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Time averages for kinetic Fokker-Planck equations

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Abstract

We consider kinetic Fokker-Planck (or Vlasov-Fokker-Planck) equations on the torus with Maxwellian or fat tail local equilibria. Results based on weak norms have recently been achieved by S. Armstrong and J.-C. Mourrat in case of Maxwellian local equilibria. Using adapted Poincaré and Lions-type inequalities, we develop an explicit and constructive method for estimating the decay rate of time averages of norms of the solutions, which covers various regimes corresponding to subexponential, exponential and superexponential (including Maxwellian) local equilibria. As a consequence, we also derive hypocoercivity estimates, which are compared to similar results obtained by other techniques.

Keywords: Kinetic Fokker-Planck equation, Ornstein-Uhlenbeck equation, time average, local equilibria, Lions’ lemma, Poincaré inequalities, hypocoercivity.

2020 MSC: Primary: 82C40; Secondary: 35B40, 35H10, 47D06, 35K65.

1. Introduction

Let us consider the *kinetic Fokker-Planck equation*

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\nabla_v f + \alpha \langle v \rangle^{\alpha-2} v f), \quad f(0, \cdot, \cdot) = f_0. \quad (1)$$

where f is a function of time $t \geq 0$, position x , velocity v , and α is a positive parameter. Here we use the notation

$$\langle v \rangle = \sqrt{1 + |v|^2}, \quad \forall v \in \mathbb{R}^d.$$

We consider the spatial domain $Q := (0, L)^d \ni x$, with *periodic boundary conditions*, and define $\Omega_t := (t, t + \tau) \times Q$, for some $\tau > 0$, $t \geq 0$ and $\Omega = \Omega_0$. The

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normalized *local equilibrium*, that is, the equilibrium of the spatially homogeneous case, is

$$\gamma_\alpha(v) = \frac{1}{Z_\alpha} e^{-(v)^\alpha}, \quad \forall v \in \mathbb{R}^d,$$

where Z_α is a non-negative normalization factor, so that $d\gamma_\alpha := \gamma_\alpha(v) dv$ is a probability measure. We shall distinguish a sublinear regime if $\alpha \in (0, 1)$, a linear regime if $\alpha = 1$ and a superlinear regime if $\alpha \geq 1$. The superlinear regime covers the Maxwellian case $\alpha = 2$. The threshold case $\alpha = 1$ corresponds to a linear growth of $\langle v \rangle^\alpha$ as $|v| \rightarrow +\infty$. The estimates in the linear case are similar to the ones of the superlinear regime. In the literature, γ_α is said to be *subexponential*, *exponential* or *superexponential* depending whether the regime is sublinear, linear or superlinear.

The mass

$$M := \iint_{Q \times \mathbb{R}^d} f(\cdot, x, v) dx dv$$

is conserved under the evolution according to the kinetic Fokker-Planck equation (1). We are interested in the convergence of the solution to the stationary solution $M L^{-d} \gamma_\alpha$. By linearity, we can assume from now on that $M = 0$ with no loss of generality. The function

$$h = \frac{f}{\gamma_\alpha}$$

solves the *kinetic-Ornstein-Uhlenbeck equation*

$$\partial_t h + v \cdot \nabla_x h = \Delta_\alpha h, \quad h(0, \cdot, \cdot) = h_0, \quad (2)$$

with

$$\Delta_\alpha h := \Delta_v h - \alpha v \langle v \rangle^{\alpha-2} \cdot \nabla_v h.$$

and *zero-average initial datum* in the sense that

$$\iint_{Q \times \mathbb{R}^d} h_0(x, v) dx d\gamma_\alpha = 0.$$

By mass conservation, solutions to (2) are zero-average for any $t > 0$.

We also consider the *time average* defined as

$$\mathop{\int}_t^{t+\tau} g(s) ds := \frac{1}{\tau} \int_t^{t+\tau} g(s) ds$$

without specifying the τ dependence when not necessary. Our first result is devoted to the decay rate of $h(t, \cdot, \cdot) \rightarrow 0$ as $t \rightarrow \infty$ using time averages.

Theorem 1. *Let $\alpha \geq 1$. Then, for all $L > 0$ and $\tau > 0$, there exists a constant $\lambda > 0$ such that, for all $h_0 \in L^2(dx d\gamma_\alpha)$ with zero-average, the solution to (2) satisfies*

$$\mathop{\int}_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds \leq \|h_0\|_{L^2(dx d\gamma_\alpha)}^2 e^{-\lambda t}, \quad \forall t \geq 0. \quad (3)$$

The expression of λ as a function of τ and L is given in Section 4. To deal with large time asymptotics in kinetic equations, it is by now standard to use hypocoercivity methods. Although not being exactly a hypocoercive method in the usual sense, Theorem 1 provides us with an hypocoercivity-type estimate.

Corollary 2. *Under the assumptions of Theorem 1, there exists an explicit constant $C > 1$ such that all solutions h to (2) fulfill*

$$\|h(t, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 \leq C \|h_0\|_{L^2(dx d\gamma_\alpha)}^2 e^{-\lambda t}, \quad \forall t \geq 0. \quad (4)$$

A typical feature of hypocoercive estimates is the factor $C > 1$ in (4) and we emphasize that there is no such constant in (3). Now, let us turn our attention to the subexponential case $0 < \alpha < 1$.

Theorem 3. *Let $\alpha \in (0, 1)$. Then, for all $L > 0$ and $\tau > 0$, for all $\sigma > 0$, there is a constant $K > 0$ such that all solutions to (2) decay according to*

$$\int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds \leq K (1+t)^{-\frac{\sigma}{2(1-\alpha)}} \iint_{Q \times \mathbb{R}^d} \langle v \rangle^\sigma h_0^2 dx d\gamma_\alpha, \quad \forall t \geq 0. \quad (5)$$

Further details will be given in Section 5. The constants λ , C in Corollary 2 and K in Theorem 3 depend on $L > 0$ and $\tau > 0$ and their values is discussed later. The rate of Theorem 3 is the same as in the homogeneous case of [18, Proposition 11]. See Section 5 for a discussion of the limit $\alpha \rightarrow 1^-$.

Equation (2) is used in physics to describe the distribution function of a system of particles interacting randomly with some background, see for instance [10]. The *kinetic Fokker-Planck equation* is the PDE version of Langevin dynamics

$$\begin{cases} dx_t = v_t dt, \\ dv_t = -v_t + \sqrt{2} W_t, \end{cases}$$

where W_t is a standard Brownian motion. See [14, Introduction] for further details on connections with probability theory. The kinetic Fokker-Planck equation (1) is a simple kinetic equation which has a long history in mathematics that we will not retrace in details here. Mathematical results go back at least to [31] and are at the basis of the theory of L. Hörmander (see, *e.g.*, [30]), at least in the case $\alpha = 2$. For the derivation of the kinetic-Fokker-Planck equation from underlying stochastic ODEs, particularly in the context of astrophysics, we can refer to [23, eq. (328)]. Modern hypoellipticity theory emerged from [29, 25] and built up in a fully developed theory in [38] with important contributions in [28, 34]. We study (1) in this paper, or actually the equivalent equation (2), because it is a useful benchmark for the comparison with other methods.

The word *hypocoercivity* was coined by T. Gallay, in analogy with the already quoted *hypoelliptic* theory of Hörmander in [30]. In [38], C. Villani distinguishes the regularity point of view for elliptic and parabolic problems driven by degenerate elliptic operators from the issue of the long-time behaviour of solutions,

which is nowadays attached to the word hypocoercivity. The underlying idea is to twist the direction in which the operator is coercive and use the transport operator to carry its coercivity properties to the directions in the phase space for which the operator is degenerate. Twisting the H^1 -norm creates equivalent norms, which are coercive along the evolution. So works the H^1 framework, see [38, 37, 26]. The H^1 -framework has been connected to the *carré du champ* method of D. Bakry and M. Emery in [9] by F. Baudoin, who proved decay also w.r.t. the Wasserstein distance, as shown in [11, 12, 13].

The H^1 hypocoercivity implies a decay rate for the L^2 norm, but the corresponding estimates turn out to be suboptimal because extra regularity is required on the initial datum. This motivates the development of direct L^2 techniques based on a perturbation of the L^2 norm. Such an approach can be found in [27] and [18], which is consistent with diffusion limits. In [17], the authors extend the technique to the subexponential case. Another possibility is to perform rotations in the phase space and use a Lyapunov inequality for matrices as in [8]. This approach gives optimal rates, but it is less general as it requires further algebraic properties for the diffusion operator and a detailed knowledge of its spectrum. The core of [8] is a spectral decomposition, that was originally understood via a toy model exposed in [27]. In a domain with periodic boundary conditions, the problem is reduced to an infinite set of ODEs corresponding to spatial modes. See [2, 3, 7] for details and extensions.

A H^{-1} hypocoercivity theory was recently proposed by S. Armstrong and J.-C. Mourrat in [6] using space-time adapted Poincaré inequalities to derive qualitative hypocoercive estimates in the case $\alpha = 2$ and bounded spatial domains. An extension to the whole space in presence of a confining potential can be found in [21]. The method of [6] is our starting point.

This paper focuses on decay rates of time averages. Hypocoercivity estimates are obtained as a consequence of these decay rates. We perform an analysis for all positive values of α , which is consistent in the threshold case $\alpha = 1$. The advantage of the simple setting of a kinetic Fokker-Planck equation on $Q \times \mathbb{R}^d$ is that constants can be computed explicitly in terms of α , τ and L (see Lemma 8), with optimal values in some cases. We make the strategy of [6] quantitative and constructive, for sake of comparison with other hypocoercivity estimates.

This paper is organized as follows. In Section 2 we collect some preliminary results: Poincaré and weighted Poincaré inequalities (Propositions 5 and 6), adapted Lions' inequality (Lemmas 7 and 8). In Section 3 we introduce an averaging lemma (Lemma 12), which is then used to prove the generalized Poincaré inequality of Proposition 13, at the core of the method. In Section 4 we use Proposition 13 and a Grönwall estimate to prove Theorem 1 and compute an explicit formula for λ (Proposition 15). Section 5 is devoted to the proof of Theorem 3, with additional details, and to the limit $\alpha \rightarrow 1^-$. Finally, in Section 6, we derive the hypocoercive estimates of Corollary 2. On the benchmark case $\alpha = 2$ in one spatial dimension, we also compare our results with those obtained by more standard methods.

2. Preliminaries

Let us start with some preliminary results.

2.1. Weighted spaces

For functions g of the variable v only, that is, of the so-called homogeneous case, we define the *weighted Lebesgue* and *Sobolev spaces*

$$L_\alpha^2 := L^2(\mathbb{R}^d, d\gamma_\alpha) \quad \text{and} \quad H_\alpha^1 := \left\{ g \in L_\alpha^2 : \nabla_v g \in (L_\alpha^2)^d \right\}.$$

We equip L_α^2 with the scalar product

$$(g_1, g_2) = \int_{\mathbb{R}^d} g_1(v) g_2(v) d\gamma_\alpha \quad (6)$$

and consider on H_α^1 the norm defined by

$$\|h\|_{H_\alpha^1}^2 := \left(\int_{\mathbb{R}^d} h d\gamma_\alpha \right)^2 + \int_{\mathbb{R}^d} |\nabla_v h|^2 d\gamma_\alpha$$

as in [6]. The duality product between $H_\alpha^{-1} \ni z$ and $H_\alpha^1 \ni g$ is given by

$$\langle z, g \rangle := \int_{\mathbb{R}^d} \nabla_v w_z \cdot \nabla_v g d\gamma_\alpha,$$

where w_z is the weak solution in H_α^1 to

$$-\Delta_\alpha w_z = z - \int_{\mathbb{R}^d} z d\gamma_\alpha, \quad \int_{\mathbb{R}^d} w d\gamma_\alpha = 0.$$

Here we write $\int_{\mathbb{R}^d} z d\gamma_\alpha$ for functions which are integrable w.r.t. $d\gamma_\alpha$ and, up to a little abuse of notations, this quantity has to be understood in the distribution sense for more general measures. As a consequence and with the above notations, we define

$$\|z\|_{H_\alpha^{-1}}^2 := \left(\int_{\mathbb{R}^d} z d\gamma_\alpha \right)^2 + \|w_z\|_{L_\alpha^2}^2.$$

With these notation, the key property of the operator Δ_α , is

$$\langle g_1, \Delta_\alpha g_2 \rangle = - \int \nabla_v g_1 \cdot \nabla_v g_2 d\gamma_\alpha$$

for any functions $g_1, g_2 \in H_\alpha^1$.

Consider next functions h of $(t, x, v) \in \mathbb{R}^+ \times Q \times \mathbb{R}^d$ and define the space

$$H_{\text{kin}} := \left\{ h \in L^2(\mathbb{R}^+ \times Q; H_\alpha^1) : \partial_t h + v \cdot \nabla_x h \in L^2(\Omega_t; H_\alpha^{-1}) \forall t \geq 0 \right\}.$$

We recall that $\Omega_t = (t, t + \tau) \times Q \subset \mathbb{R}_t^+ \times \mathbb{R}_x^d$ and that x -periodic boundary conditions are assumed. We can equip $H_{\text{kin}} \cap L^2(\Omega_t \times Q; H_\alpha^1)$ with the norm

$$\|h\|_{\text{kin}}^2 := \|h\|_{L^2(\Omega_t; L_\alpha^2)}^2 + |h|_{\text{kin}}^2$$

where the *kinetic semi-norm* is given by

$$|h|_{\text{kin}}^2 := \|\nabla_v h\|_{L^2(\Omega_t; L_\alpha^2)}^2 + \|\partial_t h + v \cdot \nabla_x h\|_{L^2(\Omega_t; H_\alpha^{-1})}^2.$$

We refer to [6, Section 7] for the proof of following result.

Proposition 4. *The embedding $H_{\text{kin}}^1 \cap L^2(\Omega_t \times Q; H_\alpha^1) \hookrightarrow \cap L^2(\Omega_t \times Q; L_\alpha^2)$ is continuous and compact for any $t \geq 0$.*

2.2. Poincaré inequalities

In this subsection, we consider functions g depending only on the variable v . Let $\alpha \geq 1$. We can state some *Poincaré inequalities*.

Proposition 5. *If $\alpha \geq 1$, there exists a constant $P_\alpha > 0$ such that, for all functions $g \in H_\alpha^1$, we have*

$$\int_{\mathbb{R}^d} |g - \rho_g|^2 d\gamma_\alpha \leq P_\alpha \int_{\mathbb{R}^d} |\nabla_v g|^2 d\gamma_\alpha \quad \text{with} \quad \rho_g := \int_{\mathbb{R}^d} g d\gamma_\alpha. \quad (7)$$

With $\alpha \geq 1$, the operator Δ_α admits a compact resolvent on $L^2(d\gamma_\alpha)$. Then, (7) holds by the standard results of [19, Chapter 6]. The best constant P_α is such that P_α^{-1} is the minimal positive eigenvalue of $-\Delta_\alpha$. See [22] and the references quoted therein for estimates on P_α . In the case of the Gaussian Poincaré inequality, it is shown in [35] that $P_2 = 1$, although the result was probably known before.

2.3. Weighted Poincaré inequalities

Here we consider again functions depending only on v . For $\alpha \in (0, 1)$, inequality (7) has to be replaced by the following *weighted Poincaré inequality*.

Proposition 6. *If $\alpha \in (0, 1)$, there exists a constant $P_\alpha > 0$ such that, for all functions $g \in H_\alpha^1$, we have*

$$\int_{\mathbb{R}^d} \langle v \rangle^{2(1-\alpha)} |g - \rho_g|^2 d\gamma_\alpha \leq P_\alpha \int_{\mathbb{R}^d} |\nabla_v g|^2 d\gamma_\alpha \quad \text{with} \quad \rho_g := \int_{\mathbb{R}^d} g d\gamma_\alpha. \quad (8)$$

For more details, we refer for instance to [17, Appendix A]. Notice that the average in the l.h.s. is taken w.r.t. $d\gamma_\alpha$, not w.r.t. $\langle v \rangle^{2(1-\alpha)} d\gamma_\alpha$.

2.4. Lions' Lemma

Let \mathcal{O} be an open, bounded and Lipschitz-regular subset of $\mathbb{R}^{d+1} \approx \mathbb{R}_t \times \mathbb{R}_x^d$. We recall that

$$\mathcal{H}^{-1}(\mathcal{O}) = \left\{ w \in \mathcal{D}^*(\mathcal{O}) : |\langle w, u \rangle_{\mathcal{O}}| \leq C \|u\|_{\mathcal{H}_0^1(\mathcal{O})}, C > 0 \right\},$$

where $\mathcal{D}^*(\mathcal{O})$ denotes the space of distributions over \mathcal{O} , equipped with the weak* topology, and $\langle w, u \rangle_{\mathcal{O}}$ is the duality product between \mathcal{H}^{-1} and \mathcal{H}_0^1 . The norm on $\mathcal{H}_0^1(\mathcal{O})$ is as usual $u \mapsto \|\nabla u\|_{L^2(\mathcal{O})}$. On $\mathcal{H}^1(\mathcal{O})$, we introduce the norm

$$\|u\|_{\mathcal{H}^1}^2 = \left| \int_{\mathcal{O}} u dt dx \right|^2 + \|\nabla u\|_{L^2(\mathcal{O})}^2.$$

The norm induced on $H^{-1}(\mathcal{O})$ is then

$$\|w\|_{H^{-1}(\mathcal{O})}^2 = \langle w, 1 \rangle_{\mathcal{O}}^2 + \|z_w\|_{L^2(\mathcal{O})}^2,$$

where z_w the solution to

$$-(\partial_{tt} + \Delta_x) z_w = w - \langle w, 1 \rangle_{\mathcal{O}}, \quad \int_{\mathcal{O}} z_w dt dx = 0.$$

Lions' Lemma gives a sufficient condition for a distribution to be an L^2 function. The following statement is taken from [5].

Lemma 7. *Let \mathcal{O} be a bounded, open and Lipschitz-regular subset in \mathbb{R}^{d+1} . Then, for all $u \in \mathcal{D}^*(\mathcal{O})$, we have that $u \in L^2(\mathcal{O})$ if and only if the weak gradient ∇u belongs to $H^{-1}(\mathcal{O})$. Moreover, there exists a constant $C_L(\mathcal{O})$ such that*

$$\left\| u - \int_{\mathcal{O}} u dx dt \right\|_{L^2(\mathcal{O})}^2 \leq C_L \|\nabla u\|_{H^{-1}(\mathcal{O})}^2,$$

for any $u \in L^2(\mathcal{O})$.

According to [20, 16, 24], if \mathcal{O} is star-shaped w.r.t. a ball, then the constant C_L has the following structure:

$$C_L = 4 |S^d| \frac{D(\mathcal{O})}{d(\mathcal{O})}, \quad (9)$$

where D is the diameter of \mathcal{O} , while $d(\mathcal{O})$ is the diameter of the largest ball one can include in \mathcal{O} . See in particular [24, Remark 9.3] and [16, Lemma 1]. As a consequence, we have the following explicit expression of C_L when $\mathcal{O} = \Omega$.

Lemma 8. *Let $L > 0$, $\tau \in (0, L)$ and $\Omega = (0, \tau) \times (0, L)^d$. Lemma 7 holds with*

$$C_L = 4 |S^d| \frac{\sqrt{dL^2 + \tau^2}}{\tau}. \quad (10)$$

2.5. The kinetic Ornstein-Uhlenbeck equation

We consider solutions to (2) in the weak sense, *i.e.*, functions h in the space $C(\mathbb{R}^+; L^2(dx d\gamma_\alpha))$ with initial datum $h_0 = h(0, \cdot, \cdot)$ in $L^2(dx d\gamma_\alpha)$ such that (2) holds in the sense of distributions on $(0, \infty) \times \mathbb{R}_x^d \times \mathbb{R}_v^d$. The following result is taken from [6] if $\alpha = 2$. The extension to $\alpha \neq 2$ is straightforward as follows from a careful reading of the proof in [6, Proposition 7.12].

Proposition 9. *Let $L > 0$ and $\alpha > 0$. With $\Omega = (0, \tau) \times (0, L)^d$, for all zero-average initial datum $h_0 \in L^2(dx d\gamma_\alpha) \cap C(\mathbb{R}^+; L^2(dx d\gamma_\alpha))$, there exists a unique solution h to (2) such that $h \in \mathbb{H}_{\text{kin}}$ for all $\tau > 0$.*

Regularity properties for (2) are collected in [6, Section 7]. In the special case $\alpha = 2$, some fractional regularity along all directions of the phase space are known. Also see [36] for further result on regularity theory for kinetic Fokker-Planck equations.

2.6. A priori estimates

We state two estimates for solutions to (2).

Lemma 10. *Let $L > 0$, $\tau > 0$, $\Omega = (0, \tau) \times (0, L)^d$, and $\alpha > 0$. If h is a solution to (2), then we have*

$$\|(\partial_t + v \cdot \nabla_x) h\|_{L^2(\Omega; H_\alpha^{-1})} \leq \|\nabla_v h\|_{L^2(\Omega; L_\alpha^2)}. \quad (11)$$

Proof. Take a test function $\phi \in L^2(H_\alpha^1)$, and write

$$\int_Q \langle (\partial_t + v \cdot \nabla_x) h, \phi \rangle dx = \int_Q \langle \Delta_\alpha h, \phi \rangle dx = - \int_Q \langle \nabla_v h, \nabla_v \phi \rangle dx,$$

from which (11) easily follows, after maximizing over $\|\nabla_v \phi\|_{L_\alpha^2} \leq 1$. \square

For completeness, let us recall the classical L^2 decay estimate for solutions to (2).

Lemma 11. *Let $L > 0$, $\tau > 0$, $\Omega = (0, \tau) \times (0, L)^d$, and $\alpha > 0$. If h is a solution to (2), then we have*

$$\frac{d}{dt} \|h\|_{L^2(dx d\gamma_\alpha)}^2 = -2 \|\nabla_v h\|_{L^2(dx d\gamma_\alpha)}^2.$$

3. An averaging lemma and a generalized Poincaré inequality

For all functions $h \in H_{\text{kin}}$, we define the *spatial density*

$$\rho_h := \int_{\mathbb{R}^d} h(\cdot, \cdot, v) d\gamma_\alpha.$$

Notice that $\int_Q \rho_h dx = 0$ whenever h is a zero-average function.

3.1. Averaging lemma

Inspired by [6, Lemma 3], the following *averaging lemma* provides a norm of the spatial density, as for instance in [36].

Lemma 12. *Let $L > \tau > 0$, $\Omega = (0, \tau) \times (0, L)^d$, and $\alpha > 0$. For all $h \in H_{\text{kin}}$, we have*

$$\|\nabla_{t,x} \rho_h\|_{H^{-1}(\Omega)}^2 \leq d_\alpha \left(\|h - \rho_h\|_{L^2(dt dx d\gamma_\alpha)}^2 + \|\partial_t h + v \cdot \nabla_x h\|_{L^2(\Omega; H_\alpha^{-1})}^2 \right) \quad (12)$$

with

$$d_\alpha = 2 \left(\|v_1 |v|^2\|_{L_\alpha^2}^2 + \left(1 + \frac{L^2}{4\pi^2}\right) \| |v|^2 \|_{L_\alpha^2}^2 + \frac{d^2 L^2}{4\pi^2} \|v\|_{L_\alpha^2}^2 \right). \quad (13)$$

Inequality (12) can be extended to any measure $d\gamma$ such that $\int_{\mathbb{R}^d} |v|^4 d\gamma < \infty$ and $\int_{\mathbb{R}^d} v d\gamma = 0$. The proof of Lemma 12 is technical, but follows in a standard way from the time-independent case, as it is common in averaging lemmas: see [36]. For sake of simplicity, we detail only the t -independent case below.

Proof of Lemma 12. Assume that $h \in \mathbf{H}_{\text{kin}}$ does not depend on t . Let $\phi \in \mathcal{D}(Q)^d$ be a smooth test-vector field with zero average on each component. We write

$$- \int_Q \rho_h \nabla_x \cdot \phi \, dx = \int_Q \nabla_x \rho_h \cdot \phi \, dx,$$

with a slight abuse of notation, since the integral of the r.h.s. is in fact a duality product. Using $\int_{\mathbb{R}^d} v_i v_j \, d\gamma_\alpha = d^{-1} \|v\|_{L_\alpha^2}^2 \delta_{ij}$, we obtain

$$\int_Q \nabla_x \rho_h \cdot \phi \, dx = d \|v\|_{L_\alpha^2}^{-2} \iint_{Q \times \mathbb{R}^d} v \cdot \nabla_x \rho_h \phi \cdot v \, dx \, d\gamma_\alpha.$$

By adding and subtracting ρ_h , and then integrating by parts, still at formal level, we obtain

$$\begin{aligned} & \iint_{Q \times \mathbb{R}^d} v \cdot \nabla_x \rho_h \phi \cdot v \, dx \, d\gamma_\alpha \\ &= - \iint_{Q \times \mathbb{R}^d} v \cdot (h - \rho_h) \nabla_x \phi \cdot v \, dx \, d\gamma - \iint_{Q \times \mathbb{R}^d} v \cdot \nabla_x h v \cdot \phi \, dx \, d\gamma_\alpha \\ &\leq \|h - \rho_h\|_{L^2(dx \, d\gamma_\alpha)} \|\nabla_x \phi\|_{L^2(dx)} \|v^2\|_{L_\alpha^2} \\ &\quad + \|v \cdot \nabla_x h\|_{L^2(H_\alpha^{-1})} \|\phi\|_{L^2(dx)} \|v\|_{L_\alpha^2} \end{aligned}$$

using Cauchy-Schwarz inequalities and duality estimates. By the Poincaré inequality, we know that

$$\frac{4\pi^2}{dL^2} \|\phi\|_{L^2(dx)}^2 \leq \|\nabla \phi\|_{L^2(dx)}^2$$

Maximizing the r.h.s. on ϕ such that $\|\nabla_x \phi\|_{L^2(dx)} \leq 1$ completes the proof of the t -independent case. When h additionally depends on t , the same scheme can be applied with $v \cdot \nabla_x$ replaced by $\partial_t + v \cdot \nabla_x$. \square

3.2. A generalized Poincaré inequality

The next *a priori* estimate is at the core of the method. It is a Poincaré type inequality in t , x and v which relies on Lemma 12 and involves derivatives of various orders.

Proposition 13. *Let $L > 0$, $\tau > 0$, $\Omega = (0, \tau) \times (0, L)^d$, and $\alpha > 0$. Then, for all $h \in \mathbf{H}_{\text{kin}}$ with zero average, we have that*

$$\|h\|_{L^2(dt \, dx \, d\gamma_\alpha)}^2 \leq C \left(\|h - \rho_h\|_{L^2(dt \, dx \, d\gamma_\alpha)}^2 + \|\partial_t h + v \cdot \nabla_x h\|_{L^2(\Omega; H_\alpha^{-1})}^2 \right) \quad (14)$$

with $C = 1 + C_L d_\alpha$, where C_L and d_α are given respectively by (10) and (13).

Proof. By orthogonality in $L^2(\Omega; L_\alpha^2)$ and because $d\gamma_\alpha$ is a probability measure, we have the decomposition

$$\|h\|_{L^2(\Omega; L_\alpha^2)}^2 = \|h - \rho_h\|_{L^2(\Omega; L_\alpha^2)}^2 + \|\rho_h\|_{L^2(\Omega)}^2.$$

The function ρ_h has zero average on Ω by construction, so that

$$\|h\|_{\mathbb{L}^2(\Omega; \mathbb{L}_\alpha^2)}^2 \leq \|h - \rho_h\|_{\mathbb{L}^2(\Omega; \mathbb{L}_\alpha^2)}^2 + C_L \|\nabla_{x,t} \rho\|_{\mathcal{H}^{-1}(\Omega)}^2$$

by Lemma 8. Hence

$$\|h\|_{\mathbb{L}^2(\Omega; \mathbb{L}_\alpha^2)}^2 \leq (1 + C_L d_\alpha) \|h - \rho_h\|_{\mathbb{L}^2(\Omega; d\gamma_\alpha)}^2 + C_L d_\alpha \|\partial_t h + v \cdot \nabla_x h\|_{\mathbb{L}^2(\Omega; H_\alpha^{-1})}^2$$

by Lemma 12. This concludes the proof. \square

4. Linear and superlinear local equilibria: exponential decay rate

In this Section, we consider the case $\alpha \geq 1$ and the domain $\Omega = (t, t + \tau) \times (0, L)^d$, for an arbitrary $t \geq 0$. Let us define $\kappa_\alpha := (1 + C_L d_\alpha)(P_\alpha + 1)$ where P_α is the Poincaré constant in (7) and where C_L and d_α are given respectively by (10) and (13).

Lemma 14. *Let $L > 0$, $\tau > 0$, $t \geq 0$, $\Omega_t = (t, t + \tau) \times (0, L)^d$, and $\alpha \geq 1$. Then, for all $h \in \mathbb{H}_{\text{kin}}$ with zero average, we have that*

$$\|h\|_{\mathbb{L}^2(dt dx d\gamma_\alpha)}^2 \leq \kappa_\alpha \|\nabla_v h\|_{\mathbb{L}^2(dt dx d\gamma_\alpha)}^2. \quad (15)$$

Proof. We know that

$$\|h\|_{\mathbb{L}^2(dt dx d\gamma_\alpha)}^2 \leq (1 + C_L d_\alpha) \left(P_\alpha \|\nabla_v h\|_{\mathbb{L}^2(dt dx d\gamma_\alpha)}^2 + \|\partial_t h + v \cdot \nabla_x h\|_{\mathbb{L}^2(H_\alpha^{-1})}^2 \right)$$

as a consequence of (7) and (14). Then (15) follows from Lemma 10. \square

We are ready to prove Theorem 1 with an explicit estimate of the constant λ .

Proposition 15. *For any $\alpha \geq 1$, Theorem 1 holds true with*

$$\frac{1}{\lambda} = \frac{1}{\tau} \left(\tau + \sqrt{dL^2 + \tau^2} \right) \left(2d_\alpha |\mathbb{S}^{d-1}| (P_\alpha + 1) \right).$$

Proof. A t integral of (15) on the interval $(t, t + \tau)$ gives

$$\int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{\mathbb{L}^2(dx d\gamma_\alpha)}^2 ds \leq \kappa_\alpha \int_t^{t+\tau} \|\nabla_v h(s, \cdot, \cdot)\|_{\mathbb{L}^2(dx d\gamma_\alpha)}^2 ds.$$

With $\lambda = 2/\kappa_\alpha$, we deduce from Lemma 11 that

$$\begin{aligned} \frac{d}{dt} \int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{\mathbb{L}^2(dx d\gamma_\alpha)}^2 ds &= -2 \int_t^{t+\tau} \|\nabla_v h(s, \cdot, \cdot)\|_{\mathbb{L}^2(dx d\gamma_\alpha)}^2 ds \\ &\leq -\lambda \int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{\mathbb{L}^2(dx d\gamma_\alpha)}^2 ds. \end{aligned}$$

Grönwall's Lemma and the monotonicity of $t \mapsto \|h(t, \cdot, \cdot)\|_{\mathbb{L}^2(dx d\gamma_\alpha)}^2$ imply

$$\begin{aligned} \int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{\mathbb{L}^2(dx d\gamma_\alpha)}^2 ds &\leq \int_0^\tau \|h(s, \cdot, \cdot)\|_{\mathbb{L}^2(dx d\gamma_\alpha)}^2 ds e^{-\lambda t} \\ &\leq \tau \|h_0\|_{\mathbb{L}^2(dx d\gamma_\alpha)}^2 e^{-\lambda t} \end{aligned}$$

for any $t \geq 0$, which proves (3), that is, Theorem 1, and gives the claimed expression for λ using the definition of κ_α . \square

Notice that the r.h.s. vanishes as $\tau \downarrow 0$, which is expected because of the degeneracy of Δ_α : an exponential decay rate of $\|h(t, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2$ cannot hold.

5. Sublinear equilibria: algebraic decay rates

5.1. Proof of Theorem 3

Assume that $\alpha \in (0, 1)$. Let us the parameter $\beta = 2(1 - \alpha)/p$ where $p, q > 1$ are Hölder conjugate exponents, i.e., $\frac{1}{p} + \frac{1}{q} = 1$ and define

$$Z_h(t) := \iint_{Q \times \mathbb{R}^d} \langle v \rangle^{\beta q} |h - \rho_h|^2 dx d\gamma_\alpha. \quad (16)$$

The following estimates replaces Proposition 13.

Proposition 16. *Let $L > 0, \tau > 0, t \geq 0, \Omega_t = (t, t + \tau) \times (0, L)^d$, and $\alpha \geq 1$. With the above notations, for all $h \in \mathbf{H}_{\text{kin}}$ with zero average, we have that*

$$\begin{aligned} & \|h\|_{L^2(dt dx d\gamma_\alpha)}^2 \\ & \leq C P_\alpha^{\frac{1}{p}} \|\nabla_v h\|_{L^2(dt dx d\gamma_\alpha)}^{\frac{2}{p}} \left(\int_t^{t+\tau} Z_h(s) ds \right)^{\frac{1}{q}} + C \|\partial_t h + v \cdot \nabla_x h\|_{L^2(\Omega; H_\alpha^{-1})}^2 \end{aligned}$$

where $C = 1 + C_L d_\alpha$ is as in Proposition 13 and P_α denotes the constant in the weighted Poincaré inequality (8).

Proof. Using (14) and Hölder's inequality w.r.t. the variable v , we find that

$$\|h - \rho_h\|_{L_\alpha^2}^2 \leq \left(\int_{\mathbb{R}^d} \langle v \rangle^{-\beta p} |h - \rho_h|^2 d\gamma_\alpha \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} \langle v \rangle^{\beta q} |h - \rho_h|^2 d\gamma_\alpha \right)^{\frac{1}{q}}.$$

The weighted Poincaré inequality (8) with $\beta p = 2(1 - \alpha)$ and an additional Hölder inequality w.r.t. the variables t and x allow us to complete the proof. \square

Lemma 17. *Let $L > 0, \tau > 0, t \geq 0, \Omega_t = (t, t + \tau) \times (0, L)^d$, and $\alpha \in (0, 1)$. There is a constant $W > 0$ such that, for all solution $h \in \mathbf{H}_{\text{kin}}$ to (2) with an initial datum h_0 with zero average, using the notation (16) as in Proposition 16, we have*

$$Z_h(t) \leq W \iint_{Q \times \mathbb{R}^d} \langle v \rangle^{\beta q} h_0^2 dx d\gamma_\alpha, \quad \forall t \geq 0.$$

Proof. An elementary computation shows that

$$\begin{aligned} \iint_{Q \times \mathbb{R}^d} \langle v \rangle^{\beta q} |h - \rho_h|^2 dx d\gamma_\alpha & \leq 2 \iint_{Q \times \mathbb{R}^d} \langle v \rangle^{\beta q} (h^2 + \rho_h^2) dx d\gamma_\alpha \\ & \leq 2 \left(1 + \int_{\mathbb{R}^d} \langle v \rangle^{\beta q} d\gamma_\alpha \right) \iint_{Q \times \mathbb{R}^d} \langle v \rangle^{\beta q} h^2 dx d\gamma_\alpha \end{aligned}$$

because $\rho_h^2 = \left(\int_{\mathbb{R}^d} h d\gamma_\alpha \right)^2 \leq \int_{\mathbb{R}^d} h^2 d\gamma_\alpha \leq \int_{\mathbb{R}^d} \langle v \rangle^{\beta q} h^2 d\gamma_\alpha$. According to [17, Proposition 4], there is a constant $\mathcal{K}_{\beta q} > 1$ such that

$$\iint_{Q \times \mathbb{R}^d} \langle v \rangle^{\beta q} |h(t, x, v)|^2 dx d\gamma_\alpha \leq \mathcal{K}_{\beta q} \iint_{Q \times \mathbb{R}^d} \langle v \rangle^{\beta q} h_0^2 dx d\gamma_\alpha, \quad \forall t \geq 0.$$

The result follows with $W = 2 \left(1 + \int_{\mathbb{R}^d} \langle v \rangle^{\beta q} d\gamma_\alpha \right) \mathcal{K}_{\beta q}$. \square

Assume that $h \in \mathbf{H}_{\text{kin}}$ solves (2) with an initial datum h_0 with zero average and let us collect our estimates. With Proposition 10, Proposition 16, and Lemma 17, the estimate of Lemma 14 is replaced by

$$\|h\|_{\mathbf{L}^2(dt dx d\gamma_\alpha)}^2 \leq A \|\nabla_v h\|_{\mathbf{L}^2(dt dx d\gamma_\alpha)}^{\frac{2}{p}} + C \|\nabla_v h\|_{\mathbf{L}^2(dt dx d\gamma_\alpha)}^2 \quad (17)$$

with $A = C P_\alpha^{\frac{1}{p}} (\tau W)^{1/q} \left(\iint_{Q \times \mathbb{R}^d} \langle v \rangle^{\beta q} h_0^2 dx d\gamma_\alpha \right)^{1/q}$.

The main result of the section is a technical version of Theorem 3. Let

$$\mathbf{x}(t) := \int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{\mathbf{L}^2(dx d\gamma_\alpha)}^2 ds \quad \text{and} \quad \mathbf{y}(t) := \int_t^{t+\tau} \|\nabla_v h(s, \cdot, \cdot)\|_{\mathbf{L}^2(dx d\gamma_\alpha)}^2 ds$$

where norms are taken on $Q \times \mathbb{R}^d$. We know from Lemma 11 and (17) that

$$\mathbf{x}' = -2\mathbf{y} \quad \text{and} \quad \mathbf{x} \leq \varphi(\mathbf{y}) := A\mathbf{y}^{1/p} + C\mathbf{y}.$$

Finally, let us denote by φ^{-1} the inverse of $\mathbf{y} \mapsto \varphi(\mathbf{y})$ and consider

$$\psi(\mathbf{z}) := \int_{\mathbf{z}}^{\mathbf{x}_0} \frac{dz}{2\varphi^{-1}(z)} \quad \text{with} \quad \mathbf{x}_0 = \|h_0\|_{\mathbf{L}^2(dx d\gamma_\alpha)}^2.$$

Theorem 18. *Let $L > 0$, $\tau > 0$, $t \geq 0$, $\Omega_t = (t, t+\tau) \times (0, L)^d$, and $\alpha \in (0, 1)$. With the above notations, for all solution $h \in \mathbf{H}_{\text{kin}}$ to (2) with an initial datum h_0 with zero average, we have*

$$\mathbf{x}(t) = \int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{\mathbf{L}^2(dx d\gamma_\alpha)}^2 ds \leq \psi^{-1}(t), \quad \forall t \geq 0.$$

Proof. The strategy goes as in [33, 17]. Everything reduces to the differential inequality

$$\mathbf{x}' \leq -2\varphi^{-1}(\mathbf{x})$$

using the monotonicity of $\mathbf{y} \mapsto \varphi(\mathbf{y})$. From by the elementary Bihari-Lasalle inequality, see [15, 32], which is obtained by a simple integration, we obtain

$$\mathbf{x}(t) = \int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{\mathbf{L}^2(dx d\gamma_\alpha)}^2 ds \leq \psi^{-1}(t + \psi(\mathbf{x}(0))).$$

Since, on the one hand

$$\mathbf{x}(0) = \int_0^\tau \|h(s, \cdot, \cdot)\|_{\mathbf{L}^2(dx d\gamma_\alpha)}^2 ds \leq \|h_0\|_{\mathbf{L}^2(dx d\gamma_\alpha)}^2 = \mathbf{x}_0$$

because $s \mapsto \|h(s, \cdot, \cdot)\|_{\mathbf{L}^2(dx d\gamma_\alpha)}^2$ is nonincreasing according to Lemma 11, and ψ is nonincreasing on the other hand, then

$$\psi^{-1}(t + \psi(\mathbf{x}(0))) \leq \psi^{-1}(t),$$

which concludes the proof. Notice that the dependence on h_0 enters in A and \mathbf{x}_0 , and henceforth in φ and ψ . \square

Proof of Theorem 3. Since $\lim_{t \rightarrow +\infty} y(t) = 0$, we have that $\varphi(y(t)) \sim Ay(t)^{1/p}$ as $t \rightarrow +\infty$, which heuristically explains the role played by p in (5). This can be made rigorous as follows. Notice that

$$\varphi(y) = Ay^{1/p} + Cy \leq A_0 y^{1/p}, \quad \forall y \leq y_0, \quad \text{with } A_0 = A + Cy_0^{1-1/p}.$$

With A replaced by A_0 and C replaced by 0, the computation of the proof of Theorem 18 is now explicit. With the choice $y_0 = \varphi^{-1}(x_0)$, we know that $y(t) \leq y_0$ for any $t \geq 0$ and obtain

$$x(t) \leq (x_0^{1-p} + 2(p-1)A_0^{-p}t)^{-\frac{1}{p-1}}, \quad \forall t \geq 0. \quad (18)$$

Using $x_0 = Ay_0^{1/p} + Cy_0 \geq Cy_0$, we know that $A_0 = x_0 y_0^{-1/p} \leq A + C^{1/p} x_0^{1-1/p}$, which proves (5) with

$$K = \max \left\{ 1, (2(p-1))^{1/(1-p)} \left(C P_\alpha^{1/p} (\tau W)^{1-1/p} + C^{1/p} \right) \right\}.$$

The conclusion holds using $\sigma = \beta q = 2(1-\alpha)/(p-1)$. \square

5.2. The linear threshold: from algebraic to exponential rates

A very natural question arises: *is the result Theorem 15 (corresponding to $\alpha \in (0, 1)$) consistent with the result of Theorem 18 (which covers any $\alpha \geq 1$)?* A first observation is that we can vary α in the assumptions concerning the initial data.

Lemma 19. *If $h_0 \in L^2(Q; L_{\alpha_0}^2)$ for some $\alpha_0 \in (0, 1)$, then $\langle v \rangle^{\sigma/2} h_0 \in L^2(Q; L_\alpha^2)$ for any $\alpha > \alpha_0$ and any $\sigma > 0$.*

The proof is a simple consequence of the fact that $v \mapsto \langle v \rangle^\sigma \exp\langle v \rangle^{\alpha-\alpha_0}$ is uniformly bounded. For any $\alpha > \alpha_0$, let us denote the corresponding solution of (2) with initial datum h_0 , of zero average, by $h^{(\alpha)}$.

If $\alpha \in (\alpha_0, 1)$, then (18) can be rewritten as

$$\int_t^{t+\tau} \|h^{(\alpha)}(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds \leq \|h_0\|_{L^2(dx d\gamma_\alpha)}^2 (1 + (p-1)\ell(\alpha)t)^{-\frac{1}{p-1}}.$$

By passing to the limit as $\alpha \rightarrow 1^-$, we recover (3) with $\lambda = \lim_{\alpha \rightarrow 1^-} \ell(\alpha)$, where $\ell(\alpha) = 2 \|h_0\|_{L^2(dx d\gamma_\alpha)}^{2(p-1)} A_0^{-p}$ and $A_0 = A_0(\alpha)$ as above. The Poincaré constant P_α in the weighted Poincaré inequality (8) admits a limit as $\alpha \rightarrow 1^-$, according to [18, Appendix A].

The limit of $\lim_{\alpha \rightarrow 1^-} \ell(\alpha)$ is certainly not optimal. By working directly on the Bihari-Lasalle estimate of Theorem 18, we can recover the value of λ in Theorem 15. Notice here that $\sigma > 0$ plays essentially no role and can be taken arbitrarily small, even depending on α , but such that $p = 2(1-\alpha)/\sigma \rightarrow 1$ as $\alpha \rightarrow 1^-$.

6. Hypocoercivity and comparison with some other methods

6.1. A hypocoercivity result

Theorem 1 implies an L^2 -hypocoercivity result in the linear and superlinear regimes $\alpha \geq 1$.

Proof of Corollary 2. For any $t \geq 0$, we know from Theorem 1 that

$$\|h(t + \tau, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 \leq \int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds \leq \|h_0\|_{L^2(dx d\gamma_\alpha)}^2 e^{-\lambda t},$$

as a consequence of the monotonicity of the L^2 norm, according to Lemma 11. This proves that

$$\|h(t, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 \leq C \|h_0\|_{L^2(dx d\gamma_\alpha)}^2 e^{-\lambda t}$$

with $C = e^{\lambda\tau}$ for any $t \geq \tau$. However, if $t \in [0, \tau)$, it turns out that $C e^{-\lambda t} \geq 1$ so that the inequality is also true by Lemma 11. This concludes the proof. \square

The remainder of this section is devoted to a comparison with earlier hypocoercivity results in a simple benchmark case: let $\alpha = 2$, $d = 1$ and $L = 2\pi$. In this case, for the choice $\tau = 2\pi$, Theorem 1 amounts to

$$\|h\|_{L^2(dx d\gamma_2)}^2 \leq e^{\frac{\pi}{8\sqrt{3}}} \|h_0\|_{L^2(dx d\gamma_2)}^2 e^{-\frac{t}{8\sqrt{3}}}, \quad \forall t \geq 0.$$

For sake of comparison, notice that $\lambda = 1/(8\sqrt{3}) \approx 0.0721688$.

6.2. The DMS method

The first comparison is with the abstract twisted L^2 hypocoercivity method of [27, 18]. Let $\|\cdot\|$ be the norm of $L^2(dx d\gamma_2)$ and (\cdot, \cdot) the associated scalar product. We consider the evolution equation

$$\partial_t h + Th = \mathcal{L}h. \tag{19}$$

Theorem 20. *Let h be a solution of (19) with initial datum $h_0 \in L^2(Q; L^2_2)$ and assume that T and \mathcal{L} are respectively anti-self-ajoint and self-adjoint operators on $L^2(Q; L^2_2)$ such that, for some positive constants λ_m , λ_M and C_M , we have*

- (A1) $(-\mathcal{L}h, h) \geq \lambda_m \|(1 - \Pi)h\|^2$ for all $h \in D(\mathcal{L})$,
- (A2) $\|T\Pi h\|^2 \geq \lambda_M \|\Pi h\|^2$ for all $h \in D(T\Pi)$,
- (A3) $\Pi T\Pi h = 0$,
- (A4) $\|AT(1 - \Pi)h\| + \|A\mathcal{L}h\| \leq C_M \|(\text{Id} - \Pi)h\|$ for all admissible $h \in L^2(Q; L^2_2)$

where $A := (\text{Id} + (T\Pi)^* T\Pi)^{-1} (T\Pi)^*$ and Π is the projection in L^2_2 onto the kernel of \mathcal{L} . Then we have

$$\|h(t, \cdot, \cdot)\|^2 \leq C \|h_0\|^2 e^{-\lambda t} \quad \forall t \geq 0$$

with $C = (1 + \delta)/(1 - \delta)$, $\delta = \frac{1}{2} \min \left\{ 1, \lambda_m, \frac{\lambda_m \lambda_M}{(1 + \lambda_M) C_M^2} \right\}$ and $\lambda = \frac{2\delta \lambda_M}{3(1 + \lambda_M)}$.

This result is taken from [18, Proposition 4]. According to [18, Corollary 9], we have the estimate $\lambda = 1/24 \approx 0.041667$. A minor improvement is obtained as follows. Using Theorem 20 applied with $d = 1$, $L = 2\pi$, $T = v \partial_x$ and $\mathcal{L} = \partial_v^2 - v \partial_v$, in Fourier variables, we obtain $\lambda_m = \lambda_M = 1$ and $C_M = (1 + \sqrt{3})/2$ according to [7, Section II.1.3.2], so that $\lambda = 1/(12 + 6\sqrt{3}) \approx 0.0446582$. Using Fourier modes, a slightly better estimate is obtained from [7, Section II.1.2] with $\lambda \approx 0.176048$.

6.3. Direct spectral methods

In a series of papers, F. Achleitner, A. Arnold, E. Carlen and several other collaborators use direct spectral methods. We refer in particular to [8, 2, 3, 4] and also [7] for an introduction to the method, which can be summarized as follows.

Let us consider (19) written after a Fourier transform in x , so that $T = i\xi \cdot v$, and acting on $L^2(dx; \mathbb{L}_2^2)$ now considered as a space of complex valued functions. Assume that for some positive definite bounded Hermitian operator P and some constant $\lambda \in (0, +\infty)$, we have

$$(L - T)^* P + P(L - T) \geq 2\lambda P.$$

Let us consider the *twisted* norm $\|\hat{f}\|_P^2 := \int_Q (\hat{f}, P \hat{f}) dx$ where (\cdot, \cdot) is the natural extension of the scalar product as defined in (6). From

$$\frac{d}{dt} \|\hat{f}\|_P^2 = -\langle \hat{f}, ((L - T)^* P + P(L - T)) \hat{f} \rangle \leq -2\lambda \|\hat{f}\|_P^2,$$

for some $C > 1$, we deduce that

$$C^{-1} \|f(t, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 \leq \|\hat{f}(t, \cdot, \cdot)\|_P^2 \leq e^{-2\lambda t} \|\hat{f}_0\|_P^2 \quad \forall t \geq 0.$$

To our knowledge, μ has not yet been computed in the case of (1). The spectral decomposition

$$h(t, x, v) = \sum_{\xi \in \mathbb{Z}^d} \sum_{k \in \mathbb{N}^d} a_{\xi, k}(t) H_k(v) e^{-i \frac{2x}{L} \xi \cdot x}$$

provides an easy framework for finite dimensional approximations using the basis of Hermite functions $(H_k)_{k \in \mathbb{N}}$ and the numerical value $\mu \approx 0.4$ has been obtained according to [1].

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References

- [1] F. ACHLEITNER, *A numerical computation of the rates for kinetic fokker-planck equations*. Personal communication.
- [2] F. ACHLEITNER, A. ARNOLD, AND E. A. CARLEN, *On linear hypocoercive BGK models*, in *From Particle Systems to Partial Differential Equations III*, Springer, 2016, pp. 1–37.
- [3] F. ACHLEITNER, A. ARNOLD, AND E. A. CARLEN, *On multi-dimensional hypocoercive BGK models*, *Kinet. Relat. Models*, 11 (2018), pp. 953–1009.
- [4] F. ACHLEITNER, A. ARNOLD, AND B. SIGNORELLO, *On optimal decay estimates for ODEs and PDEs with modal decomposition*, in *International workshop on Stochastic Dynamics out of Equilibrium*, Springer, 2017, pp. 241–264.
- [5] C. AMROUCHE, P. G. CIARLET, AND C. MARDARE, *On a lemma of Jacques-Louis Lions and its relation to other fundamental results*, *Journal de Mathématiques Pures et Appliquées*, 104 (2015), pp. 207–226.
- [6] S. ARMSTRONG AND J.-C. MOURRAT, *Variational methods for the kinetic Fokker-Planck equation*, preprint arXiv:1902.04037, (2019).
- [7] A. ARNOLD, J. DOLBEAULT, C. SCHMEISER, AND T. WÖHRER, *Sharpening of decay rates in Fourier based hypocoercivity methods*, Preprint <https://hal.archives-ouvertes.fr/hal-03078698> and <https://arxiv.org/abs/2012.09103>, to appear in *Recent Advances in Kinetic Equations and Applications* (F. Salvarani ed.), Springer INdAM Series, (2020).
- [8] A. ARNOLD AND J. ERB, *Sharp entropy decay for hypocoercive and non-symmetric Fokker-Planck equations with linear drift*, preprint arXiv:1409.5425, (2014).
- [9] D. BAKRY AND M. ÉMERY, *Diffusions hypercontractives*, in *Séminaire de Probabilités XIX 1983/84*, Springer, 1985, pp. 177–206.
- [10] R. BALIAN, *From Microphysics to Macrophysics*, *Methods and Applications of Statistical Physics*, Springer Berlin Heidelberg, 1991.
- [11] F. BAUDOIN, *Wasserstein contraction properties for hypoelliptic diffusions*, preprint arXiv:1602.04177, (2016).
- [12] ———, *Bakry-Émery meet Villani*, *J. Funct. Anal.*, 273 (2017), pp. 2275–2291.
- [13] F. BAUDOIN, M. GORDINA, AND D. P. HERZOG, *Gamma calculus beyond Villani and explicit convergence estimates for Langevin dynamics with singular potentials*, preprint arXiv:1907.03092, (2019).

- [14] E. BERNARD, M. FATHI, A. LEVITT, AND G. STOLTZ, *Hypocoercivity with Schur complements*, preprint arXiv:2003.00726, (2020).
- [15] I. BIHARI, *A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations*, Acta Mathematica Academiae Scientiarum Hungarica, 7 (1956), pp. 81–94.
- [16] M. E. BOGOVSKII, *Solution of the first boundary value problem for the equation of continuity of an incompressible medium*, in Soviet Math. Dokl., vol. 20, 1979, pp. 1094–1098.
- [17] E. BOUIN, J. DOLBEAULT, L. LAFLECHE, AND C. SCHMEISER, *Hypocoercivity and sub-exponential local equilibria*, Monatsh. Math., 194 (2021), pp. 41–65.
- [18] E. BOUIN, J. DOLBEAULT, S. MISCHLER, C. MOUHOT, AND C. SCHMEISER, *Hypocoercivity without confinement*, Pure Appl. Anal., 2 (2020), pp. 203–232.
- [19] H. BREZIS, *Opérateurs maximaux monotones, mathematics studies 5*, Notas de Matematica, 50 (1973).
- [20] A. P. CALDERÓN AND A. ZYGMUND, *On singular integrals*, American Journal of Mathematics, 78 (1956), pp. 289–309.
- [21] Y. CAO, J. LU, AND L. WANG, *On explicit L^2 -convergence rate estimate for underdamped Langevin dynamics*, preprint arXiv:1908.04746, (2019).
- [22] P. CATTIAUX, A. GUILLIN, P. MONMARCHÉ, AND C. ZHANG, *Entropic multipliers method for Langevin diffusion and weighted log-Sobolev inequalities*, Journal of Functional Analysis, 277 (2019), p. 108288.
- [23] S. CHANDRASEKHAR, *Stochastic problems in physics and astronomy*, Rev. Mod. Phys., 15 (1943), pp. 1–89.
- [24] G. CSATÓ, B. DACOROGNA, AND O. KNEUSS, *The pullback equation for differential forms*, vol. 83, Springer Science & Business Media, 2011.
- [25] L. DESVILLETES AND C. VILLANI, *On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation*, Comm. Pure Appl. Math., 54 (2001), pp. 1–42.
- [26] J. DOLBEAULT AND X. LI, *φ -entropies: convexity, coercivity and hypocoercivity for Fokker–Planck and kinetic Fokker–Planck equations*, Mathematical Models and Methods in Applied Sciences, 28 (2018), pp. 2637–2666.
- [27] J. DOLBEAULT, C. MOUHOT, AND C. SCHMEISER, *Hypocoercivity for linear kinetic equations conserving mass*, Transactions of the American Mathematical Society, 367 (2015), pp. 3807–3828.

- [28] F. HÉRAU, *Hypocoercivity and exponential time decay for the linear inhomogeneous relaxation Boltzmann equation*, *Asymptot. Anal.*, 46 (2006), pp. 349–359.
- [29] F. HÉRAU AND F. NIER, *Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with a high-degree potential*, *Arch. Ration. Mech. Anal.*, 171 (2004), pp. 151–218.
- [30] L. HÖRMANDER ET AL., *Hypoelliptic second order differential equations*, *Acta Mathematica*, 119 (1967), pp. 147–171.
- [31] A. KOLMOGOROFF, *Zufällige Bewegungen (zur Theorie der Brownschen Bewegung)*, *Annals of Mathematics*, (1934), pp. 116–117.
- [32] J. LASALLE, *Uniqueness theorems and successive approximations*, *Annals of Mathematics*, (1949), pp. 722–730.
- [33] T. M. LIGGETT, *L^2 rates of convergence for attractive reversible nearest particle systems: the critical case*, *The Annals of Probability*, (1991), pp. 935–959.
- [34] C. MOUHOT AND L. NEUMANN, *Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus*, *Nonlinearity*, 19 (2006), pp. 969–998.
- [35] J. NASH, *Continuity of solutions of parabolic and elliptic equations*, *American Journal of Mathematics*, 80 (1958), pp. 931–954.
- [36] B. PERTHAME ET AL., *Mathematical tools for kinetic equations*, *Bulletin of the American Mathematical Society*, 41 (2004), pp. 205–244.
- [37] C. VILLANI, *Hypocoercive diffusion operators*, in *International Congress of Mathematicians*, vol. 3, 2006, pp. 473–498.
- [38] ———, *Hypocoercivity*, *Memoirs of the American Mathematical Society*, 202 (2009).