



**HAL**  
open science

# Long-time dynamics of stochastic differential equations

Nils Berglund

► **To cite this version:**

| Nils Berglund. Long-time dynamics of stochastic differential equations. 2021. hal-03271001

**HAL Id: hal-03271001**

**<https://hal.science/hal-03271001v1>**

Preprint submitted on 24 Aug 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# LONG-TIME DYNAMICS OF STOCHASTIC DIFFERENTIAL EQUATIONS

Nils Berglund

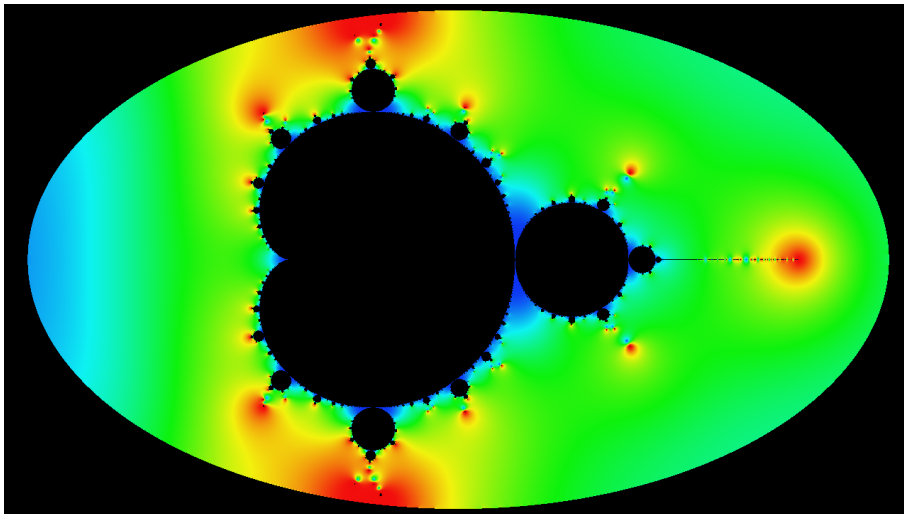
Institut Denis Poisson – UMR 7013

Universite d'Orleans, Universite de Tours, CNRS

Lecture notes

Summer School — From kinetic equations to statistical mechanics

Saint Jean de Monts, June–July 2021



— Version of August 24, 2021 —



# Contents

<b>1</b>	<b>Stochastic Differential Equations and Partial Differential Equations</b>	<b>1</b>
1.1	Brownian motion . . . . .	1
1.1.1	Construction of Brownian motion . . . . .	1
1.1.2	Basic properties of Brownian motion . . . . .	5
1.1.3	Brownian motion and heat equation . . . . .	6
1.2	Ito calculus . . . . .	9
1.2.1	Ito's integral . . . . .	9
1.2.2	Ito's formula . . . . .	11
1.2.3	Stochastic differential equations . . . . .	15
1.3	Diffusions . . . . .	18
1.3.1	The Markov property . . . . .	18
1.3.2	Semigroups and generators . . . . .	20
1.3.3	Dynkin's formula . . . . .	22
1.3.4	Kolmogorov's equations . . . . .	26
1.3.5	The Feynman–Kac formula . . . . .	29
<b>2</b>	<b>Invariant measures for SDEs</b>	<b>35</b>
2.1	Existence of invariant probability measures . . . . .	36
2.1.1	Some basic examples . . . . .	36
2.1.2	The Krylov–Bogoliubov criterion . . . . .	38
2.2	The Lyapunov function approach by Meyn and Tweedie . . . . .	38
2.2.1	Non-explosion and Harris recurrence criteria . . . . .	39
2.2.2	Positive Harris recurrence and existence of an invariant probability . . . . .	41
2.2.3	Convergence to the invariant probability measure . . . . .	42
2.2.4	A simplified proof by Hairer and Mattingly . . . . .	43
2.3	Garrett Birkhoff's approach . . . . .	49
2.3.1	Two-dimensional case . . . . .	50
2.3.2	General vector space . . . . .	52
2.3.3	Integral transformations and Jentzsch's theorem . . . . .	53
2.3.4	Banach lattices and spectral gap . . . . .	54
2.3.5	From discrete time to continuous time . . . . .	56
<b>3</b>	<b>Large deviations for stochastic differential equations</b>	<b>59</b>
3.1	Large-deviation principles . . . . .	59
3.1.1	A simple example . . . . .	59
3.1.2	Definition of large-deviation principles . . . . .	62
3.1.3	The contraction principle . . . . .	62

---

3.2	Large deviations for SDEs . . . . .	63
3.2.1	Schilder's theorem . . . . .	63
3.2.2	The Freidlin–Wentzell large-deviation principle . . . . .	66
3.3	Application to the stochastic exit problem . . . . .	69
3.3.1	The Ornstein–Uhlenbeck process . . . . .	69
3.3.2	Lagrangian and Hamiltonian formulations . . . . .	70
3.3.3	The probability of exiting a domain in a time independent of $\varepsilon$ . . . . .	72
3.3.4	Mean exit times and exit locations . . . . .	76
	<b>Bibliography</b>	<b>81</b>

# Preface

These lecture notes have been prepared for a series of lectures given at the Summer School “[From kinetic equations to statistical mechanics](#)”, organised by the Henri Lebesgue Center in Saint Jean de Monts, from June 28th to July 2nd 2021.

The author wishes to thank the organisers of the Summer School, Frédéric Hérau (Université de Nantes), Laurent Michel (Université de Bordeaux), and Karel Pravda-Starov (Université de Rennes 1) for the kind invitation that provided the motivation to compile these lecture notes.



# Stochastic Differential Equations and Partial Differential Equations

## 1.1 Brownian motion

The fundamental building block of the theory of stochastic differential equations is a mathematical object called *Wiener process*, or *Brownian motion*. This should not be confused with the physical phenomenon of Brownian motion, describing for instance the erratic movements of a small particle in a fluid, though the mathematical model has of course been introduced as a simplified description of the physical process. There is a huge literature on properties of Brownian motion. In what follows, we will focus on only a few of these properties that will be important for links between stochastic and partial differential equations.

### 1.1.1 Construction of Brownian motion

Heuristically, Brownian motion can be defined as a scaling limit of a random walk. Let  $\{X_n\}_{n \geq 0}$  be a symmetric random walk on  $\mathbb{Z}$ , defined as

$$X_n = \sum_{i=1}^n \xi_i,$$

where the  $\xi_i$  are i.i.d. (independent and identically distributed) random variables, taking values  $\pm 1$  with probability  $\frac{1}{2}$ . The following properties are easy to check:

1.  $X_n$  has zero expectation:  $\mathbb{E}[X_n] = 0$  for all  $n$ ;
2. The variance of  $X_n$  satisfies  $\text{Var}(X_n) = n$ ;
3.  $X_n$  takes values in  $\{-n, -n+2, \dots, n-2, n\}$ , with

$$\mathbb{P}\{X_n = k\} = \frac{1}{2^n} \frac{n!}{\left(\frac{n+k}{2}\right)! \left(\frac{n-k}{2}\right)!}.$$

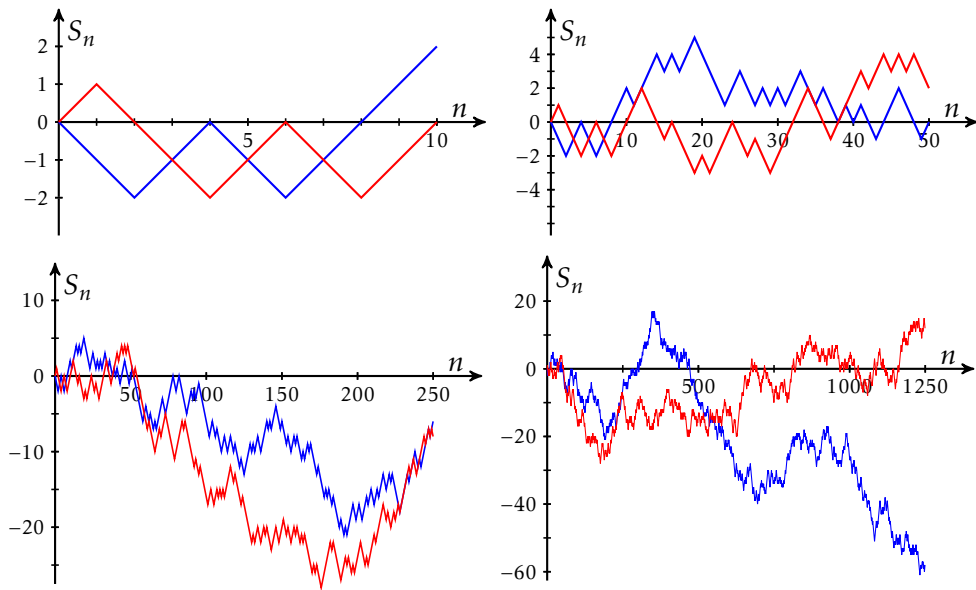
4. *Independent increments*: for all  $n > m \geq 0$ ,  $X_n - X_m$  is independent of  $X_1, \dots, X_m$ ;
5. *Stationary increments*: for all  $n > m \geq 0$ ,  $X_n - X_m$  has the same distribution as  $X_{n-m}$ .

Consider now the sequence of processes

$$W_t^{(n)} = \frac{1}{\sqrt{n}} X_{[nt]}, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

At stage  $n$ , space has been compressed by a factor  $n$ , while time has been sped up by a factor  $\sqrt{n}$  (Figure 1.1).





**Figure 1.1** – Two realisations (one in red, the other one in blue) of a symmetric random walk on  $\mathbb{Z}$ , seen at different scales. From one picture to the next, the horizontal scale is compressed by a factor 5, while the vertical scale is compressed by a factor  $\sqrt{5}$ .

Formally, as  $n \rightarrow \infty$ , the processes  $\{W_t^{(n)}\}_{t \geq 0}$  should converge to a stochastic process  $\{W_t\}_{t \geq 0}$  satisfying the following properties.

1.  $\mathbb{E}[W_t] = 0$  for all  $t \geq 0$ ;
2. The variance of  $W_t$  satisfies

$$\text{Var}(W_t) = \lim_{n \rightarrow \infty} \left( \frac{1}{\sqrt{n}} \right)^2 [nt] = t.$$

3. By the central limit theorem,  $X_{[nt]}/\sqrt{[nt]}$  converges in distribution to a standard normal random variable. Therefore, for each  $t$ ,  $W_t$  follows a normal law  $\mathcal{N}(0, t)$ .
4. *Independent increments*: for all  $t > s \geq 0$ ,  $W_t - W_s$  est independent of  $\{W_u\}_{0 \leq u \leq s}$ ;
5. *Stationary increments*: for all  $t > s \geq 0$ ,  $W_t - W_s$  has the same distribution as  $W_{t-s}$ .

This motivates the following definition.

#### Definition 1.1.1: Brownian motion

*Standard Brownian motion* (also called the *standard Wiener process*) is the stochastic process  $\{W_t\}_{t \geq 0}$  satisfying:

1.  $W_0 = 0$ ;
2. *Independent increments*: for all  $t > s \geq 0$ ,  $W_t - W_s$  est independent of  $\{W_u\}_{u \leq s}$ ;
3. *Stationary increments*: for all  $t > s \geq 0$ ,  $W_t - W_s$  follows a normal law  $\mathcal{N}(0, t - s)$ .

#### Theorem 1.1.2: Existence of Brownian motion

There exists a stochastic process  $\{W_t\}_{t \geq 0}$  satisfying Definition 1.1.1, and whose trajectories  $t \mapsto B_t(\omega)$  are continuous.

PROOF:

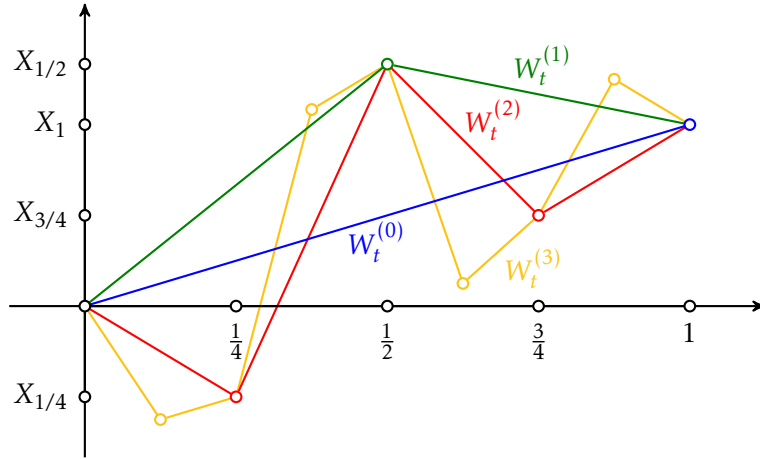


Figure 1.2 – Construction of Brownian motion by interpolation.

1. We start by constructing  $\{W_t\}_{0 \leq t \leq 1}$  from a collection of independent Gaussian random variables  $V_1, V_{1/2}, V_{1/4}, V_{3/4}, V_{1/8}, \dots$ , all with zero mean, where  $V_1$  and  $V_{1/2}$  have variance 1 and each  $V_{k2^{-n}}$  has variance  $2^{-(n-1)}$  ( $k < 2^n$  odd).

We first show that if  $X_s$  et  $X_t$  are two random variables such that  $X_t - X_s$  is centred, Gaussian with variance  $t - s$ , then there exists a random variable  $X_{(t+s)/2}$  such that the random variables  $X_t - X_{(t+s)/2}$  and  $X_{(t+s)/2} - X_s$  are i.i.d. with law  $\mathcal{N}(0, (t-s)/2)$ . If  $U = X_t - X_s$  and  $V$  is independent of  $U$ , with the same distribution, it suffices to define  $X_{(t+s)/2}$  by

$$\begin{aligned} X_t - X_{(t+s)/2} &= \frac{U + V}{2} \\ X_{(t+s)/2} - X_s &= \frac{U - V}{2}. \end{aligned} \quad (1.1.1)$$

Indeed, it is easy to check that these variables have the required distributions, and that they are independent, since  $\mathbb{E}[(U + V)(U - V)] = \mathbb{E}[U^2] - \mathbb{E}[V^2] = 0$ , and normal random variables are independent if and only if they are uncorrelated.

Let us set  $X_0 = 0$ ,  $X_1 = V_1$ , and construct  $X_{1/2}$  with the above procedure, taking  $V = V_{1/2}$ . Then we construct  $X_{1/4}$  with the help of  $X_0$ ,  $X_{1/2}$  and  $V_{1/4}$ , and so on, to obtain a family of variables  $\{X_t\}_{t=k2^{-n}, n \geq 1, k < 2^n}$  such that for  $t > s$ ,  $X_t - X_s$  is independent of  $X_s$  and has distribution  $\mathcal{N}(0, t - s)$ .

2. For  $n \geq 0$ , let  $\{W_t^{(n)}\}_{0 \leq t \leq 1}$  be the stochastic process with piecewise linear trajectories on intervals  $[k2^{-n}, (k+1)2^{-n}]$ ,  $k < 2^n$ , and such that  $W_t^{(n)}|_{k2^{-n}} = X_{k2^{-n}}$  (Figure 1.2). We want to show that the sequence  $W_t^{(n)}(\omega)$  converges uniformly on  $[0, 1]$  for any realisation  $\omega$  of the  $V_i$ . We thus have to estimate

$$\begin{aligned} \Delta^{(n)}(\omega) &= \sup_{0 \leq t \leq 1} |W_t^{(n+1)}(\omega) - W_t^{(n)}(\omega)| \\ &= \max_{0 \leq k \leq 2^{n-1}} \max_{k2^{-n} \leq t \leq (k+1)2^{-n}} |W_t^{(n+1)}(\omega) - W_t^{(n)}(\omega)| \\ &= \max_{0 \leq k \leq 2^{n-1}} \left| X_{(2k+1)2^{-(n+1)}}(\omega) - \frac{1}{2}(X_{k2^{-n}}(\omega) + X_{(k+1)2^{-n}}(\omega)) \right| \end{aligned}$$

(see Figure 1.3). The term in the absolute value is  $\frac{1}{2}V_{(2k+1)2^{-(n+1)}}$  by construction, c.f. (1.1.1),

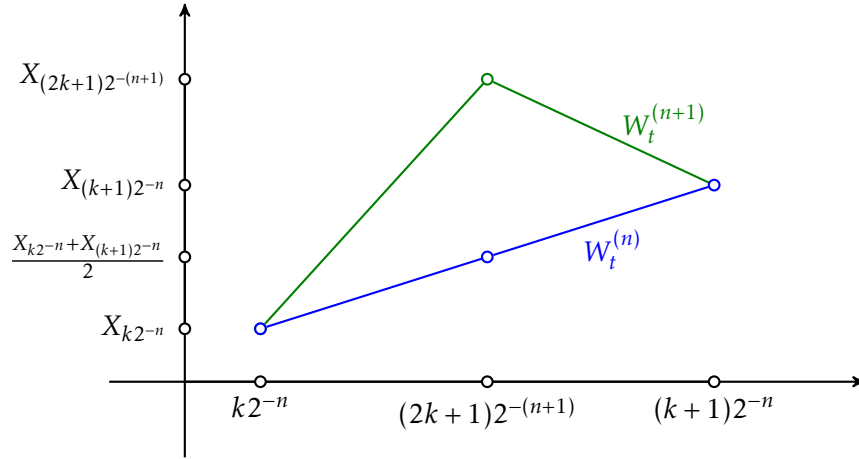


Figure 1.3 – Computation of  $\Delta^{(n)}$ .

which is Gaussian with variance  $2^{-n}$ . Therefore,

$$\begin{aligned} \mathbb{P}\{\Delta^{(n)} > \sqrt{n2^{-n}}\} &= \mathbb{P}\left\{\max_{0 \leq k \leq 2^{n-1}} |V_{(2k+1)2^{-(n+1)}}| \geq 2\sqrt{n2^{-n}}\right\} \\ &\leq 2 \cdot 2^n \int_{2\sqrt{n2^{-n}}}^{\infty} e^{-x^2/2 \cdot 2^{-n}} \frac{dx}{\sqrt{2\pi}2^{-n}} \\ &= 2 \cdot 2^n \int_{2\sqrt{n}}^{\infty} e^{-y^2/2} \frac{dx}{\sqrt{2\pi}} \leq \text{const } 2^n e^{-2n}, \end{aligned}$$

and thus

$$\sum_{n \geq 0} \mathbb{P}\{\Delta^{(n)} > \sqrt{n2^{-n}}\} \leq \text{const} \sum_{n \geq 0} (2e^{-2})^n < \infty.$$

The Borel–Cantelli lemma shows that with probability 1, there exist only finitely many  $n$  for which  $\Delta^{(n)} > \sqrt{n2^{-n}}$ . It follows that

$$\mathbb{P}\left\{\sum_{n \geq 0} \Delta^{(n)} < \infty\right\} = 1.$$

The sequence  $\{W_t^{(n)}\}_{0 \leq t \leq 1}$  is thus a Cauchy sequence for the sup norm with probability 1, and therefore converges uniformly. For  $t \in [0, 1]$  we set

$$W_t^0 = \begin{cases} \lim_{n \rightarrow \infty} W_t^{(n)} & \text{if the sequence converges uniformly} \\ 0 & \text{otherwise (with probability 0).} \end{cases}$$

It is easy to check that  $B^0$  satisfies the three properties of the definition.

3. To extend the process to all times, we build independent copies  $\{W^i\}_{i \geq 0}$  and set

$$W_t = \begin{cases} W_t^0 & 0 \leq t < 1 \\ W_1^0 + W_{t-1}^1 & 1 \leq t < 2 \\ W_1^0 + W_1^1 + W_{t-2}^2 & 2 \leq t < 3 \\ \dots & \dots \end{cases}$$

This concludes the proof. □

**Remark 1.1.3:  $n$ -dimensional Brownian motion**

For any  $n \in \mathbb{N}$ , one can define  $n$ -dimensional Brownian motion in the same way as in Definition 1.1.1, except that the normal laws are  $n$ -dimensional. Its components are then simply independent 1-dimensional Brownian motions.

**1.1.2 Basic properties of Brownian motion**

The following basic properties of Brownian motion follow more or less immediately from Definition 1.1.1.

1. *Markov property*: For any Borel set  $A \subset \mathbb{R}$ ,

$$\mathbb{P}\{W_{t+s} \in A \mid W_t = x\} = \int_A p(t+s, y|t, x) dy,$$

independently of  $\{W_u\}_{u < t}$ , with Gaussian transition probabilities

$$p(t+s, y|t, x) = \frac{e^{-(y-x)^2/2s}}{\sqrt{2\pi s}}. \quad (1.1.2)$$

The proof follows directly from the decomposition  $W_{t+s} = W_t + (W_{t+s} - W_t)$ , where the second term is independent of the first one, with distribution  $\mathcal{N}(0, s)$ . In particular, one checks the Chapman–Kolmogorov equation: For  $t > u > s$ ,

$$p(t, y|s, x) = \int_{\mathbb{R}} p(t, y|u, z)p(u, z|s, x) du. \quad (1.1.3)$$

2. *Differential property*: For all  $t \geq 0$ ,  $\{W_{t+s} - W_t\}_{s \geq 0}$  is a standard Brownian motion, independent of  $\{W_u\}_{u < t}$ .
3. *Scaling property*: For all  $c > 0$ ,  $\{cW_{t/c^2}\}_{s \geq 0}$  is a standard Brownian motion.
4. *Symmetry*:  $\{-W_t\}_{t \geq 0}$  is a standard Brownian motion.
5. *Gaussian process*: The Wiener process is Gaussian with zero mean (meaning that its finite-dimensional joint distributions  $\mathbb{P}\{W_{t_1} \leq x_1, \dots, W_{t_n} \leq x_n\}$  are centred normal), and characterised by its covariance

$$\text{Cov}\{W_t, W_s\} \equiv \mathbb{E}[W_t W_s] = s \wedge t \quad (1.1.4)$$

(where  $s \wedge t$  denotes the minimum of  $s$  and  $t$ ).

**PROOF:** For  $s < t$ , we have

$$\mathbb{E}[W_t W_s] = \mathbb{E}[W_s(W_s + W_t - W_s)] = \mathbb{E}[W_s^2] + \mathbb{E}[W_s(W_t - W_s)] = s,$$

since the second term vanishes by the independent increments property.  $\square$

In fact, one can show that a centred Gaussian process whose covariance satisfies (1.1.4) is a standard Wiener process.

One important consequence of the scaling and independent increments properties is then the following.

**Theorem 1.1.4: Non-differentiability of Brownian paths**

The paths  $t \mapsto W_t(\omega)$  are almost surely nowhere Lipschitzian, and thus nowhere differentiable.

PROOF: Fix  $C < \infty$  and introduce, for  $n \geq 1$ , the event

$$A_n = \left\{ \omega : \exists s \in [0, 1] \text{ s.t. } |W_t(\omega) - W_s(\omega)| \leq C|t - s| \text{ if } |t - s| \leq \frac{3}{n} \right\}.$$

We have to show that  $\mathbb{P}(A_n) = 0$  for all  $n$ . Observe that if  $n$  increases, the condition gets weaker, so that  $A_n \subset A_{n+1}$ . For  $n \geq 3$  and  $1 \leq k \leq n - 2$ , define

$$Y_{k,n}(\omega) = \max_{j=0,1,2} \left\{ \left| W_{(k+j)/n}(\omega) - W_{(k+j-1)/n}(\omega) \right| \right\},$$

$$B_n = \bigcup_{k=1}^{n-2} \left\{ \omega : Y_{k,n}(\omega) \leq \frac{5C}{n} \right\}.$$

The triangular inequality implies  $A_n \subset B_n$ . Indeed, let  $\omega \in A_n$ . If for instance  $s = 1$ , then for  $k = n - 2$ , one has

$$\left| W_{(n-3)/n}(\omega) - W_{(n-2)/n}(\omega) \right| \leq \left| W_{(n-3)/n}(\omega) - W_1(\omega) \right| + \left| W_1(\omega) - W_{(n-2)/n}(\omega) \right| \leq C \left( \frac{3}{n} + \frac{2}{n} \right)$$

and thus  $\omega \in B_n$ . It follows from the independent increments and scaling properties that

$$\mathbb{P}(A_n) \leq \mathbb{P}(W_n) \leq n \mathbb{P} \left( |B_{1/n}| \leq \frac{5C}{n} \right)^3 = n \mathbb{P} \left( |W_1| \leq \frac{5C}{\sqrt{n}} \right)^3 \leq n \left( \frac{10C}{\sqrt{2\pi n}} \right)^3.$$

Therefore  $\mathbb{P}(A_n) \rightarrow 0$  for all  $n \rightarrow \infty$ . But since  $\mathbb{P}(A_n) \leq \mathbb{P}(A_{n+1})$  for all  $n$ , this implies  $\mathbb{P}(A_n) = 0$  for all  $n$ .  $\square$

#### Remark 1.1.5: Hölder regularity of Brownian paths

Even though paths of Brownian motion are nowhere differentiable, one can show that they do have a regularity that is better than continuity: namely, the paths are almost surely (locally) Hölder continuous of exponent  $\alpha$  for any  $\alpha < \frac{1}{2}$ . This can be shown by applying the Kolmogorov–Centsov continuity criterion.

### 1.1.3 Brownian motion and heat equation

Observe that the Gaussian transition probabilities (1.1.3) of the Wiener process are, up to a scaling, equal to the heat kernel. In particular,  $p(t, x|0, 0)$  satisfies the heat equation

$$\frac{\partial}{\partial t} p(t, x|0, 0) = \frac{1}{2} \Delta p(t, x|0, 0)$$

$$p(0, x|0, 0) = \delta(x),$$

where we write  $\Delta$  for the second derivative with respect to  $x$ . This reflects the fact that paths of Brownian motion have the same diffusive behaviour as solutions of the heat equation.

Similarly, transition probabilities of  $n$ -dimensional Brownian motion are given by

$$p(t + s, y|t, x) = \frac{e^{-\|y-x\|^2/2s}}{(2\pi s)^{n/2}},$$

and satisfy therefore the  $n$ -dimensional heat equation.

It is, however, important to realise that Brownian motion contains much more information than the solutions of the heat equation, since it gives a probability distribution on paths  $t \mapsto W_t(\omega)$ , rather than just a collection of probability distributions for the  $W_t(\omega)$  with  $t \geq 0$ . To illustrate the difference, we discuss two examples of modifications of Brownian motion.

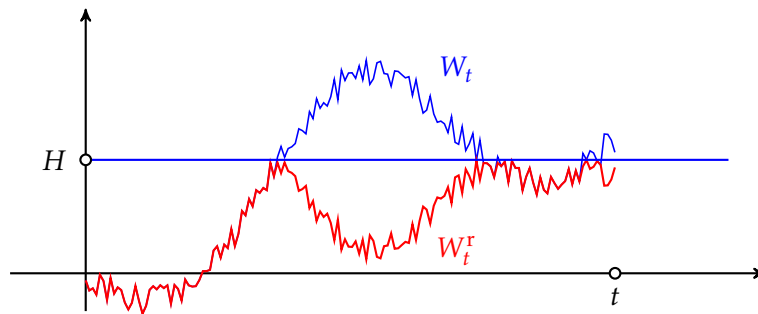


Figure 1.4 – Brownian motion reflected at level  $H$ .

### Example 1.1.6: Reflected Brownian motion

Denote by  $W_t^r$  Brownian motion reflected on the line  $x = H$  in  $(t, x)$ -space, for a constant  $H > 0$ . For any  $x \leq H$  we can write

$$\mathbb{P}\{W_t^r \leq x\} = \mathbb{P}\{W_t \leq x\} + \mathbb{P}\{W_t \geq 2H - x\}.$$

Indeed, at least heuristically, if  $W_t(\omega) \leq x$ , then one obtains a reflected path of Brownian motion from an original path simply by reflecting all parts of the path that lie above the line  $x = H$ . If  $W_t(\omega) \geq 2H - x$ , reflecting again all parts above the line also yields a reflected path ending up below  $H$ . We thus have

$$\mathbb{P}\{W_t^r \leq x\} = \Phi\left(\frac{x}{\sqrt{t}}\right) + \Phi\left(\frac{x - 2H}{\sqrt{t}}\right),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-y^2/2} dy$$

denotes the distribution function of the standard normal law. Note in particular that since  $\Phi(-x) = 1 - \Phi(x)$ , one has  $\mathbb{P}\{W_t^r \leq H\} = 1$  for all  $t \geq 0$ , as it should be. Taking the derivative with respect to  $x$ , we obtain the density

$$p^r(t, x) = \frac{1}{\sqrt{2\pi t}} \left( e^{-x^2/(2t)} + e^{-(2H-x)^2/(2t)} \right),$$

which solves

$$\begin{aligned} \frac{\partial}{\partial t} p^r(t, x) &= \frac{1}{2} \Delta p^r(t, x) & x \leq H, \\ \nabla p(t, H) &= 0, \end{aligned}$$

that is, the heat equation with Neumann boundary conditions.

To discuss our second example, we will need the so-called *reflection principle*, which we give here without proof.

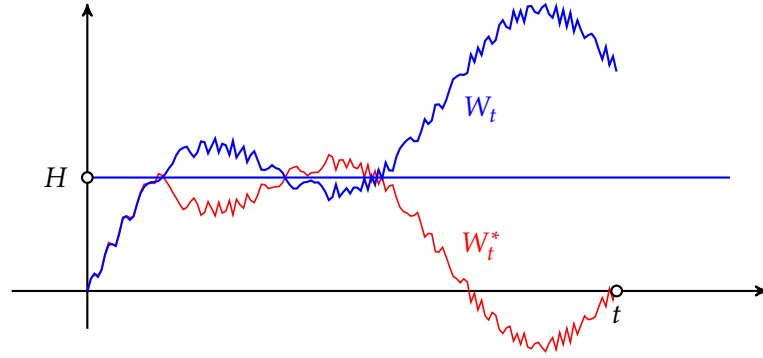


Figure 1.5 – Reflection principle.

**Proposition 1.1.7: Andre's reflection principle**

For any  $H > 0$  and setting  $\tau = \inf\{t \geq 0: W_t \geq H\}$ , the process

$$W_t^* = \begin{cases} W_t & \text{if } t \leq \tau \\ 2H - W_t & \text{if } t > \tau \end{cases}$$

is a standard Brownian motion.

**Example 1.1.8: Brownian motion killed upon reaching level  $H$** 

Denote by  $W_t^k$  Brownian motion killed at level  $H > 0$ , which is defined by

$$\mathbb{P}\{W_t^k \leq x\} = \mathbb{P}\{W_t \leq x, \tau > t\}$$

for any  $x \leq H$ . Note that  $W_t^k$  is an improper random variable for any  $t > 0$ , in the sense that the total probability is strictly smaller than 1. Since paths of Brownian motion are continuous, we can write for any  $y \geq H$

$$\begin{aligned} \mathbb{P}\{W_t \geq y\} &= \mathbb{P}\{W_t \geq y, \tau \leq t\} \\ &= \mathbb{P}\{2H - W_t \leq 2H - y, \tau \leq t\} \\ &= \mathbb{P}\{W_t^* \leq 2H - y, \tau \leq t\} \\ &= \mathbb{P}\{W_t \leq 2H - y, \tau \leq t\}. \end{aligned}$$

Setting  $y = 2H - x$ , this provides us with an expression for  $\mathbb{P}\{W_t \leq x, \tau \leq t\}$  that yields

$$\begin{aligned} \mathbb{P}\{W_t^k \leq x\} &= \mathbb{P}\{W_t \leq x\} - \mathbb{P}\{W_t \leq x, \tau \leq t\} \\ &= \mathbb{P}\{W_t \leq x\} - \mathbb{P}\{W_t \geq 2H - x\}. \end{aligned}$$

This gives the density

$$p^k(t, x) = \frac{1}{\sqrt{2\pi t}} \left( e^{-x^2/(2t)} - e^{-(2H-x)^2/(2t)} \right),$$

which solves

$$\begin{aligned}\frac{\partial}{\partial t} p^k(t, x) &= \frac{1}{2} \Delta p^k(t, x) & x \leq H, \\ p(t, H) &= 0,\end{aligned}$$

that is, the heat equation with Dirichlet boundary conditions.

### Exercise 1.1.9: 2-dimensional Brownian motion hitting a straight line

Let  $W_t = (W_t^{(1)}, W_t^{(2)})$  be a standard 2-dimensional Brownian motion, and let

$$\tau = \inf\{t > 0: W_t^{(1)} = 1\}$$

be the first-hitting time of the line  $\{x = 1\}$ . Determine the density of  $\tau$ , and use it to compute the distribution of  $W_\tau^{(2)}$ .

## 1.2 Ito calculus

While Brownian motion is a useful model with many interesting properties, one may have to deal with more general processes, such as functions of Brownian motion. The question then arises, what kinds of stochastic or partial differential equations are associated with these processes.

### Example 1.2.1: Brownian motion squared

Consider Brownian motion squared. Since

$$\mathbb{P}\{W_t^2 \leq x\} = \mathbb{P}\{|W_t| \leq \sqrt{x}\} = \Phi\left(\sqrt{\frac{x}{t}}\right) - \Phi\left(-\sqrt{\frac{x}{t}}\right),$$

we obtain that the density of  $W_t^2$  is given by

$$2 \frac{\partial}{\partial x} \Phi\left(\sqrt{\frac{x}{t}}\right) = \frac{1}{\sqrt{2\pi tx}} e^{-x/(2t)}.$$

This function should solve some PDE, which, however, is not straightforward to guess. In this section, we will develop methods that will ultimately allow to determine associated PDEs quite easily.

### 1.2.1 Ito's integral

The key notion of stochastic calculus is the Ito integral, which allows to give a meaning to the quantity

$$\int_0^t f(s) dW_s$$

for suitable, possibly random functions  $f$ . Since the Wiener process is not Lipschitzian, it does not have bounded variation, so that the above integral cannot be defined as a Riemann-Stieltjes integral. There are nowadays several (essentially equivalent) ways of defining the above integral, the oldest one going back to Kiyoshi Ito.



Fix a Brownian motion  $\{W_t\}_{t \geq 0}$ . The random variables  $\{W_s\}_{0 \leq s \leq t}$  define an increasing sequence  $\sigma$ -algebras  $\{\mathcal{F}_t\}_{t \geq 0}$  (called a *filtration*) that will play a key role in what follows. In particular, we will use the notion of random variables that are *measurable* with respect to a given  $\mathcal{F}_t$ . Intuitively, these are exactly the random variables that depend only on the behaviour of the Wiener process up to time  $t$ .

**Definition 1.2.2: Ito integral of elementary functions**

Fix a time interval  $[0, T]$ . A random function  $\{e_t\}_{t \in [0, T]}$  is called *simple* or *elementary* if there exists a partition  $0 = t_0 < t_1 < \dots < t_N = T$  of  $[0, T]$  such that

$$e_t = \sum_{k=1}^N e_{t_{k-1}} \mathbb{1}_{[t_{k-1}, t_k)}(t).$$

It is called *adapted* to the filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  if each  $e_{t_{k-1}}$  is a random variable measurable with respect to  $\mathcal{F}_{t_{k-1}}$ . For such an elementary function, *Ito's integral* is defined as

$$\int_0^t e_s dW_s = \sum_{k=1}^m e_{t_{k-1}} [W_{t_k} - W_{t_{k-1}}] + e_{t_m} [W_t - W_{t_m}] \quad (1.2.1)$$

for any  $t \in [0, T]$ , where  $m$  is such that  $t \in [t_m, t_{m+1})$ .

One easily sees that this integral is a linear functional of the integrand, and is additive with respect to time intervals. Furthermore, since each increment  $[W_{t_k} - W_{t_{k-1}}]$  is independent of  $e_{t_{k-1}}$ , the integral has zero expectation. The key property is then the following.

**Lemma 1.2.3: Ito isometry**

If  $\int_0^t \mathbb{E}[e_s^2] ds < \infty$ , then

$$\mathbb{E} \left[ \left( \int_0^t e_s dW_s \right)^2 \right] = \int_0^t \mathbb{E}[e_s^2] ds. \quad (1.2.2)$$

**PROOF:** Set  $t_{m+1} = t$ . Then

$$\begin{aligned} \mathbb{E} \left[ \left( \int_0^t e_s dW_s \right)^2 \right] &= \mathbb{E} \left[ \sum_{k,l=1}^{m+1} e_{t_{k-1}} e_{t_{l-1}} (W_{t_k} - W_{t_{k-1}})(W_{t_l} - W_{t_{l-1}}) \right] \\ &= \sum_{k=1}^{m+1} \mathbb{E}[e_{t_{k-1}}^2] \underbrace{\mathbb{E}[(W_{t_k} - W_{t_{k-1}})^2]}_{t_k - t_{k-1}} = \int_0^t \mathbb{E}[e_s^2] ds. \end{aligned}$$

We have used the property of independent increments to eliminate the terms  $k \neq l$  from the double sum, and the fact that each  $e_s$  is measurable with respect to  $\mathcal{F}_s$ .  $\square$

Ito's isometry is an isometry between the Hilbert space  $L_{\text{ad}}^2([0, T] \times \Omega, \mathbb{P})$  of adapted square-integrable processes and the Hilbert space  $L^2(\Omega, \mathbb{P})$  of square-integrable random variables. For a general square-integrable adapted process  $(X_t)_{t \geq 0}$ , one can find a sequence of elementary  $e_n$  such that

$$\lim_{n \rightarrow \infty} \int_0^T \mathbb{E}[(X_s - e_s^{(n)})^2] ds = 0.$$

The isometry (1.2.2) then shows that for any  $t \in [0, T]$ , the following limit exists in  $L^2(\mathbb{P})$ :

$$\lim_{n \rightarrow \infty} \int_0^t e_s^{(n)} dW_s =: \int_0^t X_s dW_s.$$

This is by definition the Ito integral of  $X_s$  against  $W_s$ . This integral has the same linearity and additivity properties as integrals of elementary functions, and also satisfies Ito's isometry

$$\mathbb{E} \left[ \left( \int_0^t X_s dW_s \right)^2 \right] = \int_0^t \mathbb{E}[X_s^2] ds.$$

### 1.2.2 Ito's formula

Ito's formula gives a simple answer to a question we have been asking above, namely what kind of differential relation governs functions of Brownian motion. We start by a simple example, which however contains all the essential ideas of the general case.

#### Example 1.2.4

Let us show that

$$\int_0^t W_s dW_s = \frac{1}{2} W_t^2 - \frac{t}{2}. \quad (1.2.3)$$

Define the sequence of elementary functions  $e_t^{(n)} = W_{2^{-n} \lfloor 2^n t \rfloor}$ . It is then sufficient to check that

$$\lim_{n \rightarrow \infty} \int_0^t e_s^{(n)} dW_s = \frac{1}{2} W_t^2 - \frac{t}{2}.$$

Write  $t_k = k2^{-n}$  for  $k \leq m = \lfloor 2^n t \rfloor$  and  $t_{m+1} = t$ . The definition (1.2.1) implies

$$\begin{aligned} 2 \int_0^t e_s^{(n)} dW_s &= 2 \sum_{k=1}^{m+1} W_{t_{k-1}} (W_{t_k} - W_{t_{k-1}}) \\ &= \sum_{k=1}^{m+1} [W_{t_k}^2 - W_{t_{k-1}}^2 - (W_{t_k} - W_{t_{k-1}})^2] \\ &= W_t^2 - \sum_{k=1}^{m+1} (W_{t_k} - W_{t_{k-1}})^2. \end{aligned}$$

Consider now the random variable

$$M_t^{(n)} = \sum_{k=1}^{m+1} (W_{t_k} - W_{t_{k-1}})^2 - t = \sum_{k=1}^{m+1} [(W_{t_k} - W_{t_{k-1}})^2 - (t_k - t_{k-1})]. \quad (1.2.4)$$

Since all terms of the sum are independent and have zero expectation, we obtain

$$\begin{aligned}\mathbb{E}[(M_t^{(n)})^2] &= \sum_{k=1}^{m+1} \mathbb{E}[(W_{t_k} - W_{t_{k-1}})^2 - (t_k - t_{k-1})]^2 \\ &\leq (m+1) \mathbb{E}[(W_{t_1} - W_{t_0})^2 - (t_1 - t_0)]^2 \\ &\leq \text{const } 2^n \mathbb{E}[(W_{2^{-n}})^2 - 2^{-n}]^2 \\ &= \text{const } 2^{-n} \mathbb{E}[(W_1)^2 - 1]^2 \\ &\leq \text{const } 2^{-n},\end{aligned}$$

owing to the scaling property. Therefore,  $M_t^{(n)}$  converges to zero in  $L^2$ , proving (1.2.3).

### Remark 1.2.5

Using more sophisticated tools from stochastic analysis, it is possible to prove a stronger type of convergence. Indeed,  $M_t^{(n)}$  is what is known as a submartingale, for which Doob's inequality yields

$$\mathbb{P}\left\{\sup_{0 \leq s \leq t} (M_s^{(n)})^2 > n^2 2^{-n}\right\} \leq 2^n n^{-2} \mathbb{E}[(M_t^{(n)})^2] \leq \text{const } n^{-2}.$$

The Borel–Cantelli lemma then shows that

$$\mathbb{P}\left\{\sup_{0 \leq s \leq t} |M_s^{(n)}| < n 2^{-n/2}, n \rightarrow \infty\right\} = 1,$$

proving almost sure convergence.

Consider now a stochastic integral of the form

$$X_t = X_0 + \int_0^t f_s ds + \int_0^t g_s dW_s, \quad t \in [0, T] \quad (1.2.5)$$

where  $X_0$  is a random variable independent of the Brownian motion, and  $f$  and  $g$  are two adapted processes satisfying

$$\begin{aligned}\mathbb{P}\left\{\int_0^T |f_s| ds < \infty\right\} &= 1 \\ \mathbb{P}\left\{\int_0^T g_s^2 ds < \infty\right\} &= 1.\end{aligned}$$

The process (1.2.5) can also be written in differential form as

$$dX_t = f_t dt + g_t dW_t.$$

For instance, Relation (1.2.3) is equivalent to

$$d(W_t^2) = dt + 2W_t dW_t. \quad (1.2.6)$$

Ito's formula allows to determine the effect of a change of variables on the stochastic integral (1.2.5) in a general way.

**Lemma 1.2.6: Ito's formula**

Let  $u : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $(t, x) \mapsto u(t, x)$  be continuously differentiable with respect to  $t$  and twice continuously differentiable with respect to  $x$ . Then the stochastic process  $Y_t = u(t, X_t)$  satisfies the equation

$$Y_t = Y_0 + \int_0^t \frac{\partial u}{\partial t}(s, X_s) ds + \int_0^t \frac{\partial u}{\partial x}(s, X_s) f_s ds + \int_0^t \frac{\partial u}{\partial x}(s, X_s) g_s dW_s + \frac{1}{2} \int_0^t \frac{\partial^2 u}{\partial x^2}(s, X_s) g_s^2 ds.$$

**PROOF:** It suffices to prove the result for elementary integrands, and by additivity of the integrals, one can reduce the problem to the case of constant integrands. In that case,  $X_t = f_0 t + g_0 W_t$  and  $Y_t = u(t, f_0 t + g_0 W_t)$  can be expressed as functions of  $(t, W_t)$ . It suffices thus to consider the case  $X_t = W_t$ . Now for a partition  $0 = t_0 < t_1 < \dots < t_n = t$ , one has

$$\begin{aligned} u(t, W_t) - u(0, 0) &= \sum_{k=1}^n [u(t_k, W_{t_k}) - u(t_{k-1}, W_{t_k})] + [u(t_{k-1}, W_{t_k}) - u(t_{k-1}, W_{t_{k-1}})] \\ &= \sum_{k=1}^n \frac{\partial u}{\partial t}(t_{k-1}, W_{t_k})(t_k - t_{k-1}) + \frac{\partial u}{\partial x}(t_{k-1}, W_{t_{k-1}})(W_{t_k} - W_{t_{k-1}}) \\ &\quad + \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t_{k-1}, W_{t_{k-1}})(W_{t_k} - W_{t_{k-1}})^2 + \mathcal{O}(t_k - t_{k-1}) + \mathcal{O}((W_{t_k} - W_{t_{k-1}})^2) \\ &= \int_0^t \frac{\partial u}{\partial t}(s, W_s) ds + \int_0^t \frac{\partial u}{\partial x}(s, W_s) dW_s + \frac{1}{2} \int_0^t \frac{\partial^2 u}{\partial x^2}(s, W_s) ds \\ &\quad + \sum_{k=1}^n \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t_{k-1}, W_{t_{k-1}})[(W_{t_k} - W_{t_{k-1}})^2 - (t_k - t_{k-1})] + \mathcal{O}(1). \end{aligned}$$

The sum can be dealt with as  $M_t^{(n)}$  in the above example when  $t_k - t_{k-1} \rightarrow 0$ , c.f. (1.2.4). □

**Remark 1.2.7**

1. Ito's formula can be written in differential form as

$$dY_t = \frac{\partial u}{\partial t}(t, X_t) dt + \frac{\partial u}{\partial x}(t, X_t)[f_t dt + g_t dW_t] + \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t, X_t) g_t^2 dt.$$

2. A mnemotechnic way to recover the formula is to write it in the form

$$dY_t = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} dX_t^2,$$

where  $dX_t^2$  can be computed using the rules

$$dt^2 = dt dW_t = 0, \quad dW_t^2 = dt.$$

3. The formula can be generalised to functions  $u(t, X_t^{(1)}, \dots, X_t^{(n)})$ , depending on  $n$  processes

defined by  $dX_t^{(i)} = f_t^{(i)} dt + g_t^{(i)} dW_t$ , to

$$dY_t = \frac{\partial u}{\partial t} dt + \sum_i \frac{\partial u}{\partial x_i} dX_t^{(i)} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} dX_t^{(i)} dX_t^{(j)},$$

where  $dX_t^{(i)} dX_t^{(j)} = g_t^{(i)} g_t^{(j)} dt$ .

### Example 1.2.8

1. If  $X_t = W_t$  and  $u(x) = x^2$ , one recovers Relation (1.2.6).
2. If  $dX_t = g_t dW_t - \frac{1}{2} g_t^2 dt$  and  $u(x) = e^x$ , one obtains

$$d(e^{X_t}) = g_t e^{X_t} dW_t.$$

Therefore,  $M_t = \exp\{\gamma W_t - \gamma^2 \frac{t}{2}\}$  solves the equation

$$dM_t = \gamma M_t dW_t.$$

### Exercise 1.2.9: Ornstein–Uhlenbeck process

Consider the two stochastic processes

$$X_t = \int_0^t e^s dW_s, \quad Y_t = e^{-t} X_t.$$

1. Determine  $\mathbb{E}[X_t]$ ,  $\text{Var}(X_t)$ ,  $\mathbb{E}[Y_t]$  and  $\text{Var}(Y_t)$ .
2. Specify the law of  $X_t$  and  $Y_t$ .
3. Show that  $Y_t$  converges in distribution to a random variable  $Y_\infty$  as  $t \rightarrow \infty$ , and specify its law.
4. Express  $dY_t$  as a function of  $Y_t$  and  $W_t$ .

### Exercise 1.2.10: Stratonovich integral

Let  $\{W_t\}_{t \in [0, T]}$  be a standard Brownian motion. Let  $0 = t_0 < t_1 < \dots < t_N = T$  be a partition of  $[0, T]$ , and let

$$e_t = \sum_{k=1}^N e_{t_{k-1}} \mathbb{1}_{[t_{k-1}, t_k)}(t)$$

be an elementary function, adapted to the canonical filtration of Brownian motion. The Stratonovich integral of  $e_t$  is defined by

$$\int_0^T e_t \circ dW_t = \sum_{k=1}^N \frac{e_{t_k} + e_{t_{k-1}}}{2} \Delta W_k \quad \text{where } \Delta W_k = W_{t_k} - W_{t_{k-1}}.$$

The Stratonovich integral

$$\int_0^T X_t \circ dW_t$$

of an adapted process  $X_t$  is defined as the limit of the sequence

$$\int_0^T e_t^{(n)} \circ dW_t,$$

where  $e^{(n)}$  is a sequence of elementary functions converging to  $X_t$  in  $L^2$ . Assume that this limit exists and is independent of the sequence  $e^{(n)}$ .

1. Compute

$$\int_0^T W_t \circ dW_t.$$

2. Let  $g: \mathbb{R} \rightarrow \mathbb{R}$  be a  $\mathcal{C}^2$  function, and let  $X_t$  be an adapted process satisfying

$$X_t = \int_0^t g(X_s) \circ dW_s \quad \forall t \in [0, T].$$

Let  $Y_t$  be the Ito integral

$$Y_t = \int_0^t g(X_s) dW_s.$$

Show that

$$X_t - Y_t = \frac{1}{2} \int_0^t g'(X_s) g(X_s) ds \quad \forall t \in [0, T].$$

### 1.2.3 Stochastic differential equations

A *stochastic differential equation* (SDE) is an equation of the form

$$dX_t = f(X_t, t) dt + g(X_t, t) dW_t, \quad (1.2.7)$$

where  $f, g: \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$  are deterministic measurable functions. A *strong solution* of this equation is by definition an adapted process satisfying

$$X_t = X_0 + \int_0^t f(X_s, s) ds + \int_0^t g(X_s, s) dW_s \quad (1.2.8)$$

almost surely for all  $t \in [0, T]$ , as well as the regularity conditions

$$\mathbb{P} \left\{ \int_0^T |f(X_s, s)| ds < \infty \right\} = \mathbb{P} \left\{ \int_0^T g(X_s, s)^2 ds < \infty \right\} = 1.$$

Here are two important examples of solvable SDEs.

#### Example 1.2.11: Linear SDE with additive noise

Consider the linear SDE with additive noise

$$dX_t = a(t)X_t dt + \sigma(t) dW_t, \quad (1.2.9)$$

where  $a$  and  $\sigma$  are deterministic functions. In the particular case  $\sigma \equiv 0$ , the solution can be simply written

$$X_t = e^{\alpha(t)} X_0, \quad \alpha(t) = \int_0^t a(s) ds.$$

This suggests applying the method of variation of the constant, that is, looking for a solution

of the form  $X_t = e^{\alpha(t)} Y_t$ . Ito's formula applied to  $Y_t = u(X_t, t) = e^{-\alpha(t)} X_t$  gives us

$$dY_t = -a(t)e^{-\alpha(t)} X_t dt + e^{-\alpha(t)} dX_t = e^{-\alpha(t)} \sigma(t) dW_t,$$

so that integrating and using  $Y_0 = X_0$ , one gets

$$Y_t = X_0 + \int_0^t e^{-\alpha(s)} \sigma(s) dW_s.$$

This finally gives the strong solution of equation (1.2.9)

$$X_t = X_0 e^{\alpha(t)} + \int_0^t e^{\alpha(t)-\alpha(s)} \sigma(s) dW_s.$$

One checks that this process indeed solves (1.2.8) by applying Ito's formula once again. Note in particular that if the initial condition  $X_0$  is deterministic, then  $X_t$  follows a normal law, with expectation  $\mathbb{E}[X_t] = X_0 e^{\alpha(t)}$  and variance

$$\text{Var}(X_t) = \int_0^t e^{2(\alpha(t)-\alpha(s))} \sigma(s)^2 ds,$$

as a consequence of Ito's isometry.

### Example 1.2.12: Linear SDE with multiplicative noise

Consider the linear SDE with multiplicative noise

$$dX_t = a(t)X_t dt + \sigma(t)X_t dW_t,$$

with again  $a$  and  $\sigma$  deterministic functions. We can then write

$$\frac{dX_t}{X_t} = a(t) dt + \sigma(t) dW_t.$$

Integrating the left-hand side, one should get  $\log(X_t)$ , but is this compatible with Ito calculus? To check this, set  $Y_t = u(X_t) = \log(X_t)$ . Then Ito's formula gives

$$\begin{aligned} dY_t &= \frac{1}{X_t} dX_t - \frac{1}{2X_t^2} dX_t^2 \\ &= a(t) dt + \sigma(t) dW_t - \frac{1}{2} \sigma(t)^2 dt. \end{aligned}$$

Integrating and taking the exponential, one obtains the strong solution

$$X_t = X_0 \exp \left\{ \int_0^t [a(s) - \frac{1}{2} \sigma(s)^2] ds + \int_0^t \sigma(s) dW_s \right\}.$$

In particular, if  $a \equiv 0$  and  $\sigma \equiv \gamma$ , one recovers  $X_t = X_0 \exp\{\gamma W_t - \gamma^2 t/2\}$ , which is called the *geometric (or exponential) Brownian motion*.

We now state an existence and uniqueness result of solutions for a class of SDEs.

**Theorem 1.2.13: Existence and uniqueness of a strong solution**

Assume the functions  $f$  and  $g$  satisfy the following two conditions:

1. *Global Lipschitz condition*: there exists a constant  $K$  such that

$$|f(x, t) - f(y, t)| + |g(x, t) - g(y, t)| \leq K|x - y|$$

for all  $x, y \in \mathbb{R}$  and  $t \in [0, T]$ .

2. *Bounded growth condition*: there exists a constant  $L$  such that

$$|f(x, t)| + |g(x, t)| \leq L(1 + |x|)$$

for all  $x \in \mathbb{R}$  and  $t \in [0, T]$ .

Then the SDE (1.2.7) admits, for any square-integrable initial condition  $X_0$ , a strong solution  $\{X_t\}_{t \in [0, T]}$ , which is almost surely continuous. This solution is unique in the sense that if  $\{X_t\}_{t \in [0, T]}$  and  $\{Y_t\}_{t \in [0, T]}$  are two almost surely continuous solutions, then

$$\mathbb{P}\left\{\sup_{0 \leq t \leq T} |X_t - Y_t| > 0\right\} = 0.$$

We will omit the details of the proof of this result, which is very similar to corresponding proofs in the deterministic case. Uniqueness follows by estimating the derivative of the expected difference  $\mathbb{E}[|X_t - Y_t|^2]$  and applying Gronwall's lemma, while existence is obtained by applying a fixed-point argument, or more precisely by showing that the sequence of functions

$$X_t^{(k+1)} = X_0 + \int_0^t f(X_s^{(k)}, s) ds + \int_0^t g(X_s^{(k)}, s) dW_s$$

converges to a limit which solves the SDE.

**Remark 1.2.14: Weaker conditions on drift and diffusion coefficients**

The conditions on  $f$  and  $g$  in the above result can be relaxed to the following ones:

1. *Local Lipschitz condition*: For any compact  $\mathcal{K} \in \mathbb{R}$ , there exists a constant  $K = K(\mathcal{K})$  such that

$$|f(x, t) - f(y, t)| + |g(x, t) - g(y, t)| \leq K|x - y|$$

for all  $x, y \in \mathcal{K}$  and  $t \in [0, T]$ .

2. *Bounded growth condition*: There exists a constant  $L$  such that

$$xf(x, t) + g(x, t)^2 \leq L^2(1 + x^2)$$

for all  $x, t$ .

Indeed, one can show that under the local Lipschitz condition, any solution path  $X_t(\omega)$  either exists up to time  $T$ , or leaves any compact  $\mathcal{K}$  at a time  $\tau(\omega) < T$ . Therefore, there exists a random blow-up time  $\tau$ , such that either  $\tau(\omega) = +\infty$  and then  $X_t(\omega)$  exists up to time  $T$ , or  $\tau(\omega) \leq T$ , and then  $X_t(\omega) \rightarrow \pm\infty$  as  $t \rightarrow \tau(\omega)$ .

Under the bounded growth condition, one shows that solution paths  $X_t(\omega)$  cannot blow up (because the drift term does not grow fast enough, or pulls paths back towards the origin if  $xf(x, t)$  is negative).



**Exercise 1.2.15**

Solve the SDE

$$dX_t = -\frac{1}{2}X_t dt + \sqrt{1-X_t^2} dW_t, \quad X_0 = 0$$

using the change of variables  $Y = \text{Arcsin}(X)$ .**Exercise 1.2.16**Fix  $r, \alpha \in \mathbb{R}$ . Solve the SDE

$$dY_t = r dt + \alpha Y_t dW_t, \quad Y_0 = 1$$

by using the “integrating factor”  $F_t = e^{-\alpha W_t + \frac{1}{2}\alpha^2 t}$ , and considering  $X_t = F_t Y_t$ .

### 1.3 Diffusions

A *diffusion* is a stochastic process solving an SDE of the form

$$dX_t = f(X_t)dt + g(X_t)dW_t,$$

with a drift coefficient  $f$  (modelling a deterministic force), and a diffusion coefficient  $g$  (modelling a random effect such as collisions with particles of a fluid). When speaking of diffusions, we focus on the dependence of solutions on the initial condition  $X_0 = x$ , which is one of the main mechanisms creating links between SDEs and PDEs.

**Definition 1.3.1: Ito diffusion**

A *time-homogeneous Ito diffusion* is a stochastic process  $\{X_t(\omega)\}_{t \geq 0}$  satisfying an SDE of the form

$$dX_t = f(X_t)dt + g(X_t)dW_t, \quad t \geq s > 0, \quad X_s = x, \quad (1.3.1)$$

where  $W_t$  is a standard Brownian motion of dimension  $m$ , and the *drift coefficient*  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and *diffusion coefficient*  $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$  are such that the SDE (1.3.1) admits a unique solution for all times.

We will denote the solution of (1.3.1)  $X_t^{s,x}$ .

#### 1.3.1 The Markov property

Time homogeneity, that is, the fact that  $f$  and  $g$  do not depend on time, has the following important consequence.

**Lemma 1.3.2: Time homogeneity of the law**

The processes  $\{X_{s+h}^{s,x}\}_{h \geq 0}$  and  $\{X_h^{0,x}\}_{h \geq 0}$  have the same distribution.

**PROOF:** By definition,  $X_h^{0,x}$  satisfies the integral equation

$$X_h^{0,x} = x + \int_0^h f(X_v^{0,x}) dv + \int_0^h g(X_v^{0,x}) dW_v. \quad (1.3.2)$$

Furthermore,  $X_{s+h}^{s,x}$  satisfies the equation

$$\begin{aligned} X_{s+h}^{s,x} &= x + \int_s^{s+h} f(X_u^{s,x}) du + \int_s^{s+h} g(X_u^{s,x}) dW_u \\ &= x + \int_0^h f(X_{s+v}^{s,x}) dv + \int_0^h g(X_{s+v}^{s,x}) d\widetilde{W}_v \end{aligned} \quad (1.3.3)$$

where we have used the change of variables  $u = s + v$ , and  $\widetilde{W}_v = W_{s+v} - W_s$ . By the differential property,  $\widetilde{W}_v$  is a standard Brownian motion, so that by uniqueness of solutions of the SDE (1.3.1), the integrals (1.3.3) and (1.3.2) have the same distribution.  $\square$

We will denote  $\mathbb{P}^x$  the probability measure on the  $\sigma$ -algebra generated by all random variables  $X_t^{0,x}$ ,  $t \geq 0$ ,  $x \in \mathbb{R}^n$ , defined by

$$\mathbb{P}^x \{X_{t_1} \in A_1, \dots, X_{t_k} \in A_k\} = \mathbb{P}\{X_{t_1}^{0,x} \in A_1, \dots, X_{t_k}^{0,x} \in A_k\}$$

for any choice of times  $0 \leq t_1 < t_2 < \dots < t_k$  and Borel sets  $A_1, \dots, A_k \subset \mathbb{R}^n$ . Expectations with respect to  $\mathbb{P}^x$  will be denoted  $\mathbb{E}^x$ .

### Theorem 1.3.3: Markov property for Ito diffusions

For any bounded measurable function  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$\mathbb{E}^x[\varphi(X_{t+h}) \mid \mathcal{F}_t](\omega) = \mathbb{E}^{X_t(\omega)}[\varphi(X_h)], \quad (1.3.4)$$

where the right-hand side denotes the function  $\mathbb{E}^y[\varphi(X_h)]$  evaluated at  $y = X_t(\omega)$ .

PROOF: Consider for  $y \in \mathbb{R}^n$  and  $s \geq t$  the function

$$F(y, t, s, \omega) = X_s^{t,y}(\omega) = y + \int_t^s f(X_u(\omega)) du + \int_t^s g(X_u(\omega)) dW_u(\omega).$$

Note that  $F$  is independent of  $\mathcal{F}_t$ . By uniqueness of solutions of the SDE (1.3.1), we have

$$X_s(\omega) = F(X_t(\omega), t, s, \omega).$$

Let  $g(y, \omega) = \varphi \circ F(y, t, t+h, \omega)$ . One can check that this function is measurable. Relation (1.3.4) is thus equivalent to

$$\mathbb{E}[g(X_t, \omega) \mid \mathcal{F}_t] = \mathbb{E}[\varphi \circ F(y, 0, h, \omega)] \Big|_{y=X_t(\omega)}.$$

We have

$$\mathbb{E}[g(X_t, \omega) \mid \mathcal{F}_t] = \mathbb{E}[g(y, \omega) \mid \mathcal{F}_t] \Big|_{y=X_t(\omega)}.$$

Indeed, this relation is true for functions of the form  $g(y, \omega) = \phi(y)\psi(\omega)$ , since

$$\mathbb{E}[\phi(X_t)\psi(\omega) \mid \mathcal{F}_t] = \phi(X_t)\mathbb{E}[\psi(\omega) \mid \mathcal{F}_t] = \mathbb{E}[\phi(y)\psi(\omega) \mid \mathcal{F}_t] \Big|_{y=X_t(\omega)}.$$

It can thus be extended to any bounded measurable function, by approximating it by a sequence of linear combinations of functions as above. It follows from the independence of  $F$  and  $\mathcal{F}_t$  that

$$\begin{aligned} \mathbb{E}[g(y, \omega) \mid \mathcal{F}_t] &= \mathbb{E}[g(y, \omega)] \\ &= \mathbb{E}[\varphi \circ F(y, t, t+h, \omega)] \\ &= \mathbb{E}[\varphi \circ F(y, 0, h, \omega)], \end{aligned}$$

where the last equality follows from Lemma 1.3.2. The result then follows by evaluating the last inequality at  $y = X_t$ .  $\square$

There exists an important generalisation of the Markov property to so-called stopping times. We have already encountered such a time in André's reflection principle, see Proposition 1.1.7, with the random time  $\tau = \inf\{t \geq 0: W_t \geq H\}$ . The general definition of a stopping time is as follows.

**Definition 1.3.4: Stopping time**

A *stopping time* is a random variable  $\tau : \Omega \rightarrow [0, \infty]$  such that  $\{\tau < t\} \in \mathcal{F}_t$  for all  $t \geq 0$ . For such a stopping time, the *pre- $\tau$*  sigma algebra is defined by

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \forall t \geq 0\}$$

In what follows, it will be sufficient to know that first-exit times

$$\tau = \inf\{t > 0: X_t \notin A\}$$

of an open or closed set  $A$  are stopping times. The pre- $\tau$  sigma algebra is in this case the set of all events that only depend on the behaviour of the process as long as it stays in  $A$ .

The generalisation of the Markov property to stopping times reads as follows.

**Theorem 1.3.5: Strong Markov property for Ito diffusions**

For any bounded, measurable function  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  and almost surely finite stopping time  $\tau$ ,

$$\mathbb{E}^x[\varphi(X_{\tau+h}) \mid \mathcal{F}_\tau](\omega) = \mathbb{E}^{X_\tau(\omega)}[\varphi(X_h)].$$

**PROOF:** The proof is a relatively direct adaptation of the previous proof. See for instance [Øks03, Theorem 7.2.4].  $\square$

### 1.3.2 Semigroups and generators

**Definition 1.3.6: Markov semi-group**

To any bounded measurable function  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ , one associates for all  $t \geq 0$  the function  $P_t\varphi$  defined by

$$(P_t\varphi)(x) = \mathbb{E}^x[\varphi(X_t)].$$

The linear operator  $P_t$  is called the *Markov semi-group* of the diffusion.

For instance, if  $\varphi(x) = \mathbb{1}_A(x)$  denotes the indicator function of a Borel set  $A \subset \mathbb{R}^n$ , one has

$$(P_t\mathbb{1}_A)(x) = \mathbb{P}^x\{X_t \in A\}.$$

The name semi-group is justified by the following result.

**Lemma 1.3.7: Semi-group property**

For any  $t, h \geq 0$ , one has

$$P_h \circ P_t = P_{t+h}.$$

PROOF: We have

$$\begin{aligned}
(P_h \circ P_t)(\varphi)(x) &= (P_h(P_t\varphi))(x) \\
&= \mathbb{E}^x[(P_t\varphi)(X_h)] \\
&= \mathbb{E}^x[\mathbb{E}^{X_h}[\varphi(X_t)]] \\
&= \mathbb{E}^x[\mathbb{E}^x[\varphi(X_{t+h}) \mid \mathcal{F}_t]] \\
&= \mathbb{E}^x[\varphi(X_{t+h})] \\
&= (P_{t+h}\varphi)(x),
\end{aligned}$$

where we have used the Markov property to go from the third to the fourth line.  $\square$

The following properties are easy to check:

1.  $P_t$  preserves constant functions:  $P_t(c\mathbb{1}_{\mathbb{R}^n}) = c\mathbb{1}_{\mathbb{R}^n}$ ;
2.  $P_t$  preserves non-negative functions:  $\varphi(x) \geq 0 \forall x \Rightarrow (P_t\varphi)(x) \geq 0 \forall x$ ;
3.  $P_t$  is contracting (in the non-strict sense) in the  $L^\infty$ -norm:

$$\sup_{x \in \mathbb{R}^n} |(P_t\varphi)(x)| = \sup_{x \in \mathbb{R}^n} |\mathbb{E}^x[\varphi(X_t)]| \leq \sup_{y \in \mathbb{R}^n} |\varphi(y)| \sup_{x \in \mathbb{R}^n} \mathbb{E}^x[1] = \sup_{y \in \mathbb{R}^n} |\varphi(y)|.$$

The Markov semigroup is thus a positive, linear operator, which is bounded in  $L^\infty$ -norm (in fact, it has operator norm 1).

The semi-group property implies that the behaviour of  $P_t$  on any interval  $[0, \varepsilon]$ , with  $\varepsilon > 0$  arbitrarily small, determines its behaviour for any  $t \geq 0$ . It is thus natural to consider the derivative of  $P_t$  in  $t = 0$ .

### Definition 1.3.8: Infinitesimal generator of an Ito diffusion

The *infinitesimal generator*  $\mathcal{L}$  of an Ito diffusion is defined by its action on test functions  $\varphi$  via

$$(\mathcal{L}\varphi)(x) = \lim_{h \rightarrow 0_+} \frac{(P_h\varphi)(x) - \varphi(x)}{h}. \quad (1.3.5)$$

The domain of  $\mathcal{L}$  is by definition the set of functions  $\varphi$  for which the limit (1.3.5) exists for all  $x \in \mathbb{R}^n$ .

### Remark 1.3.9

Formally, Relation (1.3.5) can be written

$$\mathcal{L} = \left. \frac{dP_t}{dt} \right|_{t=0}.$$

By the Markov property, this relation generalises to

$$\frac{d}{dt}P_t = \lim_{h \rightarrow 0_+} \frac{P_{t+h} - P_t}{h} = \lim_{h \rightarrow 0_+} \frac{P_h - \text{id}}{h} P_t = \mathcal{L}P_t,$$

and the semigroup can thus be formally written

$$P_t = e^{t\mathcal{L}}.$$

**Proposition 1.3.10**

The generator of the Ito diffusion (1.3.1) is the differential operator

$$\mathcal{L} = \sum_{i=1}^n f_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n (gg^T)_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

The domain of  $\mathcal{L}$  contains the set of twice continuously differentiable functions of compact support.

**PROOF:** Consider the case  $n = m = 1$ . Let  $\varphi$  be a twice continuously differentiable function of compact support, and let  $Y_t = \varphi(X_t)$ . By Ito's formula,

$$Y_h = \varphi(X_0) + \int_0^h \varphi'(X_s) f(X_s) ds + \int_0^h \varphi'(X_s) g(X_s) dW_s + \frac{1}{2} \int_0^h \varphi''(X_s) g(X_s)^2 ds.$$

Taking the expectation, as the expectation of the Ito integral vanishes, one gets

$$\mathbb{E}^x[Y_h] = \varphi(x) + \mathbb{E}^x \left[ \int_0^h \varphi'(X_s) f(X_s) ds + \frac{1}{2} \int_0^h \varphi''(X_s) g(X_s)^2 ds \right], \quad (1.3.6)$$

so that

$$\frac{\mathbb{E}^x[\varphi(X_h)] - \varphi(x)}{h} = \frac{1}{h} \int_0^h \mathbb{E}^x[\varphi'(X_s) f(X_s)] ds + \frac{1}{2h} \int_0^h \mathbb{E}^x[\varphi''(X_s) g(X_s)^2] ds.$$

Taking the limit  $h \rightarrow 0_+$ , we get

$$(\mathcal{L}\varphi)(x) = \varphi'(x)f(x) + \frac{1}{2}\varphi''(x)g(x)^2.$$

The cases  $n \geq 2$  or  $m \geq 2$  are treated similarly, using the multidimensional Ito formula.  $\square$

**Example 1.3.11: Generator of Brownian motion**

Let  $W_t$  be an  $m$ -dimensional Brownian motion. This is a particular case of diffusion, with  $f = 0$  and  $g = \mathbb{1}$ . Its generator is thus given by

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^m \frac{\partial^2}{\partial x_i^2} = \frac{1}{2} \Delta.$$

**1.3.3 Dynkin's formula**

Dynkin's formula is essentially a generalisation of the expression (1.3.6) to stopping times. It will yield a first important class of links between SDEs and PDEs.

**Proposition 1.3.12: Dynkin's formula**

Let  $\{X_t\}_{t \geq 0}$  be a diffusion with generator  $\mathcal{L}$ . Fix  $x \in \mathbb{R}^n$ , a stopping time  $\tau$  such that  $\mathbb{E}^x[\tau] < \infty$ , and a compactly supported, twice continuously differentiable function  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ . Then

$$\mathbb{E}^x[\varphi(X_\tau)] = \varphi(x) + \mathbb{E}^x \left[ \int_0^\tau (\mathcal{L}\varphi)(X_s) ds \right].$$

PROOF: Consider the case  $n = m = 1$ ,  $m$  being the dimension of Brownian motion. Proceeding as in the proof of Proposition 1.3.10, we obtain

$$\mathbb{E}^x[\varphi(X_\tau)] = \varphi(x) + \mathbb{E}^x\left[\int_0^\tau (\mathcal{L}\varphi)(X_s) ds\right] + \mathbb{E}^x\left[\int_0^\tau g(X_s)\varphi'(X_s) dW_s\right]. \quad (1.3.7)$$

It thus suffices to show that the expectation of the stochastic integral vanishes. For any function  $h$  bounded by  $M$  and any  $N \in \mathbb{N}$ , one has

$$\mathbb{E}^x\left[\int_0^{\tau \wedge N} h(X_s) dW_s\right] = \mathbb{E}^x\left[\int_0^N \mathbb{1}_{\{s < \tau\}} h(X_s) dW_s\right] = 0,$$

owing to the  $\mathcal{F}_s$ -measurability of  $\mathbb{1}_{\{s < \tau\}}$  and  $h(X_s)$ . Moreover,

$$\begin{aligned} \mathbb{E}^x\left[\left(\int_0^\tau h(X_s) dW_s - \int_0^{\tau \wedge N} h(X_s) dW_s\right)^2\right] &= \mathbb{E}^x\left[\int_{\tau \wedge N}^\tau h(X_s)^2 ds\right] \\ &\leq M^2 \mathbb{E}^x[\tau - \tau \wedge N], \end{aligned}$$

which goes to 0 as  $N \rightarrow \infty$ , owing to the assumption  $\mathbb{E}^x[\tau] < \infty$ , by Lebesgues' dominated convergence theorem. One can thus write

$$0 = \lim_{N \rightarrow \infty} \mathbb{E}^x\left[\int_0^{\tau \wedge N} h(X_s) dW_s\right] = \mathbb{E}^x\left[\int_0^\tau h(X_s) dW_s\right], \quad (1.3.8)$$

which finishes the proof, after plugging (1.3.8) into (1.3.7). The proof of the general case is analogous.  $\square$

Consider now the particular case where the stopping time  $\tau$  is the first-exit time from an open bounded set  $D \subset \mathbb{R}^n$ . Assume the boundary value problem

$$\begin{aligned} (\mathcal{L}u)(x) &= \theta(x) & x \in D \\ u(x) &= \psi(x) & x \in \partial D \end{aligned} \quad (1.3.9)$$

admits a unique solution. This is the case if  $D$ ,  $\theta$  and  $\psi$  are sufficiently regular. Replacing  $\varphi$  by  $u$  in Dynkin's formula, we get the relation

$$u(x) = \mathbb{E}^x\left[\psi(X_\tau) - \int_0^\tau \theta(X_s) ds\right]. \quad (1.3.10)$$

For  $\psi = 0$  and  $\theta = -1$ ,  $u(x)$  is equal to the expectation of  $\tau$ , starting from  $x$ . For  $\theta = 0$  and  $\psi$  the indicator of a subset  $A$  of the boundary  $\partial D$ ,  $u(x)$  is the probability of leaving  $D$  through  $A$ . Hence, if one can solve the problem (1.3.9), one obtains information on the first-exit time and location from  $D$ . Conversely, simulating the expression (1.3.10) by a Monte-Carlo method, one gets a numerical approximation of the solution of the boundary value problem (1.3.9).

#### Example 1.3.13: Mean exit time of Brownian motion from a ball

Let  $K = \{x \in \mathbb{R}^n : \|x\| < R\}$  be the ball of radius  $R$  centred at the origin. Given a point  $x \in K$ , let

$$\tau_K = \inf\{t > 0 : x + W_t \notin K\}$$

and let

$$\tau(N) = \tau_K \wedge N.$$

The function  $\varphi(x) = \|x\|^2 \mathbb{1}_{\{\|x\| \leq R\}}$  is compactly supported and satisfies  $\Delta\varphi(x) = 2n$  for all  $x \in K$ . One can furthermore extend it outside  $K$  in a smooth and compactly supported way.

Plugging into Dynkin's formula, one gets

$$\begin{aligned}\mathbb{E}^x[\|x + W_{\tau(N)}\|^2] &= \|x\|^2 + \mathbb{E}^x\left[\int_0^{\tau(N)} \frac{1}{2} \Delta \varphi(W_s) ds\right] \\ &= \|x\|^2 + n \mathbb{E}^x[\tau(N)].\end{aligned}$$

Since  $\|x + W_{\tau(N)}\| \leq R$ , letting  $N$  go to infinity, one obtains by dominated convergence

$$\mathbb{E}^x[\tau_K] = \frac{R^2 - \|x\|^2}{n}. \quad (1.3.11)$$

#### Example 1.3.14: Recurrence/transience of Brownian motion

Let again  $K = \{x \in \mathbb{R}^n : \|x\| < R\}$ . We now consider the case where  $x \notin K$ , and we want to determine if Brownian motion starting in  $x$  hits  $K$  almost surely, in which case it is called *recurrent*, or if it hits  $K$  with a probability strictly less than 1, in which case it is called *transient*. As for random walks, the answer depends on the dimension  $n$  of space.

We define

$$\tau_K = \inf\{t > 0 : x + W_t \in K\}.$$

For  $N \in \mathbb{N}$ , let  $A_N$  be the ring

$$A_N = \{x \in \mathbb{R}^n : R < \|x\| < 2^N R\},$$

and let  $\tau$  be the first-exit time of  $x + W_t$  from  $A_N$ . We thus have

$$\tau = \tau_K \wedge \tau', \quad \tau' = \inf\{t > 0 : \|x + W_t\| = 2^N R\}.$$

Finally, let

$$p = \mathbb{P}^x\{\tau_K < \tau'\} = \mathbb{P}^x\{\|x + W_\tau\| = R\} = 1 - \mathbb{P}^x\{\|x + W_\tau\| = 2^N R\}.$$

The spherically symmetric solutions of  $\Delta \varphi = 0$  are of the form

$$\varphi(x) = \begin{cases} |x| & \text{if } n = 1, \\ -\log|x| & \text{if } n = 2, \\ \|x\|^{2-n} & \text{if } n > 2. \end{cases}$$

For such a  $\varphi$ , Dynkin's formula yields

$$\mathbb{E}^x[\varphi(x + W_\tau)] = \varphi(x).$$

On the other hand,

$$\mathbb{E}^x[\varphi(x + W_\tau)] = \varphi(R)p + \varphi(2^N R)(1-p).$$

Solving with respect to  $p$ , one gets

$$p = \frac{\varphi(x) - \varphi(2^N R)}{\varphi(R) - \varphi(2^N R)}.$$

As  $N \rightarrow \infty$ , one has  $\tau' \rightarrow \infty$ , so that

$$\mathbb{P}^x\{\tau_K < \infty\} = \lim_{N \rightarrow \infty} \frac{\varphi(x) - \varphi(2^N R)}{\varphi(R) - \varphi(2^N R)}.$$

Consider now separately the cases  $n = 1$ ,  $n = 2$  and  $n > 2$ .

1. For  $n = 1$ , one has

$$\mathbb{P}^x\{\tau_K < \infty\} = \lim_{N \rightarrow \infty} \frac{2^N R - |x|}{2^N R - R} = 1,$$

showing that Brownian motion is recurrent in dimension 1.

2. For  $n = 2$ , one has

$$\mathbb{P}^x\{\tau_K < \infty\} = \lim_{N \rightarrow \infty} \frac{\log\|x\| + N \log 2 - \log R}{N \log 2} = 1,$$

showing that Brownian motion is also recurrent in dimension 2.

3. For  $n > 2$ , one has

$$\mathbb{P}^x\{\tau_K < \infty\} = \lim_{N \rightarrow \infty} \frac{(2^N R)^{2-n} + \|x\|^{2-n}}{(2^N R)^{2-n} + R^{2-n}} = \left(\frac{R}{\|x\|}\right)^{n-2} < 1.$$

Brownian motion is thus transient in dimension  $n > 2$ .

### Exercise 1.3.15: First-exit time of geometric Brownian motion

Consider the diffusion defined by the equation

$$dX_t = X_t dW_t.$$

1. Determine its generator  $\mathcal{L}$ .
2. Find the general solution of the equation  $\mathcal{L}u = 0$ .
3. Determine  $\mathbb{P}^x\{\tau_a < \tau_b\}$ , where  $\tau_a$  denotes the first-passage time of  $X_t$  in  $a$ .  
*Hint:* This amounts to computing  $\mathbb{E}^x[\psi(X_\tau)]$ , where  $\tau$  is the first-exit time from  $[a, b]$ , and  $\psi(a) = 1$ ,  $\psi(b) = 0$ .

### Exercise 1.3.16: First-exit time of geometric Brownian motion with linear drift

Consider more generally the diffusion defined by the equation

$$dX_t = rX_t dt + X_t dW_t, \quad r \in \mathbb{R}.$$

1. Compute its generator  $\mathcal{L}$ .
2. Show that if  $r \neq \frac{1}{2}$ , the general solution of the equation  $\mathcal{L}u = 0$  is given by

$$u(x) = c_1 x^\gamma + c_2,$$

where  $\gamma$  is a function of  $r$  to be determined.

3. Assume  $r < 1/2$ . Compute  $\mathbb{P}^x\{\tau_b < \tau_a\}$  for  $0 < a < x < b$ , and then  $\mathbb{P}^x\{\tau_b < \tau_0\}$  by letting  $a$  go to 0. Note that if  $X_{t_0} = 0$ , then  $X_t = 0$  for all  $t \geq t_0$ . Therefore, if  $\tau_0 < \tau_b$ , then  $X_t$  will never reach  $b$ . What is the probability that this happens?
4. Assume now  $r > 1/2$ .



- (a) Compute  $\mathbb{P}^x\{\tau_a < \tau_b\}$  for  $0 < a < x < b$ , and show that this probability goes to 0 as  $a \rightarrow 0_+$  for all  $x \in ]a, b[$ . Conclude that almost surely,  $X_t$  will never reach 0 in this situation.
- (b) Find  $\alpha$  and  $\beta$  such that  $u(x) = \alpha \log x + \beta$  satisfies the problem

$$\begin{cases} (Lu)(x) = -1 & \text{if } 0 < x < b, \\ u(x) = 0 & \text{if } x = b. \end{cases}$$

- (c) Use this to compute  $\mathbb{E}^x[\tau_b]$ .

### 1.3.4 Kolmogorov's equations

A second class of links between SDEs and PDEs is given by Kolmogorov's equations, which are initial value problems.

Observe that by taking the derivative of Dynkin's formula with respect to  $t$ , in the particular case  $\tau = t$ , one gets

$$\frac{\partial}{\partial t}(P_t\varphi)(x) = \frac{\partial}{\partial t}\mathbb{E}^x[\varphi(X_t)] = \mathbb{E}^x[(\mathcal{L}\varphi)(X_t)] = (P_t\mathcal{L}\varphi)(x),$$

which can be written in compact form as

$$\frac{d}{dt}P_t = P_t\mathcal{L}.$$

We have seen in Remark 1.3.9 that one can also formally write  $\frac{d}{dt}P_t = \mathcal{L}P_t$ . Therefore, the operators  $\mathcal{L}$  et  $P_t$  commute, at least formally. The next theorem makes this observation rigorous.

#### Theorem 1.3.17: Backward Kolmogorov equation

Let  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  be a compactly supported, twice continuously differentiable function.

1. The function

$$u(t, x) = (P_t\varphi)(x) = \mathbb{E}^x[\varphi(X_t)]$$

satisfies the initial value problem

$$\begin{aligned} \frac{\partial u}{\partial t}(t, x) &= (\mathcal{L}u)(t, x), & t > 0, \quad x \in \mathbb{R}^n, \\ u(0, x) &= \varphi(x), & x \in \mathbb{R}^n. \end{aligned} \quad (1.3.12)$$

2. If  $w(t, x)$  is a bounded function, which is continuously differentiable in  $t$  and twice continuously differentiable in  $x$ , satisfying the initial value problem (1.3.12), then  $w(t, x) = (P_t\varphi)(x)$ .

PROOF:

1. One has  $u(0, x) = (P_0\varphi)(x) = \varphi(x)$  and

$$\begin{aligned} (\mathcal{L}u)(t, x) &= \lim_{h \rightarrow 0_+} \frac{(P_h \circ P_t\varphi)(x) - (P_t\varphi)(x)}{h} \\ &= \lim_{h \rightarrow 0_+} \frac{(P_{t+h}\varphi)(x) - (P_t\varphi)(x)}{h} \\ &= \frac{\partial}{\partial t}(P_t\varphi)(x) = \frac{\partial}{\partial t}u(t, x). \end{aligned}$$

2. If  $w(t, x)$  satisfies (1.3.12), then one has

$$\widetilde{\mathcal{L}}w = 0 \quad \text{where} \quad \widetilde{\mathcal{L}}w = -\frac{\partial w}{\partial t} + \mathcal{L}w.$$

Fix  $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ . The process  $Y_t = (s - t, X_t^{0,x})$  admits  $\widetilde{\mathcal{L}}$  as generator. Let

$$\tau_R = \inf\{t > 0: \|X_t\| \geq R\}.$$

Dynkin's formula shows that

$$\mathbb{E}^{s,x}[w(Y_{t \wedge \tau_R})] = w(s, x) + \mathbb{E}^{s,x}\left[\int_0^{t \wedge \tau_R} (\widetilde{\mathcal{L}}w)(Y_u) du\right] = w(s, x).$$

Letting  $R$  go to infinity, one obtains

$$w(s, x) = \mathbb{E}^{s,x}[w(Y_t)] \quad \forall t \geq 0.$$

In particular, taking  $t = s$ , one has

$$w(s, x) = \mathbb{E}^{s,x}[w(Y_s)] = \mathbb{E}[w(0, X_s^{0,x})] = \mathbb{E}[\varphi(X_s^{0,x})] = \mathbb{E}^x[\varphi(X_s)],$$

as claimed. □

Note that in the case of Brownian motion, which has generator  $\mathcal{L} = \frac{1}{2}\Delta$ , Kolmogorov's backward equation (1.3.12) is nothing but the heat equation.

Since Kolmogorov's backward equation is linear, it is sufficient to solve it for a complete family of initial conditions  $\varphi$  to determine its solution for all initial conditions. A first important case occurs when one knows all eigenfunctions and eigenvalues of  $\mathcal{L}$ . Then the general solution can be decomposed on a basis of eigenfunctions, with coefficients depending exponentially on time.

#### Example 1.3.18: Brownian motion

Eigenfunctions of the generator  $\mathcal{L} = \frac{1}{2}\Delta = \frac{1}{2}\frac{d^2}{dx^2}$  of one-dimensional Brownian motion are of the form  $e^{ikx}$ . Decomposing the solution on this basis of eigenfunctions amounts to solving the heat equation by Fourier transform. One knows that the solution can be written as

$$u(t, x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-k^2 t/2} \hat{\varphi}(k) e^{ikx} dk, \quad (1.3.13)$$

where  $\hat{\varphi}(k)$  is the Fourier transform of the initial condition.

A second important case occurs when formally decomposing the initial condition on a "basis" of Dirac distributions. In practice, this amounts to using the notion of transition density.

#### Definition 1.3.19: Transition density

The diffusion  $\{X_t\}_t$  is said to admit the *transition density*  $p_t(x, y)$ , also written  $p(y, t|x, 0)$ , if

$$\mathbb{E}^x[\varphi(X_t)] = \int_{\mathbb{R}^n} \varphi(y) p_t(x, y) dy$$

for all bounded measurable functions  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$ .

By linearity, if the transition density exists and is smooth, it satisfies Kolmogorov's backward equation (the generator  $\mathcal{L}$  acting on the variable  $x$ ), with initial condition  $p_0(x, y) = \delta(x - y)$ .

**Example 1.3.20: Brownian motion and heat kernel**

In the case of one-dimensional Brownian motion, we have seen (c.f. (1.1.2)) that the transition density is given by the heat kernel

$$p(y, t|x, 0) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}.$$

This is also the value of the integral (1.3.13) with  $\hat{\phi}(k) = e^{-iky}/\sqrt{2\pi}$ , which is indeed the Fourier transform of  $\varphi(x) = \delta(x-y)$ .

The *adjoint* of the generator  $\mathcal{L}$  is by definition the linear operator  $\mathcal{L}^\dagger$  such that

$$\langle \mathcal{L}\phi, \psi \rangle = \langle \phi, \mathcal{L}^\dagger\psi \rangle \quad (1.3.14)$$

for any choice of twice continuously differentiable functions  $\phi, \psi : \mathbb{R}^n \rightarrow \mathbb{R}$ , with  $\phi$  compactly supported, where  $\langle \cdot, \cdot \rangle$  denotes the usual inner product in  $L^2$ . Integrating  $\langle \mathcal{L}\phi, \psi \rangle$  by parts twice, one obtains

$$(\mathcal{L}^\dagger\psi)(y) = \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial y_i \partial y_j} ((gg^T)_{ij}\psi)(y) - \sum_{i=1}^n \frac{\partial}{\partial y_i} (f_i\psi)(y).$$

**Theorem 1.3.21: Forward Kolmogorov equation**

If  $X_t$  admits a smooth transition density  $p_t(x, y)$ , then it satisfies the equation

$$\frac{\partial}{\partial t} p_t(x, y) = \mathcal{L}_y^\dagger p_t(x, y), \quad (1.3.15)$$

where the notation  $\mathcal{L}_y^\dagger$  means that  $\mathcal{L}^\dagger$  acts on the variable  $y$ .

**PROOF:** Dynkin's formula with  $\tau = t$  implies

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi(y) p_t(x, y) dy &= \mathbb{E}^x[\varphi(X_t)] \\ &= \varphi(x) + \int_0^t \mathbb{E}^x[(\mathcal{L}\varphi)(X_s)] ds \\ &= \varphi(x) + \int_0^t \int_{\mathbb{R}^n} (\mathcal{L}\varphi)(y) p_s(x, y) dy ds. \end{aligned}$$

Taking the derivative with respect to time, and using (1.3.14), we get

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^n} \varphi(y) p_t(x, y) dy = \int_{\mathbb{R}^n} (\mathcal{L}\varphi)(y) p_t(x, y) dy = \int_{\mathbb{R}^n} \varphi(y) (\mathcal{L}_y^\dagger p_t)(x, y) dy,$$

which implies the result.  $\square$

Assume the distribution of  $X_0$  admits a density  $\rho$  with respect to Lebesgue measure. Then  $X_t$  has the density

$$\rho(t, y) = (Q_t\rho)(y) := \int_{\mathbb{R}^n} p_t(x, y) \rho(x) dx.$$

Applying Kolmogorov's forward equation (1.3.15), one obtains the *Fokker–Planck equation*

$$\frac{\partial}{\partial t} \rho(t, y) = \mathcal{L}_y^+ \rho(t, y),$$

which can also be formally written

$$\frac{d}{dt} Q_t = \mathcal{L}^+ Q_t.$$

The adjoint generator  $\mathcal{L}^+$  is thus the generator of the adjoint semi-group  $Q_t$ .

### Corollary 1.3.22

If  $\rho_0(y)$  is the density of a probability measure satisfying  $\mathcal{L}^+ \rho_0 = 0$ , then  $\rho_0$  is a stationary measure of the diffusion. In other words, if the distribution of  $X_0$  admits the density  $\rho_0$ , then  $X_t$  admits the density  $\rho_0$  for all  $t \geq 0$ .

### Exercise 1.3.23: Invariant measure of the Ornstein–Uhlenbeck process

Consider the diffusion defined by the equation

$$dX_t = -X_t dt + dW_t.$$

1. Give its generator  $\mathcal{L}$  and its adjoint  $\mathcal{L}^+$ .
2. Let  $\rho(x) = \pi^{-1/2} e^{-x^2}$ . Compute  $\mathcal{L}^+ \rho(x)$  and interpret the result.

### 1.3.5 The Feynman–Kac formula

So far, we have encountered elliptic boundary value problems of the form  $\mathcal{L}u = \theta$ , as well as parabolic evolution equations of the form  $\partial_t u = \mathcal{L}u$ . The Feynman–Kac formula will show that one can also link properties of diffusions with those of parabolic equations containing a term linear in  $u$ . Adding a linear term to the generator can be interpreted as “killing” the diffusion at certain rate. The simplest case is that of a constant rate. Let  $\zeta$  be a random variable of exponential distribution with parameter  $\lambda$ , independent of  $W_t$ . Set

$$\tilde{X}_t = \begin{cases} X_t & \text{if } t < \zeta, \\ \Delta & \text{if } t \geq \zeta, \end{cases}$$

where  $\Delta$  is a “cemetery state” that has been added to  $\mathbb{R}^n$ . One checks that owing to the exponential distribution of  $\zeta$ ,  $\tilde{X}_t$  is a Markov process on  $\mathbb{R}^n \cup \{\Delta\}$ . If  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  is a bounded measurable test function, one has (setting  $\varphi(\Delta) = 0$ )

$$\mathbb{E}^x[\varphi(\tilde{X}_t)] = \mathbb{E}^x[\varphi(X_t) \mathbb{1}_{\{t < \zeta\}}] = \mathbb{P}\{\zeta > t\} \mathbb{E}^x[\varphi(X_t)] = e^{-\lambda t} \mathbb{E}^x[\varphi(X_t)].$$

It follows that

$$\lim_{h \rightarrow 0} \frac{\mathbb{E}^x[\varphi(\tilde{X}_h)] - \varphi(x)}{h} = -\lambda \varphi(x) + (\mathcal{L}\varphi)(x),$$

which shows that the infinitesimal generator of  $\tilde{X}$  is the differential operator

$$\tilde{\mathcal{L}} = \mathcal{L} - \lambda.$$

More generally, if  $q : \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuous function bounded from below, one can construct a random variable  $\zeta$  such that

$$\mathbb{E}^x[\varphi(\tilde{X}_t)] = \mathbb{E}^x\left[\varphi(X_t) e^{-\int_0^t q(X_s) ds}\right].$$

In this case, the generator of  $\tilde{X}_t$  is

$$\tilde{\mathcal{L}} = \mathcal{L} - q,$$

that is,  $(\tilde{\mathcal{L}}\varphi)(x) = (\mathcal{L}\varphi)(x) - q(x)\varphi(x)$ .

### Theorem 1.3.24: Feynman–Kac formula

Let  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  be a compactly supported, twice continuously differentiable function, and let  $q : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function bounded from below.

1. The function

$$v(t, x) = \mathbb{E}^x\left[e^{-\int_0^t q(X_s) ds} \varphi(X_t)\right] \quad (1.3.16)$$

solves the initial value problem

$$\begin{aligned} \frac{\partial v}{\partial t}(t, x) &= (\mathcal{L}v)(t, x) - q(x)v(t, x), & t > 0, \quad x \in \mathbb{R}^n, \\ v(0, x) &= \varphi(x), & x \in \mathbb{R}^n. \end{aligned} \quad (1.3.17)$$

2. If  $w(t, x)$  is continuously differentiable in  $t$  and twice continuously differentiable in  $x$ , bounded for  $x$  in a compact set, and satisfies (1.3.17), then  $w(t, x)$  is equal to the right-hand side of (1.3.16).

PROOF:

1. Set  $Y_t = \varphi(X_t)$  and  $Z_t = e^{-\int_0^t q(X_s) ds}$ , and let  $v(t, x)$  be given by (1.3.16). Then for  $h > 0$ ,

$$\begin{aligned} \frac{1}{h} \left[ \mathbb{E}^x[v(t, X_{t+h})] - v(t, x) \right] &= \frac{1}{h} \left[ \mathbb{E}^x \left[ \mathbb{E}^{X_{t+h}}[Y_t Z_t] \right] - \mathbb{E}^x[Y_t Z_t] \right] \\ &= \frac{1}{h} \left[ \mathbb{E}^x \left[ \mathbb{E}^x[Y_{t+h} e^{-\int_0^t q(X_{s+h}) ds} | \mathcal{F}_h] - Y_t Z_t \right] \right] \\ &= \frac{1}{h} \mathbb{E}^x \left[ Y_{t+h} Z_{t+h} e^{\int_0^h q(X_s) ds} - Y_t Z_t \right] \\ &= \frac{1}{h} \mathbb{E}^x \left[ Y_{t+h} Z_{t+h} - Y_t Z_t \right] \\ &\quad - \frac{1}{h} \mathbb{E}^x \left[ Y_{t+h} Z_{t+h} \left[ e^{\int_0^h q(X_s) ds} - 1 \right] \right]. \end{aligned}$$

As  $h$  goes to 0, the first term in the last expression converges to  $\partial_t v(t, x)$ , while the second one converges to  $q(x)v(t, x)$ .

2. If  $w(t, x)$  satisfies (1.3.17), then

$$\tilde{\mathcal{L}}w = 0 \quad \text{where} \quad \tilde{\mathcal{L}}w = -\frac{\partial w}{\partial t} + \mathcal{L}w - qw.$$

Fix  $(s, x, z) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$  and set  $Z_t = z + \int_0^t q(X_s) ds$ . The process  $Y_t = (s - t, X_t^{0,x}, Z_t)$  is a diffusion with generator

$$\widehat{\mathcal{L}} = -\frac{\partial}{\partial s} + \mathcal{L} + q \frac{\partial}{\partial z}.$$

Let  $\phi(s, x, z) = e^{-z} w(s, x)$ . Then  $\widehat{\mathcal{L}}\phi = 0$ , and Dynkin's formula shows that if  $\tau_R$  is the first-exit time from a ball of radius  $R$ , then

$$\mathbb{E}^{s,x,z}[\phi(Y_{t \wedge \tau_R})] = \phi(s, x, z).$$

It follows that

$$\begin{aligned} w(s, x) &= \phi(s, x, 0) = \mathbb{E}^{s,x,0}[\phi(Y_{t \wedge \tau_R})] \\ &= \mathbb{E}^x[\phi(s - t \wedge \tau_R, X_{t \wedge \tau_R}^{0,x}, Z_{t \wedge \tau_R})] \\ &= \mathbb{E}^x\left[e^{-\int_0^{t \wedge \tau_R} q(X_u) du} w(s - t \wedge \tau_R, X_{t \wedge \tau_R}^{0,x})\right], \end{aligned}$$

which converges to the expectation of  $e^{-\int_0^t q(X_u) du} w(s - t, X_t^{0,x})$  as  $R$  goes to infinity. In particular, for  $t = s$  one obtains

$$w(s, x) = \mathbb{E}^x\left[e^{-\int_0^s q(X_u) du} w(0, X_s^{0,x})\right],$$

which is indeed equal to the function  $v(t, x)$  defined in (1.3.16).  $\square$

In combination with Dynkin's formula, the Feynman–Kac formula can be generalised to stopping times. If for instance  $D \subset \mathbb{R}^n$  is a regular domain, and  $\tau$  denotes the first-exit time from  $D$ , then under some regularity conditions on the functions  $q, \varphi, \theta : \overline{D} \rightarrow \mathbb{R}$ , the quantity

$$v(t, x) = \mathbb{E}^x\left[e^{-\int_0^{t \wedge \tau} q(X_s) ds} \varphi(X_{t \wedge \tau}) - \int_0^{t \wedge \tau} e^{-\int_0^s q(X_u) du} \theta(X_s) ds\right]$$

satisfies the initial value problem with boundary conditions

$$\begin{aligned} \frac{\partial v}{\partial t}(t, x) &= (\mathcal{L}v)(t, x) - q(x)v(t, x) - \theta(x), & t > 0, \quad x \in D, \\ v(0, x) &= \varphi(x), & x \in D, \\ v(t, x) &= \varphi(x), & x \in \partial D. \end{aligned}$$

In particular, if  $\tau$  is almost surely finite, taking the limit  $t \rightarrow \infty$ , one obtains that

$$v(x) = \mathbb{E}^x\left[e^{-\int_0^\tau q(X_s) ds} \varphi(X_\tau) - \int_0^\tau e^{-\int_0^s q(X_u) du} \theta(X_s) ds\right]$$

satisfies

$$\begin{aligned} (\mathcal{L}v)(x) &= q(x)v(x) + \theta(x), & x \in D, \\ v(x) &= \varphi(x), & x \in \partial D. \end{aligned}$$

Note that in the case  $q = 0$ , one recovers Relations (1.3.9) and (1.3.10).

### Example 1.3.25

Let  $D = ]-a, a[$  and  $X_t = x + W_t$ . Then  $v(x) = \mathbb{E}^x[e^{-\lambda\tau}]$  satisfies

$$\begin{aligned} \frac{1}{2}v''(x) &= \lambda v(x), & x \in D, \\ v(-a) &= v(a) = 1. \end{aligned}$$

The general solution of the first equation is of the form  $v(x) = c_1 e^{\sqrt{2\lambda}x} + c_2 e^{-\sqrt{2\lambda}x}$ . The

integration constants  $c_1$  and  $c_2$  are determined by the boundary conditions, and one finds

$$\mathbb{E}^x[e^{-\lambda\tau}] = \frac{\cosh(\sqrt{2\lambda}x)}{\cosh(\sqrt{2\lambda}a)}. \quad (1.3.18)$$

Evaluating the derivative in  $\lambda = 0$ , one obtains

$$\mathbb{E}^x[\tau] = a^2 - x^2,$$

which is a particular case of (1.3.11), but (1.3.18) also determines all other moments of  $\tau$ , as well as its density.

Solving the equation with boundary conditions  $v(-a) = 0$  and  $v(a) = 1$  one obtains

$$\mathbb{E}^x[e^{-\lambda\tau} \mathbb{1}_{\{\tau_a < \tau_{-a}\}}] = \frac{\sinh(\sqrt{2\lambda}(x+a))}{\sinh(\sqrt{2\lambda} \cdot 2a)}.$$

In particular, for  $\lambda = 0$ , we find

$$\mathbb{P}^x\{\tau_a < \tau_{-a}\} = \frac{x+a}{2a},$$

which can also be obtained directly from Dynkin's formula. However, taking derivatives at  $\lambda = 0$ , we also obtain

$$\begin{aligned} \mathbb{E}^x[\tau \mathbb{1}_{\{\tau_a < \tau_{-a}\}}] &= \frac{(a^2 - x^2)(3a + x)}{6a}, \\ \mathbb{E}^x[\tau \mid \tau_a < \tau_{-a}] &= \frac{(a - x)(3a + x)}{3}. \end{aligned}$$

### Remark 1.3.26: Cover image

The cover image shows a numerical solution of the heat equation, with constant temperatures (say 1 and 0) inside the Mandelbrot set, and outside the ellipse, after a time long enough for the solution to be close to a stationary state. The colour code represents the norm of the gradient of the solution  $u(t, x)$ . By the above results (assuming regularity of the Mandelbrot set does not pose any problem),  $u(x) = \lim_{t \rightarrow \infty} u(t, x)$  is the solution of  $\Delta u = 0$  in the domain, with boundary conditions 1 on the Mandelbrot set, and 0 on the ellipse. Therefore,  $u(x)$  gives the probability, starting at  $x$ , to hit the Mandelbrot set before the ellipse. It also represents the electric potential in a capacitor formed by two conductors shaped like the boundary sets, while the colours represent the intensity of the electric field. One can observe a “knife edge effect”: the electric field is stronger near the sharp tips of the Mandelbrot set. A video of the convergence towards the equilibrium field can be found on the page <https://www.youtube.com/c/NilsBerglund>.

### Exercise 1.3.27: The arcsine law

Let  $\{W_t\}_{t \geq 0}$  be a standard Brownian motion in  $\mathbb{R}$ . Consider the process

$$X_t = \frac{1}{t} \int_0^t \mathbb{1}_{\{W_s > 0\}} ds, \quad t > 0.$$

The aim of this exercise is to prove the *arcsine law* :

$$\mathbb{P}\{X_t < u\} = \frac{2}{\pi} \text{Arcsin}(\sqrt{u}), \quad 0 \leq u \leq 1. \quad (1.3.19)$$

1. What does the variable  $X_t$  represent?
2. Show that  $X_t$  is equal in distribution to  $X_1$  for all  $t > 0$ .
3. Fix  $\lambda > 0$ . For  $t > 0$  and  $x \in \mathbb{R}$ , one defines

$$v(t, x) = \mathbb{E}\left[e^{-\lambda \int_0^t \mathbb{1}_{\{x+W_s > 0\}} ds}\right]$$

and its Laplace transform

$$g_\rho(x) = \int_0^\infty v(t, x) e^{-\rho t} dt, \quad \rho > 0.$$

Show that

$$g_\rho(0) = \mathbb{E}\left[\frac{1}{\rho + \lambda X_1}\right].$$

4. Compute  $\frac{\partial v}{\partial t}(t, x)$  using the Feynman–Kac formula.
5. Compute  $g_\rho''(x)$ . Conclude that  $g_\rho(x)$  satisfies a second-order ODE with piecewise constant coefficients. Show that its general solution is given by

$$g_\rho(x) = A_\pm + B_\pm e^{\gamma_\pm x} + C_\pm e^{-\gamma_\pm x}$$

with constants  $A_\pm, B_\pm, C_\pm, \gamma_\pm$  depending on the sign of  $x$ .

6. Determine these constants by using the fact that  $g_\rho$  should be bounded, continuous in 0, and that  $g_\rho'$  should be continuous in 0. Conclude that  $g_\rho(0) = 1/\sqrt{\rho(\lambda + \rho)}$ .
7. Prove (1.3.19) by using the identity

$$\frac{1}{\sqrt{1+\lambda}} = \sum_{n=0}^{\infty} (-\lambda)^n \frac{1}{\pi} \int_0^1 \frac{x^n}{\sqrt{x(1-x)}} dx.$$





## Invariant measures for SDEs

We consider in this chapter SDEs in  $\mathbb{R}^n$  of the form

$$dX_t = f(X_t) dt + g(X_t) dW_t, \quad (2.0.1)$$

where the drift coefficient  $f$  and the diffusion coefficient  $g$  are such that there exists a unique strong solution, which is global in time. Then it is natural to ask the following questions:

1. Does the diffusion (2.0.1) admit an invariant probability measure?
2. If so, is this measure unique?
3. If so, does any initial distribution converge to the invariant measure?
4. If so, how fast does this convergence occur? For which distance? Can one obtain explicit bounds on the speed of convergence?

Various methods have been derived to answer these questions, each one having its advantages and drawbacks. Some methods are easier to use and provide, for instance, convergence to an invariant distribution, but without any bound on the speed of convergence, while others may provide sharp bounds, but are limited to specific sets of initial distributions. In what follows, we are going to present a few selected examples of these methods, which have been chosen because they proved useful in particular applications. But one should keep in mind that there exist many more approaches.

In what follows, it will be useful to employ the notation

$$P_t(x, A) = (P_t \mathbb{1}_A)(x) = \mathbb{P}^x\{X_t \in A\}$$

for the Markov semigroup, where  $A$  is any Borel set in  $\mathbb{R}^n$ . Given a probability measure  $\mu$ , we write

$$(\mu P_t)(A) = \int_{\mathbb{R}^n} \mu(dx) P_t(x, A)$$

instead of  $(Q_t \mu)(A)$  for the action of the adjoint semigroup, because it is reminiscent of matrix multiplication used for Markov chains on finite sets. A measure is invariant if it satisfies

$$(\mu P_t)(A) = \mu(A)$$

for all  $t \geq 0$  and all Borel sets  $A \subset \mathbb{R}^n$ .

**Definition 2.0.1: Feller property**

A semigroup  $(P_t)_{t \geq 0}$  is said to have the *Feller property* if  $P_t f$  is bounded and continuous whenever  $f$  is bounded and continuous.

A useful standard result in our situation is then the following.

**Proposition 2.0.2: Condition for Feller property**

Any diffusion  $(X_t)_{t \geq 0}$  solving an SDE with globally Lipschitz coefficients has the Feller property.

For a proof, see for instance [Øks03, Lemma 8.1.4], or [RY99, Theorem IX.2.5]. The global Lipschitz condition can often be relaxed to a local condition by working with appropriate stopping times (that is, by considering the diffusion up to its first exit from a sequence of balls of growing radius).

**Remark 2.0.3: Strong Feller property**

The semigroup  $P_t$  is said to have the *strong Feller property* if  $P_t f$  is bounded and continuous whenever  $f$  is bounded and measurable, but not necessarily continuous. A sufficient condition for a diffusion to satisfy the strong Feller property is the *ellipticity* condition on the drift coefficient

$$\langle \xi, g(x)g(x)^T \xi \rangle \geq c \|\xi\|^2 \quad \forall \xi \in \mathbb{R}^d.$$

This condition can be relaxed to *hypoellipticity* (Hörmander condition).

## 2.1 Existence of invariant probability measures

### 2.1.1 Some basic examples

Corollary 1.3.22 shows that the density  $\rho$  of an invariant probability measure should satisfy  $\mathcal{L}^\dagger \rho = 0$ , where  $\mathcal{L}^\dagger$  is the adjoint generator. Cases where this equation can be solved are rare, but one important example is given by gradient SDEs.

**Example 2.1.1: Gradient system**

Consider the SDE

$$dX_t = -\nabla V(X_t) dt + \sqrt{2} dW_t, \quad (2.1.1)$$

where  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is bounded below, and satisfies

$$\int_{\mathbb{R}^n} e^{-V(x)} dx < \infty.$$

The generator of the diffusion can be written in the two equivalent ways

$$\mathcal{L} = \Delta - \nabla V \cdot \nabla = e^V \nabla \cdot e^{-V} \nabla$$

(the factor  $\sqrt{2}$  in (2.1.1) avoids a factor  $\frac{1}{2}$  in front of the Laplacian). Integrating by parts twice, we find

$$\langle f, \mathcal{L}g \rangle = - \int e^{-V(x)} \nabla(f(x)e^{V(x)}) \cdot \nabla g(x) dx = \langle \mathcal{L}^\dagger f, g \rangle,$$

with the adjoint generator given by

$$\mathcal{L}^\dagger f = \nabla \cdot (e^{-V} \nabla(e^V f)).$$

In view of Corollary 1.3.22, this shows that

$$\rho(x) = \frac{1}{\mathcal{Z}} e^{-V(x)}, \quad \mathcal{Z} = \int e^{-V(x)} dx$$

is the density of an invariant probability measure of the diffusion (2.1.1), since it satisfies  $\mathcal{L}^\dagger \rho = 0$ .

Next, we discuss two very simple examples for which existence or uniqueness of an invariant probability measure fails.

### Example 2.1.2: Brownian motion

The transition density of Brownian motion in  $\mathbb{R}^n$  is a Gaussian of variance  $t$ , as we have seen for instance in Section 1.1.3. Therefore, for any fixed  $x \in \mathbb{R}^n$ , we have

$$\lim_{t \rightarrow \infty} p(t, x | 0, 0) = 0,$$

which is not a normalisable measure. Therefore, Brownian motion in  $\mathbb{R}^n$  does not admit an invariant probability measure (though it does admit invariant measures, which are simply all multiples of Lebesgue measure).

### Example 2.1.3: Non-irreducible SDE

Consider the SDE in  $\mathbb{R}^2$

$$\begin{aligned} dX_t &= -X_t dt + dW_t \\ dY_t &= (Y_t - Y_t)^3 dt. \end{aligned}$$

The diffusion admits three extremal invariant measures, given by

$$\begin{aligned} \pi_-(dx, dy) &= \frac{1}{\sqrt{\pi}} e^{-x^2} dx \delta_{-1}(dy), \\ \pi_0(dx, dy) &= \frac{1}{\sqrt{\pi}} e^{-x^2} dx \delta_0(dy), \\ \pi_+(dx, dy) &= \frac{1}{\sqrt{\pi}} e^{-x^2} dx \delta_1(dy). \end{aligned}$$

This is because the  $x$ - and  $y$ -components do not interact with each other, and the process  $X_t$  is an Ornstein–Uhlenbeck process with Gaussian invariant measure, while  $Y_t$  is deterministic, with three invariant points located at  $\pm 1$  and  $0$ . Note that in addition, any convex combination of  $\pi_-$ ,  $\pi_0$  and  $\pi_+$  is also an invariant probability measure ( $\pi_-$ ,  $\pi_0$  and  $\pi_+$  are called extremal because they do not admit a nontrivial decomposition).

The two last examples illustrate a general fact about invariant probability measures of Markov processes, namely that their existence requires two properties to hold:

1. There should be a mechanism preventing all the probability mass of going to infinity. More precisely, a positive recurrence property is required to hold, that is, the return time to some bounded set should have finite expectation.
2. There should be a mechanism making the diffusion irreducible, that is, there should not be any non-trivial invariant sets.

These two conditions are analogous to those that one finds for Markov chains on a countable space. The main difference is that discrete-time Markov chains require an aperiodicity condition to hold in addition. However, this is specific to discrete time, and no such condition is necessary as soon as transition times between different points are sufficiently random.

### 2.1.2 The Krylov–Bogoliubov criterion

A general criterion for existence of invariant measures, going back to Krylov and Bogoliubov, is based on the notion of ergodic averages (or Cesaro means). Given an initial point  $X_0 \in \mathbb{R}^n$ , consider the family of measures

$$\left\{ \frac{1}{T} \int_0^T P_t(X_0, \cdot) dt : T \geq 1 \right\}. \quad (2.1.2)$$

In a similar way as for Markov chains, if these ergodic averages converge to a limiting probability measure, then this limit should be invariant. A convergence criterion is given by tightness.

#### Definition 2.1.4: Tightness

A family  $\{\mu_t\}$  of probability measures on  $\mathbb{R}^n$  is *tight* if for any  $\delta > 0$ , there exists a compact set  $K \subset \mathbb{R}^n$  such that  $\mu_t(K) \geq 1 - \delta$  for all  $t$ .

Then one has the following existence result, see for instance [DPZ96, Corollary 3.1.2].

#### Proposition 2.1.5: Existence of an invariant probability measure

If the family of measures (2.1.2) is tight, then there exists an invariant probability measure.

While this criterion may be used to obtain abstract existence results for invariant probability measures, it is not so easy to apply because it requires some a priori knowledge on the semi-group  $P_t$ . Therefore, in what follows, we will rather discuss more practical criteria for analysing invariant measure.

## 2.2 The Lyapunov function approach by Meyn and Tweedie

We present here some results of the approach based on Lyapunov functions, as developed in [MT93c] for continuous-time Markov processes. This approach provides a relatively easy way of proving global existence of solution, existence of an invariant measure, and some convergence results, provided one can guess an appropriate Lyapunov function.

#### Definition 2.2.1: Norm-like function

A function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is called *norm-like* if

$$\lim_{\|x\| \rightarrow \infty} V(x) = +\infty.$$

This means that the level sets  $\{x \in \mathbb{R}^n : V(x) \leq h\}$  are precompact for any  $h > 0$ .

In the case of ordinary differential equations, Lyapunov functions are norm-like functions that decrease along orbits of the dynamical system, at least when starting far away from the origin. The application to SDEs uses quite similar ideas, where the time derivative along orbits is replaced by the action of the generator. In the following, we will present several of these results, without giving detailed proofs. A detailed proof of a quite general existence and convergence result, due to Martin Hairer and Jonathan Mattingly, will be discussed in Section 2.2.4.

### 2.2.1 Non-explosion and Harris recurrence criteria

A first application of Lyapunov functions is a relatively easy criterion for existence of global in time solutions.

#### Theorem 2.2.2: Non-explosion criterion

Assume that there exist a norm-like function  $V$  and constants  $c, d > 0$  such that

$$(\mathcal{L}V)(x) \leq cV(x) + d \quad (2.2.1)$$

for all  $x \in \mathbb{R}^n$ . Then

1. The SDE (2.0.1) admits global in time solutions for any starting point  $x \in \mathbb{R}^n$ .
2. There exists an almost surely finite random variable  $D$  such that

$$V(X_t) \leq D e^{ct} \quad \forall t \geq 0.$$

The random variable  $D$  satisfies the bound

$$\mathbb{P}^x\{D \geq a\} \leq \frac{V(x)}{a} \quad \forall a > 0, \quad \forall x \in \mathbb{R}^n. \quad (2.2.2)$$

3. The expectation  $\mathbb{E}^x[V(X_t)]$  is finite for all  $x \in \mathbb{R}^n$  and all  $t \geq 0$ , and satisfies

$$\mathbb{E}^x[V(X_t)] \leq e^{ct} V(x). \quad (2.2.3)$$

SKETCH OF PROOF: Consider first the case  $c = d = 0$ . Then Ito's formula (cf. Lemma 1.2.6) yields

$$\begin{aligned} \mathbb{E}^x[V(X_t)] &= V(x) + \mathbb{E}^x\left[\int_0^t (\mathcal{L}V)(X_s) ds\right] \\ &\leq V(x). \end{aligned}$$

This proves (2.2.3), as well as global existence of the solution. The bound (2.2.2) follows from a slightly more sophisticated stochastic analysis argument, using the fact that  $e^{-ct} V(X_s)$  is a supermartingale.

The case  $c > 0$  and  $d = 0$  follows in a similar way from the Feynman–Kac formula (see Theorem 1.3.24), while the other cases can be reduced to already treated cases by changing the Lyapunov function.  $\square$

#### Remark 2.2.3: Stopping times

To rule out the possibility of finite-time blow-up, the actual argument given in [MT93c, Theorem 2.1] uses a more careful computation based on the process killed when leaving a large ball, whose radius is then sent to infinity. This requires in particular using Dynkin's formula instead of Ito's formula.

The following result gives a condition under which solutions remain bounded almost surely (a property called *non-evanescence* in [MT93c]), which is slightly stronger than (2.2.1). For a proof, see [MT93c, Theorem 3.1].

**Theorem 2.2.4: Non-evanescence condition**

Assume there exist a compact set  $C \subset \mathbb{R}^n$ , a constant  $d > 0$ , and a norm-like function  $V$  such that

$$(\mathcal{L}V)(x) \leq d \mathbb{1}_C(x) \quad (2.2.4)$$

for all  $x \in \mathbb{R}^n$ . Then

$$\mathbb{P} \left\{ \lim_{t \rightarrow \infty} \|X_t\| = \infty \right\} = 0.$$

As discussed above, in order to obtain existence of an invariant measure, we will need some stronger form of recurrence condition. Recall that a discrete-space Markov chain is said to be *recurrent* if it almost surely returns to its starting point, and thus visits this point infinitely often. Such a property cannot hold for continuous-space processes, since sets of measure 0 may never be hit. The relevant concept is given by *Harris recurrence*.

**Definition 2.2.5: Harris recurrence**

The diffusion  $(X_t)_{t \geq 0}$  is called *Harris recurrent* if there exists a  $\sigma$ -finite measure  $\mu$  such that whenever  $\mu(A) > 0$ , one has for all  $x \in \mathbb{R}^n$

$$\mathbb{P}^x \{ \tau_A < \infty \} = 1 \quad \text{where } \tau_A = \inf \{ t \geq 0 : X_t \in A \}.$$

Equivalently, the diffusion is Harris recurrent if there exists a  $\sigma$ -finite measure  $\nu$  such that whenever  $\nu(A) > 0$ , one has for all  $x \in \mathbb{R}^n$

$$\mathbb{P}^x \{ \eta_A = \infty \} = 1 \quad \text{where } \eta_A = \int_0^\infty \mathbb{1}_{X_t \in A} dt.$$

The equivalence of the two definitions is well-known for Markov chains on countable spaces, and a proof in the general case can be found for instance in [MT93a, Theorem 1.1]. The interest of this definition is the following classical result [ADR69, Get80].

**Proposition 2.2.6: Existence of an invariant measure**

If  $(X_t)_{t \geq 0}$  is Harris recurrent, then it admits an (essentially unique) invariant measure  $\pi$ .

One way of showing Harris recurrence is to use the (somewhat tricky) notion of *petite sets*.

**Definition 2.2.7: Petite set**

Let  $a$  be a probability distribution on  $\mathbb{R}_+$ , and define a Markov kernel  $K_a$  by

$$K_a(x, A) = \int_0^\infty P_t(x, A) a(dt)$$

for any Borel set  $A \subset \mathbb{R}^n$ . Let  $\varphi_a$  be a nontrivial measure on  $\mathbb{R}^n$ . Then a non-empty Borel set  $C \subset \mathbb{R}^n$  is called  $\varphi_a$ -*petite* if  $K_a(x, A) \geq \varphi_a(A)$  for all  $x \in C$ .

The intuition behind this definition is as follows. If we choose, say,  $a = \delta_1$ , then  $K_a(x, A) = P_1(x, A)$ . The condition  $K_a(x, A) \geq \varphi_a(A)$  then requires that the probability of being in the set  $A$  at time 1 is bounded below by a measure  $\varphi_a(A)$  independent of the starting point  $x$  in the petite set (the measure  $\varphi_a$  need not be a probability measure). This would be quite restrictive, but the definition allows to replace  $P_1(x, A)$  by an average of  $P_t(x, A)$  over all times  $t \geq 0$ , a much weaker requirement. Petite sets allow us to give a first condition for a diffusion to be Harris recurrent.

**Theorem 2.2.8: Harris recurrence condition**

If all compact subsets of  $\mathbb{R}^n$  are petite and (2.2.4) holds for a compact set  $C \subset \mathbb{R}^n$ , a constant  $d > 0$ , and a norm-like function  $V$ , then the process  $(X_t)_{t \geq 0}$  is Harris recurrent, and therefore admits an essentially unique invariant measure  $\pi$ .

This result follows from Theorem 2.2.4, combined with [MT93b, Theorem 5.1], which gives an analogous statement in the discrete-time case.

**2.2.2 Positive Harris recurrence and existence of an invariant probability**

Combining Theorem 2.2.8 with Proposition 2.2.6, we obtain a condition for the existence of an invariant measure. This measure need not have finite mass, preventing it from being normalisable to yield an invariant probability measure. This motivates the following definition.

**Definition 2.2.9: Positive Harris recurrence**

Let  $(X_t)_{t \geq 0}$  be a Harris recurrent diffusion, with invariant measure  $\pi$ . If  $\pi(\mathbb{R}^n) < \infty$ , then  $(X_t)_{t \geq 0}$  is said to be *positive Harris recurrent*.

**Remark 2.2.10: Link with expected return times**

A diffusion satisfying  $\mathbb{E}^x[\tau_A] < \infty$  for any  $x \in \mathbb{R}^n$  and any Borel set  $A$  such that  $\mu(A) > 0$  for a  $\sigma$ -finite measure  $\mu$  is positive Harris recurrent. Indeed, in this case,  $\tau_A$  is almost surely finite, so that the chain is Harris recurrent. Furthermore, given  $x \in \mathbb{R}^n$  and  $A$  with  $\mu(A) > 0$ , the probability measure  $\pi_A(x, \cdot)$  given by

$$\pi_A(x, B) = \frac{1}{\mathbb{E}^x[\tau_A]} \mathbb{E}^x \left[ \int_0^{\tau_A} \mathbb{1}_{X_t \in B} dt \right]$$

for any Borel set  $B \subset \mathbb{R}^n$  can easily be shown to be invariant. By essential uniqueness, any invariant measure is thus normalisable.

A sufficient condition for the process  $(X_t)_{t \geq 0}$  to be positive Harris recurrent will thus automatically be a sufficient condition for the existence of an invariant probability measure. A first such condition is provided by the following result, which is [MT93c, Theorem 4.2].

**Theorem 2.2.11: Positive Harris recurrence condition**

Assume there exist constants  $c, d > 0$ , a function  $f : \mathbb{R}^n \rightarrow [1, \infty)$ , a closed petite set  $C \subset \mathbb{R}^n$ , and a positive function  $V$  such that

$$(\mathcal{L}V)(x) \leq -cf(x) + d\mathbb{1}_C(x) \tag{2.2.5}$$

for all  $x \in \mathbb{R}^n$ . Assume furthermore that  $V$  is bounded on  $C$ . Then the process  $(X_t)_{t \geq 0}$  is positive Harris recurrent, and therefore admits an invariant probability measure  $\pi$ . Furthermore,

$$\langle \pi, f \rangle := \mathbb{E}^\pi[f] = \int_{\mathbb{R}^n} f(x)\pi(dx) < \infty.$$

While Condition (2.2.5) is usually easy to check for a given  $f$  and  $V$ , the requirement that  $C$  be petite may be harder to verify. Fortunately, for Feller diffusions, there exists an alternative criterion for the existence of an invariant measure, which avoids having to check that  $C$  is petite. The following result is [MT93c, Theorem 4.5].



**Theorem 2.2.12: Existence of invariant probability measures for Feller diffusions**

Assume that the diffusion  $(X_t)_{t \geq 0}$  has the Feller property, and that (2.2.5) holds for a compact  $C \subset \mathbb{R}^n$ . Then the diffusion admits an invariant probability measure  $\pi$ . Furthermore, any invariant probability  $\pi$  satisfies  $\langle \pi, f \rangle \leq d/c$ .

**2.2.3 Convergence to the invariant probability measure**

Once the existence of an invariant probability measure  $\pi$  is established, the next natural question is whether the distribution of  $(X_t)_{t \geq 0}$  will converge to  $\pi$ , at least under some conditions on the law of  $X_0$ . There are many choices of norms quantifying such a convergence, and results exist for several of them. Here we consider the following weighted norm on signed measures.

**Definition 2.2.13:  $f$ -norm of a signed measure**

Let  $\mu$  be a signed measure on  $(\mathbb{R}^n, \mathcal{B})$ , and let  $f : \mathbb{R}^n \rightarrow [1, \infty)$  be a measurable function. Then we define the  $f$ -norm of  $\mu$  by

$$\|\mu\|_f = \sup_{g: |g| < f} |\langle \mu, g \rangle|, \quad \langle \mu, g \rangle := \mathbb{E}^\mu[g] = \int_{\mathbb{R}^n} g(x) \mu(dx),$$

where the supremum runs over all measurable functions such that  $|g(x)| \leq f(x)$  for all  $x \in \mathbb{R}^n$ .

**Definition 2.2.14: Exponential ergodicity**

Given a measurable function  $f : \mathbb{R}^n \rightarrow [0, \infty)$ , a diffusion process  $(X_t)_{t \geq 0}$  admitting an invariant probability measure  $\pi$  is called  $f$ -exponentially ergodic if there exist  $\beta > 0$  and a function  $B : \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that

$$\|P_t(x, \cdot) - \pi\|_f \leq B(x) e^{-\beta t}$$

for any  $x \in \mathbb{R}^n$  and  $t \geq 0$ .

The following result, which is [MT93c, Theorem 6.1], provides a condition on Lyapunov functions that guarantees exponential ergodicity.

**Theorem 2.2.15: Condition for exponential ergodicity**

Assume there exist a norm-like function  $V$ , and constants  $c > 0$ ,  $d \in \mathbb{R}$  such that the diffusion  $(X_t)_{t \geq 0}$  satisfies the condition

$$(\mathcal{L}V)(x) \leq -cV(x) + d \tag{2.2.6}$$

for all  $x \in \mathbb{R}^n$ . Assume further that all compact  $K \subset \mathbb{R}^n$  are petite for some discrete-time Markov chain  $(X_{n\Delta})_{n \geq 0}$ . Then the diffusion is exponentially ergodic. More precisely, there exist constants  $\beta, b > 0$  such that

$$\|P_t(x, \cdot) - \pi\|_{1+V} \leq b(1 + V(x)) e^{-\beta t}$$

for any  $x \in \mathbb{R}^n$  and  $t \geq 0$ .

Criterion	Property
$(\mathcal{L}V)(x) \leq cV(x) + d$	Non-explosion
$(\mathcal{L}V)(x) \leq d\mathbb{1}_C(x)$ C compact C compact, all compact K petite	Non-evanescence Harris recurrence, existence of invariant measure
$(\mathcal{L}V)(x) \leq -cf(x) + d\mathbb{1}_C(x)$ C closed petite or C compact and diffusion Feller	Positive Harris recurrence and existence of invariant probability measure
$(\mathcal{L}V)(x) \leq -cV(x) + d$ all compact K petite for $(X_{n\Delta})_{n \geq 0}$	Exponential ergodicity

**Table 2.1** – Summary of the main results of the Meyn–Tweedie theory for diffusions. Here  $V$  is a norm-like function,  $c$  and  $d$  are positive constants, and  $C$  and  $K$  are subsets of  $\mathbb{R}^n$ .

Table 2.1 summarises the main results on invariant measures stated in this section. Again, while condition (2.2.6) is usually easy to check for a given Lyapunov function  $V$ , the requirement that all compact subsets be petite is often more difficult to verify. This is why we present in the next section an alternative approach, due to Martin Hairer and Jonathan Mattingly, that provides an exponential ergodicity criterion in a slightly different norm, and avoids any condition on sets being petite.

#### Exercise 2.2.16: One-dimensional diffusion

Consider a one-dimensional diffusion of the form  $dX_t = -V'(X_t)dt + \sqrt{2}dW_t$  (cf. Example 2.1.1). Assume that  $V(x)$  behaves like  $x^\alpha$  for some  $\alpha > 0$  for  $x$  large. Taking  $V$  as a Lyapunov function, discuss what one can say on convergence to the invariant probability measure depending on the value of  $\alpha$ .

#### 2.2.4 A simplified proof by Hairer and Mattingly

We present here a convergence criterion from [HM11], which applies to discrete-time Markov processes. This is, however, not really a restriction, because if  $P_t$  is the semi-group of a diffusion  $(X_t)_{t \geq 0}$ , then for any  $\delta > 0$ ,  $P_\delta$  generates the embedded discrete-time Markov chain  $(X_{\delta n})_{n \in \mathbb{N}}$  obtained by restricting  $t$  to integer multiples of  $\delta$ . An invariant measure  $\pi$  of the diffusion is clearly also invariant for the discrete-time Markov chain, and it is not difficult to convert a convergence result in discrete time to a convergence result in continuous time. To avoid confusion, we will denote the discrete semi-group by  $\mathcal{P}$ , and its actions on bounded measurable functions  $f$  and signed measures  $\mu$  by

$$(\mathcal{P}f)(x) = \mathbb{E}^x[f(X_\delta)] := \int_{\mathbb{R}^n} f(y)\mathcal{P}(x, dy),$$

$$(\mu\mathcal{P})(A) = \mathbb{P}^\mu\{X_\delta \in A\} := \int_{\mathbb{R}^n} \mathcal{P}(x, A)\mu(dx).$$

The convergence result of [HM11] requires two rather simple conditions on  $\mathcal{P}$ . The first one is a discrete-time analogue of (2.2.6), which guarantees that the Markov process does not escape to infinity.

**Assumption 2.2.17: Geometric drift condition**

There exist a function  $V : \mathbb{R}^n \rightarrow [0, \infty)$  and constants  $d \geq 0$  and  $\gamma \in (0, 1)$  such that

$$(\mathcal{P}V)(x) \leq \gamma V(x) + d \quad (2.2.7)$$

for all  $x \in \mathbb{R}^n$ .

Note that  $\gamma$  is the discrete-time analogue of  $e^{-\delta c}$  in continuous time, so that (2.2.7) is indeed the discrete analogue of (2.2.6). The second condition is a form of irreducibility condition, which is analogous to the conditions on sets being petite that we encountered above, but simpler to verify. It is also similar to what is known as *Doebelin condition* in the Markov chain literature.

**Assumption 2.2.18: Minorisation condition**

Let  $C = \{x \in \mathbb{R}^n : V(x) < R\}$  for some  $R > 2d(1 - \gamma)^{-1}$ . Then there exists  $\alpha \in (0, 1)$  and a probability measure  $\nu$  such that

$$\inf_{x \in C} \mathcal{P}(x, A) \geq \alpha \nu(A) \quad (2.2.8)$$

holds for all Borel sets  $A \subset \mathbb{R}^n$ .

Under these two conditions, the main result of [HM11] is the following statement, which provides both existence of a unique invariant measure and convergence to this measure. Convergence takes place in the weighted supremum norm defined by

$$\|f\|_{1+V} = \sup_{x \in \mathbb{R}^n} \frac{|f(x)|}{1 + V(x)}.$$

**Theorem 2.2.19: Exponential ergodicity in discrete time**

If Assumptions 2.2.17 and 2.2.18 hold, then  $\mathcal{P}$  admits a unique invariant probability measure  $\pi$ . Furthermore, there exist constants  $M > 0$  and  $\gamma \in (0, 1)$  such that

$$\|\mathcal{P}^n f - \langle \pi, f \rangle\|_{1+V} \leq M \gamma^n \|f - \langle \pi, f \rangle\|_{1+V} \quad (2.2.9)$$

holds for all measurable functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\|f\|_{1+V} < \infty$ .

Since the proof of this result is rather elementary (and also quite elegant), we will provide the details of it in the remainder of this section. Our presentation follows closely the article [HM11]. We first recall the definition of the total variation distance between probability measures.

**Definition 2.2.20: Total variation distance**

Let  $\mu$  and  $\nu$  be two probability measures on a measure space  $(\mathcal{X}, \mathcal{F})$ . The *total variation distance* between  $\mu$  and  $\nu$  is defined as

$$\|\mu - \nu\|_{\text{TV}} = 2 \sup \{ |\mu(A) - \nu(A)| : A \in \mathcal{F} \}.$$

It is known that the total variation distance between  $\mu$  and  $\nu$  coincides with the  $L^1$ -distance:

**Lemma 2.2.21**

For any probability measures  $\mu$  and  $\nu$  on  $(\mathcal{X}, \mathcal{F})$ , one has

$$\|\mu - \nu\|_{\text{TV}} = \int_{\mathcal{X}} |\mu - \nu|(\mathrm{d}x).$$

**PROOF:** Consider the case where the measures have densities with respect to Lebesgue measure, and let  $B = \{x \in \mathbb{R}^n : \mu(x) > \nu(x)\}$ . Note that since  $\mu$  and  $\nu$  are probability measures, we have

$$0 \leq \mu(B) - \nu(B) = 1 - \mu(B^c) - [1 - \nu(B^c)] = \nu(B^c) - \mu(B^c),$$

which implies

$$\int_{\mathcal{X}} |\mu - \nu|(\mathrm{d}x) = (\mu - \nu)(B) + (\nu - \mu)(B^c) = 2(\mu - \nu)(B). \quad (2.2.10)$$

For any Borel set  $A \in \mathcal{F}$ , we can write

$$\mu(A) - \nu(A) \leq \int_{A \cap B} (\mu - \nu)(\mathrm{d}x) \leq \int_B (\mu - \nu)(\mathrm{d}x) = \mu(B) - \nu(B),$$

since  $\mu - \nu$  is negative on  $A \cap B^c$ , and positive on  $A^c \cap B$ . A similar argument yields

$$\nu(A) - \mu(A) \leq \int_{B^c} (\nu - \mu)(\mathrm{d}x) = \nu(B^c) - \mu(B^c) = \mu(B) - \nu(B).$$

We have thus shown the  $|\mu(A) - \nu(A)| \leq (\mu - \nu)(B)$ , where equality holds whenever  $A = B$  or  $A = B^c$ . The result thus follows from (2.2.10).  $\square$

The main idea of the proof of Theorem 2.2.19 is to work with a whole family of equivalent norms. Instead of just  $\|f\|_{1+V}$ , we thus consider the norms  $\|f\|_{1+\beta V}$  where  $\beta > 0$  is a scale parameter. We also consider the dual metric on probability measures given by

$$\begin{aligned} \rho_{\beta}(\mu, \nu) &= \sup_{f: \|f\|_{1+\beta V} \leq 1} \int_{\mathbb{R}^n} f(x)(\mu - \nu)(\mathrm{d}x) \\ &= \sup_{f: \|f\|_{1+\beta V} \neq 0} \frac{1}{\|f\|_{1+\beta V}} \int_{\mathbb{R}^n} f(x)(\mu - \nu)(\mathrm{d}x), \end{aligned} \quad (2.2.11)$$

which is in fact equivalent to the weighted total variation distance given by

$$\rho_{\beta}(\mu, \nu) = \int_{\mathbb{R}^n} (1 + \beta V(x)) |\mu - \nu|(\mathrm{d}x).$$

In particular, for  $\beta = 0$ , we have  $\rho_0(\mu, \nu) = \|\mu - \nu\|_{\text{TV}}$ . The supremum in (2.2.11) is attained for  $f(x) = [1 + \beta V(x)][\mathbb{1}_B(x) - \mathbb{1}_{B^c}(x)]$ , with  $B$  as in the proof of the above lemma.

The key result for the distance  $\rho_{\beta}$  is the following.

**Proposition 2.2.22: Contraction estimate in  $\rho_{\beta}$  distance**

If Assumptions 2.2.17 and 2.2.18 hold, then there exist  $\bar{\alpha} \in (0, 1)$  and  $\beta > 0$  such that

$$\rho_{\beta}(\mu^{\mathcal{P}}, \nu^{\mathcal{P}}) \leq \bar{\alpha} \rho_{\beta}(\mu, \nu)$$

holds for all probability measures  $\mu, \nu$  on  $\mathbb{R}^n$ . More precisely, for any  $\alpha_0 \in (0, \alpha)$  and any  $\gamma_0 \in (\gamma + 2dR^{-1}, 1)$ , one can choose

$$\beta = \frac{\alpha_0}{d}, \quad \bar{\alpha} = \left(1 - (\alpha - \alpha_0)\right) \vee \frac{2 + R\beta\gamma_0}{2 + R\beta}.$$

To prove this result, we introduce an alternative definition of  $\rho_\beta$ . Consider first the function

$$d_\beta(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 2 + \beta V(x) + \beta V(y) & \text{if } x \neq y, \end{cases}$$

which can easily be checked to be a metric on  $\mathbb{R}^n$ . This metric induces the Lipschitz seminorm

$$\|f\|_\beta = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d_\beta(x, y)}, \quad (2.2.12)$$

and a dual metric on probability measures given by

$$\rho_\beta^*(\mu, \nu) = \sup_{f: \|f\|_\beta \leq 1} \int_{\mathbb{R}^n} f(x)(\mu - \nu)(dx).$$

Note that the supremum is taken on a different set of functions than in (2.2.11).

**Lemma 2.2.23: Equivalence of norms**

We have

$$\|f\|_\beta = \inf_{c \in \mathbb{R}} \|f + c\|_{1+\beta V}.$$

In particular,  $\rho_\beta^* = \rho_\beta$ .

**PROOF:** We first note that since  $|f(x)| \leq \|f\|_{1+\beta V}(1 + \beta V(x))$  for all  $x \in \mathbb{R}^n$ , we have

$$\frac{|f(x) - f(y)|}{2 + \beta V(x) + \beta V(y)} \leq \frac{|f(x)| + |f(y)|}{2 + \beta V(x) + \beta V(y)} \leq \|f\|_{1+\beta V}$$

for all  $x, y \in \mathbb{R}^n$ , so that  $\|f\|_\beta \leq \|f\|_{1+\beta V}$ . It follows from the definition (2.2.12) of  $\|f\|_\beta$  that this seminorm is invariant under addition of a constant to  $f$ , so that

$$\|f\|_\beta \leq \inf_{c \in \mathbb{R}} \|f + c\|_{1+\beta V}.$$

To prove the reverse inequality, it suffices by homogeneity of the norm to show that it holds for  $f$  with  $\|f\|_\beta = 1$ . We set

$$c^* = \inf_{x \in \mathbb{R}^n} (1 + \beta V(x) - f(x)).$$

For any  $x, y \in \mathbb{R}^n$ , we have

$$f(x) \leq |f(y)| + |f(x) - f(y)| \leq |f(y)| + 2 + \beta V(x) + \beta V(y),$$

which implies

$$1 + \beta V(x) - f(x) \geq -1 - \beta V(y) - |f(y)|.$$

Since  $V(y)$  is finite at one point at least,  $c^*$  is bounded below, and hence  $|c^*| < \infty$ . Now we observe that on one hand,

$$f(x) + c^* \leq f(x) + 1 + \beta V(x) - f(x) = 1 + \beta V(x),$$

while on the other hand,

$$\begin{aligned} f(x) + c^* &= \inf_{y \in \mathbb{R}^n} (f(x) + 1 + \beta V(y) - f(y)) \\ &\geq \inf_{y \in \mathbb{R}^n} (1 + \beta V(y) - \|f\|_\beta d_\beta(x, y)) \\ &\geq -1 - \beta V(x), \end{aligned}$$

where we have used the fact that  $\|f\|_\beta = 1$ . Hence  $|f(x) + c^*| \leq 1 + \beta V(x)$ , and thus

$$\inf_{c \in \mathbb{R}} \|f + c\|_{1+\beta V} \leq \|f + c^*\|_{1+\beta V} \leq 1 = \|f\|_\beta,$$

proving the reverse inequality. The equality of  $\rho_\beta^*$  and  $\rho_\beta$  follows from the fact that the unit balls  $\{f : \|f\|_{1+\beta V} \leq 1\}$  and  $\{f : \|f\|_\beta \leq 1\}$  only differ by additive constants, and homogeneity of the norms.  $\square$

**PROOF OF PROPOSITION 2.2.22.** We prove that under Assumptions 2.2.17 and 2.2.18, one has

$$\|\mathcal{P}f\|_\beta \leq \bar{\alpha} \|f\|_\beta. \quad (2.2.13)$$

Fix a test function  $f$  with  $\|f\|_\beta \leq 1$ . By Lemma 2.2.23 we can assume, without loss of generality, that one also has  $\|f\|_{1+\beta V} \leq 1$ . By homogeneity, it then suffices to show that

$$|(\mathcal{P}f)(x) - (\mathcal{P}f)(y)| \leq \bar{\alpha} d_\beta(x, y).$$

Since the claim is true for  $x = y$ , we consider the case  $x \neq y$ . We treat separately the cases  $V(x) + V(y) \geq R$  and  $V(x) + V(y) < R$ .

- If  $V(x) + V(y) \geq R$ , we note that

$$|(\mathcal{P}f)(x)| \leq \|f\|_{1+\beta V} \int_{\mathbb{R}^n} (1 + \beta V(y)) \mathcal{P}(x, dy) \leq 1 + \beta(\mathcal{P}V)(x). \quad (2.2.14)$$

Therefore, the geometric drift condition (2.2.7) yields

$$\begin{aligned} |(\mathcal{P}f)(x) - (\mathcal{P}f)(y)| &\leq 2 + \beta(\mathcal{P}V)(x) + \beta(\mathcal{P}V)(y) \\ &\leq 2 + \beta\gamma V(x) + \beta\gamma V(y) + 2\beta d \\ &\leq 2 + \beta\gamma_0 V(x) + \beta\gamma_0 V(y), \end{aligned}$$

where we have set  $\gamma_0 = \gamma + 2dR^{-1}$  and used  $V(x) + V(y) \geq R$ . We now set

$$\gamma_1 = \frac{2 + \beta R \gamma_0}{2 + \beta R}.$$

One readily checks that  $2(1 - \gamma_1) = \beta R(\gamma_1 - \gamma_0) \leq \beta(\gamma_1 - \gamma_0)(V(x) + V(y))$ , so that

$$|(\mathcal{P}f)(x) - (\mathcal{P}f)(y)| \leq \gamma_1 (2 + \beta V(x) + \beta V(y)) = \gamma_1 d_\beta(x, y). \quad (2.2.15)$$

- If  $V(x) + V(y) < R$ , then  $x, y \in C$ . We introduce the Markov kernel  $\widetilde{\mathcal{P}}$  defined by

$$\widetilde{\mathcal{P}}(x, A) = \frac{1}{1 - \alpha} \mathcal{P}(x, A) - \frac{\alpha}{1 - \alpha} \nu(A).$$

Note that  $\widetilde{\mathcal{P}}(x, \mathbb{R}^n) = 1$ , while the minorisation condition (2.2.8) implies that  $\widetilde{\mathcal{P}}(x, A)$  is always positive, as required. Then we have

$$(\mathcal{P}f)(x) = (1 - \alpha)(\widetilde{\mathcal{P}}f)(x) + \alpha \int_{\mathbb{R}^n} f(y) \nu(dy),$$

showing that

$$\begin{aligned} |(\mathcal{P}f)(x) - (\mathcal{P}f)(y)| &= (1 - \alpha) |(\widetilde{\mathcal{P}}f)(x) - (\widetilde{\mathcal{P}}f)(y)| \\ &\leq (1 - \alpha) [2 + \beta(\widetilde{\mathcal{P}}V)(x) + \beta(\widetilde{\mathcal{P}}V)(y)] \\ &\leq 2(1 - \alpha) + \beta(\mathcal{P}V)(x) + \beta(\mathcal{P}V)(y) \\ &\leq 2(1 - \alpha) + \gamma\beta[V(x) + V(y)] + 2\beta d. \end{aligned}$$

Here, to obtain the second line, we have used a similar argument as in (2.2.14), while the third line uses the fact that

$$(\widetilde{\mathcal{P}}V)(x) \leq \frac{1}{1-\alpha}(\mathcal{P}V)(x)$$

since  $V$  is non-negative. It follows that setting

$$\beta = \frac{\alpha_0}{d}, \quad \gamma_2 = \gamma \vee (1 - (\alpha - \alpha_0))$$

for some  $\alpha_0 \in (0, \alpha)$ , one obtains

$$|(\mathcal{P}f)(x) - (\mathcal{P}f)(y)| \leq \gamma_2 d_\beta(x, y). \quad (2.2.16)$$

It follows from (2.2.15) and (2.2.16) that

$$|(\mathcal{P}f)(x) - (\mathcal{P}f)(y)| \leq \bar{\alpha} d_\beta(x, y), \quad \bar{\alpha} = \gamma_1 \vee \gamma_2.$$

Since  $\gamma_1 \geq \gamma$ , this implies (2.2.13). The result then follows from the fact that  $d_\beta$  is the norm dual to  $\|\cdot\|_\beta$ . Indeed, we have

$$\begin{aligned} \rho_\beta^*(\mu^{\mathcal{P}}, \nu^{\mathcal{P}}) &= \sup_{f: \|f\|_\beta \neq 0} \frac{1}{\|f\|_\beta} \langle \mu^{\mathcal{P}} - \nu^{\mathcal{P}}, f \rangle = \sup_{f: \|f\|_\beta \neq 0} \frac{1}{\|f\|_\beta} \langle \mu - \nu, \mathcal{P}f \rangle \\ &\leq \sup_{\tilde{f}: \|\tilde{f}\|_\beta \neq 0} \frac{\bar{\alpha}}{\|\tilde{f}\|_\beta} \langle \mu - \nu, \tilde{f} \rangle = \bar{\alpha} \rho_\beta^*(\mu, \nu), \end{aligned}$$

and we obtain the conclusion since  $\rho_\beta = \rho_\beta^*$ .  $\square$

To conclude the proof of Theorem 2.2.19, it remains to prove existence of the invariant measure, which can be done by a contraction argument.

**PROOF OF THEOREM 2.2.19.** We fix some  $x \in \mathbb{R}^n$ , and define for any  $n \in \mathbb{N}$  the measure  $\mu_n^x = \delta_x \mathcal{P}^n$ . Then by Proposition 2.2.22, we have

$$\rho_\beta(\mu_{n+1}^x, \mu_n^x) \leq \bar{\alpha}^n \rho_\beta(\mu_1^x, \delta_x) \quad (2.2.17)$$

for some  $\bar{\alpha} \in (0, 1)$  and  $\beta > 0$ . Therefore,  $(\mu_n^x)_n$  is a Cauchy sequence. It is known that the total variation distance is complete for the space of measures with finite mass, implying that  $\rho_\beta$  is complete for the space of probability measures integrating  $V$ . Therefore, there exists a probability measure  $\pi = \mu_\infty$  such that  $\rho_\beta(\mu_n^x, \mu_\infty) \rightarrow 0$  as  $n \rightarrow \infty$ . This implies that  $\mu_n^x$  converges to  $\mu_\infty$  in total variation. Since  $\mathcal{P}$  is a contraction in total variation, it follows that  $\mu_\infty \mathcal{P} = \lim_{n \rightarrow \infty} \mu_n^x \mathcal{P} = \lim_{n \rightarrow \infty} \mu_{n+1}^x = \mu_\infty$ . In order to show (2.2.9), we observe that

$$(\mathcal{P}^n f)(x) - \langle \pi, f \rangle = \langle \delta_x - \pi, \mathcal{P}^n f \rangle = \langle \delta_x \mathcal{P}^n - \pi, f \rangle = \langle \mu_n^x - \pi, f \rangle.$$

Therefore, by the definition (2.2.11) of  $\rho_\beta$ , we have for any  $\beta \geq 0$ ,

$$\|\mathcal{P}^n f - \langle \pi, f \rangle\|_{1+\beta V} = \sup_{x \in \mathbb{R}^n} \frac{|\langle \mu_n^x - \pi, f \rangle|}{1 + \beta V(x)} \leq \|f\|_{1+\beta V} \sup_{x \in \mathbb{R}^n} \frac{\rho_\beta(\mu_n^x, \pi)}{1 + \beta V(x)}. \quad (2.2.18)$$

Now it follows from (2.2.17) and a telescopic sum argument that for  $\beta$  as in Proposition 2.2.22, one has  $\rho_\beta(\mu_n^x, \pi) \leq M(x) \bar{\alpha}^n$ , where  $M(x)$  is proportional to

$$\rho_\beta(\mu_1^x, \delta_x) = \int_{\mathbb{R}^n} (1 + \beta V(y)) |\mu_1^x - \delta_x|(dy) \leq \langle \mu_1^x, 1 + \beta V \rangle + 1 + \beta V(x).$$

Since  $\langle \mu_1^x, V \rangle = (\mathcal{P}V)(x) \leq \gamma V(x) + d$  by the geometric drift condition (2.2.7), the supremum in (2.2.18) has order  $\bar{\alpha}^n$ , proving the result for this particular  $\beta$ . However, all  $\|\cdot\|_{1+\beta V}$ -norms are equivalent, so that the result holds in particular for  $\beta = 1$ .  $\square$

## 2.3 Garrett Birkhoff's approach

The aim of this section is to present a slightly different approach to estimating the rate of convergence to an invariant probability distribution, due to Garrett Birkhoff [Bir57]. Compared to the approaches we have discussed so far, it has the following advantages:

1. The proof has a more transparent geometric interpretation, that helps understand the minorisation condition (2.2.8) we have seen in the last section.
2. As Theorem 2.2.19, the result provides explicit bounds on the rate of convergence to the invariant probability.
3. The result also works for submarkovian processes, that is, processes in which the total probability decreases. In that case, it provides information on the principal eigenvalue of the process, as well as on the spectral gap to the next-to-leading eigenvalue.

As in the last subsection, the approach applies to discrete-time Markov chains. For simplicity, we are going to assume that the transition kernel  $\mathcal{P}$  has a density  $p(x, y)$  defined on  $\mathcal{X} \times \mathcal{X}$  for a domain  $\mathcal{X} \subset \mathbb{R}^n$  (or possibly on a more general Banach space). The transition kernel thus acts on bounded measurable functions  $f$  and on signed measures  $\mu$  according to

$$\begin{aligned} (\mathcal{P}f)(x) &= \int_{\mathcal{X}} p(x, y)f(y) \, dy = \mathbb{E}^x[f(X_1)] , \\ (\mu\mathcal{P})(dy) &= \int_{\mathcal{X}} \mu(dx)p(x, y) \, dy = \mathbb{P}^\mu\{X_1 \in dy\} . \end{aligned} \quad (2.3.1)$$

The main property that will guarantee convergence to an invariant probability distribution (or to a so-called quasistationary distribution in the submarkovian case) is the following. Note the similarity of the lower bound with the minorisation condition (2.2.8).

### Definition 2.3.1: Uniform positivity

The transformation  $\mathcal{P}$  is called *uniformly positive* if there exist strictly positive functions  $s, m : \mathcal{X} \rightarrow (0, +\infty)$  and a constant  $L$  such that

$$s(x)m(y) \leq p(x, y) \leq Ls(x)m(y) \quad \forall x, y \in \mathcal{X} . \quad (2.3.2)$$

The main results we are going to prove are as follows. The first one is an existence result for an invariant probability measure, or for its equivalent if the chain is submarkovian. It is thus a generalisation to integral operators of the well-known Perron–Frobenius theorem, which was first obtained by Jentzsch [Jen12].

### Theorem 2.3.2: Perron–Frobenius–Jentzsch theorem

If  $\mathcal{P}$  is uniformly positive, there exist  $\lambda_0 > 0$ , a bounded measurable function  $h_0 : \mathcal{X} \rightarrow \mathbb{R}_+$ , and a probability measure  $\pi_0$  on  $\mathcal{X}$  such that

$$\begin{aligned} (\mathcal{P}h_0)(x) &= \lambda_0 h_0(x) , \\ (\pi_0\mathcal{P})(A) &= \lambda_0 \pi_0(A) \end{aligned}$$

for all  $x \in \mathcal{X}$  and all Borel sets  $A \subset \mathcal{X}$ . In particular, in the Markovian case  $\mathcal{P}(x, \mathcal{X}) = 1$  for all  $x \in \mathcal{X}$ , one has  $\lambda_0 = 1$  and  $h_0(x) = 1$  for all  $x \in \mathcal{X}$ .

The number  $\lambda_0$  is called the *principal eigenvalue* of the Markov process, the function  $h_0$  is called the *principal eigenfunction*, while  $\pi_0$  is called the *quasistationary distribution* (in the submarkovian case, when  $\lambda_0 < 1$ ), and is equal to the stationary distribution in the Markovian



case, when  $\lambda_0 = 1$ . The main interest of the uniform positivity condition is the following result on speed of convergence towards the principal eigenfunction  $h_0$ .

**Theorem 2.3.3: Spectral gap estimate**

If  $\mathcal{P}$  is uniformly positive, then for any bounded measurable  $f : \mathcal{X} \rightarrow \mathbb{R}_+$ , there exist finite constants  $M_1(f), M_2(f)$  such that

$$|\mathcal{P}^n f(x) - \lambda_0^n M_1(f) h_0(x)| \leq M_2(f) \lambda_0^n \left(1 - \frac{1}{L^2}\right)^n h_0(x) \quad (2.3.3)$$

for all  $x \in \mathcal{X}$ . In particular,

$$M_1(f) = \frac{\langle \pi_0, f \rangle}{\langle \pi_0, h_0 \rangle},$$

which reduces in the Markovian case to  $M_1(f) = \langle \pi_0, f \rangle$ .

Note that in the Markovian case  $\lambda_0 = 1$ , the bound (2.3.3) is equivalent to the exponential ergodicity result (2.2.9) of Theorem 2.2.19, but with an explicit value of the contraction constant  $\gamma$  given by  $1 - L^{-2}$ .

In what follows, we are going to provide a detailed proof of Theorem 2.3.2, starting with some simpler situations in order to build the intuition.

### 2.3.1 Two-dimensional case

The simplest case occurs when  $\mathcal{X}$  is a discrete set of cardinality 2. Then  $\mathcal{P}$  is a linear operator on  $\mathcal{E} = \mathbb{R}^2$ , that is, a  $2 \times 2$  matrix

$$\mathcal{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with strictly positive entries. Therefore  $\mathcal{P}$  maps the cone  $\mathcal{E}^+ = \mathbb{R}_+ \times \mathbb{R}_+$  strictly into itself. Iterating  $\mathcal{P}$ , the image of  $\mathcal{E}^+$  becomes thinner and thinner, and concentrates on the eigenvector of  $\mathcal{P}$  for the largest eigenvalue (Figure 2.1). However, unless the principal eigenvalue  $\lambda_0$  of  $\mathcal{P}$  is 1, iterates of a vector in  $\mathcal{E}^+$  will not converge to an eigenvector: they will shrink to 0 if  $\lambda_0 < 1$  and diverge if  $\lambda_0 > 1$ .

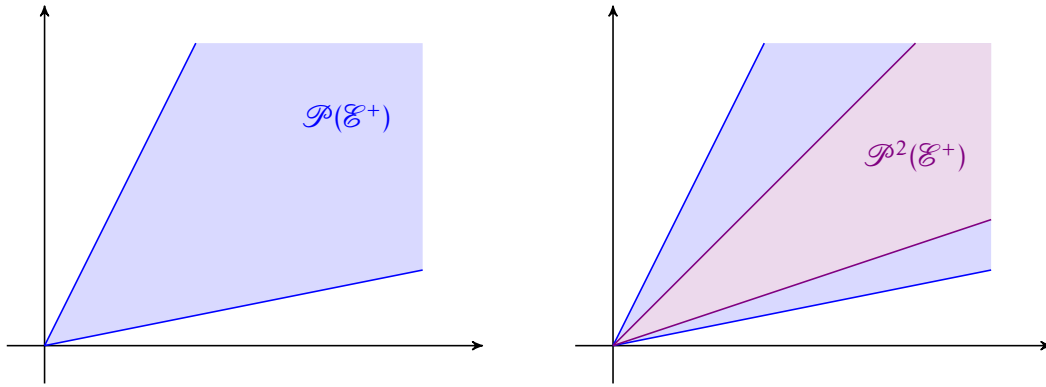
To avoid this, one can identify all vectors  $f, g$  such that  $f = \lambda g$  for some  $\lambda > 0$ . In other words, this amounts to working on the projective line. Iterates of a projective line in  $\mathcal{E}^+$  will converge to the eigenspace associated with  $\lambda_0$ .

Birkhoff introduces Hilbert's *projective metric* by defining, for  $f = (f_1, f_2)$  and  $g = (g_1, g_2) \in \mathcal{E}^+$ , the distance

$$\theta(f, g) = \left| \log \left( \frac{f_2 g_1}{f_1 g_2} \right) \right|.$$

Note that this distance is infinite if  $f$  or  $g$  belongs to a coordinate axis; in fact, it induces a hyperbolic geometry. Also note that by definition,

$$\theta(\lambda f, \mu g) = \theta(f, g) \quad \forall \lambda, \mu > 0. \quad (2.3.4)$$



**Figure 2.1** – Geometrical explanation for the convergence of iterates of a two-dimensional positive operator  $\mathcal{P}$ . The positive cone  $\mathcal{E}^+ = \mathbb{R}_+ \times \mathbb{R}_+$  is mapped to a smaller cone  $\mathcal{P}(\mathcal{E}^+)$ , strictly contained in  $\mathcal{E}_+$ . Each iterate  $\mathcal{P}^n(\mathcal{E}^+)$  has a strictly smaller diameter, and the sequence of these iterates converges to a half line.

**Lemma 2.3.4: Projective operator norm of  $\mathcal{P}$**

The operator norm of  $\mathcal{P}$  in the projective metric is given by

$$\sup_{f,g \in \mathcal{E}^+} \frac{\theta(\mathcal{P}f, \mathcal{P}g)}{\theta(f, g)} = \tanh\left(\frac{\Delta}{4}\right), \quad (2.3.5)$$

where  $\Delta = |\log(ad/bc)|$  is the diameter, in the projective norm, of  $\mathcal{P}(\mathcal{E}^+)$ .

**PROOF:** We may assume, without limitation of generality, that  $ad > bc$  (note that in the case  $ad = bc$ , the cone  $\mathcal{E}^+$  is projected to a half line by  $\mathcal{P}$ , so that the operator norm of  $\mathcal{P}$  is zero). Let  $f = (1, x)$  be a point in  $\mathcal{E}^+$ . Then

$$\mathcal{P}f = (a + bx, c + dx) = \lambda(1, \varphi(x)),$$

where  $\lambda = a + bx$ , and  $\varphi$  denotes the homographic transformation

$$x \mapsto \varphi(x) = \frac{c + dx}{a + bx}.$$

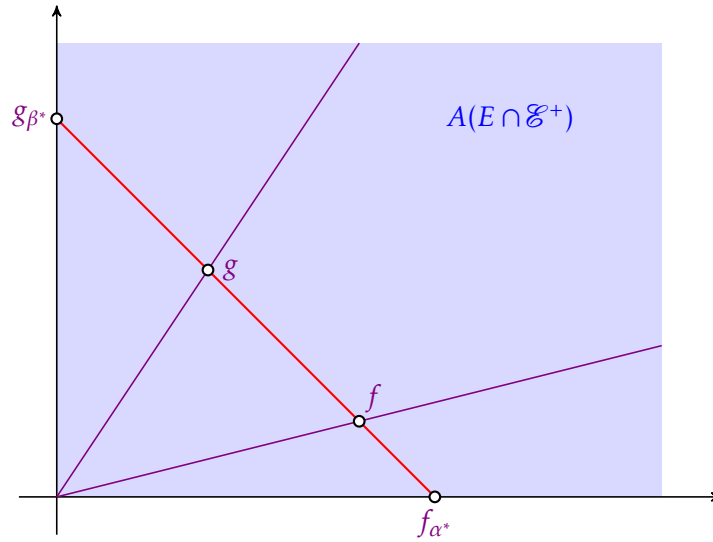
The projective distance between two infinitesimally close points  $f = (1, x)$  and  $(1, x + dx)$  is given by  $d\theta(x) = |\log(x/(x + dx))| = |dx|/x$ . Therefore, we obtain

$$\frac{d\theta(\varphi(x))}{d\theta(x)} = \frac{x\varphi'(x)}{\varphi(x)} = \frac{x(ad - bc)}{(a + bx)(c + dx)}.$$

It is straightforward to check that this expression is maximal in  $x = \sqrt{ac/(bd)}$ , where it has value

$$\frac{ad - bc}{ad + bc + 2\sqrt{abcd}} = \frac{e^\Delta - 1}{1 + e^\Delta + 2e^{\Delta/2}} = \tanh\left(\frac{\Delta}{4}\right).$$

Since this value gives the smallest rate of contraction, the claim follows.  $\square$



**Figure 2.2** – Construction of the map  $A$  via the points  $f_{\alpha^*}$  and  $g_{\beta^*}$ . The points  $f, g, \dots$  should be viewed as equivalence classes represented by violet half-lines.

### 2.3.2 General vector space

Let now  $\mathcal{E}$  be a general vector space, of finite or infinite dimension. Again, let  $\mathcal{E}^+$  be the cone of elements whose components are all non-negative.

#### Definition 2.3.5: Projective metric

Let  $f, g \in \mathcal{E}^+$ . Consider the two-dimensional vector space  $E$  spanned by  $f$  and  $g$ . The intersection  $C = E \cap \mathcal{E}^+$  is a cone (it is invariant under multiplication by positive constants). There exists a linear map  $A$ , mapping  $C$  to  $\mathbb{R}_+ \times \mathbb{R}_+$ . We define

$$\theta(f, g; \mathcal{E}^+) = \theta(Af, Ag).$$

The definition does not depend on the choice of the map  $A$  (this follows from (2.3.4)).  $\theta$  is called the *projective metric* associated with  $\mathcal{E}^+$ .

To understand this definition better, consider the line  $\{f_\alpha = f - \alpha g, \alpha \in \mathbb{R}\}$ . If  $\alpha \leq 0$ , then  $f_\alpha$ , being the sum of two positive elements, is in  $\mathcal{E}^+$ . When  $\alpha > 0$ , however, the components of  $f_\alpha$  decrease with increasing  $\alpha$ , and change sign at some point. Let

$$\alpha^* = \sup\{\alpha > 0: f - \alpha g \in \mathcal{E}^+\} \quad (2.3.6)$$

(see Figure 2.2). Similarly, we define

$$\beta^* = \sup\{\beta > 0: g - \beta f \in \mathcal{E}^+\}. \quad (2.3.7)$$

Then the linear map of matrix (in the basis  $(f, g)$ )

$$A = \begin{pmatrix} 1 & \beta^* \\ \alpha^* & 1 \end{pmatrix}$$

maps  $f_{\alpha^*}$  to a multiple of  $(1, 0)$  and  $f_{\beta^*}$  to a multiple of  $(0, 1)$ . It thus satisfies the definition. Furthermore we have  $Af = (1, \alpha^*)$  and  $Ag = (\beta^*, 1)$ , so that

$$\theta(f, g; \mathcal{E}^+) = \theta(Af, Ag) = |\log(\alpha^* \beta^*)|. \quad (2.3.8)$$

**Proposition 2.3.6: Operator norm of  $\mathcal{P}$** 

Let  $\mathcal{P} : \mathcal{E}^+ \rightarrow \mathcal{E}^+$  be a linear map. If  $\mathcal{P}(\mathcal{E}^+)$  has finite diameter  $\Delta$ , then the operator norm of  $\mathcal{P}$  is given by

$$\sup_{f, g \in \mathcal{E}^+} \frac{\theta(\mathcal{P}f, \mathcal{P}g; \mathcal{E}^+)}{\theta(f, g; \mathcal{E}^+)} = \tanh\left(\frac{\Delta}{4}\right).$$

PROOF: If  $\theta(f, g; \mathcal{E}^+) < \infty$ , let  $a, b$  be the endpoints of the intersection of  $\mathcal{E}^+$  with the line through  $f$  and  $g$ . By definition of  $\Delta$ ,  $\theta(\mathcal{P}a, \mathcal{P}b; \mathcal{E}^+) \leq \Delta$ . Thus the operator norm is bounded by  $\tanh(\Delta/4)$  as a consequence of (2.3.5). To show equality, one uses an approximation argument for a sequence of  $(f_n, g_n)$  of growing projective distance.  $\square$

**Theorem 2.3.7: Convergence of iterates in projective space**

If  $\mathcal{P}(\mathcal{E}^+)$  has finite diameter  $\Delta$  and the cone  $\mathcal{E}^+$  is complete with respect to the distance  $\theta$ , then there is a unique ray  $h$  in  $\mathcal{E}^+$  to which  $\mathcal{P}^n f$  converges for all  $f \in \mathcal{E}^+$ .

The proof is a standard contraction argument. A proof of completeness will be given in Corollary 2.3.12 below.

**2.3.3 Integral transformations and Jentzsch's theorem**

We now return to the situation described at the beginning of this section, where  $\mathcal{X}$  is a Borel set of  $\mathbb{R}^n$ , and  $\mathcal{E}$  is the Banach space of continuous functions  $f : \mathcal{X} \rightarrow \mathbb{R}$ , equipped with the supremum norm. Let  $\mathcal{E}^+$  denote the cone of positive functions  $f : \mathcal{X} \rightarrow \mathbb{R}_+$ . Consider the integral operator  $\mathcal{P} : \mathcal{E}^+ \rightarrow \mathcal{E}^+$  defined by (2.3.1).

**Proposition 2.3.8: Bound on the projective diameter**

If  $\mathcal{P}$  satisfies the uniform positivity condition (2.3.2), then the diameter of  $\mathcal{P}(\mathcal{E}^+)$  satisfies

$$\Delta \leq 2 \log L.$$

PROOF: Let  $f, g \in \mathcal{E}^+$ . Without limiting the generality, we may assume

$$\int_{\mathcal{X}} f(y)m(y)dy = \int_{\mathcal{X}} g(y)m(y)dy = 1$$

This implies

$$s(x) \leq (\mathcal{P}f)(x), (\mathcal{P}g)(x) \leq Ls(x) \quad \forall x \in \mathcal{X}.$$

It follows that  $(\mathcal{P}f) - \frac{1}{L}(\mathcal{P}g) \geq 0$  and  $(\mathcal{P}g) - \frac{1}{L}(\mathcal{P}f) \geq 0$ . Thus  $\alpha^*, \beta^*$  defined in (2.3.6) and (2.3.7) are greater or equal than  $1/L$ , that is  $1/\alpha^*, 1/\beta^* \leq L$  and the result follows from (2.3.8).  $\square$

Applying Theorem 2.3.7, we recover Jentzsch's generalisation of the Perron–Frobenius theorem to integral operators [Jen12]:

**Theorem 2.3.9: Perron–Frobenius theorem for integral operators**

If  $\mathcal{P}$  is uniformly positive, then there exists a strictly positive  $h_0 \in \mathcal{E}^+$  and  $\lambda_0 > 0$  such that  $\mathcal{P}h_0 = \lambda_0 h_0$ . Moreover, for any  $f \in \mathcal{E}^+$ , the sequence of lines spanned by  $\mathcal{P}^n f$  converges to the line spanned by  $h_0$ .

**Remark 2.3.10: Dual picture**

The dual map  $\mathcal{P}^*$  given by

$$(\mathcal{P}^*v)(y) := (v\mathcal{P})(y) = \int_{\mathcal{X}} v(x)p(x,y) dx$$

satisfies  $\langle \mathcal{P}^*v, f \rangle = \langle v, \mathcal{P}f \rangle$ , where  $\langle \cdot, \cdot \rangle$  denotes the usual inner product. Jentzsch's theorem shows the existence of a strictly positive function  $p_0$  such that  $\mathcal{P}^*p_0 = \lambda_0 p_0$ , with a similar convergence property. The eigenvalue  $\lambda_0$  is the same, since

$$\lambda_0 \langle p_0, h_0 \rangle = \langle p_0, \mathcal{P}h_0 \rangle = \langle \mathcal{P}^*p_0, h_0 \rangle$$

and  $\langle p_0, h_0 \rangle > 0$ . Furthermore, we have for any  $f \in \mathcal{E}^+$  that

$$\lambda_0^n \langle p_0, f \rangle = \langle (\mathcal{P}^*)^n p_0, f \rangle = \langle p_0, \mathcal{P}^n f \rangle,$$

which implies that

$$\lim_{n \rightarrow \infty} \frac{\mathcal{P}^n f}{\lambda_0^n} = c(f)h_0 \quad \text{where } c(f) = \frac{\langle p_0, f \rangle}{\langle p_0, h_0 \rangle}.$$

**2.3.4 Banach lattices and spectral gap**

Birkhoff extends the theory to Banach lattices, that is, Banach spaces  $\mathcal{E}$  with a (partial) order in which every pair of elements  $f, g$  admits an infimum  $f \wedge g$  and a supremum  $f \vee g$ . Examples of vector lattices include

1. the space of continuous functions  $f : \mathcal{X} \rightarrow \mathbb{R}$ , equipped with the supremum norm, with pointwise order given by

$$f \leq g \quad \Leftrightarrow \quad f(x) \leq g(x) \quad \forall x \in \mathcal{X},$$

and

$$(f \wedge g)(x) = f(x) \wedge g(x) \quad \text{and} \quad (f \vee g)(x) = f(x) \vee g(x);$$

2. the space of bounded measurable functions  $f : \mathcal{X} \rightarrow \mathbb{R}$ , with the same norm;
3. the space of finite signed measures  $\mu$  on  $\mathcal{X}$ , equipped with the  $L^1$ -norm.

The last two examples are associated with Markov kernels  $\mathcal{P}(x, A)$  and the (dual) maps introduced in (2.3.1). As in Definition 2.3.1, the Markov kernel is called uniformly positive if there exist a positive function  $f$ , a measure  $\nu$  (absolutely continuous with respect to Lebesgue measure, with strictly positive density<sup>1</sup>) and a constant  $L$  such that

$$s(x)\nu(A) \leq \mathcal{P}(x, A) \leq Ls(x)\nu(A) \quad \forall x \in \mathcal{X}, \forall A \subset \mathcal{X}.$$

A similar computation as above shows that  $\mathcal{P}(\mathcal{E}^+)$  has projective diameter  $\Delta \leq 2 \log L$ . Then similar arguments as before show that  $\mathcal{P}$  admits a unique principal eigenvalue  $\lambda_0$ , a measure  $\pi_0$  such that  $\pi_0 \mathcal{P} = \lambda_0 \pi_0$ , called the *quasistationary distribution*, and a positive function  $h_0$  such that  $\mathcal{P}h_0 = \lambda_0 h_0$ .

We now examine the speed of convergence of iterates of a positive map  $\mathcal{P}$  for a general Banach lattice. The following proposition is a key result.

<sup>1</sup>For results on more general measures, see [Num84, Ore71].

**Proposition 2.3.11: Strong comparability**

Any  $f, g \in \mathcal{E}^+$  are strongly comparable, in the sense that there exist strictly positive constants  $\alpha, \beta, R$  such that

$$\begin{aligned} \alpha f &\leq g \leq R\alpha f, \\ \beta g &\leq f \leq R\beta g. \end{aligned} \tag{2.3.9}$$

The optimal constant is  $R = e^{\theta(f, g; \mathcal{E}^+)}$ .

**PROOF:** Let  $A$  be the linear map of Definition 2.3.5, and write  $Af = (f_1, f_2)$ ,  $Ag = (g_1, g_2)$ . Assume without limiting the generality that  $f_1 g_2 \geq f_2 g_1$ . Then

$$f_1(Ag) - g_1(Af) = (0, f_1 g_2 - g_1 f_2) \in \mathbb{R}_+ \times \mathbb{R}_+$$

and thus  $f_1 g - g_1 f \in \mathcal{E}^+$ . Similarly, we have  $g_2 f - f_2 g \in \mathcal{E}^+$ . This shows that

$$\frac{g_1}{f_1} f \leq g \leq \frac{g_2}{f_2} f = e^{\theta(f, g; \mathcal{E}^+)} \frac{g_1}{f_1} f,$$

and thus (2.3.9) holds with  $\alpha = g_1/f_1$ . The proof of the second inequality is analogous.  $\square$

A first consequence of this result is that we can prove completeness.

**Corollary 2.3.12: Completeness of the metric**

If  $\|f\| = \|g\| = 1$ , then

$$\|f - g\| \leq e^{\theta(f, g; \mathcal{E}^+)} - 1.$$

As a consequence, in the metric defined by  $\theta$ , any  $\theta$ -connected component of the unit sphere is a complete metric space.

**PROOF:** If  $\|f\| = \|g\| = 1$ , then (2.3.9) holds with  $\alpha \leq 1 \leq R\alpha$  and  $R = e^{\theta(f, g; \mathcal{E}^+)}$ . Thus

$$\|f - g\| = \|f \vee g - f \wedge g\| \leq \|R\alpha f - \alpha f\| = (R - 1)\alpha \|f\| \leq R - 1,$$

as claimed.  $\square$

It follows that Theorem 2.3.7 indeed applies in this setting. Let us finally derive a spectral-gap estimate.

**PROOF OF THEOREM 2.3.3.** Denote  $\mathcal{P}^n f$  by  $f_n$ . For any  $n$  let  $\alpha_n$  and  $\beta_n$  be the optimal constants for which

$$\alpha_n h_0 \leq \frac{f_n}{\lambda_0^n} \leq \beta_n h_0.$$

Such constants exist and are positive for  $n = 1$  because  $f_1, h_0$  belong to a cone with diameter  $\Delta$ . Assuming by induction that the above inequality holds for some  $n$ , and applying  $\mathcal{P}$ , we obtain that it holds for  $n + 1$  with

$$\alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n.$$

Define

$$\begin{aligned} r_n &= f_n - \alpha_n \lambda_0^n h_0 \in \mathcal{E}^+, \\ s_n &= \beta_n \lambda_0^n h_0 - f_n \in \mathcal{E}^+. \end{aligned} \tag{2.3.10}$$

We have

$$\begin{aligned} r_n + s_n &= (\beta_n - \alpha_n)\lambda_0^n h_0, \\ \mathcal{P}r_n + \mathcal{P}s_n &= (\beta_n - \alpha_n)\lambda_0^{n+1} h_0. \end{aligned}$$

By Proposition 2.3.11, there exist positive constants  $a_n, b_n$  and  $R \leq e^\Delta$  such that

$$\begin{aligned} a_n h_0 &\leq \mathcal{P}r_n \leq R a_n h_0, \\ b_n h_0 &\leq \mathcal{P}s_n \leq R b_n h_0. \end{aligned}$$

On one hand it follows that

$$(a_n + b_n)h_0 \leq \mathcal{P}r_n + \mathcal{P}s_n = (\beta_n - \alpha_n)\lambda_0^{n+1} h_0 \leq R(a_n + b_n)h_0. \quad (2.3.11)$$

On the other hand, we conclude by applying  $\mathcal{P}$  to (2.3.10) that

$$(\alpha_n \lambda_0^{n+1} + a_n)h_0 \leq \mathcal{P}f_n = f_{n+1} \leq (\beta_n \lambda_0^{n+1} - b_n)h_0,$$

This yields

$$\alpha_{n+1} \geq \alpha_n + \frac{a_n}{\lambda_0^{n+1}}, \quad \beta_{n+1} \leq \beta_n - \frac{b_n}{\lambda_0^{n+1}}.$$

Using (2.3.11) it follows that

$$(\beta_{n+1} - \alpha_{n+1}) \leq (\beta_n - \alpha_n) - \frac{a_n + b_n}{\lambda_0^{n+1}} \leq \left(1 - \frac{1}{R}\right)(\beta_n - \alpha_n).$$

This shows that the sequences  $\alpha_n$  and  $\beta_n$  converge to a common limit  $M_1(f)$ , and thus that  $f_n/\lambda_0^n$  converges to  $M_1(f)h_0$  at rate  $(1 - R^{-1})^n = (1 - e^{-\Delta})^n$ .

Finally, the uniform positivity condition (2.3.2) implies that  $e^{-\Delta}$  is bounded below by  $1/L^2$ , which concludes the proof.  $\square$

### Remark 2.3.13: Dual picture

As in Remark 2.3.10, we have

$$\lambda_0^n \langle \pi_0, f \rangle = \langle \pi_0 \mathcal{P}^n, f \rangle = \langle \pi_0, \mathcal{P}^n f \rangle$$

for all  $n$ , which shows that

$$M_1(f) = \frac{\langle \pi_0, f \rangle}{\langle \pi_0, h_0 \rangle}.$$

### 2.3.5 From discrete time to continuous time

We provide here a simple illustration of how the discrete-time results presented in this section (and in Section 2.2.4) can be applied to continuous-time SDEs. Consider the SDE

$$dX_t = f(X_t)dt + \sigma dW_t, \quad (2.3.12)$$

where  $\sigma > 0$  is a small parameter, and  $f$  has a stable equilibrium point at the origin, that is

$$f(0) = 0,$$

and all the eigenvalues of the Jacobian matrix

$$A = \frac{\partial f}{\partial x}(0)$$

are strictly negative. The approaches we just introduced require upper and lower bounds on the transition density  $p_t(x, y)$  for some  $t > 0$ , say  $t = 1$ . A general approach for obtaining such bounds is based on Malliavin calculus, but it is also possible to obtain the required information by less elaborate methods. The approach we outline here is a simplification of the method used in [BG14, BB17].

A first point is that one can use Harnack inequalities for  $\mathcal{L}$ -harmonic functions (see [GT01, Corollaries 9.24 and 9.25]) to show that the transition density at time 1,  $p_1(x, y)$ , satisfies the following two regularity estimates on small balls. For  $x \in \mathbb{R}^n$  and  $r > 0$ , we let  $\mathcal{B}_r(x)$  denote the ball of radius  $r$  centred in  $x$ .

**Lemma 2.3.14: Harnack-type bounds on the transition density**

1. Fix  $x_0, y \in \mathbb{R}^n$ . There exists a constant  $C_0$ , independent of  $x_0$  and  $\sigma$ , such that

$$\sup_{x \in \mathcal{B}_{\sigma^2}(x_0)} p_1(x, y) \leq C_0 \inf_{x \in \mathcal{B}_{\sigma^2}(x_0)} p_1(x, y). \quad (2.3.13)$$

2. Fix  $x_0, y \in \mathbb{R}^n$  and  $r_0 > 0$ , and let  $R_0 = r_0 \sigma^2$ . Then there exist constants  $C_1 \geq 1$  and  $\alpha > 0$ , independent of  $\sigma$ , such that for any  $R \leq R_0$ , one has

$$\text{osc}_{\mathcal{B}_R(x_0)} p_1 := \sup_{x \in \mathcal{B}_R(x_0)} p_1(x, y) - \inf_{x \in \mathcal{B}_R(x_0)} p_1(x, y) \leq C_1 \left( \frac{R}{R_0} \right)^\alpha \text{osc}_{\mathcal{B}_{R_0}(x_0)} p_1. \quad (2.3.14)$$

Using (2.3.13), one can then show (cf. [BG14, Lemma 5.7]) that for  $y$  in a compact set  $D$ , one has the rough a priori bound

$$\frac{\sup_{x \in D} p_1(x, y)}{\inf_{x \in D} p_1(x, y)} \leq e^{C/\sigma^2}$$

for a constant  $C$ , depending on  $D$ , but not on  $\sigma$ . Furthermore, combining (2.3.14) with (2.3.13), one obtains (cf. [BG14, Lemma 5.8]) that for  $x_0, y$  in  $D$  and any  $\eta > 0$ , there exists  $r = r(y, \eta) > 0$ , independent of  $\sigma$ , such that

$$\sup_{x \in \mathcal{B}_{r\sigma^2}(x_0)} p_1(x, y) \leq (1 + \eta) \inf_{x \in \mathcal{B}_{r\sigma^2}(x_0)} p_1(x, y).$$

This result can now be extended to larger balls by using a coupling argument. Let  $X_t^x$  denote the solution of (2.3.12) with initial condition  $x$ , conditioned to stay in  $D$  up to time  $t$ . For  $x_1, x_2, y \in D$ , let

$$N(x_1, x_2) = \inf \left\{ n \geq 1 : |X_n^{x_2} - X_n^{x_1}| < r(\eta, y) \sigma^2 \right\}.$$

If  $p_n^D$  denotes the transition density at time  $n$  of the process conditioned to stay in  $D$ , one obtains [BG14, Proposition 5.9] that for all  $n \geq 2$ , one has

$$\frac{\sup_{x \in D} p_n^D(x, y)}{\inf_{x \in D} p_n^D(x, y)} \leq 1 + \eta + \sup_{x_1, x_2 \in D} \mathbb{P} \left\{ N(x_1, x_2) > n - 1 \right\} e^{C/\sigma^2} \quad (2.3.15)$$



for a constant  $C$  independent of  $\sigma$  and  $y \in D$ . Controlling the tails of  $N(x_1, x_2)$  thus amounts to a coupling argument, with an error of size  $r\sigma^2$ . To do this, we observe that the difference  $Y_t = X_t^{x_1} - X_t^{x_2}$  satisfies the equation

$$dY_t = AY_t dt + b(t, Y_t) dt,$$

where  $b(t, y) = \mathcal{O}(y^2)$  for small  $y$ . Using the integral representation

$$Y_t = (x_2 - x_1)e^{At} + \int_0^t \int_0^t e^{A(t-s)} b(s, Y_s) ds,$$

one can prove an estimate of the form

$$\mathbb{P}\{\|X_1^{x_1} - X_1^{x_2}\| \geq \rho \|x_2 - x_1\|\} \leq e^{-\kappa/\sigma^2}$$

uniformly over  $x_1, x_2 \in D$ , for some  $\rho \in (0, 1)$  and  $\kappa > 0$ . Using the Markov property<sup>2</sup>, one arrives at

$$\mathbb{P}\{N(x_1, x_2) > n\} \leq e^{-n\kappa/\sigma^2},$$

so that by choosing  $n$  large enough, one can make the right-hand side of (2.3.15) smaller than  $1 + 2\eta$ . We thus obtain a bound on the variation of the map  $x \mapsto p(x, y)$  inside  $D$ , yielding a uniform positivity property of the form (2.3.2) with  $m(y) = 1$  and  $L$  close to 1.

---

<sup>2</sup>We simplified the argument somewhat, because one has to account for the difference between the initial process, and the process conditioned on staying in  $D$ . See [BG14, Proposition 6.13] for the precise argument.

# Large deviations for stochastic differential equations

The theory of large deviations is a very powerful tool to analyse sequences  $(X_n)_{n \geq 0}$  or  $(X_\varepsilon)_{\varepsilon > 0}$  of random variables that obey some scaling behaviour as  $n \rightarrow \infty$  or  $\varepsilon \rightarrow 0$ . Standard textbooks on that theory are [DZ98] and [DS89]. In the particular case of scaled Brownian motion  $\sqrt{\varepsilon}W_t$ , Schilder's theorem provides a large-deviation principle, which can be transferred to SDEs using the so-called contraction principle. This is the starting point of the theory developed by Freidlin and Wentzell in the monograph [FW98].

In this chapter, we will first provide a general introduction to large-deviation principles, before specialising it to the case of SDEs, and giving some applications to the exit problem of a diffusion from a domain.

## 3.1 Large-deviation principles

### 3.1.1 A simple example

Let  $S_n$  be the number of heads obtained when throwing a fair coin  $n$  times. Then  $S_n$  follows a binomial distribution with parameters  $(n, \frac{1}{2})$ , and thus has expectation  $\frac{n}{2}$  and variance  $\frac{n}{4}$ . Therefore, the law of large numbers shows that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \left| \frac{S_n}{n} - \frac{1}{2} \right| \geq \delta \right\} = 0$$

for any  $\delta > 0$ , that is,  $\frac{S_n}{n}$  converges to  $\frac{1}{2}$  in probability. The strong law of large numbers states that this convergence also holds almost surely. The central limit theorem states that the variable

$$\widehat{S}_n = \frac{S_n - \frac{n}{2}}{\sqrt{\frac{n}{4}}}$$

converges in distribution to a standard normal law as  $n \rightarrow \infty$ . This indicates that for large  $n$ ,  $S_n$  is likely to belong to an interval of order  $\sqrt{n}$  around  $\frac{n}{2}$ .

One may be interested in more precise estimates on the probability of  $S_n$  being very far from its expected value. One way of estimating this probability is via the Bienaymé–Chebyshev inequality, which yields

$$\mathbb{P} \left\{ \left| \frac{S_n}{n} - \frac{1}{2} \right| \geq \delta \right\} \leq \frac{1}{\delta^2} \text{Var} \left( \frac{S_n}{n} \right) = \frac{1}{4n\delta^2}. \quad (3.1.1)$$

However, while this upper bound is rigorous, it may be too pessimistic for certain values of  $n$  and  $\delta$ .

**Example 3.1.1**

For  $n = 1000$  and  $\delta = 0.1$ , (3.1.1) yields

$$\mathbb{P}\left\{\left|\frac{S_{1000}}{1000} - \frac{1}{2}\right| \geq \delta\right\} = \mathbb{P}\{S_{1000} \notin [400, 600]\} \leq \frac{1}{40} = 0.025.$$

However, one would expect this probability to be much smaller. For instance, one might be tempted to infer from the central limit theorem the bound

$$\mathbb{P}\left\{\left|\frac{S_n}{n} - \frac{1}{2}\right| \geq \delta\right\} = \mathbb{P}\left\{\left|S_n - \frac{n}{2}\right| \geq n\delta\right\} \simeq \mathbb{P}\left\{|Z| \geq \sqrt{\frac{n}{4}}n\delta\right\} \leq \mathbb{P}\{|Z| \geq 1581\},$$

where  $Z$  follows a standard normal law. However, nothing guarantees that we are allowed to apply the central-limit theorem in this  $n$ -dependent way!

The Bienaymé–Chebychev inequality can be improved by using the so-called Chernoff bound. For any  $t > 0$ , one has

$$\mathbb{P}\left\{S_n - \frac{n}{2} \geq n\delta\right\} = \mathbb{P}\left\{e^{t(S_n - n/2)} \geq e^{tn\delta}\right\} \leq \frac{1}{e^{tn\delta}} \mathbb{E}[e^{t(S_n - n/2)}]. \quad (3.1.2)$$

We can write

$$S_n = \sum_{i=1}^n X_i$$

where the  $X_i$  are independent, identically distributed, taking values 0 or 1 with probability  $\frac{1}{2}$ . Setting  $Z_i = e^{t(X_i - 1/2)}$  we obtain

$$\mathbb{E}[Z_i] = e^{-t/2} \mathbb{P}\{X_i = 0\} + e^{t/2} \mathbb{P}\{X_i = 1\} = \cosh\left(\frac{t}{2}\right).$$

Since  $e^{t(S_n - n/2)} = \prod_{i=1}^n Z_i$ , independence of the  $Z_i$  implies

$$\mathbb{E}[e^{t(S_n - n/2)}] = \mathbb{E}\left[\prod_{i=1}^n Z_i\right] = \prod_{i=1}^n \mathbb{E}[Z_i] = \cosh^n\left(\frac{t}{2}\right).$$

This last expression can be written as  $\exp[n \log \cosh(\frac{t}{2})]$ . Inequality (3.1.2) thus becomes

$$\mathbb{P}\left\{S_n - \frac{n}{2} \geq n\delta\right\} \leq \exp\left\{-n\left[t\delta - \log \cosh\left(\frac{t}{2}\right)\right]\right\}. \quad (3.1.3)$$

Consider the function

$$f : t \mapsto t\delta - \log \cosh\left(\frac{t}{2}\right)$$

which is maximal for  $\tanh(\frac{t}{2}) = 2\delta$ , that is, for  $t = t^* = \log \frac{1+2\delta}{1-2\delta}$ . Let us set

$$\begin{aligned} \mathcal{F}(\delta) &= f(t^*) = \delta \log \frac{1+2\delta}{1-2\delta} - \log\left(\sqrt{\frac{1-2\delta}{1+2\delta}}\right) \\ &= \left(\frac{1}{2} + \delta\right) \log(1+2\delta) + \left(\frac{1}{2} - \delta\right) \log(1-2\delta). \end{aligned} \quad (3.1.4)$$

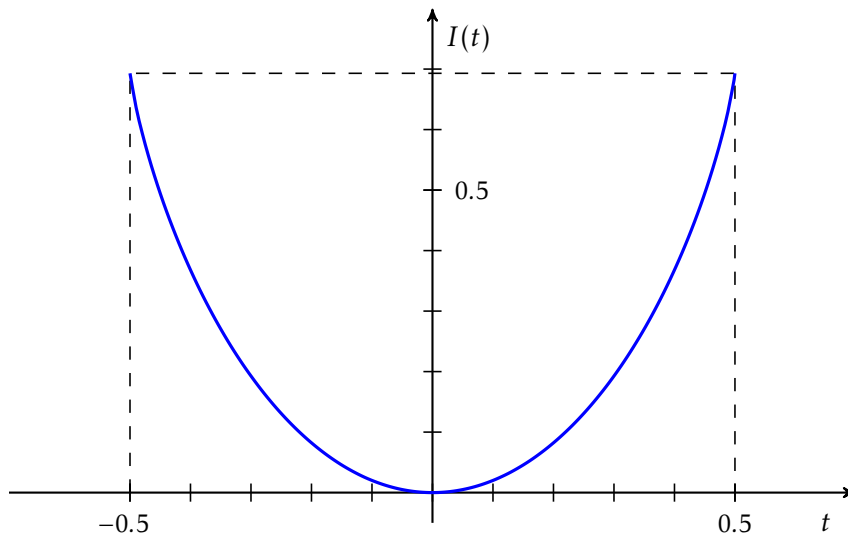


Figure 3.1 – Plot of the rate function  $\delta \mapsto \mathcal{I}(\delta)$ .

Using (3.1.3) for  $t = t^*$ , and the same estimate for  $\mathbb{P}\{S_n - \frac{n}{2} \leq -n\delta\}$ , we get

$$\mathbb{P}\left\{\left|\frac{S_n}{n} - \frac{1}{2}\right| \geq \delta\right\} \leq 2e^{-n\mathcal{I}(\delta)}. \tag{3.1.5}$$

This is a particular case of large-deviation estimate, and  $\mathcal{I}(\delta)$  is called the *rate function*, see Figure 3.1. The fact that (3.1.4) is reminiscent of entropy is no coincidence.

**Example 3.1.2**

Continuing with Example 3.1.1, since  $I(0.1) \simeq 0.02$ , we get the much smaller bound

$$\mathbb{P}\{S_{1000} \notin [400, 600]\} \leq 2e^{-1000I(0.1)} \simeq 3.6 \cdot 10^{-9}.$$

In fact, the estimate (3.1.5) is related to a particular case of Cramér’s theorem.

**Theorem 3.1.3: Cramér’s theorem**

Let  $(X_n)_{n \geq 0}$  be a sequence of independent, identically distributed random variables, such that the logarithmic moment generating function  $\Lambda(t) = \log \mathbb{E}[e^{tX_1}]$  is finite for all  $t \in \mathbb{R}$ . Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}\left\{\sum_{i=1}^n X_i \geq nx\right\} = -\Lambda^*(x)$$

for all  $x > \mathbb{E}[X_1]$ , where

$$\Lambda^*(x) = \sup_{t \in \mathbb{R}} (tx - \Lambda(t))$$

is the Legendre transform of  $\Lambda$ .

**Exercise 3.1.4**

Compute  $\Lambda^*(x)$  for a Gaussian random variable, and for an exponential random variable.

### 3.1.2 Definition of large-deviation principles

#### Definition 3.1.5: Large-deviation principle

Let  $(S, d)$  be a separable metric space. We say that a family  $\{\mu_\varepsilon\}_{\varepsilon>0}$  of probability measures on  $S$  satisfies a large deviation principle with rate  $\varepsilon$  and rate function  $\mathcal{F}$  if

1.  $\mathcal{F} : S \rightarrow [0, +\infty]$  is lower semicontinuous, has compact sublevel sets and is not identical to  $+\infty$ .
2. For every closed set  $C \subset S$  one has

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mu_\varepsilon(C) \leq -\inf_{s \in C} \mathcal{F}(s). \quad (3.1.6)$$

3. For every open set  $O \subset S$  one has

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mu_\varepsilon(O) \geq -\inf_{s \in O} \mathcal{F}(s). \quad (3.1.7)$$

Roughly speaking, the large-deviation principle says that for any sufficiently nice set  $\Gamma \subset S$ , we have

$$\mu_\varepsilon(\Gamma) \simeq e^{-\inf_\Gamma \mathcal{F}/\varepsilon}$$

in the sense of logarithmic equivalence.

#### Example 3.1.6: Cramér's theorem as a large-deviation principle

Setting  $n = \lfloor \varepsilon^{-1} \rfloor$ , Theorem 3.1.3 shows that  $S_n = \sum_{i=1}^n X_i$  satisfies a large-deviation principle with rate function

$$\mathcal{F}(s) = \Lambda^*(s).$$

### 3.1.3 The contraction principle

The contraction principle is a powerful method to generate new large-deviation principles from already known ones. The following version of this principle follows directly from [DS89, Lemma 2.1.4], see also [HW15, Lemma 3.3].

#### Lemma 3.1.7: Contraction principle

Let  $\{\mu_\varepsilon\}_{\varepsilon>0}$  be a family of probability measures on a separable metric space  $(S, d)$ , and let  $\mathcal{F} : S \rightarrow [0, \infty]$  satisfy the first condition of Definition 3.1.5. Let  $(S', d')$  be another separable metric space, and let  $\{\Psi_\varepsilon : \varepsilon \geq 0\}$  be a family of maps from  $S$  to  $S'$  which are continuous on a neighbourhood of  $\{s \in S : \mathcal{F}(s) < \infty\}$ . We assume that

1. The probability measures  $\{\mu_\varepsilon\}_{\varepsilon>0}$  satisfy a large-deviation principle on  $S$  with rate function  $\mathcal{F}$ .
2. For every  $c \in \mathbb{R}$ , there exists a neighbourhood  $O_c$  of  $\{s \in S : \mathcal{F}(s) \leq c\}$  such that the maps  $\Psi_\varepsilon$  converge uniformly on  $O_c$  to  $\Psi_0$ .

Then the image measures  $\mu_\varepsilon \circ \Psi_\varepsilon^{-1}$  satisfy a large-deviation principle on  $S'$  with rate function

$$\mathcal{F}'(s') = \inf\{\mathcal{F}(s) : s \in S, \Psi_0(s) = s'\},$$

with the convention that the infimum equals  $+\infty$  if the set is empty.

## 3.2 Large deviations for SDEs

### 3.2.1 Schilder's theorem

We consider in this section scaled Brownian motion in  $\mathbb{R}$  or  $\mathbb{R}^n$  defined for any  $\varepsilon > 0$  by

$$W_t^\varepsilon = \sqrt{\varepsilon} W_t.$$

Our aim is to obtain a large-deviation principle on sample-paths of  $W_t^\varepsilon$ , meaning that it should apply to the random functions  $\{W_t^\varepsilon\}_{0 \leq t \leq T}$  for some fixed time horizon  $T > 0$ . Let us first consider some particular cases of subsets of sample paths.

#### Example 3.2.1: Brownian motion at a fixed time

Fix a time  $T > 0$  and an open set  $A \in \mathbb{R}^n$ . Since

$$\mathbb{P}\{W_T^\varepsilon \in A\} = \int_A \frac{1}{\sqrt{(2\pi\varepsilon T)^n}} e^{-x^2/(2\varepsilon T)} dx,$$

applying the Laplace method shows that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\{W_T^\varepsilon \in A\} = -\frac{1}{2} \inf_{x \in A} \frac{\|x\|^2}{T}. \quad (3.2.1)$$

#### Example 3.2.2: 1d scaled Brownian motion staying below level $h$

The reflection principle implies

$$\mathbb{P}\left\{\sup_{0 \leq t \leq T} W_t^\varepsilon > h\right\} = 2\mathbb{P}\{W_T^\varepsilon > h\} = 2\mathbb{P}\left\{W_T > \frac{h}{\sqrt{\varepsilon}}\right\} = 2 \int_{h/\sqrt{\varepsilon}}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi\varepsilon}} dx,$$

which yields

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\left\{\sup_{0 \leq t \leq T} W_t^\varepsilon > h\right\} = -\frac{h^2}{2T}. \quad (3.2.2)$$

In both cases, we would like to write the right-hand side of (3.2.1) and (3.2.2) as the infimum of some rate function over all sample paths fulfilling the property defining the event. It turns out that such a rate function is provided by the following theorem.

#### Theorem 3.2.3: Schilder's theorem

Let  $\mathcal{C}_0$  be the space of continuous functions  $\varphi : [0, T] \rightarrow \mathbb{R}^n$  such that  $\varphi(0) = 0$ . Scaled Brownian motion on  $[0, T]$  satisfies a large-deviation principle on  $\mathcal{C}_0$  with rate function

$$\mathcal{I}_{[0, T]}(\varphi) = \begin{cases} \frac{1}{2} \|\varphi\|_{H^1}^2 = \frac{1}{2} \int_0^T \|\dot{\varphi}(s)\|^2 ds & \text{if } \varphi \in H^1 \text{ and } \varphi(0) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Before giving a proof of this result, let us look at some of its consequences. Let  $\psi : [0, T] \rightarrow \mathbb{R}^n$  be a path vanishing at times 0 and  $T$ . Then the Gateaux derivative of the rate function in the direction  $\psi$  is given by

$$\frac{d}{dt} \mathcal{I}_{[0, T]}(\varphi + t\psi) \Big|_{t=0} = \int_0^T \langle \dot{\varphi}(s), \dot{\psi}(s) \rangle ds = - \int_0^T \langle \ddot{\varphi}(s), \psi(s) \rangle ds. \quad (3.2.3)$$

Stationary points of the rate function are thus straight lines of the form  $\varphi(s) = sv$  for some  $v \in \mathbb{R}^n$ . This is nothing but the Euler–Lagrange variational principle applied to the Lagrangian of a free particle. In Example 3.2.1, we can apply the large-deviation principle to the set of paths  $\varphi$  such that  $\varphi(T) \in A$ . The infimum of the rate function is achieved by the path

$$\varphi(s) = \frac{s}{T}x_0, \quad x_0 = \operatorname{arginf}_{x \in A} \|x\|,$$

and one has indeed  $\mathcal{I}_{[0,T]}(\varphi) = \frac{1}{2T}\|x_0\|^2$ . A similar argument applies to Example 3.2.2, with  $\varphi(s) = \frac{s}{T}h$ .

Let us now outline a proof of Schilder’s theorem. Its main ingredient is the Cameron–Martin–Girsanov formula, which states that any drifted Brownian motion is equivalent to standard Brownian motion under a change of measure.

**Lemma 3.2.4: Cameron–Martin–Girsanov formula**

Let  $(W_t)_{t \geq 0}$  be a standard 1-dimensional Brownian motion on  $(\Omega, \mathcal{F}_t, \mathbb{P})$ . Then for any  $h \in L^2$ , the process defined by

$$\widehat{W}_t = W_t - \int_0^t h(s) ds$$

is a standard Brownian motion on  $(\Omega, \mathcal{F}_t, \mathbb{Q})$ , where  $\mathbb{Q}$  is defined via its Radon–Nikodym derivative

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \exp \left\{ \int_0^t h(s) dW_s - \frac{1}{2} \int_0^t h(s)^2 ds \right\}. \quad (3.2.4)$$

SKETCH OF PROOF: For  $h \in L^2$  and  $\gamma > 0$ , the processes

$$\begin{aligned} X_t &= \exp \left\{ \int_0^t h(s) dW_s - \frac{1}{2} \int_0^t h(s)^2 ds \right\}, \\ Y_t &= \exp \left\{ \int_0^t (\gamma + h(s)) dW_s - \frac{1}{2} \int_0^t (\gamma + h(s))^2 ds \right\} = X_t \exp \left\{ \gamma \widehat{W}_t - \frac{1}{2} \gamma^2 t \right\} \end{aligned}$$

are exponential martingales (similar to exponential Brownian motion) with respect to  $\mathbb{P}$ . This means that  $\mathbb{E}[X_t | \mathcal{F}_s] = X_s$  and  $\mathbb{E}[Y_t | \mathcal{F}_s] = Y_s$  whenever  $t > s$ . For any  $\mathcal{F}_s$ -measurable random variable  $Z$ , one has

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[Z e^{\gamma(\widehat{W}_t - \widehat{W}_s)}] &= \mathbb{E}_{\mathbb{P}}[Z X_t e^{\gamma(\widehat{W}_t - \widehat{W}_s)}] \\ &= \mathbb{E}_{\mathbb{P}}[Z e^{-\gamma \widehat{W}_s} Y_t e^{\frac{1}{2} \gamma^2 t}] \\ &= \mathbb{E}_{\mathbb{P}}[Z e^{-\gamma \widehat{W}_s + \frac{1}{2} \gamma^2 t} \mathbb{E}_{\mathbb{P}}[Y_t | \mathcal{F}_s]] \\ &= \mathbb{E}_{\mathbb{P}}[Z e^{-\gamma \widehat{W}_s + \frac{1}{2} \gamma^2 t} Y_s] \\ &= \mathbb{E}_{\mathbb{P}}[Z X_s e^{\frac{1}{2} \gamma^2 (t-s)}] \\ &= \mathbb{E}_{\mathbb{Q}}[Z] e^{\frac{1}{2} \gamma^2 (t-s)}, \end{aligned}$$

where the third line follows from the tower property of conditional expectations, and the fourth line from the martingale property. Since  $\widehat{W}_t - \widehat{W}_s$  is  $\mathbb{Q}$ -independent of  $\mathcal{F}_s$ , the increments of  $\widehat{W}_t$  are independent. The above computation shows that these increments are Gaussian of variance  $t - s$ , proving that indeed  $\widehat{W}_t$  is a Brownian motion with respect to the measure  $\mathbb{Q}$ .  $\square$

PROOF OF THEOREM 3.2.3. Assume for simplicity we are in dimension  $n = 1$ . We start by proving the lower bound (3.1.7). We thus have to prove that for any open set  $O \subset \mathcal{C}_0$ , one has

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\{W^\varepsilon \in O\} \geq - \inf_{\varphi \in O} \mathcal{I}_{[0,T]}(\varphi).$$

For this, it is in fact sufficient to prove that

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\{W^\varepsilon \in \mathcal{B}_\delta(\varphi)\} \geq -\mathcal{I}_{[0,T]}(\varphi) \quad (3.2.5)$$

holds for all  $\varphi \in \mathcal{C}_0$  such that  $\mathcal{I}_{[0,T]}(\varphi) < \infty$  and all  $\delta > 0$ , where  $\mathcal{B}_\delta(\varphi)$  denotes the ball of radius  $\delta$  in the sup norm, centred in  $\varphi$ .

If we set  $\widehat{W}_t = W_t - \frac{1}{\sqrt{\varepsilon}}\varphi_t$  and define  $d\mathbb{Q}$  by (3.2.4), we have

$$\begin{aligned} \mathbb{P}\{\|W^\varepsilon - \varphi\|_\infty < \delta\} &= \mathbb{P}\left\{\|\widehat{W}\|_\infty < \frac{\delta}{\sqrt{\varepsilon}}\right\} = \int_{\widehat{W} \in \mathcal{B}_{\delta/\sqrt{\varepsilon}}(0)} d\mathbb{P} \\ &= \int_{\widehat{W} \in \mathcal{B}_{\delta/\sqrt{\varepsilon}}(0)} \exp\left\{-\frac{1}{\sqrt{\varepsilon}} \int_0^T \dot{\varphi}_s d\widehat{W}_s + \frac{1}{2\varepsilon} \int_0^T \varphi_s^2 ds\right\} d\mathbb{Q} \\ &= e^{-\mathcal{I}_{[0,T]}(\varphi)/\varepsilon} \int_{\widehat{W} \in \mathcal{B}_{\delta/\sqrt{\varepsilon}}(0)} \exp\left\{-\frac{1}{\sqrt{\varepsilon}} \int_0^T \dot{\varphi}_s d\widehat{W}_s\right\} d\mathbb{Q}. \end{aligned}$$

Jensen's inequality yields

$$\begin{aligned} \frac{1}{\mathbb{Q}\{\widehat{W} \in \mathcal{B}_{\delta/\sqrt{\varepsilon}}(0)\}} \int_{\widehat{W} \in \mathcal{B}_{\delta/\sqrt{\varepsilon}}(0)} \exp\left\{-\frac{1}{\sqrt{\varepsilon}} \int_0^T \dot{\varphi}_s d\widehat{W}_s\right\} d\mathbb{Q} \\ \geq \exp\left\{-\frac{1}{\sqrt{\varepsilon}\mathbb{Q}\{\widehat{W} \in \mathcal{B}_{\delta/\sqrt{\varepsilon}}(0)\}} \int_{\widehat{W} \in \mathcal{B}_{\delta/\sqrt{\varepsilon}}(0)} \int_0^T \dot{\varphi}(s) d\widehat{W}_s d\mathbb{Q}\right\}. \end{aligned}$$

It thus follows from the Cameron–Martin–Girsanov formula that

$$\begin{aligned} \mathbb{P}\{\|W^\varepsilon - \varphi\|_\infty < \delta\} \\ \geq e^{-\mathcal{I}_{[0,T]}(\varphi)/\varepsilon} \mathbb{P}\{W \in \mathcal{B}_{\delta/\sqrt{\varepsilon}}(0)\} \exp\left\{-\frac{1}{\sqrt{\varepsilon}\mathbb{P}\{W \in \mathcal{B}_{\delta/\sqrt{\varepsilon}}(0)\}} \int_{W \in \mathcal{B}_{\delta/\sqrt{\varepsilon}}(0)} \int_0^T \dot{\varphi}(s) dW_s d\mathbb{P}\right\} \\ = e^{-\mathcal{I}_{[0,T]}(\varphi)/\varepsilon} \mathbb{P}\{W \in \mathcal{B}_{\delta/\sqrt{\varepsilon}}(0)\}. \end{aligned}$$

Noting that the probability in the last expression goes to 1 as  $\varepsilon$  decreases to 0, we obtain indeed the required lower bound (3.2.5).

Regarding the upper bound (3.1.6), we have to show that for any closed set  $C \subset \mathcal{C}_0$ , one has

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\{W^\varepsilon \in O\} \geq - \inf_{\varphi \in C} \mathcal{I}_{[0,T]}(\varphi).$$

To do this, given  $m \in \mathbb{N}$  we introduce a polygonal approximation  $W^{m,\varepsilon}$  of the Brownian path  $W^\varepsilon$ , joining the  $m + 1$  points  $(0, 0)$ ,  $(T/m, W_{T/m}^\varepsilon)$ ,  $\dots$ ,  $(T, W_T^\varepsilon)$ . For any  $\delta > 0$ , one can find a set  $C^\delta$  whose boundary is at distance  $\delta$  from the boundary of  $C$ , and such that

$$\mathbb{P}\{W^\varepsilon \in C\} \leq \mathbb{P}\{W^{m,\varepsilon} \in C^\delta\} + \mathbb{P}\{\|W^\varepsilon - W^{m,\varepsilon}\|_\infty \geq \delta\}. \quad (3.2.6)$$



We first show that the second term on the right-hand side is negligible. Indeed, by the stationary increments property of Brownian motion, we have

$$\begin{aligned} \mathbb{P}\{\|W^\varepsilon - W^{m,\varepsilon}\|_\infty \geq \delta\} &\leq m\mathbb{P}\left\{\sup_{0 \leq s \leq T/m} \|W_s^\varepsilon - W_s^{m,\varepsilon}\|_\infty \geq \delta\right\} \\ &\leq m\mathbb{P}\left\{\sup_{0 \leq s \leq T/m} \|W_s^\varepsilon\| \geq \frac{\delta}{2}\right\} \\ &= m\mathbb{P}\left\{\sup_{0 \leq s \leq T/m} \|W_s\| \geq \frac{\delta}{2\sqrt{\varepsilon}}\right\} \\ &\leq 2m e^{-m\delta^2/(8\varepsilon T)}, \end{aligned}$$

where the last bound is a standard bound for Brownian motion, that can be deduced from Doob's submartingale inequality. It follows that for any  $\delta > 0$ , one has

$$\limsup_{m \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}\{\|W^\varepsilon - W^{m,\varepsilon}\|_\infty \geq \delta\} = -\infty.$$

We now deal with the first term on the right-hand side of (3.2.6). Note that

$$\mathbb{P}\{W^{m,\varepsilon} \in C^\delta\} \leq \mathbb{P}\left\{\mathcal{J}_{[0,T]}(W^{m,\varepsilon}) \geq \inf_{\varphi \in C^\delta} \mathcal{J}_{[0,T]}(\varphi)\right\}.$$

Since  $W^{m,\varepsilon}$  is a random polygon, we have

$$\mathcal{J}_{[0,T]}(W^{m,\varepsilon}) = \frac{1}{2} \sum_{k=1}^m \frac{T}{m} \left\| \frac{m}{T} (W_{kT/m}^\varepsilon - W_{(k-1)T/m}^\varepsilon) \right\|^2 = \frac{\varepsilon}{2} \sum_{k=1}^m \xi_k^2,$$

where the  $\xi_k$  are independent, with standard normal distribution. Therefore, Markov's inequality yields for any  $\gamma < \frac{1}{2}$  and  $\ell \geq 0$  the bound

$$\mathbb{P}\left\{\mathcal{J}_{[0,T]}(W^{m,\varepsilon}) \geq \ell\right\} \leq e^{-2\gamma\ell/\varepsilon} \left(\mathbb{E}[e^{\gamma\xi_1^2}]\right)^m \leq \frac{e^{-2\gamma\ell/\varepsilon}}{(1-2\gamma)^{m/2}}.$$

Since  $\gamma < \frac{1}{2}$  is arbitrary, it follows that

$$\limsup_{m \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\left\{\mathcal{J}_{[0,T]}(W^{m,\varepsilon}) \geq \inf_{\varphi \in C^\delta} \mathcal{J}_{[0,T]}(\varphi)\right\} \leq - \inf_{\varphi \in C^\delta} \mathcal{J}_{[0,T]}(\varphi),$$

and the result follows by taking the limit  $\delta \searrow 0$ .  $\square$

### Remark 3.2.5

The proof we have given here is essentially the one found in [FW98, Chapter 3, Theorem 2.2]. There exist other proofs, however, for instance a proof based on the Wiener isometry can be found in [HW15].

## 3.2.2 The Freidlin–Wentzell large-deviation principle

We consider from now on SDEs of the form

$$dX_t^\varepsilon = f(X_t^\varepsilon) dt + \sqrt{\varepsilon} g(X_t^\varepsilon) dW_t, \quad (3.2.7)$$

where  $W_t$  denotes  $k$ -dimensional Brownian motion,  $X_t^\varepsilon$  is  $n$ -dimensional,  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a drift coefficient, and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times k}$  is a matrix-valued diffusion coefficient. We assume that  $f$  and  $g$  satisfy the Lipschitz and bounded-growth conditions seen in Section 1.2.3. Assume furthermore an ellipticity condition on the diffusion matrix  $D(x) = g(x)g(x)^T$ , requiring that

$$\langle \xi, D(x)\xi \rangle \geq c\|\xi\|^2 \quad \forall \xi \in \mathbb{R}^d \quad (3.2.8)$$

holds for some  $c > 0$ . Note that this is only possible if  $k \geq n$ , since otherwise,  $D(x)$  cannot have full rank.

Recall that the solution of (3.2.7) with initial condition  $X_0^\varepsilon = x_0$  is defined as the fixed point of the integral equation

$$X_t^\varepsilon = x_0 + \int_0^t f(X_s^\varepsilon) ds + \int_0^t g(X_s^\varepsilon) dW_s^\varepsilon, \quad (3.2.9)$$

where  $W_s^\varepsilon = \sqrt{\varepsilon}W_s$  denotes scaled Brownian motion. We view the right-hand side as a map  $\Psi$  from realisations of  $(W_t^\varepsilon)_{t \in [0, T]}$  to sample paths  $(X_t^\varepsilon)_{t \in [0, T]}$ .

Consider first the case where  $g$  is the identity matrix. Then the contraction principle, cf. Lemma 3.1.7, suggests that  $X^\varepsilon$  satisfies a large-deviation principle with rate function

$$\begin{aligned} \mathcal{I}_{[0, T]}(\varphi) &= \inf\{\mathcal{I}_{[0, T]}(\psi) : \Psi(\psi) = \varphi\} \\ &= \inf\left\{\frac{1}{2} \int_0^T \|\dot{\psi}(s)\|^2 ds : x_0 + \int_0^t f(\varphi(s)) ds + \int_0^t \dot{\psi}(s) ds = \varphi(t) \quad \forall t \in [0, T]\right\}. \end{aligned}$$

Taking the time-derivative of the condition suggests that one should take  $\dot{\psi}(t) = \dot{\varphi}(t) - f(\varphi(t))$ , yielding

$$\mathcal{I}_{[0, T]}(\varphi) = \frac{1}{2} \int_0^T \|\dot{\varphi}(s) - f(\varphi(s))\|^2 ds.$$

Consider next the case where  $f = 0$ , while  $g$  is a general matrix satisfying the ellipticity condition (3.2.8). Then (3.2.9) suggests choosing  $\psi$  in such a way that  $g(\varphi)\dot{\psi} = \dot{\varphi}$ . However, since  $g$  is not a square matrix in general, some care is required when solving the variational problem.

#### Lemma 3.2.6

The minimum of  $\|\dot{\psi}\|^2$  under the constraint  $g(\varphi)\dot{\psi} = \dot{\varphi}$  is achieved for  $\dot{\psi} = g(\varphi)^T D(\varphi)^{-1} \dot{\varphi}$ , and has the value  $\dot{\varphi}^T D^{-1} \dot{\varphi} = \langle \dot{\varphi}, D^{-1} \dot{\varphi} \rangle$ .

PROOF: The problem amounts to minimising  $p(y) = \|y\|^2$  over  $y \in \mathbb{R}^k$  under the constraint  $q(y) = g(\varphi)y - \dot{\varphi} = 0$ . By the Lagrange multiplier theorem, there exists a vector  $\lambda \in \mathbb{R}^n$  such that

$$2y_j = \frac{\partial p}{\partial y_j}(y) = \sum_{i=1}^n \lambda_i \frac{\partial q_i}{\partial y_j}(y) = \sum_{i=1}^n \lambda_i g_{ij}(\varphi), \quad j \in \{1, \dots, k\}.$$

This amounts to setting  $2y = g(\varphi)^T \lambda$ , and the constraint  $q(y) = 0$  yields  $g(\varphi)g(\varphi)^T \lambda = 2\dot{\varphi}$ , or  $\lambda = 2D(\varphi)^{-1} \dot{\varphi}$ . Therefore,  $y = g(\varphi)^T D(\varphi)^{-1} \dot{\varphi}$ , and taking the norm gives the result.  $\square$

It follows that for  $f = 0$ , one has

$$\mathcal{I}_{[0, T]}(\varphi) = \frac{1}{2} \int_0^T \langle \dot{\varphi}(s), D(\varphi(s))^{-1} \dot{\varphi}(s) \rangle ds.$$

Combining these two special cases yields the following result. Its proof is somewhat more complicated than a direct application of the contraction principle, because of the continuity properties that need to be verified. However, one can work with Euler approximations of the solutions, obtained by a time-discretisation.

**Theorem 3.2.7: Large-deviation principle for SDEs**

The processes  $(X_t^\varepsilon)_{\varepsilon>0}$  satisfy a large-deviation principle with rate function

$$\mathcal{I}_{[0,T]}(\varphi) = \begin{cases} \frac{1}{2} \int_0^T \langle \dot{\varphi}(s) - f(\varphi(s)), D(\dot{\varphi})^{-1}(\dot{\varphi}(s) - f(\varphi(s))) \rangle ds & \text{if } \varphi \in H^1 \text{ and } \varphi(0) = x_0, \\ +\infty & \text{otherwise.} \end{cases}$$

**Remark 3.2.8**

If  $D(x)$  is only positive semi-definite, the large-deviation principle remains valid with

$$\mathcal{I}_{[0,T]}(\varphi) = \inf \left\{ \mathcal{I}(\varphi) : \varphi \in H^1, \varphi(t) = x_0 + \int_0^t f(\varphi(s)) ds + \int_0^t D(\varphi(s))^{1/2} \dot{\varphi}(s) ds \quad \forall t \in [0, T] \right\}.$$

**Example 3.2.9: Interpretation of the rate function**

Fix a continuous path  $\varphi_0 : [0, T] \rightarrow \mathbb{R}^n$ , and define for  $\delta > 0$  the set

$$\Gamma_\delta = \left\{ \varphi \in \mathcal{C}([0, T], \mathbb{R}^n) : \sup_{t \in [0, T]} \|\varphi(t) - \varphi_0(t)\| > \delta \right\},$$

which one can check is open in the topology induced by the supremum norm. Then we have  $\mathbb{P}\{\Gamma_\delta\} = \mathbb{P}\{\tau(\delta) \leq T\}$ , where  $\tau(\delta)$  denotes the first-exit time from a  $\delta$ -neighbourhood of the path  $\varphi_0$ . The large-deviation principle yields

$$-\inf_{\varphi \in \Gamma_\delta} \mathcal{I}_{[0,T]}(\varphi) \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\{\tau(\delta) \leq T\} \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\{\tau(\delta) \leq T\} \leq -\inf_{\varphi \in \Gamma_\delta} \mathcal{I}_{[0,T]}(\varphi).$$

Taking the limit  $\delta \rightarrow 0$ , we obtain

$$\mathcal{I}_{[0,T]}(\varphi_0) = -\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\{\tau(\delta) \leq T\}.$$

Therefore,  $\mathcal{I}_{[0,T]}(\varphi_0)$  can be interpreted as the cost of following the path  $\varphi_0$ .

Let us also mention Varadhan's lemma, which applies to general random variables satisfying a large-deviation principle. The lemma allows to obtain results on "tilted" processes, in which unlikely outcomes are made likely by changing the probability measure.

**Theorem 3.2.10: Varadhan's lemma**

Let  $\mathcal{C}$  denote the set of continuous functions  $\varphi : [0, T] \rightarrow \mathbb{R}$ , and let  $\phi : \mathcal{C} \rightarrow \mathbb{R}$  be a continuous map. Assume the tail condition

$$\lim_{L \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \int_{\{\phi(X^\varepsilon) \geq L\}} e^{\phi(X^\varepsilon)/\varepsilon} d\mathbb{P} = -\infty.$$

Then

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \int e^{\phi(X^\varepsilon)/\varepsilon} d\mathbb{P} = \sup_{\varphi \in \mathcal{C}} [\phi(\varphi) - \mathcal{I}_{[0,T]}(\varphi)].$$

### 3.3 Application to the stochastic exit problem

One of the many applications of the large-deviation principle for SDEs is the stochastic exit problem, that we have already encountered in Section 1.3. In particular, the large-deviation principle gives information on the probability to leave a given set in finite time, on the expected time required to leave the set, and on the most probable path the process takes to do so. The approach is in fact quite similar to the geometric optics limit of the wave equation, where waves are replaced by light rays moving in straight lines. It can also be viewed as a version of Richard Feynman's path-integral approach to quantum mechanics.

#### 3.3.1 The Ornstein–Uhlenbeck process

As a relatively simple warm-up example, let us consider an Ornstein–Uhlenbeck process in  $\mathbb{R}$ , defined by the SDE

$$dX_t^\varepsilon = -X_t^\varepsilon dt + \sqrt{\varepsilon} dW_t. \quad (3.3.1)$$

Let us fix an initial value  $x_0 \geq 0$  and a level  $h > x_0$ , and ask the question of how likely it is that  $X_t^\varepsilon$  reaches the level  $h$  at or before time  $T$ , when  $\varepsilon$  is very small. For that, we define the set

$$\Gamma = \left\{ \varphi \in \mathcal{C}([0, T], \mathbb{R}) : \varphi(0) = x_0, \sup_{0 \leq t \leq T} \varphi(t) > h \right\}.$$

One checks that  $\Gamma$  is open in the topology induced by the supremum norm, and that its closure is obtained by replacing the condition  $\varphi(t) > h$  by  $\varphi(t) \geq h$ .

We have to compute the infimum over  $\Gamma$  of the rate function

$$\mathcal{I}_{[0, T]}(\varphi) = \frac{1}{2} \int_0^T [\dot{\varphi}(s) + \varphi(s)]^2 ds.$$

Proceeding as in (3.2.3), we find that the Gateaux derivative of the rate function in the direction  $\psi$ , where  $\psi(0) = \psi(T) = 0$ , is given by

$$\frac{d}{dt} \mathcal{I}_{[0, T]}(\varphi + t\psi) \Big|_{t=0} = \int_0^T [\dot{\varphi}(s) + \varphi(s)][\dot{\psi}(s) + \psi(s)] ds = \int_0^T [-\ddot{\varphi}(s) + \varphi(s)]\psi(s) ds.$$

Stationary points of the rate function thus solve the linear differential equation

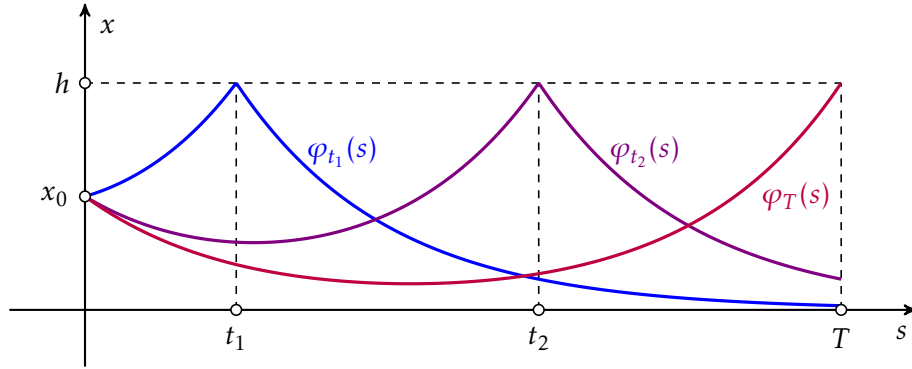
$$\ddot{\varphi}(s) = \varphi(s), \quad (3.3.2)$$

the general solution of which has the form  $\varphi(s) = Ae^s + Be^{-s}$ . We still have to deal with the boundary conditions. For that, we fix any time  $t \in (0, T)$ , and define  $\varphi_t$  by

$$\varphi_t(s) = \begin{cases} \frac{h - x_0 e^{-t}}{2 \sinh(t)} e^s + \frac{x_0 e^t - h}{2 \sinh(t)} e^{-s} & \text{for } 0 \leq s \leq t, \\ h e^{-(s-t)} & \text{for } t < s \leq T. \end{cases}$$

One readily checks (see Figure 3.2) that

- $\varphi_t$  is continuous, satisfies the equation (3.3.2) on  $[0, t)$ , the deterministic limit  $\dot{\varphi} = -\varphi$  of (3.3.1) on  $(t, T]$ , and  $\varphi_t(0) = x_0$  while  $\varphi_t(t) = h$ ;
- if  $\cosh(t) \leq h/x_0$ , then  $\varphi_t(s)$  is increasing in  $s$  on  $[0, t]$ ;
- if  $\cosh(t) > h/x_0$ , then  $\varphi_t(s)$  is first decreasing and then increasing in  $s$  on  $[0, t]$ , with a positive minimal value.



**Figure 3.2** – Several paths used to minimise the rate function for the Ornstein–Uhlenbeck process. The infimum is attained on the path  $\varphi_T$ .

In particular,  $\varphi_t$  is an element of  $\bar{\Gamma}$ , reaching its maximal value  $h$  at time  $t$ . Furthermore,

$$\begin{aligned} \mathcal{I}_{[0,T]}(\varphi_t) &= \mathcal{I}_{[0,t]}(\varphi_t) + \mathcal{I}_{[t,T]}(\varphi_t) = \frac{1}{2} \int_0^t [\dot{\varphi}_t(s) + \varphi_t(s)]^2 ds + 0 \\ &= \frac{(he^{t/2} - x_0 e^{-t/2})^2}{2 \sinh(t)} = h^2 + x_0^2 + \frac{h^2 - x_0^2}{\tanh(t)} + \frac{hx_0}{\sinh(t)}. \end{aligned} \quad (3.3.3)$$

Clearly, any modification of  $\varphi_t$  on the interval  $(t, T]$  can only increase the value of the rate function. The same is true for modifications on the interval  $[0, t)$ . One way of seeing this is to compute the second variation of the rate function, which is given by the bilinear form

$$(\psi_1, \psi_2) \mapsto \int_0^T [-\ddot{\psi}_1(s) + \psi_1(s)] \psi_2(s) ds.$$

The associated quadratic form is positive definite, because the eigenvalues of the second derivative (with Dirichlet boundary conditions on  $[0, T]$ ) are negative. Since (3.3.3) is a decreasing function of  $t$ , we have thus obtained

$$\inf_{\varphi \in \bar{\Gamma}} \mathcal{I}_{[0,T]}(\varphi) = \mathcal{I}_{[0,T]}(\varphi_T) = \frac{(he^{T/2} - x_0 e^{-T/2})^2}{2 \sinh(T)}.$$

Making the same argument with  $h_1 > h$ , and letting  $h_1$  decrease to  $h$  shows that this infimum coincides with the infimum over the open set  $\Gamma$  itself. By Theorem 3.2.7, we conclude that the first-passage time  $\tau = \inf\{t \in [0, T] : X_t^\varepsilon > h\}$  satisfies

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\{\tau < T\} = -\frac{(he^{T/2} - x_0 e^{-T/2})^2}{2 \sinh(T)}.$$

In particular, we have

$$\lim_{T \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\{\tau < T\} = -h^2, \quad (3.3.4)$$

showing that for large  $T$ , the probability of reaching level  $h$  before time  $T$  behaves roughly like  $e^{-h^2/\varepsilon}$ , independently of  $x_0 \in [0, h)$ .

### 3.3.2 Lagrangian and Hamiltonian formulations

The minimisation problems we encountered for Brownian motion and for the Ornstein–Uhlenbeck process are in fact well known in analytical mechanics. Indeed, the rate function is a

Lagrangian action, of the form

$$\mathcal{I}_{[0,T]}(\varphi) = \int_0^T L(\varphi(s), \dot{\varphi}(s)) ds,$$

where  $L$  is a Lagrangian given by

$$L(\varphi, \dot{\varphi}) = \frac{1}{2} \langle \dot{\varphi} - f(\varphi), D(\dot{\varphi})^{-1}(\dot{\varphi} - f(\varphi)) \rangle. \quad (3.3.5)$$

Let us disregard for a moment the question of boundary conditions. The stationary points of the action are known to satisfy the Euler–Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\varphi}_i} \right) = \frac{\partial L}{\partial \varphi_i}, \quad i = 1, \dots, n.$$

### Exercise 3.3.1

Write the Euler–Lagrange equations corresponding to Brownian motion, and those corresponding to the Ornstein–Uhlenbeck process. Check that their solutions are indeed of the form discussed in the previous subsections.

It is often more convenient to use the Hamiltonian formalism. This is done by introducing the conjugate momenta

$$\psi_i = \frac{\partial L}{\partial \dot{\varphi}_i}$$

and the Hamiltonian

$$H(\varphi, \psi) = \sum_{i=1}^n \psi_i \dot{\varphi}_i - L(\varphi, \dot{\varphi}),$$

where  $\dot{\varphi}$  has to be expressed in terms of the  $\psi_i$  on the right-hand side. The stationary points of the action are then solutions of the Hamilton equations

$$\frac{d\varphi_i}{dt} = \frac{\partial H}{\partial \psi_i}, \quad \frac{d\psi_i}{dt} = -\frac{\partial H}{\partial \varphi_i}. \quad (3.3.6)$$

For the Lagrangian (3.3.5), the Hamiltonian reads

$$H(\varphi, \psi) = \frac{1}{2} \langle \psi, D(\varphi)\psi \rangle + \langle \psi, f(\varphi) \rangle, \quad (3.3.7)$$

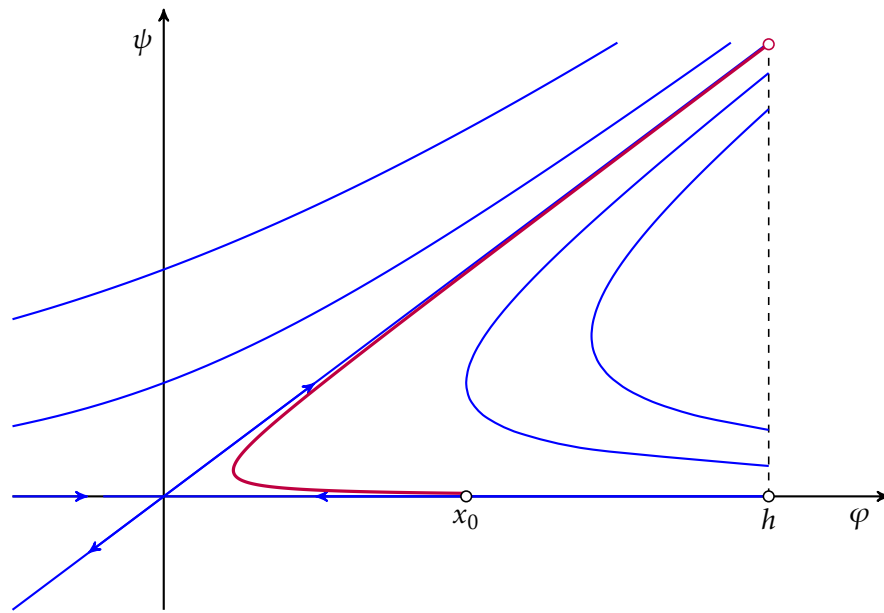
and the rate function takes the value

$$\mathcal{I}_{[0,T]}(\varphi, \psi) = \frac{1}{2} \int_0^T \langle \psi(s), D(\varphi(s))\psi(s) \rangle ds.$$

One should note that while the first term in the Hamiltonian (3.3.7) can be interpreted as a kinetic energy, the second term plays a different role from the usual potential energy in a particle system.

The Hamilton equations (3.3.6) take the form

$$\frac{d\varphi}{dt} = D(\varphi)\psi + f(\varphi), \quad \frac{d\psi}{dt} = \frac{1}{2} \nabla_{\varphi} \langle \psi, D(\varphi)\psi \rangle - \nabla_{\varphi} \langle \psi, f(\varphi) \rangle.$$



**Figure 3.3** – Hamiltonian flow corresponding to the Ornstein–Uhlenbeck process. The purple curve is the optimal trajectory allowing to reach level  $h$  starting from  $x_0$  in a given time. It corresponds to  $\varphi_T$  in Figure 3.2.

#### Example 3.3.2: Ornstein–Uhlenbeck process

For the Ornstein–Uhlenbeck process (3.3.1), the Hamiltonian has the form

$$H(\varphi, \psi) = \frac{1}{2}\psi^2 - \psi\varphi,$$

and the Hamilton equations read

$$\frac{d\varphi}{dt} = \psi - \varphi, \quad \frac{d\psi}{dt} = \psi.$$

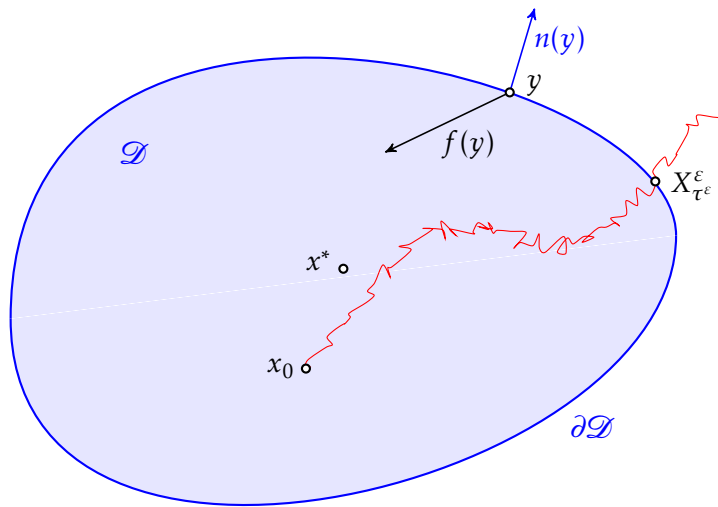
In this one-dimensional situation, one can take advantage of the fact that the Hamiltonian is a constant of motion, so that solutions of the Hamilton equations belong to level curves of  $H$ , see Figure 3.3 (this remains true for more general processes). Note that as the time  $T$  allowed to reach a given level  $h$  becomes large, the optimal path becomes closer and closer to the stable and unstable manifolds of the origin, on which the Hamiltonian has value 0. This also remains true in more generality.

### 3.3.3 The probability of exiting a domain in a time independent of $\varepsilon$

Consider now a general diffusion of the form

$$dX_t^\varepsilon = f(X_t^\varepsilon)dt + \sqrt{\varepsilon}g(X_t^\varepsilon)dW_t, \quad (3.3.8)$$

with the usual assumptions on  $f$  and  $g$  guaranteeing existence and uniqueness of a strong solution, and let  $\mathcal{D} \subset \mathbb{R}^n$  be a bounded, open set with smooth boundary. Then the large-deviation principle has the following consequence.



**Figure 3.4** – Exit of a diffusion  $(X_t^\varepsilon)_{t \geq 0}$  from a domain  $\mathcal{D}$ . The vector  $n(y)$  is the unit outward normal vector at  $y \in \partial\mathcal{D}$ , and the vector field  $f(y)$  should satisfy  $\langle f(y), n(y) \rangle < 0$  at any  $y \in \partial\mathcal{D}$ .

### Proposition 3.3.3

Fix a point  $x_0 \in \mathcal{D}$ , and define for any  $y \notin \mathcal{D}$  and  $t > 0$

$$V(x_0, y; t) = \inf \left\{ \mathcal{I}_{[0, T]}(\varphi) : \varphi \in \mathcal{C}([0, t], \mathbb{R}^n), \varphi(0) = x_0, \varphi(t) = y \right\}.$$

Let  $\tau^\varepsilon$  denote the first-exit time from  $\mathcal{D}$  of the solution of (3.3.8) starting at  $x_0$ . Then

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}^{x_0} \{ \tau^\varepsilon \leq T \} = - \inf \left\{ V(x_0, y; t) : t \in [0, T], y \notin \mathcal{D} \right\}$$

holds for any  $T > 0$ .

The proof proceeds in the same way as in the examples we have seen so far, the main point being to show that the upper and lower bound in the large-deviation principle coincide.

The main difficulty at this stage remains solving the variational problem. In general this can be quite intricate, but one can say more under some additional assumptions. We assume from now on (see Figure 3.4) that

- the deterministic equation  $\dot{x} = f(x)$  has a unique equilibrium point in  $\mathcal{D}$ , say at  $x^* = 0$ ;
- the closure  $\overline{\mathcal{D}}$  is contained in the basin of attraction of  $x^*$  for the deterministic evolution, meaning that for any initial condition  $x_0 \in \overline{\mathcal{D}}$ , the solution of  $\dot{x} = f(x)$  starting from  $x_0$  converges to  $x^*$  as time goes to infinity;
- the vector field  $f$  points inward on the boundary  $\mathcal{D}$  of the domain, meaning that if  $n(y)$  denotes the unit outward normal vector at  $y$ , then  $\langle f(y), n(y) \rangle < 0$  for any  $y \in \partial\mathcal{D}$  (the boundary  $\partial\mathcal{D}$  is called non-characteristic).

### Definition 3.3.4: Quasipotential

The quasipotential is the function defined for any  $y \notin \mathcal{D}$  by

$$V(x^*, y) = \inf_{t > 0} V(x^*, y; t).$$



**Example 3.3.5: Gradient system**

Consider a gradient SDE of the form

$$dX_t = -\nabla U(X_t) dt + \sqrt{\varepsilon} dW_t,$$

where  $U$  is a potential that admits a unique minimum in  $\mathcal{D}$  at  $x^*$ . For any  $T > 0$ , we have

$$\begin{aligned} \mathcal{I}_{[0,T]}(\varphi) &= \frac{1}{2} \int_0^T \|\dot{\varphi}(s) + \nabla U(\varphi(s))\|^2 ds \\ &= \frac{1}{2} \int_0^T \|\dot{\varphi}(s) - \nabla U(\varphi(s))\|^2 ds + 2 \int_0^T \langle \dot{\varphi}(s), \nabla U(\varphi(s)) \rangle ds \\ &= \frac{1}{2} \int_0^T \|\dot{\varphi}(s) - \nabla U(\varphi(s))\|^2 ds + 2[U(\varphi(T)) - U(\varphi(0))]. \end{aligned}$$

Taking  $\varphi(0) = x^*$  and  $\varphi(T) = y \notin \mathcal{D}$ , we see that  $V(x^*, y; T)$  is bounded below by  $2[U(y) - U(x^*)]$ . Our assumptions imply that we can make the integral on the right-hand side arbitrarily small as  $T \rightarrow \infty$ , by choosing for  $\varphi$  a path connecting  $y$  and  $x^*$  that follows the reversed deterministic dynamics  $\dot{x} = \nabla U(x)$ . It follows that the quasipotential is simply given in this case by

$$V(x^*, y) = 2[U(y) - U(x^*)]. \quad (3.3.9)$$

**Proposition 3.3.6**

Under the above assumptions, we have for any initial condition  $x_0 \in \mathcal{D}$

$$\lim_{T \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}^{x_0} \{ \tau^\varepsilon \leq T \} = -\bar{V},$$

where  $\bar{V}$  is the infimum of the quasipotential on the boundary:

$$\bar{V} = \inf_{y \in \partial \mathcal{D}} V(x^*, y).$$

**PROOF:** We will prove a slightly more quantitative result, which will be useful later on.

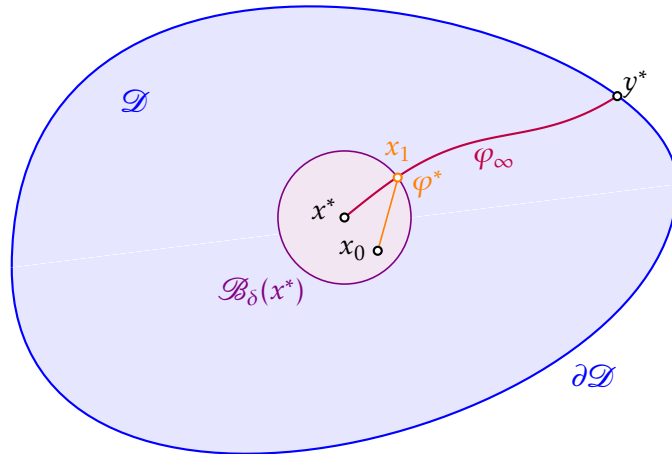
1. The first step is to show that it is costly to remain outside a neighbourhood of  $x^*$  for long. Indeed, let  $\mathcal{B}_\delta(x^*)$  be the ball of radius  $\delta > 0$  around  $x^*$ . One can show that there exist constants  $c, T_0 > 0$  such that, for any path remaining in  $\mathcal{D} \setminus \mathcal{B}_\delta(x^*)$  during a time interval  $[0, T]$ , one has  $\mathcal{I}_{[0,T]}(\varphi) \geq c(T - T_0)$ . This is due to the fact that  $\|f\|$  is bounded away from zero in  $\mathcal{D} \setminus \mathcal{B}_\delta(x^*)$ , and that the only way to keep the rate function small is to follow the deterministic dynamics that leads towards  $x^*$  (cf. [FW98, Chapter 4, Lemma 2.2]). Therefore, we have for any  $\delta > 0$

$$\lim_{T \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \left[ \sup_{x_0 \in \mathcal{D}} \mathbb{P}^{x_0} \{ X_t^\varepsilon \in \mathcal{D} \setminus \mathcal{B}_\delta(x^*) \forall t \in [0, T] \} \right] = -\infty. \quad (3.3.10)$$

This justifies restricting the argument to starting points  $x_0$  in the ball  $\mathcal{B}_\delta(x^*)$  of radius  $\delta$  centred in  $x^*$ .

2. The second step consists in showing that for any  $\eta > 0$ , there exists  $\delta_0 > 0$  such that, for any  $\delta < \delta_0$ , there is a  $T_0 > 0$  such that for any  $T \geq T_0$ , one has

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \left[ \inf_{x_0 \in \mathcal{B}_\delta(x^*)} \mathbb{P}^{x_0} \{ \tau^\varepsilon \leq T \} \right] > -(\bar{V} + \eta). \quad (3.3.11)$$



**Figure 3.5** – Construction of the path  $\varphi^*$  in the proof of Proposition 3.3.6. The path is obtained by taking a path  $\varphi_\infty$  minimising the rate function, removing its part inside a ball  $\mathcal{B}_\delta(x^*)$ , and adding a linear path connecting it to  $x_0$ .

In view of the lower bound (3.1.7) in the large-deviation principle, it is sufficient to construct, for any  $x_0 \in \mathcal{B}_\delta(x^*)$ , a particular continuous path  $\varphi^* : [0, T] \rightarrow \mathbb{R}^n$  such that  $\varphi^*(0) = x_0$ ,  $\varphi^*(T) \in \partial\mathcal{D}$ , and  $\mathcal{I}_{[0, T]}(\varphi^*) \leq \bar{V} + \eta$ . To construct this path, let  $y^*$  be a point on  $\partial\mathcal{D}$  where the quasipotential reaches its minimum (which exists by a compactness argument), and let  $\varphi_\infty$  be a path minimising the quasipotential in potentially infinite time. This path intersects the boundary of  $\mathcal{B}_\delta(x^*)$  at a point  $x_1$  (Figure 3.5). We may assume that  $\varphi_\infty$  is parametrised in such a way that  $\varphi_\infty(0) = y^*$  and  $\varphi_\infty(-T_1) = x_1$  for some  $T_1 = T_1(\delta) > 0$ . Then define for  $T > T_1$

$$\varphi^*(t) = \begin{cases} x_0 + \frac{t}{T - T_1}(x_1 - x_0) & \text{for } 0 \leq t \leq T - T_1, \\ \varphi_\infty(t - T) & \text{for } T - T_1 \leq t \leq T. \end{cases}$$

The contribution of the first part of  $\varphi^*$  to the rate function goes to 0 with  $\delta$ , while the contribution of the second part is smaller than  $\bar{V}$ . Therefore, for any  $\eta > 0$ , one has indeed  $\mathcal{I}_{[0, T]}(\varphi^*) = \bar{V} + \eta$  if  $\delta$  is small enough, as required.

3. Next we argue that for any  $\eta > 0$ , there exists  $\delta_0 > 0$  such that, for any  $\delta < \delta_0$ , there is a  $T_0 > 0$  such that

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \left[ \sup_{x_0 \in \mathcal{B}_\delta(x^*)} \mathbb{P}^{x_0} \{ \tau^\varepsilon \leq T \} \right] < -(\bar{V} - \eta) \quad (3.3.12)$$

holds for any  $T \geq T_0$ . Indeed, assume that this is not the case. Then by the upper bound (3.1.6) of the large-deviation principle, there would exist  $x_0 \in \mathcal{B}_\delta(x^*)$  and a continuous path  $\varphi : [0, T] \rightarrow \mathbb{R}^n$  such that  $\varphi(0) = x_0$  and  $\varphi(t) \notin \mathcal{D}$  for some  $t \in [0, T]$ , satisfying  $\mathcal{I}_{[0, T]}(\varphi) \leq \bar{V} - \eta$ . Adding a linear piece from  $x^*$  to  $x_0$  to this path, one would obtain a path with rate function smaller than  $\bar{V} - \frac{\eta}{2}$  if  $\delta$  is small enough, contradicting the definition of  $\bar{V}$ .

Since  $\eta > 0$  was arbitrary, combining these three steps yields the claimed result.  $\square$

#### Remark 3.3.7

For the Ornstein–Uhlenbeck process, we can take  $V(x^*, y)$  given by (3.3.9) with  $U(x) = \frac{1}{2}x^2$ , so that we recover the result (3.3.4).

### 3.3.4 Mean exit times and exit locations

So far, we have described the exit problem on timescales independent of  $\varepsilon$ . This is not quite satisfactory, however, since really interesting new phenomena appear in these systems for times that are exponentially large in  $\varepsilon$ . One of the classical results in this context is the following one.

**Theorem 3.3.8: Exit from the neighbourhood of a stable equilibrium point**

Let  $\mathcal{D} \subset \mathbb{R}^n$  be a bounded, open set with smooth boundary. Assume that the deterministic equation  $\dot{x} = f(x)$  has a unique equilibrium point  $x^* \in \mathcal{D}$ , that the closure  $\overline{\mathcal{D}}$  is contained in the deterministic basin of attraction of  $x^*$ , and that  $\langle f(y), n(y) \rangle < 0$  for any  $y \in \partial\mathcal{D}$ , where  $n(y)$  denotes the unit normal vector to the boundary at  $y$ . Then the following properties hold for any initial condition  $x_0 \in \mathcal{D}$ .

1. For any  $\eta > 0$ , one has

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}^{x_0} \left\{ e^{(\bar{V}-\eta)/\varepsilon} < \tau^\varepsilon < e^{(\bar{V}+\eta)/\varepsilon} \right\} = 1.$$

2. The mean first-exit time satisfies

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{E}^{x_0} [\tau^\varepsilon] = \bar{V}. \quad (3.3.13)$$

3. For any closed subset  $N \subset \partial\mathcal{D}$  satisfying  $\inf_{y \in N} V(x^*, y) > \bar{V}$ , one has

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}^{x_0} \{X_{\tau^\varepsilon}^\varepsilon \in N\} = 0.$$

Let us briefly comment on the meaning of these assertions.

1. The first claim means that  $\tau^\varepsilon$  concentrates around  $e^{\bar{V}/\varepsilon}$ , albeit in a rather weak sense, since there remain exponentially large windows on either side of this value.
2. The second claim says that the mean first-exit time behaves like  $e^{\bar{V}/\varepsilon}$  in the sense of logarithmic equivalence, a property known as Arrhenius' law.
3. The third claim says that the diffusion is likely to exit  $\mathcal{D}$  near those points on its boundary where the quasipotential is the smallest.

**SKETCH OF PROOF:** We will give a proof of (3.3.13), which is the most intuitive part of the result. It is based on the idea that the probabilities to exit during time intervals  $[mT, (m+1)T]$  follow approximately a geometric law with probability of success  $\mathbb{P}^{x^*} \{\tau^\varepsilon < T\}$ , which has expectation  $(\mathbb{P}^{x^*} \{\tau^\varepsilon < T\})^{-1}$ .

Fix some  $T > 0$ . For any  $x_0 \in \mathcal{D}$  and any  $m \in \mathbb{N}_0$ , we have

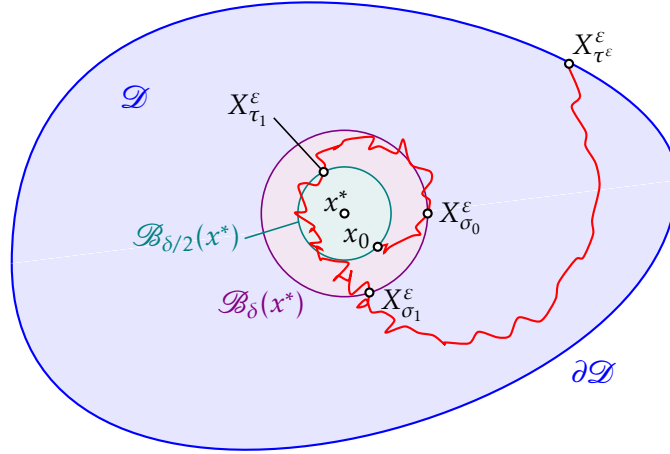
$$\begin{aligned} \mathbb{P}^{x_0} \{\tau^\varepsilon > (m+1)T\} &= \mathbb{E}^{x_0} \left[ \mathbb{1}_{\{\tau^\varepsilon > mT\}} \mathbb{P}^{X_{mT}^\varepsilon} \{\tau^\varepsilon > T\} \right] \\ &\leq \left( \sup_{x_1 \in \mathcal{D}} \mathbb{P}^{x_1} \{\tau^\varepsilon > T\} \right) \mathbb{P}^{x_0} \{\tau^\varepsilon > mT\}. \end{aligned}$$

It thus follows by induction that for all  $m \geq 0$ ,

$$\mathbb{P}^{x_0} \{\tau^\varepsilon > mT\} \leq p^m, \quad \text{where } p = \sup_{x_1 \in \mathcal{D}} \mathbb{P}^{x_1} \{\tau^\varepsilon > T\}.$$

Therefore, integration by parts shows that

$$\mathbb{E}^{x_0} [\tau^\varepsilon] = \int_0^\infty \mathbb{P}^{x_0} \{\tau^\varepsilon > t\} dt \leq T \sum_{m=0}^\infty \mathbb{P}^{x_0} \{\tau^\varepsilon > mT\} \leq T \sum_{m=0}^\infty p^m = \frac{T}{1-p}.$$



**Figure 3.6** – Construction of the Markov chain  $(Z_m)_{m \geq 0} = (X_{\tau_m}^\varepsilon)_{m \geq 0}$  used in the proof of Theorem 3.3.8.

Note that  $p$  is the complement of the probability described in Proposition 3.3.6. In particular, (3.3.11) shows that for any  $\eta > 0$ , there exists  $\varepsilon_0 > 0$  such that for any  $\varepsilon < \varepsilon_0$ , one has

$$\varepsilon \log(1 - p) \geq -(\bar{V} + \eta)$$

if  $\delta$  is small enough. Since  $\eta > 0$  is arbitrary, this proves the upper bound for (3.3.13).

This argument does not work directly for the lower bound on the expectation, because  $\mathbb{P}^{x_1}\{\tau^\varepsilon > T\}$  is not bounded below uniformly in  $\mathcal{D}$ . Instead, we introduce a discrete-time Markov chain defined as follows. Set  $\tau_0 = 0$ , and define an increasing sequence of stopping times  $\tau_0 < \sigma_1 < \tau_1 < \dots$  by

$$\begin{aligned} \sigma_m &= \inf\{t > \tau_m : X_t^\varepsilon \in \partial \mathcal{B}_\delta(x^*)\}, \\ \tau_m &= \inf\{t > \sigma_{m-1} : X_t^\varepsilon \in \partial \mathcal{B}_{\delta/2}(x^*) \cup \partial \mathcal{D}\} \end{aligned}$$

(Figure 3.6). The process  $(Z_m)_{m \geq 0}$  defined by  $Z_m = X_{\tau_m}^\varepsilon$  is a discrete-time Markov chain on the set  $\partial \mathcal{B}_{\delta/2}(x^*) \cup \partial \mathcal{D}$ , and we have  $\tau^\varepsilon = \tau_\nu$ , where  $\nu = \inf\{m \geq 1 : Z_m \in \partial \mathcal{D}\}$ . Now we note that for  $\|x_0 - x^*\| \leq \frac{\delta}{2}$ , one has

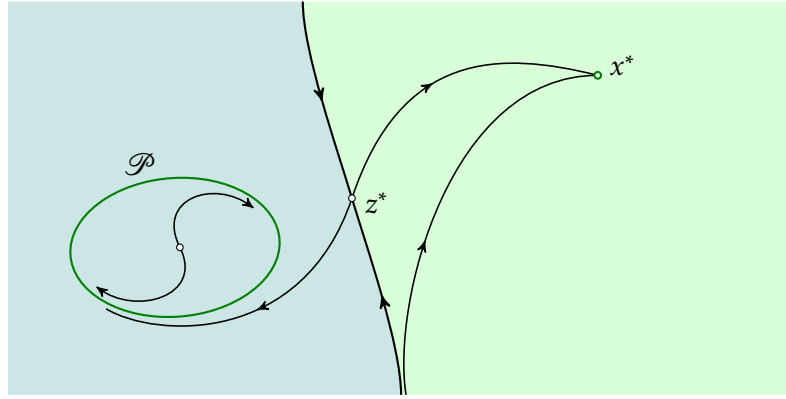
$$\begin{aligned} \mathbb{E}^{x_0}[\tau^\varepsilon] &= \mathbb{E}^{x_0} \left[ \sum_{m=1}^{\nu-1} (\tau_{m+1} - \tau_m) \right] \geq \mathbb{E}^{x_0} \left[ \sum_{m=1}^{\nu-1} (\tau_{m+1} - \sigma_m) \right] \\ &\geq \mathbb{E}^{x_0}[\nu] \inf_{x \in \partial \mathcal{B}_\delta(x^*)} \mathbb{E}^x[\tau_1]. \end{aligned}$$

The infimum is bounded below by a constant  $T_1$  independent of  $\varepsilon$  thanks to (3.3.10). To estimate  $\mathbb{E}^{x_0}[\nu]$ , we note that for any  $T > 0$ ,

$$\begin{aligned} \mathbb{P}^{x_0}\{\nu = 1\} &= \mathbb{P}^{x_0}\{\tau_1 = \tau^\varepsilon\} \\ &= \mathbb{P}^{x_0}\{\tau_1 = \tau^\varepsilon, \tau_1 < T\} + \mathbb{P}^{x_0}\{\tau_1 = \tau^\varepsilon, \tau_1 \geq T\}. \end{aligned}$$

By (3.3.12), for any  $\eta > 0$  the first term on the right-hand side can be bounded above by  $e^{-(\bar{V}-\eta)/\varepsilon}$  if  $\delta$  is small enough and  $T$  is large enough. The second term can be made smaller than the first one by taking  $T$  large enough, again thanks to (3.3.10). It follows that

$$\mathbb{E}^{x_0}[\tau^\varepsilon] \geq T_1 \sum_{m=0}^{\infty} \left( 1 - \sup_{x_0 \in \mathcal{B}_{\delta/2}(x^*)} \mathbb{P}^{x_0}\{\nu = 1\} \right)^m \leq T_1 e^{(\bar{V}-\eta)/\varepsilon},$$



**Figure 3.7** – Example of a two-dimensional vector field with two attractors (an equilibrium point  $x^*$  and a stable periodic orbit  $\mathcal{P}$ ). Their basins of attraction are shaded in different colours. When weak noise is added to the system, solutions spend long times in the neighbourhood of attractors, and make occasional transitions between attractors.

proving the lower bound for starting points in  $\mathcal{B}_{\delta/2}(x^*)$ . The result can be extended to general starting points in  $\mathcal{D}$  by using the lower bound

$$\mathbb{E}^{x_0}[\tau^\varepsilon] \geq \mathbb{E}^{x_0}[\mathbb{1}_{\{\tau_\varepsilon > \tau_1\}} \mathbb{E}^{X_{\tau_1}^\varepsilon}[\tau^\varepsilon]] \geq \mathbb{P}^{x_0}\{\tau_\varepsilon > \tau_1\} \inf_{x_1 \in \partial \mathcal{B}_\delta(x^*)} \mathbb{E}^{x_1}[\tau^\varepsilon]$$

and the fact that  $\mathbb{P}^{x_0}\{\tau_\varepsilon > \tau_1\}$  has order 1.  $\square$

One major limitation of Theorem 3.3.8 is the condition that orbits starting on  $\partial \mathcal{D}$  are attracted by the equilibrium point  $x^*$  (the condition on the vector field pointing inward on the boundary of  $\mathcal{D}$  is actually not essential for this result to hold). It is of great interest to extend this result to situations with multiple attractors, as illustrated in Figure 3.7. Theorem 3.3.8 cannot be applied, for instance, to describe transitions from the stable equilibrium  $x^*$  to the stable periodic orbit  $\mathcal{P}$ , or the other way, because this would mean leaving a set having the saddle point  $z^*$  on its boundary.

There exists a theory, discussed in [FW98, Chapter 6], dealing with such situations in great generality, by approximating the dynamics with a suitable Markov chain. This theory is quite involved, and here we will only outline the argument in a relatively simple bistable system, as shown in Figure 3.7. In particular, one would like to show that the first-hitting time  $\tau^\varepsilon$  of a neighbourhood of  $\mathcal{P}$  satisfies

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{E}^{x^*}[\tau^\varepsilon] &= \bar{V} \\ &= \inf_{T > 0} \inf \{ \mathcal{I}_{[0, T]}(\varphi) : \varphi \in \mathcal{C}([0, T], \mathbb{R}^n), \varphi(0) = x^*, \varphi(T) = z^* \}. \end{aligned} \quad (3.3.14)$$

An estimate that turns out to be useful to show such a result is the following “three-set argument” from [BG13, Corollary 5.8]. For sets  $A, B, C \subset \mathbb{R}^n$ , we denote by

$$\tau_A^+ = \inf\{t > 0 : X_t^\varepsilon \in A\}$$

the first-hitting time of  $A$ , and we write

$$\mathbb{P}^A\{\tau_B^+ < \tau_C^+\} = \sup_{x \in A} \mathbb{P}^x\{\tau_B^+ < \tau_C^+\}, \quad \mathbb{E}^A[\tau_B^+] = \sup_{x \in A} \mathbb{E}^x[\tau_B^+].$$

Then we have the following simple estimate.

**Lemma 3.3.9: Three-set argument**

Let  $A, B, C \subset \mathbb{R}^n$  be pairwise disjoint sets with smooth boundary. If  $\mathbb{P}^A\{\tau_C^+ < \tau_B^+\} < 1$ , then

$$\mathbb{E}^A[\tau_B^+] \leq \frac{\mathbb{E}^A[\tau_{B \cup C}^+] + \mathbb{P}^A\{\tau_C^+ < \tau_B^+\} \mathbb{E}^{\partial C}[\tau_{A \cup B}^+]}{1 - \mathbb{P}^A\{\tau_C^+ < \tau_B^+\}}.$$

**PROOF:** For any  $x \notin (B \cup C)$ , we have

$$\begin{aligned} \mathbb{E}^x[\tau_B^+] &= \mathbb{E}^x[\mathbb{1}_{\{\tau_B^+ < \tau_C^+\}} \tau_B^+] + \mathbb{E}^x[\mathbb{1}_{\{\tau_C^+ < \tau_B^+\}} \tau_B^+] \\ &= \mathbb{E}^x[\mathbb{1}_{\{\tau_B^+ < \tau_C^+\}} \tau_B^+] + \mathbb{E}^x[\mathbb{1}_{\{\tau_C^+ < \tau_B^+\}} (\tau_C^+ + \mathbb{E}^{X_{\tau_C^+}^\varepsilon}[\tau_B^+])] \\ &= \mathbb{E}^x[\tau_{B \cup C}^+] + \mathbb{E}^x[\mathbb{1}_{\{\tau_C^+ < \tau_B^+\}} \mathbb{E}^{X_{\tau_C^+}^\varepsilon}[\tau_B^+]], \end{aligned}$$

because  $\tau_B^+ \wedge \tau_C^+ = \tau_{B \cup C}^+$ . Since sample paths are continuous, we have  $X_{\tau_C^+}^\varepsilon \in \partial C$ , so that

$$\mathbb{E}^A[\tau_B^+] \leq \mathbb{E}^A[\tau_{B \cup C}^+] + \mathbb{P}^A\{\tau_C^+ < \tau_B^+\} \mathbb{E}^{\partial C}[\tau_B^+]. \quad (3.3.15)$$

Exchanging the roles of  $A$  and  $\partial C$ , we get

$$\mathbb{E}^{\partial C}[\tau_B^+] \leq \mathbb{E}^{\partial C}[\tau_{A \cup B}^+] + \mathbb{P}^{\partial C}\{\tau_A^+ < \tau_B^+\} \mathbb{E}^A[\tau_B^+] \leq \mathbb{E}^{\partial C}[\tau_{A \cup B}^+] + \mathbb{E}^A[\tau_B^+]. \quad (3.3.16)$$

Substituting (3.3.16) in (3.3.15) and solving for  $\mathbb{E}^A[\tau_B^+]$  gives the claimed result.  $\square$

Let us outline how this result is useful to prove the upper bound in (3.3.14). Let  $A$  and  $B$  be open sets containing respectively  $x^*$  and  $\mathcal{P}$ , and let  $C$  be the complement of a sufficiently large bounded set, containing both  $A$  and  $B$ . We further assume that there are no attractors for the deterministic dynamics in  $C$ . Then the following holds.

1. The probability  $\mathbb{P}^A\{\tau_C^+ < \tau_B^+\}$  can be shown to be smaller than, say,  $\frac{1}{2}$ , if the quasipotential on  $\partial C$  is large enough. This follows from an argument of the same type as in the proof of Theorem 3.3.8 (see [BG13, Proposition 5.14]).
2. The expectation  $\mathbb{E}^{\partial C}[\tau_{A \cup B}^+]$  can be shown to be at most of order  $\log(\varepsilon^{-1})$ , by using the fact that sample paths are likely to remain close to the deterministic orbit starting at the same point. The logarithmic behaviour is due to the time possibly spent near the saddle point.
3. It follows that  $\mathbb{E}^A[\tau_B^+]$  behaves like

$$\mathbb{E}^A[\tau_{B \cup C}^+] = \mathbb{E}^A[\tau_{(B \cup C)^c}] \leq \mathbb{E}^{C^c}[\tau_{(B \cup C)^c}^+],$$

which can be estimated by a Markov chain argument, as in the proof of Theorem 3.3.8.

**Remark 3.3.10**

The assumption that  $C$  does not contain any attractor is important to control the value of  $\mathbb{E}^{\partial C}[\tau_{A \cup B}^+]$ . Otherwise, there are cases where  $C$  contains an attractor that is much more stable than those in  $A$  and  $B$ , completely dominating the expectation of  $\tau_{A \cup B}^+$  and  $\tau_B^+$ .

**Remark 3.3.11: Eyring–Kramers law**

In the gradient case

$$dX_t = -\nabla U(X_t) dt + \sqrt{\varepsilon} dW_t,$$

the bistable situation occurs when  $U$  has exactly two local minima  $x^*$  and  $y^*$ , separated by a saddle point  $z^*$ . As pointed out in Example 3.3.5, the mean transition time from  $x^*$  to  $y^*$  obeys the Arrhenius law

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{E}^{x^*} [\tau_{\mathcal{B}_\delta(y^*)}^+] = 2[U(z^*) - U(x^*)].$$

However, using more sophisticated techniques, such as potential theoretic-methods relying on the links between PDEs and exit times seen in Chapter 1, one can prove the so-called Eyring–Kramers law

$$\mathbb{E}^{x^*} [\tau_{\mathcal{B}_\delta(y^*)}^+] = \frac{2\pi}{|\lambda_-(z^*)|} \sqrt{\frac{|\det \text{Hess } U(z^*)|}{\det \text{Hess } U(x^*)}} e^{2[U(z^*) - U(x^*)]/\varepsilon} [1 + \mathcal{O}(\varepsilon)],$$

where  $\text{Hess } U(x)$  denotes the Hessian matrix of  $U$  at  $x$ , and  $\lambda_-(z^*)$  denotes the unique negative eigenvalue of  $\text{Hess } U(z^*)$ . See for instance [Ber13] for an overview, and [BdH15] for a comprehensive account of the potential-theoretic approach allowing to prove this result. See [BR16, LMS19] for some extensions to non-gradient cases.

# Bibliography

- [ADR69] J. Azema, M. Duflo, and D. Revuz. Mesure invariante des processus de Markov recurrents. Sem. Probab. III, Univ. Strasbourg 1967/68, Lect. Notes Math. 88, 24-33 (1969)., 1969.
- [BB17] Manon Baudel and Nils Berglund. Spectral Theory for Random Poincaré Maps. *SIAM J. Math. Anal.*, 49(6):4319–4375, 2017.
- [BdH15] Anton Bovier and Frank den Hollander. *Metastability. A potential-theoretic approach*, volume 351 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Cham, 2015.
- [Ber13] Nils Berglund. Kramers’ law: Validity, derivations and generalisations. *Markov Process. Related Fields*, 19(3):459–490, 2013.
- [BG13] Nils Berglund and Barbara Gentz. Sharp estimates for metastable lifetimes in parabolic SPDEs: Kramers’ law and beyond. *Electron. J. Probab.*, 18:no. 24, 58, 2013.
- [BG14] Nils Berglund and Barbara Gentz. On the noise-induced passage through an unstable periodic orbit II: General case. *SIAM J. Math. Anal.*, 46(1):310–352, 2014.
- [Bir57] Garrett Birkhoff. Extensions of Jentzsch’s theorem. *Trans. Amer. Math. Soc.*, 85:219–227, 1957.
- [BR16] Freddy Bouchet and Julien Reygner. Generalisation of the Eyring–Kramers transition rate formula to irreversible diffusion processes. *Ann. Henri Poincaré*, 17(12):3499–3532, 2016.
- [DPZ96] G. Da Prato and J. Zabczyk. *Ergodicity for infinite-dimensional systems*, volume 229 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1996.
- [DS89] Jean-Dominique Deuschel and Daniel W. Stroock. *Large deviations*. Academic Press, Boston, 1989. Reprinted by the American Mathematical Society, 2001.
- [DZ98] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications*, volume 38 of *Applications of Mathematics*. Springer-Verlag, New York, second edition, 1998.
- [FW98] M. I. Freidlin and A. D. Wentzell. *Random Perturbations of Dynamical Systems*. Springer-Verlag, New York, second edition, 1998.
- [Get80] R. K. Gettoor. Transience and recurrence of Markov processes. Seminaire de probabilités XIV, 1978/79, Lect. Notes Math. 784, 397-409 (1980)., 1980.



- [GT01] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [HM11] Martin Hairer and Jonathan C. Mattingly. Yet another look at Harris' ergodic theorem for Markov chains. In *Seminar on Stochastic Analysis, Random Fields and Applications VI*, volume 63 of *Progr. Probab.*, pages 109–117. Birkhäuser/Springer Basel AG, Basel, 2011.
- [HW15] Martin Hairer and Hendrik Weber. Large deviations for white-noise driven, nonlinear stochastic PDEs in two and three dimensions. *Ann. Fac. Sci. Toulouse Math. (6)*, 24(1):55–92, 2015.
- [Jen12] Jentzsch. Über Integralgleichungen mit positivem Kern. *J. f. d. reine und angew. Math.*, 141:235–244, 1912.
- [LMS19] C. Landim, M. Mariani, and I. Seo. Dirichlet's and Thomson's principles for non-selfadjoint elliptic operators with application to non-reversible metastable diffusion processes. *Arch. Ration. Mech. Anal.*, 231(2):887–938, 2019.
- [MT93a] S. P. Meyn and R. L. Tweedie. Generalized resolvents and Harris recurrence of Markov processes. In *Doebelin and modern probability (Blaubeuren, 1991)*, volume 149 of *Contemp. Math.*, pages 227–250. Amer. Math. Soc., Providence, RI, 1993.
- [MT93b] Sean P. Meyn and R. L. Tweedie. Stability of Markovian processes. II. Continuous-time processes and sampled chains. *Adv. in Appl. Probab.*, 25(3):487–517, 1993.
- [MT93c] Sean P. Meyn and R. L. Tweedie. Stability of Markovian processes. III. Foster-Lyapunov criteria for continuous-time processes. *Adv. in Appl. Probab.*, 25(3):518–548, 1993.
- [Num84] Esa Nummelin. *General irreducible Markov chains and nonnegative operators*, volume 83 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1984.
- [Øks03] B. Øksendal. *Stochastic Differential Equations*. Springer, 2003.
- [Ore71] Steven Orey. *Lecture notes on limit theorems for Markov chain transition probabilities*. Van Nostrand Reinhold Co., London, 1971. Van Nostrand Reinhold Mathematical Studies, No. 34.
- [RY99] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

Nils Berglund  
Institut Denis Poisson (IDP)  
Universite d'Orleans, Universite de Tours, CNRS – UMR 7013  
Bâtiment de Mathématiques, B.P. 6759  
45067 Orleans Cedex 2, France  
E-mail address: [nils.berglund@univ-orleans.fr](mailto:nils.berglund@univ-orleans.fr)  
<https://www.idpoisson.fr/berglund>