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Convexity of elliptically distributed dependent chance constraints

Hoang Nam NGUYEN · Abdel LISSER

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Abstract In this paper, we study the problem of linear optimization with probabilistic constraints. We suppose that the constraint row vectors are elliptically distributed. Further, the dependence of the rows is modeled by a family of Archimedean copulas, namely, the Gumbel-Hougaard family of copulas. Under mild assumptions, we prove the eventual convexity of the feasibility set.

Keywords Chance constraints · Archimedean copulas · Elliptical distributions · Convex optimization

Mathematics Subject Classification (2000) MSC 90C15 · 90C25 · 90C59

1 Introduction

1.1 Chance constrained optimization

We consider the following linear optimization with joint chance constraints

$$\begin{aligned} \min \quad & c^T x \\ \text{subject to} \quad & \mathbb{P}\{Vx \leq D\} \geq p \\ & x \in Q. \end{aligned} \tag{1}$$

where Q is a closed convex subset of \mathbb{R}^n ; $c \in \mathbb{R}^n$, $D := (D_1, \dots, D_K) \in \mathbb{R}^K$ is a deterministic vector, $V := [v_1, \dots, v_K]^T$ is a random $k \times n$ -matrix where v_k is a random vector in \mathbb{R}^n , $\forall k = 1, 2, \dots, K$ and $p \in [0, 1]$. We denote $Feasi(Q)$ the feasibility set of (1).

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Chance-constrained optimization has been widely studied in the literature and plays an important role in engineering, telecommunication, etc. Chance-constrained problem was first introduced by Charnes et al. [11], they studied the individual chance constraint case. Miller and Wagner [57] dealt with joint chance constraints but only in the independent case. Prekopa [66] was the first researcher who studied joint chance constraints with random parameters on the right hand side. Joint chance-constrained problems are generally non-convex problems. Therefore, several approximations have been proposed in the literature, see for instance [15, 46, 53]. Cheng and Lissner [15] proposed piecewise tangent approximations and sequential approximations to come up with a lower and upper bounds. Luedtke and Ahmed [53] reformulated the original problem by using a mixed-integer linear optimization in finite support case. An efficient way to solve the chance constraints problem is given by the scenario approach studied in [7, 8, 17, 20, 35, 52, 63, 64]. Applications of chance-constrained optimization problems can be found in [9, 27, 28, 37, 39, 43, 48, 72, 74].

1.2 Convexity of the feasibility set

The convexity of chance constraints as well as the regularity properties of probability functions are difficult issues both from theoretical and computational perspectives. Prekopa [67] introduced the notion of logarithmic concavity to deal with dependent constraints case. Borell [7] generalized this notion to r -concavity. Prekopa et al. [69] derive a fundamental result about the convexity of the considered problem if we assume that the rows are normally distributed. Sen [71] introduced a relaxation method for chance-constrained programs with a discrete random variable. Marti [54] studied the differentiation of probability functions by an integral transformation method. The derivatives of the probability function can be obtained by applying an integral transformation to its integral representation. Some basic results on the differentiability of a probability function were studied by Kibzun et al [40]. They proposed new formulations of the gradient of probability functions in different forms. Lobo [49] studied some applications of second-order cone program leading to a new approach for solving chance constraints. A more developed direction was initialized by Henrion [32] which gave a full description of the structure (not only the convexity) of a one-row linear optimization with a chance constraint by introducing a new notion of r -decreasing function. Henrion [33] studied the convexity in the case where the constraints are independent. To deal with the dependent case, Henrion and Strugarek [34], Van Ackooij [77] and Cheng et al. [14] used the theory of copulas to model the dependence of constraints. They supposed that the distribution of the constraint row vectors are elliptically distributed. Under high probability threshold p , they prove the convexity of $Feasi(Q)$. Hong et al [35] proposed to solve joint chance-constrained programs by sequential convex approximations. They prove that the solutions of the sequence of approximations converge to a Karush-Kuhn-Tacker (KKT) point of the original problem. Farshbaf-Shaker et al [23] proved some properties of chance constraints in infinite dimensions. They supposed that the feasibility set belongs to a Banach space. Under mild conditions, they proved regularity properties of the probability function with an application to PDE constrained optimization. Wim van Ackooij [79] studied the convexity of the feasibility set in a general framework by using the radial representation of elliptical distributions.

In this paper, we deal with the convexity of $Feasi(Q)$. Firstly, we suppose that the vectors v_i are elliptical distributions. Under mild conditions, we generalize the convexity result in Cheng et al. [14].

This paper is organized as follows. In Section 2, we reformulate the probability function of problem (1). Section 3 studies sufficient conditions for the convexity of $Feasi(Q)$. In Section 4, we show the dependence structure of the constraints of problem (1) and prove the convexity of $Feasi(Q)$.

2 Elliptical constraints in linear optimization

2.1 Preliminaries

Firstly, we recall some important definitions and propositions we use in our paper.

Proposition 1 (Theorem 2.1, [21]) *Let X a random variable in \mathbb{R}^n be a spherical distribution. Hence, there exists a function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $t \in \mathbb{R}^n$, we have*

$$\phi_X(t) := \mathbb{E}(e^{it^T X}) = \psi(t^T t) = \psi(t_1^2 + \dots + t_n^2).$$

The function ψ is called a characteristic generator of X .

Definition 1 (Definition 2.2, [21]) A random variable U in \mathbb{R}^n is an elliptical distribution if we have the following representation

$$U = \mu + AX,$$

where X follows a spherical distribution with a characteristic generator ϕ , $A \in \mathbb{R}^{n \times n}$, $AA^T = \Sigma$ and $\mu \in \mathbb{R}^n$.

Definition 2 (Chapter 4.6, [68]) A function $f : \mathbb{R}^s \rightarrow (0, +\infty)$ is r -concave for a given $r \in (-\infty, +\infty)$ if $dom f$ is convex and

$$\begin{aligned} f(\alpha x + (1 - \alpha)y) &\geq [\alpha f(x)^r + (1 - \alpha)f(y)^r]^{\frac{1}{r}} \quad \text{if } r \neq 0. \\ f(\alpha x + (1 - \alpha)y) &\geq f(x)^\alpha f(y)^{1-\alpha} \quad \text{if } r = 0. \end{aligned}$$

for all $x, y \in dom f$ and $\alpha \in [0, 1]$. In the case where $r = 0$, f is called a log-concave function. Here, $dom f$ denotes the defined domain of f .

Definition 3 (Definition 2.2, [32])

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said r -decreasing for some $r \in \mathbb{R}$ if it is continuous on $(0, \infty)$ and if there exists some $t^* > 0$ such that the function $t^r f(t)$ is strictly decreasing for all $t > t^*$.

Definition 4 (Definition 1, [70]) A *copula* is the distribution function $C : [0, 1]^K \rightarrow [0, 1]$ of a random vector in dimension K whose marginals are uniformly distributed on $[0, 1]$.

Proposition 2 (Theorem of Sklar, [70]) *Given a distribution function $F : \mathbb{R}^K \rightarrow [0, 1]$ with marginals F_1, \dots, F_K , there exists a copula C such that*

$$\forall z \in \mathbb{R}^K, \quad F(z) = C(F_1(z_1), \dots, F_K(z_K)).$$

Moreover, if F_i is continuous, then C is uniquely given by

$$C(u) = F \left(F_1^{(-1)}(u_1), \dots, F_K^{(-1)}(u_K) \right).$$

In other words, C is uniquely determined on $F_1 \times \dots \times F_K$.

Proposition 3 (Fréchet-Hoeffding upper bound) For any copula C and $u = (u_1, \dots, u_K) \in [0, 1]^K$, we have

$$C(u) \leq C_M(u) := \min_{k=1, \dots, K} u_k.$$

Definition 5 (Definition 2.2, [56]) A copula C is strictly Archimedean if there exists a continuous and strictly decreasing function $\Psi : (0, 1] \rightarrow [0, \infty)$ such that $\Psi(1) = 0$, $\lim_{t \rightarrow 0} \Psi(t) = \infty$ and

$$C(u) = \Psi^{(-1)} \left(\sum_{i=1}^K \Psi(u_i) \right).$$

Ψ is called the generator of C .

Definition 6 (Definition 2.3, [56]) A real function $f : \mathbb{R} \rightarrow \mathbb{R}$ is K -monotonic on an open interval $I \subseteq \mathbb{R}$ with $K \geq 2$ if it is differentiable up to $(K - 2)$ -order and the derivatives are satisfied by

$$(-1)^k \frac{d^k}{dt^k} f(t) \geq 0, \quad 0 \leq k \leq K - 2 \text{ and } \forall t \in I.$$

and the function $(-1)^{K-2} \frac{d^{K-2}}{dt^{K-2}} f(t)$ is non increasing and convex on I . Moreover, f is called completely monotonic if f is K -monotonic, $\forall K \in \mathbb{N}$.

Proposition 4 (Theorem 2.2, [56]) Given $\psi : (0, 1] \rightarrow [0, +\infty)$ a strictly decreasing function such that $\psi(1) = 0$. Hence, it is the generator of a strictly Archimedean copula in dimension K if and only if ψ^{-1} is K -monotonic on $(0, \infty)$.

2.2 Reformulation of the probability function

In problem (1), we suppose that v_i follows an elliptical distribution such that

$$v_i = \mu_i + A_i Y_i, \quad i = 1, 2, \dots, K \quad (2)$$

where $\mu_i \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n \times n}$ such that $A_i A_i^T = \Sigma_i$ is a positive definite matrix, Y_i is a spherical distribution with a characteristic generator ψ_i . We refer the readers to [21] for more details about elliptical distributions.

Assumption 1 $0 \notin Q$.

In this paper, unless otherwise specified, we assume that Assumption 1 holds.
Let

$$\begin{aligned}\xi_i(x) &:= \frac{v_i^T x - \mu_i^T x}{\sqrt{x^T \Sigma_i x}}, \\ g_i(x) &:= \frac{D_i - \mu_i^T x}{\sqrt{x^T \Sigma_i x}}.\end{aligned}\quad (3)$$

The constraint in (1) can be rewritten as follows

$$\begin{aligned}\mathbb{P}\{Vx \leq D\} &\geq p. \\ \Leftrightarrow \mathbb{P}\{v_i^T x \leq D_i, \quad i = 1, \dots, K\} &\geq p. \\ \Leftrightarrow \mathbb{P}\{\xi_i(x) \leq g_i(x), \quad i = 1, \dots, K\} &\geq p.\end{aligned}\quad (4)$$

The characteristic function of $\xi_i(x)$ is given by

$$\mathbb{E}(e^{it\xi_i(x)}) = \psi_i(t^2) \quad \forall t \in \mathbb{R}.$$

We deduce that $\xi_i(x)$ is a 1-dimension distribution that does not depend on x . Our aim is to reformulate this function to study the convexity of $Feasi(Q)$. For this purpose, we use the Sklar Theorem (cf. Proposition 2) to rewrite constraint (4) as follows

$$\begin{aligned}\mathbb{P}\{\xi_i(x) \leq g_i(x), \quad i = 1, \dots, K\} &\geq p. \\ \Leftrightarrow C_x[F_1(g_1(x)), \dots, F_K(g_K(x))] &\geq p.\end{aligned}\quad (5)$$

where C_x denotes the Copula of the multivariate variable $\xi(x) := (\xi_1(x), \dots, \xi_K(x))$ and F_i denotes the cumulative distribution function of $\xi_i(x)$, $i = 1, \dots, K$.

We suppose a specific form of $\xi(x)$ which shows explicitly the variation of ξ with respect to x .

In particular, we suppose that for all x , C_x is a strictly Archimedean copula with generator ψ_x . We use an Archimedean copula to model the dependence as we can calculate explicitly its partial derivatives. The constraint (5) can be written as

$$\begin{aligned}C_x[F_1(g_1(x)), \dots, F_K(g_K(x))] \geq p &\Leftrightarrow \psi_x^{(-1)}\left(\sum_{i=1}^K \psi_x(F_i(g_i(x)))\right) \geq p. \\ \Leftrightarrow \sum_{i=1}^K \psi_x(F_i(g_i(x))) &\leq \psi_x(p).\end{aligned}\quad (6)$$

We summarize some selected strictly Archimedean copulas in the following table:

Now, we add auxiliary variables $\{\alpha_i \geq 0, \quad i = 1, \dots, K\}$ in order to reformulate constraint (6) into separated constraints. In particular, as ψ_x is positive, constraint (6) is equivalent to the following constraint

Type of copula	Parameter θ	Generator $\Psi_\theta(t)$
Independent	-	$-\log(t)$
Gumbel-Hougaard	$\theta \geq 1$	$[-\log(t)]^\theta$
Frank	$\theta > 0$	$-\log\left(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\right)$
Clayton	$\theta > 0$	$\frac{1}{\theta}(t^\theta - 1)$
Joe	$\theta \geq 1$	$-\log[1 - (1-t)^\theta]$

Table 1: Different types of strictly Archimedean copulas.

$$\begin{cases} \psi_x(F_i(g_i(x))) \leq \alpha_i \psi_x(p), & i = 1, \dots, K. \\ \alpha_i \geq 0, & i = 1, \dots, K. \\ \sum_{i=1}^K \alpha_i = 1. \end{cases} \quad (7)$$

This means that if $x^* \in Feasi(Q)$ then there exists $\alpha^* = (\alpha_1^*, \dots, \alpha_K^*) \in \mathbb{R}^K$ such that (x^*, α^*) satisfies constraints (7). On the other hand, if (x^*, α^*) satisfies constraints (7) and $x^* \in Q$, we deduce that $x^* \in Feasi(Q)$. Moreover, for $x^* \in Feasi(Q)$, we can choose $\alpha^* = (\alpha_1^*, \dots, \alpha_K^*)$ in order to satisfy constraints (7), i.e. ,

$$\alpha_i^* = \frac{\psi_{x^*}(F_i(g_i(x^*)))}{\sum_{j=1}^K \psi_{x^*}(F_j(g_j(x^*)))}, \quad \forall i = 1, 2, \dots, K. \quad (8)$$

By applying the decreasing monotonicity of the generator ψ_x , constraints (7) can be written as follows

$$\begin{cases} F_i(g_i(x)) \geq \psi_x^{(-1)}(\alpha_i \psi_x(p)), & i = 1, \dots, K. \\ \alpha_i \geq 0, & i = 1, \dots, K. \\ \sum_{i=1}^K \alpha_i = 1. \end{cases} \quad (9)$$

Remark 1 Our aim is to show the concavity of $F_i(g_i)$ with respect to x and the joint convexity of $\psi_x^{(-1)}(\alpha_i \psi_x(p))$ with respect to (α, x) .

2.3 Concavity of $F_i(g_i)$

In this section, we give appropriate sufficient conditions for the concavity of $F_i(g_i)$.

Lemma 1 (Lemma 2, section 3.3, Cheng et al. [14])

Let $r_i > 1$, for some $1 \leq i \leq K$. Then g_i is $(-r_i)$ -concave on any convex subset of the following subset of Q

$$\Omega^{(i)} := \left\{ x \in Q \mid D_i - \mu_i^T x > \frac{r_i + 1}{r_i - 1} \lambda_{i,\min}^{-\frac{1}{2}} \|\mu_i\| \sqrt{x^T \Sigma_i x} \right\} \quad (10)$$

where $\lambda_{i,\min}$ is the smallest eigenvalue of Σ_i .

Remark 2 It is easy to see that Ω^i is a convex set because $\frac{r_i+1}{r_i-1}\lambda_{i,\min}^{-\frac{1}{2}}\|\mu_i\| > 0$, $\mu_i^T x$ is linear and $\sqrt{x^T \Sigma_i x}$ is convex with respect to x .

Lemma 2 (Lemma 3, section 3.3, Cheng et al. [14])

If $D_i > 0$ and $\mu_i = 0$, for some $1 \leq i \leq K$, then g_i is (-1) -concave on any convex subset of the set Q .

Based on these two Lemmas, we have

Lemma 3 Let $I \subset \{1, 2, \dots, K\}$ such that $\mu_i \neq 0, \forall i \in I$ and $\mu_i = 0, \forall i \notin I$. Let $r = (r_1, \dots, r_K)$ such that $r_i > 1, \forall i \in I$ and $r_i = 1$ otherwise. Let

$$p^* := \max \left\{ \frac{1}{2}, \max_{i \in I} \left\{ F_i \left(\frac{r_i + 1}{r_i - 1} \lambda_{i,\min}^{-\frac{1}{2}} \|\mu_i\| \right) \right\} \right\}.$$

Hence, $\forall p > p^*$, we have

$$\text{Conv}(Feasi(Q)) \subset \bigcap_{i \in I} \Omega^i.$$

Moreover, $g_i > 0$ and $(-r_i)$ -concave on any convex subset of $\bigcap_{i \in I} \Omega^i$, $i=1, \dots, K$, where Conv is the convex hull.

Proof Firstly, we prove that $\forall p > p^*$, $Feasi(Q) \subset \bigcap_{i \in I} \Omega^i$.

In fact, let $x_0 \in Feasi(Q)$. In Section 2.2, we show that x_0 satisfies constraint (5). In particular, we have

$$C_{x_0}[F_1(g_1(x_0)), \dots, F_K(g_K(x_0))] \geq p. \quad (11)$$

By applying Proposition 3, we deduce that

$$1) \quad F_i(g_i(x_0)) \geq p, \quad \forall i \in I \Rightarrow F_i(g_i(x_0)) > p^* \geq F_i \left(\frac{r_i+1}{r_i-1} \lambda_{i,\min}^{-\frac{1}{2}} \|\mu_i\| \right), \quad \forall i \in I.$$

As F_i is increasing monotonic, we have

$$g_i(x_0) > \frac{r_i + 1}{r_i - 1} \lambda_{i,\min}^{-\frac{1}{2}} \|\mu_i\|, \quad \forall i \in I.$$

Therefore,

$$D_i - \mu_i^T x_0 > \frac{r_i + 1}{r_i - 1} \lambda_{i,\min}^{-\frac{1}{2}} \|\mu_i\| \sqrt{x_0^T \Sigma_i x_0}, \quad \forall i \in I.$$

$$2) \quad F_j(g_j(x_0)) \geq p, \quad \forall j \notin I \Rightarrow F_j(g_j(x_0)) > p^* \geq \frac{1}{2}, \quad j \notin I.$$

As $F_j(0) = \frac{1}{2}$, we have

$$g_j(x_0) > 0, \quad \forall j \notin I.$$

Therefore,

$$D_j > 0, \quad \forall j \notin I.$$

Hence, we deduce that $x_0 \in \bigcap_{i \in I} \Omega^i$. In other words, we have

$$Feasi(Q) \subset \bigcap_{i \in I} \Omega^i.$$

By Remark 2, $\bigcap_{i \in I} \Omega^i$ is a convex set. Hence, we have

$$Conv(Feasi(Q)) \subset \bigcap_{i \in I} \Omega^i.$$

which ends the first part of Lemma 3.

Secondly, we prove that $g_i > 0$ and $(-r_i)$ -concave on any convex subset of $\bigcap_{i \in I} \Omega^i$.

In fact, this is a direct consequence of Lemma 1 and Lemma 2. □

Lemma 4 (Lemma 3.1, Henrion, R. and Strugarek, C. [32])

Let $F : \mathbb{R} \rightarrow [0, 1]$ be a distribution function with $(r + 1)$ -decreasing density f for some $r > 0$. Hence, the function $z \mapsto F(z^{-\frac{1}{r}})$ is concave on $(0, (t^*)^{-r})$, where t^* is defined in Definition 3. Moreover, $F(t) < 1, \quad \forall t \in \mathbb{R}$.

We use Lemmas 3 and 4 to prove the following auxiliary result

Lemma 5 Let $I, \mu = (\mu_1, \dots, \mu_K), r = (r_1, \dots, r_K)$ as defined in Lemma 3. We suppose that the cumulative distribution function F_i has $(r_i + 1)$ -decreasing densities with the thresholds $t_i^*(r_i + 1), \quad \forall i = 1, \dots, K$ and $p > p^*$, where

$$p^* = \max \left\{ \frac{1}{2}, \max_{i \in I} F_i \left(\frac{r_i + 1}{r_i - 1} \lambda_{i, \min}^{-\frac{1}{2}} \|\mu_i\| \right), \max_{j=1, \dots, K} F_j [t_j^*(r_j + 1)] \right\}.$$

Hence, for all $y, z \in Feasi(Q)$ and $0 \leq a \leq 1$, we have

$$F_i(g_i(ay + (1 - a)z)) \geq aF_i(g_i(y)) + (1 - a)F_i(g_i(z)).$$

Proof By Lemma 3, we deduce that g_i is $(-r_i)$ -concave and $g_i > 0$ on $Conv(Feasi(Q))$, for all $i = 1, \dots, K$. Hence, for any $a \in [0, 1]$ and $y, z \in Feasi(Q)$, we have

$$g_i(ay + (1 - a)z) \geq [ag_i^{-r_i}(y) + (1 - a)g_i^{-r_i}(z)]^{-\frac{1}{r_i}}. \quad (12)$$

As $y \in Feasi(Q)$ and $p > p^*$, we have

$$C_y[F_1(g_1(y)), \dots, F_K(g_K(y))] > p^*. \quad (13)$$

By Proposition 3 and the definition of p^* , we deduce that

$$F_i(g_i(y)) > p^* \geq F_i[t_i^*(r_i + 1)], \quad i = 1, \dots, K. \quad (14)$$

As F_i are increasing monotonic, we deduce that

$$g_i(y) > t_i^*(r_i + 1) > 0 \Rightarrow 0 < g_i(y)^{-r_i} < (t_i^*(r_i + 1))^{-r_i}, \quad i = 1, \dots, K. \quad (15)$$

Similarly, we obtain the inequality for z .

By taking F_i on both sides of (12), we have

$$F_i(g_i(ay + (1-a)z)) \geq F_i([ag_i^{-r_i}(y) + (1-a)g_i^{-r_i}(z)]^{-\frac{1}{r_i}}). \quad (16)$$

By (15) and Lemma 4, we have

$$F_i(g_i(ay + (1-a)z)) \geq F_i([ag_i^{-r_i}(y) + (1-a)g_i^{-r_i}(z)]^{-\frac{1}{r_i}}) \geq aF_i(g_i(y)) + (1-a)F_i(g_i(z)).$$

□

In the rest of the paper, we suppose that the statements in Lemma 5 hold.

3 Convexity of $\psi_x^{(-1)}(\alpha_i \psi_x(p))$

The aim of this section is to study appropriate sufficient conditions for the joint convexity of $U(x, \alpha_i) = \psi_x^{(-1)}(\alpha_i \psi_x(p))$ with respect to (x, α_i) , $i=1, \dots, K$. For this purpose, we study the positive semidefiniteness of the Hessian matrix of $\psi_x^{(-1)}(\alpha_i \psi_x(p))$ on the convex set $Q \times [k_1, k_2]$, for given k_1, k_2 where

$$0 \leq k_1 < k_2 \leq 1. \quad (17)$$

Assumption 2 We suppose that $\psi : (x, t) \mapsto \psi_x(t)$ is a jointly continuously differentiable function with respect to (x, t) up to second-order as well as $\psi^{(-1)}$. Moreover, $\psi_x^{(-1)}$ is 4-monotonic function with respect to t , $\forall x \in Q$.

The following lemma is a reformulation of the positive semidefiniteness of the Hessian matrix of U .

Lemma 6 Suppose that Assumption 2 holds and $p < 1$. Hence, the positive semidefiniteness of the Hessian matrix of U on the convex set $Q \times [k_1, k_2]$ is equivalent to the positive semidefiniteness of the following symmetric matrix

$$\left[\frac{d^2}{dx^2} \frac{d^2}{d\alpha_i^2} - \left(\frac{d^2}{dx d\alpha_i} \right) \left(\frac{d^2}{dx d\alpha_i} \right)^T \right]_o [U(x, \alpha_i)]. \quad (18)$$

for all (x, α_i) on $Q \times [k_1, k_2]$.

Proof For $U(x, \alpha_i) = \psi_x^{(-1)}(\alpha_i \psi_x(p))$, we have

$$\frac{d^2}{d\alpha_i^2} U(x, \alpha_i) = [\psi_x(p)]^2 (\psi_x^{(-1)})''(\alpha_i \psi_x(p)).$$

As $\psi_x^{(-1)}$ is 4-monotonic, we deduce that $(\psi_x^{(-1)})''$ is decreasing. Note that $\psi_x(p) > 0$ as $p < 1$. If there exists $a \geq 0$ such that $(\psi_x^{(-1)})''(a) = 0$, by the decreasing monotonicity of $(\psi_x^{(-1)})''$, we deduce that $(\psi_x^{(-1)})''(t) = 0$ on (a, ∞) as well as $\psi_x^{(-1)}(t) = c_1 t + c_2$ on (a, ∞) where c_1, c_2 are given scalars. However, $\psi_x^{(-1)}(t) \in (0, 1]$, $\forall t \in [0, \infty]$ which means that $\psi_x^{(-1)}(t) = c_2$ on (a, ∞) , which contradicts the strict monotonicity of $\psi_x^{(-1)}$. Hence, we have $(\psi_x^{(-1)})''(\alpha_i \psi_x(p)) > 0$ as well as $\frac{d^2}{d\alpha_i^2} U(x, \alpha_i) > 0$ on $Q \times [k_1, k_2]$.

The quadratic form of the Hessian matrix of U at (x, α_i) is

$$\begin{aligned} & (s^T, r) \left\{ \frac{\delta^2}{\delta i \delta j} U(x, \alpha_i) \right\} (s, r). \\ & = r^2 \left(\frac{d^2}{d\alpha_i^2} U(x, \alpha_i) \right) + \left[s^T \frac{d^2}{dx d\alpha_i} U(x, \alpha_i) + \left(\frac{d^2}{dx d\alpha_i} U(x, \alpha_i) \right)^T s \right] r + s^T \left(\frac{d^2}{dx^2} U(x, \alpha_i) \right) s. \end{aligned}$$

$\forall (s, r) \in \mathbb{R}^N \times \mathbb{R}$, where $\left\{ \frac{\delta^2}{\delta i \delta j} U(x, \alpha_i) \right\}$ is the Hessian matrix of U at (x, α_i) .

If we consider this quadratic form as a second-order polynomial function of r with strictly positive second-order coefficient, then the positive semidefiniteness of U is equivalent to the positivity of its quadratic form for all $r \in \mathbb{R}$ which is equivalent to the following inequality

$$\begin{aligned} & \left(s^T \frac{d^2}{dx d\alpha_i} U(x, \alpha_i) \right)^2 - \left(\frac{d^2}{d\alpha_i^2} U(x, \alpha_i) \right) \cdot s^T \left(\frac{d^2}{dx^2} U(x, \alpha_i) \right) s \leq 0, \quad \forall s \in \mathbb{R}^N. \\ & \Leftrightarrow s^T \left[\left(\frac{d^2}{dx d\alpha_i} U(x, \alpha_i) \right) \left(\frac{d^2}{dx d\alpha_i} U(x, \alpha_i) \right)^T - \frac{d^2}{d\alpha_i^2} U(x, \alpha_i) \frac{d^2}{dx^2} U(x, \alpha_i) \right] s \leq 0, \quad \forall s \in \mathbb{R}^N. \\ & \Leftrightarrow \left[\left(\frac{d^2}{dx d\alpha_i} U(x, \alpha_i) \right) \left(\frac{d^2}{dx d\alpha_i} U(x, \alpha_i) \right)^T - \frac{d^2}{d\alpha_i^2} U(x, \alpha_i) \frac{d^2}{dx^2} U(x, \alpha_i) \right] - \text{is a negative} \end{aligned}$$

semi definite matrix.

□

Next, we introduce a particular family of generators which meets Assumption 2.

3.1 A particular form of the family of the generators ψ_x

We consider the following assumptions

Assumption 3 $p < 1$.

Assumption 4 Let a function $\psi : (Q \subset \mathbb{R}^n) \times (0, 1] \rightarrow [0, \infty)$ such that $(x, t) \mapsto (-\log t)^{\frac{1}{\kappa(x)}}$, where $\kappa : Q \rightarrow (0, 1]$ is a strictly positive function.

If Assumption 4 holds, we have $\psi_x^{(-1)}(t) = e^{-t^{\kappa(x)}}$. We can check that ψ_x is a strictly decreasing function and $\psi_x(1) = 0$ for all $x \in Q$ and ψ satisfies the conditions in Assumption 2. Then, by Proposition 4, we deduce that for all $k \geq 2$, ψ_x is the generator of a strictly Archimedean k -dimension copula, $\forall x \in Q$.

Remark 3 In fact, $\forall x \in Q$, ψ_x is the generator of a copula in the family of Gumbel-Hougaard copulas. The condition where $0 < \kappa(x) \leq 1$ is deduced by the defined domain of Gumbel-Hougaard copulas. We can see later that if we use a function κ to model the dependence structure of ψ with respect to x , the positive semidefiniteness of the matrix in Lemma 6 is equivalent to the following inequality

$$\kappa(x)'' \geq \omega(\kappa(x)')(\kappa(x)')^T. \quad (19)$$

where ω is a strictly positive scalar, $\kappa(x)''$ is the Hessian matrix of κ at x and $\kappa(x)'$ is the gradient vector of κ at x .

From now and on, we suppose that Assumptions 3 and 4 hold. If we apply Lemma 6, it is sufficient to study the positive semidefiniteness of the matrix

$$H(U)(x, \alpha_i) = \left[\frac{d^2}{dx^2} \frac{d^2}{d\alpha_i^2} - \left(\frac{d^2}{dx d\alpha_i} \right) \left(\frac{d^2}{dx d\alpha_i} \right)^T \right]_o [U(x, \alpha_i)]. \quad (20)$$

on the convex set $Q \times [k_1, k_2]$.

3.2 An equivalent condition of the positive semidefiniteness of $H(U)$

The main step of this section is to reformulate the positive semidefiniteness of $H(U)$ by an inequality under the form (19).

If we apply the form of ψ in Assumption (4) in $U(x, \alpha_i) = \psi_x^{(-1)}(\alpha_i \psi_x(p))$, we can rewrite $U(x, \alpha_i)$ as follows

$$\begin{aligned} U(x, \alpha_i) &= e^{-\left\{ \alpha_i (-\log p)^{\frac{1}{\kappa(x)}} \right\}^{\kappa(x)}} \\ &= p^{\alpha_i^{\kappa(x)}}. \end{aligned} \quad (21)$$

Now, by (21), we calculate explicitly $\frac{d^2}{dx^2}$, $\frac{d^2}{d\alpha_i^2}$ and $\frac{d^2}{dx d\alpha_i}$ of U .

By a direct calculation, we deduce the first formulation as follows

$$\left\{ \begin{array}{l} \frac{d}{d\alpha_i} U(x, \alpha_i) = \log(p) p^{\alpha_i^{\kappa(x)}} \kappa(x) \alpha_i^{\kappa(x)-1}. \\ \frac{d^2}{d\alpha_i^2} U(x, \alpha_i) = \kappa(x) \log(p) \alpha_i^{\kappa(x)-2} p^{\alpha_i^{\kappa(x)}} [\kappa(x) - 1 + \kappa(x) \log(p) \alpha_i^{\kappa(x)}]. \end{array} \right. \quad (22)$$

Remark 4 The first and second derivatives in (22) are deduced by the rule of differentiation for composite functions in the 1-dimension case.

Next, we calculate $\frac{d}{dx}$ of U . With the chain rule of differentiation for composite functions in the multi-dimension case, we have

$$\frac{d}{dx}U(x, \alpha_i) = \log(p)p^{\alpha_i^{\kappa(x)}} \log(\alpha_i)\alpha_i^{\kappa(x)} \kappa(x)'. \quad (23)$$

We differentiate the two sides of (23) with α_i in order to obtain $\frac{d^2}{dx d\alpha_i}$

$$\frac{d^2}{dx d\alpha_i}U(x, \alpha_i) = \log(p)\alpha_i^{\kappa(x)-1} p^{\alpha_i^{\kappa(x)}} [1 + \kappa(x) \log(\alpha_i) + \log(p) \log(\alpha_i)\alpha_i^{\kappa(x)} \kappa(x)] \kappa(x)'. \quad (24)$$

To calculate $\frac{d^2}{dx^2}$ of U , we differentiate both sides of (23) with x

$$\frac{d^2}{dx^2}U(x, \alpha_i) = p^{\alpha_i^{\kappa(x)}} (\log p)(\log \alpha_i) [\kappa''(x) + (\log \alpha_i + \log \alpha_i \log p \cdot \alpha_i^{\kappa(x)}) \kappa(x)' (\kappa(x))'^T]. \quad (25)$$

Remark 5 $\kappa(x)'$ is a column vector and $\kappa(x)''$ is a symmetric matrix. We can write

$\frac{d}{dx}U(x, \alpha_i) = \log p \cdot \log \alpha_i \cdot t_1(x) \cdot t_2(x) \cdot t_3(x)$ where $t_1(x) = p^{\alpha_i^{\kappa(x)}}$, $t_2(x) = \alpha_i^{\kappa(x)}$ and $t_3(x) = \kappa(x)'$. We use the product rule of differentiation and apply again the chain rule of differentiation for composite functions in the multi-dimension case in order to obtain (25).

We combine (22) and (25) in order to obtain $\frac{d^2}{d\alpha_i^2} \frac{d^2}{dx^2}$

$$\begin{aligned} \frac{d^2}{d\alpha_i^2} \frac{d^2}{dx^2} &= \kappa(x) \log(p)^2 \alpha_i^{2\kappa(x)-2} \log(\alpha_i) p^{2 \cdot \alpha_i^{\kappa(x)}} \times \\ &\left[\kappa(x) - 1 + \kappa(x) \log p \alpha_i^{\kappa(x)} \right] \times \left[\kappa(x)'' + \kappa(x)' (\kappa(x))'^T (\log \alpha_i + \log \alpha_i \cdot \log p \cdot \alpha_i^{\kappa(x)}) \right]. \end{aligned} \quad (26)$$

Moreover, $\frac{d^2}{d\alpha_i dx} \left(\frac{d^2}{d\alpha_i dx} \right)^T$ is deduced from (24)

$$\begin{aligned} \frac{d^2}{d\alpha_i dx} \left(\frac{d^2}{d\alpha_i dx} \right)^T &= \log(p)^2 \alpha_i^{2\kappa(x)-2} \cdot p^{2 \cdot \alpha_i^{\kappa(x)}} \times \\ &\left(1 + \kappa(x) \log \alpha_i + \log p \log \alpha_i \cdot \alpha_i^{\kappa(x)} \kappa(x) \right)^2 \kappa(x)' (\kappa(x))'^T. \end{aligned} \quad (27)$$

Note that $\log(p)^2 \cdot \alpha_i^{2\kappa(x)-2} \cdot p^{2 \cdot \alpha_i^{\kappa(x)}}$ is the common factor of (26) and (27). Then, we can reduce this term from both sides of (27) and (26) as follows

$$\begin{aligned} M &= \log(p)^{-2} \cdot \alpha_i^{-2\kappa(x)+2} \cdot p^{-2 \cdot \alpha_i^{\kappa(x)}} \cdot \left[\frac{d^2}{d\alpha_i^2} \frac{d^2}{dx^2} - \frac{d^2}{d\alpha_i dx} \left(\frac{d^2}{d\alpha_i dx} \right)^T \right] = \\ &= \left[\kappa(x) - 1 + \kappa(x) \log p \cdot \alpha_i^{\kappa(x)} \right] \cdot \left[\kappa(x)'' + \kappa(x)' (\kappa(x))'^T (\log \alpha_i + \log \alpha_i \cdot \log p \cdot \alpha_i^{\kappa(x)}) \right] \\ &\times \kappa(x) \log \alpha_i - \left(1 + \kappa(x) \log \alpha_i + \log p \log \alpha_i \cdot \alpha_i^{\kappa(x)} \kappa(x) \right)^2 \kappa(x)' (\kappa(x))'^T. \end{aligned} \quad (28)$$

Hence, in order to study the positive semidefiniteness of $\left[\frac{d^2}{d\alpha_i^2} \frac{d^2}{dx^2} - \frac{d^2}{d\alpha_i dx} \left(\frac{d^2}{d\alpha_i dx} \right)^T \right]$ on $Q \times [k_1, k_2]$, we study the positive semidefiniteness of M when $(x, \alpha) \in Q \times [k_1, k_2]$.

We rewrite M as follows

$$M = A\kappa(x)'' - B\kappa(x)'(\kappa(x)')^T. \quad (29)$$

where

$$\begin{cases} A = \kappa(x) \log \alpha_i \cdot \left[\kappa(x) - 1 + \kappa(x) \log p \cdot \alpha_i^{\kappa(x)} \right], \\ B = \kappa(x) \log(\alpha_i)^2 (1 + \log p \cdot \alpha_i^{\kappa(x)}) \left[1 - \kappa(x) - \kappa(x) \log p \cdot \alpha_i^{\kappa(x)} \right] + \\ \left(1 + \kappa(x) \cdot \log \alpha_i + \log p \cdot \log \alpha_i \cdot \alpha_i^{\kappa(x)} \cdot \kappa(x) \right)^2. \end{cases}$$

3.3 A sufficient condition of the convexity of $\psi_x^{(-1)}(\alpha_i \psi_x(p))$

In (29), A and B depend on x . The key idea of this section is to study a lower bound of A and an upper bound of B . Particularly, we will study the positive semidefiniteness of the following term

$$M^* = A_1 \kappa(x)'' - B_1 \kappa(x)'(\kappa(x)')^T. \quad (30)$$

where $A \geq A_1$, $B \leq B_1$ and A_1, B_1 do not depend on x .

Next, we consider the following assumption.

Assumption 5

1. $p \geq e^{-1}$.
2. $0 < c_l \leq \kappa(x) \leq 1$, $\forall x \in Q$.
3. $0 < h_l \leq \alpha_i \leq h_u < 1$.
4. There exists δ_l and δ_u such that $0 < \delta_l \leq \|x\| \leq \delta_u < \infty$, $\forall x \in Q$, where $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^n .

In other words, $\kappa(x)$ and α_i are bounded by strictly positive scalars c_l, c_u, h_l and h_u and Q is a bounded domain in \mathbb{R}^n . The main idea is to find a function κ such that if Assumptions 3, 4 and 5 hold, $\psi_x^{(-1)}(\alpha_i \psi_x(p))$ is a convex function with respect to (x, α) on $Q \times [h_l, h_u]$.

Now, we evaluate A and B in (29) as follows

1. We use the statements 1, 2 and 3 in Assumption 5 to deduce that $0 \leq 1 - \alpha_i^{\kappa(x)} \leq 1 + \log p \cdot \alpha_i^{\kappa(x)} \leq 1 + \log p \cdot h_l \leq 1$. Then we combine with $c_l \leq \kappa(x) \leq 1$ and $(-\log \alpha_i) \geq (-\log h_u)$ to deduce a lower bound of A

$$A = \kappa(x)(-\log \alpha_i) \cdot \left[1 - \kappa(x) \cdot (1 + \log p \cdot \alpha_i^{\kappa(x)}) \right] \geq c_l (-\log h_u) (-\log p \cdot h_l). \quad (31)$$

2. Since $0 \leq 1 + \log p \cdot \alpha_i^{\kappa(x)} \leq 1$, we apply the Cauchy-Schwarz inequality to deduce that

$$(1 + \log p \cdot \alpha_i^{\kappa(x)}) \left[1 - \kappa(x) \cdot (1 + \log p \cdot \alpha_i^{\kappa(x)}) \right] \leq \frac{1}{4 \cdot \kappa(x)} \leq \frac{1}{4c_l}. \quad (32)$$

The following inequalities are deduced by Assumption 5

$$c_l \leq \kappa(x) \leq 1. \quad (33)$$

$$-\log h_u \leq -\log \alpha_i \leq -\log h_l. \quad (34)$$

$$1 + \log p \leq 1 + \log p \cdot \alpha_i^{\kappa(x)} \leq 1 + \log p \cdot h_l. \quad (35)$$

We apply (32)-(35) in order to get

$$\begin{aligned} 1 + \log h_l(1 + \log p \cdot h_l) &\leq 1 + \kappa(x) \cdot \log \alpha_i + \log p \cdot \log \alpha_i \cdot \alpha_i^{\kappa(x)} \cdot \kappa(x) \\ &\leq 1 - c_l(-\log h_u)(1 + \log p). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &(1 + \kappa(x) \cdot \log \alpha_i + \log p \cdot \log \alpha_i \cdot \alpha_i^{\kappa(x)} \cdot \kappa(x))^2 \\ &\leq \max(|1 + \log h_l(1 + \log p \cdot h_l)|, |1 - c_l(-\log h_u)(1 + \log p)|)^2. \end{aligned}$$

We combine with $\kappa(x) \leq 1$ and $-\log \alpha_i \leq -\log h_l$ in order to deduce an upper bound of B

$$\begin{aligned} B &\leq \max(|1 + \log h_l(1 + \log p \cdot h_l)|, |1 - c_l(-\log h_u)(1 + \log p)|)^2 \\ &\quad + (\log h_l)^2 \cdot \frac{1}{4c_l}. \end{aligned} \quad (36)$$

Hence, we deduce that a sufficient condition for the positive semidefiniteness of M on $Q \times [h_l, h_u]$ is the positive semidefiniteness of the following term on Q

$$A_1 \kappa(x)'' - B_1 \kappa(x)' \cdot (\kappa(x)')^T,$$

where $B_1 = \max(|1 + \log h_l(1 + \log p \cdot h_l)|, |1 - c_l(-\log h_u)(1 + \log p)|)^2 + (\log h_l)^2 \cdot \frac{1}{4c_l}$ and $A_1 = c_l(-\log h_u)(-\log p \cdot h_l)$.

Note that $A_1 > 0$ as $-\log p \cdot h_l > 0$ by Assumption 3. Let $C = \frac{B_1}{A_1}$, we deduce that the positive semidefiniteness of M on $Q \times [h_l, h_u]$ is equivalent to the positive semidefiniteness of the following term on Q

$$\kappa(x)'' - C \cdot \kappa(x)' \cdot (\kappa(x)')^T. \quad (37)$$

3.4 Boundedness of α on $Feasi(Q)$

The main idea of this Section is to show that for all $x \in Feasi(Q)$, we can choose α by (8) and h_l, h_u defined by $D_i, \Sigma_i, \mu_i, p, c_l$ such that $h_l \leq \alpha_i \leq h_u, i = 1, \dots, K$.

In fact, by (8), if $x \in Feasi(Q)$, we can choose α_i such that

$$\alpha_i = \frac{\psi_x[F_i(g_i(x))]}{\sum_{j=1}^K \psi_x[F_j(g_j(x))]}, \quad \forall i = 1, 2, \dots, K.$$

where $\psi_x(t) = (-\log t)^{\frac{1}{\kappa(x)}}$ and $g_i(x) = \frac{D_i - \mu_i^T x}{\sqrt{x^T \Sigma_i x}}$.

We use the Cauchy-Schwarz inequality to deduce that

$$|-\mu_i^T x| \leq \|\mu_i\| \cdot \|x\|, \quad \forall i = 1, \dots, K. \quad (38)$$

From linear algebra, we have

$$\sqrt{x^T \Sigma_i x} \geq \sqrt{\lambda_{i,min}} \|x\|. \quad (39)$$

Based on inequality (6), Assumption 3, Lemma 4 and the strictly decreasing monotonicity of ψ_x , we have

$$0 < \sum_{j=1}^K \psi_x[F_j(g_j(x))] \leq \psi_x(p). \quad (40)$$

We apply (38) - (40) and $\|x\| \geq \delta_l$ to deduce the following inequality

$$g_i(x) \leq \frac{|D_i|}{\sqrt{\lambda_{i,min}} \delta_l} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,min}}}.$$

As F_i is increasing monotonic, we have

$$F_i(g_i(x)) \leq F_i\left(\frac{|D_i|}{\sqrt{\lambda_{i,min}} \delta_l} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,min}}}\right).$$

We use the decreasing monotonicity of ψ_x to deduce

$$\psi_x(F_i(g_i(x))) \geq \psi_x\left(F_i\left(\frac{|D_i|}{\sqrt{\lambda_{i,min}} \delta_l} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,min}}}\right)\right).$$

Based on the formulation of α_i in (8), we have

$$\alpha_i \geq \frac{\psi_x\left(F_i\left(\frac{|D_i|}{\sqrt{\lambda_{i,min}} \delta_l} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,min}}}\right)\right)}{\psi_x(p)}.$$

Note that $0 < -\log p \leq 1$ (by Condition 1 in Assumption 5 and Assumption 2). Hence, we have

$$\psi_x(p) \leq (-\log p). \quad (41)$$

The following inequality is deduced by Condition 2 in Assumption 5 and the form of ψ_x in Assumption 4

$$\begin{aligned} & \psi_x \left(F_i \left(\frac{|D_i|}{\sqrt{\lambda_{i,\min}} \delta_l} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}} \right) \right) \\ & \geq \min \left[\left(-\log F_i \left(\frac{|D_i|}{\sqrt{\lambda_{i,\min}} \delta_l} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}} \right) \right), \left(-\log F_i \left(\frac{|D_i|}{\sqrt{\lambda_{i,\min}} \delta_l} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}} \right) \right)^{\frac{1}{c_l}} \right] = m_i. \end{aligned} \quad (42)$$

$$\text{Let } h_l = \min_{1 \leq i \leq K} \left\{ \frac{m_i}{-\log p} \right\}. \quad (43)$$

Hence, we deduce that $\alpha_i \geq h_l$, $\forall i = 1, 2, \dots, K$ by (41) and (42).
On the other hand, we have

$$\alpha_i = 1 - \sum_{j=1, j \neq i}^n \alpha_j \leq 1 - (n-1)h_l.$$

$$\text{Let } h_u = 1 - (n-1)h_l. \quad (44)$$

We conclude that $0 < h_l \leq \alpha_i \leq h_u < 1$, $\forall i = 1, 2, \dots, K$.

4 Convexity results for elliptically distributed chance constraints

In this section, we give an example of κ that satisfies the matrix inequality (37) on Q under Assumption 5 and we show the dependence structure of the constraints. Our aim is to choose $d \in \mathbb{R}$ and a function κ^* that satisfies (37) on Q such that

$$0 < c_l - d \leq \kappa^*(x) \leq 1 - d, \quad \forall x \in Q, \quad (45)$$

and let $\kappa = \kappa^* + d$.

We have the following formulation

$$\frac{d^2}{dx^2} \log(\kappa^*(x)) = \frac{\kappa^*(x)\kappa^*(x)'' - \kappa^*(x)'(\kappa^*(x)')^T}{\kappa^*(x)^2}. \quad (46)$$

Assumption 6 $\kappa^*(x) \leq \frac{1}{C}$, $\forall x \in Q$.

If Assumption 6 holds, the positive semidefiniteness of $\frac{d^2}{dx^2} \log(\kappa^*(x))$ is a sufficient condition for the positive semidefiniteness of the following term

$$\frac{\frac{1}{C} \kappa^*(x)'' - \kappa^*(x)'(\kappa^*(x)')^T}{\kappa^*(x)^2}. \quad (47)$$

As $\kappa' = (\kappa^*)'$, $\kappa'' = (\kappa^*)''$ and $\kappa^*(x)^2 > 0$, the positive semidefiniteness of (47) is equivalent to the positive semidefiniteness of the following term

$$\frac{1}{C} \kappa(x)'' - \kappa(x)'(\kappa(x)')^T. \quad (48)$$

Hence, if $0 < c_l - d \leq \kappa^*(x) \leq \min\left(\frac{1}{C}, 1 - d\right)$, $\forall x \in Q$, we deduce that the positive semidefiniteness of $\frac{d^2}{dx^2} \log(\kappa^*(x))$ is a sufficient condition for the matrix inequality (37).

On the other hand, the positive semidefiniteness of $\frac{d^2}{dx^2} \log(\kappa^*(x))$ on Q is equivalent to the convexity of $\log(\kappa^*)$ on Q . Hence, $\log(\kappa^*)$ is a convex function on Q .

Let $\kappa^* = e^q$, we need to find a convex function $q : Q \subset \mathbb{R}^n \rightarrow \mathbb{R}$ such that q satisfies the following condition

$$\begin{aligned} 0 < c_l - d \leq e^{q(x)} &\leq \min\left(\frac{1}{C}, 1 - d\right). \\ \Leftrightarrow \log(c_l - d) \leq q(x) &\leq \log\left[\min\left(\frac{1}{C}, 1 - d\right)\right], \quad \forall x \in Q. \end{aligned} \quad (49)$$

Under Assumption 4, we can choose a function q such that

$$q(x) = \frac{1}{L} \|x\|^2 + z, \quad (50)$$

where $L > 0$, $z \in \mathbb{R}$ are real numbers such that

$$\log(c_l - d) \leq \frac{1}{L} \delta_l^2 + z \leq \frac{1}{L} \delta_u^2 + z \leq \log\left[\min\left(\frac{1}{C}, 1 - d\right)\right], \quad (51)$$

We deduce the following lemma

Lemma 7 *Let $\delta_l, \delta_u, c_l, h_u, h_l$ such that Assumptions 3,4 and 5 hold. Let $L > 0$, $d \in \mathbb{R}$, $z \in \mathbb{R}$ are real numbers such that*

$$\log(c_l - d) \leq \frac{1}{L} \delta_l^2 + z \leq \frac{1}{L} \delta_u^2 + z \leq \log\left[\min\left(\frac{1}{C}, 1 - d\right)\right].$$

where C is defined in (37). Let $\kappa : Q \rightarrow (0, 1]$ such that

$$\kappa(x) = \left(e^{\frac{1}{L} \|x\|^2 + z} + d\right). \quad (52)$$

Hence, $U(x, \alpha_i) = \psi_x^{(-1)}(\alpha_i \psi_x(p))$ is a convex function with respect to (x, α_i) on the convex set $Q \times [h_l, h_u]$.

By using the same notations as (3), the dependence structure of vectors $\xi_i(x)$ is modeled by an Archimedean copula C_x , where

$$C_x(t) = \psi_x^{(-1)} \left(\sum_{i=1}^K \psi_x(t_i) \right), \quad t = (t_1, \dots, t_K), \quad 0 < t_i \leq 1, \quad i = 1, \dots, K.$$

$$\psi_x(t_i) = (-\log t_i)^{\frac{1}{\kappa(x)}}.$$

For a fixed $x_0 \in Q$, the Archimedean copula C_{x_0} is a Gumbel-Hougaard copula with parameter $\theta = \frac{1}{\kappa(x_0)}$ and the dependence with respect to x is modeled by function κ .

Remark 6 By using $\|x\|^2$ as the form of $q(x)$ in (50) to model the dependence structure of vectors $\xi_i(x)$, we deduce that the dependence structure of $\xi_i(x)$ is unchanged on the contour lines of function $\|x\|^2$. We can replace $\|x\|^2$ by any convex function to model more general cases.

Our main result is given by the following theorem.

Theorem 1 *Consider problem (1). Let h_l, h_u defined by (43) and (44). Suppose that the statements in Lemma 7 hold and $Feasi(Q) \neq \emptyset$. Hence, $Feasi(Q)$ is a convex set.*

Proof In Section 3.4, we show that if $Feasi(Q) \neq \emptyset$, we have $0 < h_l \leq h_u < 1$. Let $x_1, x_2 \in Feasi(Q)$ and $\beta \in [0, 1]$. We show that $y := \beta x_1 + (1 - \beta)x_2 \in Feasi(Q)$.

In fact, let $\alpha^1 := (\alpha_1^1, \dots, \alpha_K^1)$ and $\alpha^2 := (\alpha_1^2, \dots, \alpha_K^2)$ given by (8). We deduce that (x^1, α^1) and (x^2, α^2) satisfy constraint (9). In Section 3.4, we show that $h_l \leq \alpha_i^1, \alpha_i^2 \leq h_u, \forall i = 1, 2, \dots, K$. By Lemma 7, we deduce that $U(x, \alpha_i) = \psi_x^{(-1)}(\alpha_i \psi_x(p))$ is a convex function with respect to (x, α_i) on the set $Q \times [h_l, h_u]$.

Hence, we deduce the two following inequalities for y :

$$F_i(g_i(y)) \geq \beta F_i(g_i(x_1)) + (1 - \beta) F_i(g_i(x_2)).$$

$$U(y, \beta \alpha^1 + (1 - \beta) \alpha^2) \leq \beta U(x_1, \alpha^1) + (1 - \beta) U(x_2, \alpha^2). \quad (53)$$

By (53), we deduce that $(y, \beta \alpha^1 + (1 - \beta) \alpha^2)$ also satisfies (9) as well as $y \in Feasi(Q)$. \square

Now, we discuss about Assumption 1. Firstly, we can state that α_i cannot go to 0 in (29). If $\alpha_i \rightarrow 0$, we have $A = \mathcal{O}(\log \alpha_i)$ and $B = \mathcal{O}(\log \alpha_i)^2$. Hence, $\frac{B}{A} \rightarrow \infty$. We deduce that $\kappa(x)' = 0, \forall x \in Q$. Therefore, κ is a scalar function and we return to the trivial case where the copula C_x does not depend on x which is the independent case. Hence, α_i must be lower bounded by a strictly positive scalar h_l . Based on (8), we deduce that $\psi_x(F_i(g_i(x)))$ is lower bounded by a strictly positive scalar as well as $g_i(x)$ is upper bounded. Therefore, $\frac{D_i - \mu_i^T x}{\sqrt{x^T \Sigma_i x}}$ is

upper bounded. If $x \rightarrow 0$, we might have $\frac{D_i - \mu_i^T x}{\sqrt{x^T \Sigma_i x}} \rightarrow \infty$ if $D_i > 0$. Hence, we do not have the convexity in this case. We deduce that there exists a strictly positive scalar δ_l such that $\|x\| \geq \delta_l, \forall x \in Q$.

5 Conclusion

In this paper, we studied the problem of linear optimization with joint probabilistic constraints in the case of elliptical distributions. Further, we modeled the dependence of random variables by a Gumbel-Hougaard copula. We studied the convexity of the feasibility set and come up with new convexity results. Further research will be dedicated to other families of copulas.

References

1. AHMED, S. Convex relaxations of chance constrained optimization problems. *Optimization Letters* 8, 1 (2014), 1–12.
2. AKHAVAN-HEJAZI, H., AND MOHSENIAN-RAD, H. Energy storage planning in active distribution grids: A chance-constrained optimization with non-parametric probability functions. *IEEE Transactions on Smart Grid* 9, 3 (2016), 1972–1985.
3. ARRIGO, A., ORDOUDIS, C., KAZEMPOUR, J., DE GRÈVE, Z., TOUBEAU, J.-F., AND VALLÉE, F. Wasserstein distributionally robust chance-constrained optimization for energy and reserve dispatch: An exact and physically-bounded formulation. *European Journal of Operational Research* (2021).
4. AYYAGARI, K. S., GATSIS, N., AND TAHA, A. F. Chance constrained optimization of distributed energy resources via affine policies. In *2017 IEEE Global Conference on Signal and Information Processing (GlobalSIP)* (2017), IEEE, pp. 1050–1054.
5. BAI, X., SUN, J., AND ZHENG, X. An augmented lagrangian decomposition method for chance-constrained optimization problems. *INFORMS Journal on Computing* (2020).
6. BLANCHET, J., ZHANG, F., AND ZWART, B. Optimal scenario generation for heavy-tailed chance constrained optimization. *arXiv preprint arXiv:2002.02149* (2020).
7. BORELL, C. Convex set functions in d-space. *Periodica Mathematica Hungarica* 6, 2 (1975), 111–136.
8. CALAFIORE, G., AND CAMPI, M. C. Uncertain convex programs: randomized solutions and confidence levels. *Mathematical Programming* 102, 1 (2005), 25–46.
9. CALAFIORE, G., AND DABBENE, F. *Probabilistic and randomized methods for design under uncertainty*. Springer, 2006.
10. CAMPI, M. C., AND GARATTI, S. The exact feasibility of randomized solutions of uncertain convex programs. *SIAM Journal on Optimization* 19, 3 (2008), 1211–1230.
11. CHARNES, A., AND COOPER, W. W. Chance-constrained programming. *Management science* 6, 1 (1959), 73–79.
12. CHARNES, A., C., W., W., AND SYMONDS, G. H. Cost horizons and certainty equivalents: an approach to stochastic programming of heating oil. *Management science* 4, 3 (1958), 256–263.
13. CHEN, P., AND GHATTAS, O. Taylor approximation for chance constrained optimization problems governed by partial differential equations with high-dimensional random parameters. *arXiv preprint arXiv:2011.09985* (2020).
14. CHENG, J., HOUDA, M., AND LISSER, A. Second-order cone programming approach for elliptically distributed joint probabilistic constraints with dependent rows. *Technical report, Optimization* (2014).
15. CHENG, J., AND LISSER, A. A second-order cone programming approach for linear programs with joint probabilistic constraints. *Operations Research Letters* 40, 5 (2012), 325–328.
16. CIFTCI, O., MEHRTASH, M., AND KARGARIAN, A. Data-driven nonparametric chance-constrained optimization for microgrid energy management. *IEEE Transactions on Industrial Informatics* 16, 4 (2019), 2447–2457.
17. DE FARIAS, D. P., AND VAN ROY, B. On constraint sampling in the linear programming approach to approximate dynamic programming. *Mathematics of operations research* 29, 3 (2004), 462–478.
18. DELAGE, E., AND MANNOR, S. Percentile optimization for markov decision processes with parameter uncertainty. *Operations research* 58, 1 (2010), 203–213.
19. DU, B., CHEN, J., SUN, D., MANYAM, S. G., AND CASBEER, D. W. Uav trajectory planning with probabilistic geo-fence via iterative chance-constrained optimization. *IEEE Transactions on Intelligent Transportation Systems* (2021).
20. ERDOĞAN, E., AND IYENGAR, G. Ambiguous chance constrained problems and robust optimization. *Mathematical Programming* 107, 1 (2006), 37–61.
21. FANG, K. W. *Symmetric multivariate and related distributions*. CRC Press, 2018.
22. FANG, S., ZHAO, T., XU, Y., AND LU, T. Coordinated chance-constrained optimization of multi-energy microgrid system for balancing operation efficiency and quality-of-service. *Journal of Modern Power Systems and Clean Energy* 8, 5 (2020), 853–862.

23. FARSHBAF-SHAKER, M. H., HENRION, R., AND HÖMBERG, D. Properties of chance constraints in infinite dimensions with an application to pde constrained optimization. *Set-Valued and Variational Analysis* 26, 4 (2018), 821–841.
24. FRAGIADAKIS, K., AND MEINTANIS, S. G. Goodness-of-fit tests for multivariate laplace distributions. *Mathematical and Computer Modelling* 53, 5-6 (2011), 769–779.
25. GELETU, A., HOFFMANN, A., SCHMIDT, P., AND LI, P. Chance constrained optimization of elliptic pde systems with a smoothing convex approximation. *ESAIM: Control, Optimisation and Calculus of Variations* 26 (2020), 70.
26. GENG, X., AND XIE, L. Chance-constrained unit commitment via the scenario approach. In *2019 North American Power Symposium (NAPS)* (2019), IEEE, pp. 1–6.
27. GENG, X., AND XIE, L. Data-driven decision making in power systems with probabilistic guarantees: Theory and applications of chance-constrained optimization. *Annual reviews in control* 47 (2019), 341–363.
28. GURVICH, I., LUEDTKE, J., AND TEZCAN, T. Staffing call centers with uncertain demand forecasts: A chance-constrained optimization approach. *Management Science* 56, 7 (2010), 1093–1115.
29. HAEUSSLING LOEWGREN, B., WEIGERT, J., ESCHE, E., AND REPKE, J.-U. Uncertainty analysis for data-driven chance-constrained optimization. *Sustainability* 12, 6 (2020), 2450.
30. HENRION, R. Structural properties of linear probabilistic constraints. *Optimization* 56, 4 (2007), 425–440.
31. HENRION, R., AND MÖLLER, A. A gradient formula for linear chance constraints under gaussian distribution. *Mathematics of Operations Research* 37, 3 (2012), 475–488.
32. HENRION, R., AND STRUGAREK, C. Convexity of chance constraints with independent random variables. *Computational Optimization and Applications* 41, 2 (2008), 263–276.
33. HENRION, R., AND STRUGAREK, C. Convexity of chance constraints with independent random variables. *Computational Optimization and Applications* 41, 2 (2008), 263–276.
34. HENRION, R., AND STRUGAREK, C. Convexity of chance constraints with dependent random variables: the use of copulae. In *Stochastic optimization methods in finance and energy*. Springer, 2011, pp. 427–439.
35. HONG, L. J., YANG, Y., AND ZHANG, L. Sequential convex approximations to joint chance constrained programs: A monte carlo approach. *Operations Research* 59, 3 (2011), 617–630.
36. HOU, W., LIU, Z., MA, L., AND WANG, L. A real-time rolling horizon chance constrained optimization model for energy hub scheduling. *Sustainable Cities and Society* 62 (2020), 102417.
37. HUANG, Y., WANG, L., GUO, W., KANG, Q., AND WU, Q. Chance constrained optimization in a home energy management system. *IEEE Transactions on Smart Grid* 9, 1 (2016), 252–260.
38. HUO, D., GU, C., GREENWOOD, D., WANG, Z., ZHAO, P., AND LI, J. Chance-constrained optimization for integrated local energy systems operation considering correlated wind generation. *International Journal of Electrical Power & Energy Systems* 132 (2021), 107153.
39. HUO, D., GU, C., MA, K., WEI, W., XIANG, Y., AND LE BLOND, S. Chance-constrained optimization for multienergy hub systems in a smart city. *IEEE Transactions on Industrial Electronics* 66, 2 (2018), 1402–1412.
40. KIBZUN, A., AND URYASEV, S. Differentiability of probability function. *Stochastic Analysis and Applications* 16, 6 (1998), 1101–1128.
41. KÜÇÜKYAVUZ, S., AND JIANG, R. Chance-constrained optimization: A review of mixed-integer conic formulations and applications. *arXiv preprint arXiv:2101.08746* (2021).
42. LAGUEL, Y., VAN ACKOOIJ, W., MALICK, J., AND RAMALHO, G. On the convexity of level-sets of probability functions. *arXiv preprint arXiv:2102.04052* (2021).
43. LE, T. A., VIEN, Q.-T., NGUYEN, H. X., NG, D. W. K., AND SCHOBER, R. Robust chance-constrained optimization for power-efficient and secure swipt systems. *IEEE Transactions on Green Communications and Networking* 1, 3 (2017), 333–346.
44. LEJEUNE, M. A., AND MARGOT, F. Solving chance-constrained optimization problems with stochastic quadratic inequalities. *Operations Research* 64, 4 (2016), 939–957.
45. LI, Z., AND LI, Z. Optimal robust optimization approximation for chance constrained optimization problem. *Computers & Chemical Engineering* 74 (2015), 89–99.
46. LIU, J., LISSER, A., AND CHEN, Z. Stochastic geometric optimization with joint probabilistic constraints. *Operations Research Letters* 44, 5 (2016), 687–691.
47. LIU, X., LIU, Q., SONG, S., AND PENG, J. A chance-constrained generative framework for sequence optimization. In *International Conference on Machine Learning* (2020), PMLR, pp. 6271–6281.
48. LIU, Z., CHAN, K. Y., MA, K., GUAN, X., ET AL. Chance-constrained optimization in d2d-based vehicular communication network. *IEEE Transactions on Vehicular Technology* 68, 5 (2019), 5045–5058.
49. LOBO, M. S., VANDENBERGHE, L., BOYD, S., AND LEBRET, H. Applications of second-order cone programming. *Linear algebra and its applications* 284, 1-3 (1998), 193–228.
50. LOBO, M. S., VANDENBERGHE, L., BOYD, S., AND LEBRET, H. Applications of second-order cone programming. *Linear algebra and its applications* 284, 1-3 (1998), 193–228.

51. LUBIN, M., BIENSTOCK, D., AND VIELMA, J. P. Two-sided linear chance constraints and extensions. *arXiv preprint arXiv:1507.01995* (2015).
52. LUEDTKE, J., AND AHMED, S. A sample approximation approach for optimization with probabilistic constraints. *SIAM Journal on Optimization* 19, 2 (2008), 674–699.
53. LUEDTKE, J., AHMED, S., AND NEMHAUSER, G. L. An integer programming approach for linear programs with probabilistic constraints. *Mathematical programming* 122, 2 (2010), 247–272.
54. MARTI, K. Differentiation of probability functions: The transformation method. *Computers & Mathematics with Applications* 30, 3-6 (1995), 361–382.
55. McNEIL, A. J., FREY, R., AND EMBRECHTS, P. *Quantitative risk management: concepts, techniques and tools-revised edition*. Princeton university press, 2015.
56. McNEIL, A. J., AND NEŠLEHOVÁ, J. Multivariate archimedean copulas, d-monotone functions and ℓ_1 -norm symmetric distributions. *The Annals of Statistics* 37, 5B (2009), 3059–3097.
57. MILLER, B. L., AND WAGNER, H. M. Chance constrained programming with joint constraints. *Operations Research* 13, 6 (1965), 930–945.
58. MINOUX, M., AND ZORGATI, R. Convexity of gaussian chance constraints and of related probability maximization problems. *Computational Statistics* 31, 1 (2016), 387–408.
59. MINOUX, M., AND ZORGATI, R. Global probability maximization for a gaussian bilateral inequality in polynomial time. *Journal of Global Optimization* 68, 4 (2017), 879–898.
60. NADEEM, Z., JAVAID, N., MALIK, A. W., AND IQBAL, S. Scheduling appliances with ga, tlbo, fa, osr and their hybrids using chance constrained optimization for smart homes. *Energies* 11, 4 (2018), 888.
61. NAZIR, F. U., PAL, B. C., AND JABR, R. A. A two-stage chance constrained volt/var control scheme for active distribution networks with nodal power uncertainties. *IEEE Transactions on Power Systems* 34, 1 (2018), 314–325.
62. NELSEN, R. B. *An introduction to copulas*. Springer Science & Business Media, 2007.
63. NEMIROVSKI, A., AND SHAPIRO, A. Scenario approximations of chance constraints. In *Probabilistic and randomized methods for design under uncertainty*. Springer, 2006, pp. 3–47.
64. PAGNONCELLI, B. K., AHMED, S., AND SHAPIRO, A. Sample average approximation method for chance constrained programming: theory and applications. *Journal of optimization theory and applications* 142, 2 (2009), 399–416.
65. PENG, S., SINGH, V. V., AND LISSER, A. General sum games with joint chance constraints. *Operations Research Letters* 46, 5 (2018), 482–486.
66. PRÉKOPA, A. On probabilistic constrained programming. In *Proceedings of the Princeton symposium on mathematical programming 113* (1970), 138.
67. PRÉKOPA, A. Logarithmic concave measures with application to stochastic programming. *Acta Scientiarum Mathematicarum* 32 (1971), 301–316.
68. PRÉKOPA, A. On logarithmic concave measures and functions. *Acta Scientiarum Mathematicarum* 34 (1973), 335–343.
69. PRÉKOPA, A., YODA, K., AND SUBASI, M. M. Uniform quasi-concavity in probabilistic constrained stochastic programming. *Operations Research Letters* 39, 3 (2011), 188–192.
70. RÜSCHENDORF, L. Copulas, sklar’s theorem, and distributional transform. In *Mathematical risk analysis*. Springer, 2013, pp. 3–34.
71. SEN, S. Relaxations for probabilistically constrained programs with discrete random variables. *Operations Research Letters* 11, 2 (1992), 81–86.
72. SHAPIRO, A., DENTCHEVA, D., AND RUSZCZYŃSKI, A. *Lectures on stochastic programming: modeling and theory*. SIAM, 2014.
73. SOARES, T., AND BESSA, R. J. Proactive management of distribution grids with chance-constrained linearized ac opf. *International Journal of Electrical Power & Energy Systems* 109 (2019), 332–342.
74. SOLTANI, N. Y., KIM, S.-J., AND GIANNAKIS, G. B. Chance-constrained optimization of ofdma cognitive radio uplinks. *IEEE transactions on wireless communications* 12, 3 (2013), 1098–1107.
75. SOLTANI, N. Y., AND NASIRI, A. Chance-constrained optimization of energy storage capacity for micro-grids. *IEEE Transactions on Smart Grid* 11, 4 (2020), 2760–2770.
76. THABANE, L., AND DREKIC, S. Hypothesis testing for the generalized multivariate modified bessel model. *Journal of multivariate analysis* 86, 2 (2017), 360–374.
77. VAN ACKOOIJ, W. Eventual convexity of chance constrained feasible sets. *Optimization* 64, 5 (2015), 1263–1284.
78. VAN ACKOOIJ, W., AND DE OLIVEIRA, W. Convexity and optimization with copulae structured probabilistic constraints. *Optimization* 65, 7 (2016), 1349–1376.
79. VAN ACKOOIJ, W., AND MALICK, J. Eventual convexity of probability constraints with elliptical distributions. *Mathematical Programming* 175, 1 (2019), 1–27.
80. WANG, B., DEGHANIAN, P., AND ZHAO, D. Chance-constrained energy management system for power grids with high proliferation of renewables and electric vehicles. *IEEE Transactions on Smart Grid* 11, 3 (2019), 2324–2336.

81. WEI, Z., AND ALI, M. M. Convex mixed integer nonlinear programming problems and an outer approximation algorithm. *Journal of Global Optimization* 63, 2 (2015), 213–227.
82. WILKS, S. S. Multivariate statistical outliers. *The Indian Journal of Statistics Series A* (1963), 407–426.
83. XIE, W., AND AHMED, S. On deterministic reformulations of distributionally robust joint chance constrained optimization problems. *SIAM Journal on Optimization* 28, 2 (2018), 1151–1182.
84. XIE, W., AND AHMED, S. On quantile cuts and their closure for chance constrained optimization problems. *Mathematical Programming* 172, 1 (2018), 621–646.
85. XIE, W., AND AHMED, S. Bicriteria approximation of chance-constrained covering problems. *Operations Research* 68, 2 (2020), 516–533.
86. XIE, Y., HARPER, O., ASSIMI, H., NEUMANN, A., AND NEUMANN, F. Evolutionary algorithms for the chance-constrained knapsack problem. In *Proceedings of the Genetic and Evolutionary Computation Conference* (2019), pp. 338–346.
87. YANG, Y., DELA ROSA, L., AND CHOW, T. Y. M. Non-convex chance-constrained optimization for blending recipe design under uncertainties. *Computers & Chemical Engineering* 139 (2020), 106868.
88. YANG, Y., VAYANOS, P., AND BARTON, P. I. Chance-constrained optimization for refinery blend planning under uncertainty. *Industrial & Engineering Chemistry Research* 56, 42 (2017), 12139–12150.
89. YAZAR, O., KESKIN, M. F., AND GEZICI, S. Power-efficient positioning for visible light systems via chance constrained optimization. *IEEE Transactions on Aerospace and Electronic Systems* 56, 5 (2020), 4124–4131.
90. ZADEH, Z. M., AND KHORRAM, E. Convexity of chance constrained programming problems with respect to a new generalized concavity notion. *Annals of Operations Research* 196, 1 (2012), 651–662.