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Time averages for kinetic Fokker-Planck equations

Giovanni M. Brigati^{a,b,1}

^a*CEREMADE, CNRS, UMR 7534, Université Paris-Dauphine, PSL University, Place du
Marechal de Lattre de Tassigny, 75016 Paris, France*

^b*Dipartimento di Matematica “F. Casorati”, Università degli Studi di Pavia, Via Ferrata 5,
27100 Pavia, Italia*

Abstract

We consider kinetic Fokker-Planck (or Vlasov-Fokker-Planck) equations on the torus with Maxwellian or fat tail local equilibria. Results based on weak norms have recently been achieved by S. Armstrong and J.-C. Mourrat in case of Maxwellian local equilibria. Using adapted Poincaré and Lions-type inequalities, we develop an explicit and constructive method for estimating the decay rate of time averages of norms of the solutions, which covers various regimes corresponding to subexponential, exponential and superexponential (including Maxwellian) local equilibria. As a consequence, we also derive hypocoercivity estimates, which are compared to similar results obtained by other techniques.

Keywords: Kinetic Fokker-Planck equation, Ornstein-Uhlenbeck equation, time average, local equilibria, Lions’ lemma, Poincaré inequalities, hypocoercivity.

2020 MSC: Primary: 82C40; Secondary: 35B40, 35H10, 47D06, 35K65.

1. Introduction

Let us consider the *kinetic Fokker-Planck equation*

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\nabla_v f + \alpha \langle v \rangle^{\alpha-2} v f), \quad f(0, \cdot, \cdot) = f_0. \quad (1)$$

where f is a function of time $t \geq 0$, position x , velocity v , and α is a positive parameter. Here we use the notation

$$\langle v \rangle = \sqrt{1 + |v|^2}, \quad \forall v \in \mathbb{R}^d.$$

We consider the spatial domain $\mathbb{T} := (0, L)^d \ni x$, with *periodic boundary conditions*, and define $\Omega_t := (t, t + \tau) \times \mathbb{T}$, for some $\tau > 0$, $t \geq 0$ and $\Omega = \Omega_0$. The

¹Email: brigati@ceremade.dauphine.fr
Web: <https://www.ceremade.dauphine.fr/~brigati/>

normalized *local equilibrium*, that is, the equilibrium of the spatially homogeneous case, is

$$\gamma_\alpha(v) = \frac{1}{Z_\alpha} e^{-(v)^\alpha}, \quad \forall v \in \mathbb{R}^d,$$

where Z_α is a non-negative normalization factor, so that $d\gamma_\alpha := \gamma_\alpha(v) dv$ is a probability measure. We shall distinguish a sublinear regime if $\alpha \in (0, 1)$, a linear regime if $\alpha = 1$ and a superlinear regime if $\alpha \geq 1$. The superlinear regime covers the Maxwellian case $\alpha = 2$. The threshold case $\alpha = 1$ corresponds to a linear growth of $\langle v \rangle^\alpha$ as $|v| \rightarrow +\infty$. The estimates in the linear case are similar to the ones of the superlinear regime. In the literature, γ_α is said to be *subexponential*, *exponential* or *superexponential* depending whether the regime is sublinear, linear or superlinear.

The mass

$$M := \iint_{\mathbb{T} \times \mathbb{R}^d} f(\cdot, x, v) dx dv$$

is conserved under the evolution according to the kinetic Fokker-Planck equation (1). We are interested in the convergence of the solution to the stationary solution $M L^{-d} \gamma_\alpha$. By linearity, we can assume from now on that $M = 0$ with no loss of generality. The function

$$h = \frac{f}{\gamma_\alpha}$$

solves the *kinetic-Ornstein-Uhlenbeck* equation

$$\partial_t h + v \cdot \nabla_x h = \Delta_\alpha h, \quad h(0, \cdot, \cdot) = h_0, \quad (2)$$

with

$$\Delta_\alpha h := \Delta_v h - \alpha v (v)^{\alpha-2} \cdot \nabla_v h,$$

and *zero-average initial datum* in the sense that

$$\iint_{\mathbb{T} \times \mathbb{R}^d} h_0(x, v) dx d\gamma_\alpha = 0.$$

By mass conservation, solutions to (2) are zero-average for any $t > 0$.

Therefore, we consider the *time average* defined as

$$\mathop{\int}_t^{t+\tau} g(s) ds := \frac{1}{\tau} \int_t^{t+\tau} g(s) ds$$

without specifying the τ dependence when not necessary. Our first result is devoted to the decay rate of $h(t, \cdot, \cdot) \rightarrow 0$ as $t \rightarrow \infty$ using time averages.

Theorem 1. *Let $\alpha \geq 1$. Then, for all $L > 0$ and $\tau > 0$, there exists a constant $\lambda > 0$ such that, for all $h_0 \in L^2(dx d\gamma_\alpha)$ with zero-average, the solution to (2) satisfies*

$$\mathop{\int}_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds \leq \|h_0\|_{L^2(dx d\gamma_\alpha)}^2 e^{-\lambda t}, \quad \forall t \geq 0. \quad (3)$$

The expression of λ as a function of τ and L is given in Section 4. To deal with large-time asymptotics in kinetic equations, it is by now standard to use hypocoercivity methods. Although not being exactly a hypocoercive method in the usual sense, Theorem 1 provides us with a hypocoercivity estimate.

Corollary 2. *Under the assumptions of Theorem 1, there exists an explicit constant $C > 1$ such that all solutions h to (2) fulfill*

$$\|h(t, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 \leq C \|h_0\|_{L^2(dx d\gamma_\alpha)}^2 e^{-\lambda t}, \quad \forall t \geq 0. \quad (4)$$

A typical feature of hypocoercive estimates is the factor $C > 1$ in (4). The prefactor $C > 1$ cannot be avoided. Otherwise, inequality (4) would be equivalent to a Poincaré inequality where the L^2 -norm of a function is controlled with the velocity gradient only. We can see explicitly that for $\alpha = 2$ the Green function of (1), computed in [39], has a built-in delay. In particular,

$$\|h_0\|_{L^2(dx d\gamma_2)}^2 - \|h(t, \cdot, \cdot)\|_{L^2(dx d\gamma_2)}^2 = O(t^3),$$

as $t \rightarrow 0^+$. Note that there is no such a constant in (3).

Now, let us turn our attention to the subexponential case $0 < \alpha < 1$.

Theorem 3. *Let $\alpha \in (0, 1)$. Then, for all $L > 0$ and $\tau > 0$, for all $\sigma > 0$, there is a constant $K > 0$ such that all solutions to (2) decay according to*

$$\int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds \leq K (1+t)^{-\frac{\sigma}{2(1-\alpha)}} \iint_{\mathbb{T} \times \mathbb{R}^d} \langle v \rangle^\sigma h_0^2 dx d\gamma_\alpha, \quad \forall t \geq 0. \quad (5)$$

Further details will be given in Section 5. The constants λ , C in Corollary 2 and K in Theorem 3 depend on $L > 0$ and $\tau > 0$ and their values are discussed later. The rate of Theorem 3 is the same as in the spatially-homogeneous case of [18, Proposition 11]. In the spatially-inhomogeneous case, rates are known, see [38, 17]. Finally, see Section 5 for a discussion of the limit $\alpha \rightarrow 1^-$.

Equation (2) is used in physics to describe the distribution function of a system of particles interacting randomly with some background, see for instance [10]. The *kinetic Fokker-Planck equation* is the Kolmogorov forward equation of Langevin dynamics

$$\begin{cases} dx_t = v_t dt, \\ dv_t = -v_t + \sqrt{2} dW_t, \end{cases}$$

where W_t is a standard Brownian motion. See [14, Introduction] for further details on connections with probability theory. The kinetic Fokker-Planck equation (1) is a simple kinetic equation which has a long history in mathematics that we will not retrace in details here. Mathematical results go back at least to [39] and are at the basis of the theory of L. Hörmander (see, *e.g.*, [37]), at least in the case $\alpha = 2$. For the derivation of the kinetic-Fokker-Planck equation from

underlying stochastic ODEs, particularly in the context of astrophysics, we can refer to [24, eq. (328)]. Modern hypoellipticity theory emerged from [36, 28] and was built up in a fully developed theory in [48] with important contributions in [35, 42]. Existence theory for solutions to the Vlasov-Fokker-Planck equation was discussed also in [26, Appendix A].

The word *hypo-coercivity* was coined by T. Gallay, in analogy with the already quoted *hypo-elliptic* theory of Hörmander in [37]. In [48], C. Villani distinguishes the regularity point of view for elliptic and parabolic problems driven by degenerate elliptic operators from the issue of the long-time behaviour of solutions, which is nowadays attached to the word hypo-coercivity. The underlying idea is to twist the reference norm, in order to carry properties (as the coercivity of the operator driving (2)) from velocity direction to space directions, thanks to commutators. Twisting the H^1 -norm creates equivalent norms, which are exponentially decaying along the evolution. So works the H^1 framework, see [48, 47, 30]. The H^1 -framework has been connected to the *carré du champ* method of D. Bakry and M. Emery in [9] by F. Baudoin, who proved decay also w.r.t. the Wasserstein distance, as shown in [11, 12, 13]. We report also the works [33, 29], where accurate convergence rates in the Wasserstein distance for (2) are computed through a coupling argument.

The H^1 hypo-coercivity implies a decay rate for the L^2 norm [42], but the corresponding estimates turn out to be sub-optimal. Moreover, kinetic equations driven by non-regularising operators are not well suited for the H^1 -framework. This motivates the development of direct L^2 techniques based on a perturbation of the L^2 norm. Such an approach can be found in [31] and [18], which is consistent with diffusion limits. In [17], the authors extend the technique to the subexponential case. Another possibility is to perform rotations in the phase space and use a Lyapunov inequality for matrices as in [8]. This approach gives optimal rates, but it is less general as it requires further algebraic properties for the diffusion operator and a detailed knowledge of its spectrum. The core of [8] is a spectral decomposition, that was originally understood via a toy model exposed in [31]. In a domain with periodic boundary conditions and no confining potential, the problem is reduced to an infinite set of ODEs corresponding to spatial modes. See [2, 3, 7] for details and extensions. Other techniques related to hypo-coercivity – involving time-integrated functionals and the application of the so called *kinetic-fluid decomposition*, appear in [45, 34] and subsequent papers.

A new hypo-coercivity theory, involving Sobolev norms with negative exponents of the transport operator, was recently proposed by S. Armstrong and J.-C. Mourrat in [5]. Using space-time adapted Poincaré inequalities they derive qualitative hypo-coercive estimates in the case $\alpha = 2$ on bounded spatial domains. The constants appearing there are not quantified. One of the difficulties lies in controlling the constant in Lions' Lemma, which is done in our Section 2. An extension to the whole space in presence of a confining potential can be found in [21]. Note that the strategy of using time-integrated functionals of the solutions to kinetic equations is present also in [46, 32].

Adopting the strategy of [5], in this paper we study the convergence to equilibrium of solutions to (1) and (2), as it is a simple benchmark in kinetic theory, [31, 47, 5], and a simplified model of the Boltzmann equation when collisions become grazing, see [27].

Our original contribution lies in making the strategy of [5] effective, and to generalise it to kinetic Fokker-Planck equations where local equilibria are not necessarily Maxwellians. First, we are able to track the Lions' constant in terms of the parameters (see Lemma 8). Moreover, we achieve a fully constructive proof of the averaging Lemma 12. This allows both for an explicit estimate of the constant and for an adaptation to more general models. One important point is the control in terms of the offset of the solution from the velocity average, without explicitly using gradients, see Proposition 13. So, we compute explicit and accurate decay rates of time averages of solutions to (2). Hypocoercivity estimates are obtained as a consequence of these decay rates, see Corollary 2. We perform an analysis for all positive values of α , which is consistent in the threshold case $\alpha = 1$. Since the estimates are explicit, we are able to compare the strategy of [5] to other L^2 -hypocoercivity methods.

This document is organized as follows. In Section 2 we collect some preliminary results: Poincaré and weighted Poincaré inequalities (Propositions 5 and 6), adapted Lions' inequality (Lemmas 7 and 8). In Section 3 we introduce an averaging lemma (Lemma 12), which is then used to prove the generalized Poincaré inequality of Proposition 13, at the core of the method. In Section 4 we use Proposition 13 and a Grönwall estimate to prove Theorem 1 and compute an explicit formula for λ (Proposition 15). Section 5 is devoted to the proof of Theorem 3, with additional details, and to the limit $\alpha \rightarrow 1^-$. Finally, in Section 6, we derive the hypocoercive estimates of Corollary 2. On the benchmark case $\alpha = 2$ in one spatial dimension, we also compare our results with those obtained by more standard methods.

2. Preliminaries

Let us start with some preliminary results.

2.1. Weighted spaces

For functions g of the variable v only, that is, of the so-called homogeneous case, we define the *weighted Lebesgue* and *Sobolev spaces*

$$L_\alpha^2 := L^2(\mathbb{R}^d, d\gamma_\alpha) \quad \text{and} \quad H_\alpha^1 := \left\{ g \in L_\alpha^2 : \nabla_v g \in (L_\alpha^2)^d \right\}.$$

We equip L_α^2 with the scalar product

$$(g_1, g_2) = \int_{\mathbb{R}^d} g_1(v) g_2(v) d\gamma_\alpha \tag{6}$$

and consider on H_α^1 the norm defined by

$$\|h\|_{H_\alpha^1}^2 := \left(\int_{\mathbb{R}^d} h d\gamma_\alpha \right)^2 + \int_{\mathbb{R}^d} |\nabla_v h|^2 d\gamma_\alpha$$

as in [5]. The duality product between $\mathbf{H}_\alpha^{-1} \ni z$ and $\mathbf{H}_\alpha^1 \ni g$ is given by

$$\langle z, g \rangle := \int_{\mathbb{R}^d} \nabla_v w_z \cdot \nabla_v g \, d\gamma_\alpha,$$

where w_z is the weak solution in \mathbf{H}_α^1 to

$$-\Delta_\alpha w_z = z - \int_{\mathbb{R}^d} z \, d\gamma_\alpha, \quad \int_{\mathbb{R}^d} w \, d\gamma_\alpha = 0.$$

Here we write $\int_{\mathbb{R}^d} z \, d\gamma_\alpha$ for functions which are integrable w.r.t. $d\gamma_\alpha$ and, up to a little abuse of notations, this quantity has to be understood in the distribution sense for more general measures. As a consequence and with the above notations, we define

$$\|z\|_{\mathbf{H}_\alpha^{-1}}^2 := \left(\int_{\mathbb{R}^d} z \, d\gamma_\alpha \right)^2 + \|w_z\|_{\mathbf{H}_\alpha^1}^2.$$

With these notation, the key property of the operator Δ_α , is

$$\langle g_1, \Delta_\alpha g_2 \rangle = - \int \nabla_v g_1 \cdot \nabla_v g_2 \, d\gamma_\alpha$$

for any functions $g_1, g_2 \in \mathbf{H}_\alpha^1$.

We recall that $\Omega_t = (t, t + \tau) \times \mathbb{T} \subset \mathbb{R}_t^+ \times \mathbb{R}_x^d$ and that x -periodic boundary conditions are assumed. Consider next functions h of $(t, x, v) \in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}^d$ and define the space

$$\mathbf{H}_{\text{kin}} := \left\{ h \in L^2 \left((t, t + \tau) \times \mathbb{T}; \mathbf{H}_\alpha^1 \right) : \partial_t h + v \cdot \nabla_x h \in L^2 \left(\Omega_t; \mathbf{H}_\alpha^{-1} \right) \forall t \geq 0 \right\}.$$

The dependence of the space on t, τ is implicit for readability purposes. We can equip \mathbf{H}_{kin} with the norm

$$\|h\|_{\text{kin}}^2 := \|h\|_{L^2(\Omega_t; L_\alpha^2)}^2 + |h|_{\text{kin}}^2$$

where the *kinetic semi-norm* is given by

$$|h|_{\text{kin}}^2 := \|\nabla_v h\|_{L^2(\Omega_t; L_\alpha^2)}^2 + \|\partial_t h + v \cdot \nabla_x h\|_{L^2(\Omega_t; \mathbf{H}_\alpha^{-1})}^2.$$

We refer to [5, Section 6] for the proof of following result.

Proposition 4. *The embedding $\mathbf{H}_{\text{kin}}^1 \hookrightarrow L^2(\Omega_t \times \mathbb{T}; L_\alpha^2)$ is continuous and compact for any $t \geq 0$.*

2.2. Poincaré inequalities

In this subsection, we consider functions g depending only on the variable v . Let $\alpha \geq 1$. We can state some *Poincaré inequalities*.

Proposition 5. *If $\alpha \geq 1$, there exists a constant $P_\alpha > 0$ such that, for all functions $g \in \mathbf{H}_\alpha^1$, we have*

$$\int_{\mathbb{R}^d} |g - \rho_g|^2 \, d\gamma_\alpha \leq P_\alpha \int_{\mathbb{R}^d} |\nabla_v g|^2 \, d\gamma_\alpha \quad \text{with} \quad \rho_g := \int_{\mathbb{R}^d} g \, d\gamma_\alpha. \quad (7)$$

With $\alpha \geq 1$, the operator Δ_α admits a compact resolvent on $L^2(d\gamma_\alpha)$. Then, (7) holds by the standard results of [19, Chapter 6]. The best constant P_α is such that P_α^{-1} is the minimal positive eigenvalue of $-\Delta_\alpha$. See [23] and the references quoted therein for estimates on P_α . In the case of the Gaussian Poincaré inequality, it is shown in [43] that $P_2 = 1$, although the result was probably known before.

2.3. Weighted Poincaré inequalities

Here we consider again functions depending only on v . For $\alpha \in (0, 1)$, inequality (7) has to be replaced by the following *weighted Poincaré inequality*.

Proposition 6. *If $\alpha \in (0, 1)$, there exists a constant $P_\alpha > 0$ such that, for all functions $g \in \mathcal{H}_\alpha^1$, we have*

$$\int_{\mathbb{R}^d} \langle v \rangle^{2(\alpha-1)} |g - \rho_g|^2 d\gamma_\alpha \leq P_\alpha \int_{\mathbb{R}^d} |\nabla_v g|^2 d\gamma_\alpha \quad \text{with} \quad \rho_g := \int_{\mathbb{R}^d} g d\gamma_\alpha. \quad (8)$$

For more details, we refer for instance to [17, Appendix A]. Notice that the average in the l.h.s. is taken w.r.t. $d\gamma_\alpha$, not w.r.t. $\langle v \rangle^{2(\alpha-1)} d\gamma_\alpha$.

2.4. Lions' Lemma

Let \mathcal{O} be an open, bounded and Lipschitz-regular subset of $\mathbb{R}^{d+1} \approx \mathbb{R}_t \times \mathbb{R}_x^d$. We recall that

$$\mathcal{H}^{-1}(\mathcal{O}) = \left\{ w \in \mathcal{D}^*(\mathcal{O}) : |\langle w, u \rangle_{\mathcal{O}}| \leq C \|u\|_{\mathcal{H}_0^1(\mathcal{O})}, C > 0 \right\},$$

where $\mathcal{D}^*(\mathcal{O})$ denotes the space of distributions over \mathcal{O} , equipped with the weak* topology, and $\langle w, u \rangle_{\mathcal{O}}$ is the duality product between \mathcal{H}^{-1} and \mathcal{H}_0^1 . The norm on $\mathcal{H}_0^1(\mathcal{O})$ is as usual $u \mapsto \|\nabla u\|_{L^2(\mathcal{O})}$. On $\mathcal{H}^1(\mathcal{O})$, we introduce the norm

$$\|u\|_{\mathcal{H}^1}^2 = \left| \int_{\mathcal{O}} u dt dx \right|^2 + \|\nabla u\|_{L^2(\mathcal{O})}^2.$$

The norm induced on $\mathcal{H}^{-1}(\mathcal{O})$ is then

$$\|w\|_{\mathcal{H}^{-1}(\mathcal{O})}^2 = \langle w, 1 \rangle_{\mathcal{O}}^2 + \|z_w\|_{\mathcal{H}^1(\mathcal{O})}^2,$$

where z_w is the solution to

$$-(\partial_{tt} + \Delta_x) z_w = w - \langle w, 1 \rangle_{\mathcal{O}}, \quad \int_{\mathcal{O}} z_w dt dx = 0.$$

Lions' Lemma gives a sufficient condition for a distribution to be an L^2 function. The following statement is taken from [6].

Lemma 7. *Let \mathcal{O} be a bounded, open and Lipschitz-regular subset in \mathbb{R}^{d+1} . Then, for all $u \in \mathcal{D}^*(\mathcal{O})$, we have that $u \in L^2(\mathcal{O})$ if and only if the weak gradient ∇u belongs to $\mathcal{H}^{-1}(\mathcal{O})$. Moreover, there exists a constant $C_L(\mathcal{O})$ such that*

$$\left\| u - \int_{\mathcal{O}} u dx dt \right\|_{L^2(\mathcal{O})}^2 \leq C_L \|\nabla u\|_{\mathcal{H}^{-1}(\mathcal{O})}^2,$$

for any $u \in L^2(\mathcal{O})$.

According to [20, 16, 25], if \mathcal{O} is star-shaped w.r.t. a ball, then the constant C_L has the following structure:

$$C_L = 4|S^d| \frac{D(\mathcal{O})}{d(\mathcal{O})}, \quad (9)$$

where D is the diameter of \mathcal{O} , while $d(\mathcal{O})$ is the diameter of the largest ball one can include in \mathcal{O} . See in particular [25, Remark 9.3] and [16, Lemma 1]. As a consequence, we have the following explicit expression of C_L when $\mathcal{O} = \Omega$.

Lemma 8. *Let $L > 0$, $\tau \in (0, L)$ and $\Omega = (0, \tau) \times (0, L)^d$. Lemma 7 holds with*

$$C_L = 4|S^d| \frac{\sqrt{dL^2 + \tau^2}}{\tau}. \quad (10)$$

2.5. The kinetic Ornstein-Uhlenbeck equation

We consider solutions to (2) in the weak sense, *i.e.*, functions h in the space $C(\mathbb{R}^+; L^2(dx d\gamma_\alpha))$ with initial datum $h_0 = h(0, \cdot, \cdot)$ in $L^2(dx d\gamma_\alpha)$ such that (2) holds in the sense of distributions on $(0, \infty) \times \mathbb{R}_x^d \times \mathbb{R}_v^d$. The following result is taken from [5] if $\alpha = 2$. The extension to $\alpha \neq 2$ is straightforward as follows from a careful reading of the proof in [5, Proposition 6.10].

Proposition 9. *Let $L > 0$ and $\alpha > 0$. With $\Omega = (0, \tau) \times (0, L)^d$, for all zero-average initial datum $h_0 \in L^2(dx d\gamma_\alpha)$, there exists a unique solution h to (2) such that $h \in H_{\text{kin}}$ for all $\tau > 0$.*

Regularity properties for (2) are collected in [5, Section 6]. In the special case $\alpha = 2$, some fractional regularity along all directions of the phase space are known. Also see [44] for further result on regularity theory for kinetic Fokker-Planck equations.

2.6. A priori estimates

We state two estimates for solutions to (2).

Lemma 10. *Let $L > 0$, $\tau > 0$, $\Omega = (0, \tau) \times (0, L)^d$, and $\alpha > 0$. If h is a solution to (2), then we have*

$$\|(\partial_t + v \cdot \nabla_x) h\|_{L^2(\Omega; H_\alpha^{-1})} \leq \|\nabla_v h\|_{L^2(\Omega; L_\alpha^2)}. \quad (11)$$

Proof. Take a test function $\phi \in L^2(H_\alpha^1)$, and write

$$\int_{\mathbb{T}} \langle (\partial_t + v \cdot \nabla_x) h, \phi \rangle dx = \int_{\mathbb{T}} \langle \Delta_\alpha h, \phi \rangle dx = - \int_{\mathbb{T}} \langle \nabla_v h, \nabla_v \phi \rangle dx,$$

from which (11) easily follows, after maximizing over $\|\nabla_v \phi\|_{L_\alpha^2} \leq 1$. \square

For completeness, let us recall the classical L^2 decay estimate for solutions to (2).

Lemma 11. *Let $L > 0$, $\tau > 0$, $\Omega = (0, \tau) \times (0, L)^d$, and $\alpha > 0$. If h is a solution to (2), then we have*

$$\frac{d}{dt} \|h\|_{L^2(dx d\gamma_\alpha)}^2 = -2 \|\nabla_v h\|_{L^2(dx d\gamma_\alpha)}^2.$$

3. An averaging lemma and a generalized Poincaré inequality

For all functions $h \in \mathbf{H}_{\text{kin}}$, we define the *spatial density*

$$\rho_h := \int_{\mathbb{R}^d} h(\cdot, \cdot, v) d\gamma_\alpha.$$

Notice that $\int_Q \rho_h dx = 0$ whenever h is a zero-average function.

3.1. Averaging lemma

Inspired by [5, Proposition 6.2], the following *averaging lemma* provides a norm of the spatial density, as for instance in [44].

Lemma 12. *Let $L > \tau > 0$, $\Omega = (0, \tau) \times (0, L)^d$, and $\alpha > 0$. For all $h \in \mathbf{H}_{\text{kin}}$, we have*

$$\|\nabla_{t,x} \rho_h\|_{\mathbf{H}^{-1}(\Omega)}^2 \leq d_\alpha \left(\|h - \rho_h\|_{L^2(dt dx d\gamma_\alpha)}^2 + \|\partial_t h + v \cdot \nabla_x h\|_{L^2(\Omega; H_\alpha^{-1})}^2 \right) \quad (12)$$

with

$$d_\alpha = 2 \left(\|v_1 |v|^2\|_{L_\alpha^2}^2 + \left(1 + \frac{L^2}{4\pi^2}\right) \| |v|^2\|_{L_\alpha^2}^2 + \frac{d^2 L^2}{4\pi^2} \|v\|_{L_\alpha^2}^2 \right). \quad (13)$$

Inequality (12) can be extended to any measure $d\gamma$ such that $\int_{\mathbb{R}^d} |v|^4 d\gamma < \infty$ and $\int_{\mathbb{R}^d} v d\gamma = 0$. The proof of Lemma 12 is technical, but follows in a standard way from the time-independent case, as it is common in averaging lemmas: see [44]. For sake of simplicity, we detail only the t -independent case below.

Proof of Lemma 12. Assume that $h \in \mathbf{H}_{\text{kin}}$ does not depend on t . Let $\phi \in \mathcal{D}(\mathbb{T})^d$ be a smooth test-vector field with zero average on each component. We write

$$- \int_{\mathbb{T}} \rho_h \nabla_x \cdot \phi dx = \int_{\mathbb{T}} (\nabla_x \rho_h) \cdot \phi dx,$$

with a slight abuse of notation, since the integral of the r.h.s. is in fact a duality product. Using $\int_{\mathbb{R}^d} v_i v_j d\gamma_\alpha = d^{-1} \|v\|_{L_\alpha^2}^2 \delta_{ij}$, we obtain

$$\int_{\mathbb{T}} \nabla_x \rho_h \cdot \phi dx = d \|v\|_{L_\alpha^2}^{-2} \iint_{\mathbb{T} \times \mathbb{R}^d} v \cdot \nabla_x \rho_h \phi \cdot v dx d\gamma_\alpha.$$

By adding and subtracting ρ_h , and then integrating by parts, still at formal level, we obtain

$$\begin{aligned} & \iint_{\mathbb{T} \times \mathbb{R}^d} v \cdot \nabla_x \rho_h \phi \cdot v dx d\gamma_\alpha \\ &= - \iint_{\mathbb{T} \times \mathbb{R}^d} v \cdot (h - \rho_h) \nabla_x \phi \cdot v dx d\gamma - \iint_{\mathbb{T} \times \mathbb{R}^d} v \cdot \nabla_x h v \cdot \phi dx d\gamma_\alpha \\ &\leq \|h - \rho_h\|_{L^2(dx d\gamma_\alpha)} \|\nabla_x \phi\|_{L^2(dx)} \|v^2\|_{L_\alpha^2} \\ &\quad + \|v \cdot \nabla_x h\|_{L^2(H_\alpha^{-1})} \|\phi\|_{L^2(dx)} \|v\|_{L_\alpha^2} \end{aligned}$$

using Cauchy-Schwarz inequalities and duality estimates. By the Poincaré inequality, we know that

$$\frac{4\pi^2}{dL^2} \|\phi\|_{L^2(dx)}^2 \leq \|\nabla\phi\|_{L^2(dx)}^2$$

Maximizing the r.h.s. on ϕ such that $\|\nabla_x\phi\|_{L^2(dx)} \leq 1$ completes the proof of the t -independent case. When h additionally depends on t , the same scheme can be applied with $v \cdot \nabla_x$ replaced by $\partial_t + v \cdot \nabla_x$. \square

3.2. A generalized Poincaré inequality

The next *a priori* estimate is at the core of the method. It is a modified Poincaré inequality in t, x and v which relies on Lemma 12 and involves derivatives of various orders.

Proposition 13. *Let $L > 0$, $\tau > 0$, $\Omega = (0, \tau) \times (0, L)^d$, and $\alpha > 0$. Then, for all $h \in \mathbf{H}_{\text{kin}}$ with zero average, we have that*

$$\|h\|_{L^2(dt dx d\gamma_\alpha)}^2 \leq C \left(\|h - \rho_h\|_{L^2(dt dx d\gamma_\alpha)}^2 + \|\partial_t h + v \cdot \nabla_x h\|_{L^2(\Omega; H_\alpha^{-1})}^2 \right) \quad (14)$$

with $C = 1 + C_L d_\alpha$, where C_L and d_α are given respectively by (10) and (13).

Proof. By orthogonality in $L^2(\Omega; L_\alpha^2)$ and because $d\gamma_\alpha$ is a probability measure, we have the decomposition

$$\|h\|_{L^2(\Omega; L_\alpha^2)}^2 = \|h - \rho_h\|_{L^2(\Omega; L_\alpha^2)}^2 + \|\rho_h\|_{L^2(\Omega)}^2.$$

The function ρ_h has zero average on Ω by construction, so that

$$\|h\|_{L^2(\Omega; L_\alpha^2)}^2 \leq \|h - \rho_h\|_{L^2(\Omega; L_\alpha^2)}^2 + C_L \|\nabla_{x,t}\rho\|_{\mathcal{H}^{-1}(\Omega)}^2$$

by Lemma 8. Hence

$$\|h\|_{L^2(\Omega; L_\alpha^2)}^2 \leq (1 + C_L d_\alpha) \|h - \rho_h\|_{L^2(\Omega; d\gamma_\alpha)}^2 + C_L d_\alpha \|\partial_t h + v \cdot \nabla_x h\|_{L^2(\Omega; H_\alpha^{-1})}^2$$

by Lemma 12. This concludes the proof. \square

4. Linear and superlinear local equilibria: exponential decay rate

In this Section, we consider the case $\alpha \geq 1$ and the domain $\Omega = (t, t + \tau) \times (0, L)^d$, for an arbitrary $t \geq 0$. Let us define $\kappa_\alpha := (1 + C_L d_\alpha)(P_\alpha + 1)$ where P_α is the Poincaré constant in (7) and where C_L and d_α are given respectively by (10) and (13).

Lemma 14. *Let $L > 0$, $\tau > 0$, $t \geq 0$, $\Omega_t = (t, t + \tau) \times (0, L)^d$, and $\alpha \geq 1$. Then, for all $h \in \mathbf{H}_{\text{kin}}$ with zero average which solve (2), we have that*

$$\|h\|_{L^2(dt dx d\gamma_\alpha)}^2 \leq \kappa_\alpha \|\nabla_v h\|_{L^2(dt dx d\gamma_\alpha)}^2. \quad (15)$$

Proof. We know that

$$\|h\|_{L^2(dt dx d\gamma_\alpha)}^2 \leq (1 + C_L d_\alpha) \left(P_\alpha \|\nabla_v h\|_{L^2(dt dx d\gamma_\alpha)}^2 + \|\partial_t h + v \cdot \nabla_x h\|_{L^2(H_\alpha^{-1})}^2 \right)$$

as a consequence of (7) and (14). Then (15) follows from Lemma 10. \square

We are ready to prove Theorem 1 with an explicit estimate of the constant λ .

Proof of Theorem 1. Inequality (15) – on the interval $(t, t + \tau)$ – gives

$$\int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds \leq \kappa_\alpha \int_t^{t+\tau} \|\nabla_v h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds.$$

With $\lambda = 2/\kappa_\alpha$, we deduce from Lemma 11 that

$$\begin{aligned} \frac{d}{dt} \int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds &= -2 \int_t^{t+\tau} \|\nabla_v h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds \\ &\leq -\lambda \int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds. \end{aligned}$$

Grönwall's Lemma and the monotonicity of $t \mapsto \|h(t, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2$ imply

$$\begin{aligned} \int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds &\leq \int_0^\tau \|h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds e^{-\lambda t} \\ &\leq \tau \|h_0\|_{L^2(dx d\gamma_\alpha)}^2 e^{-\lambda t} \end{aligned}$$

for ant $t \geq 0$, which proves (3), that is, Theorem 1. \square

Indeed, the estimate for λ is explicit, as we state in the following.

Proposition 15. *For any $\alpha \geq 1$, Theorem 1 holds true with*

$$\frac{1}{\lambda} = \frac{1}{\tau} \left(\tau + \sqrt{dL^2 + \tau^2} \right) \left(2d_\alpha |\mathbb{S}^{d-1}| (P_\alpha + 1) \right).$$

Notice that the r.h.s. vanishes as $\tau \downarrow 0$, which is expected because of the degeneracy of Δ_α : an exponential decay rate of $\|h(t, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2$ cannot hold. The section is concluded showing how the result above yields the classic hypocoercivity estimate of Corollary 2.

Proof of Corollary 2. For any $t \geq 0$, we know from Theorem 1 that

$$\|h(t + \tau, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 \leq \int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds \leq \|h_0\|_{L^2(dx d\gamma_\alpha)}^2 e^{-\lambda t},$$

as a consequence of the monotonicity of the L^2 norm, according to Lemma 11. This proves that

$$\|h(t, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 \leq C \|h_0\|_{L^2(dx d\gamma_\alpha)}^2 e^{-\lambda t}$$

with $C = e^{\lambda\tau}$ for any $t \geq \tau$. However, if $t \in [0, \tau)$, it turns out that $C e^{-\lambda t} \geq 1$ so that the inequality is also true by Lemma 11. This concludes the proof. \square

5. Sublinear equilibria: algebraic decay rates

5.1. Proof of Theorem 3

Assume that $\alpha \in (0, 1)$. Let us define the parameter $\beta = 2(1 - \alpha)/p$ where $p, q > 1$ are Hölder conjugate exponents, i.e., $\frac{1}{p} + \frac{1}{q} = 1$ and define

$$Z_h(t) := \iint_{\mathbb{T} \times \mathbb{R}^d} \langle v \rangle^{\beta q} |h - \rho_h|^2 dx d\gamma_\alpha. \quad (16)$$

The following estimates replace Proposition 13.

Proposition 16. *Let $L > 0$, $\tau > 0$, $t \geq 0$, $\Omega_t = (t, t + \tau) \times (0, L)^d$, and $\alpha \geq 1$. With the above notations, for all $h \in \mathbf{H}_{\text{kin}}$ with zero average, we have that*

$$\begin{aligned} & \|h\|_{L^2(dt dx d\gamma_\alpha)}^2 \\ & \leq C P_\alpha^{\frac{1}{p}} \|\nabla_v h\|_{L^2(dt dx d\gamma_\alpha)}^{\frac{2}{p}} \left(\int_t^{t+\tau} Z_h(s) ds \right)^{\frac{1}{q}} + C \|\partial_t h + v \cdot \nabla_x h\|_{L^2(\Omega; H_\alpha^{-1})}^2 \end{aligned}$$

where $C = 1 + C_L d_\alpha$ is as in Proposition 13 and P_α denotes the constant in the weighted Poincaré inequality (8).

Proof. Using (14) and Hölder's inequality w.r.t. the variable v , we find that

$$\|h - \rho_h\|_{L_\alpha^2}^2 \leq \left(\int_{\mathbb{R}^d} \langle v \rangle^{-\beta p} |h - \rho_h|^2 d\gamma_\alpha \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} \langle v \rangle^{\beta q} |h - \rho_h|^2 d\gamma_\alpha \right)^{\frac{1}{q}}.$$

The weighted Poincaré inequality (8) with $\beta p = 2(1 - \alpha)$ and an additional Hölder inequality w.r.t. the variables t and x allow us to complete the proof. \square

Lemma 17. *Let $L > 0$, $\tau > 0$, $t \geq 0$, $\Omega_t = (t, t + \tau) \times (0, L)^d$, and $\alpha \in (0, 1)$. There is a constant $W > 0$ such that, for all solution $h \in \mathbf{H}_{\text{kin}}$ to (2) with an initial datum h_0 with zero average, using the notation (16) as in Proposition 16, we have*

$$Z_h(t) \leq W \iint_{\mathbb{T} \times \mathbb{R}^d} \langle v \rangle^{\beta q} h_0^2 dx d\gamma_\alpha, \quad \forall t \geq 0.$$

Proof. An elementary computation shows that

$$\begin{aligned} \iint_{\mathbb{T} \times \mathbb{R}^d} \langle v \rangle^{\beta q} |h - \rho_h|^2 dx d\gamma_\alpha & \leq 2 \iint_{\mathbb{T} \times \mathbb{R}^d} \langle v \rangle^{\beta q} (h^2 + \rho_h^2) dx d\gamma_\alpha \\ & \leq 2 \left(1 + \int_{\mathbb{R}^d} \langle v \rangle^{\beta q} d\gamma_\alpha \right) \iint_{\mathbb{T} \times \mathbb{R}^d} \langle v \rangle^{\beta q} h^2 dx d\gamma_\alpha \end{aligned}$$

because $\rho_h^2 = \left(\int_{\mathbb{R}^d} h d\gamma_\alpha \right)^2 \leq \int_{\mathbb{R}^d} h^2 d\gamma_\alpha \leq \int_{\mathbb{R}^d} \langle v \rangle^{\beta q} h^2 d\gamma_\alpha$. According to [17, Proposition 4], there is a constant $\mathcal{K}_{\beta q} > 1$ such that

$$\iint_{\mathbb{T} \times \mathbb{R}^d} \langle v \rangle^{\beta q} |h(t, x, v)|^2 dx d\gamma_\alpha \leq \mathcal{K}_{\beta q} \iint_{\mathbb{T} \times \mathbb{R}^d} \langle v \rangle^{\beta q} h_0^2 dx d\gamma_\alpha, \quad \forall t \geq 0.$$

The result follows with $W = 2 \left(1 + \int_{\mathbb{R}^d} \langle v \rangle^{\beta q} d\gamma_\alpha \right) \mathcal{K}_{\beta q}$. \square

Assume that $h \in \mathbf{H}_{\text{kin}}$ solves (2) with an initial datum h_0 with zero average and let us collect our estimates. With Proposition 10, Proposition 16, and Lemma 17, the estimate of Lemma 14 is replaced by

$$\|h\|_{L^2(dt dx d\gamma_\alpha)}^2 \leq A \|\nabla_v h\|_{L^2(dt dx d\gamma_\alpha)}^{\frac{2}{p}} + C \|\nabla_v h\|_{L^2(dt dx d\gamma_\alpha)}^2 \quad (17)$$

with $A = C P_\alpha^{\frac{1}{p}} (\tau W)^{1/q} \left(\iint_{\mathbb{T} \times \mathbb{R}^d} \langle v \rangle^{\beta q} h_0^2 dx d\gamma_\alpha \right)^{1/q}$.

The main result of the section is a technical version of Theorem 3. Let

$$\mathbf{x}(t) := \int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds \quad \text{and} \quad \mathbf{y}(t) := \int_t^{t+\tau} \|\nabla_v h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds$$

where norms are taken on $\mathbb{T} \times \mathbb{R}^d$. We know from Lemma 11 and (17) that

$$\mathbf{x}' = -2\mathbf{y} \quad \text{and} \quad \mathbf{x} \leq \varphi(\mathbf{y}) := A\mathbf{y}^{1/p} + C\mathbf{y}.$$

Finally, let us denote by φ^{-1} the inverse of $\mathbf{y} \mapsto \varphi(\mathbf{y})$ and consider

$$\psi(\mathbf{z}) := \int_{\mathbf{z}}^{\mathbf{x}_0} \frac{dz}{2\varphi^{-1}(z)} \quad \text{with} \quad \mathbf{x}_0 = \|h_0\|_{L^2(dx d\gamma_\alpha)}^2.$$

Theorem 18. *Let $L > 0$, $\tau > 0$, $t \geq 0$, $\Omega_t = (t, t+\tau) \times (0, L)^d$, and $\alpha \in (0, 1)$. With the above notations, for all solution $h \in \mathbf{H}_{\text{kin}}$ to (2) with an initial datum h_0 with zero average, we have*

$$\mathbf{x}(t) = \int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds \leq \psi^{-1}(t), \quad \forall t \geq 0.$$

Proof. The strategy goes as in [41, 17]. Everything reduces to the differential inequality

$$\mathbf{x}' \leq -2\varphi^{-1}(\mathbf{x})$$

using the monotonicity of $\mathbf{y} \mapsto \varphi(\mathbf{y})$. From by the elementary Bihari-Lasalle inequality, see [15, 40], which is obtained by a simple integration, we obtain

$$\mathbf{x}(t) = \int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds \leq \psi^{-1}(t + \psi(\mathbf{x}(0))).$$

Since, on the one hand

$$\mathbf{x}(0) = \int_0^\tau \|h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds \leq \|h_0\|_{L^2(dx d\gamma_\alpha)}^2 = \mathbf{x}_0$$

because $s \mapsto \|h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2$ is nonincreasing according to Lemma 11, and ψ is nonincreasing on the other hand, then

$$\psi^{-1}(t + \psi(\mathbf{x}(0))) \leq \psi^{-1}(t),$$

which concludes the proof. Notice that the dependence on h_0 enters in A and \mathbf{x}_0 , and henceforth in φ and ψ . \square

Proof of Theorem 3. Since $\lim_{t \rightarrow +\infty} y(t) = 0$, we have that $\varphi(y(t)) \sim Ay(t)^{1/p}$ as $t \rightarrow +\infty$, which heuristically explains the role played by p in (5). This can be made rigorous as follows. Notice that

$$\varphi(y) = Ay^{1/p} + Cy \leq A_0 y^{1/p}, \quad \forall y \leq y_0, \quad \text{with} \quad A_0 = A + Cy_0^{1-1/p}.$$

With A replaced by A_0 and C replaced by 0, the computation of the proof of Theorem 18 is now explicit. With the choice $y_0 = \varphi^{-1}(x_0)$, we know that $y(t) \leq y_0$ for any $t \geq 0$ and obtain

$$x(t) \leq (x_0^{1-p} + 2(p-1)A_0^{-p}t)^{-\frac{1}{p-1}}, \quad \forall t \geq 0. \quad (18)$$

Using $x_0 = Ay_0^{1/p} + Cy_0 \geq Cy_0$, we know that $A_0 = x_0 y_0^{-1/p} \leq A + C^{1/p} x_0^{1-1/p}$, which proves (5) with

$$K = \max \left\{ 1, (2(p-1))^{1/(1-p)} \left(C P_\alpha^{1/p} (\tau W)^{1-1/p} + C^{1/p} \right) \right\}.$$

The conclusion holds using $\sigma = \beta q = 2(1-\alpha)/(p-1)$. \square

5.2. The linear threshold: from algebraic to exponential rates

A very natural question arises: *is the result Theorem 15 (corresponding to $\alpha \in (0, 1)$) consistent with the result of Theorem 18 (which covers any $\alpha \geq 1$)?* A first observation is that we can vary α in the assumptions concerning the initial data.

Lemma 19. *If $h_0 \in L^2(\mathbb{T}; L_{\alpha_0}^2)$ for some $\alpha_0 \in (0, 1)$, then $\langle v \rangle^{\sigma/2} h_0 \in L^2(\mathbb{T}; L_\alpha^2)$ for any $\alpha > \alpha_0$ and any $\sigma > 0$.*

The proof is a simple consequence of the fact that $v \mapsto \langle v \rangle^\sigma \exp\langle v \rangle^{\alpha-\alpha_0}$ is uniformly bounded. For any $\alpha > \alpha_0$, let us denote the corresponding solution of (2) with initial datum h_0 , of zero average, by $h^{(\alpha)}$.

If $\alpha \in (\alpha_0, 1)$, then (18) can be rewritten as

$$\int_t^{t+\tau} \|h^{(\alpha)}(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds \leq \|h_0\|_{L^2(dx d\gamma_\alpha)}^2 (1 + (p-1)\ell(\alpha)t)^{-\frac{1}{p-1}}.$$

By passing to the limit as $\alpha \rightarrow 1^-$, we recover (3) with $\lambda = \lim_{\alpha \rightarrow 1^-} \ell(\alpha)$, where $\ell(\alpha) = 2 \|h_0\|_{L^2(dx d\gamma_\alpha)}^{2(p-1)} A_0^{-p}$ and $A_0 = A_0(\alpha)$ as above. The Poincaré constant P_α in the weighted Poincaré inequality (8) admits a limit as $\alpha \rightarrow 1^-$, according to [18, Appendix A].

The limit of $\lim_{\alpha \rightarrow 1^-} \ell(\alpha)$ is certainly not optimal. By working directly on the Bihari-Lasalle estimate of Theorem 18, we can recover the value of λ in Theorem 15. Notice here that $\sigma > 0$ plays essentially no role and can be taken arbitrarily small, even depending on α , but such that $p = 2(1-\alpha)/\sigma \rightarrow 1$ as $\alpha \rightarrow 1^-$.

As in [22, 38], it is possible to obtain improved decay rates in (5) by picking the initial datum in a smaller space. Typically, the control of additional norms

or moments is asked. However, the strategy in the current paper is in the opposite direction. If $\alpha \in (0, 1)$ we are interested in taking the initial data in a space as large as possible so that we can compute decay rates. The additional conditions to be imposed have been shown to vanish as $\alpha \rightarrow 1^-$.

6. Hypocoercivity and comparison with some other methods

6.1. An explicit hypocoercivity result

Theorem 1 implies an L^2 -hypocoercivity result in the linear and superlinear regimes $\alpha \geq 1$, see Corollary 2. The remainder of this section is devoted to a comparison with earlier hypocoercivity results in a simple benchmark case: let $\alpha = 2$, $d = 1$ and $L = 2\pi$. In this case, for the choice $\tau = 2\pi$, Theorem 1 amounts to

$$\|h\|_{L^2(dx d\gamma_2)}^2 \leq e^{\frac{\pi}{8\sqrt{3}}} \|h_0\|_{L^2(dx d\gamma_2)}^2 e^{-\frac{t}{8\sqrt{3}}}, \quad \forall t \geq 0.$$

For sake of comparison, notice that $\lambda = 1/(8\sqrt{3}) \approx 0.0721688$. Even if we are aware of explicit or sharp results in other metrics than L^2 for (2), as [29, 42], we restrict our discussion to L^2 -hypocoercivity methods.

6.2. The DMS method

The first comparison is with the abstract twisted L^2 hypocoercivity method of [31, 18]. Let $\|\cdot\|$ be the norm of $L^2(dx d\gamma_2)$ and (\cdot, \cdot) the associated scalar product. We consider the evolution equation

$$\partial_t h + Th = \mathcal{L}h. \quad (19)$$

Theorem 20. *Let h be a solution of (19) with initial datum $h_0 \in L^2(Q; L^2_2)$ and assume that T and \mathcal{L} are respectively anti-self-adjoint and self-adjoint operators on $L^2(Q; L^2_2)$ such that, for some positive constants λ_m , λ_M and C_M , we have*

- (A1) $(-\mathcal{L}h, h) \geq \lambda_m \|(1 - \Pi)h\|^2$ for all $h \in D(\mathcal{L})$,
- (A2) $\|T\Pi h\|^2 \geq \lambda_M \|\Pi h\|^2$ for all $h \in D(T\Pi)$,
- (A3) $\Pi T \Pi h = 0$,
- (A4) $\|AT(1 - \Pi)h\| + \|A\mathcal{L}h\| \leq C_M \|(\text{Id} - \Pi)h\|$ for all admissible $h \in L^2(Q; L^2_2)$

where $A := (\text{Id} + (T\Pi)^* T\Pi)^{-1} (T\Pi)^*$ and Π is the projection in L^2_2 onto the kernel of \mathcal{L} . Then we have

$$\|h(t, \cdot, \cdot)\|^2 \leq C \|h_0\|^2 e^{-\lambda t} \quad \forall t \geq 0$$

with $C = (1 + \delta)/(1 - \delta)$, $\delta = \frac{1}{2} \min \left\{ 1, \lambda_m, \frac{\lambda_m \lambda_M}{(1 + \lambda_M) C_M^2} \right\}$ and $\lambda = \frac{2\delta \lambda_M}{3(1 + \lambda_M)}$.

This result is taken from [18, Proposition 4]. According to [18, Corollary 9], we have the estimate $\lambda = 1/24 \approx 0.041667$. A minor improvement is obtained as follows. Using Theorem 20 applied with $d = 1$, $L = 2\pi$, $T = v \partial_x$ and $\mathcal{L} = \partial_v^2 - v \partial_v$, in Fourier variables, we obtain $\lambda_m = \lambda_M = 1$ and $C_M = (1 + \sqrt{3})/2$ according to [7, Section II.1.3.2], so that $\lambda = 1/(12 + 6\sqrt{3}) \approx 0.0446582$. Using Fourier modes, a slightly better estimate is obtained from [7, Section II.1.2] with $\lambda \approx 0.176048$.

6.3. Direct spectral methods

In a series of papers, F. Achleitner, A. Arnold, E. Carlen and several other collaborators use direct spectral methods. We refer in particular to [8, 2, 3, 4] and also [7] for an introduction to the method, which can be summarized as follows.

Let us consider (19) written after a Fourier transform in x , so that $T = i\xi \cdot v$, and acting on $L^2(dx; \mathbb{L}_2^2)$ now considered as a space of complex valued functions. Assume that for some positive definite bounded Hermitian operator P and some constant $\lambda \in (0, +\infty)$, we have

$$(L - T)^* P + P(L - T) \geq 2\lambda P.$$

Let us consider the *twisted* norm $\|\hat{f}\|_P^2 := \int_Q (\hat{f}, P\hat{f}) dx$ where (\cdot, \cdot) is the natural extension of the scalar product as defined in (6). From

$$\frac{d}{dt} \|\hat{f}\|_P^2 = -\langle \hat{f}, ((L - T)^* P + P(L - T))\hat{f} \rangle \leq -2\lambda \|\hat{f}\|_P^2,$$

for some $C > 1$, we deduce that

$$C^{-1} \|f(t, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 \leq \|\hat{f}(t, \cdot, \cdot)\|_P^2 \leq e^{-2\lambda t} \|\hat{f}_0\|_P^2 \quad \forall t \geq 0.$$

To our knowledge, μ has not yet been computed in the case of (1). The spectral decomposition

$$h(t, x, v) = \sum_{\xi \in \mathbb{Z}^d} \sum_{k \in \mathbb{N}^d} a_{\xi, k}(t) H_k(v) e^{-i \frac{2\pi}{L} \xi \cdot x}$$

provides an easy framework for finite dimensional approximations using the basis of Hermite functions $(H_k)_{k \in \mathbb{N}}$ and the numerical value $\mu \approx 0.4$ has been obtained according to [1].

Proposition 15		$\lambda \approx 0.07$
DMS [31]		$\lambda \approx 0.04$
ADSW [7]		$\lambda \approx 0.17$
Achleitner (numerics) [1]		$\lambda \approx 0.4$

Table 1: Comparison among different L^2 hypocoercivity methods

6.4. Comparison for decay rates in limit regimes

Let $\alpha \geq 1$. Corollary 2 provides us with a decay estimate depending on the parameter L , which represents the length of the spatial domain \mathbb{T} . Note that (10) is meaningful if $0 < \tau < L$. We shall now consider two situations, corresponding to $L \rightarrow \infty$, where spatial diffusion dominates, and to $L \rightarrow 0$, where the dominant term is the collision operator Δ_α . In the first case, we have that the decay exponent

$$\lambda \approx \frac{\tau}{L^3} \rightarrow 0, \quad \text{as } L \rightarrow \infty.$$

The hypocoercivity constant $C \approx 1$. Hence, exponential decay is lost in the limit. On the other hand, for $L \rightarrow 0$, we have

$$\lambda \approx \frac{1}{4(P_\alpha + 1) \|v_1 |v|^2\|_{L_\alpha^2}} \approx 0.04,$$

if $\alpha = 1$. Moreover,

$$C \approx 1.$$

This rate has the wrong order once compared to [7], where the authors recover the value

$$\lambda \approx 1 - \sqrt{3/7}.$$

Our inaccuracy is mainly due to the incompatibility between (10) and Lemma 12. Moreover, the value of the Lions constant in (10) is just an estimate and it is not expected to be as accurate as something achieved by a spectral method (even if its scaling is correct).

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