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ON A PROJECTION ESTIMATOR OF THE REGRESSION FUNCTION DERIVATIVE

FABIENNE COMTE* AND NICOLAS MARIE†

ABSTRACT. In this paper, we study the estimation of the derivative of a regression function in a standard univariate regression model. The estimators are defined either by derivating nonparametric least-squares estimators of the regression function or by estimating the projection of the derivative. We prove two simple risk bounds allowing to compare our estimators. More elaborate bounds under a stability assumption are then provided. Bases and spaces on which we can illustrate our assumptions and first results are both of compact or non compact type, and we discuss the rates reached by our estimators. They turn out to be optimal in the compact case. Lastly, we propose a model selection procedure and prove the associated risk bound. To consider bases with a non compact support makes the problem difficult.

AMS 2020 classification: 62G07 ; 62G07.

Keywords: Adaptive procedure - Derivative estimation - Non compact support - Nonparametric regression - Optimal rates - Projection method

1. MOTIVATION AND BIBLIOGRAPHICAL ELEMENTS

In this paper, we consider the random design regression model

$$(1) \quad Y_i = b(X_i) + \varepsilon_i ; i \in \{1, \dots, n\}, n \geq 1,$$

where $b(\cdot)$ is the unknown continuously derivable regression function, X_1, \dots, X_n are independent and identically distributed (i.i.d.) random variables with density f with respect to Lebesgue's measure, and the errors $\varepsilon_1, \dots, \varepsilon_n$ are i.i.d, unobserved, centered with variance σ^2 , and independent of the X_i 's. The observations are $(X_i, Y_i)_{1 \leq i \leq n}$, and we assume that b is regular enough to admit a derivative. We are interested in nonparametric estimation of *the derivative* b' of b , on a compact or a non-compact support.

The question of nonparametric estimation of derivatives is not new and is studied in different contexts, such as density estimation or white noise model (see Efromovich [18]), and not only in regression. Indeed, there can be a lot of reasons for wishing to estimate not only a function but also its derivative, which may be of intrinsic interest as measure of slope for instance. Recently, Bercu *et al.* [4] studied this question in the concrete application setting of sea shores water quality. Precisely, they propose an estimator defined as the derivative of the well-known Nadaraya-Watson estimator. Dai *et al.* [16] also mention applications to the modeling of human growth data (Ramsay and Silverman [28]) or to Raman spectra of bulk materials (Charnigo *et al.* [6]).

Derivatives of a rate optimal estimate of the regression function are proved to be rate optimal estimates of the corresponding derivatives, see Stone [31, 32], who establishes optimal rates for local polynomial weighted estimators on a compact set. See also a discussion on the topic in Rice and Rosenblatt [29], for a fixed design model.

Nonparametric estimation of the regression function derivative has been studied following different methods, relying on local polynomial regression, regression by smoothing splines, or difference quotients. We emphasize that the strategy for fixed design context where $x_i = i/n$ rely on dedicated methods. Indeed, differences $Y_i - Y_{i-1}$ bring information on b' , which is not the case for random design on non compact support. In local polynomial regression, the derivative can be estimated by the coefficient of the derivative of the local polynomial regression fitted at given point x , as summarized in Tsybakov [34], Chapter 1. Stone [32] showed that derivative estimation with splines can achieve the optimal L^2 -rate of convergence

(proved in Stone [31]) under mild assumptions. Further asymptotic properties are obtained by Zhou and Wolfe [36], mainly in the fixed design setting on compact support: they rely on splines estimators, arguing that they avoid boundary problems of kernel estimators. Note that extensions to functional regressors have been conducted (see Hall *et al.* [22]).

The smoothing parameter selection problem remained unanswered in the first papers. For kernel strategies, the bandwidth choice for the derivative estimator (based on a factor rule) is discussed in Fan and Gijbels [20], but not studied from theoretical point of view. Liu and Brabanter [25] propose a methodology which generalizes the difference quotient based estimator of the first order derivative to the random design setting, when X follows a uniform distribution on $[0, 1]$. They also discuss bandwidth selection in their setting. Lastly, we mention that an adaptive method in wavelet basis is studied in Chesneau [7], but it involves an estimate of f . As a consequence, the rate of estimation depends on the regularity of this function, which we want to avoid.

In the present work, we consider a projection method and propose an estimator as a finite development in an orthonormal L^2 basis with m coefficients. We start from the least-square estimator studied by Baraud for the fixed design model in [2] and the random design model in [3]. These last results have been extended to non compact support by Comte and Genon-Catalot [11]. We show that two strategies can be considered to deduce from this an estimator of $b'(\cdot)$, which do not coincide in general. We prove risk bounds for both strategies, and to our knowledge, these are the first results allowing for non compactly supported bases and estimators. In the case of a trigonometric basis and compact support estimation, we recover the optimal rates given in Stone [31] under light assumptions. We also obtain specific rates in the non compact Hermite basis setting. Lastly, we propose a model selection strategy relying on a Goldenshluger and Lepski [21] method and prove a risk bound for the final estimator: this result holds for sub-gaussian (bounded or gaussian) errors and allows to reach automatically the optimal rate on regularity spaces, without requiring the knowledge of the regularity of b . We discuss our assumptions, which remain rather weak.

The plan of the paper is the following. We define our notation and estimators in Section 2. In Section 3, we present our assumptions and prove two simple risk bounds allowing to compare our estimators. More elaborate bounds under a stability assumption (see Cohen *et al.* [8, 9]) are also provided. Bases and spaces on which we can illustrate our assumptions and first results are described in Section 4. They are both of compact or non compact type, and we discuss the rates reached by our estimators. They are the optimal ones in the compact case. Section 5 is dedicated to the adaptive procedure: we prove a risk bound and deduce corollaries about adaptive rates. The possibility of non compact support makes the problem difficult, and even if the estimator seems to follow a standard Goldenshluger and Lepski [21] scheme, the proofs are delicate, due to an additional bias term. A short numerical study shows that the collection of estimators contains relevant proposals and suggests a simpler practical strategy.

2. ESTIMATORS AND ASSUMPTIONS

2.1. Definition of the estimators. Let $\mathcal{B} = (\varphi_j)_{j \in \mathbb{N} \setminus \{0\}}$ be a Hilbert basis of $L^2(I, dx)$ with $I \in \mathcal{B}(\mathbb{R})$. For the sake of readability, for every $j \in \mathbb{N} \setminus \{0\}$, the function $x \in \mathbb{R} \mapsto \varphi_j(x) \mathbf{1}_I(x)$ is also denoted by φ_j . The following mean squares estimator of $b_I = b \mathbf{1}_I$, which is studied in Baraud [3] and in Comte and Genon-Catalot [11], is defined by

$$\hat{b}_m(x) := \sum_{j=1}^m [\hat{\theta}_m^1]_j \varphi_j(x) ; x \in I,$$

where $m \in \mathbb{N} \setminus \{0\}$,

$$\hat{\theta}_m^1 = \hat{\theta}_m^1(\mathbf{X}, \mathbf{Y}) := \frac{1}{n} \hat{\Psi}_m^{-1} \hat{\Phi}_m^* \mathbf{Y},$$

M^* denotes the transpose of M , $\mathbf{Y} := (Y_1, \dots, Y_n)^*$, $\mathbf{X} := (X_1, \dots, X_n)^*$, $\widehat{\Phi}_m := (\varphi_j(X_i))_{1 \leq i \leq n, 1 \leq j \leq m}$ and

$$\widehat{\Psi}_m := \frac{1}{n} \widehat{\Phi}_m^* \widehat{\Phi}_m = (\langle \varphi_j, \varphi_k \rangle_n)_{1 \leq j, k \leq m}$$

with

$$\langle \varphi, \psi \rangle_n := \frac{1}{n} \sum_{i=1}^n \varphi(X_i) \psi(X_i)$$

for every $\varphi, \psi : \mathbb{R} \rightarrow \mathbb{R}$. The map $(\varphi, \psi) \mapsto \langle \varphi, \psi \rangle_n$ is the empirical scalar product, and the associated norm is denoted by $\|\cdot\|_n$ in the sequel. The theoretical analogous on $\mathbb{L}^2(\mathbb{R}, f(x)dx)$ is

$$(\varphi, \psi) \mapsto \langle \varphi, \psi \rangle_f := \int_I \varphi(z) \psi(z) f(z) dz,$$

and the associated norm is denoted by $\|\cdot\|_f$. The reader can refer to Baraud [2, 3], Cohen et al. [8, 9], and Comte and Genon-Catalot [11] for risk bounds on \widehat{b}_m and an adaptive estimator.

On the one hand, a natural estimator of b'_I is

$$(2) \quad \widehat{b}'_m(x) := \sum_{j=1}^m [\widehat{\theta}_m^1]_j \varphi'_j(x)$$

with $m \in \mathbb{N} \setminus \{0\}$. Obviously,

$$(\widehat{b}'_m(X_1), \dots, \widehat{b}'_m(X_n))^* = \widehat{\Phi}'_m \widehat{\theta}_m^1 = \frac{1}{n} \widehat{\Phi}'_m \widehat{\Psi}_m^{-1} \widehat{\Phi}_m^* \mathbf{Y}$$

with $\widehat{\Phi}'_m := (\varphi'_j(X_i))_{i,j}$. This requires to choose a regular basis. Note that, contrary to what may occur for the density estimator, this way is simpler than derivating the Nadaraya-Watson kernel based estimator as done in Bercu *et al.* [4]. Indeed, the latter involves the derivative of a quotient of two functions.

On the other hand, when $(b\varphi_j)(\inf(I)) = (b\varphi_j)(\sup(I))$ for every $j \in \{1, \dots, m\}$, $\langle b', \varphi_j \rangle = -\langle b, \varphi'_j \rangle$ and the orthogonal projection $(b')_m$ of b' on $\mathcal{S}_m := \text{span}\{\varphi_1, \dots, \varphi_m\}$ in $\mathbb{L}^2(I, dx)$ satisfies

$$(b')_m(x) = - \sum_{j=1}^m \langle b, \varphi'_j \rangle \varphi_j(x).$$

Several of the basis we have in mind are such that the derivative of φ_j can be expressed as a finite linear combination of the other φ_k 's. Thus, if there exists $\Delta_{m,m+p} \in \mathcal{M}_{m,m+p}(\mathbb{R})$ such that $\widehat{\Phi}'_m = \widehat{\Phi}_{m+p} \Delta_{m,m+p}^*$, one can consider a projection estimator of the derivative instead of derivating the projection estimator of b :

$$(3) \quad \widehat{b}'_m(x) := \sum_{j=1}^m [\widehat{\theta}_m^2]_j \varphi_j(x)$$

with

$$\widehat{\theta}_m^2 = -\frac{1}{n} \Delta_{m,m+p} \widehat{\Psi}_{m+p}^{-1} \widehat{\Phi}_{m+p}^* \mathbf{Y}.$$

Obviously,

$$(\widehat{b}'_m(X_1), \dots, \widehat{b}'_m(X_n))^* = -\frac{1}{n} \widehat{\Phi}_{m+p} \Delta_{m,m+p} \widehat{\Psi}_{m+p}^{-1} \widehat{\Phi}_{m+p}^* \mathbf{Y}.$$

We shall see in this paper that the two strategies are different and we will provide risk bounds that allow to compare the two methods.

2.2. Notations and elementary properties:

- The operator norm of a matrix M is defined by $\|M\|_{\text{op}}^2 := \lambda_{\max}(MM^*)$, where we recall that M^* is the transpose of M and $\lambda_{\max}(MM^*)$ is the largest eigenvalue of the square matrix MM^* , which are nonnegative. Note that for a square, symmetric and nonnegative matrix A , $\|A\|_{\text{op}} = \lambda_{\max}(A)$. Note also that if A and B are two matrices such that AB and BA are well defined, then $\lambda_{\max}(AB) = \lambda_{\max}(BA)$. Finally, note that if A and B are two square, symmetric and nonnegative matrices, then $\text{Tr}(AB) \leq \|A\|_{\text{op}} \text{Tr}(B) = \lambda_{\max}(A) \text{Tr}(B)$.
- The Frobenius norm of a matrix M is defined by

$$\|M\|_F^2 := \text{Tr}(MM^*) = \text{Tr}(M^*M).$$

- The natural scalar product on $\mathbb{L}^2(I, f(x)dx)$, also called f -weighted scalar product, is denoted by $\langle \cdot, \cdot \rangle_f$, and the associated norm by $\|\cdot\|_f$.
- For every $\psi \in \mathbb{L}^2(I, dx)$, its orthogonal projection on $\mathcal{S}_m = \{\varphi_1, \dots, \varphi_m\}$ in $\mathbb{L}^2(I, dx)$ is denoted by ψ_m .

3. RISK BOUNDS

3.1. Preliminary rough risk bounds on $\widehat{b}'_m{}^1$ and $\widehat{b}'_m{}^2$. In the sequel, we assume that b' exists and is square integrable on I , and that the density function f fulfills the following assumption.

Assumption 3.1. *The density function f is bounded on I .*

First, we provide the following rough but general risk bound on $\widehat{b}'_m{}^1$.

Proposition 3.2. *Under Assumption 3.1,*

$$\begin{aligned} \mathbb{E} \left[\|\widehat{b}'_m{}^1 - b'\|_n^2 \right] &\leq 3\|f\|_\infty \inf_{t \in \mathcal{S}_m} \|t - b'\|^2 + 3\mathbb{E} \left[\|\widehat{\Phi}'_m (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^*\|_{\text{op}}^2 \|b - b_m\|_n^2 \right] + 3\|b'_m - (b')_m\|_f^2 \\ &\quad + \frac{\sigma^2}{n} \mathbb{E} \left[\text{Tr} \left((\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} (\widehat{\Phi}'_m)^* \widehat{\Phi}'_m \right) \right], \end{aligned}$$

where b_m is the $\mathbb{L}^2(I, dx)$ -orthogonal projection of b on \mathcal{S}_m , and b'_m is its derivative, while $(b')_m$ is the $\mathbb{L}^2(I, dx)$ -orthogonal projection of b' on \mathcal{S}_m .

Let us comment the four terms in the previous bound:

- (1) The first term is the bias term we could expect. It can be evaluated on regularity spaces. Without Assumption 3.1, this term can be replaced by $\inf_{t \in \mathcal{S}_m} \|t - b'\|_f^2$.
- (2) The second term involves the bias related to b , which would be negligible compared to the previous one; but it is multiplied by a coefficient which has an order depending on m and will at least compensate the improvement.
- (3) The third term can be evaluated in the different bases: the procedure makes sense if the derivative of the projection and the projection of the derivative are close, for fixed m . Under Assumption 3.1, it is less than $3\|f\|_\infty \|b'_m - (b')_m\|^2$, null in trigonometric spaces with odd dimensions, and of order less or equal than the first term in Laguerre or Hermite bases (see Proposition 4.1).
- (4) The last term is the variance term, and it is established in Proposition 3.3 that it increases with m as expected.

Proposition 3.3. *The map $m \mapsto \mathbb{E} \left[\text{Tr} \left((\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} (\widehat{\Phi}'_m)^* \widehat{\Phi}'_m \right) \right]$ is increasing.*

To sum up, we will have to make a compromise between decreasing bias term (1) and increasing variance term (4), with the specific difficulty related to nuisance terms (2) and (3).

Now, we turn to the estimator $\widehat{b}'_m{}^2$ and assume that there exists $p \in \mathbb{N}$ such that $\varphi_1, \dots, \varphi_m$ fulfill the following assumption:

Assumption 3.4 (m, p). *For every $j \in \{1, \dots, m\}$, $\varphi'_j \in \mathcal{S}_{j+p}$.*

Note that Assumption 3.4(m, p) implies that there exists $\Delta_{m, m+p} \in \mathcal{M}_{m, m+p}(\mathbb{R})$ such that

$$(4) \quad \widehat{\Phi}'_m = \widehat{\Phi}_{m+p} \Delta_{m, m+p}^*.$$

Trigonometric, Laguerre, Hermite and Legendre bases satisfy Assumption 3.4(m, p), see Section 4. More precisely, we have $p = 0$ for the Laguerre and Legendre bases and $\Delta_{m, m}$ is a lower triangular square matrix. We have $p = 1$ for Hermite and trigonometric bases with $\Delta_{m, m+1}(j, k) = 0$ for $k \geq j + p$. For the trigonometric basis with an odd dimension, we can keep a square link $\Delta_{m, m}$ with a null first line followed by diagonal 2×2 blocks of type

$$\begin{pmatrix} 0 & -2\pi j \\ 2\pi j & 0 \end{pmatrix}.$$

Assume that b and $\varphi_1, \dots, \varphi_m$ fulfill also the following assumption.

Assumption 3.5 (m). For every $j \in \{1, \dots, m\}$,

$$b(\underline{a})\varphi_j(\underline{a}) = b(\bar{a})\varphi_j(\bar{a}),$$

where $\underline{a} := \inf(I)$ and $\bar{a} := \sup(I)$.

Note that, for instance, Assumption 3.5(m) holds for every $m \in \mathbb{N}$ when $b(\underline{a}) = b(\bar{a}) = 0$. Under this additional condition, by the integration by parts formula,

$$(5) \quad \langle b', \varphi_j \rangle = -\langle b, \varphi'_j \rangle; \quad \forall j \in \{1, \dots, m\}.$$

So,

$$(6) \quad (b')_m(\mathbf{X}) = -\sum_{j=1}^m \langle b, \varphi'_j \rangle \varphi_j(\mathbf{X}) = -\widehat{\Phi}_m \Delta_{m, m+p} (\langle b, \varphi_j \rangle)_{1 \leq j \leq m+p},$$

which legitimates the definition (3) of the alternative estimator \widehat{b}'_m of b' . Let us establish a risk bound for this estimator.

Proposition 3.6. Under Assumptions 3.1, 3.4(m, p) and 3.5(m),

$$\begin{aligned} \mathbb{E} \left[\|\widehat{b}'_m - b'\|_n^2 \right] &\leq 2\|f\|_\infty \inf_{t \in \mathcal{S}_m} \|t - b'\|^2 + 2\mathbb{E} \left[\|\widehat{\Phi}_m \Delta_{m, m+p} (\widehat{\Phi}_{m+p}^* \widehat{\Phi}_{m+p})^{-1} \widehat{\Phi}_{m+p}^* \|_{\text{op}}^2 \|b - b_{m+p}\|_n^2 \right] \\ &\quad + \frac{\sigma^2}{n} \mathbb{E} \left[\text{Tr} \left((\widehat{\Phi}_{m+p}^* \widehat{\Phi}_{m+p})^{-1} \Delta_{m, m+p}^* \widehat{\Phi}_m^* \widehat{\Phi}_m \Delta_{m, m+p} \right) \right]. \end{aligned}$$

Note that for $p = 0$, this bound is almost the same as in Proposition 3.2, except that the undesirable term $\|b'_m - (b')_m\|_f^2$ no longer appears. The counterpart is that the result of Proposition 3.6 requires the additional Assumptions 3.4 and 3.5. Thanks to Proposition 4.1 (see Section 4 for details):

- In the specific case of the trigonometric basis, the additional term $\|b'_m - (b')_m\|_f^2$ in the bound of Proposition 3.2 is null, and the first estimator requires less assumptions, so the first strategy is better.
- In the case of the Hermite basis, Assumption 3.4(m, p), $p = 1$ and Assumption 3.5(m) are automatically fulfilled, so the second strategy is probably better.
- In the Laguerre basis, Assumption 3.5(m) is satisfied for all m if $b(0) = 0$. If this holds, it follows from Proposition 4.1 (iii) that both strategies give the same rate.
- In the case of the Legendre basis, the additional term $\|b'_m - (b')_m\|_f^2$ is likely to be large, so the second strategy should be preferred.

3.2. Elaborate risk bounds on \widehat{b}'_m and \widehat{b}'_m . First of all, under Assumption 3.1, let us consider

$$\Psi_m := \mathbb{E}(\widehat{\Psi}_m) = (\langle \varphi_j, \varphi_k \rangle_f)_{j, k}.$$

Assume also that it fulfills the following assumption called "stability assumption" by Cohen *et al.* [8].

Assumption 3.7 (m). *The matrix Ψ_m satisfies*

$$\mathfrak{L}(m) := \sup_{x \in I} \sum_{j=1}^m \varphi_j(x)^2 < \infty \quad \text{and} \quad \mathfrak{L}(m)(\|\Psi_m^{-1}\|_{\text{op}} \vee 1) \leq \frac{\mathfrak{c}}{2} \cdot \frac{n}{\log(n)},$$

where $\mathfrak{c} = (3 \log(3/2) - 1)/9$.

Since the φ_j 's doesn't depend on m , the \mathcal{S}_m 's are nested spaces. Thus, since $m \mapsto \mathfrak{L}(m)$ and $m \mapsto \|\Psi_m^{-1}\|_{\text{op}}$ are increasing, if there exists $m_0 \in \mathbb{N} \setminus \{0\}$ such that Assumption 3.7(m_0) is fulfilled, then Assumption 3.7(m) is fulfilled for every $m \leq m_0$.

Now, consider the truncated estimators

$$\tilde{b}'_m{}^{1,1} := \tilde{b}'_m{}^{1,1} \mathbf{1}_{\Lambda_{m+p}} \quad \text{and} \quad \tilde{b}'_m{}^{1,2} := \tilde{b}'_m{}^{1,2} \mathbf{1}_{\Lambda_{m+p}},$$

where

$$\Lambda_m := \left\{ \mathfrak{L}(m)(\|\widehat{\Psi}_m^{-1}\|_{\text{op}} \vee 1) \leq \mathfrak{c} \frac{n}{\log(n)} \right\}.$$

Then, let us establish elaborate risk bounds on $\tilde{b}'_m{}^{1,1}$ and $\tilde{b}'_m{}^{1,2}$.

Proposition 3.8. *Under Assumptions 3.1, 3.4(m, p) and 3.7($m+p$), if $\mathbb{E}[b'(X_1)^4] < \infty$ and $\mathbb{E}(Y_1^4) < \infty$, then*

$$\begin{aligned} \mathbb{E} \left[\|\tilde{b}'_m{}^{1,1} - b'\|_n^2 \right] &\leq 3\|f\|_\infty \inf_{t \in \mathcal{S}_m} \|t - b'\|^2 + 9\|\Delta_{m,m+p}^{f,1}\|_{\text{op}}^2 \|b - b_m\|_f^2 + 3\|b'_m - (b')_m\|_f^2 + \frac{2\sigma^2}{n} \|\Delta_{m,m+p}^{f,1}\|_F^2 \\ &\quad + \left[\frac{2cn}{\log(n)} \|\Delta_{m,m+p}\|_{\text{op}}^2 \mathbb{E}(Y_1^4)^{1/2} + 3\mathbb{E}(b'(X_1)^4)^{1/2} \right] \frac{\mathfrak{c}_{8.1}^{1/2}}{n^4} \end{aligned}$$

with $\Delta_{m,m+p}^{f,1} := \Psi_{m+p}^{1/2} \Delta_{m,m+p}^* \Psi_m^{-1/2}$ and $\Delta_{m,m+p}$ is defined in (4).

Proposition 3.9. *Under Assumptions 3.1, 3.4(m, p), 3.5(m) and 3.7($m+p$), if $\mathbb{E}[b'(X_1)^4] < \infty$ and $\mathbb{E}(Y_1^4) < \infty$, then*

$$\begin{aligned} \mathbb{E} \left[\|\tilde{b}'_m{}^{1,2} - b'\|_n^2 \right] &\leq 2\|f\|_\infty \inf_{t \in \mathcal{S}_m} \|t - b'\|^2 + 6\|\Delta_{m,m+p}^{f,2}\|_{\text{op}}^2 \|b - b_{m+p}\|_f^2 + \frac{2\sigma^2}{n} \|\Delta_{m,m+p}^{f,2}\|_F^2 \\ &\quad + \left[\frac{2cn}{\log(n)} \|\Delta_{m,m+p}\|_{\text{op}}^2 \mathbb{E}(Y_1^4)^{1/2} + 3\mathbb{E}(b'(X_1)^4)^{1/2} \right] \frac{\mathfrak{c}_{8.1}^{1/2}}{n^4} \end{aligned}$$

with $\Delta_{m,m+p}^{f,2} := \Psi_{m+p}^{-1/2} \Delta_{m,m+p}^* \Psi_m^{1/2}$ and $\Delta_{m,m+p}$ is defined in (4).

The coefficients involved in the bounds given in Propositions 3.8 and 3.9 are the theoretical ones instead of the empirical in Propositions 3.2 and 3.6. They will allow us to evaluate rates of convergence for the estimator, provided that the basis is specified.

4. BASES EXAMPLES AND EXPLICIT RISK BOUNDS

4.1. Examples of bases. First of all, let us provide four usual bases which can be considered because the φ_j 's are derivable:

- **The trigonometric basis:** Defined on $I = [0, 1]$ by $t_1(x) := 1$, $t_{2j}(x) := \sqrt{2} \cos(2\pi jx)$ and $t_{2j+1}(x) := \sqrt{2} \sin(2\pi jx)$ for $j = 1, \dots, p$ with $m = 2p + 1$. Thus, $\mathfrak{L}(m) = m$ for $(\varphi_j)_{1 \leq j \leq m} = (t_j)_{1 \leq j \leq m}$.
- **The Laguerre basis:** Defined on $I = \mathbb{R}_+$, via Laguerre's polynomials L_j , $j \geq 0$, by

$$\ell_j(x) := \sqrt{2} L_j(2x) e^{-x} \quad \text{with} \quad L_j(x) := \sum_{k=0}^j \binom{j}{k} (-1)^k \frac{x^k}{k!}.$$

It satisfies $\langle \ell_j, \ell_k \rangle = \delta_{k,j}$ (see Abramowitz and Stegun [1], 22.2.13), where $\delta_{k,j}$ is the Kronecker symbol. Then, $(\ell_j)_{j \geq 0}$ is an orthonormal family of $\mathbb{L}^2(\mathbb{R}_+)$ such that $\ell_j(0) = \sqrt{2}$ and

$$\|\ell_j\|_\infty = \sup_{x \in \mathbb{R}_+} |\ell_j(x)| = \sqrt{2}.$$

Thus, $\mathfrak{L}(m) = 2m$ for $(\varphi_j)_{1 \leq j \leq m} = (\ell_{j-1})_{1 \leq j \leq m}$. The ℓ'_j 's satisfy the following recursive formula (see Lemma 8.1 in Comte and Genon-Catalot [12]):

$$(7) \quad \ell'_0 = -\ell_0 \quad \text{and} \quad \ell'_j = -\ell_j - 2 \sum_{k=0}^{j-1} \ell_k \quad \text{for } j \geq 1.$$

- **The Hermite basis:** Defined on $I = \mathbb{R}$, via Hermite's polynomials H_j , $j \geq 0$, by

$$h_j(x) := \mathbf{c}_h(j) H_j(x) e^{-x^2/2}$$

with

$$H_j(x) := (-1)^j e^{x^2} \frac{d^j}{dx^j} (e^{-x^2}) \quad \text{and} \quad \mathbf{c}_h(j) = (2^j j! \sqrt{\pi})^{-1/2}.$$

The family $(H_j)_{j \geq 0}$ is orthogonal for the e^{-x^2} -weighted scalar product:

$$\int_{-\infty}^{\infty} H_j(x) H_k(x) e^{-x^2} dx = 2^j j! \sqrt{\pi} \delta_{j,k} = \mathbf{c}_h^2(j) \delta_{j,k}$$

(see Abramowitz and Stegun [1], 22.2.14). Moreover,

$$\|h_j\|_\infty = \sup_{x \in \mathbb{R}} |h_j(x)| \leq \phi_0 \quad \text{with} \quad \phi_0 = \pi^{-1/4}$$

(see Abramowitz and Stegun [1], 22.14.17 and Indritz [23]). Thus, $\mathfrak{L}(m) \leq \pi^{-1/2} m$, but it is proved in [13] that there exists $K > 0$ such that

$$\sup_{x \in \mathbb{R}} \sum_{j=0}^{m-1} h_j(x)^2 \leq K \sqrt{m}.$$

Therefore, we set $\mathfrak{L}(m) = K \sqrt{m}$ for $(\varphi_j)_{1 \leq j \leq m} = (h_{j-1})_{1 \leq j \leq m}$. The h'_j 's also satisfy a recursive formula (see Comte and Genon-Catalot [12], Equation (52) in Section 8.2):

$$(8) \quad h'_0 = -\frac{1}{\sqrt{2}} h_1 \quad \text{and} \quad h'_j = \frac{1}{\sqrt{2}} (\sqrt{j} h_{j-1} - \sqrt{j+1} h_{j+1}) \quad \text{for } j \geq 1.$$

- **The Legendre basis:** Defined on $I = [-1, 1]$, via Legendre polynomials G_j , $j \geq 0$, by

$$g_j(x) := \sqrt{\frac{2j+1}{2}} G_j(x) \quad \text{with} \quad G_j(x) := \frac{1}{2^j j!} \cdot \frac{d^j}{dx^j} [(x^2 - 1)^j].$$

As

$$\int_{-1}^1 G_j(x) G_k(x) dx = \frac{2}{2j+1} \delta_{j,k},$$

the family $(g_j)_{j \geq 0}$ is an orthonormal family of $\mathbb{L}^2([-1, 1])$. For example, $g_0(x) = 1/\sqrt{2}$, $g_1(x) = \sqrt{3/2}x$, $g_2(x) = 1/2\sqrt{5/2}(3x^2 - 1)$, etc. Note that they are easy to compute numerically thanks to the recursive formula

$$g_n(x) = \frac{1}{n} [(2n-1)xg_{n-1}(x) - (n-1)g_{n-2}(x)]$$

(see Formula 2.6.2 in [19]). Moreover,

$$\|g_j\|_\infty \leq \sqrt{\frac{2j+1}{2}}, \quad \text{which gives} \quad \sum_{j=0}^{m-1} g_j(x)^2 \leq \frac{1}{2} \sum_{j=0}^{m-1} (2j+1) = \frac{m^2}{2}$$

and $\mathfrak{L}(m) = m^2/2$ (see also Cohen *et al.* [8]) for $(\varphi_j)_{1 \leq j \leq m} = (g_{j-1})_{1 \leq j \leq m}$. The g'_j 's also satisfy a recursive formula:

$$\frac{d}{dx}g_{n+1} = \sqrt{2n+3} \sum_{k=0}^{\lfloor n/2 \rfloor} \sqrt{2(n-2k)+1} g_{n-2k}(x),$$

which can be written

$$(9) \quad g'_{2p+1}(x) = \sqrt{4p+3} \sum_{k=0}^p \sqrt{4k+1} g_{2k}(x), \quad g'_{2p+2}(x) = \sqrt{4p+5} \sum_{k=0}^p \sqrt{4k+3} g_{2k+1}(x).$$

Under Assumption 3.5(m), thanks to Equality (5) and to the recursive formulas available for each basis described above, we are able to compare the derivative b'_m of b_m to the derivative of the projection $(b')_m$ of b' as follow:

Proposition 4.1. *Under Assumption 3.5(m):*

- (i) *If $I = [0, 1]$ and $\varphi_j = t_j$ (the trigonometric basis with an odd m), then $\|b'_m - (b')_m\|^2 = 0$.*
- (ii) *If $I = \mathbb{R}$ and $\varphi_j = h_{j-1}$ (the Hermite basis), then*

$$\|b'_m - (b')_m\|^2 = \frac{m}{2} (\langle b, h_{m-1} \rangle^2 + \langle b, h_m \rangle^2).$$

- (iii) *If $I = \mathbb{R}_+$ and $\varphi_j = \ell_{j-1}$ (the Laguerre basis), then*

$$\|b'_m - (b')_m\|^2 = 4m \left(\sum_{k=0}^{m-1} \langle b, \ell_k \rangle \right)^2.$$

If in addition $b(0) = 0$, then

$$\|b'_m - (b')_m\|^2 = 4m \left(\sum_{k \geq m} \langle b, \ell_k \rangle \right)^2.$$

- (iv) *If $I = [-1, 1]$, $\varphi_j = g_{j-1}$ (the Legendre basis) and $m = 2p$, then*

$$\begin{aligned} \|b'_m - (b')_m\|^2 &= 3 \left(\sum_{k=0}^{p-1} \sqrt{4k+3} \langle b, g_{2k+1} \rangle \right)^2 + (4p-1) \left(\sum_{k=0}^{p-1} \sqrt{4k+1} \langle b, g_{2k} \rangle \right)^2 \\ &+ \sum_{j=0}^{p-1} \left(\sqrt{4j+3} \sum_{k=j}^{p-1} \sqrt{4k+3} \langle b, g_{2k+1} \rangle + \sqrt{4j+1} \sum_{k=0}^j \sqrt{4k+3} \langle b, g_{2k+1} \rangle \right)^2 \\ &+ \sum_{j=0}^{p-2} \left(\sqrt{4j+5} \sum_{k=j+1}^{p-1} \sqrt{4k+1} \langle b, g_{2k} \rangle + \sqrt{4j+2} \sum_{k=0}^j \sqrt{4k+1} \langle b, g_{2k} \rangle \right)^2. \end{aligned}$$

This implies that there exists a deterministic constant $c_{4.1} > 0$, not depending on m and n , such that $\|b'_m - (b')_m\|^2 \leq c_{4.1} m^4$.

The cases are ordered from the simplest (the trigonometric one) to the most complicated (Legendre case for an even m).

4.2. Explicit risk bound for the trigonometric basis. As the trigonometric basis has compact support, say I , we estimate in fact $b := b\mathbf{1}_I$ and we can assume that $f(x) \geq f_0 > 0$ for every $x \in I$. Moreover, we assume that f is bounded (Assumption 3.1). We set $I = [0, 1]$ for simplicity and assume that $b(0) = b(1)$ (which ensures Assumption 3.5(m) for all m). Then, by considering models with an odd m , Assumption 3.4(m, p) is fulfilled for all m and $p = 0$. Moreover, we know from [11] that $\|\Psi_m^{-1}\|_{\text{op}} \leq 1/f_0$. Then, we get

$$\mathfrak{L}(m) = m, \quad \|\Delta_{m,m}\|_{\text{op}}^2 \leq \pi^2 m^2, \quad \|\Delta_{m,m}^{f,1}\|_{\text{op}}^2 = \|\Delta_{m,m}^{f,2}\|_{\text{op}}^2 \leq \frac{\|f\|_{\infty}}{f_0} m^2,$$

and

$$\|\Delta_{m,m}^{f,1}\|_F^2 = \|\Delta_{m,m}^{f,2}\|_F^2 \leq \frac{1}{f_0} m^3.$$

The last bound comes from the following inequalities

$$\begin{aligned} \|\Delta_{m,m}^{f,2}\|_F^2 &= \text{Tr}[\Psi_m^{-1} \Delta_{m,m}^* \Psi_m \Delta_{m,m}] \leq \|\Psi_m^{-1}\|_{\text{op}} \text{Tr}[\Delta_{m,m}^* \Psi_m \Delta_{m,m}] = \|\Psi_m^{-1}\|_{\text{op}} \mathbb{E} \left[\text{Tr}(\Delta_{m,m}^* \widehat{\Psi}_m \Delta_{m,m}) \right] \\ (10) \quad &= \frac{1}{n} \|\Psi_m^{-1}\|_{\text{op}} \mathbb{E} \left[\text{Tr}(\widehat{\Phi}'_m (\widehat{\Phi}'_m)^*) \right] = \frac{1}{n} \|\Psi_m^{-1}\|_{\text{op}} \mathbb{E} \left[\sum_{i=1}^n \sum_{j=0}^{m-1} \varphi'_j(X_i)^2 \right] \\ &\leq \frac{m^2}{n} \|\Psi_m^{-1}\|_{\text{op}} \mathbb{E} \left[\sum_{i=1}^n \sum_{j=0}^{m-1} \varphi_j^2(X_i) \right] = m^3 \|\Psi_m^{-1}\|_{\text{op}} \leq \frac{m^3}{f_0}, \end{aligned}$$

using that for $\varphi_j = t_j$, $\varphi'_j = \pm 2\pi j \varphi_{j\pm 1}$. So, the risk bound on $\widetilde{b}'_m{}^2$ established at Proposition 3.9 becomes

$$(11) \quad \mathbb{E} \left[\|\widetilde{b}'_m{}^2 - b'\|_n^2 \right] \leq 2\|f\|_\infty \left(\inf_{t \in \mathcal{S}_m} \|t - b'\|^2 + \frac{6}{f_0} m^2 \|b - b_m\|^2 \right) + \frac{2\sigma^2}{nf_0} m^3 + \frac{\mathbf{c}_1}{n}$$

with $\mathbf{c}_1 > 0$ and odd m . Since $p = 0$ and $b'_m = (b')_m$ for the trigonometric basis, the risk bound on $\widetilde{b}'_m{}^1$ established at Proposition 3.8 is the same up to a multiplicative constant.

Now, let us evaluate the rate of convergence of the estimator for b in some regularity space and well chosen m . Let β be a positive integer, $L > 0$ and define

$$\begin{aligned} W^{\text{per}}(\beta, L) &:= \{g \in C^\beta([0, 1]; \mathbb{R}) : g^{(\beta-1)} \text{ is absolutely continuous,} \\ &\int_0^1 g^{(\beta)}(x)^2 dx \leq L^2 \text{ and } g^{(j)}(0) = g^{(j)}(1), \forall j = 0, \dots, \beta - 1\}. \end{aligned}$$

We obtain the following result:

Corollary 4.2. *Consider the estimator $\widehat{b}'_m{}^2$ computed in the trigonometric basis on $I = [0, 1]$ with $0 < f_0 \leq f(x) \leq \|f\|_\infty < \infty$, $\mathbb{E}[b'(X_1)^4] < \infty$ and $\mathbb{E}(Y_1^4) < \infty$. If $b \in W^{\text{per}}(\beta, L)$ with $\beta > 1$, $b(0) = b(1)$ and $m_{\text{opt}} = n^{1/(2\beta+1)}$, then*

$$\mathbb{E} \left[\|\widehat{b}'_{m_{\text{opt}}}{}^2 - b'\|_n^2 \right] \leq \mathbf{c}(L, \beta, \|f\|_\infty, f_0, \sigma^2) n^{-2(\beta-1)/(2\beta+1)},$$

Proof. By Proposition 1.14 of [34], a function $f \in W^{\text{per}}(\beta, L)$ admits a development

$$f = \sum_{j=0}^{\infty} \theta_j \varphi_j \quad \text{such that} \quad \sum_{j \geq 0} \theta_j^2 \tau_j^2 \leq C(L),$$

where $\tau_j = j^\beta$ for even j , $\tau_j = (j-1)^\beta$ for odd j , and $C(L) := L^2 \pi^{-2\beta}$. Moreover, if b belongs to a the Sobolev ellipsoid $W^{\text{per}}(\beta, L)$ with $\beta > 1$, then $b' \in W^{\text{per}}(\beta-1, 2\pi L)$. So,

$$\|b - b_m\|^2 \leq \mathbf{c}(L, \beta) m^{-2\beta} \quad \text{and} \quad \|b' - (b')_m\|^2 \leq \mathbf{c}(L, \beta) m^{-2(\beta-1)}.$$

Therefore, plugging $m = m_{\text{opt}} = n^{1/(2\beta+1)}$ in (11) gives the result of Corollary 4.2. Indeed, Proposition 3.9 applies because the required conditions are automatically satisfied by the trigonometric basis. \square

Note that we obtain the optimal rate for estimating the derivative of a regression function (see Stone [32]). It coincides also with the rate of estimation for the derivative of a density (see Tsybakov [34], Efromovich [18, 19], recently Lepski [24] on general Nikolski's spaces, or Comte *et al.* [10]).

4.3. Explicit risk bound for Hermite basis. Consider $s, D > 0$ and the Sobolev-Hermite ball of regularity s

$$(12) \quad W_H^s(D) = \left\{ \theta \in \mathbb{L}^2(\mathbb{R}) : \sum_{k \geq 0} k^s a_k^2(\theta) \leq D \right\},$$

where $a_k^2(\theta) = \langle \theta, h_k \rangle$. In the Hermite case, the following bounds hold:

$$\mathfrak{L}(m) = K\sqrt{m}, \quad \|\Delta_{m,m+1}\|_{\text{op}}^2 \leq 2m, \quad \|\Delta_{m,m+1}^{f,1}\|_{\text{op}}^2 = \|\Delta_{m,m+1}^{f,2}\|_{\text{op}}^2 \leq 2\|f\|_{\infty} \|\Psi_{m+1}^{-1}\|_{\text{op}} m$$

and

$$\|\Delta_{m,m+1}^{f,1}\|_F^2 = \|\Delta_{m,m+1}^{f,2}\|_F^2 \leq 2K \|\Psi_{m+1}^{-1}\|_{\text{op}} (m+1)^{3/2}.$$

The last bound is obtained by following the line of the trigonometric case above, up to (10), using next formula (8) for the derivative of the basis functions.

In this context, it is proved in [11] that $\|\Psi_m^{-1}\|_{\text{op}}$ is increasing with m . Therefore, we can state the following result.

Corollary 4.3. *Consider the estimator $\widehat{b}_m^{\prime,2}$ computed in the Hermite basis on $I = \mathbb{R}$ under Assumptions 3.1 and 3.7($m+1$). Assume also that b' is square integrable, $\mathbb{E}[b'(X_1)^4] < \infty$ and $\mathbb{E}(Y_1^4) < \infty$. If $\|\Psi_m^{-1}\|_{\text{op}} \lesssim m^\gamma$ for all m , and if $b \in W_H^s(D)$ with $s > 1 + \gamma$, then by choosing $m_{\text{opt}} = n^{1/(s+1/2)}$ yields*

$$\mathbb{E} \left[\|\widehat{b}_{m_{\text{opt}}}^{\prime,2} - b'\|_n^2 \right] \leq \mathfrak{c}(D, s, \|f\|_{\infty}, \sigma^2) n^{-2(s-1-\gamma)/(2s+1)}.$$

The rate is deteriorated compared to $n^{-2(s-1)/(2s+1)}$, which is the optimal rate of estimation for the derivative of a density in a similar non compact setting (see bounds (15) and (16) in [10]). However, we are in the framework of an inverse problem, due both to the derivative aspect and to the non compact support feature of the basis. Moreover, we believe that the deterioration is unavoidable and is due to the term $\|\Psi_m^{-1}\|_{\text{op}}$ in the variance. Indeed, a simplification occurs when estimating b , which does not when looking for b' .

Proof. The following Lemma (Lemma 2.2 in Comte *et al.* [10]) gives a relationship between the regularity of $\theta \in W_H^s(D)$ and the regularity of its derivative.

Lemma 4.4. *Consider $s \geq 1$ and $D > 0$. If $\theta \in W_H^s(D)$ admits a square integrable derivative, then there exists a constant $D' = C(D) > D$ such that $\theta' \in W_H^{s-1}(D')$.*

By Lemma 4.4, if $b \in W_H^s(D)$, then $\|b - b_m\|^2 \leq Dm^{-s}$, $\|b' - (b')_m\|^2 \leq C(D)m^{-s+1}$, and the risk bound on $\widehat{b}_m^{\prime,2}$ established at Proposition 3.9 becomes

$$(13) \quad \begin{aligned} \mathbb{E} \left[\|\widehat{b}_m^{\prime,2} - b'\|_n^2 \right] &\leq 2\|f\|_{\infty} \left(\inf_{t \in S_m} \|t - b'\|^2 + 6\|\Psi_{m+1}^{-1}\|_{\text{op}} m \|b - b_{m+1}\|^2 \right) \\ &\quad + \frac{4K\sigma^2}{n} \|\Psi_{m+1}^{-1}\|_{\text{op}} (m+1)^{3/2} + \frac{\mathfrak{c}_1}{n} \\ &\leq C(D)\|f\|_{\infty} [m^{-(s-1)} + \|\Psi_{m+1}^{-1}\|_{\text{op}} m (m+1)^{-s}] + \frac{4K\sigma^2}{n} \|\Psi_{m+1}^{-1}\|_{\text{op}} (m+1)^{3/2} + \frac{\mathfrak{c}_1}{n} \end{aligned}$$

with $\mathfrak{c}_1 > 0$. Thus, if $\|\Psi_m^{-1}\|_{\text{op}} = O(m^\gamma)$, for $s > \gamma + 1$, the estimator is consistent, and to plug the choice $m = m_{\text{opt}} = n^{1/(s+1/2)}$ in (13) gives the result of Corollary 4.3. \square

4.4. Explicit risk bound for Legendre basis. By Proposition 2.6.1 in [19] (see also [17], Section 7.6), it is known that if $b \in C^r([-1, 1]; \mathbb{R})$ ($r \geq 1$) and if there exists $\alpha \in (0, 1]$ such that

$$|b^{(r)}(t) - b^{(r)}(s)| \leq Q|t - s|^\alpha; \quad \forall s, t \in [-1, 1],$$

then there exists $c > 0$ such that

$$\|b - b_m\|^2 \leq cm^{-2(r+\alpha)} \quad \text{and} \quad \|b' - (b')_m\|^2 \leq cm^{-2(r-1+\alpha)}.$$

The space of regularity $\beta = r + \alpha$ considered above will be called Hölder space and denoted by $\mathcal{H}(\beta, Q)$.

By Proposition 4.1, we can see that the first estimator may not be consistent as $\|(b_m)' - (b')_m\|^2$ may not tend to zero. However, Formula (9) shows that the Legendre basis satisfies Assumption 3.4(m, p) with $p = 0$ and triangular matrix $\Delta_{m,m}$ with null diagonal. As the basis is compactly supported, we can proceed as in the case of the trigonometric basis, assuming $I = [-1, 1]$, $0 < f_0 < f(x) < \|f\|_\infty < \infty$ for every $x \in I$, and $b(-1) = b(1) = 0$. Then,

$$\mathfrak{L}(m) = \frac{m^2}{2}, \quad \|\Delta_{m,m}\|_{\text{op}}^2 \leq cm^4, \quad \|\Delta_{m,m}^{f,1}\|_{\text{op}}^2 = \|\Delta_{m,m}^{f,2}\|_{\text{op}}^2 \leq c \frac{\|f\|_\infty}{f_0} m^4,$$

and

$$\|\Delta_{m,m}^{f,1}\|_F^2 = \|\Delta_{m,m}^{f,2}\|_F^2 \leq \frac{c}{f_0} m^5.$$

As a consequence, for $b \in \mathcal{H}(\beta, Q)$ with $\beta > 2$ and $b(-1) = b(1) = 0$, Proposition 3.9 implies that if $m_{\text{opt}} = n^{1/(2\beta+1)}$, then

$$\mathbb{E} \left[\|\tilde{b}'_m - b'\|_n^2 \right] \leq \mathfrak{c}(Q, \beta, \|f\|_\infty, f_0, \sigma^2) n^{-2(\beta-2)/(2\beta+1)}.$$

We mention this rate, but it is sub-optimal in the compact support case, specifically in comparison with the trigonometric basis.

5. A GOLDENSHLUGER-LEPSKI TYPE ADAPTIVE ESTIMATOR

The choice of the adequate m is crucial to reach the best order for the quadratic risk. However, this choice depends on unknown quantities, such as the order of regularity of the unknown function. This is why it is important to propose a way to select this dimension from the data. The problem is difficult, especially if we intend to bound the risk of the associated adaptive estimator. Penalty based model selection often rely on a contrast minimization, which seems not possible here. This is why we propose a Goldenshluger-Lepski type strategy, described in [21] for kernel estimators, and extended to dimension selection in Chagny [5].

More precisely, consider the random collection

$$\widehat{\mathcal{M}}_n := \left\{ m \in \{1, \dots, n\} : \mathfrak{L}(m+p) (\|\widehat{\Psi}_{m+p}^{-1}\|_{\text{op}}^2 \vee 1) \leq \mathfrak{d} \frac{n}{\log(n)} \right\}$$

where $\mathfrak{d} > 0$ is a constant depending on $\|f\|_\infty$ (see the proof of Theorem 5.2), and the random penalty

$$\widehat{V}(m) := \frac{\sigma^2 m}{n} \|(\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} (\widehat{\Phi}'_m)^* \widehat{\Phi}'_m\|_{\text{op}}.$$

This section deals with the adaptive estimator $\widehat{b}' := \widehat{b}'_{\widehat{m}}$, where

$$\widehat{m} = \arg \min_{m \in \widehat{\mathcal{M}}_n} \left\{ A(m) + \kappa_1 \widehat{V}(m) \right\}$$

with

$$A(m) := \sup_{m' \in \widehat{\mathcal{M}}_n} \left\{ \|\widehat{b}'_{m \wedge m'} - \widehat{b}'_{m'}\|_n^2 - \kappa_0 \widehat{V}(m') \right\}_+ \quad \text{and} \quad \kappa_0 \leq \kappa_1.$$

Consider

$$\mathcal{M}_n := \left\{ m \in \{1, \dots, n\} : \mathfrak{L}(m+p) (\|\Psi_{m+p}^{-1}\|_{\text{op}}^2 \vee 1) \leq \frac{\mathfrak{d}}{4} \cdot \frac{n}{\log(n)} \right\},$$

the theoretical counterpart of $\widehat{\mathcal{M}}_n$, and \mathcal{M}_n^+ with the same definition as \mathcal{M}_n but with $\mathfrak{d}/4$ replaced by $4\mathfrak{d}$. The maximal element of \mathcal{M}_n^+ is denoted by M_n^+ . Finally, let

$$V(m) := \frac{\sigma^2 m}{n} \|\Delta_{m,m+p}^{f,1}\|_{\text{op}}^2$$

be the theoretical version of $\widehat{V}(m)$.

Lemma 5.1. *The map $m \mapsto \widehat{V}(m)$ is increasing.*

Theorem 5.2. *Let Assumption 3.1 be fulfilled. Let also Assumption 3.7($m+p$) be fulfilled for every $m \in \mathcal{M}_n$. Moreover, assume that there exists $\kappa > 0$ such that $\mathbb{E}(\exp(\kappa\varepsilon_1^2)) < \infty$, that*

$$(14) \quad \sup_{n \in \mathbb{N} \setminus \{0\}} \left\{ \frac{1}{\log(n)} \sum_{m \leq n} \frac{\mathcal{L}'(m)}{\mathcal{L}(m)} [\exp(-\mathbf{a}_1 m) + \exp(-\mathbf{a}_2 \sqrt{\mathcal{L}(m)})] \right\} < \infty ; \forall \mathbf{a}_1, \forall \mathbf{a}_2 > 0$$

with

$$\mathcal{L}'(m) := \sup_{x \in I} \sum_{j=0}^{m-1} \varphi'_j(x)^2,$$

and that there exists $q \in \mathbb{N} \setminus \{0\}$ such that

$$(15) \quad \sup_{n \in \mathbb{N} \setminus \{0\}} \left\{ \frac{1}{n^{q/2} \log(n)} \sum_{m \leq n} \frac{\mathcal{L}'(m)}{\mathcal{L}(m)} \right\} < \infty.$$

Then, there exists a constant $\mathbf{c}_{5.2} > 0$, not depending on n , such that

$$\begin{aligned} \mathbb{E}(\|\widehat{b}' - b'\|_n^2) &\leq \mathbf{c}_{5.2} \inf_{m \in \mathcal{M}_n} \left\{ \mathbb{E}(\|\widehat{b}'_m - b'\|_n^2) + \kappa_1 V(m) + \|\Delta_{m,m+p}^{f,1}\|_{\text{op}}^2 \|b - b_m\|_f^2 \right. \\ &\quad \left. + \sup_{m' \in \mathcal{M}_n^+ : m' > m} \left\{ \|\Delta_{m',m'+p}^{f,1}\|_{\text{op}}^2 (\|b_{M_n^+} - b_{m'}\|_f^2 + \|b - b_{M_n^+}\|_\infty^2) \right\} \right. \\ &\quad \left. + \frac{9}{2} \|f\|_\infty \sup_{m' \in \mathcal{M}_n^+ : m' > m} \|b'_{m'} - b'_m\|^2 \right\} + \frac{\mathbf{c}_{5.2}}{n}. \end{aligned}$$

Conditions (14) and (15) are fulfilled by all the bases we mentioned (trigonometric, Laguerre, Hermite, Legendre) because $\mathcal{L}(m)$ and $\mathcal{L}'(m)$ have the order of powers of m . The condition on ε_1 is fulfilled by Gaussian random variables for any $\kappa < 1/(2\sigma^2)$, and by random variables with a compactly supported distribution. The quantity

$$\inf_{m \in \mathcal{M}_n} \left\{ \mathbb{E}(\|\widehat{b}'_m - b'\|_n^2) + \kappa_1 V(m) + \|\Delta_{m,m+p}^{f,1}\|_{\text{op}}^2 \|b - b_m\|_f^2 \right\}$$

has the order of the minimum risk over the estimators of the collection in this problem. The three additional terms are due to the bound on the *bias term*

$$\mathbb{E} \left(\sup_{m' \in \widetilde{\mathcal{M}}_n} \|\mathbb{E}_{\mathbf{X}}(\widehat{b}'_{m \wedge m'}) - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_{m'})\|_n^2 \right).$$

Concretely, Theorem 5.2 can be applied to one of our specific bases. Indeed, for the trigonometric case, we get the following result.

Corollary 5.3. *Consider the estimator \widehat{b}' computed in the trigonometric basis on $I = [0, 1]$ with $0 < f_0 \leq f(x) \leq \|f\|_\infty < \infty$ under Assumption 3.7(m). Moreover, assume that there exists $\kappa > 0$ such that $\mathbb{E}(\exp(\kappa\varepsilon_1^2)) < \infty$ and that b' is square-integrable on I . If $b \in W^{\text{per}}(\beta, L)$ with $\beta > 1$, then*

$$\mathbb{E}(\|\widehat{b}' - b'\|_n^2) \leq \mathbf{c}(f_0, \|f\|_\infty, L) n^{-2(\beta-1)/(2\beta+1)}.$$

Therefore, our adaptive estimator automatically reaches the optimal rate, up to a multiplicative constant, in the compactly supported setting associated to the trigonometric basis. For the Hermite case, we obtain the following bound.

Corollary 5.4. *Consider the estimator \widehat{b}' computed in the Hermite basis on $I = \mathbb{R}$ under Assumptions 3.1 and 3.7($m+1$). Assume also that b' is square integrable on I , $\mathbb{E}[b'(X_1)^4] < \infty$ and that there exists $\kappa > 0$ such that $\mathbb{E}(\exp(\kappa\varepsilon_1^2)) < \infty$. If $\|\Psi_m^{-1}\|_{\text{op}} \lesssim m^\gamma$ for every $m \in \{1, \dots, n\}$, and if $b \in W_H^s(D)$ with $s > 2\gamma + 9/4$, then*

$$\mathbb{E}(\|\widehat{b}' - b'\|_n^2) \leq \mathbf{c}(D, s, \|f\|_\infty, \sigma^2) n^{-2(s-1-\gamma)/(2s+1)}.$$

As a consequence, the Hermite estimator also reaches automatically the best rate we could expect, in the difficult context of non compact setting, but under stronger conditions.

6. A NUMERICAL EXPLORATION

		$n = 250$		$n = 1000$		$n = 4000$	
		Herm	Trigo	Herm	Trigo	Herm	Trigo
b_1	100 MSE	0.91 _(0.50)	0.82 _(0.46)	0.23 _(0.13)	0.23 _(0.13)	0.06 _(0.03)	0.07 _(0.03)
	dim	10.7 _(1.30)	4.94 _(0.93)	13.1 _(1.46)	6.13 _(1.30)	15.2 _(2.05)	7.40 _(1.45)
b'_1	100 MSE	11.1 _(6.27)	8.75 _(5.11)	2.97 _(1.79)	3.16 _(1.92)	0.85 _(0.47)	1.24 _(0.62)
	dim	10.8 _(1.01)	4.76 _(0.76)	12.9 _(1.37)	5.99 _(1.12)	15.2 _(2.01)	7.30 _(1.34)
b_2	100 MSE	0.07 _(0.10)	0.44 _(0.34)	0.02 _(0.03)	0.12 _(0.07)	0.005 _(0.006)	0.03 _(0.02)
	dim	1.03 _(0.22)	2.53 _(0.90)	1.03 _(0.22)	2.90 _(0.93)	1.01 _(0.15)	3.45 _(0.93)
b'_2	100 MSE	0.03 _(0.04)	1.54 _(1.22)	0.007 _(0.01)	0.50 _(0.31)	0.002 _(0.003)	0.14 _(0.10)
	dim	112 _(0.51)	2.38 _(0.64)	1.08 _(0.38)	2.75 _(0.69)	1.08 _(0.42)	3.16 _(0.55)
b_3	MSE	1.14 _(0.91)	0.64 _(0.55)	0.25 _(0.13)	0.19 _(0.10)	0.06 _(0.03)	0.06 _(0.03)
	dim	11.3 _(1.15)	3.69 _(1.16)	14.1 _(1.35)	5.12 _(1.77)	16.7 _(1.44)	7.18 _(2.07)
b'_3	100 MSE	18.8 _(24.7)	4.96 _(9.84)	3.89 _(2.69)	2.31 _(2.33)	1.08 _(0.71)	1.21 _(0.73)
	dim	11.4 _(1.15)	3.53 _(1.01)	14.2 _(1.30)	4.88 _(1.43)	16.7 _(1.42)	6.62 _(1.66)
b_4	100 MSE	0.68 _(0.37)	0.82 _(0.44)	0.21 _(0.11)	0.24 _(0.12)	0.06 _(0.03)	0.07 _(0.03)
	dim	8.71 _(2.21)	5.20 _(0.92)	11.6 _(2.21)	6.30 _(1.32)	14.9 _(2.43)	7.42 _(1.41)
b'_4	100 MSE	7.60 _(3.48)	10.8 _(5.48)	2.65 _(1.18)	3.40 _(1.57)	0.93 _(0.43)	1.17 _(0.55)
	dim	9.17 _(2.02)	5.11 _(0.75)	11.8 _(1.91)	6.17 _(1.11)	15.2 _(1.99)	7.28 _(0.94)

TABLE 1. "100 MSE": MSE of the oracle (for b and b' , defined by (16)) multiplied by 100 with standard deviations (Std) multiplied by 100 in small parenthesis. "dim": mean of the oracle dimensions with Std in small parenthesis. Columns "Herm" correspond to the Hermite basis, columns "Trigo" to the half trigonometric basis. 400 repetitions

Due to the theoretical difficulty of the question, in a model which looked rather simple at first sight, we wondered if the strategy consisting in derivating the least squares regression estimator was relevant, and if numerical investigations could bring information about a good estimation strategy. This is why we choose to look at oracles: we compute all estimators of the collection and use the knowledge of the true function to compute the error associated to all of them in order to select the best one in term of its \mathbb{L}^2 -distance to the true. We also look with careful interest at the associated dimensions. We considered the four simple functions

$$(16) \quad b_1(x) = 2 \sin(\pi x), \quad b_2(x) = \exp(-x^2/2), \quad b_3(x) = x^2, \quad b_4(x) = 4x/(1+x^2),$$

and we generated $Y_i = b(X_i) + \varepsilon_i$, $i = 1, \dots, n$, for i.i.d. $X_i \sim \mathcal{N}(0, 1)$, independent of the i.i.d $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$, with $\sigma = 0.25$ and $b = b_j$, $j = 1, \dots, 4$. For each sample, we compute the least squares estimator of b , together with its derivative, in the Hermite and in the trigonometric bases. We use what we call the "half" trigonometric basis, relying on functions $x \mapsto \sqrt{2} \sin(\pi j x)$ and $x \mapsto \sqrt{2} \cos(\pi j x)$ on $[0, 1]$, rescaled to the interval $[a, b]$. For each function b , we considered $K = 400$ repetitions, and samples of sizes $n = 250, 1000$ and 4000 . We computed the \mathbb{L}^2 -distance between each oracle estimator of b and the true b , and each oracle estimator of b' and the true b' , on an interval with bounds corresponding to the 3% and 97% quantiles of the X_i 's, and finally take the average on the 400 paths. Moreover, we average the selected dimensions for each sample. In other words, we retain the dimension and error corresponding in each case to the smallest error, and compute means and standard deviations. The results are reported in Table 1.

Table 1 shows that the MSE decreases when n increases, in all cases, and whether b or b' is estimated. The orders associated with the MSE given are more concretely illustrated in Figures 1 and 2, and we can see that the estimations are very satisfactory. We can notice that function b_1 is chosen to be easy for the trigonometric basis, but the Hermite basis performs very well in this case too. On the contrary, the function b_2 is supposed to be easy for the Hermite basis, and it is, with small selected dimensions, but the

trigonometric basis has a much worse performance. For the two other functions, the two bases perform similarly, with decreasing error when increasing n and simultaneous increase of the selected dimensions. This is expected from the theoretical formula giving the asymptotic optimal choice of m as a power of n , at least when the function under estimation does not admit a finite decomposition in the basis.

What is really puzzling in these results is the comparison of oracle dimensions for b and b' : in each case, they are almost the same. As penalization methods require an important work for calibrating constants, especially for Goldenshluger and Lepski strategies, this suggests to propose model selection for the estimation of b , and use this for b' as well. This idea is rather intuitive, but far from easy to prove. This is why we studied a different theoretical method, which reaches optimal rates for trigonometric basis, but would be difficult to calibrate. This section advocates for a useful and practical strategy, since model selection for the regression function estimation is adequately calibrated in practice. However, our numerical results consider only few elementary functions and may require to be confirmed in a larger numerical study, which beyond the scope of this paper.

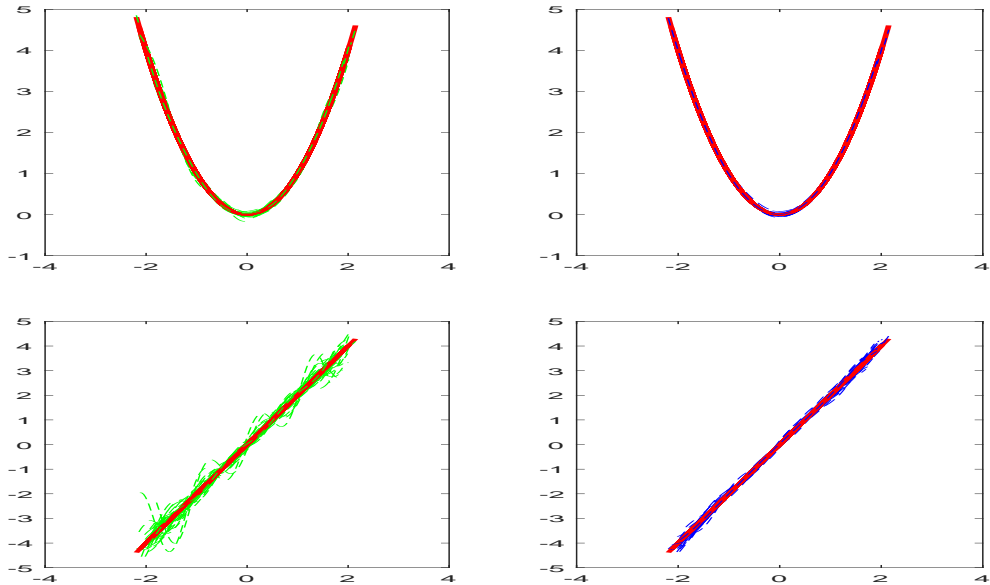


FIGURE 1. 40 oracles compared to the true, $n = 250$. First line b_3 (see (16)), 100 MSE = 1.22 and 0.51, mean selected dimensions: 11.2 and 3.6. Second line b'_3 , 100 MSE = 20.9 and 3.53, mean selected dimensions: 11.2 and 3.40. Left Hermite basis, right trigonometric basis.

7. CONCLUDING REMARKS

In this paper, we have defined two projection estimators of the derivative of b , based on observations $(X_i, Y_i)_{1 \leq i \leq n}$ drawn from Model (1). Under weak assumptions, we prove two simple risk bounds allowing to understand the differences between the two strategies. More elaborate bounds under a stability condition introduced by Cohen *et al.*[8] are also given. These results are concretely illustrated in the context of trigonometric, Legendre, Laguerre and Hermite bases, the first two ones being compactly supported, but not the last ones. Optimal rates are recovered with our method in the context of the trigonometric basis, but our setting is more general, which is a novelty. Lastly, we propose a model selection procedure and prove a general risk bound for the adaptive estimator. It automatically reaches the optimal rate in the trigonometric case. These last results are also new and not straightforward.

The method we propose should be implemented and numerical experiments conducted to compare the two estimators here proposed to other methods existing in the literature. Our elementary investigation

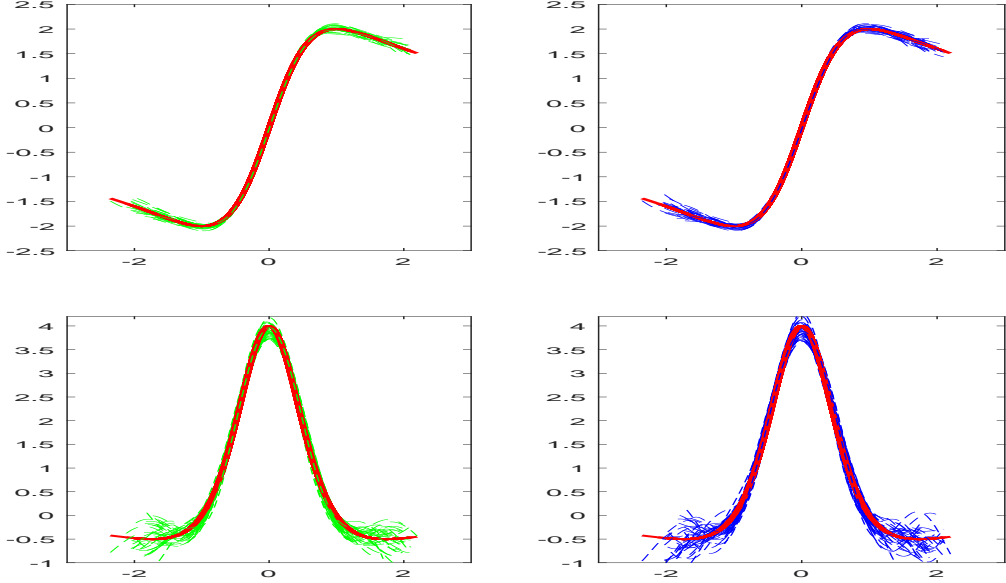


FIGURE 2. 40 oracles compared to the true, $n = 250$. First line b_4 (see (16)), 100 MSE = 0.71 and 0.89, mean selected dimensions: 8.7 and 4.9. Second line b'_4 , 100 MSE = 7.94 and 11.0, mean selected dimensions: 9.02 and 5.00. Left Hermite basis, right trigonometric basis.

for simple examples suggests that the method can work as the collection of estimators always contains a good one. It also suggests that keeping for the estimation of b' the dimension selected for b may be a safe simple strategy. Several extensions of this work may be of obvious interest: explanatory variables with higher dimensions may be studied as well as higher order of derivatives, possibly only in the compactly supported case to begin with. Extensions to dependent contexts (the case of autoregressive models or the case of diffusion models) are also to be considered. As our proofs rely on results conditionally to the X_i 's, thanks to their independence with the noise, dependency should imply theoretical difficulties.

8. PROOFS

All the properties on matrix norms used in proofs are reminded in the Subsubsection 2.2 at the end of the introduction.

8.1. **Proof of Proposition 3.2.** Note first that

$$\widehat{b}'_m{}^1(\mathbf{X}) = \widehat{\Phi}'_m (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^* \mathbf{Y},$$

and since $Y_i = b(X_i) + \varepsilon_i$, X_i is independent of ε_i , and $\mathbb{E}(\varepsilon_i) = 0$ for every $i \in \{1, \dots, n\}$,

$$\mathbb{E}_{\mathbf{X}}[\widehat{b}'_m{}^1(\mathbf{X})] = \widehat{\Phi}'_m (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^* b(\mathbf{X})$$

and

$$\mathbb{E}_{\mathbf{X}}(\langle \widehat{\Phi}'_m (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^* \varepsilon, \widehat{\Phi}'_m (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^* b(\mathbf{X}) - b'(\mathbf{X}) \rangle_n) = 0.$$

Then,

$$\begin{aligned} \mathbb{E}_{\mathbf{X}}(\|\widehat{b}'_m{}^1 - b'\|_n^2) &= \frac{1}{n} \mathbb{E}_{\mathbf{X}}[\|\widehat{b}'_m{}^1(\mathbf{X}) - \mathbb{E}_{\mathbf{X}}[\widehat{b}'_m{}^1(\mathbf{X})] + \mathbb{E}_{\mathbf{X}}[\widehat{b}'_m{}^1(\mathbf{X})] - b'(\mathbf{X})\|_{2,n}^2] \\ &= \frac{1}{n} [\mathbb{E}_{\mathbf{X}}(\|\widehat{\Phi}'_m (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^* \varepsilon\|_{2,n}^2) + \|\widehat{\Phi}'_m (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^* b(\mathbf{X}) - b'(\mathbf{X})\|_{2,n}^2] \\ &=: \frac{1}{n} (A + B). \end{aligned}$$

On the one hand,

$$\begin{aligned} A &= \mathbb{E}_{\mathbf{X}} \left[\varepsilon^* \widehat{\Phi}_m (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} (\widehat{\Phi}'_m)^* \widehat{\Phi}'_m (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^* \varepsilon \right] \\ &= \sigma^2 \text{Tr} \left[\widehat{\Phi}_m (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} (\widehat{\Phi}'_m)^* \widehat{\Phi}'_m (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^* \right] \\ &= \sigma^2 \text{Tr} \left[(\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} (\widehat{\Phi}'_m)^* \widehat{\Phi}'_m \right]. \end{aligned}$$

On the other hand,

$$B \leq 3 \|\widehat{\Phi}'_m (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^* b(\mathbf{X}) - b'_m(\mathbf{X})\|_{2,n}^2 + 3 \|b'_m(\mathbf{X}) - (b')_m(\mathbf{X})\|_{2,n}^2 + 3 \|(b')_m(\mathbf{X}) - b'(\mathbf{X})\|_{2,n}^2.$$

So,

$$\frac{1}{n} \mathbb{E}(B) \leq 3 \inf_{t \in \mathcal{S}_m} \|t - b'\|_f^2 + 3 \|b'_m - (b')_m\|_f^2 + 3 \mathbb{E}(C)$$

with

$$C = \frac{1}{n} \|\widehat{\Phi}'_m (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^* b(\mathbf{X}) - b'_m(\mathbf{X})\|_{2,n}^2.$$

In order to manage this last term, note that

$$b_m(\mathbf{X}) = \sum_{j=1}^m \langle b, \varphi_j \rangle \varphi_j(\mathbf{X}) = \widehat{\Phi}_m (\langle b, \varphi_j \rangle)_{1 \leq j \leq m}.$$

So,

$$(17) \quad (\langle b, \varphi_j \rangle)_{1 \leq j \leq m} = (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^* b_m(\mathbf{X})$$

and then,

$$b'_m(\mathbf{X}) = \sum_{j=1}^m \langle b, \varphi_j \rangle \varphi'_j(\mathbf{X}) = \widehat{\Phi}'_m (\langle b, \varphi_j \rangle)_{1 \leq j \leq m} = \widehat{\Phi}'_m (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^* b_m(\mathbf{X}).$$

Therefore,

$$C = \frac{1}{n} \|\widehat{\Phi}'_m (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^* (b(\mathbf{X}) - b_m(\mathbf{X}))\|_{2,n}^2 \leq \|\widehat{\Phi}'_m (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^*\|_{\text{op}}^2 \|b - b_m\|_n^2.$$

This concludes the proof. \square

8.2. Proof of Proposition 3.3. The φ_j 's do not depend on m , so the \mathcal{S}_m 's are nested spaces, and then to establish the following equality is sufficient in order to conclude:

$$(18) \quad \mathbb{E}_{\mathbf{X}} \left(\sup_{t \in \mathcal{S}_m: \|t\|_n = 1} \nu_n(t)^2 \right) = \frac{\sigma^2}{n} \text{Tr} \left[\widehat{\Psi}_m^{-1/2} \widehat{\Psi}'_m \widehat{\Psi}_m^{-1/2} \right]$$

with

$$\widehat{\Psi}'_m := \frac{1}{n} (\widehat{\Phi}'_m)^* \widehat{\Phi}'_m \text{ and } \nu_n(t) := \langle \varepsilon, t' \rangle_n.$$

Let us prove Equality (18). Consider $t \in \mathcal{S}_m$ such that $\|t\|_n = 1$. Necessarily (and sufficiently),

$$t = \sum_{j=1}^m a_j \varphi_j$$

with $a = \widehat{\Psi}_m^{-1/2} u$ and $u \in \mathbb{R}^m$ such that $\|u\|_{2,m} = 1$. Then,

$$t = \sum_{k=1}^m u_k \sum_{j=1}^m [\widehat{\Psi}_m^{-1/2}]_{j,k} \varphi_j$$

and, thanks to Cauchy-Schwarz's inequality,

$$\nu_n(t)^2 = \langle \varepsilon, t' \rangle_n^2 = \left[\sum_{k=1}^m u_k \left\langle \varepsilon, \sum_{j=1}^m [\widehat{\Psi}_m^{-1/2}]_{j,k} \varphi'_j \right\rangle_n \right]^2 \leq \sum_{k=1}^m \left\langle \varepsilon, \sum_{j=1}^m [\widehat{\Psi}_m^{-1/2}]_{j,k} \varphi'_j \right\rangle_n^2.$$

So,

$$\sup_{t \in \mathcal{S}_m: \|t\|_n=1} \nu_n(t)^2 = \sup_{u \in \mathbb{R}^m: \|u\|_{2,m}=1} \left[\sum_{k=1}^m u_k \left\langle \varepsilon, \sum_{j=1}^m [\widehat{\Psi}_m^{-1/2}]_{j,k} \varphi'_j \right\rangle_n \right]^2 = \sum_{k=1}^m \left\langle \varepsilon, \sum_{j=1}^m [\widehat{\Psi}_m^{-1/2}]_{j,k} \varphi'_j \right\rangle_n^2.$$

Therefore, since $\varepsilon_1, \dots, \varepsilon_n$ are i.i.d, centered, and respectively independent of X_1, \dots, X_n , and since $\widehat{\Psi}_m^{-1/2}$ and $\widehat{\Psi}'_m$ are symmetric matrices,

$$\begin{aligned} \mathbb{E}_{\mathbf{X}} \left(\sup_{t \in \mathcal{S}_m: \|t\|_n=1} \nu_n(t)^2 \right) &= \frac{\sigma^2}{n^2} \sum_{k=1}^m \sum_{i=1}^n \left(\sum_{j=1}^m [\widehat{\Psi}_m^{-1/2}]_{j,k} \varphi'_j(X_i) \right)^2 \\ &= \frac{\sigma^2}{n} \sum_{j,k,\ell=1}^m [\widehat{\Psi}_m^{-1/2}]_{j,k} [\widehat{\Psi}_m^{-1/2}]_{\ell,k} \langle \varphi'_j, \varphi'_\ell \rangle_n \\ &= \frac{\sigma^2}{n} \sum_{j,k,\ell=1}^m [\widehat{\Psi}_m^{-1/2}]_{k,j} [\widehat{\Psi}'_m]_{j,\ell} [\widehat{\Psi}_m^{-1/2}]_{\ell,k} = \frac{\sigma^2}{n} \text{Tr} \left[\widehat{\Psi}_m^{-1/2} \widehat{\Psi}'_m \widehat{\Psi}_m^{-1/2} \right]. \end{aligned}$$

This concludes the proof. \square

8.3. Proof of Proposition 3.6. As in the proof of Proposition 3.2,

$$\begin{aligned} \mathbb{E}_{\mathbf{X}} (\|\widehat{b}'_m - b'\|_n^2) &= \frac{1}{n} \left\| -\widehat{\Phi}_m \Delta_{m,m+p} (\widehat{\Phi}_{m+p}^* \widehat{\Phi}_{m+p})^{-1} \widehat{\Phi}_{m+p}^* b(\mathbf{X}) - b'(\mathbf{X}) \right\|_{2,n}^2 \\ &\quad + \frac{\sigma^2}{n} \text{Tr} \left[(\widehat{\Phi}_{m+p}^* \widehat{\Phi}_{m+p})^{-1} \Delta_{m,m+p}^* \widehat{\Phi}_m^* \widehat{\Phi}_m \Delta_{m,m+p} \right] \\ &\leq \frac{2}{n} \left[\left\| -\widehat{\Phi}_m \Delta_{m,m+p} (\widehat{\Phi}_{m+p}^* \widehat{\Phi}_{m+p})^{-1} \widehat{\Phi}_{m+p}^* b(\mathbf{X}) - (b')_m(\mathbf{X}) \right\|_{2,n}^2 + \|(b')_m(\mathbf{X}) - b'(\mathbf{X})\|_{2,n}^2 \right] \\ &\quad + \frac{\sigma^2}{n} \text{Tr} \left[(\widehat{\Phi}_{m+p}^* \widehat{\Phi}_{m+p})^{-1} \Delta_{m,m+p}^* \widehat{\Phi}_m^* \widehat{\Phi}_m \Delta_{m,m+p} \right]. \end{aligned}$$

On the one hand, as previously,

$$\frac{2}{n} \mathbb{E} (\|(b')_m(\mathbf{X}) - b'(\mathbf{X})\|_{2,n}^2) = 2 \|(b')_m - b'\|_f^2 \leq 2 \|f\|_\infty \inf_{t \in \mathcal{S}_m} \|t - b'\|^2.$$

On the other hand, thanks to Equalities (6) and (17),

$$(b')_m(\mathbf{X}) = -\widehat{\Phi}_m \Delta_{m,m+p} (\langle b, \varphi_j \rangle)_{1 \leq j \leq m+p} = -\widehat{\Phi}_m \Delta_{m,m+p} (\widehat{\Phi}_{m+p}^* \widehat{\Phi}_{m+p})^{-1} \widehat{\Phi}_{m+p}^* b_{m+p}(\mathbf{X}).$$

Then,

$$\begin{aligned} \left\| -\widehat{\Phi}_m \Delta_{m,m+p} (\widehat{\Phi}_{m+p}^* \widehat{\Phi}_{m+p})^{-1} \widehat{\Phi}_{m+p}^* b(\mathbf{X}) - (b')_m(\mathbf{X}) \right\|_{2,n}^2 \\ = \left\| -\widehat{\Phi}_m \Delta_{m,m+p} (\widehat{\Phi}_{m+p}^* \widehat{\Phi}_{m+p})^{-1} \widehat{\Phi}_{m+p}^* (b(\mathbf{X}) - b_{m+p}(\mathbf{X})) \right\|_{2,n}^2. \end{aligned}$$

This concludes the proof. \square

8.4. Proof of Proposition 3.8. Consider the following set

$$\Omega_m := \left\{ \left| \|t\|_n^2 / \|t\|_f^2 - 1 \right| \leq \frac{1}{2} ; \forall t \in \mathcal{S}_m \right\} = \left\{ \|\Psi_m^{-1/2} \widehat{\Psi}_m \Psi_m^{-1/2} - \mathbf{I}_m\|_{\text{op}} \leq \frac{1}{2} \right\}.$$

The proof relies on the following lemma, borrowed from Comte and Genon-Catalot [11, Lemma 5].

Lemma 8.1. *Under Assumption 3.7(m), there exists a deterministic constant $\mathfrak{c}_{8.1} > 0$, not depending on m and n , such that*

$$\mathbb{P}(\Omega_m^c) \leq \frac{\mathfrak{c}_{8.1}}{n^8} \quad \text{and} \quad \mathbb{P}(\Lambda_m^c) \leq \frac{\mathfrak{c}_{8.1}}{n^8}.$$

First of all,

$$\mathbb{E} \left[\|\widehat{b}'_m - b'\|_n^2 \right] = \mathbb{E} \left[\|\widehat{b}'_m - b'\|_n^2 \mathbf{1}_{\Lambda_{m+p}} \right] + \mathbb{E}(\|b'\|_n^2 \mathbf{1}_{\Lambda_{m+p}^c}).$$

Obviously, by applying Lemma 8.1, since $\mathbb{E}[b'(X_1)^4] < \infty$,

$$\mathbb{E}(\|b'\|_n^2 \mathbf{1}_{\Lambda_{m+p}^c}) \leq \mathbb{E}[b'(X_1)^4]^{1/2} \mathbb{P}(\Lambda_{m+p}^c)^{1/2} \leq \mathbf{c}_{8.1}^{1/2} \mathbb{E}[b'(X_1)^4]^{1/2} \frac{1}{n^4}.$$

Let us dissect $\|\widehat{b}'_m - b'\|_n^2 \mathbf{1}_{\Lambda_{m+p}}$ via the event Ω_{m+p} :

$$\begin{aligned} \mathbb{E} \left[\|\widehat{b}'_m - b'\|_n^2 \mathbf{1}_{\Lambda_{m+p}} \right] &= \mathbb{E} \left[\|\widehat{b}'_m - b'\|_n^2 \mathbf{1}_{\Lambda_{m+p} \cap \Omega_{m+p}} \right] + \mathbb{E} \left[\|\widehat{b}'_m - b'\|_n^2 \mathbf{1}_{\Lambda_{m+p} \cap \Omega_{m+p}^c} \right] \\ &\leq \mathbb{E} \left[\|\widehat{b}'_m - b'\|_n^2 \mathbf{1}_{\Lambda_{m+p} \cap \Omega_{m+p}} \right] \\ &\quad + 2 \left[\mathbb{E}(\|\widehat{b}'_m\|_n^4 \mathbf{1}_{\Lambda_{m+p}})^{1/2} + \mathbb{E}(\|b'\|_n^4)^{1/2} \right] \mathbb{P}(\Omega_{m+p}^c)^{1/2} =: S + T. \end{aligned}$$

On the one hand, let us find suitable bounds on the two remaining terms:

- For every measurable function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ and $q \in [1, \infty[$ such that $\mathbb{E}(\psi(X_1)^{2q}) < \infty$, by Jensen's inequality,

$$(19) \quad \mathbb{E}(\|\psi\|_n^{2q}) = \mathbb{E} \left[\left(\frac{1}{n} \sum_{i=1}^n \psi^2(X_i) \right)^q \right] \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E}[(\psi(X_i))^{2q}] = \mathbb{E}(\psi^{2q}(X_1)).$$

Then, $\mathbb{E}(\|b'\|_n^4) \leq \mathbb{E}(b'(X_1)^4)$.

- Recall that

$$\widehat{b}'_m(\mathbf{X}) = \widehat{\Phi}'_m (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^* \mathbf{Y} = \widehat{\Phi}_{m+p} \Delta_{m,m+p}^* (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^* \mathbf{Y}.$$

First,

$$\begin{aligned} \|\widehat{\Phi}_{m+p} \Delta_{m,m+p}^* (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^*\|_{\text{op}}^2 &= \lambda_{\max}(\widehat{\Phi}_{m+p} \Delta_{m,m+p}^* (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \Delta_{m,m+p} \widehat{\Phi}_m^*) \\ &= n^{-1} \|\widehat{\Phi}_{m+p} \Delta_{m,m+p}^* \widehat{\Psi}_m^{-1/2}\|_{\text{op}}^2 \\ &\leq n^{-1} \|\widehat{\Psi}_m^{-1/2}\|_{\text{op}}^2 \|\widehat{\Phi}_{m+p} \Delta_{m,m+p}^*\|_{\text{op}}^2 \\ &= \|\widehat{\Psi}_m^{-1}\|_{\text{op}} \lambda_{\max}(\Delta_{m,m+p} \widehat{\Psi}_{m+p} \Delta_{m,m+p}^*) \\ (20) \quad &\leq \|\widehat{\Psi}_m^{-1}\|_{\text{op}} \|\widehat{\Psi}_{m+p}\|_{\text{op}} \|\Delta_{m,m+p}\|_{\text{op}}^2. \end{aligned}$$

Moreover, $\|\widehat{\Psi}_{m+p}\|_{\text{op}} \leq \mathfrak{L}(m+p)$ and $\mathfrak{L}(m) \|\widehat{\Psi}_m^{-1}\|_{\text{op}} \leq \mathfrak{L}(m+p) \|\widehat{\Psi}_{m+p}^{-1}\|_{\text{op}} \leq \mathbf{c}n / \log(n)$ on Λ_{m+p} . Then,

$$\begin{aligned} \mathbb{E}(\|\widehat{b}'_m\|_n^4 \mathbf{1}_{\Lambda_{m+p}}) &\leq \frac{1}{n^2} \mathbb{E}(\|\widehat{\Psi}_m^{-1}\|_{\text{op}}^2 \|\widehat{\Psi}_{m+p}\|_{\text{op}}^2 \mathbf{1}_{\Lambda_{m+p}} \|\mathbf{Y}\|_{2,n}^4) \|\Delta_{m,m+p}\|_{\text{op}}^4 \\ &\leq \frac{\mathbf{c}^2 n^2}{\log(n)^2} \|\Delta_{m,m+p}\|_{\text{op}}^4 \mathbb{E} \left[\left(\frac{1}{n} \sum_{i=1}^n Y_i^2 \right)^2 \right] \leq \frac{\mathbf{c}^2 n^2}{\log(n)^2} \|\Delta_{m,m+p}\|_{\text{op}}^4 \mathbb{E}(Y_1^4). \end{aligned}$$

Thus, thanks to Lemma 8.1,

$$\begin{aligned} T &= 2 \left[\mathbb{E}(\|\widehat{b}'_m\|_n^4 \mathbf{1}_{\Lambda_{m+p}})^{1/2} + \mathbb{E}(\|b'\|_n^4)^{1/2} \right] \mathbb{P}(\Omega_{m+p}^c)^{1/2} \\ &\leq 2 \left[\frac{\mathbf{c}n}{\log(n)} \|\Delta_{m,m+p}\|_{\text{op}}^2 \mathbb{E}(Y_1^4)^{1/2} + \mathbb{E}(b'(X_1)^4)^{1/2} \right] \frac{\mathbf{c}_{8.1}^{1/2}}{n^4}. \end{aligned}$$

On the other hand, with the exact same ideas as in the proof of Proposition 3.2,

$$\begin{aligned} S - 3\|f\|_{\infty} \inf_{t \in \mathcal{S}_m} \|t - b'\|_f^2 - 3\|b'_m - (b')_m\|_f^2 \\ \leq 3 \mathbb{E} \left[\|\widehat{\Phi}_{m+p} \Delta_{m,m+p}^* (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^*\|_{\text{op}}^2 \|b - b_m\|_n^2 \mathbf{1}_{\Lambda_{m+p} \cap \Omega_{m+p}} \right] \\ + \frac{\sigma^2}{n} \mathbb{E} \left[\text{Tr} \left((\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \Delta_{m,m+p} \widehat{\Phi}_{m+p}^* \widehat{\Phi}_{m+p} \Delta_{m,m+p}^* \right) \mathbf{1}_{\Lambda_{m+p} \cap \Omega_{m+p}} \right] =: S_1 + S_2. \end{aligned}$$

Let us find suitable bounds on S_1 and S_2 :

- On Ω_{m+p} , the eigenvalues of $\Psi_{m+p}^{-1/2} \widehat{\Psi}_{m+p} \Psi_{m+p}^{-1/2}$ belong to $[1/2, 3/2]$. The same way, on Ω_m , the eigenvalues of $\Psi_m^{-1/2} \widehat{\Psi}_m \Psi_m^{-1/2}$ belong to $[1/2, 3/2]$ and then, those of the matrix $\Psi_m^{1/2} \widehat{\Psi}_m^{-1} \Psi_m^{1/2}$ belong to $[2/3, 2]$. So, on Ω_{m+p} , $\widehat{S}_1 := \|\widehat{\Phi}_{m+p} \Delta_{m,m+p}^* (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \widehat{\Phi}_m^*\|_{\text{op}}^2$ satisfies

$$\begin{aligned}
\widehat{S}_1 &= \lambda_{\max}(\widehat{\Phi}_{m+p} \Delta_{m,m+p}^* (\widehat{\Phi}_m^* \widehat{\Phi}_m)^{-1} \Delta_{m,m+p} \widehat{\Phi}_{m+p}^*) \\
&= \lambda_{\max}(\Delta_{m,m+p}^* \widehat{\Psi}_m^{-1} \Delta_{m,m+p} \widehat{\Psi}_{m+p}) \\
&= \lambda_{\max}(\Psi_{m+p}^{-1/2} \Delta_{m,m+p}^{f,1} \Psi_m^{1/2} \widehat{\Psi}_m^{-1} \Psi_m^{1/2} (\Delta_{m,m+p}^{f,1})^* \Psi_{m+p}^{-1/2} \widehat{\Psi}_{m+p}) \\
&= \lambda_{\max}((\Psi_{m+p}^{-1/2} \widehat{\Psi}_{m+p} \Psi_{m+p}^{-1/2})^{1/2} \Delta_{m,m+p}^{f,1} \Psi_m^{1/2} \widehat{\Psi}_m^{-1} \Psi_m^{1/2} (\Delta_{m,m+p}^{f,1})^* (\Psi_{m+p}^{-1/2} \widehat{\Psi}_{m+p} \Psi_{m+p}^{-1/2})^{1/2}) \\
&= \|(\Psi_{m+p}^{-1/2} \widehat{\Psi}_{m+p} \Psi_{m+p}^{-1/2})^{1/2} \Delta_{m,m+p}^{f,1} \Psi_m^{1/2} \widehat{\Psi}_m^{-1} \Psi_m^{1/2} (\Delta_{m,m+p}^{f,1})^* (\Psi_{m+p}^{-1/2} \widehat{\Psi}_{m+p} \Psi_{m+p}^{-1/2})^{1/2}\|_{\text{op}} \\
&\leq \|\Psi_{m+p}^{-1/2} \widehat{\Psi}_{m+p} \Psi_{m+p}^{-1/2}\|_{\text{op}} \|\Delta_{m,m+p}^{f,1}\|_{\text{op}}^2 \|\Psi_m^{1/2} \widehat{\Psi}_m^{-1} \Psi_m^{1/2}\|_{\text{op}} \\
(21) \quad &\leq 3 \|\Delta_{m,m+p}^{f,1}\|_{\text{op}}^2.
\end{aligned}$$

Thus,

$$S_1 \leq 9 \|\Delta_{m,m+p}^{f,1}\|_{\text{op}}^2 \mathbb{E}(\|b - b_m\|_n^2) = 9 \|\Delta_{m,m+p}^{f,1}\|_{\text{op}}^2 \|b - b_m\|_f^2.$$

- As previously, since the eigenvalues of $\Psi_m^{1/2} \widehat{\Psi}_m^{-1} \Psi_m^{1/2} \mathbf{1}_{\Omega_{m+p}}$ belong to $[2/3, 2]$,

$$\begin{aligned}
S_2 &= \frac{\sigma^2}{n} \mathbb{E} \left[\text{Tr} \left(\Psi_m^{1/2} \widehat{\Psi}_m^{-1} \Psi_m^{1/2} (\Delta_{m,m+p}^{f,1})^* \Psi_{m+p}^{-1/2} \widehat{\Psi}_{m+p} \Psi_{m+p}^{-1/2} \Delta_{m,m+p}^{f,1} \right) \mathbf{1}_{\Lambda_{m+p} \cap \Omega_{m+p}} \right] \\
&\leq \frac{\sigma^2}{n} \mathbb{E} \left[\|\Psi_m^{1/2} \widehat{\Psi}_m^{-1} \Psi_m^{1/2}\|_{\text{op}} \text{Tr} \left((\Delta_{m,m+p}^{f,1})^* \Psi_{m+p}^{-1/2} \widehat{\Psi}_{m+p} \Psi_{m+p}^{-1/2} \Delta_{m,m+p}^{f,1} \right) \mathbf{1}_{\Omega_{m+p}} \right] \\
&\leq \frac{2\sigma^2}{n} \text{Tr} \left[(\Delta_{m,m+p}^{f,1})^* \Psi_{m+p}^{-1/2} \mathbb{E}(\widehat{\Psi}_{m+p}) \Psi_{m+p}^{-1/2} \Delta_{m,m+p}^{f,1} \right] = \frac{2\sigma^2}{n} \|\Delta_{m,m+p}^{f,1}\|_F^2.
\end{aligned}$$

The result follows by gathering all the terms. \square

8.5. Proof of Proposition 3.9.

First of all,

$$\mathbb{E} \left[\|\widehat{b}_m'^2 - b'\|_n^2 \right] = \mathbb{E} \left[\|\widehat{b}_m'^2 - b'\|_n^2 \mathbf{1}_{\Lambda_{m+p}} \right] + \mathbb{E}(\|b'\|_n^2 \mathbf{1}_{\Lambda_{m+p}^c}).$$

Obviously, by applying Lemma 8.1, since $\mathbb{E}[b'(X_1)^4] < \infty$,

$$\mathbb{E}(\|b'\|_n^2 \mathbf{1}_{\Lambda_{m+p}^c}) \leq \mathbb{E}[b'(X_1)^4]^{1/2} \mathbb{P}(\Lambda_{m+p}^c)^{1/2} \leq \mathfrak{c}_{8.1}^{1/2} \mathbb{E}[b'(X_1)^4]^{1/2} \frac{1}{n^4}.$$

Let us dissect $\|\widehat{b}_m'^2 - b'\|_n^2 \mathbf{1}_{\Lambda_{m+p}}$ via the event Ω_{m+p} :

$$\begin{aligned}
\mathbb{E} \left[\|\widehat{b}_m'^2 - b'\|_n^2 \mathbf{1}_{\Lambda_{m+p}} \right] &= \mathbb{E} \left[\|\widehat{b}_m'^2 - b'\|_n^2 \mathbf{1}_{\Lambda_{m+p} \cap \Omega_{m+p}} \right] + \mathbb{E} \left[\|\widehat{b}_m'^2 - b'\|_n^2 \mathbf{1}_{\Lambda_{m+p} \cap \Omega_{m+p}^c} \right] \\
&\leq \mathbb{E} \left[\|\widehat{b}_m'^2 - b'\|_n^2 \mathbf{1}_{\Lambda_{m+p} \cap \Omega_{m+p}} \right] \\
&\quad + 2 \left[\mathbb{E}(\|\widehat{b}_m'^2\|_n^4 \mathbf{1}_{\Lambda_{m+p}})^{1/2} + \mathbb{E}(\|b'\|_n^4)^{1/2} \right] \mathbb{P}(\Omega_{m+p}^c)^{1/2} =: S + T.
\end{aligned}$$

On the one hand, let us find suitable bounds on the two remaining terms:

- As in the proof of Proposition 3.8, thanks to Inequality (19), $\mathbb{E}(\|b'\|_n^4) \leq \mathbb{E}(b'(X_1)^4)$.
- Recall that

$$\widehat{b}_m'^2(\mathbf{X}) = \widehat{\Phi}_m \Delta_{m,m+p} (\widehat{\Phi}_{m+p}^* \widehat{\Phi}_{m+p})^{-1} \widehat{\Phi}_{m+p}^* \mathbf{Y}.$$

First,

$$\begin{aligned}
\|\widehat{\Phi}_m \Delta_{m,m+p} (\widehat{\Phi}_{m+p}^* \widehat{\Phi}_{m+p})^{-1} \widehat{\Phi}_{m+p}^*\|_{\text{op}}^2 &= \lambda_{\max}(\widehat{\Phi}_m \Delta_{m,m+p} (\widehat{\Phi}_{m+p}^* \widehat{\Phi}_{m+p})^{-1} \Delta_{m,m+p}^* \widehat{\Phi}_m^*) \\
&= n^{-1} \|\widehat{\Phi}_m \Delta_{m,m+p} \widehat{\Psi}_{m+p}^{-1/2}\|_{\text{op}}^2 \\
&\leq n^{-1} \|\widehat{\Psi}_{m+p}^{-1/2}\|_{\text{op}}^2 \|\widehat{\Phi}_m \Delta_{m,m+p}\|_{\text{op}}^2 \\
&= \|\widehat{\Psi}_{m+p}^{-1}\|_{\text{op}} \lambda_{\max}(\Delta_{m,m+p}^* \widehat{\Psi}_m \Delta_{m,m+p}) \\
&\leq \|\widehat{\Psi}_{m+p}^{-1}\|_{\text{op}} \|\widehat{\Psi}_m\|_{\text{op}} \|\Delta_{m,m+p}\|_{\text{op}}^2.
\end{aligned}$$

Moreover, $\|\widehat{\Psi}_m\|_{\text{op}} \leq \mathfrak{L}(m) \leq \mathfrak{L}(m+p)$ and $\mathfrak{L}(m+p) \|\widehat{\Psi}_{m+p}^{-1}\|_{\text{op}} \leq cn/\log(n)$ on Λ_{m+p} . Then,

$$\begin{aligned}
\mathbb{E}(\|\widehat{b}'_m\|_n^4 \mathbf{1}_{\Lambda_{m+p}}) &\leq \frac{1}{n^2} \mathbb{E}(\|\widehat{\Psi}_{m+p}^{-1}\|_{\text{op}}^2 \|\widehat{\Psi}_m\|_{\text{op}}^2 \mathbf{1}_{\Lambda_{m+p}} \|\mathbf{Y}\|_{2,n}^4) \|\Delta_{m,m+p}\|_{\text{op}}^4 \\
&\leq \frac{cn^2}{\log(n)^2} \|\Delta_{m,m+p}\|_{\text{op}}^4 \mathbb{E}\left[\left(\frac{1}{n} \sum_{i=1}^n Y_i^2\right)^2\right] \leq \frac{cn^2}{\log(n)^2} \|\Delta_{m,m+p}\|_{\text{op}}^4 \mathbb{E}(Y_1^4).
\end{aligned}$$

Thus, thanks to Lemma 8.1,

$$\begin{aligned}
T &= 2 \left[\mathbb{E}(\|\widehat{b}'_m\|_n^4 \mathbf{1}_{\Lambda_{m+p}})^{1/2} + \mathbb{E}(\|b'\|_n^4)^{1/2} \right] \mathbb{P}(\Omega_{m+p}^c)^{1/2} \\
&\leq 2 \left[\frac{cn}{\log(n)} \|\Delta_{m,m+p}\|_{\text{op}}^2 \mathbb{E}(Y_1^4)^{1/2} + \mathbb{E}(b'(X_1)^4)^{1/2} \right] \frac{c_{8.1}^{1/2}}{n^4}.
\end{aligned}$$

On the other hand, with the exact same ideas than in the proof of Proposition 3.6,

$$\begin{aligned}
S - 2\|f\|_{\infty} \inf_{t \in \mathcal{S}_m} \|t - b'\|^2 &\leq 2\mathbb{E} \left[\|\widehat{\Phi}_m \Delta_{m,m+p} (\widehat{\Phi}_{m+p}^* \widehat{\Phi}_{m+p})^{-1} \widehat{\Phi}_{m+p}^*\|_{\text{op}}^2 \|b - b_{m+p}\|_n^2 \mathbf{1}_{\Lambda_{m+p} \cap \Omega_{m+p}} \right] \\
&\quad + \frac{\sigma^2}{n} \mathbb{E} \left[\text{Tr} \left((\widehat{\Phi}_{m+p}^* \widehat{\Phi}_{m+p})^{-1} \Delta_{m,m+p}^* \widehat{\Phi}_m^* \widehat{\Phi}_m \Delta_{m,m+p} \right) \mathbf{1}_{\Lambda_{m+p} \cap \Omega_{m+p}} \right] \\
&=: S_1 + S_2.
\end{aligned}$$

Let us find suitable bounds on S_1 and S_2 :

- On Ω_{m+p} , the eigenvalues of $\Psi_{m+p}^{-1/2} \widehat{\Psi}_{m+p} \Psi_{m+p}^{-1/2}$ belong to $[1/2, 3/2]$ and then, those of the matrix $\Psi_{m+p}^{1/2} \widehat{\Psi}_{m+p}^{-1} \Psi_{m+p}^{1/2}$ belong to $[2/3, 2]$. The same way, on Ω_m , the eigenvalues of $\Psi_m^{-1/2} \widehat{\Psi}_m \Psi_m^{-1/2}$ belong to $[1/2, 3/2]$. So, on Ω_{m+p} , $\widehat{S}_1 := \|\widehat{\Phi}_m \Delta_{m,m+p} (\widehat{\Phi}_{m+p}^* \widehat{\Phi}_{m+p})^{-1} \widehat{\Phi}_{m+p}^*\|_{\text{op}}^2$ satisfies

$$\begin{aligned}
\widehat{S}_1 &= \lambda_{\max}(\widehat{\Phi}_m \Delta_{m,m+p} (\widehat{\Phi}_{m+p}^* \widehat{\Phi}_{m+p})^{-1} \Delta_{m,m+p}^* \widehat{\Phi}_m^*) \\
&= \lambda_{\max}(\Delta_{m,m+p} \widehat{\Psi}_{m+p}^{-1} \Delta_{m,m+p}^* \widehat{\Psi}_m) \\
&= \lambda_{\max}(\Psi_m^{-1/2} (\Delta_{m,m+p}^{f,2})^* \Psi_{m+p}^{1/2} \widehat{\Psi}_{m+p}^{-1} \Psi_{m+p}^{1/2} \Delta_{m,m+p}^{f,2} \Psi_m^{-1/2} \widehat{\Psi}_m) \\
&= \lambda_{\max}((\Psi_m^{-1/2} \widehat{\Psi}_m \Psi_m^{-1/2})^{1/2} (\Delta_{m,m+p}^{f,2})^* \Psi_{m+p}^{1/2} \widehat{\Psi}_{m+p}^{-1} \Psi_{m+p}^{1/2} \Delta_{m,m+p}^{f,2} (\Psi_m^{-1/2} \widehat{\Psi}_m \Psi_m^{-1/2})^{1/2}) \\
&= \|(\Psi_m^{-1/2} \widehat{\Psi}_m \Psi_m^{-1/2})^{1/2} (\Delta_{m,m+p}^{f,2})^* \Psi_{m+p}^{1/2} \widehat{\Psi}_{m+p}^{-1} \Psi_{m+p}^{1/2} \Delta_{m,m+p}^{f,2} (\Psi_m^{-1/2} \widehat{\Psi}_m \Psi_m^{-1/2})^{1/2}\|_{\text{op}} \\
&\leq \|\Psi_m^{-1/2} \widehat{\Psi}_m \Psi_m^{-1/2}\|_{\text{op}} \|\Delta_{m,m+p}^{f,2}\|_{\text{op}}^2 \|\Psi_{m+p}^{1/2} \widehat{\Psi}_{m+p}^{-1} \Psi_{m+p}^{1/2}\|_{\text{op}} \\
&\leq 3 \|\Delta_{m,m+p}^{f,2}\|_{\text{op}}^2.
\end{aligned}$$

Thus,

$$S_1 \leq 6 \|\Delta_{m,m+p}^{f,2}\|_{\text{op}}^2 \mathbb{E}(\|b - b_{m+p}\|_n^2) = 6 \|\Delta_{m,m+p}^{f,2}\|_{\text{op}}^2 \|b - b_{m+p}\|_f^2.$$

- As previously, since the eigenvalues of $\Psi_{m+p}^{1/2} \widehat{\Psi}_{m+p}^{-1} \Psi_{m+p}^{1/2} \mathbf{1}_{\Omega_{m+p}}$ belong to $[2/3, 2]$,

$$\begin{aligned} S_2 &= \frac{\sigma^2}{n} \mathbb{E} \left[\text{Tr} \left(\Psi_{m+p}^{1/2} \widehat{\Psi}_{m+p}^{-1} \Psi_{m+p}^{1/2} \Delta_{m,m+p}^{f,2} \Psi_m^{-1/2} \widehat{\Psi}_m \Psi_m^{-1/2} (\Delta_{m,m+p}^{f,2})^* \right) \mathbf{1}_{\Lambda_{m+p} \cap \Omega_{m+p}} \right] \\ &\leq \frac{\sigma^2}{n} \mathbb{E} \left[\|\Psi_{m+p}^{1/2} \widehat{\Psi}_{m+p}^{-1} \Psi_{m+p}^{1/2}\|_{\text{op}} \text{Tr} \left(\Delta_{m,m+p}^{f,2} \Psi_m^{-1/2} \widehat{\Psi}_m \Psi_m^{-1/2} (\Delta_{m,m+p}^{f,2})^* \right) \mathbf{1}_{\Omega_{m+p}} \right] \\ &\leq \frac{2\sigma^2}{n} \text{Tr} \left[\Delta_{m,m+p}^{f,2} \Psi_m^{-1/2} \mathbb{E}(\widehat{\Psi}_m) \Psi_m^{-1/2} (\Delta_{m,m+p}^{f,2})^* \right] = \frac{2\sigma^2}{n} \|\Delta_{m,m+p}^{f,2}\|_F^2. \end{aligned}$$

The result follows by gathering all the terms. \square

8.6. Proof of Proposition 4.1: The Hermite case. Consider a square integrable function b , and

$$b_m = \sum_{j=0}^{m-1} \langle b, h_j \rangle h_j$$

its projection on $\mathcal{S}_m = \text{span}\{h_0, \dots, h_{m-1}\}$. On the one hand,

$$b'_m = \sum_{j=0}^{m-1} \langle b, h_j \rangle h'_j.$$

Then, thanks to Equality (8),

$$\begin{aligned} b'_m &= \frac{1}{\sqrt{2}} \sum_{j=0}^{m-1} \langle b, h_j \rangle (\sqrt{j} h_{j-1} - \sqrt{j+1} h_{j+1}) \\ &= \frac{1}{\sqrt{2}} \left(\sum_{j=0}^{m-2} \langle b, h_{j+1} \rangle \sqrt{j+1} h_j - \sum_{j=1}^m \langle b, h_{j-1} \rangle \sqrt{j} h_j \right) \\ &= \frac{1}{\sqrt{2}} \sum_{j=0}^{m-2} \left[\sqrt{j+1} \langle b, h_{j+1} \rangle - \sqrt{j} \langle b, h_{j-1} \rangle \right] h_j - \left(\sqrt{\frac{m-1}{2}} \langle b, h_{m-2} \rangle h_{m-1} + \sqrt{\frac{m}{2}} \langle b, h_{m-1} \rangle h_m \right). \end{aligned}$$

On the other hand, if b' is square integrable, then

$$b' = \sum_{j \geq 0} \langle b', h_j \rangle h_j.$$

The usual integration by parts gives $\langle b', h_j \rangle = -\langle b, h'_j \rangle$ as soon as $\lim_{x \rightarrow \pm\infty} b(x) h_j(x) = 0$ (this holds because the h_j 's have exponential decrease and b is square-integrable, thus bounded near infinity). So, the projection of b' is

$$\begin{aligned} (b')_m &= - \sum_{j=0}^{m-1} \langle b, h'_j \rangle h_j \\ &= - \frac{1}{\sqrt{2}} \sum_{j=0}^{m-1} \langle b, \sqrt{j} h_{j-1} - \sqrt{j+1} h_{j+1} \rangle h_j \\ &= \frac{1}{\sqrt{2}} \sum_{j=0}^{m-1} \left[\sqrt{j+1} \langle b, h_{j+1} \rangle - \sqrt{j} \langle b, h_{j-1} \rangle \right] h_j. \end{aligned}$$

All the components of b'_m and $(b')_m$ are the same on \mathcal{S}_{m-2} . So,

$$b'_m - (b')_m = -\sqrt{\frac{m}{2}} \langle b, h_{m-1} \rangle h_m - \sqrt{\frac{m}{2}} \langle b, h_m \rangle h_{m-1},$$

and then,

$$\|b'_m - (b')_m\|^2 = \frac{m}{2} (\langle b, h_{m-1} \rangle^2 + \langle b, h_m \rangle^2).$$

If b belongs to a Hermite-Sobolev space with regularity index $\alpha > 1$, then the term $\|b'_m - (b')_m\|^2$ is of order $m^{-(\alpha-1)}$, which is also the order of $\inf_{t \in \mathcal{S}_m} \|t - b'\|^2$.

The Laguerre case. As previously, on the one hand, $b_m = \sum_{j=0}^{m-1} \langle b, \ell_j \rangle \ell_j$, and thanks to (7),

$$\begin{aligned} b'_m &= \sum_{j=0}^{m-1} \langle b, \ell_j \rangle \left(-\ell_j - 2 \sum_{k=0}^{j-1} \ell_k \right) \\ &= - \sum_{j=0}^{m-1} \langle b, \ell_j \rangle \ell_j - 2 \sum_{k=0}^{m-2} \left(\sum_{j=k+1}^{m-1} \langle b, \ell_j \rangle \right) \ell_k. \end{aligned}$$

On the other hand, if b' is square integrable, then $b' = \sum_{j \geq 0} \langle b', \ell_j \rangle \ell_j$. Thus, since $\langle b', \ell_j \rangle = -\langle b, \ell'_j \rangle$ by Assumption 3.5 (true when $b(0) = 0$),

$$(b')_m = \sum_{j=0}^{m-1} \left(\langle b, \ell_j \rangle \ell_j - 2 \sum_{k=0}^{j-1} \langle b, \ell_k \rangle \right) \ell_j.$$

Consequently,

$$(b')_m - b'_m = 2 \sum_{j=0}^{m-1} \left(\sum_{k=0}^{m-1} \langle b, \ell_k \rangle \right) \ell_j = 2 \left(\sum_{k=0}^{m-1} \langle b, \ell_k \rangle \right) \sum_{j=0}^{m-1} \ell_j,$$

and then

$$\|(b')_m - b'_m\|^2 = 4m \left(\sum_{k=0}^{m-1} \langle b, \ell_k \rangle \right)^2.$$

Moreover, by assuming that $b(0) = 0$,

$$\sum_{k \geq 0} \langle b, \ell_k \rangle \ell_k(0) = \sqrt{2} \sum_{k \geq 0} \langle b, \ell_k \rangle = 0.$$

So, $\sum_{k=0}^{m-1} \langle b, \ell_k \rangle = -\sum_{k \geq m} \langle b, \ell_k \rangle$, and then

$$\|(b')_m - b'_m\|^2 = 4m \left(\sum_{k \geq m} \langle b, \ell_k \rangle \right)^2.$$

Finally, if b belongs to a Laguerre-Sobolev space with index $\alpha > 1$, then the right-hand side in the previous equality is smaller than

$$\sum_{k \geq m} k^\alpha \langle b, \ell_k \rangle^2 = O(m^{-\alpha+1}) \quad \square$$

8.7. Proof of Lemma 5.1. First,

$$\widehat{V}(m) = \sigma^2 \frac{m}{n} \|\widehat{\Psi}_m^{-1}(\widehat{\Phi}'_m)^* \widehat{\Phi}'_m\|_{\text{op}} = \sigma^2 \frac{m}{n} \|\widehat{\Psi}_m^{-1/2}(\widehat{\Phi}'_m)^* \widehat{\Phi}'_m \widehat{\Psi}_m^{-1/2}\|_{\text{op}}$$

where $\widehat{\Psi}_m^{-1/2}$ is a symmetric square root of $\widehat{\Psi}_m^{-1}$. Now, as the matrix is symmetric,

$$\|\widehat{\Psi}_m^{-1/2}(\widehat{\Phi}'_m)^* \widehat{\Phi}'_m \widehat{\Psi}_m^{-1/2}\|_{\text{op}} = \sup_{\mathbf{x} \in \mathbb{R}^m} \mathbf{x} \widehat{\Psi}_m^{-1/2}(\widehat{\Phi}'_m)^* \widehat{\Phi}'_m \widehat{\Psi}_m^{-1/2} \mathbf{x} = n \sup_{t \in \mathcal{S}_m: \|t\|_n=1} \|t'\|_n^2.$$

So, clearly, $m \mapsto \widehat{V}(m) = \sigma^2 m/n \sup_{t \in \mathcal{S}_m: \|t\|_n=1} \|t'\|_n^2$ is increasing. \square

8.8. Proof of Theorem 5.2. Throughout this subsection, for the sake of readability, we omit the superscript 1 and write \widehat{b}'_m instead of \widehat{b}'_m^1 .

Following the lines of the proof of Theorem 2 in Comte and Genon-Catalot [11], we consider the sets

$$\Xi_n = \{\omega : \mathcal{M}_n \subset \widehat{\mathcal{M}}_n(\omega) \subset \mathcal{M}_n^+\} \quad \text{and} \quad \Omega_n = \bigcap_{m \in \mathcal{M}_n^+} \Omega_m,$$

where

$$\mathcal{M}_n^+ := \left\{ m \in \{1, \dots, n\} : \mathfrak{L}(m+p)(\|\Psi_{m+p}^{-1}\|_{\text{op}}^2 \vee 1) \leq 4\mathfrak{c} \cdot \frac{n}{\log(n)} \right\}.$$

First,

$$\mathbb{E} \left[\|\widehat{b}' - b'\|_n^2 \mathbf{1}_{(\Omega_n \cap \Xi_n)^c} \right] \leq \frac{\mathfrak{c}_1}{n} \quad \text{with} \quad \mathfrak{c}_1 > 0.$$

This follows from the proof of Proposition 3.8, using that $\mathbb{P}(\Xi_n^c) \leq \mathfrak{c}_3/n^8$ and $\mathbb{P}(\Omega_n^c) \leq \mathfrak{c}_4/n^8$. For these last probabilities, we refer to Comte and Genon-Catalot [11], Lemmas 7 and 9, where the choice of $\mathfrak{d} = 1/\mathfrak{f}(\|f\|_\infty \vee 1 + 3^{-1})$ with $\mathfrak{f} = 192$ is explained. Here, the constant \mathfrak{f} has to be increased to obtain the power n^{-8} instead of n^{-2} .

Now, we control the loss of $\widehat{b}'_{\widehat{m}}$ on $\Omega_n \cap \Xi_n$. For any $m \in \mathcal{M}_n$, using that on Ξ_n it also holds that $m \in \widehat{\mathcal{M}}_n$, we have

$$\begin{aligned} \|\widehat{b}'_{\widehat{m}} - b'\|_n^2 &\leq 3(\|\widehat{b}'_{\widehat{m}} - \widehat{b}'_{\widehat{m} \wedge m}\|_n^2 + \|\widehat{b}'_{\widehat{m} \wedge m} - \widehat{b}'_m\|_n^2 + \|\widehat{b}'_m - b'\|_n^2) \\ &\leq 3(A(m) + \kappa_0 \widehat{V}(\widehat{m}) + A(\widehat{m}) + \kappa_0 \widehat{V}(m) + \|\widehat{b}'_m - b'\|_n^2) \\ (22) \quad &\leq 6(A(m) + \kappa_1 \widehat{V}(m)) + 3\|\widehat{b}'_m - b'\|_n^2 \text{ as } \kappa_0 \leq \kappa_1. \end{aligned}$$

Moreover,

$$\begin{aligned} A(m) &\leq 3 \sup_{m \in \widehat{\mathcal{M}}_n} \left\{ \|\widehat{b}'_m - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_m)\|_n^2 - \frac{\kappa_0}{6} \widehat{V}(m) \right\}_+ \\ &\quad + 3 \sup_{m' \in \widehat{\mathcal{M}}_n} \left\{ \|\widehat{b}'_{m \wedge m'} - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_{m \wedge m'})\|_n^2 - \frac{\kappa_0}{6} \widehat{V}(m') \right\}_+ \\ &\quad + 3 \sup_{m' \in \widehat{\mathcal{M}}_n} \|\mathbb{E}_{\mathbf{X}}(\widehat{b}'_{m \wedge m'}) - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_{m'})\|_n^2, \end{aligned}$$

and since

$$\sup_{m' \in \widehat{\mathcal{M}}_n} \{ \dots \} = \max \left(\sup_{m' \in \widehat{\mathcal{M}}_n : m' \leq m} \{ \dots \}; \sup_{m' \in \widehat{\mathcal{M}}_n : m' \geq m} \{ \dots \} \right),$$

by Lemma 5.1 ($m \mapsto \widehat{V}(m)$ is increasing),

$$\begin{aligned} &\sup_{m' \in \widehat{\mathcal{M}}_n} \left\{ \|\widehat{b}'_{m \wedge m'} - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_{m \wedge m'})\|_n^2 - \frac{\kappa_0}{6} \widehat{V}(m') \right\}_+ \\ &\leq \max \left(\sup_{m' \in \widehat{\mathcal{M}}_n} \left\{ \|\widehat{b}'_{m'} - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_{m'})\|_n^2 - \frac{\kappa_0}{6} \widehat{V}(m') \right\}_+ ; \left\{ \|\widehat{b}'_m - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_m)\|_n^2 - \frac{\kappa_0}{6} \widehat{V}(m) \right\}_+ \right) \\ &\leq \sup_{m \in \widehat{\mathcal{M}}_n} \left\{ \|\widehat{b}'_m - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_m)\|_n^2 - \frac{\kappa_0}{6} \widehat{V}(m) \right\}_+. \end{aligned}$$

Thus,

$$(23) \quad A(m) \leq 6 \sup_{m \in \widehat{\mathcal{M}}_n} \left\{ \|\widehat{b}'_m - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_m)\|_n^2 - \frac{\kappa_0}{6} \widehat{V}(m) \right\}_+ + 3 \sup_{m' \in \widehat{\mathcal{M}}_n} \|\mathbb{E}_{\mathbf{X}}(\widehat{b}'_{m \wedge m'}) - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_{m'})\|_n^2.$$

The following lemma provides a suitable bound on the first term in the right-hand side of Inequality (23) obtained via the conditional Talagrand inequality.

Lemma 8.2. *Let Assumption 3.1 be fulfilled. Let also Assumption 3.7($m + p$) be fulfilled for every $m \in \mathcal{M}_n$. Moreover, assume that there exists $\kappa > 0$ such that $\mathbb{E}(\exp(\kappa\varepsilon_1^2)) < \infty$, and that Conditions (14) and (15) hold. Then,*

$$\mathbb{E} \left[\sup_{m \in \widehat{\mathcal{M}}_n} \left\{ \|\widehat{b}'_m - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_m)\|_n^2 - \frac{\kappa_0}{6} \widehat{V}(m) \right\}_+ \right] \leq \frac{\mathfrak{c}_{8.2}}{n},$$

where $\mathfrak{c}_{8.2} > 0$ is a deterministic constant not depending on n .

By Inequalities (22) and (23), and then by Lemma 8.2, for any $m \in \mathcal{M}_n$,

$$\begin{aligned} \mathbb{E}(\|\widehat{b}' - b'\|_n^2 \mathbf{1}_{\Omega_n \cap \Xi_n}) &\leq 36 \mathbb{E} \left[\sup_{m \in \widehat{\mathcal{M}}_n} \left\{ \|\widehat{b}'_m - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_m)\|_n^2 - \frac{\kappa_0}{6} \widehat{V}(m) \right\}_+ \right] \\ &\quad + 18 \mathbb{E} \left[\mathbf{1}_{\Xi_n \cap \Omega_n} \sup_{m' \in \widehat{\mathcal{M}}_n} \|\mathbb{E}_{\mathbf{X}}(\widehat{b}'_{m \wedge m'}) - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_{m'})\|_n^2 \right] + 6\kappa_1 \mathbb{E}(\widehat{V}(m) \mathbf{1}_{\Omega_n}) + 3 \mathbb{E}(\|\widehat{b}'_m - b'\|_n^2) \\ &\leq 3 \mathbb{E}(\|\widehat{b}'_m - b'\|_n^2) + 6\kappa_1 \mathfrak{c}_2 V(m) + \frac{\mathfrak{c}_{8.2}}{n} \\ &\quad + 18 \mathbb{E} \left[\mathbf{1}_{\Xi_n \cap \Omega_n} \sup_{m' \in \widehat{\mathcal{M}}_n} \|\mathbb{E}_{\mathbf{X}}(\widehat{b}'_{m \wedge m'}) - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_{m'})\|_n^2 \right] \end{aligned}$$

with $\mathfrak{c}_2 > 0$. The inequality $\mathbb{E}(\widehat{V}(m) \mathbf{1}_{\Omega_n}) \leq \mathfrak{c}_2 V(m)$ is obtained via the same method than in the proof of Proposition 3.9.

Let us now control

$$B_{m,n} := \frac{1}{n} \mathbb{E} \left[\mathbf{1}_{\Xi_n \cap \Omega_n} \sup_{m' \in \widehat{\mathcal{M}}_n : m' > m} \|\mathbb{E}_{\mathbf{X}}(\widehat{b}'_{m'}^{f,1}(\mathbf{X})) - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_m^{f,1}(\mathbf{X}))\|_{2,n}^2 \right].$$

Thanks to Equality (17),

$$\begin{aligned} \mathbb{E}_{\mathbf{X}}(\widehat{b}'_{m'}^{f,1}(\mathbf{X})) - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_m^{f,1}(\mathbf{X})) &= \mathbb{E}_{\mathbf{X}}(\widehat{b}'_{m'}^{f,1}(\mathbf{X})) - b'_{m'}(\mathbf{X}) - (\mathbb{E}_{\mathbf{X}}(\widehat{b}'_m^{f,1}(\mathbf{X})) - b'_m(\mathbf{X})) + b'_{m'}(\mathbf{X}) - b'_m(\mathbf{X}) \\ &= \widehat{P}_{m'}(b(\mathbf{X}) - b_{m'}(\mathbf{X})) + \widehat{P}_m(b(\mathbf{X}) - b_m(\mathbf{X})) + b'_{m'}(\mathbf{X}) - b'_m(\mathbf{X}). \end{aligned}$$

Since we are on Ω_n , and since $m, m' \in \mathcal{M}_n^+$ on Ξ_n when $m \in \mathcal{M}_n$ and $m' \in \widehat{\mathcal{M}}_n$,

$$\text{Sp} \left[\Psi_{m'+p}^{-1/2} \widehat{\Psi}_{m'+p} \Psi_{m'+p}^{-1/2} \right] \subset [1/2, 3/2] \text{ and } \text{Sp} \left[\Psi_{m'}^{1/2} \widehat{\Psi}_{m'}^{-1} \Psi_{m'}^{1/2} \right] \subset [2/3, 2],$$

and the same for m instead of m' . So, thanks to Inequality (21),

$$\|\widehat{P}_{m'}\|_{\text{op}}^2 \leq 3 \|\Delta_{m',m'+p}^{f,1}\|_{\text{op}}^2.$$

In the same way,

$$\|\widehat{P}_m\|_{\text{op}}^2 \leq 3 \|\Delta_{m,m+p}^{f,1}\|_{\text{op}}^2.$$

Thus, on Ω_n ,

$$\begin{aligned} \frac{1}{n} \|\widehat{P}_{m'}(b(\mathbf{X}) - b_{m'}(\mathbf{X}))\|_{2,n}^2 &\leq 3 \|\Delta_{m',m'+p}^{f,1}\|_{\text{op}}^2 \|b - b_{m'}\|_n^2 \\ &\leq 6 \|\Delta_{m',m'+p}^{f,1}\|_{\text{op}}^2 (\|b - b_{M_n}\|_n^2 + \|b_{M_n} - b_{m'}\|_n^2) \\ &\leq 6 \|\Delta_{m',m'+p}^{f,1}\|_{\text{op}}^2 (\|b - b_{M_n^+}\|_{\infty}^2 + 3/2 \|b_{M_n^+} - b_{m'}\|_f^2) \end{aligned}$$

where M_n^+ is the maximal element of \mathcal{M}_n^+ , and

$$\frac{1}{n} \|\widehat{P}_m(b(\mathbf{X}) - b_m(\mathbf{X}))\|_{2,n}^2 \leq 3 \|\Delta_{m,m+p}^{f,1}\|_{\text{op}}^2 \|b - b_m\|_n^2.$$

For the last term, on Ω_n ,

$$\|b'_{m'}(\mathbf{X}) - b'_m(\mathbf{X})\|_n^2 \leq \frac{3}{2} \|b'_{m'} - b'_m\|_f^2 \leq \frac{3}{2} \|f\|_{\infty} \|b'_{m'} - b'_m\|_f^2.$$

Therefore,

$$B_{m,n} \leq 9 \|\Delta_{m,m+p}^{f,1}\|_{\text{op}}^2 \|b - b_m\|_f^2 + 18 \sup_{m' \in \mathcal{M}_n^+ : m' > m} \left\{ \|\Delta_{m',m'+p}^{f,1}\|_{\text{op}}^2 (\|b_{M_n^+} - b_{m'}\|_f^2 + \|b - b_{M_n^+}\|_\infty^2) \right\} \\ + \frac{9}{2} \|f\|_\infty \sup_{m' \in \mathcal{M}_n^+ : m' > m} \|b'_{m'} - b'_m\|^2.$$

This concludes the proof. \square

8.9. Proof of Lemma 8.2. We emphasize that the lemma would be true for $\widehat{\mathcal{M}}_n$ replaced by the weaker (and more natural)

$$\left\{ m \in \{1, \dots, n\} : \mathfrak{L}(m+p) (\|\widehat{\Psi}_{m+p}^{-1}\|_{\text{op}} \vee 1) \leq \mathfrak{c} \frac{n}{\log(n)} \right\}$$

with \mathfrak{c} defined in Assumption 3.7($m+p$). We only use this constraint in the following.

First of all, for any $m \in \widehat{\mathcal{M}}_n$, since $\|\psi\|_n^2 = \sup_{t \in \mathcal{S}_m : \|t\|_n=1} \langle t, \psi \rangle_n^2$ for every $\psi \in \mathcal{S}_m$,

$$\|\widehat{b}'_m - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_m)\|_n^2 = \sup_{t \in \mathcal{S}_m : \|t\|_n=1} \nu_n(t)^2$$

with, for any $\vec{b} = (b_1, \dots, b_m) \in \mathbb{R}^m$ and $t = \sum_{j=1}^m b_j \varphi_j$,

$$\nu_n(t) = \frac{1}{n} \langle t, \widehat{\Phi}'_m \widehat{\Psi}_m^{-1} \widehat{\Phi}_m^* \varepsilon \rangle_n = \frac{1}{n^2} \langle [\widehat{\Phi}'_m \widehat{\Psi}_m^{-1} \widehat{\Phi}_m^*]^* t(\mathbf{X}), \varepsilon \rangle_{2,n} = \frac{1}{n} \langle \Theta_t(\mathbf{X}), \varepsilon \rangle_{2,n},$$

where

$$\Theta_t(\mathbf{X}) = \frac{1}{n} \widehat{\Phi}_m \widehat{\Psi}_m^{-1} (\widehat{\Phi}'_m)^* t(\mathbf{X}) = \frac{1}{n} \widehat{\Phi}_m \widehat{\Psi}_m^{-1} (\widehat{\Phi}'_m)^* \widehat{\Phi}_m \vec{b}.$$

Note that $\nu_n(t) = \nu_n^{(1)}(t) + \nu_n^{(2)}(t)$, where $\nu_n^{(1)}(t) = n^{-1} \langle \Theta_t(\mathbf{X}), \varepsilon^{(1)} \rangle_{2,n}$ with $\varepsilon^{(1)} = (\varepsilon_i \mathbf{1}_{|\varepsilon_i| \leq \mathfrak{m}_n} - \mathbb{E}(\varepsilon_i \mathbf{1}_{|\varepsilon_i| \leq \mathfrak{m}_n}))_i$ and $\mathfrak{m}_n = (q\kappa^{-1} \log(n))^{1/2}$, and $\nu_n^{(2)}(t) = n^{-1} \langle \Theta_t(\mathbf{X}), \varepsilon^{(2)} \rangle_{2,n}$ with $\varepsilon^{(2)} = (\varepsilon_i \mathbf{1}_{|\varepsilon_i| > \mathfrak{m}_n} - \mathbb{E}(\varepsilon_i \mathbf{1}_{|\varepsilon_i| > \mathfrak{m}_n}))_i$. On the one hand, in order to apply Talagrand's inequality to $\sup_{t \in \mathcal{S}_m : \|t\|_n=1} \nu_n^{(1)}(t)^2$ conditionally to (X_1, \dots, X_n) , consider

$$A_{m,n}(\mathbf{X}) := \mathbb{E}_{\mathbf{X}} \left(\sup_{t \in \mathcal{S}_m : \|t\|_n=1} \nu_n^{(1)}(t)^2 \right), \\ B_{m,n} := \sup_{t \in \mathcal{S}_m : \|t\|_n=1} \left\{ \sup_{(e,x) \in [-2\mathfrak{m}_n, 2\mathfrak{m}_n] \times I} |e \Theta_t(x)| \right\} \text{ and} \\ C_{m,n}(\mathbf{X}) := \sup_{t \in \mathcal{S}_m : \|t\|_n=1} \left\{ \frac{1}{n} \sum_{i=1}^n \text{var}_{\mathbf{X}} \left[\varepsilon_i^{(1)} \Theta_t(X_i) \right] \right\},$$

and let us find suitable bounds on each of these random quantities.

- **Bound on $A_{m,n}(\mathbf{X})$.** Note that

$$\text{var}(\varepsilon_1^{(1)}) \leq \mathbb{E}(\varepsilon_1^2 \mathbf{1}_{|\varepsilon_1| \leq \mathfrak{m}_n}) \leq \mathbb{E}(\varepsilon_1^2) = \sigma^2.$$

Then,

$$A_{m,n}(\mathbf{X}) \leq \frac{1}{n^3} \mathbb{E}_{\mathbf{X}} (\|\widehat{\Phi}'_m \widehat{\Psi}_m^{-1} \widehat{\Phi}_m^* \varepsilon^{(1)}\|_{2,n}^2) = \frac{\text{var}(\varepsilon_1^{(1)})}{n} \text{Tr} \left[\widehat{\Phi}'_m \widehat{\Psi}_m^{-1} (\widehat{\Phi}'_m)^* \right] \\ \leq \frac{\sigma^2}{n^2} \text{Tr} \left[\widehat{\Phi}'_m \widehat{\Psi}_m^{-1} (\widehat{\Phi}'_m)^* \right] \leq \frac{\sigma^2 m}{n^2} \|\widehat{\Psi}_m^{-1} (\widehat{\Phi}'_m)^* \widehat{\Phi}'_m\|_{\text{op}} =: H^2.$$

- **Bound on $B_{m,n}$.** Since $m \in \widehat{\mathcal{M}}_n$, $m \|\widehat{\Psi}_m^{-1}\|_{\text{op}} \leq (m+p) \|\widehat{\Psi}_{m+p}^{-1}\|_{\text{op}} \leq cn/\log(n)$, and then

$$\begin{aligned}
B_{m,n} &\leq \frac{2\mathbf{m}_n}{n} \sup_{\vec{b}: \|\widehat{\Phi}_m \vec{b}\|_{2,n} = \sqrt{n}} \left| \sum_{j=1}^m [\widehat{\Psi}_m^{-1}(\widehat{\Phi}'_m)^* \widehat{\Phi}_m \vec{b}]_j \varphi_j(x) \right| \\
&\leq \frac{2\mathbf{m}_n}{n} \mathfrak{L}(m)^{1/2} \sup_{\vec{b}: \|\widehat{\Phi}_m \vec{b}\|_{2,n} = \sqrt{n}} \|\widehat{\Psi}_m^{-1}(\widehat{\Phi}'_m)^* \widehat{\Phi}_m \vec{b}\|_{m,2} \leq \frac{2\mathbf{m}_n}{\sqrt{n}} \sqrt{\mathfrak{L}(m) \|\widehat{\Psi}_m^{-1}(\widehat{\Phi}'_m)^*\|_{\text{op}}^2} \\
&\leq \frac{2\mathbf{m}_n}{\sqrt{n}} \sqrt{\mathfrak{L}(m) \|\widehat{\Psi}_m^{-1}\|_{\text{op}} \|\widehat{\Psi}_m^{-1}(\widehat{\Phi}'_m)^* \widehat{\Phi}'_m\|_{\text{op}}} \leq 2\sqrt{\mathbf{c}q\kappa^{-1}} \|\widehat{\Psi}_m^{-1}(\widehat{\Phi}'_m)^* \widehat{\Phi}'_m\|_{\text{op}} =: M.
\end{aligned}$$

Then,

$$\frac{nH}{M} = \frac{\sigma}{2\sqrt{\mathbf{c}q\kappa^{-1}}} \cdot \sqrt{m}.$$

- **Bound on $C_{m,n}(\mathbf{X})$:**

$$\begin{aligned}
C_{m,n}(\mathbf{X}) &\leq \frac{\mathbb{E}(|\varepsilon_1^{(1)}|^2)}{n} \sup_{t \in \mathcal{S}_m: \|t\|_n = 1} \sum_{i=1}^n \Theta_t(X_i)^2 \leq \frac{\sigma^2}{n^3} \sup_{\vec{b}: \|\widehat{\Phi}_m \vec{b}\|_{2,n} = \sqrt{n}} \|\widehat{\Phi}_m \widehat{\Psi}_m^{-1}(\widehat{\Phi}'_m)^* \widehat{\Phi}_m \vec{b}\|_{2,n}^2 \\
&= \frac{\sigma^2}{n^3} \sup_{\vec{b}: \|\widehat{\Phi}_m \vec{b}\|_{2,n} = \sqrt{n}} \vec{b}^* \widehat{\Phi}_m^* \widehat{\Phi}'_m \widehat{\Psi}_m^{-1} \underbrace{\widehat{\Phi}_m^* \widehat{\Phi}_m \widehat{\Psi}_m^{-1}}_{=n\mathbf{I}_m} (\widehat{\Phi}'_m)^* \widehat{\Phi}_m \vec{b} \\
&= \frac{\sigma^2}{n^2} \sup_{\vec{b}: \|\widehat{\Phi}_m \vec{b}\|_{2,n} = \sqrt{n}} \|\widehat{\Psi}_m^{-1/2}(\widehat{\Phi}'_m)^* \widehat{\Phi}_m \vec{b}\|_{2,n}^2 \leq \frac{\sigma^2}{n} \|\widehat{\Psi}_m^{-1/2}(\widehat{\Phi}'_m)^*\|_{\text{op}}^2 = \frac{\sigma^2}{n} \|\widehat{\Phi}'_m \widehat{\Psi}_m^{-1}(\widehat{\Phi}'_m)^*\|_{\text{op}} =: v.
\end{aligned}$$

Then,

$$\frac{nH^2}{v} = m.$$

So, by Talagrand's inequality,

$$\begin{aligned}
\mathbb{E}_{\mathbf{X}} \left[\left(\sup_{t \in \mathcal{S}_m: \|t\|_n = 1} \nu_n^{(1)}(t)^2 - 4H^2 \right)_+ \right] \\
\leq \mathbf{c}_1 \left[\frac{v}{n} \exp\left(-\mathbf{c}_2 \frac{nH^2}{v}\right) + \frac{M^2}{n^2} \exp\left(-\mathbf{c}_3 \frac{nH}{M}\right) \right] \\
\leq \frac{\bar{\mathbf{c}}_1}{n^2} \|\widehat{\Psi}_m^{-1}\|_{\text{op}} \mathfrak{L}'(m) \left[e^{-\mathbf{c}_2 m} + q \exp\left(-\frac{\bar{\mathbf{c}}_3}{\sqrt{q}} \cdot \sqrt{\mathfrak{L}(m)}\right) \right]
\end{aligned}$$

where $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \bar{\mathbf{c}}_1, \bar{\mathbf{c}}_3 > 0$ are universal constants, and thus

$$\begin{aligned}
S &:= \mathbb{E}_{\mathbf{X}} \left[\sup_{m \in \widehat{\mathcal{M}}_n} \left\{ \sup_{t \in \mathcal{S}_m: \|t\|_n = 1} \nu_n^{(1)}(t)^2 - \frac{\kappa_0}{6} \widehat{V}(m) \right\}_+ \right] \\
&\leq \frac{\bar{\mathbf{c}}_1}{n^2} \sum_{m \in \widehat{\mathcal{M}}_n} \left[\|\widehat{\Psi}_m^{-1}\|_{\text{op}} \mathfrak{L}'(m) \left[e^{-\mathbf{c}_2 m} + q \exp\left(-\frac{\bar{\mathbf{c}}_3}{\sqrt{q}} \cdot \sqrt{\mathfrak{L}(m)}\right) \right] \right] \\
&\leq \frac{\mathbf{c}\bar{\mathbf{c}}_1}{n \log(n)} \sum_{m \leq n} \left[\frac{\mathfrak{L}'(m)}{\mathfrak{L}(m)} \left[e^{-\mathbf{c}_2 m} + q \exp\left(-\frac{\bar{\mathbf{c}}_3}{\sqrt{q}} \cdot \sqrt{\mathfrak{L}(m)}\right) \right] \right]
\end{aligned}$$

thanks to the definition of $\widehat{\mathcal{M}}_n$. Thanks to Condition (14), this term is of order $1/n$. On the other hand, since $\mathfrak{L}(m)\|\widehat{\Psi}_m^{-1}\|_{\text{op}} \leq cn/\log(n)$ for every $m \in \widehat{\mathcal{M}}_n$, and by Markov's inequality,

$$\begin{aligned} T &:= \mathbb{E}_{\mathbf{X}} \left(\sup_{m \in \widehat{\mathcal{M}}_n} \sup_{t \in \mathcal{S}_m: \|t\|_n=1} \nu_n^{(2)}(t)^2 \right) \leq \frac{1}{n^2} \mathbb{E}(\|\varepsilon^{(2)}\|_{2,n}^2) \sup_{m \in \widehat{\mathcal{M}}_n} \sup_{t \in \mathcal{S}_m: \|t\|_n=1} \|\Theta_t(\mathbf{X})\|_{2,n}^2 \\ &\leq \frac{1}{n} \mathbb{E}(\varepsilon_1^4)^{1/2} \mathbb{P}(|\varepsilon_1| > \mathbf{m}_n)^{1/2} \sup_{m \in \widehat{\mathcal{M}}_n} \left\{ \frac{1}{n^2} \sup_{\tilde{b}: \|\widehat{\Phi}_m \tilde{b}\|_{2,n}=\sqrt{n}} \|\widehat{\Phi}_m \widehat{\Psi}_m^{-1}(\widehat{\Phi}'_m)^* \widehat{\Phi}_m \tilde{b}\|_{2,n}^2 \right\} \\ &\leq \frac{1}{n} \mathbb{E}(\varepsilon_1^4)^{1/2} \mathbb{P}(\exp(\kappa \varepsilon_1^2) > n^q)^{1/2} \sup_{m \in \widehat{\mathcal{M}}_n} \|\widehat{\Phi}'_m \widehat{\Psi}_m^{-1}(\widehat{\Phi}'_m)^*\|_{\text{op}} \\ &\leq \frac{1}{n} \mathbb{E}(\varepsilon_1^4)^{1/2} \frac{1}{n^{q/2}} \mathbb{E}(\exp(\kappa \varepsilon_1^2))^{1/2} \frac{cn}{\log(n)} \sum_{m \leq n} \frac{\mathfrak{L}'(m)}{\mathfrak{L}(m)} \leq \frac{\mathbf{c}_4}{n^{q/2} \log(n)} \sum_{m \leq n} \frac{\mathfrak{L}'(m)}{\mathfrak{L}(m)} \end{aligned}$$

with $\mathbf{c}_4 = \mathbf{c} \mathbb{E}(\varepsilon_1^4)^{1/2} \mathbb{E}(\exp(\kappa \varepsilon_1^2))^{1/2}$. Thanks to Condition (15), this term is of order $1/n$. In conclusion,

$$\mathbb{E} \left[\sup_{m \in \widehat{\mathcal{M}}_n} \left\{ \|\widehat{b}'_m - \mathbb{E}_{\mathbf{X}}(\widehat{b}'_m)\|_n^2 - \frac{\kappa_0}{3} \widehat{V}(m) \right\}_+ \right] \leq 2\mathbb{E}(S) + 2\mathbb{E}(T) \leq \frac{\mathbf{c}_5}{n}. \quad \square$$

8.10. Proof of Corollary 5.3. Here, $(\varphi_j)_{j \in \mathbb{N}^*}$ is the trigonometric basis. Thus, with $\mathfrak{L}(m)$ of order m and $\mathfrak{L}'(m)$ of order m^3 , Conditions (14) and (15) are obviously fulfilled. Moreover, under $f(x) \geq f_0 > 0$, we know that $\|\Psi_m^{-1}\|_{\text{op}} \leq 1/f_0$, and then M_n^+ has order $n/\log(n)$. We take $M_n^+ = n/\log(n)$ for simplicity. The first terms of the bound in Theorem 5.2 have been already evaluated in the proof of Corollary 4.2, so we have to study the additional ones:

$$\sup_{m < m' \leq n/\log(n)} \left\{ \|\Delta_{m',m'+p}^{f,1}\|_{\text{op}}^2 (\|b_{M_n^+} - b_{m'}\|_f^2 + \|b - b_{M_n^+}\|_{\infty}^2) \right\} \quad \text{and} \quad \sup_{m < m' \leq n/\log(n)} \|b'_{m'} - b'_m\|^2.$$

We assume that $\sum_j \langle b, \varphi_j \rangle^2 j^{2\beta} \leq L$ with $\beta \in \mathbb{N} \cap (1, \infty)$. First,

$$\begin{aligned} \sup_{m < m' \leq n/\log(n)} \|\Delta_{m',m'+p}^{f,1}\|_{\text{op}}^2 \|b_{M_n^+} - b_{m'}\|_f^2 &\leq \frac{\|f\|_{\infty}}{f_0} \sup_{m < m' \leq n/\log(n)} (m')^2 \sum_{j \geq m'} \langle b, \varphi_j \rangle^2 \\ &\leq \frac{\|f\|_{\infty}}{f_0} m^{-2(\beta-1)}. \end{aligned}$$

So, this term is of same order than the bias term. Next,

$$\sup_{m < m' \leq n/\log(n)} \|b'_{m'} - b'_m\|^2 \leq \sum_{j \geq m} [(2\pi j) \langle b, \varphi_j \rangle]^2 \leq C m^{-2(\beta-1)}.$$

Lastly, for any m and $x \in I$,

$$\begin{aligned} |(b - b_m)(x)| &\leq \sqrt{2} \sum_{j \geq m} |\langle b, \varphi_j \rangle| \leq \sqrt{2} \left(\sum_{j \geq m} j^{2\beta} \langle b, \varphi_j \rangle^2 \sum_{j \geq m} j^{-2\beta} \right)^{1/2} \\ &\leq \sqrt{\frac{2L}{2\beta-1}} m^{-\beta+1/2} \leq \mathbf{c}(\beta, L) m^{-\beta+1/2}, \end{aligned}$$

which gives $\|b - b_m\|_{\infty} \leq \mathbf{c}(\beta, L) m^{-\beta+1/2}$ and

$$\sup_{m < m' \leq n/\log(n)} \|\Delta_{m',m'+p}^{f,1}\|_{\text{op}}^2 \|b - b_{M_n^+}\|_{\infty}^2 \leq \frac{\mathbf{c}(\beta, L)}{f_0} \left(\frac{n}{\log(n)} \right)^{-2\beta+3}.$$

We have $n^{-2\beta+3} \leq n^{-2(\beta-1)/(2\beta+1)}$ as soon as $\beta \geq (3 + \sqrt{13})/4 \simeq 1.65$, which holds true when $\beta \in \mathbb{N} \cap (1, \infty)$. In conclusion, this together with Theorem 5.2 and the orders given in Section 4.2 gives the announced result. \square

8.11. Proof of Corollary 5.4. Here, $(\varphi_j)_{j \in \mathbb{N}^*}$ is the Hermite basis, and for $\|\Psi_m^{-1}\|_{\text{op}} = m^\gamma$, the constraint on the collection of models implies $M_n^+ = n^{1/(2\gamma+1/2)}$. This is compatible with the choice of $m_{\text{opt}} = 1/n^{s+1/2}$ as $s > 2\gamma + 9/4 > 2\gamma$. Again, we have to study the orders of the additional terms of the bound in Theorem 5.2:

$$\sup_{m < m' \leq n/\log(n)} \left\{ \|\Delta_{m', m'+p}^{f,1}\|_{\text{op}}^2 (\|b_{M_n^+} - b_{m'}\|_f^2 + \|b - b_{M_n^+}\|_\infty^2) \right\} \quad \text{and} \quad \sup_{m < m' \leq n/\log(n)} \|b'_{m'} - b'_m\|^2$$

under the regularity condition $b \in W_s^H(L)$ with $s > 2\gamma + 9/4 > 1$. First,

$$\begin{aligned} \sup_{m < m' \leq M_n^+} \|\Delta_{m', m'+p}^{f,1}\|_{\text{op}}^2 \|b_{M_n^+} - b_{m'}\|_f^2 &\leq \|f\|_\infty \sup_{m < m' \leq M_n^+} (m')^{\gamma+1} \sum_{j \geq m'} \langle b, \varphi_j \rangle^2 \\ &\leq \|f\|_\infty m^{-s+\gamma+1}. \end{aligned}$$

So, this term is of same order than the bias term. Next, using Formula (8),

$$\sup_{m < m' \leq M_n^+} \|b'_{m'} - b'_m\|^2 \lesssim \sum_{j \geq m} j \langle b, \varphi_j \rangle^2 \leq C m^{-s+1}.$$

This term is also of same order than the first bias term and is negligible with respect to the previous one. Lastly, $\|b - b_m\|_\infty^2 \leq C(s, L) \pi^{-1/2} m^{-s+1}$, and thus

$$\sup_{m < m' \leq M_n^+} \left\{ \|\Delta_{m', m'+p}^{f,1}\|_{\text{op}}^2 \|b - b_{M_n^+}\|_\infty^2 \right\} \lesssim n^{-(s-\gamma-2)/(2\gamma+1/2)}$$

by using the value of M_n^+ . We have

$$-\frac{s-\gamma-2}{2\gamma+1/2} \leq -\frac{s-1-\gamma}{s+1/2} \quad \text{if} \quad (s-\gamma-2) \left(s + \frac{1}{2} \right) - (s-\gamma-1) \left(2\gamma + \frac{1}{2} \right) > 0.$$

Since

$$(s-\gamma-2) \left(s + \frac{1}{2} \right) - (s-\gamma-1) \left(2\gamma + \frac{1}{2} \right) = (s-\gamma)(s-2\gamma-2) - \frac{1}{2},$$

$s-\gamma > 2$ and $s-2\gamma-2 > 1/4$, the constraint is fulfilled and this last term is negligible with respect to the rate. Considering the orders obtained in section 4.3, we get the result. \square

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