Integral representation of unbounded variational functionals on Sobolev spaces

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INTEGRAL REPRESENTATION OF UNBOUNDED VARIATIONAL FUNCTIONALS ON SOBOLEV SPACES

OMAR ANZA HAFSA AND JEAN-PHILIPPE MANDALLENA

Abstract. In this paper we establish an unbounded version of the integral representation theorem by Buttazzo and Dal Maso (see [BDM85] and also [BFLM02]). More precisely, we prove an integral representation theorem (with a formula for the integrand) for functionals defined on $W^{1,p}$ with $p > N$ (N being the dimension) that do not satisfy a standard p-growth condition from above and can take infinite values. Applications to Γ-convergence, relaxation and homogenization are also developed.

Contents

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1. Introduction and main result

Let $m, N \geqslant 1$ be two integer, let $p > 1$ be a real number, let $\Omega \subset \mathbb{R}^N$ be a bounded open set, let $\mathbb{M}$ be the space of $m \times N$ matrices and let $\mathcal{O}(\Omega)$ be the class of all open subset of Ω. In this paper we consider variational functionals[1] $\mathscr{F} : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0, \infty]$ having the following growth property:

(I_0) there exists $\mathscr{G} : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0, \infty]$ defined by

$$\mathscr{G}(u, A) := \int_\Omega G(x, \nabla u(x))dx,$$

with a Borel measurable function $G : \Omega \times \mathbb{M} \to [0, \infty]$, and there exist $\alpha, \beta > 0$ such that for every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and every $A \in \mathcal{O}(\Omega)$,

$$\alpha \mathscr{G}(u, A) \leqslant \mathscr{F}(u, A) \leqslant \beta\left(|A| + \mathscr{G}(u, A)\right).$$

In the bounded case, i.e. when $G(x, \xi) = |\xi|^p$, Buttazzo and Dal Maso (see [BDM85, Theorem 1.1] and also [But89, §4.3, pp. 148], [BD98, Chapter 9, pp. 77] and [DM93, Chapter 20, pp. 215]) and Bouchitté, Fonseca, Leoni and Mascarenhas (see [BFLM02, Theorem 2]) proved the following theorem.

Theorem 1.1 ([BDM85, BFLM02])**.** Under (I_0) with $G(x, \xi) = |\xi|^p$, if $\mathscr{F}$ satisfies the following four conditions:

(I_1) for every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$, the set function $\mathscr{F}(u, \cdot)$ is the restriction to $\mathcal{O}(\Omega)$ of a Borel measure,

(I_2) for every $u, v \in W^{1,p}(\Omega; \mathbb{R}^m)$ and every $A \in \mathcal{O}(\Omega)$, if $u(x) = v(x)$ for $\mathscr{L}_N$-a.a. $x \in A$, then $\mathscr{F}(u, A) = \mathscr{F}(v, A)$;

(I_3) for every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$, every $A \in \mathcal{O}(\Omega)$ and every $z \in \mathbb{R}^m$, $\mathscr{F}(u + z, A) = \mathscr{F}(u, A)$;

(I_4) for every $A \in \mathcal{O}(\Omega)$ the functional $\mathscr{F}(\cdot, A)$ is L^p-lower semicontinuous,

then, for every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and every $A \in \mathcal{O}(\Omega)$,

$$\mathscr{F}(u, A) = \int_A F(x, \nabla u(x))dx$$

with $F : \Omega \times \mathbb{M} \to [0, \infty]$ defined by

$$F(x, \xi) := \varlimsup_{\rho \to 0} \inf\left\{\frac{\mathscr{F}(u, Q_\rho(x))}{\rho^N} : u - l_\xi \in W_0^{1,p}(Q_\rho(x); \mathbb{R}^m)\right\}, \tag{1.1}$$

where $Q_\rho(x) := x + \rho Y$ with $Y :=]-\frac{1}{2}, \frac{1}{2}[^N$ and $l_\xi : \mathbb{R}^N \to \mathbb{R}^m$ is the linear map defined by $l_\xi(y) := \xi y$.

The object of the present paper is to deal with the problem of finding an integral representation for $\mathscr{F}$ in the unbounded case, i.e. when $G(x, \xi)$ is not necessarily equal to $|\xi|^p$. In the scalar case, i.e. when $m = 1$, integral representation problems for unbounded functional were studied by Carbone and De Arcangelis in [CDA02, Chapter 9]. Here we deal with the vectorial case.

Our main result is to establish the following unbounded version of Theorem 1.1.

[1] By a variational functional we mean a function of functions and sets.

Theorem 1.2. *Assume that $p > N$ and (I_0)–(I_4) hold. Assume further that the following five assumptions are satisfied:*

(I_5) *G is p-coercive, i.e. there exists $c > 0$ such that for every $(x, \xi) \in \Omega \times \mathbb{M}$,*
$$G(x, \xi) \geqslant c|\xi|^p;$$

(I_6) *there exists $\gamma > 0$ such that for every $x \in \Omega$, every $t \in]0, 1[$ and every $\xi, \zeta \in \mathbb{M}$,*
$$G(x, t\xi + (1-t)\zeta) \leqslant \gamma(1 + G(x, \xi) + G(x, \zeta));$$

(I_7) *there exists $r > 0$ such that*
$$\sup_{|\xi|\leqslant r} G(\cdot, \xi) \in L^1(\Omega);$$

(I_8) *for every $A \in \mathcal{O}(\Omega)$, every $u \in \mathrm{dom}(\mathcal{G}(\cdot, A))$ and $\mathcal{L}_N$-a.e. $x \in A$,*
$$\lim_{\rho\to 0} \fint_{Q_\rho(x)} \big|G(y, \nabla u(x)) - G(x, \nabla u(x)\big| dy = 0,$$
where $\mathrm{dom}(\mathcal{G}(\cdot, A))$ denotes the effective domain of $\mathcal{G}(\cdot, A)$;

(I_9) *$\mathcal{F}$ is radially uniformly upper semicontinuous (ru-usc), i.e. there exists $a \in L^1(\Omega;]0, \infty])$ such that*
$$\overline{\lim_{t\to 1^-}}\, \Delta^a_{\mathcal{F}}(t) \leqslant 0$$
with $\Delta^a_{\mathcal{F}} : [0, 1] \to]-\infty, \infty]$ defined by
$$\Delta^a_{\mathcal{F}}(t) := \sup_{A\in\mathcal{O}(\Omega)} \sup_{u\in\mathrm{dom}(\mathcal{F}(\cdot,A))} \frac{\mathcal{F}(tu, A) - \mathcal{F}(u, A)}{\int_A a(x)dx + \mathcal{F}(u, A)},$$
where $\mathrm{dom}(\mathcal{F}(\cdot, A))$ denotes the effective domain of $\mathcal{F}(\cdot, A)$[2].

Then, for every $A \in \mathcal{O}(\Omega)$,
$$\mathcal{F}(u, A) = \begin{cases} \displaystyle\int_A \widehat{F}(x, \nabla u(x))dx & \text{if } u \in \mathrm{dom}(\mathcal{G}(\cdot, A)) \\ \infty & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^m)\backslash\mathrm{dom}(\mathcal{G}(\cdot, A)), \end{cases}$$
where $\widehat{F} : \Omega \times \mathbb{M} \to [0, \infty]$ is defined by
$$\widehat{F}(x, \xi) = \lim_{t\to 1^-} F(x, t\xi)$$
with F given by (1.1).

Remark 1.3.

(i) If $G(x, \xi) = G_1(x) + G_2(\xi)$ or $G(x, \xi) = G_1(x)G_2(\xi)$ and if $G_1 \in L^1_{\mathrm{loc}}(\Omega)$, then (I_8) holds.

(ii) (I_6) implies that for every $x \in \mathbb{M}$, the effective domain of $G(x, \cdot)$ is convex.

[2]Note that under (I_0) we have $\mathrm{dom}(\mathcal{F}(\cdot, A)) = \mathrm{dom}(\mathcal{G}(\cdot, A))$ for all $A \in \mathcal{O}(\Omega)$.

(iii) When G does not depend on x, i.e. $G(x,\xi) = G(\xi)$, (I_8) can be dropped and (I_7) means that G is bounded at the neighborhood of the null matrix. It is proved in [AHM12, Lemma 4.1] that such a boundlessness condition holds if (I_6) is satisfied and if the null matrix belongs to the interior of the effective domain of G. So, for $G(x,\xi) = G(\xi)$, under (I_6), (I_7) can be replaced by the following "simpler" assumption:

($\mathrm{I}_{7'}$) *the null matrix belongs to the interior of the effective domain of G.*

(iv) When G is convex with respect to ξ, (I_6) can be dropped and, since a convex function is continuous in the interior of its effective domain, (I_7) can be replaced by the following "simpler"assumption:

($\mathrm{I}_{7''}$) *the null matrix belongs to the interior of the effective domain of the convex function $\xi \mapsto \sup_{x\in\Omega} G(x,\xi)$.*

Integral representation theorems for variational functionals are part of a general method, usually called "the localization method[3]", which was introduced by the Italian school at the end of the seventies (see [DGL77, DM78, DMM81]) for dealing with Γ-convergence, relaxation and homogenization of integral functionals of the Calculus of Variations. In the bounded case, i.e. under standard p-growth conditions, the method is well-developed (see the books [But89, DM93, BD98]). On the other hand, in the unbounded case the method does not work satisfactory and from [Bra06, Remark 4.1] and [BD98, Remark 12.7] a long-standing conjecture of De Giorgi is that it should be possible to deal with the G-growth case where G is such that $\mathscr{G}(u, A) = \int_A G(x, \nabla u(x))dx$ is lower semicontinuous. Theorem 1.2 and its corollaries (see Theorem 4.3 and Corollaries 4.9 and 4.10) gives a partial answer to this conjecture, the main ingredient being the condition of ru-usc (see §2.5 for more details on this notion) that plays a fundamental role in the proof of Theorem 1.2 and its applications to Γ-convergence, relaxation and homogenization. Convexity implies ru-usc (see [AHM14]) but, in the nonconvex and vectorial case, ru-usc seems to be essential to develop the localization method beyond the p-growth case.

The plan of the paper is as follows. Section 2 contains auxiliary results for proving Theorem 1.2 (see §2.1) and for dealing with applications (see §2.2, §2.3, §2.4 and 2.5). Section 3 is devoted to the proof of Theorem 1.2. Finally, in Section 4, applications to $\Gamma(L^p)$-convergence (see Theorem 4.3), relaxation (see Corollary 4.9) and homogenization (see Corollary 4.10) are developed.

Throughout the paper, we will use the following notation and terminology.

- The Lebesgue measure is denoted by dx, dy or $\mathscr{L}_N$ and the Lebesgue measure of any Borel measurable set $Q \subset \mathbb{R}^N$ is denoted by $|Q|$.
- The interior (resp. closure) of a set $B \subset \mathbb{R}^N$ is denoted by $\mathring{B}$ (resp. $\overline{B}$).
- The interior of any subset $\mathbb{U}$ of the set $\mathbb{M}$ of $m \times N$ matrices will be denoted by $\mathrm{int}(\mathbb{U})$.
- The symbol $⨍$ stands for the mean-value integral, i.e. $⨍_Q f dx = \frac{1}{|Q|}\int_Q f dx.$

[3]From [Bra93, Lesson Two, pp. 51] the localization method consists of, firstly, proving a compactness theorem which allows to obtain for each sequence of integral functionals a subsequence Γ-converging to an abstract limit functional, secondly, proving an integral representation result which allows us to write the limit functional as an integral and, thirdly, proving a representation formula for the limit integrand which does not depend on the subsequence, showing thus that the limit is well-defined.

- By the effective domain of a function $L : \mathbb{M} \to [0,\infty]$ we mean $\mathbb{L} \subset \mathbb{M}$ given by $\mathbb{L} := \{\xi \in \mathbb{M} : L(\xi) < \infty\}$.
- Given a variational functional $\mathscr{J} : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$, by the effective domain $\mathscr{J}(\cdot,A)$ with $A \in \mathcal{O}(\Omega)$, where $\mathcal{O}(\Omega)$ denotes the class of all open subset of Ω, we mean $\mathrm{dom}(\mathscr{J}(\cdot,A))$ given by $\mathrm{dom}(\mathscr{J}(\cdot,A)) := \{u \in W^{1,p}(\Omega;\mathbb{R}^m) : \mathscr{J}(u,A) < \infty\}$.

2. Auxiliary results

2.1. Integral representation of the Vitali envelope of a set function. What follows was first developed in [BFM98, BB00] (see also [AHCM17]). Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $\mathcal{O}(\Omega)$ be the class of open subsets of Ω. We begin with the concept of the Vitali envelope of a set function.

For each $\varepsilon > 0$ and each $A \in \mathcal{O}(\Omega)$, denote the class of countable families $\{Q_i = Q_{\rho_i}(x_i)\}_{i\in I}$ (where $Q_{\rho_i}(x_i) := x_i + \rho_i Y$ where $Y :=]-\frac{1}{2},\frac{1}{2}[^N$) of disjoint open cubes of A with $x_i \in A$, $\rho_i > 0$ and $\mathrm{diam}(Q_i) \in]0,\varepsilon[$ such that $|A \backslash \cup_{i\in I} Q_i| = 0$ by $\mathscr{V}_\varepsilon(A)$.

Definition 2.1. Given $\mathcal{S} : \mathcal{O}(\Omega) \to [0,\infty]$, for each $\varepsilon > 0$ we define $\mathcal{S}^\varepsilon : \mathcal{O}(\Omega) \to [0,\infty]$ by

$$\mathcal{S}^\varepsilon(A) := \inf\left\{\sum_{i\in I} \mathcal{S}(Q_i) : \{Q_i\}_{i\in I} \in \mathscr{V}_\varepsilon(A)\right\}. \tag{2.1}$$

By the Vitali envelope of $\mathcal{S}$ we call the set function $\mathcal{S}^* : \mathcal{O}(\Omega) \to [-\infty,\infty]$ defined by

$$\mathcal{S}^*(A) := \sup_{\varepsilon>0} \mathcal{S}^\varepsilon(A) = \lim_{\varepsilon\to 0} \mathcal{S}^\varepsilon(A). \tag{2.2}$$

The interest of Definition 2.1 comes from the following integral representation result. (For a proof we refer to [AHCM17, §A.4].)

Theorem 2.2. *Let $\mathcal{S} : \mathcal{O}(\Omega) \to [0,\infty]$ be a set function satisfying the following two conditions*:

(a) *there exists a finite Borel measure ν on Ω which is absolutely continuous with respect to $\mathscr{L}_N$ such that $\mathcal{S}(A) \leqslant \nu(A)$ for all $A \in \mathcal{O}(\Omega)$;*
(b) *$\mathcal{S}$ is subadditive, i.e. $\mathcal{S}(A) \leqslant \mathcal{S}(B) + \mathcal{S}(C)$ for all $A, B, C \in \mathcal{O}(\Omega)$ with $B, C \subset A$, $B \cap C = \emptyset$ and $|A\backslash B \cup C| = 0$.*

Then $\lim_{\rho\to 0} \frac{\mathcal{S}(Q_\rho(\cdot))}{\rho^N} \in L^1(\Omega)$ *and for every* $A \in \mathcal{O}(\Omega)$,

$$\mathcal{S}^*(A) = \int_A \lim_{\rho\to 0} \frac{\mathcal{S}(Q_\rho(x))}{\rho^N} dx.$$

2.2. Compactness theorem with respect to $\Gamma(L^p)$-convergence. Let $p > 1$, let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $\mathcal{O}(\Omega)$ denote the class of all open subset of Ω. We begin by recalling the definition of $\Gamma(L^p)$-convergence (see [DM93, BD98, Bra06] for more details).

Definition 2.3. For each $\varepsilon > 0$, let $\mathscr{J}_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$ be a variational functional and let $\Gamma(L^p)\text{-}\varliminf_{\varepsilon\to 0} \mathscr{J}_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$ and $\Gamma(L^p)\text{-}\varlimsup_{\varepsilon\to 0} \mathscr{J}_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \times$

$\mathscr{O}(\Omega) \to [0, \infty]$ be respectively defined by:

$$\Gamma(L^p)\text{-}\varliminf_{\varepsilon\to 0} \mathscr{I}_\varepsilon(u, A) := \inf\left\{\varliminf_{\varepsilon\to 0} \mathscr{I}_\varepsilon(u_\varepsilon, A) : u_\varepsilon \xrightarrow{L^p} u\right\};$$
$$\Gamma(L^p)\text{-}\varlimsup_{\varepsilon\to 0} \mathscr{I}_\varepsilon(u, A) := \inf\left\{\varlimsup_{\varepsilon\to 0} \mathscr{I}_\varepsilon(u_\varepsilon, A) : u_\varepsilon \xrightarrow{L^p} u\right\}.$$

Let $\mathscr{I}_0 : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathscr{O}(\Omega) \to [0, \infty]$ be a variational functional. We say that $\{\mathscr{I}_\varepsilon\}_{\varepsilon>0}$ $\Gamma(L^p)$-converges to $\mathscr{I}_0$, and we write $\mathscr{I}_0 = \Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} \mathscr{I}_\varepsilon$, if the following two inequalities hold:

$$\mathscr{I}_0 \leqslant \Gamma(L^p)\text{-}\varliminf_{\varepsilon\to 0} \mathscr{I}_\varepsilon;$$
$$\Gamma(L^p)\text{-}\varlimsup_{\varepsilon\to 0} \mathscr{I}_\varepsilon \leqslant \mathscr{I}_0.$$

For families of increasing variational functionals (typically variational integrals), the following theorem isolates a condition on the $\Gamma(L^p)$-limit inf and the $\Gamma(L^p)$-limit sup to have compactness with respect to $\Gamma(L^p)$-convergence. (For a proof we refer to [DM93, Theorem 16.9, pp. 184] and also [BD98, Theorem 10.3, pp. 84].)

Theorem 2.4. *For each $\varepsilon > 0$, let $\mathscr{I}_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathscr{O}(\Omega) \to [0, \infty]$ be an increasing variational functional, i.e. $\mathscr{I}_\varepsilon(u, \cdot)$ is increasing[4] for all $u \in W^{1,p}(\Omega; \mathbb{R}^m)$. Suppose that for every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$,*

$$\Gamma(L^p)\text{-}\varliminf_{\varepsilon\to 0} \mathscr{I}_\varepsilon(u, \cdot) \textit{ and } \Gamma(L^p)\text{-}\varlimsup_{\varepsilon\to 0} \mathscr{I}_\varepsilon(u, \cdot) \textit{ are inner regular.}^5 \tag{2.3}$$

Then, every sequence $\{\mathscr{I}_\varepsilon\}_{\varepsilon>0}$ has a $\Gamma(L^p)$-convergent subsequence.

Remark 2.5. To verify (2.3) we will use Lemma 2.9 (see §2.3) which will allow to establish, under additional assumptions, the stronger property that the $\Gamma(L^p)$-limit inf and the $\Gamma(L^p)$-limit sup are restrictions to $\mathscr{O}(\Omega)$ of Borel measures.

The following proposition, combined with Theorem 2.4, furnishes a useful tool to deal with the $\Gamma(L^p)$-convergence of increasing variational functionals. (For a proof we refer to [DM93, Proposition 16.8, pp. 183].)

Proposition 2.6. *For each $\varepsilon > 0$, let $\mathscr{I}_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathscr{O}(\Omega) \to [0, \infty]$ be an increasing variational functional and let $\mathscr{I}_0 : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathscr{O}(\Omega) \to [0, \infty]$ be an increasing variational functional. If* (2.3) *holds then $\{\mathscr{I}_\varepsilon\}_{\varepsilon>0}$ $\Gamma(L^p)$-converges to $\mathscr{I}_0$ if and only every subsequence of $\{\mathscr{I}_\varepsilon\}_{\varepsilon>0}$ contains a further subsequence which $\Gamma(L^p)$-converges to $\mathscr{I}_0$.*

[4] A set function $\mathcal{S} : \mathscr{O}(\Omega) \to [0, \infty]$ is said to be increasing if $\mathcal{S}(A) \leqslant \mathcal{S}(B)$ for all $A, B \in \mathscr{O}(\Omega)$ such that $A \subset B$.

[3] An increasing set function $\mathcal{S} : \mathscr{O}(\Omega) \to [0, \infty]$ is said to be inner regular if for every $A \in \mathscr{O}(\Omega)$, $\mathcal{S}(A) = \sup\{\mathcal{S}(U) : U \in \mathscr{O}(\Omega) \text{ and } \overline{U} \subset A\}$. (Note that $\Gamma(L^p)\text{-}\varliminf_{\varepsilon\to 0} \mathscr{I}_\varepsilon(u, \cdot)$ and $\Gamma(L^p)\text{-}\varlimsup_{\varepsilon\to 0} \mathscr{I}_\varepsilon(u, \cdot)$ are increasing whenever every $\mathscr{I}_\varepsilon(u, \cdot)$ is increasing.)

2.3. **Increasing set functions.** Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, let $\mathcal{O}(\Omega)$ be the class of open subsets of Ω and let $\mathcal{B}(\Omega)$ be the class of Borel subsets of Ω, i.e. the smallest σ-algebra containing the open (or equivalently the closed) subsets of Ω. The following result is due to De Giorgi and Letta (see [DGL77] and also [DM93, Theorem 14.23, pp. 172]).

Theorem 2.7. *Let $\mathcal{S} : \mathcal{O}(\Omega) \to [0, \infty]$ be an increasing set function such that $\mathcal{S}(\varnothing) = 0$. Then, $\mathcal{S}$ is the restriction to $\mathcal{O}(\Omega)$ of a Borel measure, i.e. there exists a measure $\mu : \mathcal{B}(\Omega) \to [0, \infty]$ such that $\mathcal{S}(A) = \mu(A)$ for all $A \in \mathcal{O}(\Omega)$, if and only if the following three conditions hold*:

- (a_1) *$\mathcal{S}$ is subadditive, i.e. $\mathcal{S}(A \cup B) \leqslant \mathcal{S}(A) + \mathcal{S}(B)$ for all $A, B \in \mathcal{O}(\Omega)$;*
- (a_2) *$\mathcal{S}$ is superadditive, i.e. $\mathcal{S}(A \cup B) \geqslant \mathcal{S}(A) + \mathcal{S}(B)$ for all $A, B \in \mathcal{O}(\Omega)$ such that $A \cap B = \varnothing$;*
- (a_3) *$\mathcal{S}$ is inner regular, i.e. $\mathcal{S}(A) = \sup\left\{\mathcal{S}(U) : U \in \mathcal{O}(\Omega) \text{ and } \overline{U} \subset A\right\}$ for all $A \in \mathcal{O}(\Omega)$.*

To establish inner regularity we have the following proposition due to Carbone and De Arcangelis (see [CDA02, Proposition 2.6.10, pp. 74]).

Proposition 2.8. *Let $\mathcal{S} : \mathcal{O}(\Omega) \to [0, \infty]$ be an increasing set function satisfying* (a_1) *and the following two additional conditions*:

- (a_4) *for every $A \in \mathcal{O}(\Omega)$ with $\mathcal{S}(A) < \infty$, $\lim_{n\to\infty} \mathcal{S}(A_n) = 0$ for all $\{A_n\}_n \subset \mathcal{O}(\Omega)$ such that $\overline{A\backslash A_n} \subset A\backslash A_{n+1}$ and $\cup_n A\backslash A_n = A$;*
- (a_5) *for every $A \in \mathcal{O}(\Omega)$ with $\mathcal{S}(A) = \infty$, $\lim_{n\to\infty} \mathcal{S}(A_n) = \infty$ for all $\{A_n\}_n \subset \mathcal{O}(\Omega)$ such that $\overline{A_n} \subset A_{n+1}$ and $\cup_n A_n = A$.*

Then (a_3) *holds.*

It is easily seen that the condition (a_6) below implies (a_4) and (a_5). The following result is then a straightforward consequence of Theorem 2.7 and Proposition 2.8.

Lemma 2.9. *Let $\mathcal{S} : \mathcal{O}(\Omega) \to [0, \infty]$ be an increasing set function satisfying* (a_1) *and* (a_2) *and the following additional condition*:

- (a_6) *there exist $\alpha, \beta > 0$, a measure $\nu : \mathcal{B}(\Omega) \to [0, \infty]$ and a finite measure $\lambda : \mathcal{B}(\Omega) \to [0, \infty[$ such that $\alpha\nu(A) \leqslant \mathcal{S}(A) \leqslant \beta(\lambda(A) + \nu(A))$ for all $A \in \mathcal{O}(\Omega)$.*

Then, $\mathcal{S}$ is the restriction to $\mathcal{O}(\Omega)$ of a Borel measure.

2.4. **A subadditive theorem.** Let $\mathcal{O}_{\rm b}(\mathbb{R}^N)$ be the class of all bounded open subsets of $\mathbb{R}^N$. We begin with the following definition.

Definition 2.10. Let $\mathcal{S} : \mathcal{O}_{\rm b}(\mathbb{R}^N) \to [0, \infty]$ be a set function.

- (i) We say that $\mathcal{S}$ is subadditive if
$$\mathcal{S}(A) \leqslant \mathcal{S}(B) + \mathcal{S}(C)$$
for all $A, B, C \in \mathcal{O}_{\rm b}(\mathbb{R}^N)$ with $B, C \subset A$, $B \cap C = \varnothing$ and $|A\backslash B \cup C| = 0$.
- (ii) We say that $\mathcal{S}$ is $\mathbb{Z}^N$-invariant if
$$\mathcal{S}(A + z) = \mathcal{S}(A)$$
for all $A \in \mathcal{O}_{\rm b}(\mathbb{R}^N)$ and all $z \in \mathbb{Z}^N$.

Let $\mathrm{Cub}(\mathbb{R}^N)$ be the class of all open cubes in $\mathbb{R}^N$. The following theorem is due to Akcoglu and Krengel (see [AK81] and also [LM02] and [AHM11, Theorem 3.11]).

Theorem 2.11. *Let $\mathcal{S} : \mathcal{O}_{\rm b}(\mathbb{R}^N) \to [0,\infty]$ be a subadditive and $\mathbb{Z}^N$-invariant set function for which there exists $C \in]0,\infty[$ such that for every $A \in \mathcal{O}_{\rm b}(\mathbb{R}^N)$,*

$$\mathcal{S}(A) \leqslant C|A|.$$

Then, for every $Q \in \mathrm{Cub}(\mathbb{R}^N)$,

$$\lim_{\varepsilon\to 0} \frac{\mathcal{S}\left(\frac{1}{\varepsilon}Q\right)}{\left|\frac{1}{\varepsilon}Q\right|} = \inf_{k\geqslant 1} \frac{\mathcal{S}(]0,k[^N)}{k^N}.$$

2.5. **Ru-usc property.** We begin by recalling the concept of ru-usc function which was introduced in [AH10] (see also [AHM14] and [AHM11, §3.1]).

2.5.1. *Ru-usc function.* Let $\Omega \subset \mathbb{R}^N$ be an open set and let $L : \Omega \times \mathbb{M} \to [0,\infty]$ is a Borel measurable function, where $\mathbb{M}$ denotes the space of $m \times N$ matrices. For each $x \in \Omega$, we denote the effective domain of $L(x,\cdot)$ by $\mathbb{L}_x$ and, for each $a \in L^1_{\rm loc}(\Omega;]0,\infty])$, we consider $\delta^a_L : [0,1] \to]-\infty,\infty]$ defined by

$$\delta^a_L(t) := \sup_{x\in\Omega} \sup_{\xi\in\mathbb{L}_x} \frac{L(x,t\xi) - L(x,\xi)}{a(x) + L(x,\xi)}.$$

Definition 2.12. We say that $L : \Omega \times \mathbb{M} \to [0,\infty]$ is ru-usc if there exists $a \in L^1_{\rm loc}(\Omega;]0,\infty])$ such that

$$\overline{\lim_{t\to 1^-}} \delta^a_L(t) \leqslant 0. \tag{2.4}$$

The interest of Definition 2.12 comes from the following theorem. (For a proof we refer to [AHM11, Theorem 3.5] and also [AHM12, §4.2]) Let $\widehat{L} : \Omega \times \mathbb{M} \to [0,\infty]$ be defined by

$$\widehat{L}(x,\xi) := \underline{\lim_{t\to 1^-}} L(x,t\xi).$$

Theorem 2.13. *If $L : \Omega \times \mathbb{M} \to [0,\infty]$ is ru-usc and if for every $x \in \Omega$,*

$$t\overline{\mathbb{L}_x} \subset \mathrm{int}(\mathbb{L}_x) \text{ for all } t \in]0,1[, \tag{2.5}$$

then:

(a) $\widehat{L}$ *is ru-usc*;
(b) $\widehat{L}(x,\xi) = \lim_{t\to 1^-} L(x,t\xi)$ *for all* $(x,\xi) \in \Omega \times \mathbb{M}$.

If moreover, for every $x \in \Omega$, $L(x,\cdot)$ is lsc on $\mathrm{int}(\mathbb{L}_x)$ then:

(c) $$\widehat{L}(x,\xi) = \begin{cases} L(x,\xi) & \text{if } \xi \in \mathrm{int}(\mathbb{L}_x) \\ \lim_{t\to 1^-} L(x,t\xi) & \text{if } \xi \in \partial\mathbb{L}_x \\ \infty & \text{otherwise;} \end{cases}$$
(d) *for every $x \in \Omega$, $\widehat{L}(x,\cdot)$ is the lsc envelope of $L(x,\cdot)$.*

2.5.2. *Ru-usc variational functional.* Let $p > 1$, let $\mathcal{O}(\Omega)$ be the class of all open subsets of Ω. Let $\mathcal{J} : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$ be a variational functional and let $\Delta^a_{\mathcal{J}} : [0,1] \to]-\infty,\infty]$ be defined by

$$\Delta^a_{\mathcal{J}}(t) := \sup_{A\in\mathcal{O}(\Omega)} \sup_{u\in\mathrm{dom}(\mathcal{J}(\cdot,A))} \frac{\mathcal{J}(tu,A) - \mathcal{J}(u,A)}{\int_A a(x)dx + \mathcal{J}(u,A)},$$

where $\mathrm{dom}(\mathcal{J}(\cdot,A))$ denotes the effective domain of $\mathcal{G}(\cdot,A)$. From now on, Ω is bounded. Analogously to the case of a function, the concept of ru-usc for a variational functional is defined as follows.

Definition 2.14. We say that $\mathcal{J} : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$ is ru-usc if there exists $a \in L^1(\Omega;]0,\infty])$ such that

$$\varlimsup_{t\to 1^-} \Delta^a_{\mathcal{J}}(t) \leqslant 0.$$

Definition 2.14 is motivated by the following result which asserts that "the variational integrals whose integrand is ru-usc are ru-usc variational functionals".

Proposition 2.15. *If $L : \Omega \times \mathbb{M} \to [0,\infty]$ is a ru-usc function with $a \in L^1(\Omega;]0,\infty])$ and if $\mathcal{J} : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$ is such that*

$$\mathcal{J}(u,A) = \int_A L(x,\nabla u(x))dx$$

for all $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and all $A \in \mathcal{O}(\Omega)$, then $\mathcal{J}$ is a ru-usc variational integral with the same function a.

Proof of Proposition 2.15. Fix any $t \in [0,1]$, any $A \in \mathcal{O}(\Omega)$ and any $u \in \mathrm{dom}(\mathcal{J}(\cdot,A))$. Then $\nabla u(x) \in \mathbb{L}_x$ for $\mathscr{L}_N$-a.e. $x \in A$, and so

$$L(x,\nabla(tu)(x)) = L(x,t\nabla u(x)) \leqslant \delta^a_L(t)(a(x) + L(x,\nabla u(x)) + L(x,\nabla u(x))$$

for $\mathscr{L}_N$-a.a. $x \in A$. Hence

$$\mathcal{J}(tu,A) \leqslant \delta^a_L(t)\left(\int_A a(x)dx + \mathcal{J}(u,A)\right) + \mathcal{J}(u,A)$$

for all $A \in \mathcal{O}(\Omega)$ and all $u \in \mathrm{dom}(\mathcal{J}(\cdot,A))$, and consequently

$$\sup_{A\in\mathcal{O}(\Omega)} \sup_{u\in\mathrm{dom}(\mathcal{J}(\cdot,A))} \frac{\mathcal{J}(tu,A) - \mathcal{J}(u,A)}{\int_A a(x)dx + \mathcal{J}(u,A)} \leqslant \delta^a_L(t), \text{ i.e. } \Delta^a_{\mathcal{J}}(t) \leqslant \delta^a_L(t),$$

for all $t \in [0,1]$, and the proposition follows because L is ru-usc. ■

2.5.3. *Family of ru-usc variational functionals.* The following definition generalizes Definition 2.14 to the case of a family of variational functionals.

Definition 2.16. For each $\varepsilon > 0$, let $\mathcal{J}_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$ be a variational integral. We say that the family $\{\mathcal{J}_\varepsilon\}_{\varepsilon>0}$ is ru-usc if there exist $\{a_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega;]0,\infty])$ and $a_0 \in L^1(\Omega;]0,\infty])$ such that:

$$\lim_{\varepsilon\to 0} \int_A a_\varepsilon(x)dx = \int_A a_0(x)dx \text{ for all } A \in \mathcal{O}(\Omega); \tag{2.6}$$

$$\varlimsup_{t\to 1^-} \sup_{\varepsilon>0} \Delta^{a_\varepsilon}_{\mathcal{J}_\varepsilon}(t) \leqslant 0. \tag{2.7}$$

The interest of Definition 2.16 comes from the following result.

Proposition 2.17. *For each $\varepsilon > 0$, let $\mathcal{J}_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$ be a variational integral. If $\{\mathcal{J}_\varepsilon\}_{\varepsilon>0}$ is ru-usc with $\{a_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega;]0,\infty])$ and $a_0 \in L^1(\Omega;]0,\infty])$ and if $\{\mathcal{J}_\varepsilon\}_{\varepsilon>0}$ $\Gamma(L^p)$-converges to $\mathcal{J}_0 : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$ then $\mathcal{J}_0$ is ru-usc with the function a_0.*

Proof of Proposition 2.17. Fix any $A \in \mathcal{O}(\Omega)$ and any $u \in \mathrm{dom}(\mathcal{J}_0(\cdot,A))$. Since $\{\mathcal{J}_\varepsilon\}_{\varepsilon>0}$ $\Gamma(L^p)$-converges to $\mathcal{J}_0$, there exists $\{u_\varepsilon\}_{\varepsilon>0} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ with $u_\varepsilon \in \mathrm{dom}(\mathcal{J}_\varepsilon(\cdot,A))$ such that:

$$u_\varepsilon \xrightarrow{L^p} u; \tag{2.8}$$

$$\lim_{\varepsilon\to 0} \mathcal{J}_\varepsilon(u_\varepsilon,A) = \mathcal{J}_0(u,A). \tag{2.9}$$

Fix any $t \in [0,1]$. For every $\varepsilon > 0$, we have

$$\begin{aligned}\mathcal{J}_\varepsilon(tu_\varepsilon,A) &\leqslant \Delta^{a_\varepsilon}_{\mathcal{J}_\varepsilon}(t)\left(\int_A a_\varepsilon(x)dx + \mathcal{J}_\varepsilon(u_\varepsilon,A)\right) + \mathcal{J}_\varepsilon(u_\varepsilon,A)\\ &\leqslant \sup_{\varepsilon'>0} \Delta^{a_{\varepsilon'}}_{\mathcal{J}_{\varepsilon'}}(t)\left(\int_A a_\varepsilon(x)dx + \mathcal{J}_\varepsilon(u_\varepsilon,A)\right) + \mathcal{J}_\varepsilon(u_\varepsilon,A). \end{aligned}\tag{2.10}$$

From (2.8) we see that $tu_\varepsilon \xrightarrow{L^p} tu$, and so, since $\{\mathcal{J}_\varepsilon\}_{\varepsilon>0}$ $\Gamma(L^p)$-converges to $\mathcal{J}_0$,

$$\mathcal{J}_0(tu,A) \leqslant \varliminf_{\varepsilon\to 0} \mathcal{J}_\varepsilon(tu_\varepsilon,A). \tag{2.11}$$

As $\{\mathcal{J}_\varepsilon\}_{\varepsilon>0}$ is ru-usc, (2.6) and (2.7) hold. Letting $\varepsilon \to 0$ in (2.10) and using (2.6), (2.9) and (2.11) we deduce that

$$\mathcal{J}_0(tu,A) \leqslant \sup_{\varepsilon'>0} \Delta^{a_{\varepsilon'}}_{\mathcal{J}_{\varepsilon'}}(t)\left(\int_A a_0(x)dx + \mathcal{J}_0(u,A)\right) + \mathcal{J}_0(u,A).$$

Hence, for every $A \in \mathcal{O}(\Omega)$ and every $u \in \mathrm{dom}(\mathcal{J}_0(\cdot,A))$,

$$\frac{\mathcal{J}_0(tu,A) - \mathcal{J}_0(u,A)}{\int_A a(x)dx + \mathcal{J}_0(u,A)} \leqslant \sup_{\varepsilon>0} \Delta^{a_\varepsilon}_{\mathcal{J}_\varepsilon}(t).$$

Consequently

$$\sup_{A\in\mathcal{O}(\Omega)} \sup_{u\in\mathrm{dom}(\mathcal{J}_0(\cdot,A))} \frac{\mathcal{J}_0(tu,A) - \mathcal{J}_0(u,A)}{\int_A a_0(x)dx + \mathcal{J}_0(u,A)} \leqslant \sup_{\varepsilon>0} \Delta^{a_\varepsilon}_{\mathcal{J}_\varepsilon}(t), \text{ i.e. } \Delta^{a_0}_{\mathcal{J}_0}(t) \leqslant \sup_{\varepsilon>0} \Delta^{a_\varepsilon}_{\mathcal{J}_\varepsilon}(t)$$

for all $t \in [0,1]$, and the proposition follows by using (2.7). ■

Remark 2.18. From the proof of Proposition 2.17 we see that we are in fact proved that *if $\{\mathcal{J}_\varepsilon\}_{\varepsilon>0}$ is ru-usc with $\{a_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega;]0,\infty])$ and $a_0 \in L^1(\Omega;]0,\infty])$ then both $\Gamma(L^p)\text{-}\varliminf_{\varepsilon\to 0}\mathcal{J}_\varepsilon$ and $\Gamma(L^p)\text{-}\varliminf_{\varepsilon\to 0}\mathcal{J}_\varepsilon$ are ru-usc with the function a_0.*

The following result is a direct consequence of Proposition 2.17.

Corollary 2.19. *Let $\mathcal{J} : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$. If $\mathcal{J}$ is ru-usc with $a \in L^1(\Omega;]0,\infty])$ then its L^p-lower semicontinuous envelope, i.e. $\overline{\mathcal{J}} : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$ defined by*

$$\overline{\mathcal{J}}(u,A) := \Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} \mathcal{J}(u,A) = \inf\left\{\varliminf_{\varepsilon\to 0} \mathcal{J}(u_\varepsilon,A) : u_\varepsilon \xrightarrow{L^p} u\right\},$$

is ru-usc with the same function a.

2.5.4. *Family of ru-usc functions.* Analogously to the case of a family of variational functionals, the concept of ru-usc for a family of functions is defined as follows.

Definition 2.20. For each $\varepsilon > 0$, let $L_\varepsilon : \Omega \times \mathbb{M} \to [0, \infty]$ be a Borel measurable function. We say that the family $\{L_\varepsilon\}_{\varepsilon>0}$ is ru-usc if there exist $\{a_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega;]0, \infty])$ and $a_0 \in L^1(\Omega;]0, \infty])$ such that:

$$\lim_{\varepsilon\to 0} \int_A a_\varepsilon(x)dx = \int_A a_0(x)dx \text{ for all } A \in \mathcal{O}(\Omega);$$
$$\overline{\lim_{t\to 1^-}} \sup_{\varepsilon>0} \delta^{a_\varepsilon}_{L_\varepsilon}(t) \leqslant 0. \tag{2.12}$$

The following lemma will be useful for dealing with $\Gamma(L^p)$-convergence (see §4.1).

Lemma 2.21. *For each $\varepsilon > 0$, let $L_\varepsilon : \Omega \times \mathbb{M} \to [0, \infty]$ be a Borel measurable function and, for each $\rho > 0$, let $\mathscr{H}^\rho L_\varepsilon : \Omega \times \mathbb{M} \to [0, \infty]$ be defined by*

$$\mathscr{H}^\rho L_\varepsilon(x, \xi) := \inf \left\{ \fint_{Q_\rho(x)} L_\varepsilon(y, \nabla u(y))dy : u - l_\xi \in W^{1,p}_0(Q_\rho(x); \mathbb{R}^m) \right\}.$$

If $\{L_\varepsilon\}_{\varepsilon>0}$ is ru-usc with $\{a_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega;]0, \infty])$ and $a_0 \in L^\infty(\Omega;]0, \infty])$ then

$$L_0 := \overline{\lim_{\rho\to 0}}\, \overline{\lim_{\varepsilon\to 0}}\, \mathscr{H}^\rho L_\varepsilon : \Omega \times \mathbb{M} \to [0, \infty]$$

is ru-usc with the constant function $\|a_0\|_{L^\infty}$.

Proof of Lemma 2.21. Fix any $t \in [0, 1]$, any $x \in \Omega$ and any $\xi \in \mathbb{L}_{0,x}$ where $\mathbb{L}_{0,x}$ is the effective domain of $L_0(x, \cdot)$. Then $L_0(x, \xi) := \overline{\lim}_{\rho\to 0} \overline{\lim}_{\varepsilon\to 0} \mathscr{H}^\rho L_\varepsilon(x, \xi) < \infty$ and without loss of generality we can suppose that $\mathscr{H}^\rho L_\varepsilon(x, \xi) < \infty$ for all $\rho > 0$ and all $\varepsilon > 0$. Fix any $\rho > 0$ and any $\varepsilon > 0$. By definition of $\mathscr{H}^\rho L_\varepsilon(x, \xi)$ there exists $\{u_n\}_n \subset W^{1,p}(\Omega; \mathbb{R}^m)$ with $u_n - l_\xi \in W^{1,p}_0(Q_\rho(x); \mathbb{R}^m)$ such that:

$$\mathscr{H}^\rho L_\varepsilon(x, \xi) = \lim_{n\to\infty} \fint_{Q_\rho(x)} L_\varepsilon(y, \nabla u_n(y))dy; \tag{2.13}$$
$$\nabla u_n(y) \in \mathbb{L}_{\varepsilon,y} \text{ for all } n \geqslant 1 \text{ and } \mathscr{L}_N\text{-a.a. } y \in Q_\rho(x), \tag{2.14}$$

where $\mathbb{L}_{\varepsilon,y}$ denotes the effective domain of $L_\varepsilon(y, \cdot)$. Moreover, for every $n \geqslant 1$,

$$\mathscr{H}^\rho L_\varepsilon(x, t\xi) \leqslant \fint_{Q_\rho(x)} L_\varepsilon(y, t\nabla u_n(y))dy \tag{2.15}$$

since $tu_n - tl_\xi = tu_n - l_{t\xi} \in W^{1,p}_0(Q_\rho(x); \mathbb{R}^m)$. Taking (2.14) into account, we see that for every $n \geqslant 1$ and $\mathscr{L}_N$-a.e. $y \in Q_\rho(x)$,

$$L_\varepsilon(y, t\nabla u_n(y)) \leqslant \delta^{a_\varepsilon}_{L_\varepsilon}(t)\big(a_\varepsilon(x) + L_\varepsilon(y, \nabla u_n(y))\big) + L_\varepsilon(y, \nabla u_n(y)).$$

Hence

$$\begin{aligned}\fint_{Q_\rho(x)} L_\varepsilon(y, t\nabla u_n(y))dy \;&\leqslant\; \delta^{a_\varepsilon}_{L_\varepsilon}(t) \left(\fint_{Q_\rho(x)} a_\varepsilon(y)dy + \fint_{Q_\rho(x)} L_\varepsilon(y, \nabla u_n(y))dy \right) \\ &\quad + \fint_{Q_\rho(x)} L_\varepsilon(y, \nabla u_n(y))dy,\end{aligned}$$

and so, by using (2.15),

$$\mathscr{H}^\rho L_\varepsilon(x,t\xi) \leqslant \delta^{a_\varepsilon}_{L_\varepsilon}(t)\left(\fint_{Q_\rho(x)} a_\varepsilon(y)dy + \fint_{Q_\rho(x)} L_\varepsilon(y,\nabla u_n(y))dy\right) + \fint_{Q_\rho(x)} L_\varepsilon(y,\nabla u_n(y))dy$$

for all $n \geqslant 1$. Letting $n \to \infty$ and using (2.13), it follows that

$$\begin{aligned}\mathscr{H}^\rho L_\varepsilon(x,t\xi) &\leqslant \delta^{a_\varepsilon}_{L_\varepsilon}(t)\left(\fint_{Q_\rho(x)} a_\varepsilon(y)dy + \mathscr{H}^\rho L_\varepsilon(x,\xi)\right) + \mathscr{H}^\rho L_\varepsilon(x,\xi)\\ &\leqslant \sup_{\varepsilon'>0}\delta^{a_{\varepsilon'}}_{L_{\varepsilon'}}(t)\left(\fint_{Q_\rho(x)} a_\varepsilon(y)dy + \mathscr{H}^\rho L_\varepsilon(x,\xi)\right) + \mathscr{H}^\rho L_\varepsilon(x,\xi), \qquad (2.16)\end{aligned}$$

for all $\varepsilon > 0$. Letting $\varepsilon \to 0$ and noticing that $a_0 \in L^\infty(\Omega;]0,\infty])$, we get

$$\begin{aligned}\overline{\lim_{\varepsilon\to0}}\mathscr{H}^\rho L_\varepsilon(x,t\xi) &\leqslant \sup_{\varepsilon'>0}\delta^{a_{\varepsilon'}}_{L_{\varepsilon'}}(t)\left(\fint_{Q_\rho(x)} a_0(y)dy + \overline{\lim_{\varepsilon\to0}}\mathscr{H}^\rho L_\varepsilon(x,\xi)\right) + \overline{\lim_{\varepsilon\to0}}\mathscr{H}^\rho L_\varepsilon(x,\xi)\\ &\leqslant \sup_{\varepsilon'>0}\delta^{a_{\varepsilon'}}_{L_{\varepsilon'}}(t)\left(\|a_0\|_{L^\infty} + \overline{\lim_{\varepsilon\to0}}\mathscr{H}^\rho L_\varepsilon(x,\xi)\right) + \overline{\lim_{\varepsilon\to0}}\mathscr{H}^\rho L_\varepsilon(x,\xi)\end{aligned}$$

for all $\rho > 0$. Hence, by letting $\rho \to 0$,

$$L_0(x,t\xi) \leqslant \sup_{\varepsilon>0}\delta^{a_\varepsilon}_{L_\varepsilon}(t)\big(\|a_0\|_{L^\infty} + L_0(x,\xi)\big) + L_0(x,\xi), \text{ i.e. } \frac{L_0(x,t\xi)-L_0(x,\xi)}{\|a_0\|_{L^\infty}+L_0(x,\xi)} \leqslant \sup_{\varepsilon>0}\delta^{a_\varepsilon}_{L_\varepsilon}(t)$$

for all $x \in \Omega$ and all $\xi \in \mathbb{L}_{0,x}$. Consequently

$$\sup_{x\in\Omega}\sup_{\xi\in\mathbb{L}_{0,x}}\frac{L_0(x,t\xi)-L_0(x,\xi)}{\|a_0\|_{L^\infty}+L_0(x,\xi)} \leqslant \sup_{\varepsilon>0}\delta^{a_\varepsilon}_{L_\varepsilon}(t), \text{ i.e. } \delta^{\|a_0\|_{L^\infty}}_{L_0}(t) \leqslant \sup_{\varepsilon>0}\delta_{L_\varepsilon}(t)$$

for all $t \in [0,1]$, and the lemma follows by using (2.12). ■

As a direct consequence of Lemma 2.21, we have the following result.

Corollary 2.22. *Let $L : \Omega \times \mathbb{M} \to [0,\infty]$ be a Borel measurable function and let $\mathscr{H}^\rho L : \Omega\times\mathbb{M} \to [0,\infty]$ be defined by*

$$\mathscr{H}^\rho L(x,\xi) := \inf\left\{\fint_{Q_\rho(x)} L(y,\nabla u(y))dy : u - l_\xi \in W^{1,p}_0(Q_\rho(x);\mathbb{R}^m)\right\}.$$

If L is ru-usc with $a \in L^\infty(\Omega;]0,\infty])$ then

$$\overline{\lim_{\rho\to0}}\mathscr{H}^\rho L : \Omega\times\mathbb{M} \to [0,\infty]$$

is ru-usc with the constant function $\|a\|_{L^\infty}$.

The following proposition makes clear the link between Definition 2.16 and Definition 2.20.

Proposition 2.23. *For each $\varepsilon > 0$, let $L_\varepsilon : \Omega\times\mathbb{M} \to [0,\infty]$ be a Borel measurable function and let $\mathscr{I}_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m)\times\mathscr{O}(\Omega) \to [0,\infty]$ be defined by*

$$\mathscr{I}_\varepsilon(u,A) := \int_A L_\varepsilon(x,\nabla u(x))dx.$$

If $\{L_\varepsilon\}_{\varepsilon>0}$ is ru-usc with $\{a_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega;]0,\infty])$ and $a_0 \in L^1(\Omega;]0,\infty])$ then $\{\mathscr{I}_\varepsilon\}_{\varepsilon>0}$ is ru-usc with the same family of functions $\{a_\varepsilon\}_{\varepsilon>0}$ and the same function a_0.

Proof of Proposition 2.23. It suffices to prove (2.7). Fix any $t \in [0,1]$, any $\varepsilon > 0$, any $A \in \mathcal{O}(\Omega)$ and any $u \in \mathrm{dom}(\mathscr{I}_\varepsilon(\cdot, A))$. Then $\nabla u(x) \in \mathbb{L}_{\varepsilon,x}$ for $\mathscr{L}_N$-a.a. $x \in A$, where $\mathbb{L}_{\varepsilon,x}$ denotes the effective domain of $L_\varepsilon(x,\cdot)$, and so

$$L_\varepsilon(x, \nabla(tu)(x)) = L_\varepsilon(x, t\nabla u(x)) \leqslant \delta^{a_\varepsilon}_{L_\varepsilon}(t)(a_\varepsilon(x) + L_\varepsilon(x, \nabla u(x)) + L_\varepsilon(x, \nabla u(x))$$

for $\mathscr{L}_N$-a.a. $x \in A$. Hence

$$\mathscr{I}_\varepsilon(tu, A) \leqslant \delta^{a_\varepsilon}_{L_\varepsilon}(t)\left(\int_A a_\varepsilon(x)dx + \mathscr{I}_\varepsilon(u, A)\right) + \mathscr{I}_\varepsilon(u, A)$$

for all $A \in \mathcal{O}(\Omega)$ and all $u \in \mathrm{dom}(\mathscr{I}_\varepsilon(\cdot, A))$, and consequently

$$\sup_{A\in\mathcal{O}(\Omega)} \sup_{u\in\mathrm{dom}(\mathscr{I}_\varepsilon(\cdot,A))} \frac{\mathscr{I}_\varepsilon(tu, A) - \mathscr{I}_\varepsilon(u, A)}{\int_A a_\varepsilon(x)dx + \mathscr{I}_\varepsilon(u, A)} \leqslant \delta^{a_\varepsilon}_{L_\varepsilon}(t), \text{ i.e. } \Delta^{a_\varepsilon}_{L_\varepsilon}(t) \leqslant \delta^{a_\varepsilon}_{L_\varepsilon}(t),$$

for all $\varepsilon > 0$. Thus

$$\sup_{\varepsilon>0} \Delta^{a_\varepsilon}_{L_\varepsilon}(t) \leqslant \sup_{\varepsilon>0} \delta^{a_\varepsilon}_{L_\varepsilon}(t)$$

for all $t \in [0,1]$, and (2.7) follows from (2.12). ■

The following lemma, which motivates Definition 2.20 with respect to Definition 2.12, will be useful for homogenization (see §4.3).

Lemma 2.24. *Let $L : \mathbb{R}^N \times \mathbb{M} \to [0,\infty]$ be Borel measurable function such that $L(\cdot,\xi)$ is 1-periodic for all $\xi \in \mathbb{M}$, i.e. for every $(x,z) \in \mathbb{R}^N \times \mathbb{Z}^N$, $L(x+z,\xi) = L(x,\xi)$, and, for each $\varepsilon > 0$, let $L_\varepsilon : \Omega \times \mathbb{M} \to [0,\infty]$ be defined by*

$$L_\varepsilon(x,\xi) := L\left(\frac{x}{\varepsilon}, \xi\right).$$

Let $a \in L^1_{\mathrm{loc}}(\mathbb{R}^N;]0,\infty])$ be a 1-periodic function and, for each $\varepsilon > 0$, let $a_\varepsilon \in L^1_{\mathrm{loc}}(\mathbb{R}^N;]0,\infty])$ be defined by

$$a_\varepsilon(x) := a\left(\frac{x}{\varepsilon}\right).$$

If L is ru-usc with the function a then $\{L_\varepsilon\}_{\varepsilon>0}$ is ru-usc with the family of functions $\{a_\varepsilon\}_{\varepsilon>0}$ and the constant function $\langle a\rangle := \int_Y a(y)dy$.

Proof of Lemma 2.24. First of all, it is clear that $\lim_{\varepsilon\to 0}\int_A a_\varepsilon(x)dx = |A|\langle a\rangle$ for all $A \in \mathcal{O}(\Omega)$. So, it suffices to prove (2.12). For any $t \in [0,1]$, any $\varepsilon > 0$, any $x \in \Omega$ and any $\xi \in \mathbb{L}_{\varepsilon,x}$, we have

$$\frac{L_\varepsilon(x,t\xi) - L_\varepsilon(x,\xi)}{a_\varepsilon(x) + L_\varepsilon(x,\xi)} = \frac{L\left(\frac{x}{\varepsilon}, t\xi\right) - L\left(\frac{x}{\varepsilon}, \xi\right)}{a\left(\frac{x}{\varepsilon}\right) + L\left(\frac{x}{\varepsilon}, \xi\right)}. \tag{2.17}$$

As $\mathbb{L}_{\varepsilon,x} = \mathbb{L}_{\frac{x}{\varepsilon}}$ we see that

$$\frac{L\left(\frac{x}{\varepsilon}, t\xi\right) - L\left(\frac{x}{\varepsilon}, \xi\right)}{a\left(\frac{x}{\varepsilon}\right) + L\left(\frac{x}{\varepsilon}, \xi\right)} \leqslant \sup_{y\in\mathbb{R}^N}\sup_{\zeta\in\mathbb{L}_y} \frac{L(y,t\zeta) - L(y,\zeta)}{a(y) + L(y,\zeta)} = \delta^a_L(t),$$

and from (2.17) we deduce that

$$\sup_{\varepsilon>0} \delta^{a_\varepsilon}_{L_\varepsilon}(t) \leqslant \delta^a_L(t) \tag{2.18}$$

for all $t \in [0,1]$, and (2.12) follows from (2.4) because L is ru-usc. ■

3. Proof of the integral representation theorem

The proof of Theorem 1.2 is divided into five steps.

Step 1: a simple integral representation for $\mathscr{F}$. We begin by proving the following elementary lemma.

Lemma 3.1. *If* (I_0)–(I_1) *hold then for every* $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, $\mathscr{F}(u,\cdot)$ *is the restriction to* $\mathcal{O}(\Omega)$ *of a Borel measure which absolutely continuous with respect to* $\mathscr{L}_N$. *More precisely, for every* $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ *and every* $A \in \mathcal{O}(\Omega)$,

$$\mathscr{F}(u,A) = \int_A \lambda_u(x)dx$$

with $\lambda_u : \Omega \to [0,\infty]$ *Borel measurable given by*

$$\lambda_u(x) = \lim_{\rho\to 0} \frac{\mathscr{F}(u, Q_\rho(x))}{\rho^N}.$$

Proof of Lemma 3.1. By (I_0) and (I_1) we see that for every $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, the set function $\mathscr{F}(u,\cdot)$ is the restriction to $\mathcal{O}(\Omega)$ of a Borel measure which is absolutely continuous with respect to $\mathscr{L}_N$, and the lemma follows by using Radon-Nikodym's theorem and then Lebesgue's differentiation theorem. ■

From now on, we fix $A \in \mathcal{O}(\Omega)$.

Step 2: using the Vitali envelope. For every $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ we consider the set function $\mathrm{m}_u : \mathcal{O}(A) \to [0,\infty]$ defined by

$$\mathrm{m}_u(U) := \inf\left\{\mathscr{F}(v,U) : v-u \in W^{1,p}_0(U;\mathbb{R}^m)\right\}. \tag{3.1}$$

For every $\varepsilon > 0$ and every $U \in \mathcal{O}(A)$, we denote the class of countable families $\{Q_i := Q_{\rho_i}(x_i)\}_{i\in I}$ of disjoint open cubes of U with $x_i \in U$, $\rho_i > 0$ and $\mathrm{diam}(Q_i) \in]0,\varepsilon[$ such that $|U\setminus \cup_{i\in I} Q_i| = 0$ by $\mathscr{V}_\varepsilon(U)$, we consider $\mathrm{m}^\varepsilon_u : \mathcal{O}(A) \to [0,\infty]$ given by

$$\mathrm{m}^\varepsilon_u(U) := \inf\left\{\sum_{i\in I} \mathrm{m}_u(Q_i) : \{Q_i\}_{i\in I} \in \mathscr{V}_\varepsilon(U)\right\},$$

and we define $\mathrm{m}^*_u : \mathcal{O}(A) \to [0,\infty]$ by

$$\mathrm{m}^*_u(U) := \sup_{\varepsilon>0} \mathrm{m}^\varepsilon_u(U) = \lim_{\varepsilon\to 0} \mathrm{m}^\varepsilon_u(U),$$

i.e. m^*_u is the Vitali envelope of m_u (see §2.1).

Step 2 consists of proving the following lemma.

Lemma 3.2. *If* (I_0)–(I_2) *and* (I_4)–(I_5) *hold then for every* $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ *and every* $U \in \mathcal{O}(A)$,

$$\mathscr{F}(u,U) = \mathrm{m}^*_u(U). \tag{3.2}$$

Proof of Lemma 3.2. Fix $u \in W^{1,p}(\Omega;\mathbb{R}^m)$. Given any $Q \in \mathcal{O}(A)$, it is easy to see that $\mathrm{m}_u(Q) \leqslant \mathscr{F}(u,Q)$. Hence, for every $U \in \mathcal{O}(A)$,

$$\mathrm{m}_u^*(U) \leqslant \mathscr{F}(u,U)$$

because, by (I_1), $\mathscr{F}(\cdot,U)$ is the restriction to $\mathcal{O}(A)$ of a Borel measure. So, to establish (3.2), it remains to prove that for every $U \in \mathcal{O}(A)$,

$$\mathscr{F}(u,U) \leqslant \mathrm{m}_u^*(U). \tag{3.3}$$

Fix $U \in \mathcal{O}(A)$ with $\mathrm{m}_u^*(U) < \infty$. Fix any $\varepsilon > 0$. By definition of $\mathrm{m}_u^\varepsilon(U)$ there exists $\{Q_i\}_{i\in I} \in \mathscr{V}_\varepsilon(U)$ such that

$$\sum_{i\in I} \mathrm{m}_u(Q_i) \leqslant \mathrm{m}_u^\varepsilon(U) + \frac{\varepsilon}{2}. \tag{3.4}$$

For each $i \in I$, by definition of $\mathrm{m}_u(Q_i)$ there exists $v_i \in W^{1,p}(Q_i;\mathbb{R}^m)$ such that $v_i - u \in W_0^{1,p}(Q_i;\mathbb{R}^m)$ and

$$\mathscr{F}(v_i,Q_i) \leqslant \mathrm{m}_u(Q_i) + \frac{\varepsilon|Q_i|}{2|U|}. \tag{3.5}$$

Define $u_\varepsilon : \Omega \to \mathbb{R}^m$ by

$$u_\varepsilon := \begin{cases} u & \text{in } \Omega\backslash U \\ v_i & \text{in } Q_i. \end{cases}$$

Then $u_\varepsilon - u \in W_0^{1,p}(U;\mathbb{R}^m)$. Taking ($\mathrm{I}_2$) into account, from (3.5) we see that

$$\mathscr{F}(u_\varepsilon,U) \leqslant \sum_{i\in I} \mathrm{m}_u(Q_i) + \frac{\varepsilon}{2},$$

hence $\mathscr{F}(u_\varepsilon,U) \leqslant \mathrm{m}_u^\varepsilon(U) + \varepsilon$ by using (3.4), and consequently

$$\overline{\lim_{\varepsilon\to 0}}\, \mathscr{F}(u_\varepsilon,U) \leqslant \mathrm{m}_u^*(U). \tag{3.6}$$

On the other hand, we have

$$\|u_\varepsilon - u\|_{L^p}^p = \int_U |u_t^\varepsilon - tu|^p dx = \sum_{i\in I}\int_{Q_i} |v_i - u|^p dx.$$

As $\mathrm{diam}(Q_i) \in]0,\varepsilon[$ for all $i \in I$, by using Poincaré inequality we see that

$$\sum_{i\in I}\int_{Q_i} |v_i - u|^p dx \leqslant \varepsilon^p C \sum_{i\in I}\int_{Q_i} |\nabla v_i - \nabla u|^p dx,$$

where $C > 0$ is independent of ε, t and i. Hence

$$\sum_{i\in I}\int_{Q_i} |v_i - u|^p dx \leqslant 2^p\varepsilon^p C\left(\sum_{i\in I}\int_{Q_i} |\nabla v_i|^p dx + \int_U |\nabla u|^p dx\right),$$

and consequently

$$\|u_\varepsilon - u\|_{L^p}^p \quad \leqslant \quad 2^p\varepsilon^p C\left(\sum_{i\in I}\int_{Q_i} |\nabla v_i|^p dx + \int_U |\nabla u|^p dx\right). \tag{3.7}$$

Taking (I_5), (I_0), (3.5) and (3.4) into account, from (3.7) we deduce that

$$\|u_\varepsilon - u\|_{L^p}^p \leqslant 2^p\varepsilon^p C\left(\frac{1}{\alpha c}(\mathrm{m}_u^\varepsilon(U)+\varepsilon) + \int_A |\nabla u|^p dx\right),$$

which gives

$$\lim_{\varepsilon\to 0}\|u_\varepsilon - u\|_{L^p}^p = 0 \tag{3.8}$$

because $\lim_{\varepsilon\to 0}\mathrm{m}_u^\varepsilon(U) = \mathrm{m}_u^*(U) < \infty$, and (3.3) follows from (3.6) because $\mathscr{F}(u,U) \leqslant \underline{\lim}_{\varepsilon\to 0}\mathscr{F}(w_\varepsilon, U)$ by (I_4). ■

Step 3: differentiation with respect to $\mathscr{L}_N$. This step consists of applying Theorem 2.2 with $\mathcal{S} = \mathrm{m}_u$ where $u \in \mathrm{dom}(\mathscr{G}(\cdot, A))$. More precisely, Step 3 consists of proving the following lemma.

Lemma 3.3. *If* (I_0)–(I_2) *and* (I_4)–(I_6) *then for every* $u \in \mathrm{dom}(\mathscr{G}(\cdot, A))$ *and every* $U \in \mathcal{O}(A)$,

$$\mathrm{m}_u^*(U) = \int_U \lim_{\rho\to 0}\frac{\mathrm{m}_u(Q_\rho(x))}{\rho^N}dx. \tag{3.9}$$

As a direct consequence we have

$$\mathscr{F}(u,U) = \int_U \lim_{\rho\to 0}\frac{\mathrm{m}_u(Q_\rho(x))}{\rho^N}dx \tag{3.10}$$

for all $u \in \mathrm{dom}(\mathscr{G}(\cdot, A))$ *and all* $U \in \mathcal{O}(A)$.

Proof of Lemma 3.3. Fix $u \in \mathrm{dom}(\mathscr{G}(\cdot, A))$. The integral representation of $\mathscr{F}(u,\cdot)$ in (3.10) follows from (3.9) by using Lemma 3.2 and the definition of m_u in (3.1). So, we only need to establish (3.9). For this, it is sufficient to prove that m_u is subadditive and there exists a finite Borel measure ν on $\mathcal{O}(A)$ which is absolutely continuous with respect to $\mathscr{L}_N$ such that

$$\mathrm{m}_u(U) \leqslant \nu(U) \tag{3.11}$$

for all $U \in \mathcal{O}(A)$, and then to apply Theorem 2.2. From the definition of m_u, it is easy to see that for every $U, V, W \in \mathcal{O}(A)$ with $V, W \subset U$, $V \cap W = \varnothing$ and $|U\backslash V \cup W| = 0$,

$$\mathrm{m}_u(U) \leqslant \mathrm{m}_u(V) + \mathrm{m}_u(W),$$

which shows the subadditivity of m_u. On the other hand, by (I_0) we have

$$\begin{aligned}\mathrm{m}_u(U) &\leqslant \mathscr{F}(u,U)\\ &\leqslant \beta\left(|U| + \mathscr{G}(u,U)\right)\\ &= \beta|U| + \beta\int_U G(x,\nabla u(x))dx.\end{aligned}$$

Thus (3.11) is satisfied with the Borel measure $\nu := \beta(1 + G(\cdot, \nabla u(\cdot))\mathscr{L}_N$ which is finite since $u \in \mathrm{dom}(\mathscr{G}(\cdot, A))$. ■

Step 4: formula for the integrand. According to (3.10), the proof of Theorem 1.2 will be completed (see Substep 5-2 and also Step 6) if we prove that for every $u \in \mathrm{dom}(\mathscr{G}(\cdot, A))$ and $\mathscr{L}_N$-a.e. $x \in A$, we have:

$$\lim_{\rho\to 0}\frac{\mathrm{m}_u(Q_\rho(x))}{\rho^N}\geqslant \overline{\lim_{t\to 1^-}}\ \overline{\lim_{\rho\to 0}}\frac{\mathrm{m}_{tu_x}(Q_\rho(x))}{\rho^N};\tag{3.12}$$

$$\lim_{\rho\to 0}\frac{\mathrm{m}_u(Q_\rho(x))}{\rho^N}\leqslant \underline{\lim_{t\to 1^-}}\ \overline{\lim_{\rho\to 0}}\frac{\mathrm{m}_{tu_x}(Q_\rho(x))}{\rho^N},\tag{3.13}$$

where $u_x(y) := u(x) + \nabla u(x)(y - x)$.

Substep 4-1: proof of (3.12) and (3.13). We only give the proof of (3.12). As the proof of (3.13) uses the same method, the details are left to the reader. Fix any $\varepsilon > 0$. Since, by (I_9), $\overline{\lim}_{t\to 1^-}\Delta^a_{\mathscr{F}}(t)\leqslant 0$, there exists $t_0\in]0,1[$ such that

$$\Delta^a_{\mathscr{F}}(t)\leqslant\varepsilon\tag{3.14}$$

for all $t\in[t_0,1[$. Fix $u\in\mathrm{dom}(\mathscr{G}(\cdot,A))$. Fix any $\tau\in]0,1[$, any $\rho\in]0,\varepsilon[$ and any $t\in[t_0,1[$. By definition of $\mathrm{m}_u(Q_{\tau\rho}(x))$ there exists $w\in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $w-u\in W^{1,p}_0(Q_{\tau\rho}(x);\mathbb{R}^m)$ and

$$\mathscr{F}(w,Q_{\tau\rho}(x))\leqslant \mathrm{m}_u(Q_{\tau\rho}(x))+\varepsilon(\tau\rho)^N.\tag{3.15}$$

Let $\varphi\in C^\infty(\Omega)$ be a cut-off function for the pair $(\Omega\backslash Q_\rho(x),\overline{Q}_{\tau\rho}(x))$, i.e. $\varphi(x)\in[0,1]$ for all $x\in\Omega$, $\varphi(x)=0$ for all $x\in\Omega\backslash Q_\rho(x)$ and $\varphi(x)=1$ for all $\overline{Q}_{\tau\rho}(x)$, such that

$$\|\nabla\varphi\|_{L^\infty}\leqslant\frac{c}{\rho(1-\tau)}\tag{3.16}$$

for some $c>0$ (which does not depend on ρ and τ). Define $v\in W^{1,p}(Q_\rho(x);\mathbb{R}^m)$ by

$$v:=\varphi tu+(1-\varphi)tu_x.\tag{3.17}$$

Then $v-tu_x\in W^{1,p}_0(Q_\rho(x);\mathbb{R}^m)$ and we have

$$\nabla v=\begin{cases}\nabla(tu) & \text{in } \overline{Q}_{\tau\rho}(x)\\ (1-t)\frac{t}{1-t}\nabla\varphi\otimes(u-u_x)+t\big(\varphi\nabla u+(1-\varphi)\nabla u(x)\big) & \text{in } Q_\rho(x)\backslash\overline{Q}_{\tau\rho}(x).\end{cases}\tag{3.18}$$

As $tw-tu\in W^{1,p}_0(Q_{\tau\rho}(x);\mathbb{R}^m)$ we have $v+(tw-tu)-tu_x\in W^{1,p}_0(Q_\rho(x);\mathbb{R}^m)$. Taking Lemma 3.1 into account we see that

$$\begin{aligned}\frac{\mathrm{m}_{tu_x}(Q_\rho(x))}{(\tau\rho)^N}&\leqslant\frac{\mathscr{F}(v+tw-tu,Q_\rho(x))}{(\tau\rho)^N}\\&=\frac{1}{(\tau\rho)^N}\int_{Q_\rho(x)}\lambda_{v+tw-tu}(y)dy\\&=\frac{1}{(\tau\rho)^N}\left(\int_{\overline{Q}_{\tau\rho}(x)}\lambda_{v+tw-tu}(y)dy+\int_{Q_\rho(x)\backslash\overline{Q}_{\tau\rho}(x)}\lambda_{v+tw-tu}(y)dy\right)\\&=\frac{1}{(\tau\rho)^N}\int_{Q_{\tau\rho}(x)}\lambda_{v+tw-tu}(y)dy+\frac{\mathscr{F}(v+tw-tu,Q_\rho(x)\backslash\overline{Q}_{\tau\rho}(x))}{(\tau\rho)^N}\\&=\frac{\mathscr{F}(v+tw-tu,Q_{\tau\rho}(x))}{(\tau\rho)^N}+\frac{\mathscr{F}(v+tw-tu,Q_\rho(x)\backslash\overline{Q}_{\tau\rho}(x))}{(\tau\rho)^N}.\end{aligned}$$

But, by (3.17) and (I_2), we have $\mathscr{F}(v + tw - tu, Q_{\tau\rho}(x)) = \mathscr{F}(tw, Q_{\tau\rho}(x))$ and $\mathscr{F}(v + tw - tu, Q_\rho(x)\backslash\overline{Q}_{\tau\rho}(x)) = \mathscr{F}(v, Q_\rho(x)\backslash\overline{Q}_{\tau\rho}(x))$ because $tw - tu = 0$ in $Q_\rho(x)\backslash\overline{Q}_{\tau\rho}(x)$, and so, by using ($\mathrm{I}_1$),

$$\frac{\mathrm{m}_{tu_x}(Q_\rho(x))}{(\tau\rho)^N} \leqslant \frac{\mathscr{F}(tw, Q_{\tau\rho}(x))}{(\tau\rho)^N} + \frac{\mathscr{F}(v, Q_\rho(x)\backslash\overline{Q}_{\tau\rho}(x))}{(\tau\rho)^N}.$$

Consequently, since $\mathscr{F}(tw, Q_{\tau\rho}(x)) \leqslant (1+\Delta^a_{\mathscr{F}}(t))\mathscr{F}(w, Q_{\tau\rho}(x)) + \Delta^a_{\mathscr{F}}(t)\int_{Q_{\tau\rho}(x)} a(y)dy$, we get

$$\begin{aligned}\frac{\mathrm{m}_{tu_x}(Q_\rho(x))}{(\tau\rho)^N} &\leqslant (1+\Delta^a_{\mathscr{F}}(t))\frac{\mathscr{F}(w, Q_{\tau\rho}(x))}{(\tau\rho)^N} + \Delta^a_{\mathscr{F}}(t)\fint_{Q_{\tau\rho}(x)} a(y)dy\\ &\quad + \frac{\mathscr{F}(v, Q_\rho(x)\backslash\overline{Q}_{\tau\rho}(x))}{(\tau\rho)^N}.\end{aligned}$$

From (3.14), (3.15), (3.18), (I_0) and (I_6) we deduce that

$$\begin{aligned}\frac{\mathrm{m}_{tu_x}(Q_\rho(x))}{\rho^N} \leqslant \frac{\mathrm{m}_{tu_x}(Q_\rho(x))}{(\tau\rho)^N} &\leqslant (1+\varepsilon)\left(\frac{\mathrm{m}_u(Q_{\tau\rho}(x))}{(\tau\rho)^N} + \varepsilon\right) + \varepsilon\fint_{Q_{\tau\rho}(x)} a(y)dy\\ &\quad + \frac{C}{(\tau\rho)^N}\int_{Q_\rho(x)\backslash Q_{\tau\rho}(x)} G\left(y, \frac{t}{1-t}\nabla\varphi\otimes(u-u_x)\right)dy\\ &\quad + \frac{C}{(\tau\rho)^N}\int_{Q_\rho(x)\backslash Q_{\tau\rho}(x)} \big[G(y,\nabla u) + G(y,\nabla u(x))\big]dy\\ &\quad + C\left(\frac{1}{\tau^N} - 1\right)\end{aligned}$$

with $C := \beta + \beta\gamma + \beta\gamma^2$. On the other hand, by (3.16) we have

$$\begin{aligned}\left|\frac{t}{1-t}\nabla\varphi(y)\otimes(u(y)-u_x(y))\right| &\leqslant \left|\frac{t}{1-t}\right|\|\nabla\varphi\|_{L^\infty}\|u-u_x\|_{L^\infty(Q_\rho(x);\mathbb{R}^m)}\\ &\leqslant \frac{tc}{(1-t)(1-\tau)}\frac{1}{\rho}\|u-u_x\|_{L^\infty(Q_\rho(x);\mathbb{R}^m)}\end{aligned}$$

for $\mathscr{L}_N$-a.a. $y \in Q_\rho(x)\backslash Q_{\tau\rho}(x)$. But $\lim_{\rho\to 0}\frac{1}{\rho}\|u-u_x\|_{L^\infty(Q_\rho(x);\mathbb{R}^m)} = 0$ because $p > N$, hence there exists $\rho_0 > 0$ (which depends on t and τ) such that

$$\left|\frac{t}{1-t}\nabla\varphi(y)\otimes(u(y)-u_x(y))\right| \leqslant r$$

for $\mathscr{L}_N$-a.a. $y \in Q_\rho(x)\backslash Q_{\tau\rho}(x)$ and all $\rho \in]0,\rho_0[$ with $r > 0$ given by (I_7). Consequently

$$\int_{Q_\rho(x)\backslash Q_{\tau\rho}(x)} G\left(y, \frac{t}{1-t}\nabla\varphi\otimes(u-u_x)\right)dy \leqslant \int_{Q_\rho(x)\backslash Q_{\tau\rho}(x)} \sup_{|\xi|\leqslant r} G(y,\xi)dy \tag{3.19}$$

for all $\rho \in]0,\rho_0[$. Moreover, it easy to see that:

$$\begin{aligned}\int_{Q_\rho(x)\setminus Q_{\tau\rho}(x)}\sup_{|\xi|\leqslant r}G(y,\xi)dy &\leqslant \rho^N\fint_{Q_\rho(x)}\left|\sup_{|\xi|\leqslant r}G(y,\xi)-\sup_{|\xi|\leqslant r}G(x,\xi)\right|dy\\&\quad+\rho^N(1-\tau^N)\sup_{|\xi|\leqslant r}G(x,\xi);\end{aligned}\tag{3.20}$$

$$\begin{aligned}\int_{Q_\rho(x)\setminus Q_{\tau\rho}(x)}G(y,\nabla u(y))y &\leqslant \rho^N\fint_{Q_\rho(x)}\big|G(y,\nabla u(y))-G(x,\nabla u(x))\big|dy\\&\quad+\rho^N(1-\tau^N)G(x,\nabla u(x));\end{aligned}\tag{3.21}$$

$$\begin{aligned}\int_{Q_\rho(x)\setminus Q_{\tau\rho}(x)}G(y,\nabla u(x))dy &\leqslant \rho^N\fint_{Q_\rho(x)}\big|G(y,\nabla u(x))-G(x,\nabla u(x))\big|dy\\&\quad+\rho^N(1-\tau^N)G(x,\nabla u(x)).\end{aligned}\tag{3.22}$$

Combining (3.19) with (3.20), (3.21) and (3.22) we deduce that

$$\begin{aligned}\frac{\mathrm{m}_{tu_x}(Q_\rho(x))}{\rho^N} &\leqslant (1+\varepsilon)\left(\frac{\mathrm{m}_u(Q_{\tau\rho}(x))}{(\tau\rho)^N}+\varepsilon\right)+\varepsilon\fint_{Q_{\tau\rho}(x)}a(y)dy\\&\quad+\frac{C}{\tau^N}\fint_{Q_\rho(x)}\left|\sup_{|\xi|\leqslant r}G(y,\xi)-\sup_{|\xi|\leqslant r}G(x,\xi)\right|dy\\&\quad+\frac{C}{\tau^N}\fint_{Q_\rho(x)}\big|G(y,\nabla u(y))-G(x,\nabla u(x))\big|dy\\&\quad+\frac{C}{\tau^N}\fint_{Q_\rho(x)}\big|G(y,\nabla u(x))-G(x,\nabla u(x))\big|dy\\&\quad+C\left(\frac{1}{\tau^N}-1\right)\left(\sup_{|\xi|\leqslant r}G(x,\xi)+2G(x,\nabla u(x))+1\right).\end{aligned}\tag{3.23}$$

As $\sup_{|\xi|\leqslant r}G(\cdot,\xi)\in L^1(\Omega)$ by (I_7), we have

$$\lim_{\rho\to0}\fint_{Q_\rho(x)}\left|\sup_{|\xi|\leqslant r}G(y,\xi)-\sup_{|\xi|\leqslant r}G(x,\xi)\right|dy=0.\tag{3.24}$$

In the same way, as $u\in\mathrm{dom}(\mathscr{G}(\cdot,A))$, i.e. $G(\cdot,\nabla u(\cdot))\in L^1(A)$, we can assert that

$$\lim_{\rho\to0}\fint_{Q_\rho(x)}\big|G(y,\nabla u(y))-G(x,\nabla u(x))\big|dy=0,\tag{3.25}$$

and by (I_8) we have

$$\lim_{\rho\to0}\fint_{Q_\rho(x)}\big|G(y,\nabla u(x))-G(x,\nabla u(x))\big|dy=0.\tag{3.26}$$

Moreover, as $a\in L^1(\Omega;]0,\infty])$ it is clear that

$$\lim_{\rho\to0}\fint_{Q_{\tau\rho}(x)}a(y)dy=a(x).\tag{3.27}$$

Letting $\rho \to 0$ in (3.23) and using (3.24), (3.25), (3.26) and (3.27) we see that

$$\begin{aligned}\overline{\lim_{\rho\to 0}} \frac{\mathrm{m}_{tu_x}(Q_\rho(x))}{\rho^N} &\leqslant (1+\varepsilon)\left(\lim_{\rho\to 0}\frac{\mathrm{m}_u(Q_\rho(x))}{\rho^N}+\varepsilon\right)+\varepsilon a(x)\\ &\quad +C\left(\frac{1}{\tau^N}-1\right)\left(\sup_{|\xi|\leqslant r} G(x,\xi)+2G(x,\nabla u(x))+1\right). \qquad (3.28)\end{aligned}$$

Letting $t \to 1^-$ and $\tau \to 1^-$ in (3.28) we deduce that

$$\overline{\lim_{t\to 1^-}}\,\overline{\lim_{\rho\to 0}} \frac{\mathrm{m}_{tu_x}(Q_\rho(x))}{\rho^N} \leqslant (1+\varepsilon)\left(\lim_{\rho\to 0}\frac{\mathrm{m}_u(Q_\rho(x))}{\rho^N}+\varepsilon\right)+\varepsilon a(x)$$

and (3.12) follows by letting $\varepsilon \to 0$.

Substep 4-2: establishing the formula for the integrand. By (3.12) and (3.13) we have

$$\lim_{\rho\to 0}\frac{\mathrm{m}_u(Q_\rho(x))}{\rho^N} = \lim_{t\to 1^-}\overline{\lim_{\rho\to 0}}\frac{\mathrm{m}_{tu_x}(Q_\rho(x))}{\rho^N}. \qquad (3.29)$$

On the other hand, it is easily seen that

$$\begin{aligned}\lim_{t\to 1^-}\overline{\lim_{\rho\to 0}}\frac{\mathrm{m}_{tu_x}(Q_\rho(x))}{\rho^N} &= \lim_{t\to 1^-}\overline{\lim_{\rho\to 0}}\inf\left\{\frac{\mathscr{F}(v,Q_\rho(x))}{\rho^N} : v - tu_x \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)\right\}\\ &= \lim_{t\to 1^-}\overline{\lim_{\rho\to 0}}\inf\left\{\frac{\mathscr{F}(v+tu_x,Q_\rho(x))}{\rho^N} : v \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)\right\}.\end{aligned}$$

But by (I_3) we have

$$\mathscr{F}(v+tu_x,Q_\rho(x)) = \mathscr{F}(v+t\nabla u(x)+tu(x)-\nabla u(x)x,Q_\rho(x)) = \mathscr{F}(v+t\nabla u(x),Q_\rho(x)),$$

and so

$$\begin{aligned}\lim_{t\to 1^-}\overline{\lim_{\rho\to 0}}\frac{\mathrm{m}_{tu_x}(Q_\rho(x))}{\rho^N} &= \lim_{t\to 1^-}\overline{\lim_{\rho\to 0}}\inf\left\{\frac{\mathscr{F}(v+t\nabla u(x),Q_\rho(x))}{\rho^N} : v \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)\right\}\\ &= \lim_{t\to 1^-}\overline{\lim_{\rho\to 0}}\inf\left\{\frac{\mathscr{F}(v,Q_\rho(x))}{\rho^N} : v - l_{t\nabla u(x)} \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)\right\}\\ &= \lim_{t\to 1^-} F(x,t\nabla u(x)) \qquad (3.30)\end{aligned}$$

with $F : \Omega\times\mathbb{M}\to[0,\infty]$ defined by (1.1). Combining (3.10), (3.29) and (3.30) we conclude that

$$\mathscr{F}(u,U) = \int_U \lim_{t\to 1^-} F(x,t\nabla u(x))dx$$

for all $u \in \mathrm{dom}(\mathscr{G}(\cdot,A))$ and all $U \in \mathcal{O}(A)$.

Step 5: end of the proof. From Steps 2, 3 and 4, we have proved that for every $A \in \mathcal{O}(\Omega)$ and every $u \in \mathrm{dom}(\mathscr{G}(\cdot,A))$, $\mathscr{F}(u,A) = \int_A \lim_{t\to 1^-} F(x,t\nabla u(x))dx$. On the other hand, by (I_0) we see that for each $A \in \mathcal{O}(\Omega)$, if $u \in W^{1,p}(\Omega;\mathbb{R}^m)\backslash\mathrm{dom}(\mathscr{G}(\cdot,A))$ then $\mathscr{F}(u,A) = \infty$. ■

4. Applications

In what follows, $p > 1$ is a real number, $\Omega \subset \mathbb{R}^N$ is a bounded open set, $\mathcal{O}(\Omega)$ denotes the class of all open subset of Ω and $\mathbb{M}$ is the space of $m \times N$ matrices.

4.1. $\Gamma(L^p)$**-convergence.** Let $G : \Omega \times \mathbb{M} \to [0, \infty]$ be a Borel measurable function satisfying (I_5)–(I_8) and the following two additional assumptions:

- (C_0) for every $A \in \mathcal{O}(\Omega)$, $\mathcal{G}(\cdot, A)$ is L^p-lower semicontinuous with $\mathcal{G} : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0, \infty]$ defined by
$$\mathcal{G}(u, A) := \int_A G(x, \nabla u(x))dx;$$
- (C_1) $t\overline{\mathbb{G}_{0,x}} \subset \mathrm{int}(\mathbb{G}_{0,x})$ for all $x \in \Omega$ and all $t \in]0, 1[$ with $\mathbb{G}_{0,x}$ denoting the effective domain of $G_0(x, \cdot)$, where $G_0 : \Omega \times \mathbb{M} \to [0, \infty]$ is defined by
$$G_0(x, \xi) := \varlimsup_{\rho \to 0} \mathcal{H}^\rho G(x, \xi) \tag{4.1}$$
with $\mathcal{H}^\rho G : \Omega \times \mathbb{M} \to [0, \infty]$ given by
$$\mathcal{H}^\rho G(x, \xi) := \inf \left\{ \fint_{Q_\rho(x)} G(y, \nabla u(y))dy : u - l_\xi \in W_0^{1,p}(Q_\rho(x); \mathbb{R}^m) \right\},$$
where $Q_\rho(x) := x + \rho Y$ with $Y :=] - \frac{1}{2}, \frac{1}{2}[^N$ and $l_\xi : \mathbb{R}^N \to \mathbb{R}^m$ is the linear map defined by $l_\xi(y) := \xi y$.

Remark 4.1.

- (i) Under (I_5), if G is convex and lower semicontinuous with respect to ξ, then (C_0) can be dropped.
- (ii) Under (I_6) it is easy to see that for every $x \in \Omega$, $\mathbb{G}_{0,x}$ is convex, and so if $0 \in \mathrm{int}(\mathbb{G}_{0,x})$ then (C_1) holds.
- (iii) When G does not depend on x, i.e. $G(x, \xi) = G(\xi)$, G_0 is the $W^{1,p}$-quasiconvex envelope of G. Hence, $G_0 = G$ if moreover (C_0) holds since G is then $W^{1,p}$-quasiconvex.

For each $\varepsilon > 0$, let $L_\varepsilon : \Omega \times \mathbb{M} \to [0, \infty]$ be a Borel measurable function. We assume that the family $\{L_\varepsilon\}_{\varepsilon > 0}$ satisfies the following three conditions:

- (C_2) there exist $\alpha, \beta > 0$ such that for every $\varepsilon > 0$ and every $(x, \xi) \in \Omega \times \mathbb{M}$,
$$\alpha G(x, \xi) \leqslant L_\varepsilon(x, \xi) \leqslant \beta(1 + G(x, \xi));$$
- (C_3) $\{L_\varepsilon\}_{\varepsilon > 0}$ is ru-usc with $\{a_\varepsilon\}_{\varepsilon > 0} \subset L^1(\Omega;]0, \infty])$ and $a_0 \in L^\infty(\Omega;]0, \infty])$, i.e.
$$\varlimsup_{t \to 1^-} \sup_{\varepsilon > 0} \delta_{L_\varepsilon}^{a_\varepsilon}(t) \leqslant 0$$
with $\delta_{L_\varepsilon}^{a_\varepsilon} : [0, 1] \to] - \infty, \infty]$ defined by
$$\delta_{L_\varepsilon}^{a_\varepsilon}(t) := \sup_{x \in \Omega} \sup_{\xi \in \mathbb{L}_{\varepsilon,x}} \frac{L_\varepsilon(x, t\xi) - L_\varepsilon(x, \xi)}{a_\varepsilon(x) + L_\varepsilon(x, \xi)},$$
where $\mathbb{L}_{\varepsilon,x}$ denotes the effective domain of $L_\varepsilon(x, \cdot)$;
- (C_4) for every $x \in \Omega$ and every $\xi \in \mathbb{G}_{0,x}$,
$$\varlimsup_{\rho \to 0} \varliminf_{\varepsilon \to 0} \mathcal{H}^\rho L_\varepsilon(x, \xi) \geqslant \varlimsup_{\rho \to 0} \varlimsup_{\varepsilon \to 0} \mathcal{H}^\rho L_\varepsilon(x, \xi)$$

with $\mathscr{H}^\rho L_\varepsilon : \Omega \times \mathbb{M} \to [0, \infty]$ given by

$$\mathscr{H}^\rho L_\varepsilon(x, \xi) := \inf \left\{ \fint_{Q_\rho(x)} L_\varepsilon(y, \nabla u(y))dy : u - l_\xi \in W_0^{1,p}(Q_\rho(x); \mathbb{R}^m) \right\}. \tag{4.2}$$

Remark 4.2. When $\{L_\varepsilon\}_{\varepsilon>0} = L$, (C$_3$) means that L is ru-usc and (C$_4$) can be dropped.

For each $\varepsilon > 0$, let $\mathscr{I}_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0, \infty]$ be defined by

$$\mathscr{I}_\varepsilon(u, A) := \int_A L_\varepsilon(x, \nabla u(x))dx.$$

The following $\Gamma(L^p)$-convergence theorem is a consequence of Theorem 1.2.

Theorem 4.3. *Assume that $p > N$ and* (C$_0$)–(C$_4$) *and* (I$_5$)–(I$_8$) *hold. Then, for every $A \in \mathcal{O}(\Omega)$,*

$$\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} \mathscr{I}_\varepsilon(u, A) = \begin{cases} \displaystyle\int_A \lim_{t\to 1^-} \overline{\lim_{\rho\to 0}}\, \overline{\lim_{\varepsilon\to 0}}\, \mathscr{H}^\rho L_\varepsilon(x, t\nabla u(x))dx & \text{if } u \in \mathrm{dom}(\mathscr{G}(\cdot, A)) \\ \infty & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^m)\setminus\mathrm{dom}(\mathscr{G}(\cdot, A)). \end{cases}$$

Proof of Theorem 4.3. The proof is divided into three steps.

Step 1: proving that $\Gamma(L^p)$-$\underline{\lim}_{\varepsilon\to 0} \mathscr{I}_\varepsilon(u, \cdot)$ and $\Gamma(L^p)$-$\overline{\lim}_{\varepsilon\to 0} \mathscr{I}_\varepsilon(u, \cdot)$ are restrictions to $\mathcal{O}(\Omega)$ of Borel measures. For each $u \in W^{1,p}(\Omega; \mathbb{R}^m)$, let $\mathcal{S}_u^-, \mathcal{S}_u^+ : \mathcal{O}(\Omega) \to [0, \infty]$ be given by:

$$\mathcal{S}_u^-(A) := \Gamma(L^p)\text{-}\underline{\lim}_{\varepsilon\to 0} \mathscr{I}_\varepsilon(u, A);$$
$$\mathcal{S}_u^+(A) := \Gamma(L^p)\text{-}\overline{\lim}_{\varepsilon\to 0} \mathscr{I}_\varepsilon(u, A).$$

From (C$_0$) and (C$_2$) we see that:

$$\alpha \int_A G(x, \nabla u(x))dx \leqslant \mathcal{S}_u^-(A) \leqslant \beta \left(|A| + \int_A G(x, \nabla u(x))dx \right); \tag{4.3}$$

$$\alpha \int_A G(x, \nabla u(x))dx \leqslant \mathcal{S}_u^+(A) \leqslant \beta \left(|A| + \int_A G(x, \nabla u(x))dx \right) \tag{4.4}$$

for all $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and all $A \in \mathcal{O}(\Omega)$. Step 1 consists of proving the following lemma.

Lemma 4.4. *Assume that $p > N$ and* (C$_0$), (C$_2$)–(C$_3$) *and* (I$_5$)–(I$_6$) *hold. Then, for every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$, $\mathcal{S}_u^-$ and $\mathcal{S}_u^+$ are restrictions to $\mathcal{O}(\Omega)$ of Borel measures.*

Proof of Lemma 4.4. Fix $u \in W^{1,p}(\Omega; \mathbb{R}^m)$. The proof consists of applying Lemma 2.9 with $\mathcal{S} = \mathcal{S}_u^-$ (resp. $\mathcal{S} = \mathcal{S}_u^+$). By (4.3) and (4.4) we see that the condition (a$_6$) of Lemma 2.9 is satisfied with $\lambda = \mathscr{L}_N$ and $\nu = G(\cdot, \nabla u(\cdot))\mathscr{L}_N$. On the other hand, it is easily seen that the condition (a$_2$) of Lemma 2.9 is verified. Hence, the proof is completed if we prove the condition (a$_1$) of Lemma 2.9, i.e.

$$\mathcal{S}_u^-(A \cup B) \leqslant \mathcal{S}_u^-(A) + \mathcal{S}_u^-(B) \text{ for all } A, B \in \mathcal{O}(\Omega); \tag{4.5}$$

$$\mathcal{S}_u^+(A \cup B) \leqslant \mathcal{S}_u^+(A) + \mathcal{S}_u^+(B) \text{ for all } A, B \in \mathcal{O}(\Omega). \tag{4.6}$$

Substep 1-1: an auxiliary result for proving Lemma 4.4. To show (4.5) and (4.6) we need the following lemma.

Lemma 4.5. *Assume that $p > N$ and* (C_0), (C_2)–(C_3) *and* (I_5)–(I_6) *hold. If $U, V, Z, T \in \mathcal{O}(\Omega)$ are such that $\overline{Z} \subset U$ and $T \subset V$, then*:

$$\mathcal{S}_u^-(Z \cup T) \leqslant \mathcal{S}_u^-(U) + \mathcal{S}_u^-(V); \tag{4.7}$$

$$\mathcal{S}_u^+(Z \cup T) \leqslant \mathcal{S}_u^+(U) + \mathcal{S}_u^+(V). \tag{4.8}$$

Proof of Lemma 4.5. As the proofs of (4.7) and (4.8) are the same, we only give the proof of (4.7). Let $\{u_\varepsilon\}_{\varepsilon>0}$ and $\{v_\varepsilon\}_{\varepsilon>0}$ be two sequences in $W^{1,p}(\Omega;\mathbb{R}^m)$ such that:

$$u_\varepsilon \xrightarrow{L^p} u; \tag{4.9}$$

$$v_\varepsilon \xrightarrow{L^p} u; \tag{4.10}$$

$$\lim_{\varepsilon\to 0} \int_U L_\varepsilon(x, \nabla u_\varepsilon(x))dx = \mathcal{S}_u^-(U) < \infty; \tag{4.11}$$

$$\lim_{\varepsilon\to 0} \int_V L_\varepsilon(x, \nabla v_\varepsilon(x))dx = \mathcal{S}_u^-(V) < \infty. \tag{4.12}$$

By (C_2) and (I_5) we have $\sup_{\varepsilon>0} \|\nabla u_\varepsilon\|_{L^p(U)} < \infty$ and $\sup_{\varepsilon>0} \|\nabla v_\varepsilon\|_{L^p(V)} < \infty$. Taking (4.9) and (4.10) into account, as $p > N$, up to a subsequence, we have:

$$u_\varepsilon \xrightarrow{L^\infty(U)} u; \tag{4.13}$$

$$v_\varepsilon \xrightarrow{L^\infty(V)} u. \tag{4.14}$$

Fix $\delta \in]0, \mathrm{dist}(Z, \partial U)[$ with $\partial U := \overline{U}\backslash U$, fix any $q \geqslant 1$ and consider $W_i^-, W_i^+ \subset \Omega$ given by:

$$W_i^- := \left\{x \in \Omega : \mathrm{dist}(x, Z) \leqslant \tfrac{\delta}{3} + \tfrac{(i-1)\delta}{3q}\right\};$$
$$W_i^+ := \left\{x \in \Omega : \tfrac{\delta}{3} + \tfrac{i\delta}{3q} \leqslant \mathrm{dist}(x, Z)\right\},$$

where $i \in \{1, \cdots, q\}$. For every $i \in \{1, \cdots, q\}$ there exists a cut-off function $\varphi_i \in C^\infty(\Omega)$ for the pair (W_i^+, W_i^-), i.e. $\varphi(x) \in [0, 1]$ for all $x \in \Omega$, $\varphi(x) = 0$ for all $x \in W_i^+$ and $\varphi(x) = 1$ for all $x \in W_i^-$. Fix any $\varepsilon > 0$ and define $w_\varepsilon^i \in W^{1,p}(\Omega;\mathbb{R}^m)$ by

$$w_\varepsilon^i := \varphi_i u_\varepsilon + (1 - \varphi_i)v_\varepsilon. \tag{4.15}$$

Fix any $t \in]0, 1[$. Setting $W_i := \Omega\backslash(W_i^- \cup W_i^+)$ we have

$$\nabla(tw_\varepsilon^i) = t\nabla w_\varepsilon^i = \begin{cases} t\nabla u_\varepsilon & \text{in } W_i^- \\ (1-t)\frac{t}{1-t}\nabla\varphi_i \otimes (u_\varepsilon - v_\varepsilon) + t\big(\varphi_i \nabla u_\varepsilon + (1-\varphi_i)\nabla v_\varepsilon\big) & \text{in } W_i \\ t\nabla v_t & \text{in } W_i^+. \end{cases}$$

Noticing that $Z \cup T = ((Z \cup T) \cap W_i^-) \cup (W \cap W_i) \cup (T \cap W_i^+)$ with $(Z \cup T) \cap W_i^- \subset U$, $T \cap W_i^+ \subset V$ and $W := T \cap \{x \in U : \frac{\delta}{3} < \mathrm{dist}(x, Z) < \frac{2\delta}{3}\}$ we deduce that for every $i \in \{1, \cdots, q\}$,

$$\begin{aligned}\int_{Z\cup T} L_\varepsilon(x, t\nabla w_\varepsilon^i)dx \;\leqslant\; & \int_U L_\varepsilon(x, t\nabla u_\varepsilon)dx + \int_V L_\varepsilon(x, t\nabla v_\varepsilon)dx \\ & + \int_{W\cap W_i} L_\varepsilon(x, t\nabla w_\varepsilon^i)dx.\end{aligned} \tag{4.16}$$

Fix any $i \in \{1, \cdots, q\}$. From (C_2) and (I_6) we see that

$$\begin{aligned}\int_{W\cap W_i} L_\varepsilon(x, t\nabla w_\varepsilon^i)dx &\leqslant \beta|W\cap W_i| + \beta\int_{W\cap W_i} G(x, t\nabla w_\varepsilon^i)dx \\ &\leqslant \beta(1+\gamma)|W\cap W_i| \\ &\quad +\beta\gamma\int_{W\cap W_i} G(x, \varphi_i\nabla u_\varepsilon + (1-\varphi_i)\nabla v_\varepsilon)dx \\ &\quad +\beta\gamma\int_{W\cap W_i} G\left(x, \frac{t}{1-t}\nabla\varphi_i\otimes(u_\varepsilon - v_\varepsilon)\right)dx,\end{aligned}$$

and so, by using again (C_2) and (I_6),

$$\begin{aligned}\int_{W\cap W_i} L_\varepsilon(x, t\nabla w_\varepsilon^i)dx &\leqslant \beta(1+\gamma+\gamma^2)|W\cap W_i| \\ &\quad +\frac{\beta\gamma^2}{\alpha}\left(\int_{W\cap W_i} L_\varepsilon(x,\nabla u_\varepsilon)dx + \int_{W\cap W_i} L_\varepsilon(x,\nabla v_\varepsilon)dx\right) \\ &\quad +\beta\gamma\int_{W\cap W_i} G\left(x, \frac{t}{1-t}\nabla\varphi_i\otimes(u_\varepsilon - v_\varepsilon)\right)dx. \end{aligned}\tag{4.17}$$

On the other hand, we have

$$\left|\frac{t}{1-t}\nabla\varphi_i(x)\otimes(u_\varepsilon(x) - v_\varepsilon(x))\right| \leqslant \left|\frac{t}{1-t}\right| \|\nabla\varphi_i\|_{L^\infty}\|u_\varepsilon - v_\varepsilon\|_{L^\infty(U\cap V)}$$

for $\mathscr{L}_N$-a.a. $x \in W\cap W_i \subset U\cap V$. But $\lim_{\varepsilon\to 0}\|u_\varepsilon - v_\varepsilon\|_{L^\infty(U\cap V)} = 0$ by (4.13) and (4.14), hence for each $t \in]0,1[$ and each $i \in \{1,\cdots,q\}$ there exists $\varepsilon_{t,i} > 0$ such that

$$\left|\frac{t}{1-t}\nabla\varphi_i(x)\otimes(u_\varepsilon(x) - v_\varepsilon(x))\right| \leqslant r$$

for $\mathscr{L}_N$-a.a. $x \in W\cap W_i$ and all $\varepsilon \in]0, \varepsilon_{t,i}]$ with $r > 0$ given by (I_7). Hence

$$\int_{W\cap W_i} G\left(x, \frac{t}{1-t}\nabla\varphi_i\otimes(u_\varepsilon - v_\varepsilon)\right)dx \leqslant \int_{W\cap W_i}\sup_{|\xi|\leqslant r} G(x,\xi)dx \tag{4.18}$$

for all $\varepsilon \in]0, \overline{\varepsilon}_{t,q}]$ with $\overline{\varepsilon}_{t,q} := \min\{\varepsilon_{t,i} : i \in \{1,\cdots,q\}\}$. Moreover, we have:

$$\int_U L_\varepsilon(x, t\nabla u_\varepsilon)dx \leqslant \left(1 + \sup_{\varepsilon'>0}\delta_{L_{\varepsilon'}}^{a_{\varepsilon'}}(t)\right)\int_U L_\varepsilon(x,\nabla u_\varepsilon)dx + \sup_{\varepsilon'>0}\delta_{L_{\varepsilon'}}^{a_{\varepsilon'}}(t)\int_U a_\varepsilon(x)dx; \tag{4.19}$$

$$\int_V L_\varepsilon(x, t\nabla v_\varepsilon)dx \leqslant \left(1 + \sup_{\varepsilon'>0}\delta_{L_{\varepsilon'}}^{a_{\varepsilon'}}(t)\right)\int_V L_\varepsilon(x,\nabla v_\varepsilon)dx + \sup_{\varepsilon'>0}\delta_{L_{\varepsilon'}}^{a_{\varepsilon'}}(t)\int_V a_\varepsilon(x)dx, \tag{4.20}$$

where $\delta_{L_{\varepsilon'}}^{a_{\varepsilon'}} : [0,1] \to]-\infty,\infty]$ is defined in (C_4). Taking (4.18) into account and substituting (4.17), (4.19) and (4.20) into (4.16) and then averaging these inequalities, it follows that for every $q \geqslant 1$, every $t \in]0,1[$ and every $\varepsilon \in]0, \overline{\varepsilon}_{t,q}]$, there exists $i_{\varepsilon,t,q} \in \{1, \cdots, q\}$ such that

$$\begin{aligned}\int_{Z\cup T} L_\varepsilon(x, \nabla(tw_\varepsilon^{i_{\varepsilon,t,q}}))dx \leqslant & \left(1 + \sup_{\varepsilon'>0} \delta_{L_{\varepsilon'}}^{a_{\varepsilon'}}(t)\right) \int_U L_\varepsilon(x, \nabla u_\varepsilon)dx + \sup_{\varepsilon'>0} \delta_{L_{\varepsilon'}}^{a_{\varepsilon'}}(t) \int_U a_\varepsilon(x)dx \\ & + \left(1 + \sup_{\varepsilon'>0} \delta_{L_{\varepsilon'}}^{a_{\varepsilon'}}(t)\right) \int_V L_\varepsilon(x, \nabla v_\varepsilon)dx + \sup_{\varepsilon'>0} \delta_{L_{\varepsilon'}}^{a_{\varepsilon'}}(t) \int_V a_\varepsilon(x)dx \\ & + \frac{C}{q}\left(\int_\Omega \sup_{|\xi|\leqslant r} G(x,\xi)dx + \int_U L_\varepsilon(x, \nabla u_\varepsilon)dx + \int_V L_\varepsilon(x, \nabla v_\varepsilon)dx\right)\end{aligned}$$

with $C = \max\left\{\beta(1+\gamma+\gamma^2)+1, \frac{\beta\gamma^2}{\alpha}\right\}$, where $\int_\Omega \sup_{|\xi|\leqslant r} G(x,\xi)dx < \infty$ by (I$_7$). Thus, letting $\varepsilon \to 0$, $q \to \infty$ and $t \to 1^-$ and using (C$_4$), (4.11) and (4.12), we get

$$\overline{\lim_{t\to 1^-}}\ \overline{\lim_{q\to\infty}}\ \overline{\lim_{\varepsilon\to 0}} \int_{Z\cup T} L_\varepsilon(x, \nabla(tw_\varepsilon^{i_{\varepsilon,t,q}}))dx \leqslant \mathcal{S}_u^-(U) + \mathcal{S}_u^-(V). \tag{4.21}$$

On the other hand, taking (4.15) into account and using (4.9) and (4.10) we see that

$$\lim_{t\to 1^-} \lim_{q\to\infty} \lim_{\varepsilon\to 0} \|tw_\varepsilon^{i_{\varepsilon,t,q}} - u\|_{L^p} = 0.$$

By diagonalization, there exist increasing mappings $\varepsilon \mapsto t_\varepsilon$ and $\varepsilon \mapsto q_\varepsilon$ with $t_\varepsilon \to 1^-$ and $q_\varepsilon \to \infty$ such that:

$$\begin{aligned}&\underline{\lim_{\varepsilon\to 0}} \int_{Z\cup T} L_\varepsilon(x, \nabla \hat{w}_\varepsilon)dx \leqslant \overline{\lim_{t\to 1^-}}\ \overline{\lim_{q\to\infty}}\ \overline{\lim_{\varepsilon\to 0}} \int_{Z\cup T} L_\varepsilon(x, \nabla(tw_\varepsilon^{i_{\varepsilon,t,q}}))dx;\\ &\lim_{\varepsilon\to 0} \|\hat{w}_\varepsilon - u\|_{L^p} = 0,\end{aligned}$$

where $\hat{w}_\varepsilon := t_\varepsilon w_\varepsilon^{i_{\varepsilon,t_\varepsilon,q_\varepsilon}}$. Hence

$$\mathcal{S}_u^-(Z\cup T) \leqslant \overline{\lim_{t\to 1^-}}\ \overline{\lim_{q\to\infty}}\ \overline{\lim_{\varepsilon\to 0}} \int_{Z\cup T} L_\varepsilon(x, \nabla(tw_\varepsilon^{i_{\varepsilon,t,q}}))dx,$$

and (4.7) follows from (4.21). ■

Substep 1-2: end of the proof of Lemma 4.4. We now prove (4.5). Fix $A, B \in \mathcal{O}(\Omega)$ such that $\mathcal{S}_u^-(A) < \infty$ and $\mathcal{S}_u^-(B) < \infty$. Then, by (4.3), $\int_{A\cup B} G(x, \nabla u(x))dx < \infty$. Fix any $\eta > 0$ and consider $C_0, D_0 \in \mathcal{O}(\Omega)$ such that $\overline{C}_0 \subset A$, $\overline{D}_0 \subset B$ and

$$\beta|E| + \beta \int_E G(x, \nabla u(x))dx < \eta$$

with $E := A \cup B \backslash \overline{C_0 \cup D_0}$. Then $\mathcal{S}_u^-(E) \leqslant \eta$ by (4.3). Let $\hat{C}, \hat{D} \in \mathcal{O}(\Omega)$ be such that $\overline{C}_0 \subset C$, $\overline{C} \subset \hat{C}$, $\overline{\hat{C}} \subset A$, $\overline{D}_0 \subset D$, $\overline{D} \subset \hat{D}$ and $\overline{\hat{D}} \subset B$. Applying Lemma 4.5 with $U = \hat{C} \cup \hat{D}$, $V = T = E$ and $Z = C \cup D$ (resp. $U = A$, $V = B$, $Z = \hat{C}$ and $T = \hat{D}$) we obtain

$$\mathcal{S}_u^-(A\cup B) \leqslant \mathcal{S}_u^-(\hat{C}\cup\hat{D}) + \eta \ \big(\text{resp. } \mathcal{S}_u^-(\hat{C}\cup\hat{D}) \leqslant \mathcal{S}_u^-(A) + \mathcal{S}_u^-(B)\big),$$

i.e. $\mathcal{S}_u^-(A\cup B) \leqslant \mathcal{S}_u^-(A) + \mathcal{S}_u^-(B) + \eta$, and (4.5) follows by letting $\eta \to 0$. ■

Step 2: applying Theorem 1.2. For each $\varepsilon > 0$ and each $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, $\mathcal{I}_\varepsilon(u,\cdot)$ is an increasing set function. Moreover, from Lemma 4.4 we can assert that for every $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, $\Gamma(L^p)$-$\underline{\lim}_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u,\cdot)$ and $\Gamma(L^p)$-$\overline{\lim}_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u,\cdot)$ are inner regular. Hence, by

Theorem 2.4, every sequence $\{\mathscr{I}_\varepsilon\}_{\varepsilon>0}$ has a $\Gamma(L^p)$-convergent subsequence. So, without loss of generality we can assume that $\{\mathscr{I}_\varepsilon\}_{\varepsilon>0}$ $\Gamma(L^p)$-converges. Taking Proposition 2.6 into account, to establish Theorem 4.3 it suffices to prove that

$$\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0}\mathscr{I}_\varepsilon(u,A)=\mathscr{I}_0(u,A) \tag{4.22}$$

for all $u\in W^{1,p}(\Omega;\mathbb{R}^m)$ and all $A\in\mathcal{O}(\Omega)$ with $\mathscr{I}_0: W^{1,p}(\Omega;\mathbb{R}^m)\times\mathcal{O}(\Omega)\to[0,\infty]$ given by

$$\mathscr{I}_0(u,A):=\begin{cases}\displaystyle\int_A \lim_{t\to 1^-}\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\mathscr{H}^\rho L_\varepsilon(x,t\nabla u(x))dx & \text{if } u\in\mathrm{dom}(\mathscr{G}(\cdot,A))\\ \infty & \text{if } u\in W^{1,p}(\Omega;\mathbb{R}^m)\backslash\mathrm{dom}(\mathscr{G}(\cdot,A)).\end{cases}$$

For this, we are going to apply Theorem 1.2 with $\mathscr{F}: W^{1,p}(\Omega;\mathbb{R}^m)\times\mathcal{O}(\Omega)\to[0,\infty]$ defined by

$$\mathscr{F}(u,A):=\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0}\mathscr{I}_\varepsilon(u,A). \tag{4.23}$$

First of all, it is clear that (I_2)–(I_4) hold and (I_0) follows from (C_0) and (C_2). On the other hand, (I_1) follows from Lemma 4.4, and by (C_3), Propositions 2.23 and 2.17 we see that (I_9) is verified. So, since (I_5)–(I_8) are assumed to be satisfied, from Theorem 1.2 we deduce that for every $A\in\mathcal{O}(\Omega)$,

$$\mathscr{F}(u,A)=\begin{cases}\displaystyle\int_A \widehat{F}(x,\nabla u(x))dx & \text{if } u\in\mathrm{dom}(\mathscr{G}(\cdot,A))\\ \infty & \text{if } u\in W^{1,p}(\Omega;\mathbb{R}^m)\backslash\mathrm{dom}(\mathscr{G}(\cdot,A))\end{cases}$$

where $\mathrm{dom}(\mathscr{G}(\cdot,A))$ denotes the effective domain of $\mathscr{G}(\cdot,A)$ and $\widehat{F}:\Omega\times\mathbb{M}\to[0,\infty]$ is defined by

$$\widehat{F}(x,\xi)=\lim_{t\to 1^-}F(x,t\xi)$$

with F given by (1.1).

Step 3: refining the formula for the limit integrand. Taking (4.23) into account we have

$$\begin{aligned}\widehat{F}(x,\xi)&=\lim_{t\to 1^-}F(x,t\xi)\\ &=\lim_{t\to 1^-}\overline{\lim_{\rho\to 0}}\inf\left\{\frac{\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0}\mathscr{I}_\varepsilon(u,Q_\rho(x))}{\rho^N}: u-l_{t\xi}\in W^{1,p}_0(Q_\rho(x);\mathbb{R}^m)\right\}.\end{aligned} \tag{4.24}$$

In what follows, we are going to refine the formula for $\widehat{F}$ in (4.24).

Substep 3-1: an intermediate lemma. Let $\widetilde{\mathscr{I}}: W^{1,p}(\Omega;\mathbb{R}^m)\times\mathcal{O}(\Omega)\to[0,\infty]$ be defined by

$$\widetilde{\mathscr{I}}(u,A):=\inf\left\{\underline{\lim_{\varepsilon\to 0}}\mathscr{I}_\varepsilon(u_\varepsilon,A): W^{1,p}_0(A;\mathbb{R}^m)\ni u_\varepsilon-u\xrightarrow{L^p}0\right\}.$$

Substep 3-1 consists of proving the following lemma.

Lemma 4.6. *Assume that $p > N$ and* (C_0), (C_2)–(C_3) *and* (I_5)–(I_6) *hold. Then, for every $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, every $A \in \mathcal{O}(\Omega)$ and every $t \in]0,1[$,*

$$\widetilde{\mathcal{F}}(tu,A) \leqslant \left(1+\sup_{\varepsilon>0}\delta^{a_\varepsilon}_{L_\varepsilon}(t)\right)\Gamma(L^p)\text{-}\lim_{\varepsilon\to0}\mathcal{I}_\varepsilon(u,A)+\sup_{\varepsilon>0}\delta^{a_\varepsilon}_{L_\varepsilon}(t)\int_A a_0(x)dx, \tag{4.25}$$

where $\{a_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega;]0,\infty])$, $a_0 \in L^\infty(\Omega;]0,\infty])$ and $\delta^{a_\varepsilon}_{L_\varepsilon} : [0,1] \to]-\infty,\infty]$ are given in (C_3).

Proof of Lemma 4.6. Fix $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $A \in \mathcal{O}(\Omega)$. Without loss of generality, we can assume that $\Gamma(L^p)\text{-}\lim_{\varepsilon\to0}\mathcal{I}_\varepsilon(u,A) < \infty$, and so, by (C_0) and (C_2),

$$\mathcal{G}(u,A) < \infty. \tag{4.26}$$

By definition of $\Gamma(L^p)\text{-}\lim_{\varepsilon\to0}\mathcal{I}_\varepsilon(u,A)$ there exists $\{u_\varepsilon\}_\varepsilon \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that:

$$u_\varepsilon \xrightarrow{L^p} u; \tag{4.27}$$

$$\lim_{\varepsilon\to0}\mathcal{I}_\varepsilon(u_\varepsilon,A) = \Gamma(L^p)\text{-}\lim_{\varepsilon\to0}\mathcal{I}_\varepsilon(u,A). \tag{4.28}$$

Since $\Gamma(L^p)\text{-}\lim_{\varepsilon\to0}\mathcal{I}_\varepsilon(u,A) < \infty$, by (C_2) and (I_5) we see that $\sup_{\varepsilon>0}\|\nabla u_\varepsilon\|_{L^p(A)} < \infty$. As $p > N$, up to a subsequence, we have

$$u_\varepsilon \xrightarrow{L^\infty(A)} u. \tag{4.29}$$

Fix $\delta > 0$ and set $A_\delta := \{x \in A : \mathrm{dist}(x,\partial A) > \delta\}$. Fix any $\varepsilon > 0$ and any $q \geqslant 1$ and consider $W_i^-, W_i^+ \subset \Omega$ given by

$$W_i^- := \left\{x \in \Omega : \mathrm{dist}(x,A_\delta) \leqslant \tfrac{\delta}{3}+\tfrac{(i-1)\delta}{3q}\right\};$$
$$W_i^+ := \left\{x \in \Omega : \tfrac{\delta}{3}+\tfrac{i\delta}{3q} \leqslant \mathrm{dist}(x,A_\delta)\right\},$$

where $i \in \{1,\cdots,q\}$. (Note that $W_i^- \subset A$.) For every $i \in \{1,\cdots,q\}$ there exists a cut-off function $\varphi_i \in C^\infty(\Omega)$ for the pair (W_i^+,W_i^-), i.e. $\varphi(x) \in [0,1]$ for all $x \in \Omega$, $\varphi(x) = 0$ for all $x \in W_i^+$ and $\varphi(x) = 1$ for all $x \in W_i^-$. Define $w_\varepsilon^i : \Omega \to \mathbb{R}^m$ by

$$w_\varepsilon^i := \varphi_i u_\varepsilon + (1-\varphi_i)u. \tag{4.30}$$

Then $w_\varepsilon^i - u \in W_0^{1,p}(A;\mathbb{R}^m)$. Fix any $t \in]0,1[$. Setting $W_i := \Omega\backslash(W_i^- \cup W_i^+) \subset A$ we have

$$\nabla(tw_\varepsilon^i) = t\nabla w_\varepsilon^i = \begin{cases} t\nabla u_\varepsilon & \text{in } W_i^- \\ (1-t)\frac{t}{1-t}\nabla\varphi_i \otimes (u_\varepsilon - u) + t\big(\varphi_i\nabla u_\varepsilon + (1-\varphi_i)\nabla u\big) & \text{in } W_i \\ t\nabla u & \text{in } W_i^+. \end{cases}$$

Noticing that $A = W_i^- \cup W_i \cup (A \cap W_i^+)$ we deduce that for every $i \in \{1,\cdots,q\}$,

$$\mathcal{I}_\varepsilon(tw_\varepsilon^i,A) = \int_A L_\varepsilon(x,t\nabla w_\varepsilon^i)dx \leqslant \int_A L_\varepsilon(x,t\nabla u_\varepsilon)dx + \int_{A\cap W_i^+} L_\varepsilon(x,t\nabla u)dx + \int_{W_i} L_\varepsilon(x,t\nabla w_\varepsilon^i)dx. \tag{4.31}$$

Fix any $i \in \{1, \cdots, q\}$. From (C_2) and (I_6) we see that

$$\begin{aligned}\int_{W_i} L_\varepsilon(x, t\nabla w_\varepsilon^i)dx &\leqslant \beta\left(|W_i| + \int_{W_i} G(x, t\nabla w_\varepsilon^i)dx\right)\\ &\leqslant \beta(1+\gamma)|W_i|\\ &\quad +\beta\gamma\int_{W_i} G(x, \varphi_i\nabla u_\varepsilon + (1-\varphi_i)\nabla u)dx\\ &\quad +\beta\gamma\int_{W_i} G\left(x, \frac{t}{1-t}\nabla\varphi_i \otimes (u_\varepsilon - u)\right)dx,\end{aligned}$$

and we obtain, by using again (C_2) and (I_6),

$$\begin{aligned}\int_{W_i} L_\varepsilon(x, t\nabla w_\varepsilon^i)dx &\leqslant \beta(1+\gamma+\gamma^2)|W_i|\\ &\quad +\frac{\beta\gamma^2}{\alpha}\left(\int_{W_i} L_\varepsilon(x, \nabla u_\varepsilon(x))dx + \int_{W_i} L_\varepsilon(x, \nabla u(x))dx\right)\\ &\quad +\beta\gamma\int_{W_i} G\left(x, \frac{t}{1-t}\nabla\varphi_i\otimes(u_\varepsilon - u)\right)dx. \end{aligned}\tag{4.32}$$

On the other hand, we have

$$\left|\frac{t}{1-t}\nabla\varphi_i(x)\otimes(u_\varepsilon(x)-u(x))\right| \leqslant \left|\frac{t}{1-t}\right| \|\nabla\varphi_i\|_{L^\infty}\|u_\varepsilon - u\|_{L^\infty(A)}$$

for $\mathscr{L}_N$-a.a. $x \in W_i \subset A$. But $\lim_{\varepsilon\to 0}\|u_\varepsilon - u\|_{L^\infty(A)} = 0$ by (4.29), hence for each $i \in \{1, \cdots, q\}$ there exists $\varepsilon_i > 0$ such that

$$\left|\frac{t}{1-t}\nabla\varphi_i(x)\otimes(u_\varepsilon(x)-u(x))\right| \leqslant r$$

for $\mathscr{L}_N$-a.a. $x \in W_i$ and all $\varepsilon \in]0, \varepsilon_i]$ with $r > 0$ given by (I_7). Consequently

$$\int_{W_i} G\left(x, \frac{t}{1-t}\nabla\varphi_i\otimes(u_\varepsilon - u)\right)dx \leqslant \int_{W_i}\sup_{|\xi|\leqslant r} G(x,\xi)dx \tag{4.33}$$

for all $\varepsilon \in]0, \overline{\varepsilon}_q]$ with $\overline{\varepsilon}_q = \min\{\varepsilon_i : i \in \{1, \cdots, q\}\}$. Moreover, we have:

$$\int_A L_\varepsilon(x, t\nabla u_\varepsilon)dx \leqslant \left(1 + \sup_{\varepsilon'>0}\delta_{L_{\varepsilon'}}^{a_{\varepsilon'}}(t)\right)\int_A L_\varepsilon(x, \nabla u_\varepsilon)dx + \sup_{\varepsilon'>0}\delta_{L_{\varepsilon'}}^{a_{\varepsilon'}}(t)\int_A a_\varepsilon(x)dx; \tag{4.34}$$

$$\int_{A\cap W_i^+} L_\varepsilon(x, t\nabla u)dx \leqslant \left(1 + \sup_{\varepsilon'>0}\delta_{L_{\varepsilon'}}^{a_{\varepsilon'}}(t)\right)\int_{A\cap W_i^+} L_\varepsilon(x, \nabla u)dx + \sup_{\varepsilon'>0}\delta_{L_{\varepsilon'}}^{a_{\varepsilon'}}(t)\int_A a_\varepsilon(x)dx. \tag{4.35}$$

Taking (C_3) and (4.33) into account and substituting (4.32), (4.34) and (4.35) into (4.31) and then averaging these inequalities, it follows that for every $q \geqslant 1$ and every $\varepsilon \in]0, \overline{\varepsilon}_q]$, there exists

$i_{\varepsilon,q} \in \{1, \cdots, q\}$ such that

$$\begin{aligned} \mathscr{I}_\varepsilon(tw_\varepsilon^{i_{\varepsilon,q}}, A) \quad \leqslant \quad & \left(1 + \sup_{\varepsilon'>0} \delta_{L_{\varepsilon'}}^{a_{\varepsilon'}}(t)\right) \mathscr{I}_\varepsilon(u_\varepsilon, A) + \sup_{\varepsilon'>0} \delta_{L_{\varepsilon'}}^{a_{\varepsilon'}}(t) \int_A a_\varepsilon(x)dx \\ & + \frac{1}{q}\left[\left(1 + \sup_{\varepsilon'>0} \delta_{L_{\varepsilon'}}^{a_{\varepsilon'}}(t)\right) \mathscr{I}_\varepsilon(u, A) + \sup_{\varepsilon'>0} \delta_{L_{\varepsilon'}}^{a_{\varepsilon'}}(t) \int_A a_\varepsilon(x)dx\right] \\ & + \frac{C}{q}\left(\int_A \sup_{|\xi|\leqslant r} G(x,\xi)dx + \mathscr{I}_\varepsilon(u_\varepsilon, A) + \mathscr{I}_\varepsilon(u, A)\right) \end{aligned}$$

with $C = \max\left\{\beta(1+\gamma+\gamma^2)+1, \frac{\beta\gamma^2}{\alpha}\right\}$ where $\int_A \sup_{|\xi|\leqslant r} G(x,\xi)dx < \infty$ by (I_7). Moreover, by (C_2) we have $\mathscr{I}_\varepsilon(u, A) \leqslant \beta(|A| + \mathscr{G}(u, A))$, and so $\overline{\lim}_{\varepsilon\to 0} \mathscr{I}_\varepsilon(u, A) < \infty$ by (4.26). Thus, letting $\varepsilon \to 0$ and $q \to \infty$ and using (4.28), we get

$$\overline{\lim_{q\to\infty}}\, \overline{\lim_{\varepsilon\to 0}}\, \mathscr{I}_\varepsilon(tw_\varepsilon^{i_{\varepsilon,q}}, A) \leqslant \left(1 + \sup_{\varepsilon>0} \delta_{L_\varepsilon}^{a_\varepsilon}(t)\right) \Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} \mathscr{I}_\varepsilon(u, A) + \sup_{\varepsilon>0} \delta_{L_\varepsilon}^{a_\varepsilon}(t) \int_A a_0(x)dx. \quad (4.36)$$

On the other hand, taking (4.30) into account and using (4.27) we see that

$$\lim_{q\to\infty} \lim_{\varepsilon\to 0} \|tw_\varepsilon^{i_{\varepsilon,q}} - tu\|_{L^p} = 0.$$

By diagonalization, there exists an increasing mapping $\varepsilon \mapsto q_\varepsilon$ with $q_\varepsilon \to \infty$ such that:

$$\begin{aligned} & \underline{\lim}_{\varepsilon\to 0}\, \mathscr{I}_\varepsilon(\hat{w}_\varepsilon, A) \leqslant \overline{\lim_{q\to\infty}}\, \overline{\lim_{\varepsilon\to 0}}\, \mathscr{I}_\varepsilon(tw_\varepsilon^{i_{\varepsilon,q}}, A); \\ & \lim_{\varepsilon} \|\hat{w}_\varepsilon - tu\|_{L^p} = 0, \end{aligned}$$

where $\hat{w}_\varepsilon := tw_\varepsilon^{i_{\varepsilon,q_\varepsilon}}$ is such that $\hat{w}_\varepsilon - tu \in W_0^{1,p}(A;\mathbb{R}^m)$. Hence

$$\widetilde{\mathscr{I}}(tu, A) \leqslant \overline{\lim_{q\to\infty}}\, \overline{\lim_{\varepsilon\to 0}}\, \mathscr{I}(tw_\varepsilon^{i_{\varepsilon,q}}, A),$$

and (4.25) follows from (4.36). ■

Substep 3-2: a first estimate for F. Substep 3-2 consists of proving the following lemma.

Lemma 4.7. *Assume that $p > N$. If* (C_0), (C_2)–(C_3) *and* (I_5)–(I_6) *hold then for every* $(x,\xi) \in \Omega \times \mathbb{M}$,

$$\overline{\lim_{t\to 1^-}}\, \overline{\lim_{\rho\to 0}} \inf\left\{\frac{\widetilde{\mathscr{I}}(tu, Q_\rho(x))}{\rho^N} : u - l_\xi \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)\right\} \leqslant F(x,\xi).$$

Proof of Lemma 4.7. Fix $(x,\xi) \in \Omega \times \mathbb{M}$. Fix any $\eta > 0$. By (C_3), $\overline{\lim}_{t\to 1^-} \sup_{\varepsilon>0} \delta_{L_\varepsilon}^{a_\varepsilon}(t) \leqslant 0$, and so there exists $t_0 \in]0,1[$ such that for every $t \in [t_0, 1[$,

$$\sup_{\varepsilon>0} \delta_{L_\varepsilon}^{a_\varepsilon}(t) \leqslant \eta. \quad (4.37)$$

Fix any $t \in [t_0, 1[$, any $\rho > 0$ and any $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $u - l_\xi \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)$. Taking (4.37) into account, from Lemma 4.6 we see that

$$\widetilde{\mathscr{I}}(tu, Q_\rho(x)) \leqslant (1+\eta)\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} \mathscr{I}_\varepsilon(u, Q_\rho(x)) + \eta \int_{Q_\rho(x)} a_0(y)dy.$$

Hence, since $a_0 \in L^\infty(\Omega;]0,\infty])$,

$$\begin{aligned}\frac{\widetilde{\mathscr{I}}(tu, Q_\rho(x))}{\rho^N} &\leqslant (1+\eta)\frac{\Gamma(L^p)\text{-}\lim\limits_{\varepsilon\to 0}\mathscr{I}_\varepsilon(u,Q_\rho(x))}{\rho^N} + \eta \fint_{Q_\rho(x)} a_0(y)dy \\ &\leqslant (1+\eta)\frac{\Gamma(L^p)\text{-}\lim\limits_{\varepsilon\to 0}\mathscr{I}_\varepsilon(u,Q_\rho(x))}{\rho^N} + \eta\|a_0\|_{L^\infty}\end{aligned}$$

for all $t \in [t_0, 1[$, all $\rho > 0$ and all $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $u - l_\xi \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)$. Passing to the infimum and letting $\rho \to 0$ and $t \to 1^-$ we deduce that

$$\varlimsup_{t\to 1^-}\varlimsup_{\rho\to 0}\inf\left\{\frac{\widetilde{\mathscr{I}}(tu, Q_\rho(x))}{\rho^N} : u - l_\xi \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)\right\} \leqslant (1+\eta)F(x,\xi) + \eta\|a_0\|_{L^\infty},$$

and the result follows by letting $\eta \to 0$. ■

Substep 3-3: further estimates for *F*. Substep 3-3 consists of proving the following lemma.

Lemma 4.8. *Assume that $p > N$. If* (C_0), (C_2)–(C_3) *and* (I_5)–(I_6) *hold then for every* $(x,\xi) \in \Omega \times \mathbb{M}$,

$$\varlimsup_{t\to 1^-}\varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0}\mathscr{H}^\rho L_\varepsilon(x,t\xi) \leqslant F(x,\xi) \leqslant \varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0}\mathscr{H}^\rho L_\varepsilon(x,\xi),$$

where $\mathscr{H}^\rho L_\varepsilon : \Omega \times \mathbb{M} \to [0,\infty]$ is defined by (4.2).

Proof of Lemma 4.8. Fix $(x,\xi) \in \Omega \times \mathbb{M}$.

First of all, since $\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0}\mathscr{I}_\varepsilon(u,A) \leqslant \varlimsup_{\varepsilon\to 0}\mathscr{I}_\varepsilon(u,A)$ for all $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and all $A \in \mathcal{O}(\Omega)$, we have

$$\frac{\Gamma(L^p)\text{-}\lim\limits_{\varepsilon\to 0}\mathscr{I}_\varepsilon(u,Q_\rho(x))}{\rho^N} \leqslant \varlimsup_{\varepsilon\to 0}\frac{\mathscr{I}_\varepsilon(u,Q_\rho(x))}{\rho^N}$$

for all $\rho > 0$ and all $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $u - l_\xi \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)$. Passing to the infimum and letting $\rho \to 0$, we conclude that

$$\varlimsup_{\rho\to 0}\inf\left\{\frac{\Gamma(L^p)\text{-}\lim\limits_{\varepsilon\to 0}\mathscr{I}_\varepsilon(u,Q_\rho(x))}{\rho^N} : u - l_\xi \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)\right\} \leqslant \varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0}\mathscr{H}^\rho L_\varepsilon(x,\xi),$$

i.e. $F(x,\xi) \leqslant \varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0}\mathscr{H}^\rho L_\varepsilon(x,\xi)$.

Fix any $t \in]0,1[$, any $\rho > 0$ and any $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $u - l_\xi \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)$. By definition of $\mathscr{I}_0(tu, Q_\rho(x))$ there exists $\{u_\varepsilon\}_\varepsilon \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that:

$$u_\varepsilon - tu \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m) \text{ for all } \varepsilon > 0; \tag{4.38}$$

$$\lim_{\varepsilon\to 0}\mathscr{I}_\varepsilon(u_\varepsilon, Q_\rho(x)) = \mathscr{I}_0(tu, Q_\rho(x)). \tag{4.39}$$

As $u - l_\xi \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)$ we have $tu - l_{t\xi} \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)$, and so, by (4.38),

$$u_\varepsilon - l_{t\xi} \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m) \text{ for all } \varepsilon > 0. \tag{4.40}$$

From (4.39) and (4.38) we deduce that

$$\varliminf_{\varepsilon\to0}\inf\left\{\frac{\mathscr{I}_\varepsilon(v,Q_\rho(x))}{\rho^N}:v-l_{t\xi}\in W^{1,p}_0(Q_\rho(x);\mathbb{R}^m)\right\}\leqslant\frac{\widetilde{\mathscr{F}}(tu,Q_\rho(x))}{\rho^N}$$

for all $\rho>0$ and all $u\in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $u-l_\xi\in W^{1,p}_0(Q_\rho(x);\mathbb{R}^m)$. Passing to the infimum and letting $\rho\to0$ we deduce that

$$\varlimsup_{\rho\to0}\varliminf_{\varepsilon\to0}\mathscr{H}^\rho L_\varepsilon(x,t\xi)\leqslant\varlimsup_{\rho\to0}\inf\left\{\frac{\widetilde{\mathscr{F}}(tu,Q_\rho(x))}{\rho^N}:u-l_\xi\in W^{1,p}_0(Q_\rho(x);\mathbb{R}^m)\right\}\tag{4.41}$$

for all $t\in]0,1[$. But, by (C_2) we have

$$\mathrm{dom}\left(\varlimsup_{\rho\to0}\varliminf_{\varepsilon\to0}\mathscr{H}^\rho L_\varepsilon(x,\cdot)\right)=\mathrm{dom}\left(\varlimsup_{\rho\to0}\varlimsup_{\varepsilon\to0}\mathscr{H}^\rho L_\varepsilon(x,\cdot)\right)=\mathrm{dom}\left(\varlimsup_{\rho\to0}\mathscr{H}^\rho G(x,\cdot)\right)=:\mathbb{G}_{0,x},$$

hence, by (C_4),

$$\varlimsup_{\rho\to0}\varlimsup_{\varepsilon\to0}\mathscr{H}^\rho L_\varepsilon(x,\zeta)\leqslant\varlimsup_{\rho\to0}\varliminf_{\varepsilon\to0}\mathscr{H}^\rho L_\varepsilon(x,\zeta)\text{ for all }\zeta\in\mathbb{M},$$

and so

$$\varlimsup_{\rho\to0}\varlimsup_{\varepsilon\to0}\mathscr{H}^\rho L_\varepsilon(x,t\xi)\leqslant\varlimsup_{\rho\to0}\varliminf_{\varepsilon\to0}\mathscr{H}^\rho L_\varepsilon(x,t\xi)\text{ for all }t\in]0,1[.\tag{4.42}$$

Letting $t\to1^-$, from (4.41) and (4.42) it follows that

$$\varlimsup_{t\to1^-}\varlimsup_{\rho\to0}\varlimsup_{\varepsilon\to0}\mathscr{H}^\rho L_\varepsilon(x,t\xi)\leqslant\varlimsup_{t\to1^-}\varlimsup_{\rho\to0}\inf\left\{\frac{\widetilde{\mathscr{F}}(tu,Q_\rho(x))}{\rho^N}:u-l_\xi\in W^{1,p}_0(Q_\rho(x);\mathbb{R}^m)\right\},$$

and consequently $\varlimsup_{t\to1^-}\varlimsup_{\rho\to0}\varlimsup_{\varepsilon\to0}\mathscr{H}^\rho L_\varepsilon(x,t\xi)\leqslant F(x,\xi)$ by using Lemma 4.7. ■

Substep 3-4: end of the proof. By (C_3) and Lemma 2.21, $L_0:=\varlimsup_{\rho\to0}\varlimsup_{\varepsilon\to0}\mathscr{H}^\rho L_\varepsilon$ is ru-usc. Moreover, from (C_2) we see that

$$\mathbb{L}_{0,x}=\mathbb{G}_{0,x}\text{ for all }x\in\Omega,$$

where $\mathbb{L}_{0,x}$ denotes the effective domain of $L_0(x,\cdot)$. Hence, by (C_1),

$$t\overline{\mathbb{L}_{0,x}}\subset\mathrm{int}(\mathbb{L}_{0,x})\text{ for all }x\in\Omega\text{ and all }t\in]0,1[,$$

and consequently, by using Theorem 2.13(b), for every $(x,\xi)\in\Omega\times\mathbb{M}$,

$$\widehat{L}_0(x,\xi):=\varliminf_{\tau\to1^-}\varlimsup_{\rho\to0}\varlimsup_{\varepsilon\to0}\mathscr{H}^\rho L_\varepsilon(x,\tau\xi)=\lim_{\tau\to1^-}\varlimsup_{\rho\to0}\varlimsup_{\varepsilon\to0}\mathscr{H}^\rho L_\varepsilon(x,\tau\xi).\tag{4.43}$$

Fix $(x,\xi)\in\Omega\times\mathbb{M}$. From Lemma 4.8 we deduce that

$$\varlimsup_{s\to1^-}\lim_{t\to1^-}\varlimsup_{\rho\to0}\varlimsup_{\varepsilon\to0}\mathscr{H}^\rho L_\varepsilon(x,ts\xi)\leqslant\lim_{s\to1^-}F(x,s\xi)\leqslant\lim_{s\to1^-}\varlimsup_{\rho\to0}\varlimsup_{\varepsilon\to0}\mathscr{H}^\rho L_\varepsilon(x,s\xi).\tag{4.44}$$

By diagonalization, there exist increasing mappings $t\mapsto s_t$, with $s_t\to1^-$ as $t\to1^-$, such that

$$\varlimsup_{t\to1^-}\varlimsup_{\rho\to0}\varlimsup_{\varepsilon\to0}\mathscr{H}^\rho L_\varepsilon(x,ts_t\xi)\leqslant\varlimsup_{s\to1^-}\lim_{t\to1^-}\varlimsup_{\rho\to0}\varlimsup_{\varepsilon\to0}\mathscr{H}^\rho L_\varepsilon(x,ts\xi).$$

But, since $ts_t \to 1^-$ as $t \to 1^-$ and by using (4.43),

$$\begin{aligned}\overline{\lim_{t\to1^-}}\,\overline{\lim_{\rho\to0}}\,\overline{\lim_{\varepsilon\to0}}\,\mathscr{H}^\rho L_\varepsilon(x,ts_t\xi) &\geqslant \underline{\lim_{t\to1^-}}\,\overline{\lim_{\rho\to0}}\,\overline{\lim_{\varepsilon\to0}}\,\mathscr{H}^\rho L_\varepsilon(x,ts_t\xi)\\ &\geqslant \underline{\lim_{\tau\to1^-}}\,\overline{\lim_{\rho\to0}}\,\overline{\lim_{\varepsilon\to0}}\,\mathscr{H}^\rho L_\varepsilon(x,\tau\xi)\\ &= \lim_{\tau\to1^-}\overline{\lim_{\rho\to0}}\,\overline{\lim_{\varepsilon\to0}}\,\mathscr{H}^\rho L_\varepsilon(x,\tau\xi),\end{aligned}$$

hence

$$\lim_{\tau\to1^-}\overline{\lim_{\rho\to0}}\,\overline{\lim_{\varepsilon\to0}}\,\mathscr{H}^\rho L_\varepsilon(x,\tau\xi) \leqslant \lim_{s\to1^-} F(x,s\xi) \leqslant \lim_{s\to1^-}\overline{\lim_{\rho\to0}}\,\overline{\lim_{\varepsilon\to0}}\,\mathscr{H}^\rho L_\varepsilon(x,s\xi). \tag{4.45}$$

From (4.44) and (4.45) we conclude that $\lim_{t\to1^-} F(x,t\xi) = \lim_{t\to1^-}\overline{\lim}_{\rho\to0}\overline{\lim}_{\varepsilon\to0}\mathscr{H}^\rho L_\varepsilon(x,t\xi)$, i.e.

$$\widehat{F}(x,\xi) = \lim_{t\to1^-}\overline{\lim_{\rho\to0}}\,\overline{\lim_{\varepsilon\to0}}\,\mathscr{H}^\rho L_\varepsilon(x,t\xi).$$

We have thus established (4.22), which finishes the proof of Theorem 4.3. ■

4.2. **Relaxation.** Let $G : \Omega\times\mathbb{M} \to [0,\infty]$ be a Borel measurable function satisfying (C_0)–(C_1) and (I_5)–(I_8) and let $L : \Omega\times\mathbb{M} \to [0,\infty]$ be a Borel measurable such that:

(R_0) there exist $\alpha,\beta > 0$ such that for every $(x,\xi) \in \Omega\times\mathbb{M}$,

$$\alpha G(x,\xi) \leqslant L(x,\xi) \leqslant \beta(1 + G(x,\xi));$$

(R_1) L is ru-usc with $a \in L^\infty(\Omega;]0,\infty])$, i.e.

$$\overline{\lim_{t\to1^-}}\,\delta^a_L(t) \leqslant 0$$

with $\delta^a_L : [0,1] \to]-\infty,\infty]$ defined by

$$\delta^a_L(t) := \sup_{x\in\Omega}\sup_{\xi\in\mathbb{L}_x}\frac{L(x,t\xi) - L(x,\xi)}{a(x) + L(x,\xi)},$$

where $\mathbb{L}_x$ denotes the effective domain of $L(x,\cdot)$.

Let $\mathscr{I} : W^{1,p}(\Omega;\mathbb{R}^m)\times\mathscr{O}(\Omega) \to [0,\infty]$ be defined by

$$\mathscr{I}(u,A) := \int_A L(x,\nabla u(x))dx$$

and let $\overline{\mathscr{I}} : W^{1,p}(\Omega;\mathbb{R}^m)\times\mathscr{O}(\Omega) \to [0,\infty]$ be the L^p-lower semicontinuous envelope of $\mathscr{I}$, i.e. for every $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and every $A \in \mathscr{O}(\Omega)$,

$$\overline{\mathscr{I}}(u,A) := \Gamma(L^p)\text{-}\underline{\lim_{\varepsilon\to0}}\,\mathscr{I}(u,A) = \inf\left\{\underline{\lim_{\varepsilon\to0}}\,\mathscr{I}(u_\varepsilon,A) : u_\varepsilon \xrightarrow{L^p} u\right\}.$$

Applying Theorem 4.3 with $\{L_\varepsilon\}_{\varepsilon>0} = L$ and taking Remark 4.2 into account, as a straightforward consequence, we obtain the following relaxation result.

Corollary 4.9. *Assume that $p > N$ and* (R$_0$)–(R$_1$), (C$_0$)–(C$_1$) *and* (I$_5$)–(I$_8$) *hold. Then, for every $A \in \mathcal{O}(\Omega)$,*

$$\overline{\mathcal{F}}(u, A) = \begin{cases} \displaystyle\int_A \lim_{t\to 1^-} \overline{\lim_{\rho\to 0}} \mathcal{H}^\rho L(x, t\nabla u(x))dx & \text{if } u \in \mathrm{dom}(\mathcal{G}(\cdot, A)) \\ \infty & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^m)\backslash \mathrm{dom}(\mathcal{G}(\cdot, A)). \end{cases}$$

4.3. **Homogenization.** Let $G : \mathbb{M} \to [0, \infty]$ be a Borel measurable function (which does not depend on x) satisfying (C$_0$), (I$_5$)–(I$_6$) and

(I$_{7'}$) $0 \in \mathrm{int}(\mathbb{G})$ where $\mathbb{G}$ denotes the effective domain of G.

Let $L : \mathbb{R}^N \times \mathbb{M} \to [0, \infty]$ be a Borel measurable function with the following properties:

(H$_0$) for every $\xi \in \mathbb{M}$, $L(\cdot, \xi)$ is 1-periodic, i.e. for every $(x, z) \in \mathbb{R}^N \times \mathbb{Z}^N$,

$$L(x + z, \xi) = L(x, \xi);$$

(H$_1$) there exist $\alpha, \beta > 0$ such that for every $(x, \xi) \in \Omega \times \mathbb{M}$,

$$\alpha G(\xi) \leqslant L(x, \xi) \leqslant \beta(1 + G(\xi));$$

(H$_2$) L is ru-usc with a 1-periodic function $a \in L^1_{\mathrm{loc}}(\mathbb{R}^N;]0, \infty])$, i.e.

$$\overline{\lim_{t\to 1^-}} \delta^a_L(t) \leqslant 0$$

with $\delta^a_L : [0, 1] \to] - \infty, \infty]$ defined by

$$\delta^a_L(t) := \sup_{x\in\mathbb{R}^N} \sup_{\xi\in\mathbb{L}_x} \frac{L(x, t\xi) - L(x, \xi)}{a(x) + L(x, \xi)},$$

where $\mathbb{L}_x$ denotes the effective domain of $L(x, \cdot)$.

For each $\varepsilon > 0$, let $\mathcal{I}_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0, \infty]$ be defined by

$$\mathcal{I}_\varepsilon(u, A) := \int_A L\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx.$$

As a consequence of Theorem 4.3, we obtain the following homogenization result (which is a variant of [AHMZ15, Theorem 1.1]).

Corollary 4.10. *Assume that $p > N$ and* (H$_0$)–(H$_2$), (C$_0$), (I$_5$)–(I$_6$) *and* (I$_{7'}$) *hold. Then, for every $A \in \mathcal{O}(\Omega)$,*

$$\overline{\mathcal{I}}(u, A) = \begin{cases} \displaystyle\int_A \lim_{t\to 1^-} L_{\mathrm{hom}}(t\nabla u(x))dx & \text{if } u \in \mathrm{dom}(\mathcal{G}(\cdot, A)) \\ \infty & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^m)\backslash \mathrm{dom}(\mathcal{G}(\cdot, A)) \end{cases}$$

with $L_{\mathrm{hom}} : \mathbb{M} \to [0, \infty]$ given by

$$L_{\mathrm{hom}}(\xi) := \inf_{k\in\mathbb{N}^*} \inf \left\{ \fint_{]0,k[^N} L(y, \nabla u(y))dy : u - l_\xi \in W^{1,p}_0(]0, k[^N; \mathbb{R}^m) \right\}.$$

Proof of Corollary 4.10. It suffices to apply Theorem 4.3 with $\{L_\varepsilon\}_{\varepsilon>0} = \left\{L(\frac{\cdot}{\varepsilon},\cdot)\right\}_{\varepsilon>0}$ and $G(x,\xi) = G(\xi)$. First of all, taking Remark 1.3(iii) and Remarks 4.1(ii)-(iii) into account, from (C_0), (I_6) and ($I_{7'}$) we see (C_1) and (I_7)–(I_8) are verified. On the other hand, it is clear that by (H_1) we have (C_2) and from (H_2) and Lemma 2.24 we deduce that (C_3) holds. So, to apply Theorem 4.3, we only need to show that (C_4) is satisfied. For each $\xi \in \mathbb{M}$, let $\mathcal{S}_\xi : \mathcal{O}_{\rm b}(\mathbb{R}^N) \to [0,\infty]$ be defined by

$$\mathcal{S}_\xi(A) := \inf\left\{\int_A L(x,\nabla u(x))dx : u - l_\xi \in W_0^{1,p}(A;\mathbb{R}^m)\right\}.$$

It is easily seen that $\mathcal{S}_\xi$ is subadditive and, by (H_0), $\mathcal{S}_\xi$ is $\mathbb{Z}^N$-invariant. Moreover, from (H_1) we can assert that for every $\xi \in \mathbb{G}$ and every $A \in \mathcal{O}_{\rm b}(\mathbb{R}^N)$,

$$\mathcal{S}_\xi(A) \leqslant C_\xi|A|$$

with $C_\xi \in]0,\infty[$ given by $C_\xi := \beta(1+G(\xi))$. By Theorem 2.11 it follows that

$$\lim_{\varepsilon\to0}\frac{\mathcal{S}_\xi\left(\frac{1}{\varepsilon}Q\right)}{\left|\frac{1}{\varepsilon}Q\right|} = L_{\rm hom}(\xi) \text{ for all } \xi \in \mathbb{G} \text{ and all } Q \in \mathrm{Cub}(\mathbb{R}^N),$$

where $\mathrm{Cub}(\mathbb{R}^N)$ denotes the class of all open cubes in $\mathbb{R}^N$. Hence

$$\lim_{\varepsilon\to0}\mathcal{H}^\rho L_\varepsilon(x,\xi) = \lim_{\varepsilon\to0}\frac{\mathcal{S}_\xi\left(\frac{1}{\varepsilon}Q_\rho(x)\right)}{\left|\frac{1}{\varepsilon}Q_\rho(x)\right|} = L_{\rm hom}(\xi) \text{ for all } x \in \Omega \text{ and all } \xi \in \mathbb{G}$$

which shows that (C_4) is verified, and the proof is complete. ■

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