



HAL
open science

Convexity of elliptically distributed dependent chance constraints

Hoang Nam Nguyen, Abdel Lisser

► **To cite this version:**

Hoang Nam Nguyen, Abdel Lisser. Convexity of elliptically distributed dependent chance constraints. 2022. hal-03269257v2

HAL Id: hal-03269257

<https://hal.science/hal-03269257v2>

Preprint submitted on 26 Jul 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Convexity of elliptically distributed dependent linear joint chance constraints

Hoang Nam NGUYEN · Abdel LISSER

Received: date / Accepted: date

Abstract In this paper, we study the convexity of the linear joint chance constraints. We assume that the constraint row vectors are elliptically distributed. Further, the dependence of the rows is modeled by a family of Archimedean copulas, namely, the Gumbel-Hougaard family of copulas. Under mild assumptions, we prove the eventual convexity of the feasibility set.

Keywords Linear joint chance constraints · Gumbel-Hougaard copulas · Elliptical distributions · Eventual convexity

Mathematics Subject Classification (2000) MSC 90C15 · 90C25 · 90C59

1 Introduction

We consider the following linear optimization with joint chance constraints

$$\begin{aligned} \min \quad & c^T x \\ \text{subject to} \quad & \mathbb{P}\{Vx \leq D\} \geq p \\ & x \in Q. \end{aligned} \tag{1}$$

where $Q = \{x \in \mathbb{R}^n \mid \delta_l \leq \|x\| \leq \delta_u\}$ is a closed convex subset of \mathbb{R}^n , δ_l and δ_u are strictly positive real numbers, $D := (D_1, \dots, D_K) \in \mathbb{R}^K$ is a deterministic vector, $V := [v_1, \dots, v_K]^T$ is a random matrix with size $K \times n$, where v_k is a random vector in \mathbb{R}^n , $\forall k = 1, 2, \dots, K$ and $p \in (0, 1)$. We denote $S(p)$ the feasibility set of (1). Let $J := \{1, 2, \dots, K\}$.

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Corresponding author : Abdel Lisser.

Hoang Nam NGUYEN · Abdel LISSER
Universite Paris-Saclay, CNRS, CentraleSupélec, Laboratory of signals and systems, 3 rue Joliot-Curie, 91190 Gif-sur-Yvette, France
E-mail: hoang-nam.nguyen3@centralesupelec.fr , abdel.lisser@centralesupelec.fr

The convexity of chance constraints as well as the analytical properties of the probability function play an important role in convex optimisation which are difficult issues and scarcely studied in the literature. This problem was first introduced by Prékopa [23]. He considers the following form of chance constraints

$$\mathbb{P}(h(x, \xi) \geq 0) \geq p, \quad (2)$$

where $x \in \mathbb{R}^n$ is a decision vector, $\xi : \Omega \rightarrow \mathbb{R}^m$ is an m - random variable vector defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, $h : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^s$ and $p \in [0, 1]$ is a given probability threshold.

Theorem 10.2.1 in [23] states that the feasibility set of (2) is convex if $\mathbb{P} \circ \xi^{-1}$ of ξ is a log-concave probability measure on \mathbb{R}^m and the components of h are quasi-concave. Henrion and Strugarek [8] studied a particular form of (2) by taking $h(x, \xi) = g(x) - \xi$ where $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Hence, they consider the following form

$$\mathbb{P}(\xi \leq g(x)) \geq p, \quad (3)$$

where they suppose that the components of ξ are independent. They prove that if the cumulative distribution functions of the components of ξ have r - decreasing densities and the components of g are r - concave, the feasibility set of (3) is convex. Henrion [10] generalized this result to the case where the components of ξ are dependent. He uses the theory of copulas to model the dependence of the components of ξ .

Marti [16] studied the differentiation of probability functions by an integral transformation method. The derivatives of the probability function can be obtained by applying an integral transformation to its integral representation. Some basic results on the differentiability of a probability function were studied by Kibzun et al [12]. They proposed new formulations of the gradient of probability functions in different forms, i.e., integral over the surface, volume, or sum of surface and volume integrals. Lobo [14] studied some applications of second-order cone program leading to a new approach for solving chance constraints. A more developed direction was initialized by Henrion [9] which gave a full description of the structure (not only the convexity) of a one-row linear optimization with a chance constraint by introducing a new notion of r -decreasing function. Henrion [8] studied the convexity in the case where the constraints are independent. To deal with the dependent case, Henrion and Strugarek [10], Cheng et al. [2] and Van Ackooij [28] used the theory of copulas to model the dependence of the constraints. They supposed that the distribution of the constraint row vectors are elliptically distributed. Under high probability threshold p , they prove the convexity of $S(p)$. Hong et al [11] proposed to solve joint chance-constrained programs by sequential convex approximations. They proved that the solutions of the sequence of approximations converge to a Karush-Kuhn-Tacker (KKT) point of the original problem. Farshbaf-Shaker et al [5] proved some properties of chance constraints in infinite dimensions. They supposed that the feasibility set belongs to a Banach space. Under mild conditions, they proved regularity properties of the probability function with an application to PDE constrained optimization. Wim van Ackooij [30] studied the convexity of the feasibility set in a general framework by using the radial representation of elliptical distributions.

In this paper, we deal with the convexity of $S(p)$. We assume that the row vectors v_i follow an elliptical distribution. We derive a reformulation of the joint chance constraints and assume that the dependence of the random constraint vector is captured by a Gumbel-Hougaard copula. Under mild conditions, we prove a new convexity result of the feasibility set $S(p)$.

This paper is organized as follows. In Section 2, we recall some basic concepts and preliminary results. In Section 2.1, we present a reformulation of the joint chance constraints. In Sections 2.2 and 3, we prove the concavity of the left-hand side and the convexity of the right-hand side of the reformulated constraint. In Section 4, we prove the convexity of the feasibility set $S(p)$ under mild conditions. We conclude the paper in Section 5.

2 Basic concepts and preliminary results

First, we recall some important definitions and propositions that we use in our paper.

Definition 1 An n -dimensional random vector X follows a spherical distribution if there exists a function $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ such that the characteristic function $\phi_X(t)$ of X is given by

$$\phi_X(t) = \mathbb{E}(e^{it^T X}) = \Psi(t^T t).$$

The function Ψ is called a characteristic generator of the spherical distribution.

Definition 2 An n -dimensional random vector U follows an elliptical distribution with location parameter μ , positive definite scale matrix Σ and characteristic generator Ψ (in short $U \sim \text{Ellip}(\mu, \Sigma, \Psi)$), if we have the following representation

$$U \hat{=} \mu + AX,$$

where X follows a spherical distribution with a characteristic generator Ψ , $A \in \mathbb{R}^{n \times n}$ such that $AA^T = \Sigma$ and $\mu \in \mathbb{R}^n$; $\hat{=}$ implies that the both sides have the same distribution.

The probability density function of all the distributions from elliptical family does not always exist. Whenever it exists, it is of the form

$$f_U(z) = \frac{c}{\sqrt{\det(\Sigma)}} g_{den} \left(\sqrt{(z - \mu)^T \Sigma^{-1} (z - \mu)} \right),$$

where g_{den} is a nonnegative function called *radial density* and $c > 0$ is a normalization factor which makes f_U a probability density function.

Definition 3 A function $f : Q \rightarrow (0, +\infty)$ is r -concave on a set $Q \subset \mathbb{R}^s$ for a given $r \in (-\infty, +\infty)$ if for any $x, y \in Q$ and $y \in [0, 1]$,

$$\begin{aligned} f(yx + (1 - y)y) &\geq [yf(x)^r + (1 - y)f(y)^r]^{\frac{1}{r}}, \text{ when } r \neq 0, \\ f(yx + (1 - y)y) &\geq f(x)^y f(y)^{1-y}, \text{ otherwise.} \end{aligned}$$

Definition 4 (Definition 2.2, [9]) A real function $f : \mathbb{R} \rightarrow \mathbb{R}$ is r -decreasing for some real number $r \in \mathbb{R}$, if f is continuous on $(0, +\infty)$ and there exists some strictly positive real number t^* such that the function $t \mapsto t^r f(t)$ is strictly decreasing on $(t^*, +\infty)$.

Table 1 presents some 1-dimensional spherical distributions with r -decreasing densities for some values of r and their thresholds t^* .

Definition 5 A function $C : [0, 1]^K \rightarrow [0, 1]$ is a K -dimensional copula if C is a joint CDF of a K -dimensional random vector, on the unit cube $[0, 1]^K$, whose marginals are uniformly distributed on $[0, 1]$.

Distribution	Radial density	r	t^*
Normal	$e^{-\frac{1}{2}u^2}$	$r > 0$	\sqrt{r}
t	$\left(1 + \frac{1}{\nu}u^2\right)^{-(1+\nu)/2}$, $\nu > 0, \nu$ integer	$0 < r < \nu + 1$	$\sqrt{\frac{r\nu}{\nu+1-r}}$
Laplace	$e^{- u }$	$r > 0$	$\frac{r}{\sqrt{2}}$
Kotz type	$u^{2(N-1)}e^{-qu^{2s}}$, $q, s > 0, N > \frac{1}{2}$	$r > 2(1-N)$	$\sqrt{\frac{2s\sqrt{2(N-1)+r}}{2qs}}$
Pearson type VII	$\left(1 + \frac{u^2}{m}\right)^{-N}$, $m > 0, N > \frac{1}{2}$	$0 < r < 2N$	$\sqrt{\frac{rm}{2N-r}}$

Table 1: Selected 1-dimensional spherical distributions with r -decreasing densities and their thresholds t^* .

Proposition 1 (Sklar's Theorem) Let $F : \mathbb{R}^K \rightarrow [0, 1]$ be a joint CDF of a K -dimensional random vector and F_1, \dots, F_K are the marginal CDFs. Then, there exists a K -dimensional copula C such that

$$F(z) = C(F_1(z_1), \dots, F_K(z_K)).$$

Moreover, if F_i is continuous for any $i = 1, \dots, K$, then C is uniquely given by

$$C(u) = F\left(F_1^{-1}(u_1), \dots, F_K^{-1}(u_K)\right).$$

Proposition 2 (Fréchet-Hoeffding upper bound) For any K -dimensional copula C and $u = (u_1, \dots, u_K) \in [0, 1]^K$, we have

$$C(u) \leq C_M(u) := \min_{k=1, \dots, K} u_k.$$

Definition 6 A K -dimensional copula C is strictly Archimedean if there exists a continuous and strictly decreasing function $\psi : (0, 1] \rightarrow [0, +\infty)$, such that $\psi(1) = 0$, $\lim_{t \rightarrow 0} \psi(t) = +\infty$, and for any K -dimensional vector $u = (u_1, \dots, u_K) \in [0, 1]^K$, we have

$$C(u) = \psi^{(-1)}\left(\sum_{i=1}^K \psi(u_i)\right).$$

The function ψ is called a generator of the copula C .

Table 2 presents a selection of some strictly Archimedean copulas with their generators.

Definition 7 A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is K -monotonic on an open interval $I \subseteq \mathbb{R}$ for some positive integer $K \geq 2$, if the following three conditions hold:

1. f is differentiable up to the order $(K - 2)$ on I ,
2. The derivatives of f are satisfied by

$$(-1)^k \frac{d^k}{dt^k} f(t) \geq 0, \quad 0 \leq k \leq K - 2,$$

for all $t \in I$,

Type of copula	Parameter θ	Generator $\psi_\theta(t)$
Independent	-	$-\log(t)$
Gumbel-Hougaard	$\theta \geq 1$	$[-\log(t)]^\theta$
Frank	$\theta > 0$	$-\log\left(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\right)$
Clayton	$\theta > 0$	$\frac{1}{\theta}(t^\theta - 1)$
Joe	$\theta \geq 1$	$-\log[1 - (1-t)^\theta]$

Table 2: Different types of strictly Archimedean copulas.

3. The function $t \mapsto (-1)^{K-2} \frac{d^{K-2}}{dt^{K-2}} f(t)$ is nonincreasing and convex on I .

Proposition 3 (Theorem 2.2, [18]) *Let $\psi : (0, 1] \rightarrow [0, +\infty)$ be a strictly decreasing function such that $\psi(1) = 0$ and $\lim_{t \rightarrow 0} \psi(t) = +\infty$. Then, ψ is the generator of a K -dimensional strictly Archimedean copula if and only if the inverse function $\psi^{(-1)}$ is K -monotonic on $(0, +\infty)$ and continuous on $[0, +\infty)$.*

2.1 Reformulation of the probability function

In problem (1), we assume that the random constraint vector $v_i \sim \text{Ellip}(\mu_i, \Sigma_i, \Psi_i)$, $\forall i \in J$. Let $\lambda_{i,\min}$ and $\lambda_{i,\max}$ be the smallest and the largest eigenvalues of the positive definite matrix Σ_i respectively.

Assume that $0 \notin Q$. Let

$$\begin{aligned}\xi_i(x) &:= \frac{v_i^T x - \mu_i^T x}{\sqrt{x^T \Sigma_i x}} \\ g_i(x) &:= \frac{D_i - \mu_i^T x}{\sqrt{x^T \Sigma_i x}}\end{aligned}\quad (4)$$

The constraint in (1) can be rewritten as follows

$$\begin{aligned}\mathbb{P}\{Vx \leq D\} &\geq p. \\ \Leftrightarrow \mathbb{P}\{v_i^T x \leq D_i, i \in J\} &\geq p. \\ \Leftrightarrow \mathbb{P}\{\xi_i(x) \leq g_i(x), i \in J\} &\geq p.\end{aligned}\quad (5)$$

It is easy to see that $\xi_i(x)$ follows 1-dimensional spherical distribution with characteristic generator Ψ_i . Our aim is to reformulate this function to study the convexity of $S(p)$. For this purpose, we use the Sklar's Theorem (cf. Proposition 1) to rewrite constraint (5) as follows

$$\mathbb{P}\{\xi_i(x) \leq g_i(x), i \in J\} \geq p \Leftrightarrow C_x[F_1(g_1(x)), \dots, F_K(g_K(x))] \geq p, \quad (6)$$

where C_x is the K -dimensional copula of the K -dimensional random vector $\xi(x) := (\xi_1(x), \dots, \xi_K(x))$ and F_i is the marginal cumulative distribution function of $\xi_i(x)$, $i = 1, \dots, K$.

Remark 1 The copula C_x depends on x while the marginal distribution of $\xi_i(x)$ does not depend on x .

We suppose that for all x , C_x is a strictly Archimedean copula with generator ψ_x . The constraint (6) can be written as

$$\begin{aligned} C_x[F_1(g_1(x)), \dots, F_K(g_K(x))] \geq p &\Leftrightarrow \psi_x^{(-1)}\left(\sum_{i=1}^K \psi_x(F_i(g_i(x)))\right) \geq p. \\ \Leftrightarrow \sum_{i=1}^K \psi_x(F_i(g_i(x))) &\leq \psi_x(p). \end{aligned} \quad (7)$$

We add auxiliary variables $\{y_i \geq 0, i \in J\}$ in order to reformulate constraint (7) into individual chance constraints [2,3]. In particular, as ψ_x is positive, constraint (7) is equivalent to the following constraint

$$\begin{cases} \psi_x(F_i(g_i(x))) \leq y_i \psi_x(p), & i \in J. \\ y_i \geq 0, & i \in J. \\ \sum_{i=1}^K y_i = 1. \end{cases} \quad (8)$$

This means that if $x^* \in S(p)$ then there exists $y^* := (y_1^*, \dots, y_K^*) \in \mathbb{R}^K$ such that (x^*, y^*) satisfies constraints (8). On the other hand, if (x^*, y^*) is a feasible solution for constraints (8) and $x^* \in Q$, we deduce that $x^* \in S(p)$. Moreover, for $x^* \in S(p)$, we can choose y^* in order to satisfy constraints (8) as

$$y_i^* = \frac{\psi_{x^*}(F_i(g_i(x^*)))}{\sum_{j=1}^K \psi_{x^*}(F_j(g_j(x^*)))}, \quad \forall i \in J. \quad (9)$$

By applying the decreasing monotonicity of the generator ψ_x , constraints (8) can be written as follows

$$\begin{cases} F_i(g_i(x)) \geq \psi_x^{(-1)}(y_i \psi_x(p)), & i \in J. \\ y_i \geq 0, & i = 1, \dots, K. \\ \sum_{i=1}^K y_i = 1. \end{cases} \quad (10)$$

In the rest of the paper, we consider the Gumbel-Hougaard family of copulas where the generator ψ_x is given by

$$\psi_x(t) = (-\log t)^{\frac{1}{\kappa(x)}}, \quad \forall (x, t) \in Q \times (0, 1], \quad (11)$$

where $\kappa(x) : Q \rightarrow (0, 1]$ is a strictly positive function.

Remark 2 Our aim is to show the concavity of $F_i(g_i)$ with respect to x and the joint convexity of $\psi_x^{(-1)}(y_i \psi_x(p))$ with respect to (y_i, x) .

2.2 Concavity of $F_i(g_i(\cdot))$

Define an index set $I^\mu \subset J$ such that $\mu_i \neq 0$ for any $i \in I^\mu$ and $\mu_i = 0$ otherwise. Define a set of real numbers $\{r_i \mid i \in J\}$ such that

$$\begin{cases} r_i > 1, & \text{if } i \in I^\mu, \\ r_i = 1, & \text{if } i \notin I^\mu. \end{cases}$$

Assumption 1

1. The cumulative distribution function F_i has $(r_i + 1)$ - decreasing densities with the thresholds t_i^* , $\forall i \in J$.
2. $p > p^*$, where

$$p^* := \max \left\{ \frac{1}{2}, \max_{j \in I^\mu} F_j \left(\frac{r_j + 1}{r_j - 1} \lambda_{j, \min}^{-\frac{1}{2}} \|\mu_j\| \right), \max_{j \in J} F_j [t_j^*(r_j + 1)] \right\}. \quad (12)$$

Lemma 1 *If Assumption 1 holds, then we have*

$$\text{Conv}(S(p)) \subset \bigcap_{j \in I^\mu} \Omega^j.$$

where

$$\Omega^j := \left\{ x \in Q \mid D_j - \mu_j^T x > \frac{r_j + 1}{r_j - 1} \lambda_{j, \min}^{-\frac{1}{2}} \|\mu_j\| \sqrt{x^T \Sigma_j x} \right\},$$

and Conv is the convex hull. Moreover, for any $i = 1, 2, \dots, K$, $g_i > 0$ and $(-r_i)$ -concave on any convex subset Q^i of $\bigcap_{j \in I^\mu} \Omega^j$.

The proof is given in Appendix A.

Lemma 2 *If Assumption 1 holds, then $F_i(g_i(\cdot))$ is concave on $S(p)$, $\forall i \in J$.*

Proof By Lemma 1, we deduce that g_i is $(-r_i)$ -concave and $g_i > 0$ on $\text{Conv}(S(p))$, for $i \in J$. Hence, for $a \in [0, 1]$ and $x_1, x_2 \in S(p)$, we have

$$g_i(ax_1 + (1-a)x_2) \geq [ag_i^{-r_i}(x_1) + (1-a)g_i^{-r_i}(x_2)]^{-\frac{1}{r_i}}. \quad (13)$$

As $x_1 \in S(p)$ and $p > p^*$, we have

$$C_{x_1} [F_1(g_1(x_1)), \dots, F_K(g_K(x_1))] > p^*.$$

By Proposition 2 and the definition of p^* , we deduce that

$$F_i(g_i(x_1)) > p^* \geq F_i[t_j^*(r_i + 1)], \quad i \in J.$$

Since, $F_i(\cdot)$ is monotonically increasing, we deduce that

$$g_i(x_1) > t_i^*(r_i + 1) > 0 \Rightarrow 0 < g_i(x_1)^{-r_i} < (t_i^*(r_i + 1))^{-r_i}, \quad i \in J.$$

Similarly, we obtain the inequality for x_2 . By taking F_i on both sides of (13), we have

$$F_i(g_i(ax_1 + (1-a)x_2)) \geq F_i([ag_i^{-r_i}(x_1) + (1-a)g_i^{-r_i}(x_2)]^{-\frac{1}{r_i}}). \quad (14)$$

Since, $F_i(\cdot)$ has $(r_i + 1)$ -decreasing density, from Lemma 3.1 of [9], the function $t \mapsto F_i\left(t^{-\frac{1}{r_i}}\right)$ is concave on $(0, (t_i^*)^{-r_i})$. Therefore, we can write

$$F_i([ag_i^{-r_i}(x_1) + (1-a)g_i^{-r_i}(x_2)]^{-\frac{1}{r_i}}) \geq aF_i(g_i(x_1)) + (1-a)F_i(g_i(x_2)). \quad (15)$$

From (14) and (15), we have

$$(F_i(g_i(ax_1 + (1-a)x_2))) \geq a((F_i(g_i(x_1)))) + (1-a)((F_i(g_i(x_2)))).$$

Therefore, $F_i(g_i(\cdot))$ is concave on $S(p)$. □

3 Convexity of $\psi_x^{(-1)}(y_i\psi_x(p))$

In this section, we use the following notations. Given arbitrarily positive real numbers $0 < c_l \leq 1$ and $0 < h_l \leq h_u < 1$. Let

$$\begin{aligned} \varphi_1^* &:= c_l \cdot \log h_u \cdot \log p \cdot h_l, \\ \varphi_2^* &:= \frac{(\log h_l)^2}{4c_l} + \max\left([1 + \log h_l(1 + \log p \cdot h_l)]^2, [1 + \log h_u \cdot c_l(1 + \log p)]^2\right), \\ \omega &:= \frac{\varphi_2^*}{\varphi_1^*}. \end{aligned} \quad (16)$$

Remark 3 Since $\varphi_1^* > 0$, then ω is well-defined. Moreover, ω does not depend on (x, y_i) .

In order to show the convexity of $U(x, y_i) = \psi_x^{(-1)}(y_i\psi_x(p))$, we first show that the Hessian matrix of U is positive semidefinite. The following lemma is a reformulation of the positive semidefiniteness of the Hessian matrix of U .

Lemma 3 *The positive semidefiniteness of the Hessian matrix of U on the convex set $Q \times [h_l, h_u]$ is equivalent to the positive semidefiniteness of the following n -dimensional symmetric matrix*

$$N(x, y_i) := \frac{\partial^2}{\partial y_i^2} U(x, y_i) \times H_x U(x, y_i) - \left(\nabla_x \frac{\partial}{\partial y_i} U(x, y_i) \right) \left(\nabla_x \frac{\partial}{\partial y_i} U(x, y_i) \right)^T, \quad (17)$$

for all (x, y_i) on $Q \times [h_l, h_u]$, where

$$H_x U = \begin{bmatrix} \frac{\partial^2 U}{\partial x_1^2} & \frac{\partial^2 U}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 U}{\partial x_1 \partial x_n} \\ \frac{\partial^2 U}{\partial x_2 \partial x_1} & \frac{\partial^2 U}{\partial x_2^2} & \cdots & \frac{\partial^2 U}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 U}{\partial x_n \partial x_1} & \frac{\partial^2 U}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 U}{\partial x_n^2} \end{bmatrix}, \quad \nabla_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)^T.$$

The proof is given in Appendix B.

Assumption 2 We assume the following statements:

1. $p \geq e^{-1}$
2. $0 < c_l \leq \kappa(x) \leq 1, \forall x \in Q$.
3. $H_\kappa(x) - \omega \nabla_x \kappa(x) (\nabla_x \kappa(x))^T$ is a positive semidefinite matrix for any $x \in Q$, where H_κ is the Hessian matrix of κ ; $\nabla_x \kappa$ is the gradient vector of κ .

Lemma 4 Suppose that Assumption 2 holds. Let $\varphi_1(x, y_i) := \kappa(x) \log y_i \cdot \left[\kappa(x) - 1 + \kappa(x) \log p \cdot y_i^{\kappa(x)} \right]$. Then, $\varphi_1(x, y_i) \geq \varphi_1^* > 0$, for any $(x, y_i) \in Q \times [h_l, h_u]$, where φ_1^* is defined in (16).

Proof Since $0 < y_i \leq h_u < 1$, then

$$(-\log y_i) \geq (-\log h_u) > 0. \quad (18)$$

As $0 < h_l \leq y_i < 1$ and $0 < \kappa(x) \leq 1$, we deduce that $1 \geq y_i^{\kappa(x)} \geq y_i \geq h_l > 0$. Note that $1 \geq -\log p > 0$ (because $e^{-1} \leq p < 1$). Then, $1 \geq -\log p \cdot y_i^{\kappa(x)} \geq -\log p \cdot h_l > 0$. We have:

$$\begin{aligned} 1 - \kappa(x) - \kappa(x) \log(p) \cdot y_i^{\kappa(x)} &= 1 - \kappa(x)(1 + \log p \cdot y_i^{\kappa(x)}) \\ &\geq 1 - 1 \cdot (1 + \log p \cdot y_i^{\kappa(x)}) \text{ (since } 0 < \kappa(x) \leq 1 \text{ and } 1 + \log p \cdot y_i^{\kappa(x)} \geq 0) \\ &= -\log p \cdot y_i^{\kappa(x)} \geq -\log p \cdot h_l > 0, \quad \forall (x, y_i) \in Q \times [h_l, h_u]. \end{aligned} \quad (19)$$

Moreover,

$$\kappa(x) \geq c_l > 0, \quad \forall x \in Q. \quad (20)$$

Therefore, it suffices to multiply term by term the inequalities (18)-(20) to complete the proof. \square

Lemma 5 Suppose that Assumption 2 is fulfilled.

$$\begin{aligned} \text{Let } \varphi_2(x, y_i) &:= \kappa(x) \log(y_i)^2 (1 + \log p \cdot y_i^{\kappa(x)}) \left[1 - \kappa(x) - \kappa(x) \log p \cdot y_i^{\kappa(x)} \right] + \\ &\quad \left(1 + \kappa(x) \cdot \log y_i + \log p \cdot \log y_i \cdot y_i^{\kappa(x)} \cdot \kappa(x) \right)^2. \end{aligned} \quad (21)$$

Then, $0 < \varphi_2(x, y_i) \leq \varphi_2^*$, for any $(x, y_i) \in Q \times [h_l, h_u]$, where φ_2^* is defined in (16).

Proof Since $p \geq e^{-1}$, then $0 > \log p \geq -1$. Using $0 < y_i^{\kappa(x)} < 1$, we have $0 < (1 + \log p \cdot y_i^{\kappa(x)}) < 1$. Moreover, as $0 < \kappa(x) \leq 1$, then $0 < \kappa(x) \cdot (1 + \log p \cdot y_i^{\kappa(x)}) < 1$. Let $s := \kappa(x) \cdot (1 + \log p \cdot y_i^{\kappa(x)})$. By applying the Cauchy-Schwarz inequality, we deduce that $s(1-s) \leq \frac{1}{4}$ (the equality holds if and only if $s = \frac{1}{2}$). Hence,

$$\begin{aligned} \kappa(x)(1 + \log p \cdot y_i^{\kappa(x)}) \left[1 - \kappa(x) \cdot (1 + \log p \cdot y_i^{\kappa(x)}) \right] &\leq \frac{1}{4}. \\ \Rightarrow 0 < (1 + \log p \cdot y_i^{\kappa(x)}) \left[1 - \kappa(x) \cdot (1 + \log p \cdot y_i^{\kappa(x)}) \right] &\leq \frac{1}{4 \cdot \kappa(x)} \leq \frac{1}{4c_l}. \end{aligned} \quad (22)$$

Since $1 > y_i \geq h_l > 0$, we have $0 > \log y_i \geq \log h_l$. Then,

$$0 < (\log y_i)^2 \leq (\log h_l)^2. \quad (23)$$

Moreover,

$$0 < \kappa(x) \leq 1 \quad (24)$$

Hence, by multiplying term by term the inequalities (22)- (24), we get an upper bound for the first term of (21):

$$0 < \kappa(x) \log(y_i)^2 (1 + \log p \cdot y_i^{\kappa(x)}) \left[1 - \kappa(x) - \kappa(x) \log p \cdot y_i^{\kappa(x)} \right] \leq \frac{(\log h_l)^2}{4c_l}, \quad \forall (x, y_i) \in Q \times [h_l, h_u]. \quad (25)$$

In order to get an upper bound for the second term of $\varphi_2(x, y_i)$, we need some following inequalities:

$$0 < c_l \leq \kappa(x) \leq 1. \quad (26)$$

$$0 < -\log h_u \leq -\log y_i \leq -\log h_l. \quad (27)$$

$$0 \leq 1 + \log p \leq 1 + \log p \cdot y_i^{\kappa(x)} \leq 1 + \log p \cdot h_l. \quad (28)$$

Note that (28) holds since $1 > p$ then $\log p < 0$ and $1 \geq y_i^{\kappa(x)} \geq h_l$. By multiplying the inequalities (26)-(28) term by term, we get

$$-\log h_l (1 + \log p \cdot h_l) \geq -\log y_i \cdot \kappa(x) (1 + \log p \cdot y_i^{\kappa(x)}) \geq -\log h_u \cdot c_l (1 + \log p)$$

Or equivalently,

$$1 + \log h_l (1 + \log p \cdot h_l) \leq 1 + \log y_i \cdot \kappa(x) (1 + \log p \cdot y_i^{\kappa(x)}) \leq 1 + \log h_u \cdot c_l (1 + \log p). \quad (29)$$

We deduce from (29) that

$$0 \leq [1 + \log y_i \cdot \kappa(x) (1 + \log p \cdot y_i^{\kappa(x)})]^2 \leq \max \left((1 + \log h_l (1 + \log p \cdot h_l))^2, (1 + \log h_u \cdot c_l (1 + \log p))^2 \right). \quad (30)$$

Adding (25) and (30) together, completes the proof. \square

Lemma 6 *If Assumption 2 holds, then U is jointly convex on $\mathcal{Q} \times [h_l, h_u]$.*

Proof Using Lemma 3, it suffices to show the positive semidefiniteness of $N(x, y_i)$ in (17) for any $(x, y_i) \in \mathcal{Q} \times [h_l, h_u]$. Since $\psi_x(t) = (-\log t)^{\frac{1}{\kappa(x)}}$, we have $\psi_x^{(-1)}(t) = e^{-t^{\kappa(x)}}$.

As $U(x, y_i) = \psi_x^{(-1)}(y_i \psi_x(p))$, we can rewrite $U(x, y_i)$ as follows:

$$U(x, y_i) = e^{-\left\{y_i(-\log p)^{\frac{1}{\kappa(x)}}\right\}^{\kappa(x)}} = p^{y_i^{\kappa(x)}}. \quad (31)$$

By (31), we calculate explicitly the partial derivatives of U as follows

$$\left\{ \begin{array}{l} \frac{\partial}{\partial y_i} U(x, y_i) = \log(p) p^{y_i^{\kappa(x)}} \kappa(x) y_i^{\kappa(x)-1} \\ \frac{\partial^2}{\partial y_i^2} U(x, y_i) = \kappa(x) \log(p) y_i^{\kappa(x)-2} p^{y_i^{\kappa(x)}} [\kappa(x) - 1 + \kappa(x) \log(p) y_i^{\kappa(x)}] \\ \nabla_x U(x, y_i) = \log(p) p^{y_i^{\kappa(x)}} \log(y_i) y_i^{\kappa(x)} \nabla_x \kappa(x) \\ \nabla_x \frac{\partial}{\partial y_i} U(x, y_i) = \log(p) y_i^{\kappa(x)-1} p^{y_i^{\kappa(x)}} [1 + \kappa(x) \log(y_i) + \log(p) \log(y_i) y_i^{\kappa(x)} \kappa(x)] \nabla_x \kappa(x) \\ H_x U(x, y_i) = p^{y_i^{\kappa(x)}} y_i^{\kappa(x)} (\log p) (\log y_i) [H_\kappa(x) + (\log y_i + \log p \log y_i \log p \cdot y_i^{\kappa(x)}) \nabla_x \kappa(x) (\nabla_x \kappa(x))^T]. \end{array} \right.$$

Hence, we obtain the following formulations:

$$\left\{ \begin{array}{l} \frac{\partial^2}{\partial y_i^2} U(x, y_i) \times H_x U(x, y_i) = \kappa(x) \log(p)^2 y_i^{2\kappa(x)-2} \log(y_i) p^{2 \cdot y_i^{\kappa(x)}} \times \\ \left[\kappa(x) - 1 + \kappa(x) \log p y_i^{\kappa(x)} \right] \times \left[H_\kappa(x) + \nabla_x \kappa(x) (\nabla_x \kappa(x))^T (\log y_i + \log p \cdot \log p \cdot y_i^{\kappa(x)}) \right] \\ \left(\nabla_x \frac{\partial}{\partial y_i} U(x, y_i) \right) \left(\nabla_x \frac{\partial}{\partial y_i} U(x, y_i) \right)^T = \log(p)^2 y_i^{2\kappa(x)-2} \cdot p^{2 \cdot y_i^{\kappa(x)}} \times \\ \left(1 + \kappa(x) \log y_i + \log p \log y_i \cdot y_i^{\kappa(x)} \kappa(x) \right)^2 \nabla_x \kappa(x) (\nabla_x \kappa(x))^T. \end{array} \right. \quad (32)$$

Note that $\log(p)^2 \cdot y_i^{2\kappa(x)-2} \cdot p^{2 \cdot y_i^{\kappa(x)}}$ is a positive common factor of $\frac{\partial^2}{\partial y_i^2} U(x, y_i) \times H_x U(x, y_i)$ and $\left(\nabla_x \frac{\partial}{\partial y_i} U(x, y_i) \right) \left(\nabla_x \frac{\partial}{\partial y_i} U(x, y_i) \right)^T$. Then, using (17) and (32), the positive semidefiniteness of $N(x, y_i)$ is equivalent to the positive semidefiniteness of the following matrix:

$$M(x, y_i) = \left[\kappa(x) - 1 + \kappa(x) \log p \cdot y_i^{\kappa(x)} \right] \times \left[H_\kappa(x) + \nabla_x \kappa(x) (\nabla_x \kappa(x))^T (\log y_i + \log p \cdot \log p \cdot y_i^{\kappa(x)}) \right] \\ \times \kappa(x) \log y_i - \left(1 + \kappa(x) \log y_i + \log p \log y_i \cdot y_i^{\kappa(x)} \kappa(x) \right)^2 \nabla_x \kappa(x) (\nabla_x \kappa(x))^T. \quad (33)$$

We reformulate (33) as follows:

$$M(x, y_i) = \varphi_1(x, y_i) H_\kappa(x) - \varphi_2(x, y_i) \nabla_x \kappa(x) (\nabla_x \kappa(x))^T$$

where $\varphi_1(x, y_i)$ and $\varphi_2(x, y_i)$ are defined in Lemmas 4 and 5.

We have $H_\kappa(x) - \omega \nabla_x \kappa(x) (\nabla_x \kappa(x))^T$ is a positive semidefinite matrix for all $x \in \mathcal{Q}$ where ω is defined in (16). Since $\varphi_1^*, \varphi_2^* > 0$, we have $\omega > 0$. Moreover, $\nabla_x \kappa(x) (\nabla_x \kappa(x))^T$ is a positive semidefinite matrix. Hence, $H_\kappa(x)$ is also a positive semidefinite matrix. Therefore, it suffices to prove that $\varphi_1(x, y_i) \geq \varphi_1^* > 0$ and $\varphi_2(x, y_i) \leq \varphi_2^*$, for all $(x, y_i) \in \mathcal{Q} \times [h_l, h_u]$. Using Lemmas 4 and 5 completes the proof. \square

4 Convexity of the feasibility set $S(p)$

In section 2.2, we have shown that if Assumption 1 holds, then Lemma 2 holds. In section 3, we have shown that given arbitrarily numbers c_l and h_l, h_u which are defined in (16), if Assumption 2 holds, then Lemma 6 holds. We will apply the results of Lemma 2 and Lemma 6 to prove the convexity of the feasibility set $S(p)$.

Define an index set $I^D \subset J$ such that $D_j > 0$ for any $j \in I^D$ and $D_j \leq 0$ otherwise. In this section, let c_l be a number chosen arbitrarily in $(0, 1]$ as defined in (16). However, h_l, h_u are not chosen arbitrarily, but defined in the following lemma.

Lemma 7 For all $j \in J$, let:

$$G_j := \begin{cases} \left(\frac{\log \left(F_j \left(\frac{D_j}{\sqrt{\lambda_{j,\min}} \cdot \delta_l} + \frac{\|\mu_j\|}{\sqrt{\lambda_{j,\min}}} \right) \right)}{\log p} \right)^{\frac{1}{c_l}}, & \text{if } j \in I^D. \\ \left(\frac{\log \left(F_j \left(\frac{D_j}{\sqrt{\lambda_{j,\max}} \cdot \delta_u} + \frac{\|\mu_j\|}{\sqrt{\lambda_{j,\min}}} \right) \right)}{\log p} \right)^{\frac{1}{c_l}}, & \text{if } j \in J \setminus I^D. \end{cases}$$

Let $h_l := \min_{1 \leq j \leq K} (G_j)$ et $h_u := 1 - (n-1) \cdot h_l$. Assume that $S(p) \neq \emptyset$. Given $x \in S(p)$ and y_i defined in (9). Hence, $0 < h_l \leq y_i \leq h_u < 1, \forall i \in J$.

The proof is given in Appendix C.

Remark 4 Notice that the inequality $0 < h_l \leq h_u < 1$ defined in Lemma 7 is a necessary condition for the nonemptiness property of $S(p)$. It is natural to assume that $S(p) \neq \emptyset$ since the case $S(p) = \emptyset$ is trivial.

Based on Lemma 7, we prove the convexity of $S(p)$.

Theorem 1 Let c_l be defined in (16) and h_l, h_u defined in Lemma 7. Suppose that Assumptions 1 and 2 hold and $S(p) \neq \emptyset$. Then, $S(p)$ is a convex set.

Proof Let $x_1, x_2 \in S(p)$ and $\beta \in [0, 1]$. We show that $x^* := \beta x_1 + (1 - \beta)x_2 \in S(p)$.

In fact, let $y^1 := (y_1^1, \dots, y_K^1)$ and $y^2 := (y_1^2, \dots, y_K^2)$ given by (9). Using Lemma 7, $0 < h_l \leq y_i^1, y_i^2 \leq h_u < 1$, for all $i \in J$. Moreover, we deduce from Lemma 2 and Lemma 6 that $F_i(g_i(\cdot))$ is concave on $S(p)$ and $U(\cdot, \cdot)$ is jointly convex on $Q \times [h_l, h_u]$, i.e.,

$$\begin{aligned} F_i(g_i(x^*)) &\geq \beta F_i(g_i(x_1)) + (1 - \beta) F_i(g_i(x_2)), \quad \forall i \in J \\ U(x^*, \beta y_i^1 + (1 - \beta) y_i^2) &\leq \beta U(x_1, y_i^1) + (1 - \beta) U(x_2, y_i^2), \quad \forall i \in J. \end{aligned}$$

Therefore, $(x^*, \beta y^1 + (1 - \beta) y^2)$ satisfies (10). In other word, $x^* \in S(p)$. \square

We are interested in finding an example which fits all the Assumptions in Theorem 1. Finding a necessary and sufficient condition for the nonemptiness of $S(p)$ is a hard problem. In the following, we will study some cases which fit Assumption 2 and Assumption 1.

4.1 An example of the function κ

In this section, we give an example of κ which satisfies all the conditions in Assumption 2. Let c_l be defined in (16) and h_l, h_u defined in Lemma 7. Assume that $S(p) \neq \emptyset$.

Remark 5 It is not necessary to verify the condition $p \geq e^{-1} \approx 0.37$ since we study an eventual convexity of the feasibility set, i.e., under high value of the probability threshold p .

Lemma 8 *Let d be a real number such that $d < c_l$ and $q : Q \rightarrow \mathbb{R}$ be a real-valued function which satisfies the two following conditions.*

1. q is a twice continuously differentiable and convex function on Q .
2. $\log(c_l - d) \leq q(x) \leq \log\left[\min\left(\frac{1}{\omega}, 1 - d\right)\right]$, $\forall x \in Q$, where ω is defined in (16).

Hence, $\kappa(x) := e^{q(x)} + d$ is a function which satisfies Assumption 2.

Proof First we check if the condition $c_l \leq \kappa(x) \leq 1, \forall x \in Q$ is true.

In fact, since $\log(c_l - d) \leq q(x) \leq \log(1 - d)$, then we have $c_l \leq \kappa(x) \leq 1, \forall x \in Q$. Then, we prove the third condition of Assumption 2.

Since $q(x) \leq \log\left(\frac{1}{\omega}\right)$, then $e^{q(x)} \leq \frac{1}{\omega}, \forall x \in Q$. Let $\kappa^*(x) := e^{q(x)}$. We have the following formulation:

$$H_x \log(\kappa^*(x)) = \frac{\kappa^*(x)H_x \kappa^*(x) - \nabla_x \kappa^*(x)(\nabla_x \kappa^*(x))^T}{\kappa^*(x)^2}, \quad (34)$$

where $H_x \log(\kappa^*(\cdot))$ is the Hessian matrix of the function $\log(\kappa^*(\cdot))$.

As q is a convex function on Q and $\log(\kappa^*(\cdot)) = q(\cdot)$, we deduce that $H_x(\log(\kappa^*(x)))$ is a positive semidefinite matrix for any $x \in Q$. Using (34), $\kappa^*(x)H_x \kappa^*(x) - \nabla_x \kappa^*(x)(\nabla_x \kappa^*(x))^T$ is a positive semidefinite matrix. Since $0 < \kappa^*(x)$ and $\nabla_x \kappa^*(x)(\nabla_x \kappa^*(x))^T$ is positive semidefinite, we deduce that $H_x \kappa^*(x)$ is also positive semidefinite, for all $x \in Q$. Moreover, $\kappa^*(x) \leq \frac{1}{\omega}$. Hence,

$$\frac{1}{\omega} H_x \kappa^*(x) - \nabla_x \kappa^*(x)(\nabla_x \kappa^*(x))^T$$

is a positive semidefinite matrix.

On the other hand, since $\kappa(x) = \kappa^*(x) + d$, then $H_x \kappa^*(x) = H_x \kappa(x)$ and $\nabla_x \kappa^*(x) = \nabla_x \kappa(x), \forall x \in Q$. Therefore, the third condition of Assumption 2 holds. \square

Notice that we are interested to find a function q which meets the two conditions in Lemma 8.

Lemma 9 *Let $q(x) := \frac{\|x\|^2}{L} + z$ where $\|\cdot\|$ denotes the Euclidean norm, $L > 0, z \in \mathbb{R}$ are real numbers such that:*

$$\log(c_l - d) \leq \frac{1}{L} \delta_l^2 + z \leq \frac{1}{L} \delta_u^2 + z \leq \log\left[\min\left(\frac{1}{\omega}, 1 - d\right)\right], \quad (35)$$

Hence, $q(x)$ satisfies the two conditions in Lemma 8.

Proof The first condition is trivial. Since $\frac{\delta_l^2}{L} + z = \min_{x \in Q} q(x)$ and $\frac{\delta_u^2}{L} + z = \max_{x \in Q} q(x)$, we deduce that the second condition holds. \square

Using Lemma 9, the function $\kappa(x) = e^{\frac{\|x\|^2}{L} + z} + d$ which satisfies (35) fits all conditions in Assumption 2. It suffices to choose appropriate parameters L, z and d . In fact, (35) is a mild condition. Let d be an arbitrary real number in $(c_l - \frac{1}{\omega}, c_l)$. We deduce from this condition that $\log(c_l - d)$ is well defined and

$$\log(c_l - d) \leq \log \left[\min \left(\frac{1}{\omega}, 1 - d \right) \right].$$

Let L be an arbitrary real number in $\left[\frac{\delta_u^2 - \delta_l^2}{\log \left[\min \left(\frac{1}{\omega}, 1 - d \right) \right] - \log(c_l - d)}, +\infty \right)$ which depends on d . We deduce from this condition that

$$\log \left[\min \left(\frac{1}{\omega}, 1 - d \right) \right] - \log(c_l - d) \geq \left(\frac{1}{L} \delta_u^2 + z \right) - \left(\frac{1}{L} \delta_l^2 + z \right) > 0.$$

Let z be an arbitrary real number in $\left[\log(c_l - d) - \frac{\delta_l^2}{L}, \log \left[\min \left(\frac{1}{\omega}, 1 - d \right) \right] - \frac{\delta_u^2}{L} \right]$ which depends on d and L . We deduce from this condition that:

$$\begin{cases} \log(c_l - d) \leq \frac{1}{L} \delta_l^2 + z. \\ \frac{1}{L} \delta_u^2 + z \leq \log \left[\min \left(\frac{1}{\omega}, 1 - d \right) \right]. \end{cases}$$

Therefore, we can easily verify that (35) holds.

4.2 Numerical experiments

To test the r -decreasing property for a differentiable density $f(t)$, we verify if the derivative of $t^r f(t)$ is strictly negative for $t > t^*(r) > 0$. This is equivalent to check the inequality $r \cdot f(t) + t \cdot f'(t) < 0$.

The results for different 1-dimensional spherical distributions are given in Table 1. In this section, we study different values of p^* for some popular distributions. All the numerical results are performed using Python 3.8.8 on an Intel Core i5-1135G7, Processor 2.4 GHz (8M Cache, up to 4.2 GHz), RAM 16G, 512G SSD. In Assumption 1, we suppose that $\forall j \in J, F_j$ has the same density. For the term $F_j \left(\frac{r_j+1}{r_j-1} \lambda_{j,min}^{-\frac{1}{2}} \|\mu_j\| \right)$ in (12), we choose the

parameters $\lambda_{j,min}$ and μ_j such that $\frac{r_j+1}{r_j-1} \lambda_{j,min}^{-\frac{1}{2}} \|\mu_j\|$ is small. For the sake of illustration, we set $p^* = F_j[t_j^*(r_j + 1)]$. As $r_j \geq 1$, then $p^* = F_j[t_j^*(2)]$ is the best value we can obtain. Table 3 gives numerical values of p^* for some commonly used distributions.

We consider the case where $n = 2$ and $K = 2$, i.e., 2 constraints in 2-dimensional case. We choose the parameters as follows: $p = 0.95, c_l = 0.9, D_1 = 1, D_2 = 0.85, \mu_1 = (1, 20)^T, \mu_2 = (7, 2)^T, \delta_l = 0.2, \delta_u = 1.5, \Sigma_1 = 3 \times \mathbb{I}_2$, and $\Sigma_2 = 30 \times \mathbb{I}_2$, where \mathbb{I}_2 is the 2×2 -identity matrix. h_l and h_u are calculated using Lemma 7. φ_1^*, φ_2^* and ω are calculated using (16). Let $d = c_l - \frac{1}{2\omega}, L = \frac{\delta_u^2 - \delta_l^2}{\log \left(\min \left(\frac{1}{\omega}, 1 - d \right) \right) - \log(c_l - d)}$ and $z = \log \left(\min \left(\frac{1}{\omega}, 1 - d \right) \right) - \frac{\delta_u^2}{L}$. It is easy to see that this choice of parameters fits the condition in Lemma 9.

Distribution	p^*
Normal	0.92
t with 2–degrees of freedom	0.84
t with 3–degrees of freedom	0.87
t with 4–degrees of freedom	0.88
Laplace	0.88

Table 3: Selected 1-dimensional spherical distributions with r -decreasing densities and the value of the threshold p^* respectively.

In our simulations, we consider the probability function $f_{proba}(x) = \psi_x^{(-1)} \left(\sum_{i=1}^2 \psi_x(F_i(g_i(x))) \right)$ as in (7), where ψ_x is defined in (11), F_1 is the cumulative distribution function of an 1–dimensional standard t –distribution with 3–degrees of freedom and F_2 is the cumulative distribution function of an 1–dimensional standard t –distribution with 4–degrees of freedom, $x = (x_1, x_2)^T$ where $x_1, x_2 \in [-1, 1]$. Figures 1 and 2 show the function $z = f_{proba}(x_1, x_2)$ on the domain $[-1, 1]^2$, and its contour lines with four different levels 0.6, 0.7, 0.8 and 0.9, respectively.

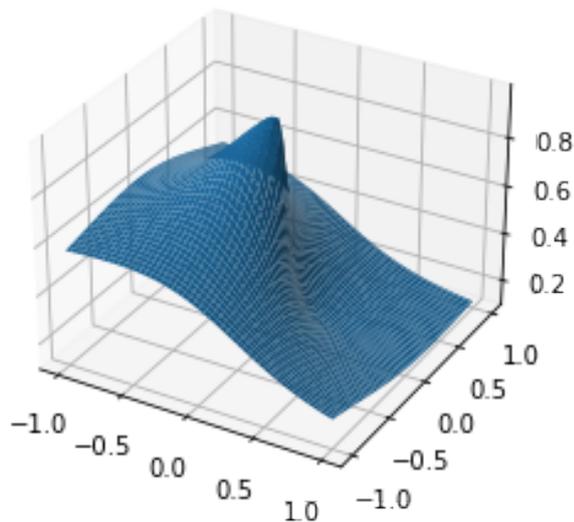


Figure 1 : Graphical representation of the probability function f_{proba} on $[-1, 1]^2$.

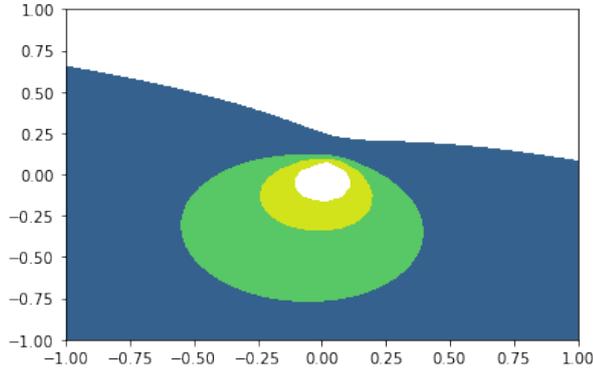


Figure 2 : Contour line of the probability function f_{proba} with 4 levels (0.6, 0.7, 0.8, 0.9).

5 Conclusion

In this paper, we studied the convexity of joint chance constraints in the case of elliptical distributions. Further, we modeled the dependence of random variables by a Gumbel-Hougaard copula. We come up with new convexity results of the feasibility set. Further research will be dedicated to other families of copulas.

Appendices

Appendix A: Proof of Lemma 1

Let $i \in I$ and $x_0 \in S(p)$. In Section 2.1, we show that x_0 satisfies constraint (6). In particular, we have

$$C_{x_0}[F_1(g_1(x_0)), \dots, F_K(g_K(x_0))] \geq p. \quad (36)$$

where g_i is defined in (4), for all $i \in J$.
By applying Proposition 2, we deduce that

$$\begin{aligned} F_i(g_i(x_0)) &\geq p. \\ \Rightarrow F_i(g_i(x_0)) &> p^* \geq F_i\left(\frac{r_i+1}{r_i-1} \lambda_{i,min}^{-\frac{1}{2}} \|\mu_i\|\right). \end{aligned} \quad (37)$$

Since, $F_i(\cdot)$ is an increasing function, we have

$$g_i(x_0) > \frac{r_i+1}{r_i-1} \lambda_{i,min}^{-\frac{1}{2}} \|\mu_i\|.$$

Therefore,

$$D_i - \mu_i^T x_0 > \frac{r_i+1}{r_i-1} \lambda_{i,min}^{-\frac{1}{2}} \|\mu_i\| \sqrt{x_0^T \Sigma_i x_0}.$$

We deduce that $S(p) \subset \bigcap_{j \in I} \Omega^j$. For each $j \in I$, Ω^j is a convex set which implies that $\text{Conv}(S(p)) \subset \bigcap_{j \in I} \Omega^j$. We prove the second part of Lemma 1 by considering the following two cases:

Case 1: Let $i \notin I$, then $\mu_i = 0$. From the definition of p^* , we have $p > \frac{1}{2}$. We prove that $D_i > 0$. In fact, let $x_0 \in S(p)$. By applying Proposition 2 on (36), we have

$$(F_i(g_i(x_0))) \geq p > \frac{1}{2}. \quad (38)$$

Since, F_i is the CDF of an 1-dimensional real-valued random variable which is symmetric at 0, $F_i(0) = \frac{1}{2}$. From (38) we get $g_i(x_0) > 0$ which in turn implies that $D_i - (\mu_i)^T x_0 > 0$. Since $\mu_i = 0$, we get $D_i > 0$. In this case, the proof follows directly from Lemma 3 of [2].

Case 2: Let $i \in I$. It follows from Lemma 2 of [2] that the function

$$f_i(x) = \left(\frac{\sqrt{(x)^T \Sigma_i x}}{D_i - (\mu_i)^T x} \right)^{r_i}.$$

is defined and a convex function on $\bigcap_{j \in I} \Omega^j$. Therefore, for any $y, z \in Q^i$ and $\lambda \in [0, 1]$, we have

$$f_i[\lambda y + (1 - \lambda)z] \leq \lambda f_i(y) + (1 - \lambda)f_i(z). \quad (39)$$

Note that $g_i(x) = (f_i(x))^{\frac{1}{r_i}}$ on Q^i . From (39), we can write

$$g_i[\lambda y + (1 - \lambda)z] \geq (\lambda(g_i(y))^{-r_i} + (1 - \lambda)(g_i(z))^{-r_i})^{\frac{1}{r_i}}.$$

Hence, g_i is defined and $(-r_i)$ -concave on Q^i .

Appendix B: Proof of Lemma 3

The Hessian matrix of U at a point (x, y_i) is an $(n + 1)$ -dimensional symmetric matrix which has the form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A = H_x U(x, y_i)$, $B = \nabla_x \frac{\partial}{\partial y_i} U(x, y_i)$, $C = B^T$, $D = \frac{\partial^2}{\partial y_i^2} U(x, y_i)$.

The proof is an application of the Schur's complement. Hence, it suffices to show that $\frac{\partial^2}{\partial y_i^2} U(x, y_i) > 0$, $\forall (x, y_i) \in Q \times [h_l, h_u]$.

In fact, for $U(x, y_i) = \psi_x^{(-1)}(y_i \psi_x(p))$, we have

$$\frac{\partial^2}{\partial y_i^2} U(x, y_i) = [\psi_x(p)]^2 (\psi_x^{(-1)})''(y_i \psi_x(p)).$$

Since $\psi_x(t) = (-\log t)^{\frac{1}{\kappa(x)}}$, we deduce that $\psi_x^{(-1)}(t) = e^{-t^{\kappa(x)}}$ and $(\psi_x^{(-1)})''(t) = e^{-t^{\kappa(x)}} t^{\kappa(x)-2} \kappa(x) [\kappa(x) t^{\kappa(x)} - \kappa(x) + 1]$.

Using the above formulations, we have $\psi_x(p) > 0$ and $(\psi_x^{(-1)})''(t) > 0$, $\forall t > 0$.

Hence, $\frac{\partial^2}{\partial y_i^2} U(x, y_i) > 0$, $\forall (x, y_i) \in Q \times [h_l, h_u]$.

Appendix C: Proof of Lemma 7

Let $x \in S(p)$ and

$$y_i = \frac{\psi_x[F_i(g_i(x))]}{\sum_{j=1}^K \psi_x[F_j(g_j(x))]}, \quad \forall i \in J. \quad (40)$$

where $\psi_x(\cdot)$ is defined in (11).

It is easy to see that $\sum_{i \in J} y_i = 1$ and $y_i \geq 0, \forall i \in J$. We prove that $y_i \geq G_i, \forall i \in J$. Consider two cases as follows.

Case 1: $i \in I^D$.

We use the Cauchy-Schwarz inequality to deduce that

$$|-\mu_i^T x| \leq \|\mu_i\| \cdot \|x\|, \quad \forall i \in J. \quad (41)$$

From linear algebra, we have

$$\sqrt{\lambda_{i,max}} \|x\| \geq \sqrt{x^T \Sigma_i x} \geq \sqrt{\lambda_{i,min}} \|x\|. \quad (42)$$

Since $x \in S(p)$, using (7), we get:

$$0 < \sum_{j=1}^K \psi_x [F_j(g_j(x))] \leq \psi_x(p). \quad (43)$$

We apply (41) - (42) and $\|x\| \geq \delta_i$ to deduce the following inequality

$$\begin{aligned} g_i(x) &\leq \frac{D_i}{\sqrt{x^T \Sigma_i x}} + \frac{|-\mu_i^T x|}{\sqrt{x^T \Sigma_i x}} \\ &\leq \frac{D_i}{\sqrt{\lambda_{i,min}} \delta_i} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,min}}}. \end{aligned}$$

As F_i is increasing monotonic, we have

$$F_i(g_i(x)) \leq F_i\left(\frac{D_i}{\sqrt{\lambda_{i,min}} \delta_i} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,min}}}\right).$$

We use the decreasing monotonicity of ψ_x to deduce

$$\psi_x(F_i(g_i(x))) \geq \psi_x\left(F_i\left(\frac{D_i}{\sqrt{\lambda_{i,min}} \delta_i} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,min}}}\right)\right).$$

Combining with (43) and (40), we have

$$y_i \geq \frac{\psi_x\left(F_i\left(\frac{D_i}{\sqrt{\lambda_{i,min}} \delta_i} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,min}}}\right)\right)}{\psi_x(p)} = \left(\frac{\log\left(F_i\left(\frac{D_i}{\sqrt{\lambda_{i,min}} \delta_i} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,min}}}\right)\right)}{\log p}\right)^{\frac{1}{\kappa(x)}} \quad (44)$$

As $0 \leq y_i \leq 1$, we deduce that:

$$0 < \frac{\log\left(F_i\left(\frac{D_i}{\sqrt{\lambda_{i,min}} \delta_i} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,min}}}\right)\right)}{\log p} \leq 1$$

Therefore,

$$\left(\frac{\log\left(F_i\left(\frac{D_i}{\sqrt{\lambda_{i,min}} \delta_i} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,min}}}\right)\right)}{\log p}\right)^{\frac{1}{\kappa(x)}} \geq \left(\frac{\log\left(F_i\left(\frac{D_i}{\sqrt{\lambda_{i,min}} \delta_i} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,min}}}\right)\right)}{\log p}\right)^{\frac{1}{c_i}} = G_i. \quad (45)$$

Using (44) and (45), we have $y_i \geq G_i$.

Case 2: $i \in J \setminus J^D$.

In this case, $D_i \leq 0$. Then, using (41) and (42), we have:

$$g_i(x) \leq \frac{D_i}{\sqrt{\lambda_{i,max}} \delta_u} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,min}}}.$$

Following the same proof as Case 1, we have $y_i \geq G_i$. Therefore, using Case 1 and Case 2, $y_i \geq h_l > 0$, $\forall i \in J$. Since, $\sum_{i \in J} y_i = 1$, $\forall i \in J$, we have

$$y_i = 1 - \sum_{j \in J, j \neq i} y_j \leq 1 - (n-1) \cdot h_l = h_u < 1. \quad (46)$$

Hence, $0 < h_l \leq y_i \leq h_u < 1$, $\forall i \in J$.

References

1. BORELL, C. Convex set functions in d-space. *Periodica Mathematica Hungarica* 6, 2 (1975), 111–136.
2. CHENG, J., HOUDA, M., AND LISSER, A. Second-order cone programming approach for elliptically distributed joint probabilistic constraints with dependent rows. *Technical report, Optimization* (2014).
3. CHENG, J., AND LISSER, A. A second-order cone programming approach for linear programs with joint probabilistic constraints. *Operations Research Letters* 40, 5 (2012), 325–328.
4. FANG, K. W. *Symmetric multivariate and related distributions*. CRC Press, 2018.
5. FARSHBAF-SHAKER, M. H., HENRION, R., AND HÖMBERG, D. Properties of chance constraints in infinite dimensions with an application to pde constrained optimization. *Set-Valued and Variational Analysis* 26, 4 (2018), 821–841.
6. HENRION, R. Structural properties of linear probabilistic constraints. *Optimization* 56, 4 (2007), 425–440.
7. HENRION, R., AND MÖLLER, A. A gradient formula for linear chance constraints under gaussian distribution. *Mathematics of Operations Research* 37, 3 (2012), 475–488.
8. HENRION, R., AND STRUGAREK, C. Convexity of chance constraints with independent random variables. *Computational Optimization and Applications* 41, 2 (2008), 263–276.
9. HENRION, R., AND STRUGAREK, C. Convexity of chance constraints with independent random variables. *Computational Optimization and Applications* 41, 2 (2008), 263–276.
10. HENRION, R., AND STRUGAREK, C. Convexity of chance constraints with dependent random variables: the use of copulae. In *Stochastic optimization methods in finance and energy*. Springer, 2011, pp. 427–439.
11. HONG, L. J., YANG, Y., AND ZHANG, L. Sequential convex approximations to joint chance constrained programs: A monte carlo approach. *Operations Research* 59, 3 (2011), 617–630.
12. KIBZUN, A., AND URYASEV, S. Differentiability of probability function. *Stochastic Analysis and Applications* 16, 6 (1998), 1101–1128.
13. LAGUEL, Y., VAN ACKOOIJ, W., MALICK, J., AND RAMALHO, G. On the convexity of level-sets of probability functions. *arXiv preprint arXiv:2102.04052* (2021).
14. LOBO, M. S., VANDENBERGHE, L., BOYD, S., AND LEBRET, H. Applications of second-order cone programming. *Linear algebra and its applications* 284, 1-3 (1998), 193–228.
15. LOBO, M. S., VANDENBERGHE, L., BOYD, S., AND LEBRET, H. Applications of second-order cone programming. *Linear algebra and its applications* 284, 1-3 (1998), 193–228.
16. MARTI, K. Differentiation of probability functions: The transformation method. *Computers & Mathematics with Applications* 30, 3-6 (1995), 361–382.
17. McNEIL, A. J., FREY, R., AND EMBRECHTS, P. *Quantitative risk management: concepts, techniques and tools-revised edition*. Princeton university press, 2015.
18. McNEIL, A. J., AND NEŠLEHOVÁ, J. Multivariate archimedean copulas, d-monotone functions and ℓ_1 -norm symmetric distributions. *The Annals of Statistics* 37, 5B (2009), 3059–3097.
19. MINOUX, M., AND ZORGATI, R. Convexity of gaussian chance constraints and of related probability maximization problems. *Computational Statistics* 31, 1 (2016), 387–408.
20. NELSEN, R. B. *An introduction to copulas*. Springer Science & Business Media, 2007.
21. PREKOPA, A. On probabilistic constrained programming. In *Proceedings of the Princeton symposium on mathematical programming* 113 (1970), 138.
22. PRÉKOPA, A. Logarithmic concave measures with application to stochastic programming. *Acta Scientiarum Mathematicarum* 32 (1971), 301–316.

23. PRÉKOPA, A. *Stochastic programming*, vol. 324. Springer Science & Business Media, 2013.
24. PRÉKOPA, A. On logarithmic concave measures and functions. *Acta Scientiarum Mathematicarum* 34 (1973), 335–343.
25. PRÉKOPA, A., YODA, K., AND SUBASI, M. M. Uniform quasi-concavity in probabilistic constrained stochastic programming. *Operations Research Letters* 39, 3 (2011), 188–192.
26. RÜSCHENDORF, L. Copulas, sklar’s theorem, and distributional transform. In *Mathematical risk analysis*. Springer, 2013, pp. 3–34.
27. SEN, S. Relaxations for probabilistically constrained programs with discrete random variables. *Operations Research Letters* 11, 2 (1992), 81–86.
28. VAN ACKOOIJ, W. Eventual convexity of chance constrained feasible sets. *Optimization* 64, 5 (2015), 1263–1284.
29. VAN ACKOOIJ, W., AND DE OLIVEIRA, W. Convexity and optimization with copulae structured probabilistic constraints. *Optimization* 65, 7 (2016), 1349–1376.
30. VAN ACKOOIJ, W., AND MALICK, J. Eventual convexity of probability constraints with elliptical distributions. *Mathematical Programming* 175, 1 (2019), 1–27.
31. ZADEH, Z. M., AND KHORRAM, E. Convexity of chance constrained programming problems with respect to a new generalized concavity notion. *Annals of Operations Research* 196, 1 (2012), 651–662.