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Polynomial continued fractions for $\exp(\pi)$

Tanguy Rivoal

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Abstract

We present two (inequivalent) polynomial continued fraction representations of the number e^π with all their elements in \mathbb{Q} ; no such representation was seemingly known before. The proof uses a classical polynomial continued fraction representation of α^β , for $|\arg(\alpha)| < \pi$ and $\beta \in \mathbb{C} \setminus \mathbb{Z}$, of which we offer a proof using a complex contour integral originating from interpolation theory. We also deduce some consequences of arithmetic interest concerning the elements of certain polynomial continued fraction representations of the (transcendental) Gel'fond-Schneider numbers α^β , where $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$ and $\beta \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$.

1 Introduction

To prove the irrationality of some classical constant ξ , a standard method is to construct two sequences of integers $(p_n)_{n \geq 0}$ and $(q_n)_{n \geq 0}$ such that $0 \neq q_n \xi - p_n \rightarrow 0$ as $n \rightarrow +\infty$. One of the most celebrated example is Apéry's explicit construction [1] of two sequences of rational numbers whose quotient tends to $\zeta(3)$ and prove its irrationality. Apéry's sequences are P -recursive of order 2: they are both solutions of the linear recurrence $(n+1)^3 u_{n+1} - (34n^3 + 51n^2 + 27n + 5)u_n + n^3 u_{n-1} = 0$. A sequence $(u_n)_{n \geq 0}$ is said to be P -recursive of order d when it satisfies a linear recurrence relation of the form $\sum_{j=0}^d p_j(n) u_{n+j} = 0$, $p_j(x) \in \mathbb{C}[x]$, $p_d(x) \neq 0$, and sufficiently many initial values $u_0, u_1, \dots, u_{d'} \in \mathbb{C}$ (for some $d' \geq d-1$) to compute u_n from the recurrence for all $n \geq 0$. Following a terminology used in [3] in a particular situation, we shall say that a number ξ is an *Apéry limit* if there exist two P -recursive sequences $(p_n)_{n \geq 0} \in \overline{\mathbb{Q}}^{\mathbb{N}}$ and $(q_n)_{n \geq 0} \in \overline{\mathbb{Q}}^{\mathbb{N}}$ (not necessarily of the same order) such that $p_n/q_n \rightarrow \xi$ as $n \rightarrow +\infty$ and whose underlying recurrences have coefficients in $\overline{\mathbb{Q}}[x]$. Notice that given two P -recursive sequences solutions of linear recurrences R_1 and R_2 , it is always possible to assume that $R_1 = R_2$, by taking the left lowest common multiple of R_1 and R_2 in the non-commutative ring $\overline{\mathbb{Q}}(n)[S]$, where S is the usual “shift by +1”; this procedure increases the order of the recurrence in general.

The termwise sum and product of P -recursive sequences are P -recursive. Hence Apéry limits form a countable subfield of \mathbb{C} we shall denote by $\mathbf{A}(\overline{\mathbb{Q}})$: indeed, if $p_n/q_n \rightarrow \xi_1$ and $\tilde{p}_n/\tilde{q}_n \rightarrow \xi_2$, we have $(p_n \tilde{p}_n)/(q_n \tilde{q}_n) \rightarrow \xi_1 \xi_2$, $(p_n \tilde{q}_n + q_n \tilde{p}_n)/(q_n \tilde{q}_n) \rightarrow \xi_1 + \xi_2$ and

$q_n/p_n \rightarrow 1/\xi_1$. Given \mathbb{K} a subfield of $\overline{\mathbb{Q}}$, the subfield $\mathbf{A}(\mathbb{K})$ of $\mathbf{A}(\overline{\mathbb{Q}})$ corresponds to Apéry limits for which the underlying linear recurrences can be found with coefficients in $\mathbb{K}[x]$ (but initial conditions of their solutions are not assumed to be necessarily in \mathbb{K}). We also consider the sets $\mathbf{A}_d(\mathbb{K})$ of Apéry limits for which the two underlying recurrences are exactly of order $d \geq 1$ with coefficients in $\mathbb{K}[x]$; these sets are not known to form a subfield of $\mathbf{A}(\overline{\mathbb{Q}})$. The latter contains two important subrings introduced and studied in [7, 8], namely the ring \mathbf{G} of G -values (which contains $\overline{\mathbb{Q}}$, π , Catalan's constant G , $\zeta(3)$, multiple zeta values, Beta values $B(a, b)$ with $a, b \in \mathbb{Q}$, Gamma values $\Gamma(a/b)^b$ with $a, b \in \mathbb{N}$) and the ring \mathbf{E} of E -values (which contains e^a and Bessel's $J_0(a)$, $a \in \overline{\mathbb{Q}}$). $\mathbf{A}(\overline{\mathbb{Q}})$ also contains elements which are conjecturally neither in \mathbf{G} nor in \mathbf{E} , like Euler's constant γ and more generally $\gamma + \log(x)$ ($x \in \overline{\mathbb{Q}}$), $\Gamma(a/b)$ with $a, b \in \mathbb{N}$, and Gompertz constant $\delta := \int_0^\infty e^{-x}/(x+1)dx$; see [4, 18, 19] and [9, p. 424]. It is not known if periods in Konsevich-Zagier's sense are in $\mathbf{A}(\overline{\mathbb{Q}})$ but this is very likely.

Because of Apéry's example, important efforts have been devoted to prove that certain classical numbers, generically denoted ξ here, are Apéry limits with underlying sequences of order 2; this then proves ξ to be in $\mathbf{A}_2(\overline{\mathbb{Q}})$. Besides an obvious arithmetic motivation, another motivation is that this immediately yields a continued fraction representation for ξ with *eventually polynomial elements* (see Lemma 1 in §2), ie we have

$$\xi = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \dots}} = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots \quad (1.1)$$

where for any $n \geq N_0$, $a_n = A(n)$, $b_n = B(n)$ for some $A(x), B(x) \in \mathbb{C}[x]$ and some integer N_0 ; the a_n 's and b_n 's are called the elements of the continued fraction [16, p. 5]. We say that ξ is represented by a *polynomial continued fraction*, following the terminology of [5]. Recall that the sequence of convergents $(p_n/q_n)_n$ of the continued fraction on the right-hand side of (1.1) is $\frac{p_n}{q_n} := b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n}$ where $p_0 = b_0$, $q_0 = 1$ and (by convention) $p_{-1} = 1$, $q_{-1} = 0$, so that the sequences $(p_n)_{n \geq -1}$ and $(q_n)_{n \geq -1}$ both satisfy the linear recurrence $u_n = b_n u_{n-1} + a_n u_{n-2}$, $n \geq 1$. Amongst interesting numbers already known to be in $\mathbf{A}_2(\overline{\mathbb{Q}})$, we have algebraic numbers of the form a^b ($a \in \mathbb{Q}^+$, $b \in \mathbb{Q}$), $\log(2)$, π , π^2 , π^4 , $\zeta(3)$, G , e , δ , $\pi \coth(\pi)$, $\Gamma(1/3)^3/(\pi\sqrt{3})$; see [1, 17, 20, 22], [9, pp. 15, 23, 46, 57, 426] and [16, pp. 266-268]. It is not known if all algebraic numbers, $\zeta(5)$, γ , all integral powers of π , $e\pi$ or $e + \pi$ are in $\mathbf{A}_2(\overline{\mathbb{Q}})$.

The main result of the paper is that the number e^π is in $\mathbf{A}_2(\overline{\mathbb{Q}})$ and more precisely that it can be represented by explicit polynomial continued fractions with elements in \mathbb{Q} . This was seemingly not known before.

Theorem 1. (i) *We have*

$$e^\pi = 1 - \frac{4}{1} - \frac{A(0)}{2B(0)} + \frac{A(1)}{B(1)} + \frac{A(2)}{2B(2)} + \dots + \frac{A(2n-1)}{B(2n-1)} + \frac{A(2n)}{2B(2n)} + \dots \quad (1.2)$$

where $A(x) := 2(x+1)^2 + 8$ and $B(x) := 2x + 3$.

(ii) The number e^π is representable by the two (inequivalent) polynomial continued fractions:

$$e^\pi = 1 + \cfrac{6}{1} - \cfrac{560}{800} - \cfrac{C(2)}{D(2)} - \cfrac{C(3)}{D(3)} - \dots - \cfrac{C(n)}{D(n)} - \dots \quad (1.3)$$

$$= -3 + \cfrac{200}{4} + \cfrac{3744}{1064} - \cfrac{E(2)}{F(2)} - \cfrac{E(3)}{F(3)} - \dots - \cfrac{E(n)}{F(n)} - \dots, \quad (1.4)$$

where

$$\begin{aligned} C(x) &:= 4A(2x-2)A(2x-1)B(2x-4)B(2x), \\ D(x) &:= 2B(2x-2)A(2x) + 4B(2x-2)B(2x-1)B(2x) + 2A(2x-1)B(2x), \\ E(x) &:= A(2x-1)A(2x)B(2x-3)B(2x+1), \\ F(x) &:= B(2x-1)A(2x+1) + 2B(2x-1)B(2x)B(2x+1) + A(2x)B(2x+1). \end{aligned}$$

In particular, e^π is in $\mathbf{A}_2(\mathbb{Q})$.

By inequivalent, we mean that the continued fractions (1.3) and (1.4) don't have the same sequences of convergents $(\widehat{p}_n/\widehat{q}_n)_n$ and $(\widetilde{p}_n/\widetilde{q}_n)_n$. Moreover, letting $(p_n/q_n)_n$ denote the sequence of convergents $(p_n/q_n)_n$ of (1.2), we have

$$\lim_{n \rightarrow +\infty} \left| e^\pi - \frac{p_n}{q_n} \right|^{1/n} = \lim_{n \rightarrow +\infty} \left| e^\pi - \frac{\widehat{p}_n}{\widehat{q}_n} \right|^{1/(2n)} = \lim_{n \rightarrow +\infty} \left| e^\pi - \frac{\widetilde{p}_n}{\widetilde{q}_n} \right|^{1/(2n)} = \frac{1}{2}(2 - \sqrt{2})^2$$

and

$$\lim_{n \rightarrow +\infty} |q_n|^{1/n} = \lim_{n \rightarrow +\infty} |\widehat{q}_n|^{1/(2n)} = \lim_{n \rightarrow +\infty} |\widetilde{q}_n|^{1/(2n)} = 2 + \sqrt{2}.$$

(See Lemma 6 with $\alpha = e^{i\pi/2}$ and $\beta = -2i$.)

To prove Theorem 1, we shall need the following proposition. For $\alpha \in \mathbb{C}^*$, we set $\log(\alpha) := \ln|\alpha| + i \arg(\alpha)$ where $-\pi < \arg(\alpha) < \pi$ and for any $\beta \in \mathbb{C}$, $\alpha^\beta := \exp(\beta \log(\alpha))$.

Proposition 1. *Let $\alpha \in \mathbb{C}^*$ be such that $|\arg(\alpha)| < \pi$ and $\beta \in \mathbb{C} \setminus \mathbb{Z}$. Set $A(x) := (x+1)^2 - \beta^2$ and $B(x) := (\alpha+1)(2x+3)$. Then*

$$\begin{aligned} \alpha^\beta = 1 + & \cfrac{2\beta(\alpha-1)}{\alpha + \beta + 1 - \alpha\beta} - \cfrac{(\alpha-1)^2 A(0)}{B(0)} \\ & - \cfrac{(\alpha-1)^2 A(1)}{B(1)} - \cfrac{(\alpha-1)^2 A(2)}{B(2)} - \dots - \cfrac{(\alpha-1)^2 A(n)}{B(n)} - \dots \quad (1.5) \end{aligned}$$

That result is not new, see for instance [12, p. 105] where it is attributed to Euler. We shall give a proof of Proposition 1 different of that of [12] (based on an analysis of solutions of Riccati equation). The method directly provides explicit closed form formulas for the convergents as well as exact rates of convergence, not given in [12]. It can also be adapted to produce other continued fractions for α^β ; see §3.4 for some details.

Proposition 1 implies that, for every $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$, $\beta \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$, the Gel'fond-Schneider number α^β is in $\mathbf{A}_2(\mathbb{Q}(\alpha, \beta^2))$. Observe that if $\alpha \in \mathbb{Q}$ and $\beta^2 \in \mathbb{Q}$, then $A(x)$ and $B(x)$ are in $\mathbb{Q}[x]$, so that the elements of the continued fractions for these numbers are rational numbers, except possibly the second and third ones. This applies for instance to $2^{\sqrt{2}}$ which is thus proved to be in $\mathbf{A}_2(\mathbb{Q})$: taking $\alpha = 2$ and $\beta = \sqrt{2}$ in (1.5), we get

$$2^{\sqrt{2}} = 1 - \frac{2\sqrt{2}}{\sqrt{2}-3} + \frac{1}{9} - \frac{2}{15} - \frac{7}{21} - \dots - \frac{n^2 + 2n - 1}{3(2n+3)} - \dots$$

It would be interesting to obtain a polynomial continued fraction for $2^{\sqrt{2}}$ with all its elements in \mathbb{Q} ; none of those listed in [16, pp. 269-270] seems to provide one. The case of e^π is particularly remarkable. Taking $\alpha = e^{i\pi/2}$ and $\beta = -2i$, or $\alpha = e^{-i\pi/2}$ and $\beta = 2i$ (both choices lead in the end to the fractions in Theorem 1), Proposition 1 provides for it a polynomial continued fraction with elements in $\mathbb{Q}(i)$. This *a priori* only proves that e^π is in $\mathbf{A}_2(\mathbb{Q}(i))$ and surprisingly a more careful analysis leads to Theorem 1. This choice of parameters is very special in the sense that, taking $\alpha = e^{i\pi/k}$ and $\beta = -ki$ for some integer k such that $|k| \geq 3$, we apparently do not get a continued fraction for e^π with all its elements in \mathbb{Q} .

We conclude this introduction with the following simple considerations. Euler gave a formal transformation of a series into a continued fraction, the sequence of convergents of which coincide with the sequence of partial sums of the series (so that the convergence of the series is equivalent to that of the continued fraction); see [15, p. 458, Eq. (12)]. It is more easily stated as

$$\sum_{n=0}^{\infty} \left(\prod_{j=0}^n a_j \right) = \frac{a_0}{1} - \frac{a_1}{1+a_1} - \frac{a_2}{1+a_2} - \dots - \frac{a_n}{1+a_n} - \dots \quad (1.6)$$

That transformation is at the origin of numerous polynomial continued fractions for classical numbers such as e and π (see for instance [5, 15]). It can be applied as well to the series $\sum_{n=0}^{\infty} \binom{\beta}{n} (\alpha - 1)^n$ which converges (for any $\alpha \in \mathbb{C}$ such that $|\alpha - 1| < 1$ and any $\beta \in \mathbb{C}$) to α^β with $|\arg(\alpha)| < \pi$. In this case, the corresponding sequence $(a_n)_{n \geq 0}$ in (1.6) is $a_0 = 1$ and $a_n = (\alpha - 1)(\beta - n + 1)/n$ for $n \geq 1$. This leads to the continued fraction (this is [16, p. 269, Eq. (2.3.5)] that holds for $|\alpha - 1| < 1$ and any $\beta \in \mathbb{C} \setminus \mathbb{Z}$:

$$\alpha^\beta = \frac{1}{1} - \frac{(\alpha - 1)\beta}{1 + (\alpha - 1)\beta} - \frac{(\alpha - 1)1(\beta - 1)}{2 + (\alpha - 1)(\beta - 1)} - \dots - \frac{(\alpha - 1)n(\beta - n)}{n + 1 + (\alpha - 1)(\beta - n)} - \dots \quad (1.7)$$

Applying (1.7) with $\alpha = 1/2$ and $\beta = -\sqrt{2}$, we obtain a polynomial continued fraction for $2^{\sqrt{2}}$ with elements in $\mathbb{Q}(\sqrt{2})$. We can also apply it with $\alpha = e^{i\pi\ell/k}$ for any rational number ℓ/k such that $0 < |\ell/k| < 1/3$ (so that $0 < |\alpha - 1| < 1$) and $\beta = -ik/\ell$; this yields a polynomial continued fraction for e^π with elements in $\mathbb{Q}(i, e^{i\pi\ell/k})$. These are weaker results than those obtained above using Proposition 1 but they are much simpler to obtain.

The proof of Theorem 1 is given in §2 and that of Proposition 1 in §3.

2 Proof of Theorem 1

We first state and prove a useful lemma that make the connection between linear recurrences of order 2 and continued fractions.

Lemma 1. *Let $(p_n)_{n \geq -1} \in \mathbb{C}^{\mathbb{N}}$ and $(q_n)_{n \geq -1} \in \mathbb{C}^{\mathbb{N}}$ be two solutions of a recurrence*

$$u_n = B_n u_{n-1} + A_n u_{n-2}, \quad n \geq 1 \quad (2.1)$$

with initial conditions $p_{-1} = a \neq 0$, $p_0 = b$, $q_{-1} = c$, $q_0 = d$ such that $ad \neq bc$. Assume that $\xi := \lim_{n \rightarrow +\infty} p_n/q_n$ exists and is finite.

If $c\xi \neq a$, then

$$\frac{A_1}{B_1} + \frac{A_2}{B_2} + \frac{A_3}{B_3} + \dots = \frac{(ad - bc)\xi}{a(a - c\xi)} - \frac{b}{a}.$$

A continued fraction representation for ξ is readily obtained.

Proof. The sequences $\hat{p}_n := \frac{1}{a}p_n$ and $\hat{q}_n = \frac{1}{ad-bc}(aq_n - cp_n)$ are solutions of (2.1) with initial conditions

$$\hat{p}_{-1} = 1, \quad \hat{p}_0 = \frac{b}{a}, \quad \hat{q}_{-1} = 0, \quad \hat{q}_0 = 1.$$

Moreover,

$$\lim_{n \rightarrow +\infty} \frac{\hat{p}_n}{\hat{q}_n} = \frac{(ad - bc)\xi}{a(a - c\xi)}.$$

The theory of continued fractions [16, p. 6] then ensures that

$$\frac{(ad - bc)\xi}{a(a - c\xi)} = \frac{b}{a} + \frac{A_1}{B_1} + \frac{A_2}{B_2} + \frac{A_3}{B_3} + \dots.$$

This completes the proof. \square

Proof of Theorem 1. (i) In Proposition 1, we take $\alpha = e^{i\pi/2}$ and $\beta = 2i$, so that $\alpha^\beta = e^\pi$. To avoid a clash of notations between Theorem 1 and Proposition 1, we temporarily set $a(x) := 2(x+1)^2 + 8$ and $b(x) := 2x+3$. Hence $a(x) = 2A(x)$ and $b(x) = B(x)/(i+1)$ where A and B are as in Proposition 1, that yields

$$e^\pi = 1 + \frac{-4(i+1)}{i+1} - \frac{-ia(0)}{(i+1)b(0)} - \frac{-ia(1)}{(i+1)b(1)} - \frac{-ia(2)}{(i+1)b(2)} - \dots - \frac{-ia(n)}{(i+1)b(n)} + \dots \quad (2.2)$$

Recall that we want to prove that

$$e^\pi = 1 - \frac{4}{1} + \frac{a(0)}{2b(0)} + \frac{a(1)}{b(1)} + \frac{a(2)}{2b(2)} + \dots + \frac{a(2n-1)}{b(2n-1)} + \frac{a(2n)}{2b(2n)} + \dots \quad (2.3)$$

To do that, we shall prove that (2.2) and (2.3) are *equivalent* continued fractions in the sense that they have the same sequence of convergents, so that they both converge to e^π because the right-hand side of (2.2) does. By [14, p. 235, Theorem 2.1], two continued fractions $b_0 + \frac{a_1}{|b_1|} + \frac{a_2}{|b_2|} + \dots$ and $d_0 + \frac{c_1}{|d_1|} + \frac{c_2}{|d_2|} + \dots$ are equivalent if and only if there exists a sequence $(\rho_n)_{n \geq 0}$ such that $\rho_0 = 1$, $\rho_n \neq 0$ for all $n \geq 0$ and $c_n = \rho_n \rho_{n-1} a_n$ ($n \geq 1$), $d_n = \rho_n b_n$ ($n \geq 0$). Here, considering that $b_0 + \frac{a_1}{|b_1|} + \frac{a_2}{|b_2|} + \dots$ is the right-hand side of (2.3) and $d_0 + \frac{c_1}{|d_1|} + \frac{c_2}{|d_2|} + \dots$ is the right-hand side of (2.2), we see that taking $\rho_0 = 1$, $\rho_{2n+1} = i+1$ and $\rho_{2n+2} = (i+1)/2$ for $n \geq 0$, we have $c_n = \rho_n \rho_{n-1} a_n$ ($n \geq 1$) and $d_n = \rho_n b_n$ ($n \geq 0$).

(ii) We come back to the notations of Theorem 1. Observe the alternance of the factors 1 and 2 in front of $B(n)$: the continued fraction (1.2) is formally not a polynomial continued fraction. Consider now the sequence $(p_n/q_n)_n$ of its convergents. The sequences $(p_n)_n$ and $(q_n)_n$ are solutions of the same linear recurrences:

$$\begin{aligned} u_{2n} &= \mu_{2n} u_{2n-1} + \lambda_{2n} u_{2n-2}, \\ u_{2n+1} &= \delta_{2n+1} u_{2n} + \lambda_{2n+1} u_{2n-1} \end{aligned}$$

where, for any n large enough, $\lambda_n = A(n-2)$, $\mu_n = 2B(n-2)$, $\delta_n = B(n-1)$. A quick computation then shows that

$$\begin{aligned} \delta_{2n+1} u_{2n+3} &= \\ &((\delta_{2n+3} \mu_{2n+2} + \lambda_{2n+3}) \delta_{2n+1} + \delta_{2n+3} \lambda_{2n+2}) u_{2n+1} - \delta_{2n+3} \lambda_{2n+2} \lambda_{2n+1} u_{2n-1}. \end{aligned} \quad (2.4)$$

This is a linear recurrence of order 2 with coefficients in $\mathbb{Q}[n]$. Hence, not only (2.4) implies that $e^\pi = \lim_n \frac{p_{2n+1}}{q_{2n+1}}$ is in $\mathbf{A}_2(\mathbb{Q})$, but Lemma 1 applied to (2.4) shows that e^π can be represented by a polynomial continued fraction with elements in $\mathbb{Q}[n]$. We can similarly obtain the linear recurrence of order 2 satisfied by $(p_{2n})_n$ and $(q_{2n})_n$:

$$\mu_{2n} u_{2n+2} = ((\delta_{2n+1} \mu_{2n+2} + \lambda_{2n+2}) \mu_{2n} + \mu_{2n+2} \lambda_{2n+1}) u_{2n} - \mu_{2n+2} \lambda_{2n+1} \lambda_{2n} u_{2n-2}$$

and make the same deductions.

In fact, this process is classical: it consists in “taking the odd (respectively even) part” of a continued fraction. The two resulting continued fractions are consequences of the general formulas given in [16, p. 86-87, Theorems 2.19 and 2.20], which we repeat below. For the even part, we have

$$\begin{aligned} &\frac{a_1}{|b_1|} + \frac{a_2}{|b_2|} + \frac{a_3}{|b_3|} + \dots \\ &= \frac{b_2 a_1}{|b_2 b_1 + a_2|} - \frac{a_2 a_3 b_4}{|b_2(a_4 + b_3 b_4) + a_3 b_4|} - \frac{a_4 a_5 b_6 b_2}{|b_4(a_6 + b_5 b_6) + a_5 b_6|} - \frac{a_6 a_7 b_8 b_4}{|b_6(a_8 + b_7 b_8) + a_7 b_8|} - \dots \end{aligned}$$

Applying this formula to (1.2), we obtain (1.3) after some simplifications, which is obviously a polynomial continued fraction. The formula for the odd part is

$$\begin{aligned} & \left| \frac{a_1}{b_1} \right| + \left| \frac{a_2}{b_2} \right| + \left| \frac{a_3}{b_3} \right| + \dots \\ &= \frac{a_1}{b_1} - \frac{a_1 a_2 b_3 / b_1}{b_1(a_3 + b_2 b_3) + a_2 b_3} - \frac{a_3 a_4 b_5 b_1}{b_3(a_5 + b_4 b_5) + a_4 b_5} - \frac{a_5 a_6 b_7 b_3}{b_5(a_7 + b_6 b_7) + a_6 b_7} - \dots, \end{aligned}$$

from which we compute the odd part of (1.2), given by the polynomial continued fraction (1.4).

Finally, let us prove that the continued fractions (1.3) and (1.4) are not equivalent. Indeed, if they were equivalent, by the characterization of equivalence used in (i) above, the quantity $\rho_n := D(n)/F(n)$ (well defined for all integer $n \geq 2$ because F vanishes at no integer point) would satisfy

$$\frac{C(n)}{E(n)} = \rho_n \rho_{n-1} = \frac{D(n)D(n-1)}{F(n)F(n-1)}$$

for every integer $n \geq 3$, hence for all $n \in \mathbb{C}$ (but finitely many points) because this is an equality involving rational functions of n . But

$$\frac{C(n)}{E(n)} = \frac{(5 + 4n + 4n^2)(5 + 4n)(4n - 3)}{4(4n^2 - 4n + 5)(4n + 3)(4n - 5)}$$

and

$$\frac{D(n)D(n-1)}{F(n)F(n-1)} = \frac{(43n + 54n^2 + 24n^3 + 12)(24n^3 + 7n - 18n^2 - 1)}{4(24n^3 + 18n^2 + 7n + 1)(24n^3 - 54n^2 + 43n - 12)}$$

are not equal because $5/4$ is a pole of the former and not of the latter. This completes the proof of Theorem 1. \square

Remark. We can produce many more inequivalent polynomial continued fractions for e^π by taking the odd or even part of (1.3) and (1.4), and so on and so forth.

3 Proof of Proposition 1

We first need to specify the branch of logarithm. It will be necessary below to have $|\sqrt{\alpha} + 1| \neq |\sqrt{\alpha} - 1|$, ie that $\alpha \notin \mathbb{R}^-$. Thus we consider \mathbb{R}^- as a cut and we set $\log(\alpha) := \ln|\alpha| + i \arg(\alpha)$, where $-\pi + 2k_0\pi < \arg(\alpha) < \pi + 2k_0\pi$ for some $k_0 \in \mathbb{Z}$. Moreover, we shall first prove Proposition 1 under the following supplementary technical assumption (which will then be lifted):

$$|e^{|\log \alpha|/2} - 1| < \max|\sqrt{\alpha} \pm 1|, \quad (3.1)$$

where for simplicity, we set $\max|\sqrt{\alpha} \pm 1| := \max(|\sqrt{\alpha} + 1|, |\sqrt{\alpha} - 1|)$, and similarly for min. A necessary condition for Eq. (3.1) to hold is that $k_0 = 0$. Hence, from now on, we

assume that $\log(\alpha)$ is defined with its principal determination. Accordingly, for any $z \in \mathbb{C}$, we set $\alpha^z := \exp(z \log(\alpha))$.

Note that this weaker version of Proposition 1 is enough to prove Theorem 1, with $\alpha = i$ and $\beta = -2i$. Moreover, (3.1) is always satisfied if $\alpha > 1$ because $|e^{|\log(\alpha)|/2} - 1| = |\sqrt{\alpha} - 1|$ in this case.

3.1 Construction of the convergents

For every integers $m, n \geq 0$, let

$$I(m, n) := \frac{m!n!}{2i\pi} \int_{\mathcal{C}} \frac{\alpha^z dz}{(\prod_{j=0}^m (z - j)) (\prod_{j=0}^n (z - j - \beta))}$$

where \mathcal{C} is any closed direct curve surrounding the poles of the integrand. If $\alpha = 1$, then $I(n, m) = 0$ for all $m, n \geq 0$; this case is not interesting even though most lemmas below are valid for $\alpha = 1$. The integral $I(m, n)$ is a variation of a family of integrals appearing in interpolation theory, especially for functions of exponential type, where the interpolating sets are here \mathbb{N} and $\mathbb{N} + \beta$. When $\alpha = e^{i\pi/2}$, $\beta = -2i$ or $\alpha = 2$, $\beta = \sqrt{2}$, suitable generalisations of $I(m, n)$ enabled Gel'fond [10] and Kuzmin [13] to prove the transcendence of e^π and $2^{\sqrt{2}}$ respectively ⁽¹⁾ by interpolating $e^{\pi z}$ and 2^z on $\mathbb{Z}[i]$ and $\mathbb{Z}[\sqrt{2}]$ respectively. Taking $n = 0$ and allowing multiplicities for the poles at $0, 1, \dots, m$, the case $\alpha = e$ leads to a proof of the transcendence of e .

Lemma 2. *Let $\alpha \in \mathbb{C}^*$ such that $|\arg(\alpha)| < \pi$ and $\beta \in \mathbb{C} \setminus \mathbb{Z}$. For every integers $m, n \geq 0$, we have*

$$I(m, n) = Q(m, n)\alpha^\beta - P(m, n),$$

where

$$Q(m, n) := (-1)^n \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} m!}{\prod_{j=0}^m (k - j + \beta)} \alpha^k$$

and

$$P(m, n) := (-1)^{m+1} \sum_{k=0}^m \frac{(-1)^k \binom{m}{k} n!}{\prod_{j=0}^n (k - j - \beta)} \alpha^k.$$

are polynomials in α of respective degree n and m .

Proof. This is an immediate application of the residue theorem applied to $I(m, n)$ because all the poles of the integrand are simple. \square

Note that $(P(m, n))_{m, n \geq 0}$ and $(Q(m, n))_{m, n \geq 0}$ are in fact well-defined for any $\alpha \in \mathbb{C}$ and any $\beta \in \mathbb{C} \setminus \mathbb{Z}$. Note also that

$$\int_{\mathcal{C}} \frac{z^k dz}{(\prod_{j=0}^m (z - j)) (\prod_{j=0}^n (z - j - \beta))} = 0$$

¹The interpolating integral is implicit in Kuzmin's paper.

for any integer $k \in \{0, \dots, m+n\}$; hence the expansion $\alpha^z = \sum_{k=0}^{\infty} \frac{\log(\alpha)^k}{k!} z^k$ shows that $I(m, n)$ is the remainder term of the $[m/n]$ Padé approximant of α^β at $\alpha = 1$. See [2] for an alternative approach (and different goals) to the computation of the $[n/n]$ Padé approximants, with more complicated expressions and no recurrence given as in the next lemma.

We shall now only be interested in the case $m = n$ and for $S \in \{I, P, Q\}$, we set $S_n := S(n, n)$.

Lemma 3. *Let $\alpha \in \mathbb{C}^*$ such that $|\arg(\alpha)| < \pi$ and $\beta \in \mathbb{C} \setminus \mathbb{Z}$. The three sequences $(P_n)_{n \geq 0}$, $(Q_n)_{n \geq 0}$ and $(I_n)_{n \geq 0}$ are solutions of the linear recurrence of order 2:*

$$((n+2)^2 - \beta^2)u_{n+2} - (\alpha+1)(n+2)(2n+3)u_{n+1} + (\alpha-1)^2(n+1)(n+2)u_n = 0. \quad (3.2)$$

The roots of the characteristic equation associated to this recurrence are $(\sqrt{\alpha} \pm 1)^2$. They are distinct and such that $|\sqrt{\alpha} + 1| \neq |\sqrt{\alpha} - 1|$.

Note that

$$P_0 = \frac{1}{\beta}, \quad P_1 = -\frac{\alpha\beta + \alpha - \beta + 1}{\beta(\beta^2 - 1)}, \quad Q_0 = \frac{1}{\beta}, \quad Q_1 = \frac{\alpha\beta - \alpha - \beta - 1}{\beta(\beta^2 - 1)}.$$

Proof. We apply Zeilberger's algorithm (as implemented in Maple) to the two sequences $(P_n)_{n \geq 0}$ and $(Q_n)_{n \geq 0}$. They are both found to satisfy (3.2) (under even less restrictive assumptions on α and β). By linearity, the same holds for the sequence $(I_n)_{n \geq 0}$.

Let us give the details for $P_n = \sum_{k=0}^n T(n, k)$ where

$$T(n, k) := (-1)^{n+1} \frac{\binom{n}{k} n!}{\prod_{j=0}^n (k - j - \beta)} (-\alpha)^k.$$

Note that $P_n = \sum_{k=0}^{n+\ell} T(n, k)$ for any integer $\ell \geq 0$ because $\binom{n}{k} = 0$ for any integer $k \geq n+1$. Zeilberger's algorithm (as implemented in Maple 16) shows that

$$((n+2)^2 - \beta^2)T(n+2, k) - (\alpha+1)(n+2)(2n+3)T(n+1, k) + (\alpha-1)^2(n+1)(n+2)T(n, k) = G(n, k+1) - G(n, k) \quad (3.3)$$

where the certificat

$$G(n, k) = (4n^2 - \alpha n^2 - \alpha\beta n + 13n + 2kn\alpha - 4kn - 4n\alpha + 2n\beta - k\beta + 10 + k^2 - 4\alpha - 2\alpha\beta - 6k + 3\beta + 4k\alpha + k\alpha\beta - k^2\alpha) \frac{(-\alpha)^k \Gamma(1-k+\beta) n! (n+2)!}{\Gamma(n+3-k+\beta) \Gamma(k) \Gamma(n+3-k)}$$

Since $G(n, 0) = G(n, n+3) = 0$ (because of the factor $\Gamma(k)\Gamma(n+3-k)$ at the denominator), summing both sides of (3.3) for k from 0 to $n+2$ proves that $(P_n)_{n \geq 0}$ is solution of (3.2).

The sequence $(Q_n)_{n \geq 0}$ is also one of its solution because Q_n is obtained from $-P_n$ by changing β to $-\beta$, and the polynomial coefficients of (3.2) depends on β only through β^2 .

The characteristic equation of the recurrence (3.2) is $x^2 - 2(\alpha + 1)x + (\alpha - 1)^2 = 0$, with solutions given by $(\sqrt{\alpha} \pm 1)^2$, obviously distinct because $\alpha \neq 0$. Finally, $|\sqrt{\alpha} + 1| = |\sqrt{\alpha} - 1|$ implies that $\alpha \in \mathbb{R}^-$, which is excluded. \square

Lemma 4. *Let us assume that $\alpha \in \mathbb{C} \setminus \{1\}$ and $\beta \in \mathbb{C} \setminus \mathbb{Z}$. Then, the sequences $(P_n)_{n \geq 0}$ and $(Q_n)_{n \geq 0}$ are linearly independent over \mathbb{C} .*

Proof. We define the Casoratian $C_n := P_n Q_{n+1} - P_{n+1} Q_n$. It is readily checked that C_n is the solution of the recurrence

$$(\beta + n + 2)(\beta - n - 2)C_{n+1} = -(\alpha - 1)^2(n + 1)(n + 2)C_n$$

with initial condition $C_0 = \frac{2(\alpha-1)}{\beta(\beta^2-1)}$. It follows that

$$C_n = (-1)^n \frac{2(\alpha - 1)^{2n+1} n! (n + 1)!}{(\beta)_{n+2} (\beta - n - 1)_{n+1}}$$

and this implies that $C_n \neq 0$ for every $n \geq 0$. We then use the fact that the sequences $(P_n)_{n \geq 0}$ and $(Q_n)_{n \geq 0}$ are linearly independent over \mathbb{C} if and only if their Casoratian does not vanish for every $n \geq n_0$ (for some integer $n_0 \geq 0$); see [6, p. 71, Theorem 2.15]. \square

3.2 Convergence of the sequence of convergents

In this section, we determine the behavior of the sequences P_n, Q_n and I_n as $n \rightarrow +\infty$.

Lemma 5. *Let $\alpha \in \mathbb{C}^*$ such that $|\arg(\alpha)| < \pi$ and $\beta \in \mathbb{C} \setminus \mathbb{Z}$. We have*

$$\limsup_{n \rightarrow +\infty} |I_n|^{1/n} \leq (e^{|\log \alpha|/2} - 1)^2. \quad (3.4)$$

Observe that both sides of (3.4) vanish when $\alpha = 1$. Moreover, as we shall see in Lemma 6 below, Eq. (3.4) is sharp when $\alpha \geq 1$.

Proof. In the integral for I_n , we choose \mathcal{C} as the circle of center 0 and radius $R := vn$ for some $v > 1$ to be specified later, with n large enough to ensure that the poles of the integrand are all inside the open disk delimited by \mathcal{C} . For $z \in \mathcal{C}$, we have

$$\prod_{j=0}^n |z - j| \geq c_{v,n} \frac{\Gamma(vn)}{\Gamma((v-1)n)} \quad \text{and} \quad \prod_{j=0}^n |z - j - \beta| \geq d_{v,\beta,n} \frac{\Gamma(vn)}{\Gamma((v-1)n)},$$

where $c_{v,n}^{1/n}$ and $d_{v,\beta,n}^{1/n}$ both tend to 1 as $n \rightarrow +\infty$. Moreover, for $z \in \mathcal{C}$, we have $|\alpha^z| \leq e^{|\log(\alpha)|vn}$. Hence, using Stirling's formula $\Gamma(x) = x^x x^{x-1/2} \sqrt{2\pi} (1 + O(1/x))$ (valid for $|\arg(x)| < \pi$), we see that

$$\limsup_{n \rightarrow +\infty} |I_n|^{1/n} \leq e^{|\log \alpha|v} (v - 1)^{2(v-1)} v^{-2v}.$$

It remains to minimize the right-hand side with respect to $v > 1$, which is immediate: it is minimal for $v = 1/(e^{-|\log \alpha|/2} - 1) > 1$, which leads to (3.4). \square

Lemma 6. *Let $\alpha \in \mathbb{C}^*$ such that $|\arg(\alpha)| < \pi$ and $\beta \in \mathbb{C} \setminus \mathbb{Z}$. Assume that (3.1) holds. Then*

$$\lim_{n \rightarrow +\infty} |P_n|^{1/n} = \lim_{n \rightarrow +\infty} |Q_n|^{1/n} = \max |\sqrt{\alpha} \pm 1|^2$$

and

$$\lim_{n \rightarrow +\infty} |I_n|^{1/n} = \min |\sqrt{\alpha} \pm 1|^2.$$

Proof. We recall that the Birkhoff-Trjitzinsky theory (see [18, 21] for the principal results) ensures the existence of two independent solutions $(U_n)_{n \geq 0}$ and $(V_n)_{n \geq 0}$ of (3.2) such that

$$\lim_{n \rightarrow +\infty} U_n^{1/n} = (\sqrt{\alpha} + 1)^2 \quad \text{and} \quad \lim_{n \rightarrow +\infty} V_n^{1/n} = (\sqrt{\alpha} - 1)^2.$$

Any other solution $(W_n)_{n \geq 0}$ of (3.2) is a \mathbb{C} -linear combination of $(U_n)_{n \geq 0}$ and $(V_n)_{n \geq 0}$ and is such that

$$\lim_{n \rightarrow +\infty} W_n^{1/n} = (\sqrt{\alpha} + 1)^2 \quad \text{or} \quad \lim_{n \rightarrow +\infty} W_n^{1/n} = (\sqrt{\alpha} - 1)^2.$$

because $|\sqrt{\alpha} + 1| \neq |\sqrt{\alpha} - 1|$ by the assumption $|\arg(\alpha)| < \pi$ (ie, there is no oscillating behavior, which typically happens when the modulus of characteristic roots are equal). By Lemma 4, the sequences $(P_n)_{n \geq 0}$ and $(Q_n)_{n \geq 0}$ are \mathbb{C} -independent (and thus generate the space of solutions of (3.2)), so that I_n is not 0 for any large enough n . Hence, Assumption (3.1) and Lemma 5 together imply that

$$\lim_{n \rightarrow +\infty} |I_n|^{1/n} = \min |\sqrt{\alpha} \pm 1|^2.$$

We now assume that $\max |\sqrt{\alpha} \pm 1| = |\sqrt{\alpha} + 1|$; the other possibility would be dealt with in a similar way. There exist $a, b, c, d \in \mathbb{C}$ such that for all $n \geq 0$

$$P_n = aU_n + bV_n \quad \text{and} \quad Q_n = cU_n + dV_n.$$

We cannot have $a = 0$ and $c = 0$ because by Lemma 4, the sequences $(P_n)_{n \geq 0}$ and $(Q_n)_{n \geq 0}$ are \mathbb{C} -independent. Assume that $a \neq 0$, so that $|P_n|$ behave like $|\sqrt{\alpha} + 1|^{2n}$. Then, $c \neq 0$ as well because $|I_n| = |Q_n \alpha^\beta - P_n|$ behaves like $|\sqrt{\alpha} - 1|^{2n}$. Similarly, $c \neq 0$ forces $a \neq 0$. Hence we always $ac \neq 0$, so that

$$\lim_{n \rightarrow +\infty} |P_n|^{1/n} = \lim_{n \rightarrow +\infty} |Q_n|^{1/n} = \lim_{n \rightarrow +\infty} |U_n|^{1/n} = |\sqrt{\alpha} + 1|^2$$

as expected. □

3.3 Completion of the proof of Proposition 1

We first assume that (3.1) holds, and we recall that $|\arg(\alpha)| < \pi$ is a necessary condition for that. Consider the sequences $(P_n)_{n \geq 0}$ and $(Q_n)_{n \geq 0}$ as in Lemma 3 with the same initial conditions. Under the assumptions of Lemma 6, we have

$$\lim_{n \rightarrow +\infty} \left(\alpha^\beta - \frac{P_n}{Q_n} \right) = \lim_{n \rightarrow +\infty} \frac{I_n}{Q_n} = 0,$$

and even more precisely

$$\lim_{n \rightarrow +\infty} \left| \alpha^\beta - \frac{P_n}{Q_n} \right|^{1/n} = \frac{\min |\sqrt{\alpha} \pm 1|^2}{\max |\sqrt{\alpha} \pm 1|^2} < 1.$$

Set $p_n := P_{n+1}$ and $q_n := Q_{n+1}$ for all $n \geq -1$. We have

$$p_{-1} = \frac{1}{\beta}, p_0 = -\frac{\alpha\beta + \alpha - \beta + 1}{\beta(\beta^2 - 1)}, q_{-1} = \frac{1}{\beta}, q_0 = \frac{\alpha\beta - \alpha - \beta - 1}{\beta(\beta^2 - 1)}.$$

To apply Lemma 1 with $\xi := \alpha^\beta$, we first need to ensure that $\alpha^\beta \neq p_{-1}/q_{-1} = 1$ and that

$$0 \neq p_{-1}q_0 - p_0q_{-1} = \frac{2(\alpha - 1)}{\beta(\beta^2 - 1)}$$

ie that $\alpha \neq 1$. With our assumptions on α, β and the definition of α^β for $|\arg(\alpha)| < \pi$, the condition $\alpha^\beta \neq 1$ is equivalent to $\alpha \neq 1$. Then, by Lemma 1, we have that

$$\frac{2\beta(\alpha - 1)}{\beta^2 - 1} \cdot \frac{1}{\alpha^{-\beta} - 1} = \frac{\beta - \alpha - \alpha\beta - 1}{\beta^2 - 1} + \frac{C(1)/A(1)}{D(1)/A(1)} + \frac{C(2)/A(2)}{D(2)/A(2)} + \dots$$

where $A(x) := (x+1)^2 - \beta^2$, $C(x) := -(\alpha-1)^2x(x+1)$, and $D(x) := (\alpha+1)(x+1)(2x+1)$. We also need $B(x) := (\alpha+1)(2x+3)$ because after some straightforward simplifications, we obtain

$$\alpha^{-\beta} = 1 + \frac{2\beta(\alpha - 1)}{\beta - \alpha - \alpha\beta - 1} + \frac{(\alpha - 1)^2 A(0)}{B(0)} - \frac{(\alpha - 1)^2 A(1)}{B(1)} - \frac{(\alpha - 1)^2 A(2)}{B(2)} - \dots - \frac{(\alpha - 1)^2 A(n)}{B(n)} + \dots \quad (3.5)$$

This is the statement of the Proposition 1 with $-\beta$ instead of β . Observe now that the assumptions of Proposition 1 (even under (3.1)) are the same for β and $-\beta$, and that $A(x)$ and $B(x)$ depend on β only through β^2 . We thus obtain (1.5) by simply changing β to $-\beta$ in (3.5). We can get rid of the assumption $\alpha \neq 1$ because (3.5) holds for $\alpha = 1$ and $\beta \in \mathbb{C} \setminus \mathbb{Z}$.

It remain to get rid of the technical assumption (3.1). For this, we remark that in [12, pp. 102-106] Khovanskii proved that the continued fraction on the right-hand side of (1.5) defines an analytic function in $\mathbb{C} \setminus \mathbb{R}^-$ (because its elements satisfy a general property implying that). Now, Assumption (3.1) holds for $\alpha > 1$. Hence, Identity (1.5) holds for $\alpha > 1$ and then for $|\arg(\alpha)| < \pi$ by analytic continuation of both sides.

3.4 Variations on Proposition 1

The method used to prove Proposition 1 is flexible. We started with the integral $I(n, n)$ but we can also consider an integral of the form $I(kn, \ell n)$, where k, ℓ are positive integers.

By Lemma 2, we have $I(kn, \ell n) = Q(kn, \ell n)\alpha^\beta - P(kn, \ell n)$. Our method then rests on the explicit computation of the linear recurrence satisfied by $P(kn, \ell n)$ and $Q(kn, \ell n)$. Provided k and ℓ are given specific values, Zeilberger's algorithm can again be used. For instance, for $k = 2$ and $\ell = 1$, we find that both satisfy the following linear recurrence:

$$\begin{aligned} &(\beta - 3 - 2n)(\beta - 4 - 2n)(n + 2 + \beta)(\alpha\beta - \beta + \alpha n + 8n + 5 + \alpha)u_{n+2} \\ &+ (n + 2)(86 + 260n + 246\alpha - 23\beta - 27\alpha\beta n + 69\alpha^2 n + 51\alpha^2 n^2 + 612\alpha n^2 + 30\alpha^2 \\ &+ 64n^3 + 10\beta^2 + 702\alpha n - 15\alpha\beta + 232n^2 - 28n\beta + 39\alpha^2\beta - 8n^2\beta + 8n\beta^2 + 57\alpha^2\beta n \\ &+ 3\alpha\beta^3 - 3\alpha^2\beta^3 + \alpha^3\beta^3 - 2\alpha^3 - 5\alpha^3 n - 4\alpha^3 n^2 - \alpha^3 n^3 - \alpha^3\beta + 2\alpha^3\beta^2 + 6\alpha^2\beta^2 \\ &- 18\alpha\beta^2 + 12\alpha^2 n^3 + 168\alpha n^3 - b^3 + 21\alpha^2 n^2\beta - 12\alpha n^2\beta + 6\alpha^2 n\beta^2 - 15\alpha n\beta^2 - 2\alpha^3 n\beta \\ &- \alpha^3 n^2\beta + \alpha^3\beta^2 n)u_{n+1} + 2(1 - \alpha)^3(2n + 1)(n + 2)(n + 1)(\alpha n + 8n + \alpha\beta + 2\alpha + 13 - \beta)u_n = 0. \end{aligned}$$

It is then possible to perform the same study done in the previous sections and in the end we obtain a new polynomial continued fraction for α^β , though obviously quite cumbersome to write down explicitly.

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