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A VERY EASY HIGH-ORDER WELL-BALANCED RECONSTRUCTION FOR HYPERBOLIC SYSTEMS WITH SOURCE TERMS*

CHRISTOPHE BERTHON †, SOLÈNE BULTEAU ‡, FRANÇOISE FOUCHER*§, MEISSA M'BAYE*¶, AND VICTOR MICHEL-DANSAC ‖

Abstract. When adopting high-order finite-volume schemes based on MUSCL reconstruction techniques to approximate the weak solutions of hyperbolic systems with source terms, the preservation of the steady states turns out to be very challenging. Indeed, the designed reconstruction must preserve the steady states under consideration in order to get the required well-balancedness property. A priori, to capture such a steady state, one needs to solve some strongly nonlinear equations. Here, in order to preserve the required well-balancedness property to be satisfied by finite volume methods, we design a very easy correction. This correction can be applied to any scheme of order greater than or equal to 2, such as a MUSCL-type scheme, and ensures that this scheme exactly preserves the steady solutions. The main discrepancy with usual techniques lies in never having to invert the nonlinear function governing the steady solutions. Moreover, for under-determined steady solutions, several nonlinear functions must be considered simultaneously. Since the derived correction only considers the evaluation of the governing nonlinear functions, we are able to deal with under-determined stationary systems. Several numerical experiments illustrate the relevance of the proposed well-balanced correction.

Key words. Hyperbolic conservation laws, Balance laws, Well-balanced schemes, High-order reconstruction techniques

AMS subject classifications. 65M08, 65M12

1. Introduction.

1.1. General framework.

The present work is devoted to the numerical approximation of the weak solutions of an evolution law of the form

$$\partial_t w + \partial_x f(w) = S(w,x), \quad x \in \mathbb{R}, \quad t > 0, \tag{1.1}$$

where $w : \mathbb{R} \times \mathbb{R}^+ \to \Omega \subset \mathbb{R}^N$ denotes the unknown vector. The set Ω stands for the set of the admissible states and it is assumed to be convex. The flux function $f : \Omega \to \mathbb{R}^n$ and the source term $S : \Omega \times \mathbb{R} \to \mathbb{R}^N$ are assumed to be smooth enough, say C^1. For stability reasons, in the present work, the matrix $\nabla_w f(w)$ is assumed to be diagonalizable in $\mathbb{R}$ so that the homogeneous system extracted from (1.1) is hyperbolic. The PDE system (1.1) is endowed with initial data $w(x, t = 0) = w^0(x)$, where $w^0(x) \in \Omega$ for all $x \in \mathbb{R}$ is a given function.

Because of the source term $S(w,x)$, there exists non-trivial steady solutions gov-

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erned by

$$\begin{cases} \partial_x f(w) = S(w,x), \\ w(x_0) = w_0, \end{cases} \tag{1.2}$$

where w_0 is a given state in Ω and x_0 a given point in $\mathbb{R}$.

Now, if the above system can be integrated, there exists $G : \Omega \times \mathbb{R} \to \mathbb{R}^N$ such that the stationary solutions are governed by

$$\begin{cases} \partial_x G(w,x) = 0, \\ w(x_0) = w_0. \end{cases}$$

In fact, it is not always possible to integrate (1.2). Even then, G is not necessarily unique and, in general, it must be restricted according to some invariant domains. Usually, the steady solutions are restricted to some particular definition of G. Then, we have to deal with a sequence $(G_\ell)_{1\leq \ell \leq L}$ with $L < +\infty$ given.

Equipped with these comments, in this work, we only consider steady solutions defined by

$$\begin{cases} \partial_x G_\ell(w,x) = 0, \quad 1 \leq \ell \leq L, \\ w(x_0) = \Pi_\ell^{eq}(w_0), \end{cases} \tag{1.3}$$

where we have denoted by $\Pi^{eq}(w)$ the projection of w over the invariant domain under consideration.

1.2. Illustrating models. In order to illustrate the relevance of such a definition of the steady states, let us present some examples of particular interest. First, let us adopt the well-known shallow water model given by

$$w = \begin{pmatrix} h \\ q \end{pmatrix}, \quad f(w) = \begin{pmatrix} q \\ \dfrac{q^2}{h} + g\dfrac{h^2}{2} \end{pmatrix}, \quad S(w,x) = \begin{pmatrix} 0 \\ -gh\partial_x z \end{pmatrix}, \tag{1.4}$$

where $g > 0$ is the gravity constant, $z(x)$ the given smooth topography function, h is the water height and q is the water discharge. The smooth steady solutions (see [4, 11, 25]) are given by

$$\begin{cases} \partial_x q = 0, \\ \partial_x \left(\dfrac{q^2}{2h^2} + g(h+z) \right) = 0, \end{cases} \tag{1.5}$$

with $w(x_0) = w_0$ for a given w_0 in Ω, where the set Ω of admissible states is defined as follows:

$$\Omega = \left\{ {}^t(h,q) \in \mathbb{R}^2;\ h \geq 0,\ q \in \mathbb{R} \right\}. \tag{1.6}$$

As a consequence, we immediately obtain

$$G(w,x) = \begin{pmatrix} q \\ \dfrac{q^2}{2h^2} + g(h+z) \end{pmatrix} \quad \text{and} \quad \Pi^{eq}(w) = w. \tag{1.7}$$

Now, let us restrict the definition of the steady states just considering the usual lake at rest given by $q = 0$ and $\partial_x(h + z) = 0$, then we easily get

$$G(w,x) = \begin{pmatrix} 0 \\ h+z \end{pmatrix} \quad \text{and} \quad \Pi^{eq}(w) = \begin{pmatrix} h \\ 0 \end{pmatrix}. \tag{1.8}$$

The second example we present is given by the shallow water equations with a Manning friction source term (see [36, 34, 18]). This model reads

$$w = \begin{pmatrix} h \\ q \end{pmatrix}, \quad f(w) = \begin{pmatrix} q \\ \dfrac{q^2}{h} + g\dfrac{h^2}{2} \end{pmatrix}, \quad S(w,x) = \begin{pmatrix} 0 \\ -\kappa \dfrac{q|q|}{h^\eta} \end{pmatrix}, \tag{1.9}$$

where $\kappa > 0$ is the friction coefficient and $\eta \neq 1$ is the Manning exponent. The set of the admissible states is given by (1.6). After [36], the stationary solutions are given as follows:

$$\begin{cases} \partial_x q = 0, \\ \partial_x \left(-q^2 \dfrac{h^{\eta-1}}{\eta-1} + g\dfrac{h^{\eta+2}}{\eta+2} + \kappa x q|q| \right) = 0, \end{cases}$$

with $w(x_0) = w_0$ for a given $w_0 \in \Omega$, so that we immediately obtain

$$G(w,x) = \begin{pmatrix} q \\ -q^2 \dfrac{h^{\eta-1}}{\eta-1} + g\dfrac{h^{\eta+2}}{\eta+2} + \kappa x q|q| \end{pmatrix} \quad \text{and} \quad \Pi^{eq}(w) = w. \tag{1.10}$$

Next, we present a system involving a non-unique definition of G, the Euler model with gravity (see [19, 45]). This model reads as follows:

$$w = \begin{pmatrix} \rho \\ q \\ E \end{pmatrix}, \quad f(w) = \begin{pmatrix} q \\ \dfrac{q^2}{\rho} + p \\ (E+p)\dfrac{q}{\rho} \end{pmatrix}, \quad S(w,x) = \begin{pmatrix} 0 \\ -\rho\partial_x\varphi \\ -q\partial_x\varphi \end{pmatrix}, \tag{1.11}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ stands for a given smooth gravitational potential and $p := p(\rho, E - \frac{1}{2}\frac{q^2}{\rho})$ denotes the pressure law, with E the total energy. The set of admissible states is defined here by

$$\Omega = \left\{ {}^t(\rho, q, E) \in \mathbb{R}^3;\ \rho > 0,\ q \in \mathbb{R},\ E - \frac{1}{2}\frac{q^2}{\rho} > 0 \right\}.$$

Concerning the steady solutions, we are concerned by steady solutions at rest governed by (see [19, 45])

$$\begin{cases} q = 0, \\ \partial_x p + \rho\partial_x\varphi = 0, \end{cases} \quad \text{with} \quad w(x_0) = \begin{pmatrix} \rho_0 \\ 0 \\ E_0 \end{pmatrix}.$$

Once again, the system to govern the steady state solutions turns out to be underdetermined and we have to focus on particular families of steady solutions. According

to [19, 45], three families of steady states are of prime importance. The first family is given by

$$\begin{cases} q = 0, \\ \partial_x \rho = 0, \\ \partial_x (p + \rho\varphi) = 0, \end{cases} \quad \text{with} \quad w(x_0) = \begin{pmatrix} \rho_0 \\ 0 \\ E_0 \end{pmatrix},$$

for all $\rho_0 > 0$ and $E_0 > 0$.

In order to define both second and third families of steady states, we have to impose that the function $E \mapsto p(\rho, E)$ is invertible and we denote by $p_E^{-1}(\rho, .)$ this inverse function so that $p_E^{-1}(\rho, p(\rho, E)) = E$. Now, the second steady state family reads

$$\begin{cases} q = 0, \\ \partial_x (p - \kappa\rho) = 0, \\ \partial_x (\varphi + \kappa \ln \rho) = 0, \end{cases} \quad \text{with} \quad w(x_0) = \begin{pmatrix} \rho_0 \\ 0 \\ p_E^{-1}(\rho_0, \kappa\rho_0) \end{pmatrix},$$

where $\kappa > 0$ is a given parameter. The last steady state family is defined by

$$\begin{cases} q = 0, \\ \partial_x (p - \kappa\rho^\gamma) = 0, \\ \partial_x \left(\dfrac{\kappa\gamma}{\gamma - 1} \rho^{\gamma - 1} + \varphi \right) = 0, \end{cases} \quad \text{with} \quad w(x_0) = \begin{pmatrix} \rho_0 \\ 0 \\ p_E^{-1}(\rho_0, \kappa\rho_0^\gamma) \end{pmatrix}, \tag{1.12}$$

where $\gamma > 1$ is a given parameter.

As a consequence, we get

$$G_1(w, x) = \begin{pmatrix} q \\ \rho \\ p + \rho\varphi \end{pmatrix}, \quad G_2(w, x) = \begin{pmatrix} q \\ p - \kappa\rho \\ \varphi + \kappa \ln \rho \end{pmatrix}, \quad G_3(w, x) = \begin{pmatrix} q \\ p - \kappa\rho^\gamma \\ \dfrac{\kappa\gamma}{\gamma - 1} \rho^{\gamma - 1} + \varphi \end{pmatrix}, \tag{1.13}$$

with

$$\Pi_1^{eq}(w) = \begin{pmatrix} \rho \\ 0 \\ E \end{pmatrix}, \quad \Pi_2^{eq}(w) = \begin{pmatrix} \rho \\ 0 \\ p_E^{-1}(\rho, \kappa\rho) \end{pmatrix} \quad \text{and} \quad \Pi_3^{eq}(w) = \begin{pmatrix} \rho \\ 0 \\ p_E^{-1}(\rho, \kappa\rho^\gamma) \end{pmatrix}. \tag{1.14}$$

1.3. Main motivations. Now, considering the derivation of numerical schemes approximating the solutions of (1.1) and able to accurately, or even exactly, capture the steady solutions defined by (1.3) has been an important challenge during the two last decades. Numerous techniques have been designed for the shallow water model with topography (1.4) supplemented by the lake at rest (1.8). For a non-exhaustive bibliography, the reader is referred to [2, 11, 4, 9, 32, 38, 22, 16, 13, 17, 8]. More recently, in [5, 6, 36, 35, 37, 42, 44, 25], extensions are given in order to deal with moving steady states given by (1.7). In [36, 28, 12], the Manning-type friction source term is adopted and suitable discretizations are introduced to capture steady states given by (1.10). Regarding the discretization of the Euler model with gravity (1.11), the reader is referred to [19, 3, 14, 29, 31, 33, 43, 45, 15] where numerical strategies

are developed to capture steady states according to the pairs (G_ℓ, Π_ℓ^{eq}) defined by (1.13) – (1.14).

In the present work, we are not interested in the derivation of well-balanced schemes, namely schemes able to capture steady solutions given by (1.3). Here, our purpose concerns the high-order extensions obtained by involving a polynomial reconstruction procedure. Indeed, as soon as the well-balancedness property must be preserved, the reconstruction may involve strong difficulties. In particular, to be well-balanced, the usual reconstruction approaches need to invert $G_\ell(w, x)$ with respect to w for one given ℓ as long as the function $w \mapsto G_\ell(w, x)$ is invertible. We immediately understand that dealing simultaneously with distinct functions $G_\ell(w, x)$ does not seem reachable. Moreover, imposing that the application $w \mapsto G_\ell(w, x)$ is invertible is a strong assumption, not satisfied in general by physical models.

In this work, we present a very easy strategy to force any reconstruction procedure to preserve the steady solutions given by (1.3) just evaluating the applications $G_\ell(w, x)$ according to the projection $\Pi_\ell^{eq}(w)$. To address such an issue, the present work is organized as follows. In order to set the framework and the main notations, in section 2 we introduce the numerical schemes and the usual MUSCL second-order strategy [41, 30, 40]. In addition, we present the main issues when enforcing the polynomial reconstruction to be well-balanced. Section 3 is then devoted to the strategy designed here, which ensures that the expected well-balancedness property is satisfied by any reconstruction. The proposed improvement comes from a suitable evaluation of the pairs $(G_\ell, \Pi_\ell^{eq})_{1\leq\ell\leq L}$; G_ℓ is never inverted. In section 4, we present a high-order well-balanced extension. To conclude the paper, section 5 is devoted to several numerical experiments to illustrate the relevance of the designed high-order reconstruction improvement.

2. Issues of the well-balanced second-order MUSCL schemes. To approximate the solutions of (1.1), the space is discretized by introducing a sequence of cells $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, for all $i \in \mathbb{Z}$, with a constant size Δx. We denote by $x_i = (x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}})/2$ the center of each cell. We set $t^{n+1} = t^n + \Delta t$ to discretize the time domain with a time step Δt. In general, Δt is restricted according to a CFL condition.

At time t^n, we denote by w_i^n a constant approximation of the solution of (1.1) over the cell $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$. To evolve this approximation in time, we adopt a finite volume scheme of the form

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(f_{i+\frac{1}{2}}^n - f_{i-\frac{1}{2}}^n\right) + \frac{\Delta x}{2}\left(S_{i-\frac{1}{2}}^n + S_{i+\frac{1}{2}}^n\right), \tag{2.1}$$

where we have set

$$f_{i+\frac{1}{2}}^n = f_\Delta(w_i^n, w_{i+1}^n) \quad \text{and} \quad S_{i+\frac{1}{2}}^n = S_\Delta(w_i^n, w_{i+1}^n, x_i, x_{i+1}, \Delta x).$$

In order to get a consistent scheme, the numerical flux function f_Δ and the discrete source term S_Δ are assumed to be Lipschitz-continuous and to verify

$$f_\Delta(w, w) = f(w) \quad \text{and} \quad S_\Delta(w, w, x, x, 0) = S(w, x). \tag{2.2}$$

At this level, the reader is referred to the large literature devoted to the derivation of a well-balanced scheme according to the system of interest. Here, we have imposed the well-balancedness property according to the definition (1.3) of the steady states. As a consequence, we get $w_i^{n+1} = w_i^n$ as long as, for all i in $\mathbb{Z}$, we have

$$G_\ell(w_i^n, x_i) = G_\ell(w_{i+1}^n, x_{i+1}) \quad \text{and} \quad w_i^n = \Pi_\ell^{eq}(w_i^n) \quad \text{with} \quad 1 \leq \ell \leq L. \tag{2.3}$$

Now, we focus on a second-order extension, see for instance [40, 41, 7]. To address such an issue, we have to introduce suitable reconstructed states, denoted by $w^{\pm}_{i+\frac{1}{2}}$, on each side of the interface $x_{i+\frac{1}{2}}$. This reconstruction is said to be second-order accurate in space if, for all $i \in \mathbb{Z}$, we have

$$w^{-}_{i+\frac{1}{2}} = w(x_{i+\frac{1}{2}}, t^n) + \mathcal{O}(\Delta x^2) \quad \text{and} \quad w^{+}_{i+\frac{1}{2}} = w(x_{i+\frac{1}{2}}, t^n) + \mathcal{O}(\Delta x^2), \tag{2.4}$$

for some smooth function $x \mapsto w(x, t^n)$ such that $w^n_i = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x, t^n)dx$.

Equipped with this second-order reconstruction, from the first-order scheme (2.1), we define a second-order scheme as follows:

$$w^{n+1}_i = w^n_i - \frac{\Delta t}{\Delta x}\left(f^{\pm}_{i+\frac{1}{2}} - f^{\pm}_{i-\frac{1}{2}}\right) + \Delta t S^{\pm}_i, \tag{2.5}$$

where we have set $f^{\pm}_{i+\frac{1}{2}} = f_\Delta(w^{-}_{i+\frac{1}{2}}, w^{+}_{i+\frac{1}{2}})$, and where $S^{\pm}_i$ is a second-order approximation of the source term average, i.e.

$$S^{\pm}_i = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} S(w(x, t^n), x)\, dx + \mathcal{O}(\Delta x^2). \tag{2.6}$$

A classical choice for such second-order accurate schemes is to use the second-order midpoint approximation:

$$S^{\pm}_i = S\left(\frac{1}{2}\left(w^{+}_{i-\frac{1}{2}} + w^{-}_{i+\frac{1}{2}}\right), x_i\right). \tag{2.7}$$

It is clear that second-order accuracy is achieved as soon as the reconstructed states are defined. At the interface $x_{i+\frac{1}{2}}$, the reconstructed states read (for instance, see [7, 40, 41])

$$\begin{aligned} w^{-}_{i+\frac{1}{2}} &= w^n_i + \frac{1}{2}\mathcal{L}(w^n_i - w^n_{i-1}, w^n_{i+1} - w^n_i), \\ w^{+}_{i+\frac{1}{2}} &= w^n_{i+1} - \frac{1}{2}\mathcal{L}(w^n_{i+1} - w^n_i, w^n_{i+2} - w^n_{i+1}), \end{aligned} \tag{2.8}$$

where $\mathcal{L} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ are Lipschitz-continuous functions, which satisfy

$$\begin{aligned} &\mathcal{L}(w, w) = w \quad \text{for all } w \in \mathbb{R}^N, \\ &\exists M > 0 \text{ such that } \|\mathcal{L}(w_L, w_R)\| \le M \max(\|w_L\|, \|w_R\|), \quad \forall\ w_L, w_R \in \mathbb{R}^N. \end{aligned}$$

A large body of literature is devoted to introduce suitable definition of $\mathcal{L}$ (for instance, see [30] and references therein).

Now, by adopting (2.8), the steady states are, in general, not preserved by such a reconstruction. Indeed, in order to recover the expected well-balancedness property, we require

$$w^{-}_{i+\frac{1}{2}} = w^n_i \quad \text{and} \quad w^{+}_{i+\frac{1}{2}} = w^n_{i+1} \quad \text{for all } i \in \mathbb{Z}, \tag{2.9}$$

as soon as $(w^n_i)_{i\in\mathbb{Z}}$ defines a steady state according to (2.3). Except for linear steady states, the steady condition (2.9) is lost. As a consequence, a particular attention must be paid on the definition of $\mathcal{L}$ to preserves the steady states.

Currently, the reconstruction on G_ℓ instead of w is preferred (see [2, 24]). For a fixed ℓ, denoted by $\ell^\star$, we perform the reconstruction as follows:

$$
\begin{aligned}
G^-_{i+\frac12} &= G_{\ell^\star}(w_i^n, x_i) \\
&\quad + \frac12 \mathcal{L}\Big(G_{\ell^\star}(w_i^n, x_i) - G_{\ell^\star}(w_{i-1}^n, x_{i-1}), G_{\ell^\star}(w_{i+1}^n, x_{i+1}) - G_{\ell^\star}(w_i^n, x_i)\Big), \\
G^+_{i+\frac12} &= G_{\ell^\star}(w_{i+1}^n, x_{i+1}) \\
&\quad - \frac12 \mathcal{L}\Big(G_{\ell^\star}(w_{i+1}^n, x_{i+1}) - G_{\ell^\star}(w_i^n, x_i), G_{\ell^\star}(w_{i+2}^n, x_{i+2}) - G_{\ell^\star}(w_{i+1}^n, x_{i+1})\Big),
\end{aligned}
$$

The reconstructed states $w^\pm_{i+\frac12}$ at the interface $x_{i+\frac12}$ are then defined by

$$
\begin{cases}
G_{\ell^\star}(w^-_{i+\frac12}, x_{i+\frac12}) = G^-_{i+\frac12}, \\
G_{\ell^\star}(w^+_{i+\frac12}, x_{i+\frac12}) = G^+_{i+\frac12}.
\end{cases}
\tag{2.10}
$$

We immediately remark that (2.9) holds as soon as $(w_i^n)_{i\in\mathbb{Z}}$ defines a steady state for $G_{\ell^\star}$ according to (2.3). However, the function $w \mapsto G_{\ell^\star}(w, x)$ must be inverted. Such a procedure may be very costly, or even impossible to carry out if $G_{\ell^\star}(., x)$ is not invertible.

In fact, in the simpler situation of the lake at rest for the shallow water equation, where G is given by (1.8), we have to solve a linear 2×2 system. But for a moving steady state, i.e. with G defined by (1.7), the uniqueness of the reconstructed states $w^\pm_{i+\frac12}$ is no longer ensured. Next, considering (1.10), neither the existence nor the uniqueness of $w^\pm_{i+\frac12}$ is ensured.

Moreover, adopting such a procedure needs to fix ℓ. As a consequence, it is not possible to deal with steady states governed by several families $G_\ell(w, x)$ with $1 \le \ell \le L$ for $L \ge 2$. Such a restriction arises for instance for the Euler equations with gravity, where we consider three steady state families.

To summarize the failure of the usual well-balanced reconstruction technique, the reconstructed states, solution of (2.10), may not exist or not be unique. Moreover, since we have to solve a nonlinear system, the evaluation of the reconstructed states turns out to be computationally expensive. In addition, such a reconstruction technique preserves only one steady state family while some systems involve several steady state families.

3. A very easy well-balanced MUSCL reconstruction.

The objective is now to derive a reconstruction technique able to preserve the steady states but never involving an inversion of G_ℓ. To address such an issue, we suggest to improve the usual reconstruction (2.8) as follows:

$$
\begin{aligned}
\tilde w^-_{i+\frac12} &= w_i^n + \frac12 \theta^n_{i+\frac12} \mathcal{L}(w_i^n - w_{i-1}^n, w_{i+1}^n - w_i^n), \\
\tilde w^+_{i+\frac12} &= w_{i+1}^n - \frac12 \theta^n_{i+\frac12} \mathcal{L}(w_{i+1}^n - w_i^n, w_{i+2}^n - w_{i+1}^n),
\end{aligned}
\tag{3.1}
$$

where the correction $\theta^n_{i+\frac12}$ must be an approximation of 1 at least with second-order of accuracy which vanishes for pairs (w_i^n, w_{i+1}^n) satisfying (2.3). We propose the

following formulation of $\theta^n_{i+\frac{1}{2}}$:

$$\theta^n_{i+\frac{1}{2}} = \frac{\varepsilon^n_{i+\frac{1}{2}}}{\varepsilon^n_{i+\frac{1}{2}} + \left(\dfrac{\Delta x}{C^n_{i+\frac{1}{2}}}\right)^k}, \text{ with} \tag{3.2}$$

$$\varepsilon^n_{i+\frac{1}{2}} = \prod_{\ell=1}^{L} \Big(\|G_\ell(w^n_{i+1}, x_{i+1}) - G_\ell(w^n_i, x_i)\| + \|w^n_{i+1} - \Pi^{eq}_\ell(w^n_{i+1})\| + \|w^n_i - \Pi^{eq}_\ell(w^n_i)\| \Big), \tag{3.3}$$

where $k \geq 2$ must be selected and where $C^n_{i+\frac{1}{2}} \neq 0$ is any expression independent from Δx. We shall suggest an expression of $C^n_{i+\frac{1}{2}}$ in the numerical experiments. From now on, it is worth noting that $\varepsilon^n_{i+\frac{1}{2}} = 0$ if and only if the pair (w^n_i, w^n_{i+1}) defines a local steady state, according to (2.3), at the interface $x_{i+\frac{1}{2}}$.

Concerning the source term discretization, we adopt the following definition:

$$\tilde{S}^{\pm}_i = \frac{1}{2}\left(\left(1-\theta^n_{i-\frac{1}{2}}\right) S^n_{i-\frac{1}{2}} + \left(1-\theta^n_{i+\frac{1}{2}}\right) S^n_{i+\frac{1}{2}}\right) + \frac{1}{2}\left(\theta^n_{i-\frac{1}{2}} + \theta^n_{i+\frac{1}{2}}\right) S^{\pm}_i, \tag{3.4}$$

where $S^{\pm}_i$ is given by (2.7) and where $S^n_{i\pm\frac{1}{2}}$ comes from the first-order discretization (2.1). As a consequence, the second-order MUSCL scheme now reads

$$w^{n+1}_i = w^n_i - \frac{\Delta t}{\Delta x}\left(f^{\pm}_{i+\frac{1}{2}} - f^{\pm}_{i-\frac{1}{2}}\right) + \Delta t \tilde{S}^{\pm}_i, \tag{3.5}$$

where we have set

$$f^{\pm}_{i+\frac{1}{2}} = f_\Delta(\tilde{w}^-_{i+\frac{1}{2}}, \tilde{w}^+_{i+\frac{1}{2}}). \tag{3.6}$$

Before we establish the main properties satisfied by the second-order MUSCL scheme (3.5) – (3.6) with the reconstructed states (3.1) and the source term discretization (3.4), let us recall the definition of the order of accuracy that is adopted here (for instance, see [11]).

DEFINITION 3.1. *For some smooth solution $w(x,t)$ of (1.1), let us consider*

$$w^n_i = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x,t^n)\,dx. \tag{3.7}$$

Define w^{n+1}_i by (2.5). The scheme (2.5) is said of order τ in time and δ in space if, for all i in $\mathbb{Z}$, we have

$$w^{n+1}_i = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x,t^n+\Delta t)\,dx - \frac{\Delta t}{\Delta x}\left(\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}}\right) + \Delta t \mathcal{S}_i, \tag{3.8}$$

where $\mathcal{F}_{i+\frac{1}{2}} = \mathcal{O}(\Delta t^\tau) + \mathcal{O}(\Delta x^\delta)$ and $\mathcal{S}_i = \mathcal{O}(\Delta t^\tau) + \mathcal{O}(\Delta x^\delta)$.

Now, arguing the above definition of the order of accuracy, the improved reconstruction technique based on $\theta_{i+\frac{1}{2}}$ is established to yield a second-order accurate and well-balanced scheme.

THEOREM 3.2. *The scheme (3.5) – (3.6), with reconstructed states given by (3.1) and a source term discretization given by (3.4), satisfies the following properties:*

(i) it is second-order accurate in space for unsteady solutions;

(ii) it is well-balanced, i.e. it exactly preserves steady solutions: if $(w_i^n)_{i\in\mathbb{Z}}$ defines a steady state according to (2.3), then $w_i^{n+1} = w_i^n$ for all i in $\mathbb{Z}$;

(iii) it is robust, i.e. if the original reconstruction (2.8) preserves Ω, then Ω remains invariant by the improved reconstruction (3.1).

Proof. We establish properties (i), (ii) and (iii) in order.

(i) To establish the order of accuracy, let us consider $w(x,t)$ a smooth unsteady solution of (1.1). By integration of (1.1) over $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (t^n, t^n + \Delta t)$, we get

(3.9)
$$\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x,t^n+\Delta t)\,dx - \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x,t^n)\,dx + \frac{\Delta t}{\Delta x}\left(\frac{1}{\Delta t}\int_{t^n}^{t^n+\Delta t} f\left(w(x_{i+\frac{1}{2}},t)\right)\,dt - \frac{1}{\Delta t}\int_{t^n}^{t^n+\Delta t} f\left(w(x_{i-\frac{1}{2}},t)\right)\,dt\right) = \Delta t \frac{1}{\Delta t \Delta x}\int_{t^n}^{t^n+\Delta t}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} S(w(x,t),x)\,dx\,dt.$$

With $(w_i^n)_{i\in\mathbb{Z}}$ given by (3.7) and $(w_i^{n+1})_{i\in\mathbb{Z}}$ given by (3.5) – (3.6), a straightforward computation gives the expected relation (3.8), where we have set

$$\mathcal{F}_{i+\frac{1}{2}} = f^{\pm}_{i+\frac{1}{2}} - \frac{1}{\Delta t}\int_0^{\Delta t} f\left(w(x_{i+\frac{1}{2}},t^n+t)\right)\,dt,$$

$$\mathcal{S}_i = \tilde{S}_i^{\pm} - \frac{1}{\Delta t \Delta x}\int_0^{\Delta t}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} S(w(x,t^n+t),x)\,dx\,dt,$$

where $f^{\pm}_{i+\frac{1}{2}}$ and $\tilde{S}_i^{\pm}$ are respectively given by (3.6) and (3.4).

We first treat the approximation of the flux function. By definition of $\theta^n_{i+\frac{1}{2}}$, given by (3.2), as long as $\varepsilon^n_{i+\frac{1}{2}}$ does not vanish, we have $\theta^n_{i+\frac{1}{2}} = 1 + \mathcal{O}(\Delta x^k)$. As a consequence, in the current unsteady context, we get

$$\tilde{w}^-_{i+\frac{1}{2}} = w^-_{i+\frac{1}{2}} + \mathcal{O}(\Delta x^k) \quad \text{and} \quad \tilde{w}^+_{i+\frac{1}{2}} = w^+_{i+\frac{1}{2}} + \mathcal{O}(\Delta x^k),$$

where $w^{\pm}_{i+\frac{1}{2}}$ are given by (2.8), and with $k \geq 2$. Since (2.4) holds for the second-order polynomial reconstruction, we immediately obtain

$$\tilde{w}^-_{i+\frac{1}{2}} = w(x_{i+\frac{1}{2}},t^n) + \mathcal{O}(\Delta x^2) \quad \text{and} \quad \tilde{w}^+_{i+\frac{1}{2}} = w(x_{i+\frac{1}{2}},t^n) + \mathcal{O}(\Delta x^2).$$

Assuming a Lipschitz-continuous numerical flux function such that the consistency condition (2.2) holds, we have

$$f^{\pm}_{i+\frac{1}{2}} = f(w(x_{i+\frac{1}{2}},t^n)) + \mathcal{O}(\Delta x^2),$$

and we get $\mathcal{F}_{i+\frac{1}{2}} = \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2)$.

Next, we study the accuracy of the source term discretization. By definition of the source term reconstruction (3.4), we have

$$\tilde{S}_i^{\pm} = S_i^{\pm} + \mathcal{O}(\Delta x^2),$$

and arguing (2.6) immediately yields $\mathcal{S}_i = \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t)$.
Arguing Definition 3.1, the second-order space accuracy is thus established.

(ii) Concerning the preservation of the steady states, as soon as $(w_i^n)_{i\in\mathbb{Z}}$ satisfy (2.3), we easily get $\varepsilon^n_{i+\frac{1}{2}} = 0$ for all i in $\mathbb{Z}$. As a consequence, we have $\theta^n_{i+\frac{1}{2}} = 0$, which leads to

$$\tilde{w}^-_{i+\frac{1}{2}} = w_i^n \text{ and } \tilde{w}^+_{i+\frac{1}{2}} = w^n_{i+1},$$

while $\tilde{S}_i^{\pm} = \frac{1}{2}(S^n_{i-\frac{1}{2}} + S^n_{i+\frac{1}{2}})$. Put in other words, the reconstruction vanishes for steady states. Then, the original well-balanced first-order scheme (2.1) is recovered and the steady states are preserved.

(iii) We finally turn to the robustness of the improved reconstructed states $\tilde{w}^{\pm}_{i+\frac{1}{2}}$. We remark that

$$\tilde{w}^-_{i+\frac{1}{2}} = (1-\theta^n_{i+\frac{1}{2}})w_i^n + \theta^n_{i+\frac{1}{2}} w^-_{i+\frac{1}{2}} \text{ and } \tilde{w}^+_{i+\frac{1}{2}} = (1-\theta^n_{i+\frac{1}{2}})w^n_{i+1} + \theta^n_{i+\frac{1}{2}} w^+_{i+\frac{1}{2}},$$

where $\theta^n_{i+\frac{1}{2}}$, defined by (3.2), belongs to $[0,1]$, and where $w^{\pm}_{i+\frac{1}{2}}$ are given by the initial reconstruction (2.8). Since the states w_i^n, w^n_{i+1} and $w^{\pm}_{i+\frac{1}{2}}$ belong to Ω, the states $\tilde{w}^{\pm}_{i+\frac{1}{2}}$ turn out to be convex combinations of states in Ω. With Ω a convex set, we immediately deduce that $\tilde{w}^{\pm}_{i+\frac{1}{2}}$ are in Ω.

The proof is thus completed. ∎

To conclude this section, we emphasize that we have designed a well-balanced reconstruction procedure by only evaluating $(G_\ell(w_i^n, x_i))_{1\le\ell\le L}$ and never solving some nonlinear equations. Moreover, the introduced procedure simultaneously deals with all the involved families of steady states and it is not necessary to give more importance to one than to another.

4. Well-balanced high-order extension. The above well-balanced improvement for the second-order MUSCL scheme is easily extended to yield a well-balanced and high-order accurate scheme. To address such an issue, let us first introduce a high-order reconstruction according to existing works, see for instance [20, 21]. With $w(x,t)$ a given smooth function, we define w_i^n by adopting (3.7). Now, we consider the following polynomial reconstruction of degree d in space:

$$p_w^n(x;i) = w_i^n + \pi_i^w(x - x_i), \tag{4.1}$$

where π_i^w is a polynomial function of degree d such that, for all $x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, we have

$$p_w^n(x;i) = w(x,t^n) + \mathcal{O}(\Delta x^{d+1}) \quad \text{and} \quad \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} p_w^n(x;i)\,dx = w_i^n. \tag{4.2}$$

Equipped with this reconstruction of degree d, a scheme of space order $\delta = d+1$ is derived. The reader is referred to [20, 21] where such reconstruction techniques are derived.

From this high-order reconstruction, we now give the associated high-order well-balanced scheme to approximate the weak solutions of (1.1) as follows:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(f^{\pm}_{i+\frac{1}{2}} - f^{\pm}_{i-\frac{1}{2}}\right) + \Delta t \bar{S}_i^{\pm}, \tag{4.3}$$

with the numerical flux function given by (3.6), with

$$w^{-}_{i+\frac{1}{2}} = (\tilde{p}_w^n)_i^{+} \quad \text{and} \quad w^{+}_{i+\frac{1}{2}} = (\tilde{p}_w^n)_{i+1}^{-}, \tag{4.4}$$

where $(\tilde{p}_w^n)_i^{+}$ is the following well-balanced modification of the high-order polynomial reconstruction (4.1) evaluated at the interface point $x_{i\pm\frac{1}{2}}$:

$$(\tilde{p}_w^n)_i^{\pm} = w_i^n + \theta^n_{i\pm\frac{1}{2}} \pi_i^w\left(\pm\frac{\Delta x}{2}\right), \tag{4.5}$$

with $\theta^n_{i+\frac{1}{2}}$ defined by (3.2) – (3.3). In (3.2), the parameter k must be fixed larger than $\delta = d+1$ in order to preserve the order δ of the polynomial reconstruction. Concerning the source term approximation, we start with an approximation of order δ of the source term average, as follows:

$$S_i^{\pm} = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} S(w(x,t^n),x)\,dx + \mathcal{O}(\Delta x^{\delta}). \tag{4.6}$$

In practice, this approximation is nothing but a quadrature formula of order δ, see for instance [1]. We then adopt the following well-balanced modification of this approximation:

$$\bar{S}_i^{\pm} = \frac{1}{2}\left(\left(1-\theta^n_{i-\frac{1}{2}}\right)S^n_{i-\frac{1}{2}} + \left(1-\theta^n_{i+\frac{1}{2}}\right)S^n_{i+\frac{1}{2}}\right) + \frac{1}{2}\left(\theta^n_{i-\frac{1}{2}} + \theta^n_{i+\frac{1}{2}}\right)S_i^{\pm}. \tag{4.7}$$

At this level, it is worth noting that the δ-order numerical scheme designed here is obtained arguing a very easy modification (4.5) and (4.7) of any polynomial reconstruction (4.1) of degree d and any source term integration (4.6) of order δ. However, this minor correction ensures that the scheme is well-balanced, is of order δ in space, and preserves the set of admissible states as soon as the original high-order scheme does.

THEOREM 4.1. *The scheme (4.3), with the reconstructed states given by (4.4) and the source term approximation (4.7), satisfies the following properties:*

(i) it is of order $\delta = d+1$ in space for unsteady solutions;

(ii) it is well-balanced, i.e. it exactly preserves steady solutions: $w_i^{n+1} = w_i^n$ for all $i \in \mathbb{Z}$ if $(w_i^n)_{i\in\mathbb{Z}}$ define a steady state according to (2.3);

(iii) it is robust, i.e. if the original reconstruction (4.1) of degree d preserves Ω, then Ω remains invariant by the well-balanced improvement of the reconstruction (4.5).

Proof. We establish properties (i), (ii) and (iii) in order.

(i) We first establish the order of accuracy, as defined in Definition 3.1. To address such an issue, we consider $w(x,t)$ a smooth unsteady solution of (1.1) so that the relation (3.9) holds. Next, with $(w_i^n)_{i\in\mathbb{Z}}$ given by (3.7) and $(w_i^{n+1})_{i\in\mathbb{Z}}$

given by the high-order scheme (4.3), the relation (3.8) holds for

$$\mathcal{F}_{i+\frac{1}{2}} = f^{\pm}_{i+\frac{1}{2}} - \frac{1}{\Delta t}\int_0^{\Delta t} f(w(x_{i+\frac{1}{2}}, t^n + t))\, dt,$$

$$\mathcal{S}_i = \bar{S}^{\pm}_i - \frac{1}{\Delta t \Delta x}\int_0^{\Delta t}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} S(w(x, t^n + t), x)\, dx\, dt,$$

where $\bar{S}^{\pm}_i$ is defined by (4.7) and $f^{\pm}_{i+\frac{1}{2}}$ by (3.6), with the high-order reconstructed states $w^{\pm}_{i+\frac{1}{2}}$ given by (4.4).

Next, we establish the order of accuracy associated with the numerical flux function. First, because of the definition (3.2) of $\theta^n_{i+\frac{1}{2}}$, as long as $\varepsilon^n_{i+\frac{1}{2}}$ does not vanish, a Taylor expansion yields

$$\theta^n_{i+\frac{1}{2}} = 1 + \mathcal{O}(\Delta x^k), \qquad \text{with } k \geq \delta = d + 1.$$

As a consequence, in the current unsteady context, by definition of the polynomial reconstruction according to (4.2), we obtain

$$w^{\pm}_{i+\frac{1}{2}} = w(x_{i+\frac{1}{2}}, t^n) + \mathcal{O}(\Delta x^{\delta}).$$

Next, from (2.2), we know that the numerical flux function is Lipschitz-continuous and consistent. Therefore,

$$f_{\Delta}(w^{-}_{i+\frac{1}{2}}, w^{+}_{i+\frac{1}{2}}) = f(w(x_{i+\frac{1}{2}}, t^n)) + \mathcal{O}(\Delta x^{\delta}),$$

and we get $\mathcal{F}_{i+\frac{1}{2}} = \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^{\delta})$.

Concerning the order of accuracy of the source term, since (4.7) reduces to $\bar{S}^{\pm}_i = S^{\pm}_i + \mathcal{O}(\Delta x^{\delta})$, arguing (4.6) yields

$$\bar{S}^{\pm}_i = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} S(w(x, t^n), x)\, dx + \mathcal{O}(\Delta x^{\delta}). \tag{4.8}$$

Plugging (4.8) into the definition of $\mathcal{S}_i$, we get $\mathcal{S}_i = \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^{\delta})$.

The establishment of the order of accuracy is thus completed.

(ii) For the proof of the well-balancedness property, let us consider $(w^n_i)_{i\in\mathbb{Z}}$ to define a steady state according to (2.3). By definition of the correction, given by (3.2) – (3.3), we easily obtain $\theta^n_{i+\frac{1}{2}} = 0$ for all i in $\mathbb{Z}$ so that the reconstructed states now read

$$w^{-}_{i+\frac{1}{2}} = w^n_i \text{ and } w^{+}_{i+\frac{1}{2}} = w^n_{i+1}.$$

Similarly, regarding the source term reconstruction given by (4.7), we now have $\bar{S}^{\pm}_i = \frac{1}{2}(S^n_{i-\frac{1}{2}} + S^n_{i+\frac{1}{2}})$. As a consequence, the high-order scheme (4.3) coincides with the first-order well-balanced scheme (2.1), and the preservation of the steady states immediately follows.

(iii) To conclude the proof, we now establish that the improvement (4.5) preserves the convex set Ω as long as the original polynomial reconstruction (4.1) preserves Ω. Indeed, we have

$$(\tilde{p}^n_w)^{\pm}_i = \left(1 - \theta^n_{i\pm\frac{1}{2}}\right) w^n_i + \theta^n_{i\pm\frac{1}{2}} p^n_w\left(x \pm \frac{\Delta x}{2}; i\right).$$

Since $p_w^n(x;i) \in \Omega$ for all $x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, $w_i^n \in \Omega$ and $\theta_i^n \in [0,1]$, we immediately get $(\tilde{p}_w^n)_i^{\pm} \in \Omega$.

The proof is thus completed. ∎

5. Numerical experiments. To assess the performance of the scheme developed above, we now perform several numerical experiments. First, we describe in subsection 5.1 the setup used to assess both the order of accuracy and the well-balancedness property of the schemes under consideration. Then, we apply the high-order well-balanced strategy to several systems, namely the shallow water equations with topography (1.4) in subsection 5.2, the shallow water equations with friction (1.9) in subsection 5.3, and the Euler equations with gravity (1.11) in subsection 5.4.

5.1. Setup. To justify the relevance of the procedure outlined in Sections 3 and 4, we wish to compare the results of a given first-order well-balanced scheme to the ones produced by its second-order and high-order extensions, with and without the well-balancedness correction. For the sake of simplicity, we introduce the following notations:

- the $\mathbb{P}_d$ scheme is the scheme of order $d+1$ without the well-balancedness correction,
- the $\mathbb{P}_d^{\mathrm{WB}}$ scheme is the scheme of order $d+1$ with the well-balancedness correction.

Note that the $\mathbb{P}_0$ and $\mathbb{P}_0^{\mathrm{WB}}$ schemes are identical. Furthermore, note that forcing $\theta = 1$ on the whole space-time domain in the $\mathbb{P}_d^{\mathrm{WB}}$ scheme is enough to yield the $\mathbb{P}_d$ scheme. In this paper, we consider high-order schemes up to a third-order accurate $\mathbb{P}_2$ scheme. This is enough to justify both the high-order accuracy and the steady state preservation.

To use the $\mathbb{P}_d$ scheme, we need to define three elements: the polynomial reconstruction from (4.1), the approximation of the source term average from (4.6), and the time integration. These elements are summarized in Table 1.

TABLE 1
Polynomial reconstruction from (4.1), source term average from (4.6), and time integrator for the $\mathbb{P}_1$ and $\mathbb{P}_2$ schemes.

	polynomial reconstruction	source term average	time integration
$\mathbb{P}_1$ scheme	MUSCL [41]	midpoint method	SSPRK2 [26, 27]
$\mathbb{P}_2$ scheme	third-order [39]	Simpson's method	SSPRK3 [26, 27]

Moreover, recall that the $\mathbb{P}_d^{\mathrm{WB}}$ scheme is defined up to the choice of $C_{i+\frac{1}{2}}^n$ in the definition (3.2) of $\theta_{i+\frac{1}{2}}^n$. Heuristically, a good choice uses the numerical time derivative of the solution. Let us define, for $n \geq 1$,

$$C_{i+\frac{1}{2}}^n = C_\theta \frac{1}{2} \left(\frac{\|W_{i+1}^n\| - \|W_{i+1}^{n-1}\|}{\Delta t} + \frac{\|W_i^n\| - \|W_i^{n-1}\|}{\Delta t} \right),$$

with C_θ a constant parameter, which can be interpreted as a normalization of the time derivative. The choice of C_θ depends on the numerical experiment under consideration (unless otherwise mentioned, we take $C_\theta = 1$). We also take $C_{i+\frac{1}{2}}^0 = 1$. Note that,

equipped with this definition of $C^n_{i+\frac{1}{2}}$, the expression (3.2) of $\theta^n_{i+\frac{1}{2}}$ reads:

$$\theta^n_{i+\frac{1}{2}} = \frac{\varepsilon^n_{i+\frac{1}{2}} (C^n_{i+\frac{1}{2}})^k}{\varepsilon^n_{i+\frac{1}{2}} (C^n_{i+\frac{1}{2}})^k + (\frac{\Delta x}{C_\theta})^k}.$$

Therefore, we get $\theta^n_{i+\frac{1}{2}} = 0$ if $\varepsilon^n_{i+\frac{1}{2}} = 0$ or if $C^n_{i+\frac{1}{2}} = 0$. This is justified in each case, as follows.

- If $\varepsilon^n_{i+\frac{1}{2}} = 0$, then a steady solution of the equations has been reached, and the first-order well-balanced scheme should be used to ensure the preservation of this solution. Taking $\theta^n_{i+\frac{1}{2}} = 0$ enables this behavior.
- If $C^n_{i+\frac{1}{2}} = 0$, then a local steady solution of the $\mathbb{P}_d$ scheme has been reached. Regardless of whether $\varepsilon^n_{i+\frac{1}{2}} = 0$, we should get $\theta^n_{i+\frac{1}{2}} = 0$ in this case, since a steady solution for the $\mathbb{P}_d$ scheme will not, in general, be a steady solution for the equations. Setting $\theta^n_{i+\frac{1}{2}} = 0$ for such cases perturbs the steady numerical solution and allows it to converge towards the real steady solution.

We shall perform two experiments for each system, in order to validate both the high-order accuracy and the well-balancedness property. These experiments are detailed below; system-specific parameters (such as the final physical time, for instance) will be given in the relevant sections.

In each experiment, the space domain is $(0, 1)$ and the simulation is run until some final time t_{end}. Each experiment relies on the following compactly supported $\mathcal{C}^\infty$ bump function:

$$\omega(x) = \begin{cases} \exp\left(1 - \dfrac{1}{1 - (4(x - 1/2))^2}\right) & \text{if } |x - 1/2| < 1/4, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, all errors computed in the remainder of the text are L^2 errors between the approximate solution and the exact or reference solution.

The first experiment we perform yields a measure of the order of accuracy of the schemes, and it is designed to show that the well-balancedness correction does not reduce the accuracy for unsteady solutions. To correctly measure the order of accuracy, no slope limitation is added to the $\mathbb{P}_d$ and $\mathbb{P}^{\text{WB}}_d$ schemes. Since we do not necessarily know an exact solution of the system under consideration, we compute a reference solution using a very fine grid made of 20×2^{12} cells. Then, after computing the approximate solution on a coarser dyadic grid made of $N = 20 \times 2^k$ cells, $0 \leq k < 12$, the fine solution is projected onto the coarser grid to measure the error between the reference solution and its approximation. To ensure that no shock waves form, the initial condition is smooth; its expression is given for each system. The initial condition is then evolved until the final time $t_{\text{end}} = 5 \cdot 10^{-3}$. Periodic boundary conditions are prescribed for this experiment.

The second experiment assesses the well-balancedness property. To that end, we study the dissipation of a perturbation applied to an initially steady solution. Here, we add a slope limitation to the $\mathbb{P}_d$ and $\mathbb{P}^{\text{WB}}_d$ schemes, namely the MC limiter from [30] for $d = 1$ and the limiter from [39] for $d = 2$. The initial condition is a steady solution W, computed by solving the nonlinear system (1.3). This steady solution is then perturbed using the bump function ω. Namely, each variable in W is multiplied by $(1 + \omega(x)/4)$. The steady solution is imposed on the boundaries, and the experiment

is run until the numerical solution becomes steady; this final time is given for each system. We take 50 discretization cells for this experiment.

5.2. Application: the shallow water equations with topography. The first application concerns the shallow water equations with topography (1.4). The first-order well-balanced scheme comes from [35, 10]. It contains a parameter C, set here to ∞, or rather to the upper bound of the double-precision floating-point numbers in practice. In addition, the topography function is set to $Z(x) = \omega(x)$ and we take $g = 9.81$.

5.2.1. Order of accuracy assessment. For this experiment, the initial condition is given by $h_0(x) = 2 - Z(x) + \cos^2(2\pi x)$ and $q_0(x) = \sin(2\pi x)$.

In Figure 1, we display the reference solution and the approximations given by the $\mathbb{P}_0$, $\mathbb{P}_2$ and $\mathbb{P}_2^{\mathrm{WB}}$ schemes with 40 discretization cells. We observe that the third-order schemes are very close to the reference solution, even with such a few cells.

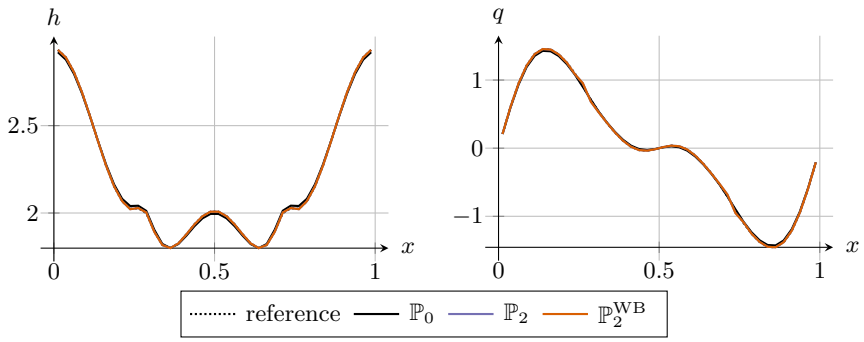

FIG. 1. *Shallow water equations with topography: comparison between the reference and approximate solutions for the dyadic experiment with* 40 *discretization cells, at time* $t_{end} = 5 \cdot 10^{-3}$. *Left panel: water height* h*; right panel: discharge* q.

This observation is confirmed in Table 2 and Figure 2, where we report the errors on h and q, as well as the orders of accuracy. As expected, the well-balancedness procedure does not alter the order of accuracy of the scheme, since the solution produced by the $\mathbb{P}_2^{\mathrm{WB}}$ scheme is almost the same as the one produced by the $\mathbb{P}_2$ scheme in this unsteady context. We even observe a slight over-convergence, possibly explained by the use of the fourth-order accurate Simpson's method in the source term approximation.

5.2.2. Well-balancedness property. The initial condition is a perturbation of the steady solution implicitly given by (1.5), with $q = 1$ and $\frac{q^2}{2h^2} + g(h + Z) = 2$. We take $C_\theta = 9 \cdot 10^{-3}$, and the final physical time is $t_{\mathrm{end}} = 20$.

In Figure 3, we display the initial condition, as well as the approximations given by the $\mathbb{P}_0$, $\mathbb{P}_2$ and $\mathbb{P}_2^{\mathrm{WB}}$ schemes at the time $t = 2.5 \cdot 10^{-2}$. We observe that the solutions of the $\mathbb{P}_2$ and $\mathbb{P}_2^{\mathrm{WB}}$ are quite close, even in this case of a perturbed steady solution, and that they are less diffusive than the solution given by the $\mathbb{P}_0$ scheme.

Then, in Figure 4 and Table 3, we report the errors on h and q at the final time t_{end}. We observe that the $\mathbb{P}_0$, $\mathbb{P}_1^{\mathrm{WB}}$ and $\mathbb{P}_2^{\mathrm{WB}}$ schemes have all converged towards the exact steady solution up to machine precision, while a non-zero error remains for the $\mathbb{P}_1$ and $\mathbb{P}_2$ schemes. These observations validate the proposed well-balancedness correction.

TABLE 2

Shallow water equations with topography: errors and orders of accuracy for the dyadic experiment. For the sake of conciseness, only the errors on h are reported in this table; the reader is referred to Figure 2 for a visualization of the errors on q.

N	error, $\mathbb{P}_0$	order, $\mathbb{P}_0$	error, $\mathbb{P}_2$	order, $\mathbb{P}_2$	error, $\mathbb{P}_2^{\mathrm{WB}}$	order, $\mathbb{P}_2^{\mathrm{WB}}$
40	$1.04 \cdot 10^{-2}$	—	$1.12 \cdot 10^{-3}$	—	$1.12 \cdot 10^{-3}$	—
80	$5.24 \cdot 10^{-3}$	0.99	$3.25 \cdot 10^{-4}$	1.78	$3.26 \cdot 10^{-4}$	1.78
160	$2.58 \cdot 10^{-3}$	1.02	$4.06 \cdot 10^{-5}$	3.00	$4.08 \cdot 10^{-5}$	3.00
320	$1.29 \cdot 10^{-3}$	1.00	$2.74 \cdot 10^{-6}$	3.89	$2.73 \cdot 10^{-6}$	3.90
640	$6.42 \cdot 10^{-4}$	1.00	$1.76 \cdot 10^{-7}$	3.96	$1.76 \cdot 10^{-7}$	3.96
1280	$3.21 \cdot 10^{-4}$	1.00	$1.34 \cdot 10^{-8}$	3.72	$1.36 \cdot 10^{-8}$	3.69
2560	$1.60 \cdot 10^{-4}$	1.00	$1.50 \cdot 10^{-9}$	3.16	$1.54 \cdot 10^{-9}$	3.14

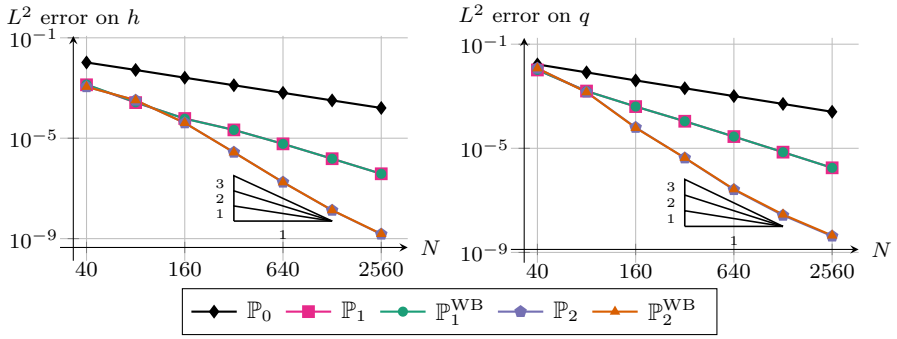

FIG. 2. *Shallow water equations with topography: error lines for the dyadic experiment. Left panel: error on h; right panel: error on q.*

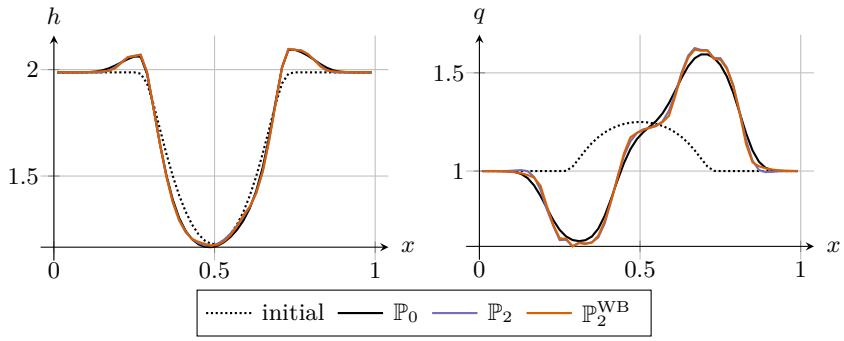

FIG. 3. *Shallow water equations with topography: comparison between the initial condition and the approximate solutions at the time $t = 2.5 \cdot 10^{-2}$, for the perturbed steady state experiment with 50 cells. Left panel: water height h; right panel: discharge q.*

5.3. Application: the shallow water equations with Manning friction. The next application concerns the shallow water equations with Manning friction, governed by (1.9). The first-order well-balanced scheme comes from [36]. It also contains a parameter C, also set here to ∞. We take the friction exponent $\eta = 7/3$ according to Manning's model [34, 18], and we set $\kappa = 1$ as well as $g = 9.81$.

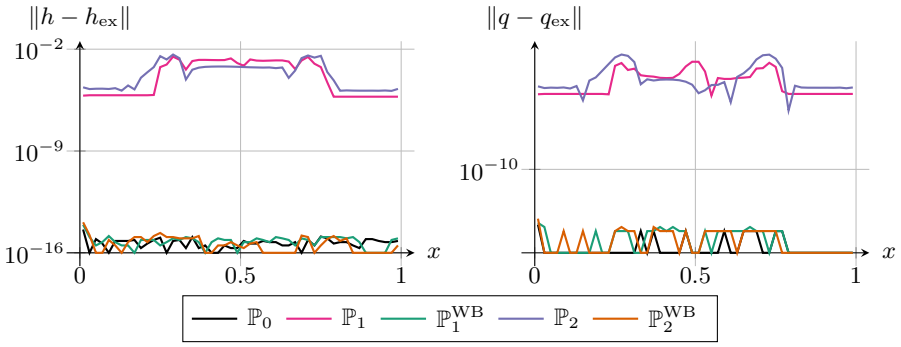

FIG. 4. *Shallow water equations with topography: errors between the steady solution and the approximate solutions at the time t_{end}, for the perturbed steady state experiment with 50 cells. Left panel: error on h; right panel: error on q.*

TABLE 3
Shallow water equations with topography: errors between the steady solution and the approximate solutions at the time t_{end}, for the perturbed steady state experiment with 50 cells.

	error, $\mathbb{P}_0$	error, $\mathbb{P}_1$	error, $\mathbb{P}_1^{WB}$	error, $\mathbb{P}_2$	error, $\mathbb{P}_2^{WB}$
h	$7.55 \cdot 10^{-16}$	$1.22 \cdot 10^{-3}$	$1.40 \cdot 10^{-15}$	$1.30 \cdot 10^{-3}$	$1.92 \cdot 10^{-15}$
q	$2.15 \cdot 10^{-15}$	$1.67 \cdot 10^{-3}$	$3.62 \cdot 10^{-15}$	$5.43 \cdot 10^{-3}$	$4.87 \cdot 10^{-15}$

5.3.1. Order of accuracy assessment. The initial condition for this experiment is given by $h_0(x) = 2 + \cos^2(2\pi x)$ and $q_0(x) = \sin(2\pi x)$.

In Figure 5, we display the reference solution and the approximations given by the $\mathbb{P}_0$, $\mathbb{P}_2$ and $\mathbb{P}_2^{WB}$ schemes with 40 discretization cells. We once again observe that the third-order schemes are very close to the reference solution.

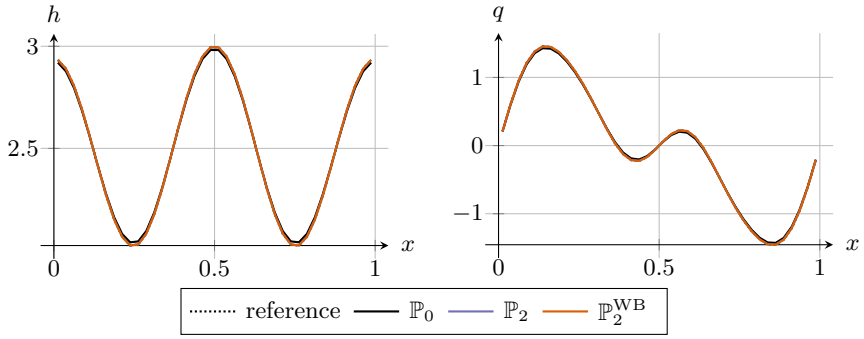

FIG. 5. *Shallow water equations with Manning friction: comparison between the reference and approximate solutions for the dyadic experiment with 40 discretization cells, at time $t_{end} = 5 \cdot 10^{-3}$. Left panel: water height h; right panel: discharge q.*

We report the errors on h and q, as well as the orders of accuracy, in Table 4 and Figure 6. The previous observation is once again confirmed since the $\mathbb{P}_d$ and $\mathbb{P}_d^{WB}$ schemes produce almost exactly the same solution for this experiment.

5.3.2. Well-balancedness property. The initial condition is a perturbation of the steady solution implicitly given by (1.2), with $q = 1$ and $-q^2 \frac{h^{\eta-1}}{\eta-1} + g\frac{h^{\eta+2}}{\eta+2} +$

TABLE 4

Shallow water equations with Manning friction: errors and orders of accuracy for the dyadic experiment. For the sake of conciseness, only the errors on h are reported in this table; the reader is referred to Figure 6 for a visualization of the errors on q.

N	error, $\mathbb{P}_0$	order, $\mathbb{P}_0$	error, $\mathbb{P}_2$	order, $\mathbb{P}_2$	error, $\mathbb{P}_2^{\mathrm{WB}}$	order, $\mathbb{P}_2^{\mathrm{WB}}$
40	$1.04 \cdot 10^{-2}$	—	$2.82 \cdot 10^{-4}$	—	$2.82 \cdot 10^{-4}$	—
80	$5.23 \cdot 10^{-3}$	1.00	$3.64 \cdot 10^{-5}$	2.95	$3.64 \cdot 10^{-5}$	2.95
160	$2.57 \cdot 10^{-3}$	1.03	$4.62 \cdot 10^{-6}$	2.98	$4.62 \cdot 10^{-6}$	2.98
320	$1.28 \cdot 10^{-3}$	1.00	$5.80 \cdot 10^{-7}$	2.99	$5.80 \cdot 10^{-7}$	2.99
640	$6.40 \cdot 10^{-4}$	1.00	$7.28 \cdot 10^{-8}$	3.00	$7.28 \cdot 10^{-8}$	3.00
1280	$3.20 \cdot 10^{-4}$	1.00	$9.11 \cdot 10^{-9}$	3.00	$9.11 \cdot 10^{-9}$	3.00
2560	$1.60 \cdot 10^{-4}$	1.00	$1.14 \cdot 10^{-9}$	3.00	$1.14 \cdot 10^{-9}$	3.00

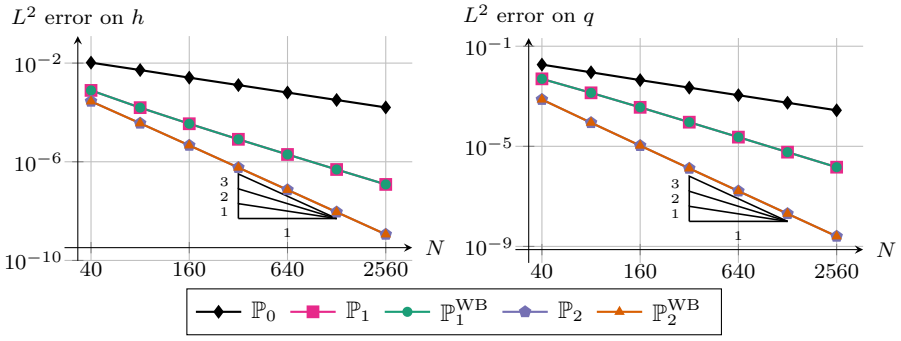

FIG. 6. *Shallow water equations with Manning friction: error lines for the dyadic experiment. Left panel: error on h; right panel: error on q.*

$kq|q|x = 3$. The final physical time is $t_{\mathrm{end}} = 20$.

We display the initial condition, as well as the approximations produced by the $\mathbb{P}_0$, $\mathbb{P}_2$ and $\mathbb{P}_2^{\mathrm{WB}}$ schemes at the time $t = 2.5 \cdot 10^{-2}$, in Figure 7. We observe that the solutions of the $\mathbb{P}_2$ and $\mathbb{P}_2^{\mathrm{WB}}$ are indistinguishable, and that they are less diffusive than the solution given by the $\mathbb{P}_0$ scheme.

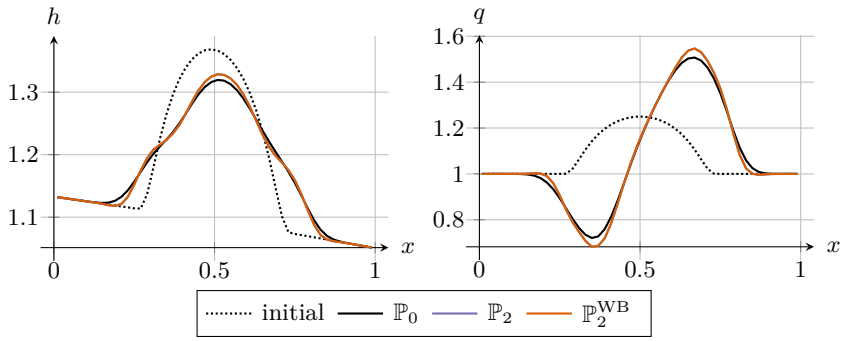

FIG. 7. *Shallow water equations with Manning friction: comparison between the initial condition and the approximate solutions at the time $t = 2.5 \cdot 10^{-2}$, for the perturbed steady state experiment with 50 cells. Left panel: water height h; right panel: discharge q.*

In Figure 8 and Table 5, the errors on h and q at the final time t_{end} are reported.

Once again, the $\mathbb{P}_0$, $\mathbb{P}_1^{\mathrm{WB}}$ and $\mathbb{P}_2^{\mathrm{WB}}$ schemes have all converged towards the exact steady solution, and the $\mathbb{P}_1$ and $\mathbb{P}_2$ schemes produce a non-zero error.

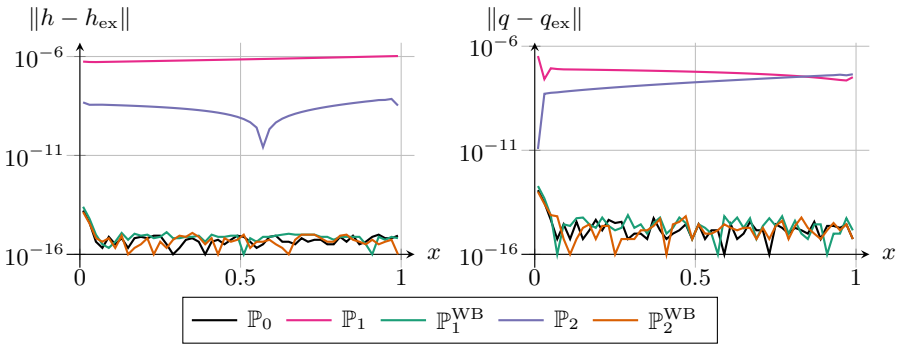

FIG. 8. *Shallow water equations with Manning friction: errors between the steady solution and the approximate solutions at the time t_{end}, for the perturbed steady state experiment with 50 cells. Left panel: error on h; right panel: error on q.*

TABLE 5

Shallow water equations with Manning friction: errors between the steady solution and the approximate solutions at the time t_{end}, for the perturbed steady state experiment with 50 cells.

	error, $\mathbb{P}_0$	error, $\mathbb{P}_1$	error, $\mathbb{P}_1^{\mathrm{WB}}$	error, $\mathbb{P}_2$	error, $\mathbb{P}_2^{\mathrm{WB}}$
h	$2.43 \cdot 10^{-15}$	$7.63 \cdot 10^{-7}$	$3.79 \cdot 10^{-15}$	$3.24 \cdot 10^{-9}$	$2.08 \cdot 10^{-15}$
q	$1.79 \cdot 10^{-14}$	$7.53 \cdot 10^{-8}$	$2.84 \cdot 10^{-14}$	$2.40 \cdot 10^{-8}$	$1.51 \cdot 10^{-14}$

5.4. Application: the Euler equations with gravity. For the last application, we turn to another system to highlight the genericness of our method. We choose the Euler equations with gravity (1.11). The first-order well-balanced scheme is based on the strategy from [35, 36, 23]. It is designed to exactly preserve and capture the family of steady states given by (1.12). We consider an ideal gas pressure law, where the pressure p is given by $p = (\gamma - 1)(E - \frac{1}{2}\frac{q^2}{\rho})$. Furthermore, we take the parameters $\gamma = 1.4$ and $\kappa = 1$. The gravity potential is given by $\varphi(x) = \omega(x)$.

5.4.1. Order of accuracy assessment. For this experiment, we take the initial condition $\rho_0(x) = 2 + \cos^2(2\pi x), q_0(x) = \sin(2\pi x), E_0(x) = 5 + \cos^2(2\pi x)$.

In Figure 9, we display the reference solution and the approximations given by the $\mathbb{P}_0$, $\mathbb{P}_2$ and $\mathbb{P}_2^{\mathrm{WB}}$ schemes with 40 discretization cells. As usual, the third-order schemes are very close to the reference solution.

We also report the errors and the orders of accuracy for ρ, q and E in Table 6 and Figure 10. The same conclusion as for the shallow water system with topography or Manning friction is drawn.

5.4.2. Well-balancedness property. The initial condition is a perturbation of the steady solution implicitly given by (1.12), with $q = 0$, $p - \kappa\rho = 1$ and $\kappa\gamma\frac{\rho^{\gamma-1}}{\gamma-1} + \varphi = 5$. The final physical time is $t_{\mathrm{end}} = 300$.

We display the initial condition, as well as the approximations produced by the $\mathbb{P}_0$, $\mathbb{P}_2$ and $\mathbb{P}_2^{\mathrm{WB}}$ schemes at the time $t = 1 \cdot 10^{-1}$, in Figure 11. The solutions of the

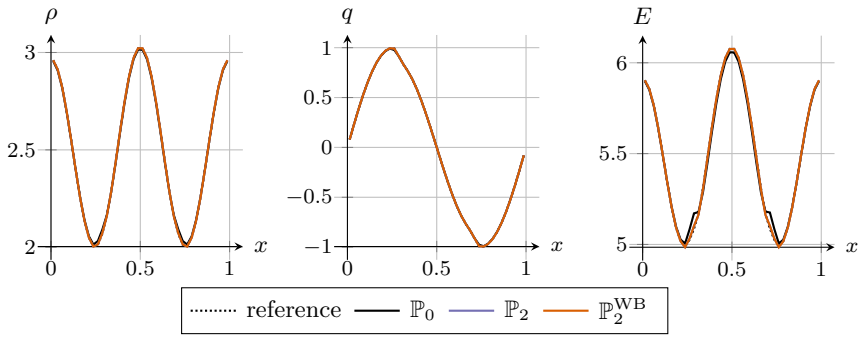

FIG. 9. *Euler equations with gravity: comparison between the reference and approximate solutions for the dyadic experiment with* 40 *discretization cells, at time* $t_{end} = 5 \cdot 10^{-3}$. *Left panel: density* ρ*; middle panel: momentum* q*; right panel: energy* E.

TABLE 6

Euler equations with gravity: errors and orders of accuracy for the dyadic experiment. For the sake of conciseness, only the errors on ρ *are reported in this table; the reader is referred to Figure 6 for a visualization of the errors on* q *and* E.

N	error, $\mathbb{P}_0$	order, $\mathbb{P}_0$	error, $\mathbb{P}_2$	order, $\mathbb{P}_2$	error, $\mathbb{P}_2^{\mathrm{WB}}$	order, $\mathbb{P}_2^{\mathrm{WB}}$
80	$4.18 \cdot 10^{-3}$	—	$2.61 \cdot 10^{-4}$	—	$2.61 \cdot 10^{-4}$	—
160	$2.07 \cdot 10^{-3}$	1.01	$5.59 \cdot 10^{-5}$	2.22	$5.59 \cdot 10^{-5}$	2.22
320	$1.03 \cdot 10^{-3}$	1.01	$1.09 \cdot 10^{-5}$	2.36	$1.09 \cdot 10^{-5}$	2.36
640	$5.14 \cdot 10^{-4}$	1.00	$1.72 \cdot 10^{-6}$	2.66	$1.72 \cdot 10^{-6}$	2.66
1280	$2.56 \cdot 10^{-4}$	1.00	$2.31 \cdot 10^{-7}$	2.89	$2.31 \cdot 10^{-7}$	2.89
2560	$1.28 \cdot 10^{-4}$	1.00	$2.95 \cdot 10^{-8}$	2.97	$2.95 \cdot 10^{-8}$	2.97
5120	$6.41 \cdot 10^{-5}$	1.00	$3.70 \cdot 10^{-9}$	2.99	$3.70 \cdot 10^{-9}$	2.99

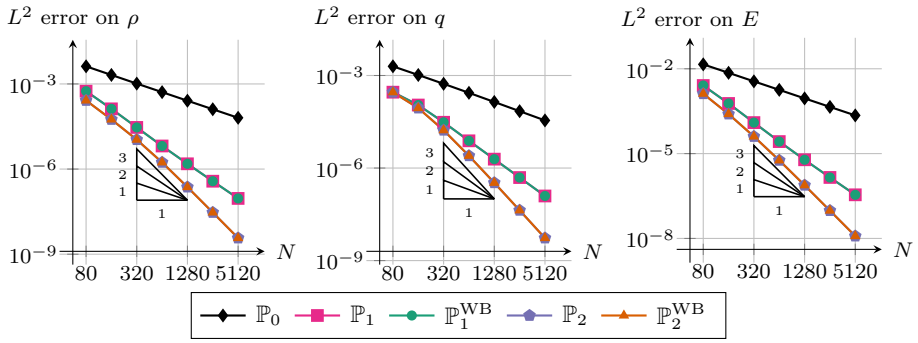

FIG. 10. *Euler equations with gravity: error lines for the dyadic experiment. Left panel: error on* ρ*; middle panel: error on* q*; right panel: error on* E.

$\mathbb{P}_2$ and $\mathbb{P}_2^{\mathrm{WB}}$ are once again indistinguishable and less diffusive than the one given by the $\mathbb{P}_0$ scheme.

In Figure 12 and Table 7, the errors on ρ, q and E at the final time t_{end} are reported. The same conclusions as in the shallow water case are reached.

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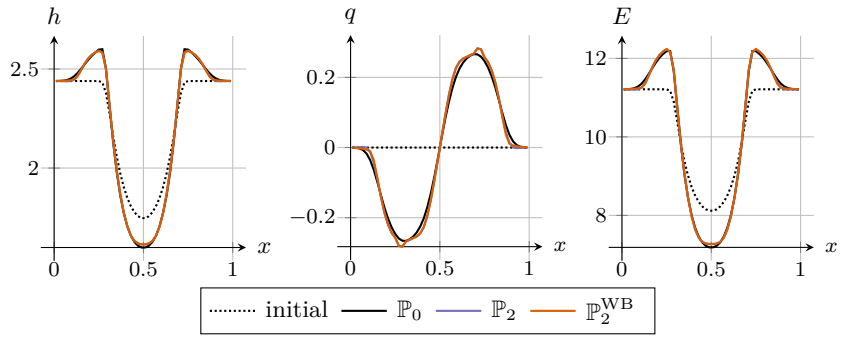

FIG. 11. *Euler equations with gravity: comparison between the initial condition and the approximate solutions at the time* $t = 1 \cdot 10^{-1}$*, for the perturbed steady state experiment with* 50 *cells. Left panel: density* ρ*; middle panel: momentum* q*; right panel: energy* E*.*

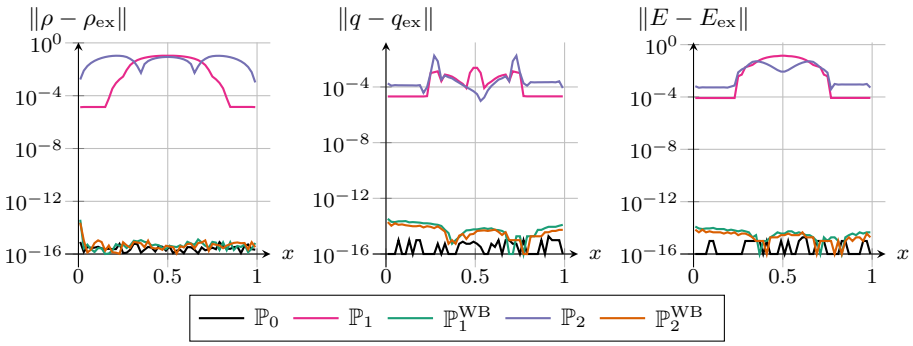

FIG. 12. *Euler equations with gravity: errors between the steady solution and the approximate solutions at the time* t_{end}*, for the perturbed steady state experiment with* 50 *cells. Left panel: error on* ρ*; middle panel: error on* q*; right panel: error on* E*.*

TABLE 7

Euler equations with gravity: errors between the steady solution and the approximate solutions at the time t_{end}*, for the perturbed steady state experiment with* 50 *cells.*

	error, $\mathbb{P}_0$	error, $\mathbb{P}_1$	error, $\mathbb{P}_1^{\mathrm{WB}}$	error, $\mathbb{P}_2$	error, $\mathbb{P}_2^{\mathrm{WB}}$
ρ	$2.64 \cdot 10^{-16}$	$5.44 \cdot 10^{-2}$	$5.75 \cdot 10^{-15}$	$6.77 \cdot 10^{-2}$	$3.42 \cdot 10^{-15}$
q	$5.60 \cdot 10^{-16}$	$7.14 \cdot 10^{-4}$	$1.20 \cdot 10^{-14}$	$3.83 \cdot 10^{-3}$	$7.01 \cdot 10^{-15}$
E	$6.47 \cdot 10^{-16}$	$6.19 \cdot 10^{-2}$	$5.12 \cdot 10^{-15}$	$2.31 \cdot 10^{-2}$	$3.05 \cdot 10^{-15}$

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