y(tF+H)) = v + tr_1$, for all $t \geqslant 0$, we obtain:

$$\begin{aligned} h^0(\mathcal{Q}_\sigma) &= h^0(\mathcal{O}_Y(F+H)) + h^0(\mathcal{O}_{\hat{\Xi}}(1)) = \\ &= \sum_{i=1}^{r_1} (c_i + 2) + h^0(\mathcal{C}_{\mathrm{t}}) = (v + 2r_1) + (u - v). \end{aligned}$$

We are reduced to show $u+2r_1 < n+1$ and, since $n+1 = u+v+r_1$, this amounts to $v > r_1$. But the inequality $v = \sum_{i=1}^{r_1} c_i > r_1$ takes place precisely if there is $i \in \{1,\dots,r_1\}$ such that $c_i \geqslant 2$, so the non-vanishing $H^1(\mathcal{T}_\sigma) \neq 0$ is established. The proof of the theorem is achieved. □

Example 4.14. Let σ be a regular pencil of quadrics with $r_0 = 1$. Then the module of global sections T_σ of $\mathcal{T}_\sigma$ has the Buchsbaum–Rim type resolution:

$$0 \leftarrow T_\sigma \leftarrow R(-2)^{\binom{n+1}{3}} \leftarrow \cdots \leftarrow R(-1-j)^{j\binom{n+1}{j+2}} \leftarrow \cdots \leftarrow R(-n)^{n-1} \leftarrow 0,$$

with $j \in \{1,\dots,n-1\}$. This resolution is minimal and linear of length $n-2$.

For a regular pencil of quadrics, the Buchsbaum–Rim complex in the above display is exact if and only if $\dim(\mathcal{Q}_\sigma) = 0$. This happens if and only if $r_0 = 1$.

5. Pencils of quadrics in dimension 3

We can arrive at a full classification for pencils of quadrics in $\mathbb{P}^3$ over an algebraically closed field κ of characteristic different from 2. The result is the following.

Theorem 5.1. *Let σ be a pencil of quadrics in $\mathbb{P}^3$. Then the following holds.*

- *i) The pencil σ is free if and only if it is locally free. This happens:*
  - *a) If σ has Segre symbol $[(1,1),(1,1)]$, in which case $\mathcal{T}_\sigma \simeq \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2}$;*
  - *b) If σ has Segre symbol $[(2,2)]$, in which case $\mathcal{T}_\sigma \simeq \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2}$;*
  - *c) If σ is irregular and incompressible, in which case $\mathcal{T}_\sigma \simeq \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2}$;*
  - *d) If σ is compressible, in which case $\mathcal{T}_\sigma \simeq \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(e-2)$, where $e \in \{0,1,2\}$ is the number of double planes in the pencil.*
- *ii) In all other cases σ is regular,* $\operatorname{pdim}(\mathcal{T}_\sigma) = 1$ *and the sheafified minimal graded free resolution of $\mathcal{T}_\sigma$ reads:*
  - *a) If $r_0 = 1$:*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 4} \to \mathcal{T}_\sigma \to 0.$$

  - *b) If $r_0 = 2$ and the Segre symbol is not $[(1,1),(1,1)]$ or $[(2,2)]$:*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{T}_\sigma \to 0.$$

The proof of the theorem is by inspection of the different Segre symbols. It follows from the analysis appearing in the next subsubsections.

5.1. **Regular pencils.** Let us write the table of possible Segre symbols of regular pencils, together with the description of Ξ_σ arising from the previous sections.

Segre	ℓ	Ξ_σ	r_0	Chern	stable	pdim
$[1,1,1,1]$	4	4 simple points	1	$(-2,3,4)$	s	1
$[2,1,1]$	3	double point & 2 simple points	1	$(-2,3,4)$	s	1
$[2,2]$	2	2 double points	1	$(-2,3,4)$	s	1
$[3,1]$	2	triple point & simple point	1	$(-2,3,4)$	s	1
$[4]$	3	quadruple point	1	$(-2,3,4)$	s	1
$[(1^2),1,1]$	2	line & 2 simple points	2	$(-2,2,2)$	sss	1
$[(1^2),2]$	2	line & double point	2	$(-2,2,2)$	sss	1
$[(2,1),1]$	2	line $\Upsilon^{(2)}$ & simple point $\Upsilon^{(1)}$ & simple point	2	$(-2,2,2)$	sss	1
$[(3,1)]$	1	line $\Upsilon^{(2)}$ & double point $\Upsilon^{(1)}$	2	$(-2,2,2)$	sss	1
$[(1^2),(1^2)]$	2	2 disjoint lines	2	$(-2,1,0)$	free	0
$[(2^2)]$	1	double line $\Upsilon^{(1)}$	2	$(-2,1,0)$	free	0
$[(1^3),1]$	2	plane & simple point	3	$(-1,1,1)$	s	1
$[(2,1^2)]$	1	plane $\Upsilon^{(2)}$ & simple point $\Upsilon^{(1)}$	3	$(-1,1,1)$	s	1

In the column labelled *stable*, we wrote *s* or *sss* according to whether $\mathcal{T}_\sigma$ is stable or strictly semi stable (in the sense of the slope), and *free* when $\mathcal{T}_\sigma$ is split. In the description of Ξ_σ we let the subschemes $\Upsilon^{(k)}$ show up when a primary component of Ξ_σ has a non trivial filtration as in the proof of Theorem 4.2. In the column labelled *Chern* we write the triple $(c_1(\mathcal{T}_\sigma), c_2(\mathcal{T}_\sigma), c_3(\mathcal{T}_\sigma))$.

Some comments are in order.

i) When σ is free, we have $\mathcal{T}_\sigma \simeq \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2}$.

ii) When $r_0 = 1$, the sheaf $\mathcal{Q}_\sigma$ has a Buchsbaum–Rim resolution that induces a sheafified minimal graded free resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 4} \to \mathcal{T}_\sigma \to 0.$$

This gives the Chern classes of $\mathcal{T}_\sigma$ when $r_0 = 1$.

iii) When $r_0 = 2$, there are lines $M, L \subset \mathbb{P}^3$, not necessarily distinct, with $L \subset \Xi_\sigma$, and a finite length subscheme $W \subset M$, such that $\mathcal{T}_\sigma$ fits into (18). Note that, since $r_0 = 2$, there is a quadric in the pencil, say f_2, which is a rank-2 quadric in the coordinates x_0, x_1, up to homography. So, setting $L = V(x_0, x_1)$ and composing the Jacobian matrix with the projection $\mathcal{O}_{\mathbb{P}^3}^{\oplus 2}(1) \to \mathcal{O}_{\mathbb{P}^3}(1)$ onto the second factor and with the obvious quotient $\mathcal{O}_{\mathbb{P}^3}(1) \to \mathcal{O}_L(1)$ we obtain explicitly the morphism $\mathcal{Q}_\sigma \to \mathcal{O}_L(1)$ required to get (18), so Lemma 3.4 holds also for $(n, \varrho) = (3, 2)$.

We have $\mathcal{R}_M \simeq \mathcal{R}_L \simeq \mathcal{O}_{\mathbb{P}^3}(-2)$ and the length of W is either 2 or 0, according to whether $\operatorname{pdim}(\mathcal{T}_\sigma)$ is 1 or 0. So $\mathcal{T}_\sigma$ is polystable in the free case, otherwise it is strictly slope semistable and Gieseker-unstable.

In the latter case, we have $\mathcal{I}_{W/M} \simeq \mathcal{O}_M(-2)$ and the morphism $\mathcal{R}_L(1) \to \mathcal{I}_{W/M}(1)$ of (18) is the natural surjection $\mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_M(-1)$. Therefore the term $\mathcal{G}_2$ appearing in the sequence (20) fits into:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2} \to \mathcal{G}_2 \to 0.$$

Then we have a sheafified minimal graded free resolution of $\mathcal{T}_\sigma$ of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{T}_\sigma \to 0.$$

This gives the Chern classes of $\mathcal{T}_\sigma$ when $r_0 = 2$.

iv) Stability of $\mathcal{T}_\sigma$ for $r_0 = 3$ follows from Proposition 3.1.

v) One can put two quadrics of σ in normal form. This is done in [6].

5.2. **Irregular pencils.** We gave in Example 4.12 the list of numerical invariants of irregular pencils in $\mathbb{P}^3$. We observed that there is only one irregular incompressible pencil in $\mathbb{P}^3$. Note that in any case the number of points in the support of $\mathcal{C}_{\mathrm{t}}$ is at most 3, so irregular pencils have a normal form (f_1, f_2) which is completely determined up to SL_2-action as SL_2 is 3-transitive on $\mathbb{P}^1$. Note that we can assume that this support is contained in $\{(1:0),(0:1),(1:-1)\}$. The column labelled m displays the compressibility of the pencil.

We are going to see that, for irregular pencils of quadrics σ in $\mathbb{P}^3$, the sheaf $\mathcal{T}_\sigma$ is always free, with exponents as in the following table. The pair (a, b) in the column *exponents* indicates that $\mathcal{T}_\sigma \simeq \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b)$.

r_1	(u, v)	exponents	m	Segre	f_1	f_2
1	$(3, 0)$	$(0, -2)$	1	$[1, 1, 1]$	$x_0^2 + x_2^2$	$x_1^2 + x_2^2$
1	$(3, 0)$	$(0, -2)$	1	$[2, 1]$	x_0x_1	$x_0^2 + x_2^2$
1	$(3, 0)$	$(0, -2)$	1	$[3]$	$2x_0x_2 + x_1^2$	x_0x_1
1	$(3, 0)$	$(0, -1)$	1	$[(2, 1)]$	$2x_0x_1 + x_2^2$	x_0^2
1	$(3, 0)$	$(0, -1)$	1	$[1^2, 1]$	x_0^2	$x_1^2 + x_2^2$
1	$(2, 1)$	$(-1, -1)$	0	$[1]$	x_0x_2	$2x_0x_1 + x_3^2$
2	$(2, 0)$	$(0, 0)$	2	$[1, 1]$	x_0^2	x_1^2
2	$(2, 0)$	$(0, 0)$	2	$[2]$	x_0^2	x_0x_1
2	$(1, 1)$	$(0, -1)$	1	$[\varnothing]$	x_0x_2	x_0x_1

5.2.1. *Irregular incompressible pencils.* The unique irregular incompressible pencil on $\mathbb{P}^3$ has $r_1 = 1$ so according to Proposition 4.6 we have $\mathcal{T}_\sigma$ locally free.

The splitting type of σ is $(2, 1)$ and the regular part of σ vanishes at a single point $\lambda \in \mathbb{P}^1$ which gives a single quadric of corank 2 in the pencil, so $r_0 = 2$. This gives a component $\Xi^{(1)} \subset \Xi_\sigma$ which is a reduced line. The component Y of Ξ_σ is a line which meets $\Xi^{(1)}$ at λ. In the normalized form appearing in the proof of Theorem 4.2, we have $\lambda = (0:1)$ and the matrix ρ reads:

$$\left(\begin{array}{ccc|c} 0 & z_1 & z_2 & 0 \\ z_1 & 0 & 0 & 0 \\ z_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & z_1 \end{array}\right).$$

The associated pencil is $(2x_0x_2, 2x_0x_1 + x_3^2)$ and, up to dividing by 2, the Jacobian matrix reads:

$$\begin{pmatrix} x_2 & 0 & x_0 & 0 \\ x_1 & x_0 & 0 & x_3 \end{pmatrix}$$

The kernel of this matrix is $\mathcal{T}_\sigma \simeq \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2}$, the syzygy map being:

$$\begin{pmatrix} x_0 & 0 \\ -x_1 & -x_3 \\ -x_2 & 0 \\ 0 & x_0 \end{pmatrix}.$$

5.2.2. *Compressible pencils.* Assume σ is compressible. Then the sheaf $\mathcal{T}_\sigma$ contains a copy of the trivial sheaf $\mathcal{O}_{\mathbb{P}^3}$ which is thus a direct summand of $\mathcal{T}_\sigma$. Therefore $\mathcal{T}_\sigma/\mathcal{O}_{\mathbb{P}^3}$ is a reflexive sheaf of rank one and thus isomorphic to $\mathcal{O}_{\mathbb{P}^3}(-c_1(\mathcal{T}_\sigma))$. For compressible pencils, we get the following.

i) If σ contains no double plane, then $\mathcal{T}_\sigma \simeq \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-2)$.
ii) If σ contains precisely one double plane, then $\mathcal{T}_\sigma \simeq \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)$.
iii) If σ contains two double planes, then $\mathcal{T}_\sigma \simeq \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}$.

6. Locally free pencils of quadrics

Let us conclude the analysis of freeness and local freeness of pencils of quadrics over an algebraically closed field κ of characteristic different from 2. In the next table, the column labelled Σ displays the Segre symbol of the regular part of σ. In the column labelled *exponents* we write the sequence of degrees of the line bundles which are direct summands of $\mathcal{T}_\sigma$, for instance the sequence $(0^{n-3}, -1^2)$ means that $\mathcal{T}_\sigma \simeq \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-3)} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus 2}$. Recall from Section 2.5 that $\hat{n} = n - h^0(\mathcal{T}_\sigma)$. When $n = 2$ one should not consider the first three lines.

Theorem 6.1. *Let $n \geqslant 2$. A pencil of quadrics σ is free if and only if σ is locally free. This happens if and only if, up to homography, $\sigma = (f_1, f_2)$ is:*

(52)

f_1	f_2	exponents	$\hat{n}$	r_0	r_1	Σ
x_0x_1	x_2x_3	$(0^{n-3}, -1^2)$	3	$n-2$	$n-3$	$[(1^2), (1^2)]$
$x_0x_1 + x_2x_3$	$x_0^2 + x_1^2$	$(0^{n-3}, -1^2)$	3	$n-2$	$n-3$	$[(2^2)]$
x_0x_2	$x_0x_1 + x_3^2$	$(0^{n-3}, -1^2)$	3	$n-1$	$n-2$	$[1]$
$x_0^2 + x_2^2$	$x_1^2 + x_2^2$	$(0^{n-2}, -2)$	2	$n-1$	$n-2$	$[1,1,1]$
x_0x_1	$x_0^2 + x_2^2$	$(0^{n-2}, -2)$	2	$n-1$	$n-2$	$[2,1]$
$2x_0x_2 + x_1^2$	x_0x_1	$(0^{n-2}, -2)$	2	$n-1$	$n-2$	$[3]$
$2x_0x_1 + x_2^2$	x_0^2	$(0^{n-2}, -1)$	2	n	$n-2$	$[(2,1)]$
x_0^2	$x_1^2 + x_2^2$	$(0^{n-2}, -1)$	2	n	$n-2$	$[1^2,1]$
x_0^2	x_1^2	(0^{n-1})	1	n	$n-1$	$[1,1]$
x_0^2	x_0x_1	(0^{n-1})	1	n	$n-1$	$[2]$

Proof. Let σ be a locally free pencil of quadrics. Following the notation of Section 2.5, set m for the compressibility of σ and $\hat{n} = n - m$. By Lemma 2.14, the sheaf $\mathcal{T}_\sigma$ decomposes as $\mathcal{O}_{\mathbb{P}^n}^{\oplus m} \oplus \mathcal{E}$, thus $\mathcal{E}$ must be locally free. In addition, the associated incompressible pencil $\hat{\sigma}$ is also locally free, since $\mathcal{T}_{\hat{\sigma}}$ coincides with the restriction of $\mathcal{E}$ to some $\hat{n}$-dimensional linear space.

If $\hat{\sigma}$ is regular, then Corollary 4.9 says that $\hat{n} \leqslant 2$ or $\hat{n} = 3$ and $\hat{\sigma}$ has Segre symbol $[1^2, 1^2]$ or $[2^2]$. In the latter case the normal form of the quadrics of $\hat{\sigma}$ obtained as in the proof of Theorem 4.2 is the one displayed in the first two lines of the table in display (52). Since σ depends only on $x_0, \ldots, x_{\hat{n}}$, this is actually the normal form of the quadrics of σ.

On the other hand, if $\hat{\sigma}$ is irregular, then, since $\hat{\sigma}$ is locally free, setting $\hat{r}_1$ for the generic rank of $\hat{\sigma}$, we must have $\hat{r}_1 \geqslant \hat{n} - 2$ by Corollary 4.6. Combining this with Lemma 3.10 gives $\hat{n} + 1 \geqslant 3\hat{n} - 6$ which implies $\hat{n} \leqslant 3$. If $\hat{n} = 3$, we are in the situation of Subsection 5.2.1 and we obtain the third line of the above table.

It remains to treat the cases $\hat{n} \leqslant 2$. Let us assume $\hat{n} = 2$. Since $\hat{\sigma}$ is incompressible, $\mathcal{T}_{\hat{\sigma}}$ is a reflexive sheaf of rank 1 with determinant equal to $e - 2$ where e is the number of double lines in $\hat{\sigma}$. Note that $e \in \{0, 1\}$ as $\hat{\sigma}$ is incompressible.

Also, $e = 0$ if and only if $r_0 = n - 1$ and this corresponds to the Segre symbols $[1, 1, 1]$, $[2, 1]$ and $[3]$, while $e = 1$ takes place when $r_0 = n$ and the Segre symbol is $[(2, 1)]$ or $[(1, 1), 1]$. These Segre symbols are associated to unique pencils up to homography since $\rho_{\tilde{\sigma}}$ defines at most 3 distinct points in support of $\mathcal{C}_{\mathrm{t}}$ and $\mathrm{PGL}_2(\kappa)$ acts transitively on triplets of points of $\mathbb{P}^1$.

Finally, assume $\hat{n} = 1$, so that $\mathcal{T}_\sigma \simeq \mathcal{O}_{\mathbb{P}^n}$. Then there are only two possible Segre symbols, whose normal forms give the pencils (x_0^2, x_1^2) and (x_0x_1, x_0^2). □

7. Locally free pencils of higher degree

In this section, κ is any field of characteristic 0. In contrast with the case of pencils of quadrics seen in the previous section, we will now show that there are locally free pencils of higher degree that are not free.

Before stepping into the general case, let us take a look at the case of pencils of cubics in detail.

Over the complex projective space $\mathbb{P}^3$, there are, according to [2], two non-normal cubic surfaces up to homography. In the homogeneous variables $(x_0, \ldots, x_3)$, the equations of these surfaces are:

$$f = x_1^3 + x_0^2x_2 + x_1^2x_3, \ , \ g = x_1^3 + x_0^2x_2 + x_0x_1x_3.$$

Both surfaces are singular along the line $L = V(x_0, x_1)$. The Jacobian matrix of the pencil of cubics $\sigma = (f, g)$ reads:

$$\mathcal{J}_\sigma = \begin{pmatrix} 2x_0x_2 & 3x_1^2 + 2x_1x_3 & x_0^2 & x_1^2 \\ 2x_0x_2 + x_1x_3 & 3x_1^2 + x_0x_3 & x_0^2 & x_0x_1 \end{pmatrix}.$$

The sheaf $\mathcal{Q}_\sigma$ has rank two over L and admits no zero-dimensional subsheaf. The first part of Lemma 2.2 implies that $\mathcal{T}_\sigma$ is locally free.

However, the scheme-theoretic locus where $\mathcal{Q}_\sigma$ has rank two has an embedded point at $p = (0 : 0 : 1 : 0)$. In fact, the Jacobian scheme Ξ_σ has 4 primary components $P_1, \ldots, P_4$ described by the next table.

Dimension	degree	radical ideal
1	5	(x_0, x_1)
1	1	(x_1, x_3)
1	1	$(x_0 - x_1, x_3)$
0	20	(x_0, x_1, x_3).

Note that $(\Xi_\sigma)_{\mathrm{red}}$ consists of the union of the 3 lines $V(x_0, x_1)$, $V(x_1, x_3)$ and $V(x_0 - x_1, x_3)$; the first one appears with multiple structure of degree 5.

In this example, $\mathcal{T}_\sigma(2)$ has $c_1(\mathcal{T}_\sigma(2)) = 0$ and $c_2(\mathcal{T}_\sigma(2)) = 1$, $c_3(\mathcal{T}_\sigma(2)) = 0$. Also, we have $H^0(\mathcal{T}_\sigma(2)) = 0$. Therefore $\mathcal{T}_\sigma(2)$ is a null correlation bundle.

This example is generalized to degree $k + 3$, for any $k \geqslant 0$, in our next result.

Theorem 7.1. *For any $k \geqslant 0$, define the pencil $\sigma = (f, g)$ as:*

$$f = x_0x_1^{k+2} + x_2^{k+3} + x_2^{k+2}x_3, \ g = x_2x_3(x_2^{k+1} - x_1^{k+1}).$$

Then we have $\mathcal{T}_\sigma \simeq N(-k - 2)$, where N is a null correlation bundle.

Proof. Set $f = x_0x_1^{k+2} + x_2^{k+3} + x_2^{k+2}x_3$ and $g = x_2x_3(x_2^k - x_1^k)$. Observe that, in the algebraic closure of κ, the divisor $V(g)$ is an arrangement of planes consisting of $k + 2$ planes $H_1, \ldots, H_{k+2}$ passing through the line $L = V(x_1, x_2)$ together with

an extra plane H not containing L, namely the plane $V(x_3)$. This arrangement is free, more precisely, we have :

$$\mathcal{T}_g \simeq \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-k-1).$$

Factoring out the trivial summand $\mathcal{O}_{\mathbb{P}^3}$ of $\mathcal{T}_g$, we write explicitly the syzygy $\phi : \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-k-1) \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 3}$, independently on whether κ is closed or not:

$$\phi = \begin{pmatrix} x_1 & x_2^{k+1} \\ x_2 & x_1^k x_2 \\ -(k+2)\, x_3 & -x_1^k x_3 \end{pmatrix}.$$

The the Jacobian matrix $\mathcal{J}_\sigma : \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 2}(k+2)$ reads:

$$\mathcal{J}_\sigma = \begin{pmatrix} x_1^{k+2} & (k+2)x_0 x_1^{k+1} & (k+3)x_2^{k+2} + (k+2)x_2^{k+1}x_3 & x_2^{k+2} \\ 0 & (k+1)x_1^k x_2 x_3 & x_1^{k+1}x_3 - (k+2)x_2^{k+1}x_3 & x_1^{k+1}x_2 - x_2^{k+2} \end{pmatrix}.$$

Note that this matrix has a vanishing entry at the bottom left corner and that the vanishing of this entry corresponds to the trivial summand of $\mathcal{T}_g$. Therefore, projecting onto the last three factors of $\mathcal{O}_{\mathbb{P}^3}^{\oplus 4}$ and onto the second factor of $\mathcal{O}_{\mathbb{P}^3}^{\oplus 2}(k+2)$ we get a commutative diagram, which is essentially a particular case of the diagram in display (13):

$$\begin{array}{ccccccc} & & & & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(k+2) \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{T}_\sigma & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} & \xrightarrow{\mathcal{J}_\sigma} & \mathcal{O}_{\mathbb{P}^3}^{\oplus 2}(k+2) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-k-1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^{\oplus 3} & \xrightarrow{\mathcal{J}_g} & \mathcal{O}_{\mathbb{P}^3}(k+2) \end{array} \tag{53}$$

The top arrow defines a surface $D \subset \mathbb{P}^3$ of degree $k+2$, whose equation must sit in the top left corner of $\mathcal{J}_\sigma$. In other words, $D = V(x_1^{k+2})$ is the $(k+2)$-tuple structure over the plane $V(x_1)$.

Also, we observe that the image of $\mathcal{J}_g$ is $\mathcal{I}_{C/\mathbb{P}^3}(k+2)$, where the curve C is the scheme-theoretic singular locus of $V(g)$ and is defined by the 3 minors of order 2 of ϕ. Incidentally, over the algebraic closure of κ, the curve C consists of $k+2$ reduced lines $L_1, \ldots, L_{k+2}$, with $L_i = H \cap H_i$ for all $i \in \{1, \ldots, k+2\}$, together with a $(k+1)$-tuple complete intersection structure over L of degree $(k+1)^2$ defined by $V(x_0^k x_1, x_0^{k+1} - (k+2)x_1^{k+1})$.

The rightmost column of diagram (53) gives a surjection $\mathcal{M}_\sigma \to \mathcal{I}_{C/\mathbb{P}^3}(k+2)$, whose kernel is a torsion free sheaf of rank 1, isomorphic to $\mathcal{I}_{B/\mathbb{P}^3}(k+2)$, where the subscheme $B \subset D \subset \mathbb{P}^3$ is defined in $\mathbb{P}^3$ by the homogeneous polynomials h of the form $h = a_0 f_0 + \cdots + a_3 f_3$ with $a_i \in R = \kappa[x_0, \ldots, x_3]$ and satisfying $a_1 g_1 + a_2 g_2 + a_3 g_3 = 0$, where we put $f_i = \partial f / \partial x_i$ and $g_i = \partial g / \partial x_i$, for $i \in \{0, 1, 2, 3\}$. Since ϕ accounts for all relations of the homogeneous ideal of C, the homogeneous ideal of B is thus generated by $(f_0, f_1\phi_{1,1} + f_2\phi_{2,1} + f_3\phi_{3,1}, f_1\phi_{1,2} + f_2\phi_{2,2} + f_3\phi_{3,2})$ and the matrix of these generators is:

$$\begin{pmatrix} x_1^{k+2} \\ (k+2)x_0 x_1^{k+2} + (k+3)x_2^{k+3} \\ (k+2)x_0 x_1^{k+1} x_2^{k+1} + (k+3)x_1^k x_2^{k+3} + (k+1)x_1^k x_2^{k+2} x_3 \end{pmatrix}.$$

Therefore, the homogeneous ideal $I_{B/D}$ of B in $D = V(x_1^{k+2})$ is :

$$\left(x_2^{k+3}, (k+2)x_0x_1^{k+1}x_2^{k+1} + (k+1)x_1^k x_2^{k+2}x_3\right). \tag{54}$$

We have an exact sequence:

$$0 \to \mathcal{I}_{B/\mathbb{P}^3}(k+2) \to \mathcal{M}_\sigma \to \mathcal{I}_{C/\mathbb{P}^3}(k+2) \to 0,$$

and thus, from the leftmost column of diagram (53):

$$0 \to \mathcal{T}_\sigma \to \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-k-1) \to \mathcal{I}_{B/D}(k+2) \to 0.$$

The morphism $\mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{I}_{B/D}(k+2)$ is given by the generator x_2^{k+3} of the ideal of B in D as in (54). It defines a curve section A of D of degree $k+3$ which contains B. We get an exact sequence:

$$0 \to \mathcal{I}_{E/\mathbb{P}^3}(-k-1) \to \mathcal{O}_{\mathbb{P}^3}(-1-k) \to \mathcal{I}_{B/A}(k+2) \to 0,$$

where the curve $E \subset \mathbb{P}^3$ is defined by the sequence and is cut in D as the residual scheme of B with respect to the complete intersection $A = V(x_1^{k+2}, x_2^{k+3})$. From the diagram (53), using the snake lemma we also get:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-k-3) \to \mathcal{T}_\sigma \to \mathcal{I}_{E/\mathbb{P}^3}(-1-k) \to 0. \tag{55}$$

We compute the equations of E from (54) as $\left(I_{B/D} : (x_2^{k+3})\right)$ and get:

$$E = V\left(x_1^2, x_1x_2, x_2^2, (k+2)x_0x_1 - (k+1)x_2x_3\right).$$

Therefore, the curve E is a double structure of arithmetic genus -1 over the line L. We conclude from (55) that $\mathcal{T}_\sigma(k+2)$ is a null correlation bundle. □

Remark 7.2. By the previous theorem, for any degree $k+3$ there is a pencil σ which is locally free but not free. Also, we have $\operatorname{gpdim}(\mathcal{T}_\sigma) = 2$. More precisely, the sheafified minimal graded free resolution of $\mathcal{T}_\sigma$ reads:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-k-5) \to \mathcal{O}_{\mathbb{P}^3}(-k-4)^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^3}(-k-3)^{\oplus 5} \to \mathcal{T}_\sigma \to 0.$$

This is in contrast with the case of pencils of quadrics, where local freeness is equivalent to freeness and where $\operatorname{gpdim}(\mathcal{T}_\sigma) \leqslant n-2$.

8. Regular sequences of length 2 and rational foliations

We complete this paper by looking at arbitrary regular sequences of length 2 and showing how these are related to rational 1-forms, which we now introduce.

Let $\omega \in H^0(\Omega^1_{\mathbb{P}^n}(d+2))$ be a *rational 1-form* of type (d_1+1, d_2+1), where $0 \leqslant d_1 \leqslant d_2$, given by

$$\omega = af_1 \cdot \mathrm{d}f_2 - bf_2 \cdot \mathrm{d}f_1,$$

where f_1 and f_2 are homogeneous polynomials with no common factors of degree d_1+1 and d_2+1, respectively, with $d_1+d_2 = d$, and a and b are relatively prime integers such that $(d_1+1)b = (d_2+1)a$. Remark that $\sigma := (f_1, f_2)$ is a regular sequence in $R = \kappa[x_0, \ldots, x_n]$.

Regarding ω as an element of $\operatorname{Hom}_{\mathbb{P}^n}(\mathrm{T}\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d+2))$, we set $\mathcal{K}_\omega := \ker(\omega)$. Since ω vanishes along the complete intersection scheme $C := V(\sigma)$, the image of the morphism $\omega : \mathrm{T}\mathbb{P}^n \to \mathcal{O}_{\mathbb{P}^n}(d+2)$ is actually contained in the ideal sheaf $\mathcal{I}_C(d+2)$. Applying the functor $\operatorname{Hom}_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1}, -)$ to the resolution of $\mathcal{I}_C(d+2)$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\tilde{\eta}} \mathcal{O}_{\mathbb{P}^n}(d_1+1) \oplus \mathcal{O}_{\mathbb{P}^n}(d_2+1) \longrightarrow \mathcal{I}_C(d+2) \longrightarrow 0,$$

where $\tilde{\eta} = ((d_1+1)f_1 \quad (d_2+1)f_2)^{\mathrm{t}}$, we check that the composed morphism

$$\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \twoheadrightarrow \mathrm{T}\mathbb{P}^n \to \mathcal{I}_C(d+2)$$

lifts to a unique morphism $\mu : \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n}(d_1+1) \oplus \mathcal{O}_{\mathbb{P}^n}(d_2+1)$, since

$$\mathrm{Hom}_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1}, \mathcal{O}_{\mathbb{P}^n}) = \mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1}, \mathcal{O}_{\mathbb{P}^n}) = 0.$$

Therefore we obtain the commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{\mathbb{P}^n} & = & \mathcal{O}_{\mathbb{P}^n} & & \\
 & & & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \tilde{\eta}} & & \\
0 & \to & \mathcal{K}_\omega & \to & \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} & \xrightarrow{\mu} & \mathcal{O}_{\mathbb{P}^n}(d_1+1) \oplus \mathcal{O}_{\mathbb{P}^n}(d_2+1) & & \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{K}_\omega & \to & \mathrm{T}\mathbb{P}^n & \xrightarrow{\omega} & \mathcal{I}_C(d+2) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{56}$$

This proves that $\mathcal{K}_\omega \simeq \ker(\mu)$. We argue that $\mu = \mathcal{J}_\sigma$, thus in fact $\mathcal{K}_\omega \simeq \mathcal{T}_\sigma(1)$. Indeed, note that

$$\omega = \sum_{i=0}^{n} (pf_1\partial_i f_2 - qf_2\partial_i f_1) \cdot dx_i,$$

which means that the entries of the morphism $\alpha : \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n}(d+2)$ given by the composition

$$\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \twoheadrightarrow \mathrm{T}\mathbb{P}^n \to \mathcal{I}_C(d+2) \hookrightarrow \mathcal{O}_{\mathbb{P}^n}(d+2)$$

are precisely $\alpha_i = p\cdot f_1\partial_i f_2 - q\cdot f_2\partial_i f_1$. Since, on the other hand, $\alpha = (-qf_2 \quad pf_1)\circ\mu$, we conclude that $\mu = \mathcal{J}_\sigma$, as desired.

Conversely, given a regular sequence $\sigma = (f_1, f_2)$ with $\deg(f_i) = d_i + 1$, we follow the proof of Lemma 2.3 in Section 2.2 and consider the associated codimension 1 distribution $\mathscr{D}_\sigma$ as presented in display (8); in the case at hand, this simplifies to (setting $d = d_1 + d_2$)

$$0 \longrightarrow \mathcal{T}_\sigma(1) \longrightarrow \mathrm{T}\mathbb{P}^n \xrightarrow{\omega} \mathcal{I}_{\Gamma_\sigma}(d-l+2) \longrightarrow 0, \tag{57}$$

where $\Gamma_\sigma \subset \mathbb{P}^n$ is a (possibly not pure) 2-codimensional subscheme of $\mathbb{P}^n$, and $l = c_1(\mathcal{Q}_\sigma)$; this is precisely the codimension one distribution associated to the (possibly non saturated) twisted rational 1-form

$$\omega = (d_1+1)f_1\cdot \mathrm{d}f_2 - (d_2+1)f_2\cdot \mathrm{d}f_1 \in \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{T}\mathbb{P}^n, \mathcal{I}_\Gamma(d-l+2)) \subset H^0(\Omega^1_{\mathbb{P}^n}(d+2)).$$

Moreover, the bottom line of the diagram in display (7) yields the following description for the singular scheme Γ_σ of ω:

$$0 \longrightarrow \mathcal{I}_{\Gamma_\sigma}(d-l+1) \longrightarrow \mathcal{I}_C(d+1) \longrightarrow \mathcal{Q}_\sigma \longrightarrow 0. \tag{58}$$

In particular, we have that

$$\deg(\Gamma_\sigma) = \deg(\mathcal{Q}_\sigma) + \deg(C) = \deg(\mathcal{Q}_\sigma) + (d_1+1)(d_2+1).$$

Summarizing, we have established a natural 1-1 correspondence between regular sequences of length 2 and rational codimension one foliations as follows.

Lemma 8.1. *There exists a 1-1 correspondence between regular sequences* $\sigma = (f_1, f_2)$ *on* R *and rational codimension one foliations* $\mathscr{D}$ *on* $\mathbb{P}^n$ *of type* $(\deg(f_1), \deg(f_2))$ *such that* $\mathcal{T}_\sigma(1) = \mathcal{T}_{\mathscr{D}}$ *and* $\mathrm{Sing}(\mathscr{D}) = \Gamma_\sigma$.

The previous statement has the following two important applications when $n = 3$. First, as an immediate consequence of [4, Theorem 6.3], we obtain the following stability result for logarithmic sheaves associated to regular sequences of length 2 on $\mathbb{P}^3$.

Corollary 8.2. *Let* $\sigma = (f_1, f_2)$ *be a regular sequence in* $\kappa[x_0, x_1, x_2, x_3]$ *and let* $d_i := \deg(f_i) - 1$*; assume that* $d_1 + d_2 > 0$ *and* $c_1(\mathcal{Q}_\sigma) = 0$.

(1) If $d_1 + d_2$ *is even, then*
- *if* $\deg(\mathcal{Q}_\sigma) < (d_1^2 + d_2^2 - d_1 - d_2 - 2)/2$*, then* $\mathcal{T}_\sigma$ *is slope-stable;*
- *if* $\deg(\mathcal{Q}_\sigma) < (d_1^2 + d_2^2 + d_1 + d_2)/2$*, then* $\mathcal{T}_\sigma$ *is slope-semistable;*

(2) If $d_1 + d_2$ *is odd and* $\deg(\mathcal{Q}_\sigma) < (d_1^2 + d_2^2 - 1)/2$*, then* $\mathcal{T}_\sigma$ *is slope-stable.*

In particular, if the Jacobian scheme is 0-dimensional, then $\mathcal{T}_\sigma$ *is slope-stable.*

We remark that the previous result is not sharp, and it is not hard to find examples of regular sequences with slope-stable logarithmic sheaves whose degrees do not satisfy the numerical inequalities above. Indeed, if σ corresponds to a pencil of quadrics, so that $d_1 = d_2 = 1$, with $\dim \mathcal{G}_\sigma = 0$, then Corollary 8.2 only implies that $\mathcal{T}_\sigma$ is slope-semistable; however, as we have seen in Section 5.1, $\mathcal{T}_\sigma$ is actually slope-stable in this case. Note that the case $d_1 = d_2 = 1$ is the only one for which the right hand sides of the inequalities is not positive.

In addition, the higher degree pencils provided in Theorem 7.1 yield yet another set of examples showing that the converse of Corollary 8.2 does not hold.

Finally, as a second application, we give a negative answer to a problem posed by Calvo-Andrade, Cerveau, Giraldo and Lins Neto, see [3, Problem 2]. To be preceise, these authors asked whether the tangent sheaf of a codimension one foliation on $\mathbb{P}^3$ splits as a sum of line bundles whenever it is locally free. Indeed, in light of the proof of Lemma 8.1, the pencils presented in Theorem 7.1 provide examples, for each $k \geqslant 0$, of rational foliations of type $(k+3, k+3)$ on $\mathbb{P}^3$ whose tangent sheaves are slope-stable locally free sheaves.

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