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LARGE DEVIATIONS OF ℓ^p -BLOCKS OF REGULARLY VARYING TIME SERIES AND APPLICATIONS TO CLUSTER INFERENCE

GLORIA BURITICÁ, THOMAS MIKOSCH, AND OLIVIER WINTENBERGER

ABSTRACT. In the regularly varying time series setting, a cluster of exceedances is a short period for which the supremum norm exceeds a high threshold. We propose to study a generalization of this notion considering short periods, or blocks, with ℓ^p -norm above a high threshold. Our main result derives new large deviation principles of extremal ℓ^p -blocks. We show an application to cluster inference to motivate our result, where we design consistent disjoint blocks estimators. We focus on inferring important indices in extreme value theory, e.g., the *extremal index*. Our approach motivates cluster inference based on extremal ℓ^p -blocks with $p < \infty$ rather than the classical one with $p = \infty$ where the bias is more difficult to control as illustrated in simulations from an example. Our estimators also promote the use of large empirical quantiles from the ℓ^p -norm of blocks as threshold levels which eases implementation and facilitates comparison.

1. INTRODUCTION

For various applications of extreme value statistics with stationary time series, it is natural to wonder how a recorded high level can affect the future behavior of the sequence or how often extreme events occur. However, the extremal dependencies of the time series can disturb the inference procedures tailored for independent observations; cf. Leadbetter [24], Embrechts *et al.* [18]. We consider the setting of \mathbb{R}^d -valued stationary regularly varying time series $(\mathbf{X}_t)_{t \in \mathbb{Z}}$ with generic element \mathbf{X}_0 ; see Section 2.1 for a definition, cf. Basrak and Segers [4]. In this framework, an exceedance of a high threshold by the norm $|\mathbf{X}_t|$ at time t might trigger consecutive exceedances in some small time interval around t . To model these short periods with at least one exceedance Davis and Hsing introduced the *clusters (of exceedances)* implicitly in the seminal paper [11]. These were further reviewed in Basrak and Segers [4] and Basrak *et al.* [3].

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The main motivation for studying clusters (of exceedances) can be traced back to Theorem 2.5. in Davis and Hsing [11]. Roughly speaking, for weakly dependent regularly varying time series, the limit distribution of $a_n^{-1}\mathbf{X}_{[0,n]} = a_n^{-1}(\mathbf{X}_0, \dots, \mathbf{X}_n)$ is fully characterized in terms of the index of regular variation, the cluster (of exceedances), and the *extremal index* of $(|\mathbf{X}_t|)$, denoted by $\theta_{|\mathbf{X}|}$, where $n\mathbb{P}(|\mathbf{X}_0| > a_n) \rightarrow 1$. In this representation, the notion of a cluster aims at modeling how rare events of $\mathbf{X}_{[0,n]}$ happen, i.e., its behavior if its supremum norm exceeds the high level x_n , and as the probability of recording a cluster tends to zero, i.e., $\mathbb{P}(\|\mathbf{X}_{[1,n]}\|_\infty > x_n) \sim \theta_{|\mathbf{X}|} n \mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$. From this last equivalence, we can also see that $\theta_{|\mathbf{X}|}$ summarizes the ratio between the expected number of exceedances in the stationary time-dependent scenario and the independent setting.

By the previous discussion, the cluster (of exceedances) is tied together with the extremal index by the supremum norm. Our main theoretical result extends the aforementioned ideas from the ℓ^∞ -norm to ℓ^p -norms. In Theorem 2.1 we investigate the behavior of $\mathbf{X}_{[0,n]}$ when its ℓ^p -norm exceeds high levels (x_n) satisfying $\mathbb{P}(\|\mathbf{X}_{[0,n]}\|_p > x_n) = \mathbb{P}(\sum_{t=1}^n |\mathbf{X}_t|^p > x_n^p) \rightarrow 0$ as $n \rightarrow +\infty$. We call this a large deviation result since it describes the block behavior when the probability that the partial sums of the norm p th powers of \mathbb{R}^d -valued regularly varying vectors exceed the extreme threshold x_n^p . This leads us to a new definition of a *cluster process* in the space $\ell^p = \ell^p(\mathbb{R}^d)$ and, in the limiting case $p = \infty$, one recovers the classical clusters (of exceedances). Similarly, large deviation principles for sums were considered by Nagaev [31], Cline and Hsing [9] in the independent heavy-tailed case, and by Mikosch and Wintenberger [27, 28, 29], Mikosch and Rodionov [30] in the dependent heavy-tailed case. Instead, we derive large deviation principles for blocks in ℓ^p and focus on ℓ^p -norms: they increase when p decreases, hence the proportion of extremal clusters increases. This fact points at an advantageous road for improving inference procedures which we will follow in the second part of this article.

For inference purposes we divide a sample $\mathbf{X}_1, \dots, \mathbf{X}_n$ into disjoint blocks $(\mathbf{B}_t)_{1 \leq t \leq \lfloor n/b_n \rfloor}$, $\mathbf{B}_t := \mathbf{X}_{(t-1)b_n + [1, b_n]}$, for a sequence of block lengths (b_n) such that $b_n \rightarrow \infty$ and $b_n/n \rightarrow 0$ as $n \rightarrow \infty$. To apply our findings we study cluster inference. Typically, inference on clusters (of exceedances) starts with selecting blocks whose supremum norm exceeds a high threshold x_{b_n} ; cf. Kulik and Soulier [23]. Instead, we aim at considering blocks whose ℓ^p -norms exceed a high threshold. As a matter of fact, for cluster inference the thresholds (x_{b_n}) should adapt to the block lengths (b_n) and take into account the value of p . In the existing literature for $p = \infty$ no detailed advice is given of how the couple (b_n) and (x_{b_n}) must be chosen; see for example Drees, Rootzén [16], Drees, Neblung [15], Cissokho, Kulik [8], Drees *et al.* [14], who assume bias conditions on the sequence (x_n) such as $n\mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$ as $n \rightarrow \infty$. It is common practice to replace x_{b_n} by an upper order statistic of $(|\mathbf{X}_t|)_{1 \leq t \leq n}$, turning the choice of b_n into a delicate problem which has not

been studied carefully. This problem occurs for example in the context of the blocks estimator of the *extremal index* proposed by Hsing [20].

To address the aforementioned threshold/block length difficulties of cluster inference, we design consistent disjoint blocks methods in Theorem 4.1 with thresholds chosen as order statistics of ℓ^p -norms that adapt to the length of the block. In this way one deals with the classical bias-variance trade-off in extreme value statistics (see Resnick [33]) where choosing a small number of order statistics increases the variance while a large number might lead to biased estimates. We retrieve the need for a rigorous definition of extremal ℓ^p -blocks for cluster inference to reveal how p can play a key role. Indeed, we can infer the same quantity using extremal ℓ^p -blocks, for different values of p , by applying a change of norms functional. When comparing these estimators, our large deviation principles support our approach for $p < \infty$ reduces the bias of the classical perspective based on $p = \infty$ since we can consider a more significant number of order statistics without introducing any bias. A similar change of norms approach was investigated in Drees *et al.* [17] and Davis *et al.* [10] for inference of the tail process. In this paper, we focus on inference of cluster processes and show consistency of the blocks estimators. Asymptotic normality of our estimators can be derived by combining arguments from Theorem 4.3 in Cissokho and Kulik [8] and the large deviation arguments developed below; this topic is the subject of ongoing work.

We apply our inference procedure to estimate the extremal index using extremal ℓ^α -blocks, with α equal to the index of regular variation of (\mathbf{X}_t) . We also consider inference of *cluster indices* as defined by Mikosch and Wintenberger [28] on partial sum functionals by considering extremal ℓ^1 -blocks. These functionals are shift-invariant with respect to the backward shift in sequence spaces; see Kulik and Soulier [23] for details. To define extremal ℓ^p -blocks for $p \leq \alpha$, we require a so-called vanishing-small-values condition too; see Section 5. Another advantage of this consideration is that, using a simple continuity argument, we can also study α th-power sum functionals acting on ℓ^p . Then, coupled with the random shift analysis of Janssen [22] based on the α th moment of the cluster process, we also extend cluster inference to functionals acting on ℓ^p rather than on shift-invariant spaces. We heavily rely on Theorem 2.1 to provide the main argument for comparing inference methods as we build therein on the definition of cluster processes from the new large deviation principles.

1.1. Outline of the paper. Section 2 states the main large deviation principle in Theorem 2.1 after introducing the preliminaries on regular variation and fixing the notation. In Section 3 we study the cluster processes in the space ℓ^p introduced in Theorem 2.1. In Section 4 we apply Theorem 2.1 to inference for shift-invariant functionals acting on these cluster processes through Theorem 4.1, choosing thresholds as empirical quantiles of the ℓ^p -norms of blocks. In an example, we illustrate our approach for $p < \infty$ and

compare it with the setting $p = \infty$. This is done at the end of Section 4. In Section 5 we supplement the discussion on Theorem 2.1 and analyze in depth its assumptions. In Section 6 we provide inference for non-shift-invariant functionals. We defer all proofs to Section 7.

2. PRELIMINARIES AND MAIN RESULT

2.1. About regular variation of time series. We consider an \mathbb{R}^d -valued stationary process (\mathbf{X}_t) . Following Davis and Hsing [11], we call it regularly varying if the finite-dimensional distributions of the process are regularly varying. This notion involves the vague convergence of certain tail measures; see Resnick [33]. Avoiding the concept of vague convergence and infinite limit measures, Basrak and Segers [4] showed that regular variation of (\mathbf{X}_t) is equivalent to the weak convergence relations: for any $h \geq 0$,

$$\mathbb{P}(x^{-1}(\mathbf{X}_t)_{|t| \leq h} \in \cdot \mid |\mathbf{X}_0| > x) \xrightarrow{w} \mathbb{P}(Y(\Theta_t)_{|t| \leq h} \in \cdot), \quad x \rightarrow \infty, \quad (2.1)$$

where Y is Pareto(α)-distributed, i.e., it has tail $\mathbb{P}(Y > y) = y^{-\alpha}$, $y > 1$, independent of the vector $(\Theta_t)_{|t| \leq h}$ and $|\Theta_0| = 1$. According to Kolmogorov's consistency theorem, one can extend the latter finite-dimensional vectors to a sequence $\Theta = (\Theta_t)_{t \in \mathbb{Z}}$ in $(\mathbb{R}^d)^{\mathbb{Z}}$ called the *spectral tail process* of (\mathbf{X}_t) . The regular variation property, say \mathbf{RV}_α , of (\mathbf{X}_t) is determined by the (tail)-index $\alpha > 0$ and the spectral tail process.

Extending vague convergence to M_0 -convergence, Hult and Lindskog [21] introduced regular variation for random elements assuming values in a general complete separable metric space; see also Lindskog *et al.* [26]. Segers *et al.* [34] proved regular variation of random elements with values in star-shaped metric spaces. Their results are based on weak convergence in the spirit of (2.1). Our focus will be on a special star-shaped space: the sequence space ℓ^p , $p \in (0, \infty]$ equipped with the metric

$$d_p(\mathbf{x}, \mathbf{y}) := \begin{cases} \|\mathbf{x} - \mathbf{y}\|_p = \left(\sum_{t \in \mathbb{Z}} |\mathbf{x}_t - \mathbf{y}_t|^p \right)^{1/p}, & p \geq 1, \\ \|\mathbf{x} - \mathbf{y}\|_p^p, & p \in (0, 1), \end{cases} \quad \mathbf{x}, \mathbf{y} \in \ell^p,$$

with the usual convention in the case $p = \infty$. We know that d_p makes ℓ^p a separable Banach space for $p \geq 1$, and a separable complete metric space for $p \in (0, 1)$. Using the p -modulus function $\|\cdot\|_p$, the ℓ^p -valued stationary process (\mathbf{X}_t) has the property \mathbf{RV}_α if and only if relation (2.1) holds with $|\mathbf{X}_0|$ replaced by $\|\mathbf{X}_{[0,h]}\|_p$. Equivalently, (Proposition 3.1 in Segers *et al.* [34]), for any $h \geq 0$,

$$\mathbb{P}(x^{-1}\mathbf{X}_{[0,h]} \in \cdot \mid \|\mathbf{X}_{[0,h]}\|_p > x) \xrightarrow{w} \mathbb{P}(Y \mathbf{Q}^{(p)}(h) \in \cdot), \quad x \rightarrow \infty, \quad (2.2)$$

where for $a \leq b$, $\mathbf{X}_{[a,b]} = (\mathbf{X}_a, \dots, \mathbf{X}_b)$, and the Pareto(α) variable Y is independent of $\mathbf{Q}^{(p)}(h) \in \mathbb{R}^{d(h+1)}$, and $\|\mathbf{Q}^{(p)}(h)\|_p = 1$ a.s. We call $\mathbf{Q}^{(p)}(h)$ the *spectral component* of $\mathbf{X}_{[0,h]}$.

2.2. Notation. For integers i and $a < b$ we write $i + [a, b] = \{i + a, \dots, i + b\}$. It is convenient to embed the vectors $\mathbf{x}_{[a,b]} \in \mathbb{R}^{d(b-a+1)}$ in $(\mathbb{R}^d)^\mathbb{Z}$ by assigning zeros to indices $i \notin [a, b]$, and we then also write $\mathbf{x}_{[a,b]} \in (\mathbb{R}^d)^\mathbb{Z}$. We write $\mathbf{x} := (\mathbf{x}_t) = (\mathbf{x}_t)_{t \in \mathbb{Z}}$, and define truncation at level $\varepsilon > 0$ from above and below by $\underline{\mathbf{x}}_\varepsilon = (\underline{\mathbf{x}}_{t_\varepsilon})_{t \in \mathbb{Z}}$, $\bar{\mathbf{x}}^\varepsilon = (\bar{\mathbf{x}}_{t^\varepsilon})_{t \in \mathbb{Z}}$, where $\underline{\mathbf{x}}_{t_\varepsilon} = \mathbf{x}_t \mathbb{1}(|\mathbf{x}_t| > \varepsilon)$, $\bar{\mathbf{x}}_{t^\varepsilon} = \mathbf{x}_t \mathbb{1}(|\mathbf{x}_t| \leq \varepsilon)$.

Recall the *backshift operator* acting on $\mathbf{x} \in (\mathbb{R}^d)^\mathbb{Z}$: $B^k \mathbf{x} = (\mathbf{x}_{t-k})_{t \in \mathbb{Z}}$, $k \in \mathbb{Z}$. Let $\tilde{\ell}^p = \ell^p / \sim$ be the quotient space with respect to the equivalence relation \sim in ℓ^p : $\mathbf{x} \sim \mathbf{y}$ holds if there exists $k \in \mathbb{Z}$ such that $B^k \mathbf{x} = \mathbf{y}$. An element of $\tilde{\ell}^p$ is denoted by $[\mathbf{x}] = \{B^k \mathbf{x} : k \in \mathbb{Z}\}$. For ease of notation, we often write \mathbf{x} instead of $[\mathbf{x}]$, and we notice that any element in ℓ^p can be embedded in $\tilde{\ell}^p$ by using the equivalence relation. We define for $[\mathbf{x}], [\mathbf{y}] \in \tilde{\ell}^p$,

$$\tilde{d}_p([\mathbf{x}], [\mathbf{y}]) := \inf_{k \in \mathbb{Z}} \{d_p(B^k \mathbf{a}, \mathbf{b}) : \mathbf{a} \in [\mathbf{x}], \mathbf{b} \in [\mathbf{y}]\}.$$

For $p > 0$, \tilde{d}_p is a metric on $\tilde{\ell}^p$ and turns it into a complete metric space; see Basrak *et al.* [3].

2.3. Main result. We start by giving our main result on large deviations of the sequence $\mathbf{X}_{[0,n]}$, that we embed in the space $(\tilde{\ell}^p, \tilde{d}_p)$. The proof is postponed to Section 7.1.

Theorem 2.1. *Consider an \mathbb{R}^d -valued stationary time series (\mathbf{X}_t) satisfying \mathbf{RV}_α for some $\alpha > 0$. For a given $p > 0$, assume that there exist a sequence (x_n) such that $n \mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$ and some $\kappa > 0$ such that $n/x_n^{p \wedge (\alpha - \kappa)} \rightarrow 0$ as $n \rightarrow +\infty$. Furthermore, assume that for all $\delta > 0$,*

$$\mathbf{AC} : \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(\|\mathbf{X}_{[k,n]}\|_\infty > \delta x_n \mid |\mathbf{X}_0| > \delta x_n) = 0,$$

$$\mathbf{CS}_p : \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\|\overline{x_n^{-1} \mathbf{X}_{[1,n]}}^\varepsilon\|_p > \delta)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} = 0.$$

Then, there exists $c(p) > 0$ such that

$$(2.3) \quad \lim_{n \rightarrow +\infty} \frac{\mathbb{P}(\|\mathbf{X}_{[0,n]}\|_p > x_n)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} = c(p).$$

Moreover, $c(p) < \infty$ if $p \geq \alpha$, in particular, $c(\infty) \leq c(p) \leq c(\alpha) = 1$. If $c(p) < \infty$ there exists $\mathbf{Q}^{(p)} \in \ell^p$ such that $\|\mathbf{Q}^{(p)}\|_p = 1$ a.s. and

$$(2.4) \quad \mathbb{P}(x_n^{-1} \mathbf{X}_{[0,n]} \in \cdot \mid \|\mathbf{X}_{[0,n]}\|_p > x_n) \xrightarrow{w} \mathbb{P}(Y \mathbf{Q}^{(p)} \in \cdot), \quad n \rightarrow \infty,$$

in the space $(\tilde{\ell}^p, \tilde{d}_p)$ where Y is Pareto(α) distributed, independent of $\mathbf{Q}^{(p)}$.

Since $\|\mathbf{Q}^{(p)}\|_p = 1$ a.s., $\mathbf{Q}^{(p)}$ has interpretation as the spectral component of a regularly varying ℓ^p -valued random element. From (2.3) we infer that $c(p)$ is a non-decreasing function of p . Letting $p = \infty$, one recovers the classical definition of clusters (of exceedances) in (2.4). From Theorem 2.1

we also have $c(\alpha) = 1 < \infty$ which motivates the study of extremal ℓ^α -blocks and, for the purposes of statistical inference, suggests $\mathbf{Q}^{(\alpha)}$ as a potential competitor of $\mathbf{Q}^{(\infty)}$.

We refer to a relation of the type (2.3) as *large deviation probabilities* motivated by the following observation. Write $S_k^{(p)} = \sum_{t=1}^k |\mathbf{X}_t|^p$ for $k \geq 1$. Then $|\mathbf{X}|^p$ is regularly varying with index α/p . Relation (2.3) implies that

$$\begin{aligned} \mathbb{P}(\|\mathbf{X}_{[0,n]}\|_p > x_n) &= \mathbb{P}(S_n^{(p)} > x_n^p) \\ &\sim c(p) n \mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Thus the left-hand probability describes the rare event that the sum process $S_n^{(p)}$ exceeds the extreme threshold x_n^p .

Equation (2.4) extends the large deviation result for $\|\mathbf{X}_{[0,n]}\|_p$ in (2.3) to one for the process $\mathbf{X}_{[0,n]}$ in the sequence space ℓ^p . Motivated by inference for the spectral cluster process $\mathbf{Q}^{(p)}$, we establish (2.4) employing weak convergence in the spirit of the polar decomposition given in Segers *et al.* [34].

We work under the one-sided anti-clustering condition **AC** together with a telescoping sum argument to compensate for the classical two-sided condition (5.1) used in Kulik and Soulier [23]. Conditions similar to **CS_p** are standard when dealing with sum functionals acting on (\mathbf{X}_t) and hold for any time series satisfying **RV_α** for $p > \alpha$ by a Karamata-type argument. In Section 5 we further discuss the assumptions of Theorem 2.1.

Remark 2.2. Equation (2.4) provides a family of Borel sets in $(\tilde{\ell}^p, \tilde{d}_p)$ for which the weak limit of the self-normalized blocks $\mathbf{X}_{[0,n]}/\|\mathbf{X}_{[0,n]}\|_p$ exists. This result implies that the sequence of measures

$$\begin{aligned} \mu_n(\cdot) &:= \mathbb{P}(x_n^{-1} \mathbf{X}_{[0,n]} \in \cdot) / \mathbb{P}(\|\mathbf{X}_{[0,n]}\|_p > x_n) \\ \rightarrow \mu(\cdot) &:= \int_0^\infty \mathbb{P}(y \mathbf{Q}^{(p)} \in \cdot) d(-y^{-\alpha}), \quad n \rightarrow \infty, \end{aligned}$$

in the M_0 -sense in $(\tilde{\ell}^p, \tilde{d}_p)$. By the portmanteau theorem for measures (Theorem 2.4. in Hult and Lindskog [21])

$$\mu_n(A) = \mathbb{P}(x_n^{-1} \mathbf{X}_{[0,n]} \in A) / \mathbb{P}(\|\mathbf{X}_{[0,n]}\|_p > x_n) \rightarrow \mu(A),$$

for all Borel sets A in $(\tilde{\ell}^p, \tilde{d}_p)$ satisfying $\mu(\partial A) = 0$ and $\mathbf{0} \notin \bar{A}$. This approach is discussed in Kulik and Soulier [23] where similar conditions are stated for obtaining limit results for $\tilde{\ell}^1$ -functionals.

3. SPECTRAL CLUSTER PROCESS REPRESENTATION

3.1. The spectral cluster process in ℓ^p . From (2.1) recall the spectral tail process Θ of a stationary sequence (\mathbf{X}_t) satisfying **RV_α**. We start with a representation of the spectral cluster process $\mathbf{Q}^{(p)}$ from (2.4) in terms of Θ . The proof is deferred to Section 7.3.

Proposition 3.1. *For $p, \alpha > 0$ assume $\|\Theta\|_p + \|\Theta\|_\alpha < \infty$ a.s. Under the assumptions of Theorem 2.1, the distribution of the spectral cluster process $\mathbf{Q}^{(p)}$ in the space (ℓ^p, d_p) is given by*

$$(3.1) \quad \mathbb{P}(\mathbf{Q}^{(p)} \in \cdot) = c(p)^{-1} \mathbb{E}[\|\Theta/\|\Theta\|_\alpha\|_p^\alpha \mathbf{1}(\Theta/\|\Theta\|_p \in \cdot)],$$

where the constant $c(p)$ defined in (2.3) admits the representation

$$(3.2) \quad c(p) = \mathbb{E}[\|\Theta/\|\Theta\|_\alpha\|_p^\alpha].$$

This result provides a new representation of the distribution of $\mathbf{Q}^{(p)}$ for fixed p and relates distinct spectral cluster processes to each other by the change of norms transform in (3.1). In the next sections we consider the cases $p \in \{\alpha, \infty\}$ in detail.

3.1.1. *The spectral cluster process in ℓ^α .* In view of (3.1) the process $\Theta/\|\Theta\|_\alpha$ is the candidate for the ℓ^α -spectral cluster process $\mathbf{Q}^{(\alpha)}$ introduced in (2.4), and it plays a key role for characterising $\mathbf{Q}^{(p)}$ in general. The following result shows that $\Theta/\|\Theta\|_\alpha$ is well defined under **AC**.

Proposition 3.2. *Let (\mathbf{X}_t) be a stationary sequence satisfying \mathbf{RV}_α with spectral tail process (Θ_t) . Then the following statements are equivalent:*

- i) $\|\Theta\|_\alpha < \infty$ a.s. and $\Theta/\|\Theta\|_\alpha$ is well defined in ℓ^α .
- ii) $|\Theta_t| \rightarrow 0$ a.s. as $t \rightarrow \infty$.
- iii) The time of the largest record $T^* := \inf\{s : s \in \mathbb{Z} \text{ such that } |\Theta_s| = \sup_{t \in \mathbb{Z}} |\Theta_t|\}$ is finite a.s.

Moreover, these statements hold under **AC**.

A proof of Proposition 3.2 is given in Lemma 3.6 of Buriticá *et al.* [7], appealing to results by Janssen [22].

From (2.2) recall the sequence of spectral components $(\mathbf{Q}^{(\alpha)}(h))_{h \geq 0}$ of the vectors $(\mathbf{X}_{[0,h]})_{h \geq 0}$ with the property $\|\mathbf{Q}^{(\alpha)}(h)\|_\alpha = 1$ a.s. Our next result relates the sequence $(\mathbf{Q}^{(\alpha)}(h))_{h \geq 0}$ to $\Theta/\|\Theta\|_\alpha$, the candidate process for $\mathbf{Q}^{(\alpha)}$.

Proposition 3.3. *Let (\mathbf{X}_t) be a stationary time series satisfying \mathbf{RV}_α and $\lim_{t \rightarrow \infty} |\Theta_t| = 0$ a.s. Then $\mathbf{Q}^{(\alpha)}(h) \xrightarrow{d} \mathbf{Q}^{(\alpha)}(\infty)$ as $h \rightarrow \infty$ in $(\tilde{\ell}^\alpha, \tilde{d}_\alpha)$ with $\mathbf{Q}^{(\alpha)}(\infty) \stackrel{d}{=} \Theta/\|\Theta\|_\alpha$.*

This result gives raise to the interpretation of $\Theta/\|\Theta\|_\alpha$ as *the spectral component of (\mathbf{X}_t) in ℓ^α* . The proof is given in Section 7.3. Under the assumptions of Theorem 2.1 we also have the a.s. representations $\mathbf{Q}^{(\alpha)} = \Theta/\|\Theta\|_\alpha$ and $\Theta = \mathbf{Q}^{(\alpha)}/|\mathbf{Q}_0^{(\alpha)}|$.

3.1.2. *The spectral cluster process in ℓ^∞ .* Letting $p = \infty$ in Theorem 2.1, we retrieve the classical definition of the cluster (of exceedances) by embedding

$\mathbf{Q}^{(\infty)}$ from (2.4) in $(\tilde{\ell}^\infty, \tilde{d}_\infty)$. From Proposition 3.2, assuming **AC**, we can relate the cluster (of exceedances) to $\Theta/\|\Theta\|_\alpha$ using (3.1) by

$$(3.3) \quad \mathbb{P}(Y\mathbf{Q}^{(\infty)} \in \cdot) = \mathbb{P}(Y\Theta/\|\Theta\|_\infty \in \cdot \mid \|Y\Theta/\|\Theta\|_\alpha\|_\infty > 1),$$

where Y is a Pareto(α) random variable independent of $\mathbf{Q}^{(\infty)}$ and Θ . In particular, $\|\mathbf{Q}^{(\infty)}\|_\infty = 1$ a.s. and $\mathbb{P}(\|Y\Theta/\|\Theta\|_\alpha\|_\infty > 1) = \mathbb{E}[\|\Theta/\|\Theta\|_\alpha\|_\infty^\alpha] = c(\infty)$.

Assuming **AC** and additional mixing assumptions (see e.g. Theorem 2.3. in [7]), the extremal index $\theta_{|\mathbf{X}|}$ of $(|\mathbf{X}_t|)$ exists and equals $c(\infty)$. Under **AC**, Basrak and Segers [4] proved that $|\Theta_t| \rightarrow 0$ as $|t| \rightarrow \infty$ and $c(\infty)$ is given by

$$(3.4) \quad c(\infty) = \mathbb{E}[\|(\Theta_t)_{t \geq 0}\|_\infty^\alpha - \|(\Theta_t)_{t \geq 1}\|_\infty^\alpha].$$

Following Planinić and Soulier [32], the spectral tail process (Θ_t) satisfies the *time-change formula*: for any measurable bounded function $f : (\tilde{\ell}^p, \tilde{d}_p) \rightarrow \mathbb{R}$ such that $f(\lambda \mathbf{x}) = f(\mathbf{x})$ for all $\lambda > 0$, we have for all $t, s \in \mathbb{Z}$,

$$(3.5) \quad \mathbb{E}[f(B^s(\Theta_t))\mathbb{1}(\Theta_{-s} \neq \mathbf{0})] = \mathbb{E}[|\Theta_s|^\alpha f((\Theta_t))].$$

An application of this formula and a telescoping sum argument yield

$$(3.6) \quad \begin{aligned} c(p) &= \mathbb{E}[\|\Theta\|_p^\alpha / \|\Theta\|_\alpha^\alpha] \\ &= \sum_{s \in \mathbb{Z}} \mathbb{E}[\|(\Theta_t)_{t \geq -s}\|_p^\alpha / \|\Theta\|_\alpha^\alpha - \|(\Theta_t)_{t \geq -s+1}\|_p^\alpha / \|\Theta\|_\alpha^\alpha] \\ &= \sum_{s \in \mathbb{Z}} \mathbb{E}[|\Theta_s|^\alpha (\|(\Theta_t)_{t \geq 0}\|_p^\alpha / \|\Theta\|_\alpha^\alpha - \|(\Theta_t)_{t \geq 1}\|_p^\alpha / \|\Theta\|_\alpha^\alpha)] \\ &= \mathbb{E}[\|\Theta\|_\alpha^\alpha (\|(\Theta_t)_{t \geq 0}\|_p^\alpha / \|\Theta\|_\alpha^\alpha - \|(\Theta_t)_{t \geq 1}\|_p^\alpha / \|\Theta\|_\alpha^\alpha)] \\ &= \mathbb{E}[\|(\Theta_t)_{t \geq 0}\|_p^\alpha - \|(\Theta_t)_{t \geq 1}\|_p^\alpha]. \end{aligned}$$

Thus we deduce the representation in (3.4) for $c(\infty)$ by letting $p = \infty$ in (3.6).

Furthermore the representation in (3.6) extends from $p = \infty$ to $0 < p \leq \infty$ under **AC**. We notice that under **AC** it is in general not obvious whether $c(p)$ is finite or not for $0 < p < \alpha$. A Taylor expansion shows that (3.6) is finite if $\mathbb{E}[\|(\Theta_t)_{t \geq 0}\|_p^{\alpha-p}] < \infty$ and thus $c(p) < \infty$, which is a necessary condition for defining $\mathbf{Q}^{(p)}$ as in (2.4).

4. CONSISTENT CLUSTER INFERENCE BASED ON SPECTRAL CLUSTER PROCESSES

Let $\mathbf{X}_1, \dots, \mathbf{X}_n$ be a sample from a stationary sequence (\mathbf{X}_t) satisfying **RV** $_\alpha$ for some $\alpha > 0$ and choose $p > 0$. We split the sample into disjoint blocks $\mathbf{B}_t := \mathbf{X}_{(t-1)b+1:b}$, $t = 1, \dots, m_n$, where $b = b_n \rightarrow \infty$ and $m = m_n := \lfloor n/b_n \rfloor \rightarrow \infty$. Throughout we assume the conditions of Theorem 2.1

for p . Then, in particular, $\mathbb{P}(\|\mathbf{B}_1\|_p > x_b) \rightarrow 0$ and $k = k_n := \lceil m_n \mathbb{P}(\|\mathbf{B}\|_p > x_{b_n}) \rceil \rightarrow \infty$.

The key message from Proposition 3.1 is that the same quantity, say e.g., $c(q)$, for $q > 0$, can be obtained from letting different functionals act on $\mathbf{Q}^{(p)}$ for different p , in particular for $p < \infty$. In this context, we want to compare the inference procedures of $p < \infty$ and $p = \infty$, and promote the use of the order statistics of $\|\mathbf{B}_t\|_p$, $t = 1, \dots, m_n$, i.e., $\|\mathbf{B}\|_{p,(1)} \geq \|\mathbf{B}\|_{p,(2)} \geq \dots \geq \|\mathbf{B}\|_{p,(m)}$.

4.1. Cluster functionals and mixing. The real-valued function g on $\tilde{\ell}^p$ is a *cluster functional* if it vanishes in some neighborhood of the origin and $\mathbb{P}(Y\mathbf{Q}^{(p)} \in D(g)) = 0$ where $D(g)$ denotes the set of discontinuity points of g . In what follows, it will be convenient to write $\mathcal{G}_+(\tilde{\ell}^p)$ for the class of non-negative functions on $\tilde{\ell}^p$ which vanish in some neighborhood of the origin.

For asymptotic theory we will need the following *mixing condition*.

Condition \mathbf{MX}_p . There exists an integer sequence $b_n \rightarrow \infty$ such that $m_n \rightarrow \infty$, and for any Lipschitz-continuous $f \in \mathcal{G}_+(\tilde{\ell}^p)$, the sequence (x_n) satisfies

$$\mathbb{E}\left[e^{-\frac{1}{k} \sum_{t=1}^m f(x_b^{-1} \mathbf{B}_t)}\right] = \left(\mathbb{E}\left[e^{-\frac{1}{k} \sum_{t=1}^{\lfloor m/k \rfloor} f(x_b^{-1} \mathbf{B}_t)}\right]\right)^k + o(1), \quad n \rightarrow \infty. \quad (4.1)$$

If \mathbf{MX}_p is required in the sequel we will refer to the sequences (b_n) , (m_n) and (k_n) chosen in this condition. \mathbf{MX}_p is similar to the mixing conditions \mathcal{A} , \mathcal{A}' in Davis and Hsing [11], Basrak *et al.* [2], respectively. These are defined in terms of sequences $(f(\mathbf{X}_t))$ while our functionals f act on blocks. \mathbf{MX}_p holds under mild conditions, for example, under strong mixing with quite general mixing rate; cf. Lemma 6.2. in Basrak *et al.* [3].

4.2. Consistent cluster inference. The following result is the basis for an empirical procedure for spectral cluster inference built on disjoint blocks. The proof is given in Section 7.4.1.

Theorem 4.1. *Assume the conditions of Theorem 2.1 hold for $p > 0$ with $c(p) < \infty$ together with \mathbf{MX}_p . Then $\|\mathbf{B}\|_{p,(k)}/x_b \xrightarrow{\mathbb{P}} 1$ and for all $g \in \mathcal{G}_+(\tilde{\ell}^p)$,*

$$(4.2) \quad \frac{1}{k} \sum_{t=1}^m g(\|\mathbf{B}\|_{p,(k)}^{-1} \mathbf{B}_t) \xrightarrow{\mathbb{P}} \int_0^\infty \mathbb{E}[g(y\mathbf{Q}^{(p)})] d(-y^{-\alpha}), \quad n \rightarrow \infty.$$

The sequence (k_n) in (4.2) depends on p , but the choice of p is flexible due to Proposition 3.1 relating spectral cluster processes by a change of norms functional. The constant at the right-hand side of (4.2) can be written in different ways by combining the choice of p with the choice of the functional g . The choice of p, g are left to the practitioner. Theorem 4.1 points to the following two advantages.

Notice Theorem 4.1 promotes the use of order statistics of the sample of ℓ^p -norms where thresholds adapt automatically to block lengths. This allows us to exploit the link between the block length and the threshold through the sequence (k_n) that must satisfy

$$(4.3) \quad k_n = [m_n \mathbb{P}(\|\mathbf{B}\|_p > x_{b_n})] \sim c(p) n \mathbb{P}(|\mathbf{X}_0| > x_{b_n}) = o(n/b_n^{(\alpha/p)\vee 1}), \\ n \rightarrow +\infty.$$

The asymptotic equivalence follows from the restriction on the thresholds in Theorem 2.1 where for $p < \alpha$, $n/x_n^p \rightarrow 0$, and for $p = \alpha$, $n/x_n^{\alpha-\kappa} \rightarrow 0$ for some $\kappa > 0$.

Also from (4.2) we see that k_n corresponds to the total number of extreme blocks used for inference. Then, for inference through $\mathbf{Q}^{(p)}$, equation (4.3) justifies taking k_n larger if $p < \infty$ than if $p = \infty$ since $c(\cdot)$ is a non-increasing function of p .

In the next section we illustrate these two points.

4.3. Applications. In this section we apply Theorem 4.1 to inference on some indices related to the extremes in a time-dependent sample and focus on cluster inference using $\mathbf{Q}^{(\alpha)}$ and $\mathbf{Q}^{(1)}$.

4.3.1. The extremal index. The extremal index of a regularly varying stationary time series has interpretation as a measure of clustering of serial exceedances, and was originally introduced in Leadbetter [24] and Leadbetter *et al.* [25]. If (\mathbf{X}'_t) is iid with the same marginal distribution as (\mathbf{X}_t) then the extremal index $\theta_{|\mathbf{X}|}$ relates the expected number of serial exceedances of $(|\mathbf{X}_t|)$ with the serial exceedances of $(|\mathbf{X}'_t|)$.

We aim to apply Theorem 4.1 with $p = \alpha$. In this setting, the change of norms formula in (3.2) and the discussion in Section 3.1.2 lead to the identities

$$\theta_{|\mathbf{X}|} = c(\infty) = \mathbb{E} \left[\frac{\|\Theta\|_\infty^\alpha}{\|\Theta\|_\alpha^\alpha} \right] = \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_\infty^\alpha].$$

Then letting $p = \alpha$ and $g(\mathbf{x}) = (\|\mathbf{x}\|_\infty^\alpha / \|\mathbf{x}\|_\alpha^\alpha) \mathbf{1}(\|\mathbf{x}\|_\alpha > 1)$ on the right-hand side of (4.2), we obtain

$$\int_0^\infty \mathbb{E}[g(y\mathbf{Q}^{(\alpha)})] d(-y^{-\alpha}) = \int_0^\infty \mathbb{E} \left[\frac{\|\mathbf{Q}^{(\alpha)}\|_\infty^\alpha}{\|\mathbf{Q}^{(\alpha)}\|_\alpha^\alpha} \mathbf{1}(\|\mathbf{Q}^{(\alpha)}\|_\alpha^\alpha > y^{-\alpha}) \right] d(-y^{-\alpha}) \\ = \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_\infty^\alpha] = c(\infty).$$

Next we introduce a new consistent disjoint blocks estimator of the extremal index defined from exceedances of ℓ^α -norm blocks.

Corollary 4.2. *Assume the conditions of Theorem 4.1 for $p = \alpha$. Then*

$$(4.4) \quad \frac{1}{k} \sum_{t=1}^m \frac{\|\mathbf{B}_t\|_\infty^\alpha}{\|\mathbf{B}_t\|_\alpha^\alpha} \mathbf{1}(\|\mathbf{B}_t\|_\alpha > \|\mathbf{B}\|_{\alpha,(k)}) \xrightarrow{\mathbb{P}} c(\infty), \quad n \rightarrow \infty.$$

To motivate this estimator of $c(\infty)$ we compare it to one based on the cluster (of exceedances) following a more classical approach. Motivated by the blocks estimator in Hsing [20] for the extremal index, we let $g(\mathbf{x}) := \sum_{j \in \mathbb{Z}} \mathbb{1}(|\mathbf{x}_t| > 1)$ act on large ℓ^∞ -blocks. Choosing $p = \infty$ and fixing g on the right-hand side of (4.2), we can find an integer sequence $k' = k'_n \rightarrow \infty$ such that

$$(4.5) \quad \left(\frac{1}{k'} \sum_{t=1}^n \mathbb{1}(|\mathbf{X}_t| \geq \|\mathbf{B}\|_{\infty, (k')}) \right)^{-1} \xrightarrow{\mathbb{P}} c(\infty), \quad n \rightarrow \infty.$$

Arguing as for (4.3), we obtain $k'_n \sim m_n \mathbb{P}(\|\mathbf{B}\|_\infty > x_b) \sim c(\infty) n \mathbb{P}(|\mathbf{X}_0| > x_b) \sim c(\infty) m_n \mathbb{P}(\|\mathbf{B}\|_\alpha > x_b) \sim c(\infty) k_n$. Thus, the proportion of extreme blocks used in (4.5) is as small as the extremal index itself, compared with the number of extreme blocks used in (4.4).

This example justifies inference of the extremal index based on extremal ℓ^α -blocks as we can consider a larger number of order statistics than for extremal ℓ^∞ -blocks without introducing a bias term.

4.3.2. A cluster index for sums. In this section we assume $\alpha \in (0, 2)$ and $\mathbb{E}[\mathbf{X}] = \mathbf{0}$ for $\alpha \in (1, 2)$. We study the partial sums $\mathbf{S}_n := \sum_{t=1}^n \mathbf{X}_t$, $n \geq 1$, and introduce a normalizing sequence (a_n) such that $n \mathbb{P}(|\mathbf{X}_0| > a_n) \rightarrow 1$. Starting with Davis and Hsing [11], α -stable central limit theory for (\mathbf{S}_n/a_n) was proved under suitable anti-clustering and mixing conditions.

In this setting, the quantity $c(1)$ appears naturally and was coined cluster index in Mikosch and Wintenberger [28]. For $d = 1$ it can be interpreted as an equivalent of the extremal index for partial sums rather than maxima. Indeed, consider a real-valued regularly varying stationary sequence (X_t) with index of regular variation $\alpha \in (0, 2)$ satisfying $\mathbb{P}(X \leq -x) = o(\mathbb{P}(X > x))$ or $X \stackrel{d}{=} -X$. Consider an iid sequence (X'_t) with $X \stackrel{d}{=} X'$ and partial sums (S'_n) . Then $a_n^{-1} S_n \xrightarrow{d} \xi_\alpha$ and $a_n^{-1} S'_n \xrightarrow{d} \xi'_\alpha$, both ξ_α and ξ'_α are α -stable and

$$\mathbb{E}[e^{iu\xi_\alpha}] = (\mathbb{E}[e^{iu\xi'_\alpha}])^{c(1)}.$$

Letting $p = 1$ and $g(\mathbf{x}) = (\|\mathbf{x}\|_\alpha^\alpha / \|\mathbf{x}\|_1^\alpha) \mathbb{1}(\|\mathbf{x}\|_1 > 1)$ on the right-hand side of (4.2), we obtain

$$\begin{aligned} \int_0^\infty \mathbb{E}[g(y\mathbf{Q}^{(1)})] d(-y^{-\alpha}) &= \int_0^\infty \mathbb{E}\left[\frac{\|\mathbf{Q}^{(1)}\|_\alpha^\alpha}{\|\mathbf{Q}^{(1)}\|_1^\alpha} \mathbb{1}(\|\mathbf{Q}^{(1)}\|_1^\alpha > y^{-\alpha})\right] d(-y^{-\alpha}) \\ &= \mathbb{E}[\|\mathbf{Q}^{(1)}\|_\alpha^\alpha] = (c(1))^{-1}, \end{aligned}$$

where the last identity follows from Proposition 3.1. Then Theorem 4.1 for $p = 1$ and g as mentioned yields a consistent estimator of $c(1)$.

Corollary 4.3. *Assume the conditions of Theorem 4.1 for $p = 1$. Then*

$$(4.6) \quad \left(\frac{1}{k} \sum_{t=1}^m \frac{\|\mathbf{B}_t\|_\alpha^\alpha}{\|\mathbf{B}_t\|_1^\alpha} \mathbb{1}(\|\mathbf{B}_t\|_1 > \|\mathbf{B}\|_{1, (k)}) \right)^{-1} \xrightarrow{\mathbb{P}} c(1), \quad n \rightarrow \infty.$$

As we have argued above, such an estimator is appealing since it is based on large ℓ^1 -norms of blocks instead of ℓ^∞ -norms. Moreover, arguing as in Cissokho and Kulik [8], Kulik and Soulier [23], we can extend Theorem 4.1 for $p = \infty$ to hold for ℓ^1 -functionals. Then, for $g(\mathbf{x}) := \mathbb{1}(\|\mathbf{x}\|_1 > 1)$ and $p = \infty$ on the right-hand side of (4.2), we find $k' = k'_n \rightarrow \infty$ such that

$$(4.7) \quad \frac{\sum_{t=1}^m \mathbb{1}(\|\mathbf{B}_t\|_1 > \|\mathbf{B}\|_{\infty, (k')})}{\sum_{t=1}^n \mathbb{1}(\|\mathbf{X}_t\| > \|\mathbf{B}\|_{\infty, (k')})} \xrightarrow{\mathbb{P}} c(1), \quad n \rightarrow \infty,$$

where as in (4.3), $k'_n \sim c(\infty) n \mathbb{P}(\|\mathbf{X}_0\| > x_b) \sim c(\infty)(c(1))^{-1} m_n \mathbb{P}(\|\mathbf{B}\|_1 > x_b) \sim c(\infty)(c(1))^{-1} k_n$. In particular, for $\alpha \in (1, 2)$, $c(1) \geq 1$, thus a small proportion (even smaller than $c(\infty)$) of blocks should be used to infer on $c(1)$. For $\alpha \in (0, 1)$ we can argue similarly and improve inference by proposing an estimator based on ℓ^α -norm order statistics.

As in the extremal index example, we promote inference based on extremal ℓ^p -blocks with $p < \infty$ for a better control of the bias compared to inference with extremal ℓ^∞ -blocks.

Furthermore, Theorem 4.1 yields estimators of the parameters of the α -stable limit for (\mathbf{S}_n/a_n) denoted $\boldsymbol{\xi}_\alpha$. Indeed, following the theory in Bartkiewicz *et al.* [1], we characterize the α -stable limit in terms of $\mathbf{Q}^{(1)}$; the proof is given Section 7.4.2.

Proposition 4.4. *Assume that (\mathbf{X}_t) is a regularly varying stationary sequence with $\alpha \in (0, 1) \cup (1, 2)$ together with the mixing condition:*

$$\mathbb{E}[e^{i\mathbf{u}^\top \mathbf{S}_n/a_n}] = (\mathbb{E}[e^{i\mathbf{u}^\top \mathbf{S}_{b_n}/a_n}])^{m_n} + o(1), \quad n \rightarrow \infty, \quad \mathbf{u} \in \mathbb{R}^d,$$

and the anti-clustering condition: for every $\delta > 0$,

$$(4.8) \quad \lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} n \sum_{t=l}^{b_n} \mathbb{E}[(\|\mathbf{X}_t/a_n\| \wedge \delta) (\|\mathbf{X}_0/a_n\| \wedge \delta)] = 0.$$

Then $\mathbf{S}_n/a_n \xrightarrow{d} \boldsymbol{\xi}_\alpha$ for an α -stable random vector $\boldsymbol{\xi}_\alpha$ with characteristic function $\mathbb{E}[\exp(i\mathbf{u}^\top \boldsymbol{\xi}_\alpha)] = \exp(-c_\alpha \sigma_\alpha(\mathbf{u}) (1 - i\beta(\mathbf{u}) \tan(\alpha\pi/2)))$, $\mathbf{u} \in \mathbb{R}^d$, where $c_\alpha := (\Gamma(2-\alpha)/|1-\alpha|)(1 \wedge \alpha) \cos(\alpha\pi/2)$, and the scale and skewness parameters have representation

$$\begin{aligned} \sigma_\alpha(\mathbf{u}) &:= c(1) \mathbb{E}[|\mathbf{u}^\top \sum_{t \in \mathbb{Z}} \mathbf{Q}_t^{(1)}|^\alpha], \\ \beta(\mathbf{u}) &:= (\mathbb{E}[(\mathbf{u}^\top \sum_{t \in \mathbb{Z}} \mathbf{Q}_t^{(1)})_+^\alpha] - (\mathbf{u}^\top \sum_{t \in \mathbb{Z}} \mathbf{Q}_t^{(1)})_-^\alpha] / \mathbb{E}[|\mathbf{u}^\top \sum_{t \in \mathbb{Z}} \mathbf{Q}_t^{(1)}|^\alpha]). \end{aligned}$$

As for $c(1)$, an application of Theorem 4.1 with $p = 1$ for $\alpha \in (1, 2)$ and $p = \alpha$ for $\alpha \in (0, 1)$ yields natural estimators of the parameters $(\sigma_\alpha(\mathbf{u}), \beta(\mathbf{u}))$ in the central limit theorem of Proposition 4.4

4.3.3. An example: a regularly varying linear process. We illustrate the index estimators of Corollaries 4.2 and 4.3 for a regularly varying linear process $X_t := \sum_{j \in \mathbb{Z}} \varphi_j Z_{t-j}$, $t \in \mathbb{Z}$, where (Z_t) is an iid real-valued regularly varying sequence (Z_t) with (tail)-index $\alpha > 0$, and (φ_j) are real coefficients such that $\sum_{j \in \mathbb{Z}} |\varphi_j|^{1 \wedge (\alpha - \varepsilon)} < \infty$ for some $\varepsilon > 0$.

In this setting, (X_t) is regularly varying with the same (tail)-index $\alpha > 0$, and the distributions of Z_t and X_t are tail-equivalent; see Davis and Resnick [12]. The spectral cluster process of (X_t) is given by $Q_t^{(\alpha)} = (\varphi_{t+J}/\|(\varphi_t)\|_\alpha) \Theta_0^Z$, $t \in \mathbb{Z}$, where $\lim_{x \rightarrow \infty} \mathbb{P}(\pm Z_0 > x)/\mathbb{P}(|Z_0| > x) = \mathbb{P}(\Theta_0^Z = \pm 1)$, Θ_0^Z is independent of a random shift J with distribution $\mathbb{P}(J = j) = |\varphi_j|^\alpha / \|(\varphi_t)\|_\alpha^\alpha$; see Kulik and Soulier [23], (15.3.9). Then

$$c(\infty) = \max_{t \in \mathbb{Z}} |\varphi_t|^\alpha / \|\varphi\|_\alpha^\alpha, \quad c(1) = \left(\sum_{t \in \mathbb{Z}} |\varphi_t| \right)^\alpha / \|\varphi\|_\alpha^\alpha.$$

For the causal AR(1) model given by $X_t = \varphi X_{t-1} + Z_t$, $t \in \mathbb{Z}$, $|\varphi| < 1$, one retrieves $\theta_{|X|} = c(\infty) = 1 - |\varphi|^\alpha$ and $c(1) = (1 - |\varphi|^\alpha)/(1 - |\varphi|)^\alpha$.

We aim at comparing estimators of $\theta_{|X|}$ and $c(1)$ built on extremal ℓ^p -blocks, $p < \infty$, against inference over ℓ^∞ -blocks, for the causal AR(1) model with student(α) noise for $\alpha = 1.3$. Guided by (4.3), we take $k = k_n := \max\{2, \lfloor n/b_n^{(1+\kappa)} \rfloor\}$ with $\kappa = 1$. For estimation of α , we follow the bias-correction procedure in de Haan *et al.* [19]. This estimator is plugged into (4.4), (4.6), resulting in the estimators $\hat{\theta}_{|X|}$, $1/\hat{c}(1)$, based on ℓ^α - and ℓ^1 -extremal blocks, respectively. Figures 4.5 and 4.6 present boxplots of these estimators as a function of b_n and for different sample sizes n in blue. For comparison, we also show boxplots of the estimators in (4.5) and (4.7) based on extremal ℓ^∞ -blocks in white. Estimation based on ℓ^p -block for $p < \infty$ outperforms in simulation the ℓ^∞ -blocks approach in terms of bias for fixed k . As expected from (4.3), we should use a smaller k for ℓ^∞ -block based inference to reduce the bias. Also, notice the bias for large block lengths decreases as n increases. Indeed, if we fix n , the relation $\lfloor n/b^{(1+\kappa)} \rfloor \rightarrow 0$ as $b \rightarrow \infty$ restricts the block length for small sample sizes. We have fixed the tuning parameter $\kappa = 1$, though improvement can be achieved by fine-tuning $\kappa > 0$. We also refer to Buriticá *et al.* [7] for further simulation experiences showing that the estimator of the extremal index in (4.4) compares favorably with various classical estimators as regards bias.

5. A DISCUSSION OF THE ASSUMPTIONS OF THE LARGE DEVIATION PRINCIPLE IN THEOREM 2.1

Consider a stationary sequence (\mathbf{X}_t) satisfying \mathbf{RV}_α and let (x_n) be a threshold sequences such that $n \mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$. In the conditions **AC** and **CS_p** below we refer to the same sequence (x_n) . We will discuss the conditions of Theorem 2.1.

5.1. Anti-clustering condition AC. For any $\delta > 0$,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(\|\mathbf{X}_{[k,n]}\|_\infty > \delta x_n \mid |\mathbf{X}_0| > \delta x_n) = 0.$$

Condition **AC** ensures that a large value at present time does not persist indefinitely in the extreme future of the time series. This anti-clustering is weaker than the more common two-sided one:

$$(5.1) \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\max_{k \leq |t| \leq n} |\mathbf{X}_t| > \delta x_n \mid |\mathbf{X}_0| > \delta x_n \right) = 0.$$

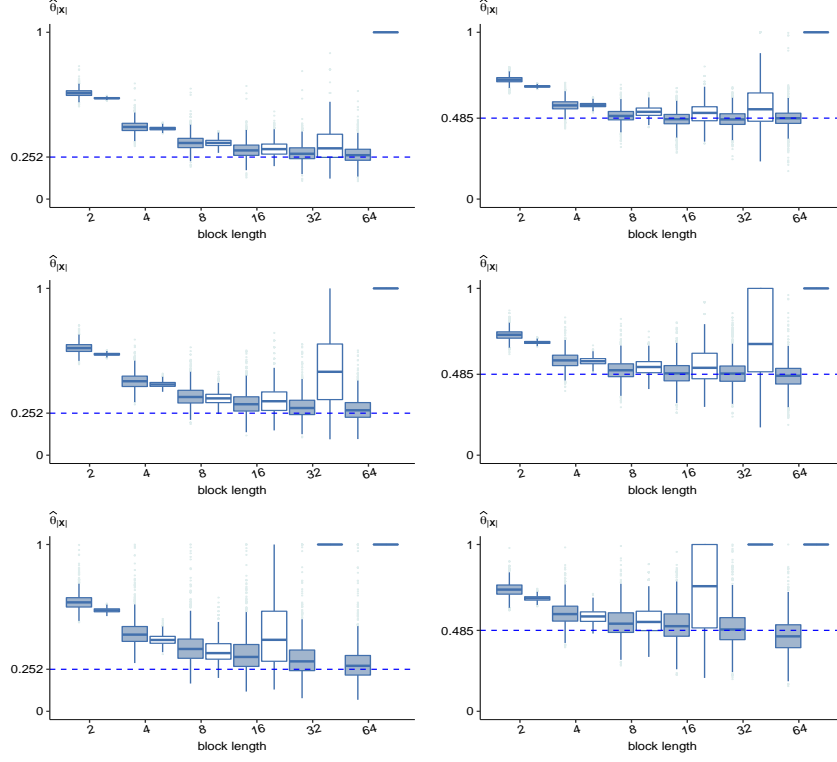


Figure 4.5. Boxplot of estimates $\hat{\theta}_{|X|}$ as a function of b_n from (4.4) for inference through $\mathbf{Q}^{(\alpha)}$ (in blue) and from (4.5) through $\mathbf{Q}^{(\infty)}$ (in white). 1 000 simulated samples $(X_t)_{t=1,\dots,n}$ from a causal AR(1) model with student(α) noise with $\alpha = 1.3$ and $\varphi = 0.8$ (left column), $\varphi = 0.6$ (right column) were considered. Rows correspond to results for $n = 8\,000$, $3\,000$, $1\,000$ from top to bottom.

A simple sufficient condition, which breaks block-wise extremal dependence into pair-wise, is given by

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{t=k}^n \mathbb{P}(|\mathbf{X}_t| > \delta x_n \mid |\mathbf{X}_0| > \delta x_n).$$

For m -dependent (\mathbf{X}_t) the latter condition turns into $n \mathbb{P}(|\mathbf{X}_0| > \delta x_n) \rightarrow 0$ which is always satisfied.

If $p \leq \alpha$ an extra assumption is required for controlling the accumulation of moderate extremes within a block.

5.2. Vanishing-small-values condition \mathbf{CS}_p . For $p \in (0, \alpha]$ we assume that for a sequence (x_n) satisfying $n \mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$ and for any $\delta > 0$,

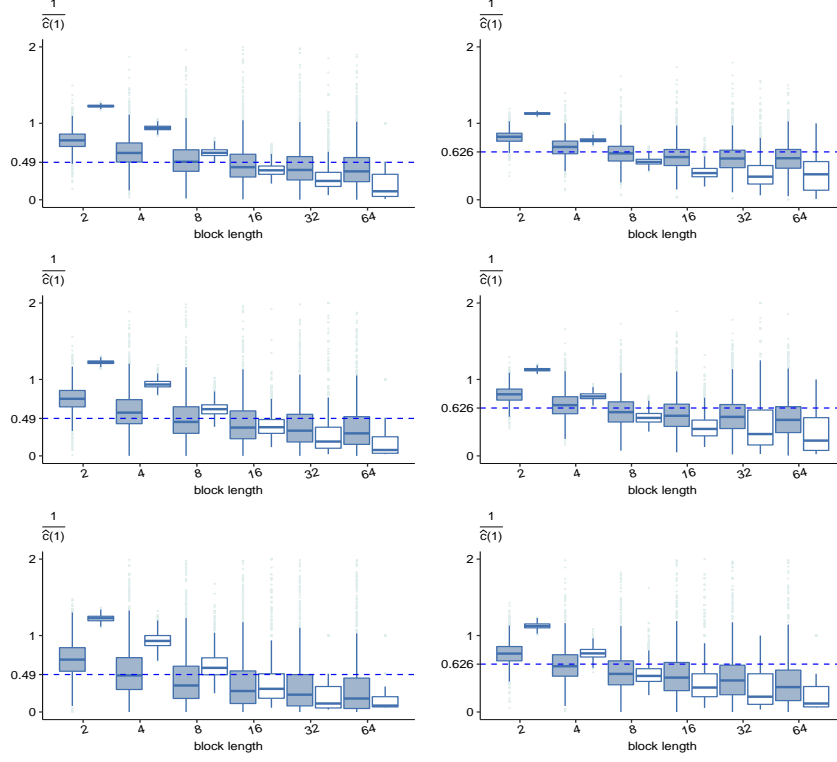


Figure 4.6. Boxplot of estimates $1/\widehat{c}(1)$ as a function of b_n from (4.6) for inference through $\mathbf{Q}^{(1)}$ (in blue) and from (4.7) for inference through $\mathbf{Q}^{(\infty)}$ (in white). We simulate 1000 samples $(X_t)_{t=1,\dots,n}$ from an AR(1) model with student(α) noise; models are the same as in Figure 4.5. Rows correspond to $n = 8000, 3000, 1000$ from top to bottom.

we have

$$(5.2) \quad \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\left| \overline{\|x_n^{-1} \mathbf{X}_{[1,n]}\|_p^\epsilon} - \mathbb{E}[\overline{\|x_n^{-1} \mathbf{X}_{[1,n]}\|_p^\epsilon}] \right| > \delta)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} = 0.$$

We refer to (5.2) as condition \mathbf{CS}_p in what follows. Indeed, if $\alpha < p < \infty$ then by Karamata's theorem (see Bingham *et al.* [5]) and since $n \mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$,

$$\mathbb{E}[\overline{\|x_n^{-1} \mathbf{X}_{[1,n]}\|_p^\epsilon}^p] = n \mathbb{E}[\overline{\|x_n^{-1} \mathbf{X}\|_p^\epsilon}^p] = o(1), \quad n \rightarrow \infty.$$

Also, if $p < \alpha$, then $\mathbb{E}[|\mathbf{X}|^p] < \infty$. If we also have $n/x_n^p \rightarrow 0$ then

$$\mathbb{E}[\overline{\|x_n^{-1} \mathbf{X}_{[1,n]}\|_p^\epsilon}^p] \leq n x_n^{-p} \mathbb{E}[|\mathbf{X}|^p] \rightarrow 0, \quad n \rightarrow \infty.$$

If $p = \alpha$, $\mathbb{E}[|\mathbf{X}|^\alpha] < \infty$ and $n/x_n^\alpha \rightarrow 0$ then the latter relation remains valid. If $\mathbb{E}[|\mathbf{X}|^\alpha] = \infty$ then $\mathbb{E}[\overline{\|x_n^{-1} \mathbf{X}\|_p^\epsilon}^\alpha] = x_n^{-\alpha} \ell(x_n)$ for some slowly varying function ℓ depending on ϵ , hence for any small $\kappa > 0$ and large n , $\ell(x_n) \leq$

x_n^κ . Then the condition $nx_n^{-\alpha+\kappa} \rightarrow 0$ also implies that $\mathbb{E}[\|\overline{x_n^{-1}\mathbf{X}_{[1,n]}}^\epsilon\|_\alpha^\alpha] = o(1)$. Thus we retrieve \mathbf{CS}_p as used in Theorem 2.1. In sum, under the aforementioned additional growth conditions on (x_n) , if (5.2) holds then it holds without the centering term holds too as is the case for condition \mathbf{CS}_p written in Theorem 2.1.

We mentioned that conditions of a similar type as \mathbf{CS}_p are standard when dealing with sum functionals acting on (\mathbf{X}_t) (see for example Davis and Hsing [11], Bartkiewicz *et al.* [1], Mikosch and Wintenberger [27, 28, 29]), and are also discussed in Kulik and Soulier [23].

Remark 5.1. Assume $\alpha < p < \infty$. Then by an applications of Markov's inequality of order 1 and Karamata's theorem yield for $\delta > 0$, as $n \rightarrow \infty$,

$$\begin{aligned} \frac{\mathbb{P}(\|\overline{x_n^{-1}\mathbf{X}_{[1,n]}}^\epsilon\|_p^p > \delta)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} &= \frac{\mathbb{P}(\sum_{t=1}^n |\overline{x_n^{-1}\mathbf{X}_t}^\epsilon|^p > \delta)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ &\leq \frac{\mathbb{E}[\|\overline{x_n^{-1}\mathbf{X}_0}^\epsilon\|_p^p]}{\delta \mathbb{P}(|\mathbf{X}_0| > \epsilon x_n)} \frac{\mathbb{P}(|\mathbf{X}_0| > \epsilon x_n)}{\mathbb{P}(|\mathbf{X}_0| > x_n)} \rightarrow c \epsilon^{p-\alpha}. \end{aligned}$$

The right-hand side converges to zero as $\epsilon \rightarrow 0$. Here and in what follows, c denotes any positive constant whose value is not of interest. We conclude that (5.2) is automatic for $p > \alpha$.

Remark 5.2. Condition \mathbf{CS}_p is challenging to check for $p \leq \alpha$. For $p/\alpha \in (1/2, 1]$, by Čebyšev's inequality,

$$\begin{aligned} &\mathbb{P}(\|\overline{x_n^{-1}\mathbf{X}_{[1,n]}}^\epsilon\|_p^p - \mathbb{E}[\|\overline{x_n^{-1}\mathbf{X}_{[1,n]}}^\epsilon\|_p^p] > \delta) / [n \mathbb{P}(|\mathbf{X}_0| > x_n)] \\ &\leq \delta^{-2} \text{var}(\|\overline{x_n^{-1}\mathbf{X}_{[1,n]}}^\epsilon\|_p^p) / [n \mathbb{P}(|\mathbf{X}_0| > x_n)] \\ &\leq \delta^{-2} \frac{\mathbb{E}[\|\overline{x_n^{-1}\mathbf{X}_0}^\epsilon\|_p^{2p}]}{\mathbb{P}(|\mathbf{X}_0| > x_n)} \left[1 + 2 \sum_{h=1}^{n-1} |\text{corr}(\|\overline{x_n^{-1}\mathbf{X}_0}^\epsilon\|_p^p, \|\overline{x_n^{-1}\mathbf{X}_h}^\epsilon\|_p^p)| \right]. \end{aligned}$$

Now assume that (\mathbf{X}_t) is ρ -mixing with summable rate function (ρ_h) ; cf. Bradley [6]. Then the right-hand side is bounded by

$$\delta^{-2} \frac{\mathbb{E}[\|\overline{x_n^{-1}\mathbf{X}}^\epsilon\|_p^{2p}]}{\mathbb{P}(|\mathbf{X}_0| > x_n)} \left[1 + 2 \sum_{h=1}^{\infty} \rho_h \right] \sim \delta^{-2} \epsilon^{2p-\alpha} \left[1 + 2 \sum_{h=1}^{\infty} \rho_h \right], \quad \epsilon \rightarrow 0,$$

where we applied Karamata's theorem in the last step, and \mathbf{CS}_p follows. For Markov chains weaker assumptions such as the drift condition (DC) in Mikosch and Wintenberger [28, 29] can be used for checking \mathbf{CS}_p .

Remark 5.3. Condition \mathbf{CS}_p not only restricts the serial dependence of the time series (\mathbf{X}_t) but also the level of thresholds (x_n) . Indeed for $p/\alpha < 1/2$ and (\mathbf{X}_t) iid, since $(\|n^{-1/2}\mathbf{X}'_{[1,n]}\|_p^p - \mathbb{E}[\|n^{-1/2}\mathbf{X}'_{[1,n]}\|_p^p])$ converges in distribution to a Gaussian limit by virtue of the central limit theorem, \mathbf{CS}_p implies necessarily that $x_n/\sqrt{n} \rightarrow \infty$ as $n \mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$.

5.3. Threshold condition. In Theorem 2.1 we assume growth conditions on (x_n) : $n/x_n^p \rightarrow 0$ if $p < \alpha$ and $n/x_n^{\alpha-\kappa} \rightarrow 0$ for some $\kappa > 0$ if $p = \alpha$.

For inference purposes it is tempting to decrease the threshold level x_n such that more exceedances are included in the estimators. Indeed, the assumptions on (x_n) can be relaxed, justified by results such as Nagaev's large deviation principle in [31], by adding a centering term as we will show in Lemma 5.4. However, in this section we aim at pointing at the difficulties that might arise while doing so in practice.

To motivate the results of this section we start by considering an iid sequence (\mathbf{X}_t) satisfying \mathbf{RV}_α for some $\alpha > 0$. Then, for $p > \alpha$, (2.3) holds with limit $c(p) = 1$ and $S_n^{(p)} = \sum_{t=1}^n |\mathbf{X}_t|^p$ has infinite expectation. If $p < \alpha$ the process $(S_n^{(p)})$ has finite expectation and by the law of large numbers, for $n/x_n^p \rightarrow 0$,

$$(5.3) \quad \begin{aligned} & \mathbb{P}(\|\mathbf{X}_{[0,n]}\|_p > x_n (n x_n^{-p} \mathbb{E}[|\mathbf{X}|^p] + 1)^{1/p}) \\ &= \mathbb{P}(S_n^{(p)} - \mathbb{E}[S_n^{(p)}] > x_n^p (1 + o(1))) \rightarrow 0. \end{aligned}$$

Following Nagaev [31], a large deviation result for the centered process holds:

$$\mathbb{P}(S_n^{(p)} - \mathbb{E}[S_n^{(p)}] > x_n^p) \sim n \mathbb{P}(|\mathbf{X}_0| > x_n), \quad n \rightarrow \infty,$$

provided $n/x_n^{\alpha-\kappa} \rightarrow 0$ for $p/\alpha \in (1/2, 1)$ and some $\kappa > 0$, and $\sqrt{n \log n}/x_n^p \rightarrow 0$ for $p/\alpha < 1/2$. These conditions are satisfied for extreme thresholds satisfying $n/x_n^p \rightarrow 0$: in this case the centering term $\mathbb{E}[S_n^{(p)}]$ in (5.3) is always negligible which allows us to derive (2.3). Next, we extend the previous ideas to regularly varying time series.

Lemma 5.4. *Consider an \mathbb{R}^d -valued stationary process (\mathbf{X}_t) satisfying the conditions \mathbf{RV}_α , \mathbf{AC} , \mathbf{CS}_p and $c(p) < \infty$ for some $p > 0$. If $p < \alpha$ then*

$$(5.4) \quad \lim_{n \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{X}_{[0,n]}\|_p > x_n (n x_n^{-p} \mathbb{E}[|\mathbf{X}|^p] + 1)^{1/p})}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} = c(p).$$

If $p = \alpha$ then

$$(5.5) \quad \lim_{n \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{X}_{[0,n]}\|_\alpha > x_n (n \mathbb{E}[|\overline{\mathbf{X}}/x_n^{-1}|^\alpha] + 1)^{1/\alpha})}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} = c(\alpha) = 1.$$

Moreover, if also $\mathbb{E}[|\mathbf{X}|^\alpha] < \infty$ then equation (5.4) holds for $p = \alpha$.

The proof is given in Section 7.2. Now the restrictions on the level of the thresholds (x_n) are the ones implicitly implied by condition \mathbf{CS}_p in (5.2); see Remark 5.3.

We define an auxiliary sequence of levels:

$$z_n := z_n(p) = \begin{cases} x_n (n x_n^{-p} \mathbb{E}[|\mathbf{X}|^p] + 1)^{1/p} & \text{if } p < \alpha, \\ x_n (n \mathbb{E}[|\overline{\mathbf{X}}/x_n^{-1}|^\alpha] + 1)^{1/\alpha} & \text{if } p = \alpha, \\ x_n & \text{if } p > \alpha. \end{cases}$$

For thresholds satisfying the growth conditions $n/x_n^p \rightarrow 0$ we have $z_n \sim x_n$, while for moderate thresholds verifying \mathbf{CS}_p and $n/x_n^p \rightarrow \infty$ this is no longer the case.

For the purposes of inference Lemma 5.4 is not as satisfactory as (2.3) in Theorem 2.1. Indeed, the level z_n in the selection of the exceedances is not the original threshold x_n . For any moderate threshold x_n with $z_n/x_n \rightarrow \infty$ the use of x_n instead of z_n might include a bias. As a toy example, consider the problem of inferring the constant $c(q)/c(p)$ for $p < \alpha$, $q > p$. Then an application of Lemma 5.4 ensures that

$$\begin{aligned} & \mathbb{P}(\|\mathbf{X}_{[0,n]}\|_q > z_n(q) \mid \|\mathbf{X}_{[0,n]}\|_p > z_n(p)) \\ & \rightarrow \mathbb{P}(\|Y\mathbf{Q}^{(p)}\|_q > 1) = \mathbb{E}[\|\mathbf{Q}^{(p)}\|_q^\alpha] = c(q)/c(p), \quad n \rightarrow \infty. \end{aligned}$$

However, choosing moderate thresholds (x_n) , we would have

$$\begin{aligned} \mathbb{P}(\|\mathbf{X}_{[0,n]}\|_q > x_n \mid \|\mathbf{X}_{[0,n]}\|_p > x_n) & \sim \frac{\mathbb{P}(\|\mathbf{X}_{[0,n]}\|_q > x_n)}{\mathbb{P}(n^{1/p}\mathbb{E}[\|\mathbf{X}\|_p^{1/p}] > x_n)} \\ & \rightarrow \begin{cases} 1 & \text{if } q < \alpha, \\ 0 & \text{if } q > \alpha, \end{cases} \quad n \rightarrow \infty. \end{aligned}$$

By this argument, the assumption $n/x_n^{p \wedge (\alpha - \kappa)} \rightarrow 0$ is justified to simplify inference procedures. Otherwise, the choice of the threshold sequence becomes complicated.

6. INFERENCE BEYOND SHIFT-INVARIANT FUNCTIONALS

So far we only considered inference for shift-invariant functionals acting on $(\tilde{\ell}^p, \tilde{d}_p)$ such as maxima and sums. Following the shift-projection ideas in Janssen [22], jointly with continuous mapping arguments, we extend inference to functionals on (ℓ^p, d_p) .

6.1. Inference for cluster functionals in (ℓ^p, d_p) . Let $g : (\ell^p, d_p) \rightarrow \mathbb{R}$ be a bounded measurable function. We define the functional $\psi_g : (\tilde{\ell}^p, \tilde{d}_p) \rightarrow \mathbb{R}$ by

$$(6.1) \quad [\mathbf{z}] \mapsto \psi_g([\mathbf{z}]) := \sum_{j \in \mathbb{Z}} |\mathbf{z}_{-j}^*|^\alpha g((B^j \mathbf{z}_t^*)_{t \in \mathbb{Z}}),$$

where $\mathbf{z}_t^* := \mathbf{z}_{t-T^*(\mathbf{z})}$, for $t \in \mathbb{Z}$, such that $T^*(\mathbf{z}) := \inf\{s \in \mathbb{Z} : |\mathbf{z}_s| = \|\mathbf{z}\|_\infty\}$ and $B : \ell^p \rightarrow \ell^p$ is the backward-shift map.

We link the distribution of the spectral cluster process $\mathbf{Q}^{(\alpha)}$ and the distribution of the class $[\mathbf{Q}^{(\alpha)}]$ through the mappings (6.1) in the next proposition whose proof is given in Section 7.5.1.

Proposition 6.1. *The following relation holds for any real-valued bounded measurable function g on ℓ^α*

$$\mathbb{E}[g(\mathbf{Q}^{(\alpha)})] = \mathbb{E}[\psi_g([\mathbf{Q}^{(\alpha)}])].$$

where ψ_g is as in (6.1). This relation remains valid if α is replaced by p , whenever the spectral cluster process in ℓ^p is well defined.

For $p \leq \alpha$ the mappings in (6.1) are continuous functionals on $(\tilde{\ell}^p, \tilde{d}_p)$ and we can extend Theorem 4.1 to continuous functionals on (ℓ^p, d_p) evaluated at the spectral cluster process $\mathbf{Q}^{(p)}$.

Theorem 6.2. *Assume the conditions of Theorem 4.1 for $p \leq \alpha$. Then for any continuous bounded function $g : \ell^p \cap \{\mathbf{x} : \|\mathbf{x}\|_p = 1, |\mathbf{x}_0| > 0\} \rightarrow \mathbb{R}$,*

$$(6.2) \quad \hat{g}^{(p)} := \frac{1}{k} \sum_{t=1}^m \underbrace{\sum_{j=1}^b W_{j,t}(p) g\left(\frac{B^{j-1}\mathbf{B}_t}{\|\mathbf{B}_t\|_p}\right)}_{=: \psi_g(\mathbf{B}_t)} \mathbf{1}(\|\mathbf{B}_t\|_p > \|\mathbf{B}\|_{p,(k)})$$

$$\xrightarrow{\mathbb{P}} \mathbb{E}[g(\mathbf{Q}^{(p)})], \quad n \rightarrow \infty,$$

where $W_{j,t}(p) = |\mathbf{X}_{(t-1)b+j}|^\alpha / \|\mathbf{B}_t\|_p^\alpha$ for all $1 \leq j \leq b$.

The proof is given in Section 7.5.2.

6.2. Applications. Examples of non-shift-invariant functionals on (ℓ^p, d_p) are measures of serial dependence, probabilities of large deviations such as the supremum of a random walk and ruin probabilities, and functionals of the spectral tail process Θ . We study these examples in the remainder of this section.

6.2.1. Measures of serial dependence. Define $g_h(\mathbf{x}_t) = |\mathbf{x}_h|^\alpha \frac{\mathbf{x}_0^\top \mathbf{x}_h}{|\mathbf{x}_0| |\mathbf{x}_h|}$. Then the following result is straightforward from Theorem 6.2.

Corollary 6.3. *Assume the conditions of Theorem 6.2 for $p = \alpha$. Then*

$$\hat{g}_h^{(\alpha)} := \frac{1}{k} \sum_{t=1}^m \underbrace{\sum_{j=1}^{b-h} W_{j,t} W_{j+h,t} \frac{\mathbf{X}_{j,t}^\top \mathbf{X}_{j+h,t}}{|\mathbf{X}_{j,t}| |\mathbf{X}_{j+h,t}|}}_{=: \psi_{g_h}(\mathbf{B}_t)} \mathbf{1}(\|\mathbf{B}_t\|_\alpha > \|\mathbf{B}\|_{\alpha,(k)}).$$

$$\xrightarrow{\mathbb{P}} \mathbb{E}[g_h(\mathbf{Q}^{(\alpha)})], \quad n \rightarrow +\infty,$$

where the weights $W_{j,t} = W_{j,t}(\alpha)$ are defined in Theorem 6.2, satisfying $\sum_{j=1}^b W_{j,t} = 1$, and $\mathbf{X}_{j,t} := \mathbf{X}_{(t-1)b+j}$ for $1 \leq j \leq b$.

The function g_h gives a summary of the magnitude and direction of the time series h lags after recording a high-level exceedance of the norm, and satisfies the relation $\sum_{h \in \mathbb{Z}} \mathbb{E}[g_h(\mathbf{Q}^{(\alpha)})] = 1$.

Example 6.4. Let (X_t) be a linear process satisfying the assumptions in Example 4.3.3, then

$$\mathbb{E}[g_h(Q^{(\alpha)})] = \frac{\sum_{t \in \mathbb{Z}} |\varphi_t|^\alpha |\varphi_{t+h}|^\alpha \text{sign}(\varphi_t) \text{sign}(\varphi_{t+h})}{(\|\varphi\|_\alpha^\alpha)^2}, \quad h \in \mathbb{Z}.$$

This function is proportional to the autocovariance function of a finite variance linear process with coefficients $(|\varphi_t|^\alpha \text{sign}(\varphi_t))$. In particular, for $\alpha = 1$ it is proportional to the autocovariance function of a finite variance linear process with coefficients (φ_t) .

6.2.2. *Large deviations for the supremum of a random walk.* We start by reviewing Theorem 4.5 in Mikosch and Wintenberger [29]; the proof is given in Section 7.5.3.

Proposition 6.5. *Consider a univariate stationary sequence (X_t) satisfying \mathbf{RV}_α for some $\alpha \geq 1$, \mathbf{AC} , \mathbf{CS}_1 , and $c(1) < \infty$. Then for all $p \geq 1$,*

$$(6.3) \quad \left| \frac{\mathbb{P}(\sup_{1 \leq t \leq n} S_t > x_n)}{n\mathbb{P}(|X| > x_n)} - c(p) \mathbb{E} \left[\lim_{s \rightarrow \infty} \left(\sup_{t \geq -s} \sum_{i=-s}^t Q_i^{(p)} \right)_+^\alpha \right] \right| \rightarrow 0, \quad n \rightarrow \infty.$$

If $\alpha \geq 1$, then $\|Q^{(1)}\|_\alpha^\alpha \leq \|Q^{(1)}\|_1^\alpha = 1$ and a consistent estimator of $c(1) = 1/\mathbb{E}[\|Q^{(1)}\|_\alpha^\alpha]$ was suggested in Corollary 4.3. A consistent estimator of the term in (6.3) is given next.

Corollary 6.6. *Assume the conditions of Theorem 6.2 for $p = 1$. Then*

$$\left| \frac{\sum_{t=1}^m \left(\sup_{1 \leq j \leq b} \frac{X_{t,j}}{\|\mathbf{B}_t\|_1} \right)_+^\alpha \mathbf{1}(\|\mathbf{B}_t\|_1 > \|\mathbf{B}\|_{1,(k)})}{\sum_{t=1}^m \frac{\|\mathbf{B}_t\|_\alpha^\alpha}{\|\mathbf{B}_t\|_1^\alpha} \mathbf{1}(\|\mathbf{B}_t\|_1 > \|\mathbf{B}\|_{1,(k)})} - c(1) \mathbb{E} \left[\lim_{s \rightarrow \infty} \left(\sup_{t \geq -s} \sum_{i=-s}^t Q_i^{(1)} \right)_+^\alpha \right] \right| \xrightarrow{\mathbb{P}} 0, \quad n \rightarrow \infty,$$

where $X_{t,j} := X_{(t-1)b+j}$, for $1 \leq j \leq b$, $1 \leq t \leq m$.

Following the same ideas and using Theorem 4.9 in [29], one can also derive a consistent estimator for the constant in the related ruin problem.

6.2.3. *Application: a cluster-based method for inferring on (Θ_t) .* Exploiting the relation $(\mathbf{Q}_t^{(\alpha)})/|\mathbf{Q}_0^{(\alpha)}| \stackrel{d}{=} (\Theta_t)$ discussed in Section 3.1.1, we propose cluster-based estimation methods for the spectral tail process.

Cluster-based approaches with the goal to improve inference on Θ_1 for Markov chains were considered in Drees *et al.* [17]; see also Davis *et al.* [10], Drees *et al.* [14] for related cluster-based procedures on $(\Theta_t)_{|t| \leq h}$ for fixed $h \geq 0$. Our approach can be seen as an extension for inference on the ℓ^α -valued sequence (Θ_t) .

Consider the continuous re-normalization function $\zeta(\mathbf{x}) = \mathbf{x}/|\mathbf{x}_0|$ on $\{\mathbf{x} \in \ell^\alpha : |\mathbf{x}_0| > 0\}$. We derive the following result from Theorem 6.2; the proof is given in Section 7.5.4.

Proposition 6.7. *Assume the conditions of Theorem 6.2 for $p = \alpha$. Let $\rho : (\ell^\alpha, d_\alpha) \rightarrow \mathbb{R}$ be a homogeneous continuous function and $\rho_\zeta(\mathbf{x}) := (\rho^\alpha \wedge$*

1) $\circ \zeta(\mathbf{x})$. Then

$$\widehat{\rho}_\zeta^{(\alpha)} := \frac{1}{k} \sum_{t=1}^m \psi_{\rho_\zeta}(\mathbf{B}_t) \mathbf{1}(\|\mathbf{B}_t\|_p > \|\mathbf{B}\|_{p,(k)}) \xrightarrow{\mathbb{P}} \mathbb{P}(\rho(Y \Theta) > 1), \quad n \rightarrow \infty,$$

where $\psi_{\rho_\zeta}(\mathbf{B}_t)$ is defined in (6.2) and the Pareto(α) random variable Y is independent of Θ .

Classical examples of such functionals are $\rho(\mathbf{x}) = \max_{i \geq 0, j \geq i} (x_i - x_j)_+$, functionals related to large deviations such as $\rho(\mathbf{x}) = \sup_{t \geq 0} (\sum_{i=0}^t x_i)_+$, or measures of serial dependence such as $\rho(\mathbf{x}) = |\mathbf{x}_h|$.

7. PROOFS

7.1. Proof of Theorem 2.1. Recall the properties of the sequence (x_n) from Section 5, in particular $n \mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$. The main result in Theorem 2.1 follows by applications of Lemma 7.1 and Proposition 7.2 below; their proofs are given at the end of this section.

Lemma 7.1. *Consider an \mathbb{R}^d -valued stationary time series (\mathbf{X}_t) satisfying the conditions \mathbf{RV}_α , \mathbf{AC} , \mathbf{CS}_p . If $p < \alpha$, assume also $n/x_n^p \rightarrow 0$ and if $p = \alpha$ also $n/x_n^{\alpha-\kappa} \rightarrow 0$ for some $\kappa > 0$. Then the following relation holds*

$$(7.1) \quad \lim_{n \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{X}_{[0,n]}\|_p > x_n)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} = c(p),$$

where $c(p)$ is given in (3.2).

We recall from Remark 5.1 that (5.2) in \mathbf{CS}_p is always satisfied for $p > \alpha$. Moreover, for $p \leq \alpha$, under the growth conditions on (x_n) in Theorem 2.1, centering with the expectation in (5.2) is not necessary.

Proposition 7.2. *Assume the conditions of Lemma 7.1. Then,*

$$(7.2) \quad \mathbb{P}(x_n^{-1} \mathbf{X}_{[0,n]} \in \cdot \mid \|\mathbf{X}_{[0,n]}\|_p > x_n) \xrightarrow{w} \mathbb{P}(Y \mathbf{Q}^{(p)} \in \cdot), \quad n \rightarrow \infty,$$

in the space $(\tilde{\ell}^p, \tilde{d}_p)$ where the Pareto(α) random variable Y and $\mathbf{Q}^{(p)}$ are independent.

Proof of Lemma 7.1. Choose some $\epsilon > 0$, $\delta \in (0, 1)$. Since $\|x_n^{-1} \mathbf{X}_{[0,n]}\|_p^p$ is a sum of non-negative random variables we have the following bounds via truncation

$$(7.3) \quad \begin{aligned} & \mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p^p > 1) \leq \mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0,n]}\|_p^p > 1) \\ & \leq \mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p^p > (1 - \delta^p)) + \mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0,n]}\|_p^p > \delta^p). \end{aligned}$$

By \mathbf{CS}_p and in view of Remark 5.1 we have

$$(7.4) \quad \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p^p > \delta^p) / (n \mathbb{P}(|\mathbf{X}_0| > x_n)) = 0.$$

Now, for any choice of $u > 0$, it remains to determine the limits of the terms $\mathbb{P}(\|\underline{x}_n^{-1}\mathbf{X}_{[0,n]_\epsilon}\|_p^p > u)/(n\mathbb{P}(|\mathbf{X}_0| > x_n))$. We start with a telescoping sum representation

$$\begin{aligned} & \mathbb{P}(\|\underline{x}_n^{-1}\mathbf{X}_{[0,n]_\epsilon}\|_p^p > u) - \mathbb{P}(\|\underline{x}_n^{-1}\mathbf{X}_0\|_p^p > u) \\ &= \sum_{i=1}^n (\mathbb{P}(\|\underline{x}_n^{-1}\mathbf{X}_{[0,i]_\epsilon}\|_p^p > u) - \mathbb{P}(\|\underline{x}_n^{-1}\mathbf{X}_{[0,i-1]_\epsilon}\|_p^p > u)) \\ &= \sum_{i=1}^n \mathbb{E}[(\mathbb{1}(\|\underline{x}_n^{-1}\mathbf{X}_{[0,i]_\epsilon}\|_p^p > u) - \mathbb{1}(\|\underline{x}_n^{-1}\mathbf{X}_{[1,i]_\epsilon}\|_p^p > u)) \mathbb{1}(|\mathbf{X}_0| > \epsilon x_n)], \end{aligned}$$

where we used stationarity in the last step and the fact that the difference of the indicator functions vanishes on $\{|\mathbf{X}_0| \leq \epsilon x_n\}$. We also observe that the second term on the left-hand side is of the order $o(n\mathbb{P}(|\mathbf{X}_0| > x_n))$. For any fixed k write $A_k = \{\max_{k \leq t \leq n} |\mathbf{X}_t| > \epsilon x_n\}$. Regular variation of (\mathbf{X}_t) ensures that, as $n \rightarrow \infty$,

$$\begin{aligned} & \frac{\mathbb{P}(\|\underline{x}_n^{-1}\mathbf{X}_{[0,n]_\epsilon}\|_p^p > u)}{n\mathbb{P}(|\mathbf{X}_0| > x_n)} \\ & \sim \epsilon^{-\alpha} \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\mathbb{1}(\|\underline{x}_n^{-1}\mathbf{X}_{[0,i]_\epsilon}\|_p^p > u) - \mathbb{1}(\|\underline{x}_n^{-1}\mathbf{X}_{[1,i]_\epsilon}\|_p^p > u) \mid |\mathbf{X}_0| > \epsilon x_n] \\ & \sim \epsilon^{-\alpha} \mathbb{E}[(\mathbb{1}(\|\underline{x}_n^{-1}\mathbf{X}_{[0,k-1]_\epsilon}\|_p^p > u) - \mathbb{1}(\|\underline{x}_n^{-1}\mathbf{X}_{[1,k-1]_\epsilon}\|_p^p > u)) \\ & \quad \times \mathbb{1}(A_k^c) \mid |\mathbf{X}_0| > \epsilon x_n] + \epsilon^{-\alpha} O(\mathbb{P}(A_k \mid |\mathbf{X}_0| > \epsilon x_n)), \end{aligned}$$

where the second term vanishes, first letting $n \rightarrow \infty$ and then $k \rightarrow \infty$, by virtue of **AC**. Now the regular variation property of (\mathbf{X}_t) implies that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{\mathbb{P}(\|\underline{x}_n^{-1}\mathbf{X}_{[0,n]_\epsilon}\|_p^p > u)}{n\mathbb{P}(|\mathbf{X}_0| > x_n)} \\ &= \lim_{k \rightarrow \infty} \epsilon^{-\alpha} (\mathbb{P}(\sum_{t=0}^{k-1} |\epsilon Y \Theta_t|^p \mathbb{1}(|Y \Theta_t| > 1) > u) \\ & \quad - \mathbb{P}(\sum_{t=1}^{k-1} |\epsilon Y \Theta_t|^p \mathbb{1}(|Y \Theta_t| > 1) > u)), \end{aligned}$$

by a change of variable this term equals

$$\begin{aligned} &= \lim_{k \rightarrow \infty} \mathbb{E}[\int_\epsilon^\infty (\mathbb{1}(\sum_{t=0}^{k-1} |y \Theta_t|^p \mathbb{1}(|y \Theta_t| > \epsilon) > u) \\ & \quad - \mathbb{1}(\sum_{t=1}^{k-1} |y \Theta_t|^p \mathbb{1}(|y \Theta_t| > \epsilon) > u)) d(-y^{-\alpha})] \\ &= \lim_{k \rightarrow \infty} \mathbb{E}[\int_0^\infty (\mathbb{1}(\|\underline{y}\Theta_{[0,k-1]_\epsilon}\|_p^p > u) - \mathbb{1}(\|\underline{y}\Theta_{[1,k-1]_\epsilon}\|_p^p > u)) d(-y^{-\alpha})]. \end{aligned}$$

In the last step we used the fact that the integrand vanishes for $0 \leq y \leq \epsilon$. The integrand is non-negative and bounded by $\mathbb{1}(y > \epsilon)$ which is integrable. Thus we may take the limit as $k \rightarrow \infty$ inside the integral to derive the quantity

$$\mathbb{E}[\int_0^\infty (\mathbb{1}(\|\underline{y}\Theta_{[0,\infty]_\epsilon}\|_p^p > u) - \mathbb{1}(\|\underline{y}\Theta_{[1,\infty]_\epsilon}\|_p^p > u)) d(-y^{-\alpha})].$$

By monotone convergence as $\epsilon \downarrow 0$ we get the limit

$$(7.5) \quad u^{-\alpha/p} \mathbb{E} [\|\Theta_{[0,\infty]}\|_p^\alpha - \|\Theta_{[1,\infty]}\|_p^\alpha] = u^{-\alpha/p} c(p);$$

see (3.6). Now an appeal to (7.3) with $u = 1$ and $u = 1 - \delta^p$ yields

$$\begin{aligned} c(p) &\leq \liminf_{n \rightarrow \infty} \frac{\mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0,n]}\|_p^p > 1)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ &\leq \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0,n]}\|_p^p > 1)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \leq (1 - \delta^p)^{-\alpha/p} c(p). \end{aligned}$$

The limit relation (7.1) follows as $\delta \downarrow 0$.

Proof of Proposition 7.2. Consider any bounded Lipschitz-continuous function $f : (\tilde{\ell}^p, \tilde{d}_p) \rightarrow \mathbb{R}$. The statement is proved if we can show that

$$\begin{aligned} &\lim_{n \rightarrow \infty} \mathbb{E}[f(x_n^{-1} \mathbf{X}_{[0,n]}) \mid \|\mathbf{X}_{[0,n]}\|_p > x_n] \\ &= c(p)^{-1} \mathbb{E}[\|\Theta / \|\Theta\|_\alpha\|_p^\alpha f(Y \Theta / \|\Theta\|_p)]. \end{aligned}$$

In view of Lemma 7.1 it suffices to show

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{\mathbb{E}[f(x_n^{-1} \mathbf{X}_{[0,n]}) \mathbf{1}(\|\mathbf{X}_{[0,n]}\|_p > x_n)]}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ &= \mathbb{E}[\|\Theta / \|\Theta\|_\alpha\|_p^\alpha f(Y \Theta / \|\Theta\|_p)]. \end{aligned}$$

In these limit relations we may replace $f(x_n^{-1} \mathbf{X}_{[0,n]})$ by $f(\underline{x_n^{-1} \mathbf{X}_{[0,n]}}_\epsilon)$ since by (7.4) for any $\delta > 0$, some $K_f > 0$,

$$\begin{aligned} &\lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(|f(x_n^{-1} \mathbf{X}_{[0,n]}) - f(\underline{x_n^{-1} \mathbf{X}_{[0,n]}}_\epsilon)| > \delta, \|\mathbf{X}_{[0,n]}\|_p > x_n)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ &\leq \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(K_f d_p(\underline{x_n^{-1} \mathbf{X}_{[0,n]}}_\epsilon, \mathbf{0}) > \delta)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ &\leq \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(K_f \|\underline{x_n^{-1} \mathbf{X}_{[0,n]}}_\epsilon\|_p^p > \delta(p))}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} = 0, \end{aligned}$$

where $\delta(p) = \delta^p$ for $p \geq 1$ and $= \delta$ for $p \in (0, 1)$. We also have for $\delta \in (0, 1)$,

$$\begin{aligned} G_{n,\epsilon} &= \frac{\mathbb{E}[f(\underline{x_n^{-1} \mathbf{X}_{[0,n]}}_\epsilon) \mid \mathbf{1}(\|\mathbf{X}_{[0,n]}\|_p > 1) - \mathbf{1}(\|\underline{x_n^{-1} \mathbf{X}_{[0,n]}}_\epsilon\|_p > 1)]}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ &\leq c \frac{\mathbb{P}(\|\mathbf{X}_{[0,n]}\|_p > 1 \geq \|\underline{x_n^{-1} \mathbf{X}_{[0,n]}}_\epsilon\|_p)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ &\leq c \frac{\mathbb{P}(\|\underline{x_n^{-1} \mathbf{X}_{[0,n]}}_\epsilon\|_p > \delta)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} + c \frac{\mathbb{P}(1 \geq \|\underline{x_n^{-1} \mathbf{X}_{[0,n]}}_\epsilon\|_p > 1 - \delta)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ &= G_{n,\epsilon,\delta}^{(1)} + G_{n,\epsilon,\delta}^{(2)}. \end{aligned}$$

Applying (7.4) to $G_{n,\epsilon,\delta}^{(1)}$ and using the calculations in the proof of Lemma 7.1 leading to (7.5) for $G_{n,\epsilon,\delta}^{(2)}$, we conclude that

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} G_{n,\epsilon} &\leq \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} G_{n,\epsilon,\delta}^{(1)} + \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} G_{n,\epsilon,\delta}^{(2)} \\ &= 0 + c \left((1 - \delta)^{-\alpha/p} - 1 \right) \downarrow 0, \quad \delta \downarrow 0. \end{aligned}$$

Thus it suffices to show

$$(7.6) \quad \lim_{\epsilon \downarrow 0} \lim_{n \rightarrow \infty} \frac{\mathbb{E} \left[f(\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon}) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p > 1) \right]}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} = \mathbb{E} \left[\|\Theta / \|\Theta\|_\alpha\|_\alpha^\alpha f(Y \Theta / \|\Theta\|_p) \right].$$

This is the goal of the remaining proof.

Choose any $\epsilon > 0$. Noticing that $\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon} = \underline{x}_n^{-1} \mathbf{X}_{[1,n]_\epsilon}$ on $\{|\underline{x}_n^{-1} \mathbf{X}_0| \leq \epsilon\}$, we have

$$\begin{aligned} I &:= \mathbb{E} \left[f(\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon}) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p > 1) \right] \\ &= \mathbb{E} \left[\left(f(\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon}) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p > 1) \right. \right. \\ &\quad \left. \left. - f((0, \underline{x}_n^{-1} \mathbf{X}_{[1,n]_\epsilon})) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[1,n]_\epsilon}\|_p > 1) \right) \mathbb{1}(|\mathbf{X}_0| > \epsilon x_n) \right] \\ &\quad + \mathbb{E} \left[f((0, \underline{x}_n^{-1} \mathbf{X}_{[1,n]_\epsilon})) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[1,n]_\epsilon}\|_p > 1) \right] \\ &= \mathbb{E} \left[\left(f(\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon}) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p > 1) \right. \right. \\ &\quad \left. \left. - f((0, \underline{x}_n^{-1} \mathbf{X}_{[1,n]_\epsilon})) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[1,n]_\epsilon}\|_p > 1) \right) \mathbb{1}(|\mathbf{X}_0| > \epsilon x_n) \right] \\ &\quad + \mathbb{E} \left[f((0, \underline{x}_n^{-1} \mathbf{X}_{[0,n-1]_\epsilon})) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[0,n-1]_\epsilon}\|_p > 1) \right]. \end{aligned}$$

where we used the stationarity in the last step. Using the same idea recursively, we obtain

$$\begin{aligned} I &= \sum_{j=1}^n \mathbb{E} \left[\left(f((\mathbf{0}^{n-j}, \underline{x}_n^{-1} \mathbf{X}_{[0,j]_\epsilon})) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[0,j]_\epsilon}\|_p > 1) \right. \right. \\ &\quad \left. \left. - f((\mathbf{0}^{n-j+1}, \underline{x}_n^{-1} \mathbf{X}_{[1,j]_\epsilon})) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[1,j]_\epsilon}\|_p > 1) \right) \mathbb{1}(|\mathbf{X}_0| > \epsilon x_n) \right] \\ &\quad + \mathbb{E} \left[f((\mathbf{0}^n, \underline{x}_n^{-1} \mathbf{X}_0)) \mathbb{1}(|\mathbf{X}_0| > \epsilon x_n) \right], \end{aligned}$$

where $\mathbf{0}^k := \{0\}^k$ for $k \geq 1$. By regular variation of \mathbf{X}_0 the last right-hand term is $o(n \mathbb{P}(|\mathbf{X}_0| > x_n))$. Therefore by regular variation of (\mathbf{X}_t) we obtain as $n \rightarrow \infty$,

$$\begin{aligned} &I / (n \mathbb{P}(|\mathbf{X}_0| > x_n)) \\ &\sim \frac{\epsilon^{-\alpha}}{n} \sum_{j=1}^n \mathbb{E} \left[\left(f((\mathbf{0}^{n-j}, \underline{x}_n^{-1} \mathbf{X}_{[0,j]_\epsilon})) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[0,j]_\epsilon}\|_p > 1) \right. \right. \\ &\quad \left. \left. - f((\mathbf{0}^{n-j+1}, \underline{x}_n^{-1} \mathbf{X}_{[1,j]_\epsilon})) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[1,j]_\epsilon}\|_p > 1) \right) \mathbb{1}(|\mathbf{X}_0| > \epsilon x_n) \right] \\ &=: II. \end{aligned}$$

Write $A_k = \{\|\mathbf{X}_{[k,n]}\|_\infty > \epsilon x_n\}$ for fixed $k \geq 1$. By **AC**, $\mathbb{P}(A_k \mid |\mathbf{X}_0| > \epsilon x_n)$ vanishes by first letting $n \rightarrow \infty$ then $k \rightarrow \infty$. Since each of the summands in II is uniformly bounded in absolute value we may restrict the summation to $j \in \{k-1, \dots, n\}$ for any fixed $k \geq 1$. Therefore we have as $n \rightarrow \infty$,

$$\begin{aligned}
& II - O(\mathbb{P}(A_k \mid |\mathbf{X}_0| > \epsilon x_n)) \\
& \sim \frac{\epsilon^{-\alpha}}{n} \sum_{j=k-1}^n \mathbb{E} [(f(\mathbf{0}^{n-j}, \underline{x_n^{-1} \mathbf{X}_{[0,j]_\epsilon}})) \mathbb{1}(\{\|\underline{x_n^{-1} \mathbf{X}_{[0,j]_\epsilon}\|_p > 1\}) \\
& \quad - f(\mathbf{0}^{n-j+1}, \underline{x_n^{-1} \mathbf{X}_{[1,j]_\epsilon}}) \mathbb{1}(\{\|\underline{x_n^{-1} \mathbf{X}_{[1,j]_\epsilon}\|_p > 1\}) \mathbb{1}(A_k^c) \mid |\mathbf{X}_0| > \epsilon x_n] \\
& = \frac{\epsilon^{-\alpha}}{n} \sum_{j=k-1}^n \mathbb{E} [f(\mathbf{0}^{n-j}, \underline{x_n^{-1} \mathbf{X}_{[0,k-1]_\epsilon}}, \mathbf{0}^{j-k}) \mathbb{1}(\|\underline{x_n^{-1} \mathbf{X}_{[0,k-1]_\epsilon}\|_p > 1) \\
& \quad - f(\mathbf{0}^{n-j+1}, \underline{x_n^{-1} \mathbf{X}_{[1,k-1]_\epsilon}}, \mathbf{0}^{j-k}) \mathbb{1}(\|\underline{x_n^{-1} \mathbf{X}_{[1,k-1]_\epsilon}\|_p > 1) \mid |\mathbf{X}_0| > \epsilon x_n].
\end{aligned}$$

Next we apply shift-invariance and regular variation in $\tilde{\ell}^p$:

$$\begin{aligned}
& \sim \epsilon^{-\alpha} \mathbb{E} [f(\underline{x_n^{-1} \mathbf{X}_{[0,k-1]_\epsilon}}) \mathbb{1}(\|\underline{x_n^{-1} \mathbf{X}_{[0,k-1]_\epsilon}\|_p > 1) \\
& \quad - f(\underline{x_n^{-1} \mathbf{X}_{[1,k-1]_\epsilon}}) \mathbb{1}(\|\underline{x_n^{-1} \mathbf{X}_{[1,k-1]_\epsilon}\|_p > 1) \mid |\mathbf{X}_0| > \epsilon x_n] \\
& \rightarrow \epsilon^{-\alpha} \mathbb{E} [f(\underline{\epsilon Y \Theta_{[0,k-1]_\epsilon}}) \mathbb{1}(\|\underline{\epsilon Y \Theta_{[0,k-1]_\epsilon}\|_p > 1) \\
& \quad - f(\underline{\epsilon Y \Theta_{[1,k-1]_\epsilon}}) \mathbb{1}(\|\underline{\epsilon Y \Theta_{[1,k-1]_\epsilon}\|_p > 1)] \\
& = \mathbb{E} [f(\underline{\epsilon Y \Theta_{[0,k-1]_1}}) \mathbb{1}(\|\underline{\epsilon Y \Theta_{[0,k-1]_1}\|_p > 1) \\
& \quad - f(\underline{\epsilon Y \Theta_{[1,k-1]_1}}) \mathbb{1}(\|\underline{\epsilon Y \Theta_{[1,k-1]_1}\|_p > 1)] =: J_{k,\epsilon}.
\end{aligned}$$

By Proposition 3.2 we have $\|\Theta\|_\alpha < \infty$ a.s., $|\Theta_t| \xrightarrow{\text{a.s.}} 0$ as $|t| \rightarrow \infty$, hence $T := \inf_{t \geq 0} \{t : Y|\Theta_t| < 1\} < \infty$ a.s. Then by monotone convergence as $k \rightarrow \infty$,

$$\begin{aligned}
J_{k,\epsilon} & = \epsilon^{-\alpha} \mathbb{E} [(f(\underline{\epsilon Y \Theta_{[0,\infty]_1}}) \mathbb{1}(\|\underline{\epsilon Y \Theta_{[0,\infty]_1}\|_p > 1) \\
& \quad - f(\underline{\epsilon Y \Theta_{[1,\infty]_1}}) \mathbb{1}(\|\underline{\epsilon Y \Theta_{[1,\infty]_1}\|_p > 1)) \mathbb{1}(T < k)] \\
& \quad + O(\mathbb{P}(T \geq k)) \\
& \rightarrow \epsilon^{-\alpha} \mathbb{E} [f(\underline{\epsilon Y \Theta_{[0,\infty]_1}}) \mathbb{1}(\|\underline{\epsilon Y \Theta_{[0,\infty]_1}\|_p > 1) \\
& \quad - f(\underline{\epsilon Y \Theta_{[1,\infty]_1}}) \mathbb{1}(\|\underline{\epsilon Y \Theta_{[1,\infty]_1}\|_p > 1)] \\
& = \int_0^\infty \mathbb{E} [f(\underline{y \Theta_{[0,\infty]_\epsilon}}) \mathbb{1}(\|\underline{y \Theta_{[0,\infty]_\epsilon}\|_p > 1) \\
& \quad - f(\underline{y \Theta_{[1,\infty]_\epsilon}}) \mathbb{1}(\|\underline{y \Theta_{[1,\infty]_\epsilon}\|_p > 1)] d(-y^{-\alpha}) =: J_\epsilon.
\end{aligned}$$

In the last step we changed variables, $u = \epsilon y$, and observed that the integrand vanishes for $y < \epsilon$.

Finally, we want to let $\epsilon \downarrow 0$. We start by inter-changing expectation and integral in J_ϵ , and change variables, $u = y\|\Theta_{[0,\infty]}\|_\alpha$, in the first term

of the integrand and then proceed similarly for the second term with the convention that it is zero on $\{\|\Theta_{[1,\infty]}\|_\alpha = 0\}$:

$$\begin{aligned}
J_\epsilon &= \mathbb{E} \left[\int_0^\infty \left(\|\Theta_{[0,\infty]}\|_\alpha^\alpha f \left(\frac{y \Theta_{[0,\infty]}}{\|\Theta_{[0,\infty]}\|_\alpha} \right) \mathbb{1} \left(\left\| \frac{y \Theta_{[0,\infty]}}{\|\Theta_{[0,\infty]}\|_\alpha} \right\|_p > 1 \right) \right. \right. \\
&\quad \left. \left. - \|\Theta_{[1,\infty]}\|_\alpha^\alpha f \left(\frac{y \Theta_{[1,\infty]}}{\|\Theta_{[1,\infty]}\|_\alpha} \right) \mathbb{1} \left(\left\| \frac{y \Theta_{[1,\infty]}}{\|\Theta_{[1,\infty]}\|_\alpha} \right\|_p > 1 \right) \right) d(-y^{-\alpha}) \right] \\
&= \sum_{t=1}^\infty \int_0^\infty \mathbb{E} \left[\|\Theta_t\|_\alpha^\alpha \left(f \left(\frac{y \Theta_{[0,\infty]}}{\|\Theta_{[0,\infty]}\|_\alpha} \right) \mathbb{1} \left(\left\| \frac{y \Theta_{[0,\infty]}}{\|\Theta_{[0,\infty]}\|_\alpha} \right\|_p > 1 \right) \right. \right. \\
&\quad \left. \left. - f \left(\frac{y \Theta_{[1,\infty]}}{\|\Theta_{[1,\infty]}\|_\alpha} \right) \mathbb{1} \left(\left\| \frac{y \Theta_{[1,\infty]}}{\|\Theta_{[1,\infty]}\|_\alpha} \right\|_p > 1 \right) \right) \right] d(-y^{-\alpha}) \\
&\quad + \mathbb{E} \left[\int_0^\infty f \left(\frac{y \Theta_{[0,\infty]}}{\|\Theta_{[0,\infty]}\|_\alpha} \right) \mathbb{1} \left(\left\| \frac{y \Theta_{[0,\infty]}}{\|\Theta_{[0,\infty]}\|_\alpha} \right\|_p > 1 \right) d(-y^{-\alpha}) \right].
\end{aligned}$$

Next we apply the time-change formula (3.5) to each summand.

$$\begin{aligned}
J_\epsilon &= \sum_{t=1}^\infty \int_0^\infty \mathbb{E} \left[f \left(\frac{y \Theta_{[-t,\infty]}}{\|\Theta_{[-t,\infty]}\|_\alpha} \right) \mathbb{1} \left(\left\| \frac{y \Theta_{[-t,\infty]}}{\|\Theta_{[-t,\infty]}\|_\alpha} \right\|_p > 1 \right) \right. \\
&\quad \left. - f \left(\frac{y \Theta_{[1-t,\infty]}}{\|\Theta_{[1-t,\infty]}\|_\alpha} \right) \mathbb{1} \left(\left\| \frac{y \Theta_{[1-t,\infty]}}{\|\Theta_{[1-t,\infty]}\|_\alpha} \right\|_p > 1 \right) \right] d(-y^{-\alpha}) \\
&\quad + \mathbb{E} \left[\int_0^\infty f \left(\frac{y \Theta_{[0,\infty]}}{\|\Theta_{[0,\infty]}\|_\alpha} \right) \mathbb{1} \left(\left\| \frac{y \Theta_{[0,\infty]}}{\|\Theta_{[0,\infty]}\|_\alpha} \right\|_p > 1 \right) d(-y^{-\alpha}) \right].
\end{aligned}$$

This is a telescoping sum in t with value

$$J_\epsilon = \mathbb{E} \left[\int_0^\infty f \left(\frac{y \Theta}{\|\Theta\|_\alpha} \right) \mathbb{1} \left(\left\| \frac{y \Theta}{\|\Theta\|_\alpha} \right\|_p > 1 \right) d(-y^{-\alpha}) \right].$$

By monotone convergence we have

$$(7.7) \quad \lim_{\epsilon \downarrow 0} J_\epsilon = \mathbb{E} \left[\int_0^\infty f \left(y \frac{\Theta}{\|\Theta\|_\alpha} \right) \mathbb{1} \left(\left\| y \frac{\Theta}{\|\Theta\|_\alpha} \right\|_p > 1 \right) d(-y^{-\alpha}) \right].$$

Combining the arguments above, we proved (7.6) as desired. \square

7.2. Proof of Lemma 5.4. The case $p < \alpha$. Choose some $\epsilon > 0$, $\delta \in (0, 1)$. We have the following bounds via truncation

$$\begin{aligned}
I_1 - I_2 &:= \mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0,n]}\|_\epsilon^p - \mathbb{E}[\|x_n^{-1} \mathbf{X}_{[0,n]}\|_\epsilon^p] > 1 + \delta^p) \\
&\quad - \mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0,n]}\|_\epsilon^p - \mathbb{E}[\|x_n^{-1} \mathbf{X}_{[0,n]}\|_\epsilon^p] \leq -\delta^p) \\
&\leq \mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0,n]}\|_\epsilon^p - \mathbb{E}[\|x_n^{-1} \mathbf{X}_{[0,n]}\|_\epsilon^p] > 1) \\
&\leq \mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0,n]}\|_\epsilon^p - \mathbb{E}[\|x_n^{-1} \mathbf{X}_{[0,n]}\|_\epsilon^p] > 1 - \delta^p) \\
&\quad + \mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0,n]}\|_\epsilon^p - \mathbb{E}[\|x_n^{-1} \mathbf{X}_{[0,n]}\|_\epsilon^p] > \delta^p) =: I_3 + I_4.
\end{aligned}$$

Taking into account \mathbf{CS}_p for $p < \alpha$, we have

$$\lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} (I_2 + I_4) / (n \mathbb{P}(|\mathbf{X}_0| > x_n)) = 0.$$

Moreover, we observe that by Karamata's theorem for $p < \alpha$

$$\begin{aligned} \mathbb{E}\left[\|x_n^{-1}\mathbf{X}_{[0,n]}\|_p^p\right] &= n\mathbb{E}\left[|\mathbf{X}_0/x_n|^p \mathbf{1}(|\mathbf{X}_0| > \epsilon x_n)\right] \\ &= O(n\mathbb{P}(|\mathbf{X}_0| > \epsilon x_n)) = o(1). \end{aligned}$$

Thus centering in I_1 and I_3 is not needed, and one can follow the lines of the proof of Lemma 7.1 to conclude.

The case $p = \alpha$. It requires only slight changes; we omit details. \square

7.3. Proofs of the results of Section 3.

7.3.1. *Proof of Proposition 3.1.* The representation (3.1) follows by identifying $\lim_{\epsilon \downarrow 0} J_\epsilon$ as on the right-hand side of (7.7). In particular, taking f as the constant map $(\mathbf{x}_t) \mapsto 1$ in (3.1) we obtain the representation of the constant $c(p)$ in (3.2).

7.3.2. *Proof of Proposition 3.3.* Our goal is first to relate the sequence of spectral components $(\mathbf{Q}^{(p)}(h))_{h \geq 0}$ to (Θ_t) . We start with two auxiliary results whose proofs are given at the end of this section.

Lemma 7.3. *Let (\mathbf{X}_t) be a stationary time series satisfying \mathbf{RV}_α . Then for $h \geq 0$,*

$$\mathbb{P}(\mathbf{Q}^{(p)}(h) \in \cdot) = \frac{1}{c(p, h)} \sum_{k=0}^h \mathbb{E}\left[\frac{\|\Theta_{-k+[0, h]}\|_p^\alpha}{\|\Theta_{-k+[0, h]}\|_\alpha^\alpha} \mathbf{1}\left(\frac{\Theta_{-k+[0, h]}}{\|\Theta_{-k+[0, h]}\|_p} \in \cdot\right)\right], \quad (7.8)$$

where $c(p, h) := \sum_{k=0}^h \mathbb{E}[\|\Theta_{-k+[0, h]}\|_p^\alpha / \|\Theta_{-k+[0, h]}\|_\alpha^\alpha]$. In particular, $c(\alpha, h) = h + 1$ and

$$\mathbb{P}(\mathbf{Q}^{(\alpha)}(h) \in \cdot) = \mathbb{P}(\Theta_{-U^{(h)}+[0, h]} / \|\Theta_{-U^{(h)}+[0, h]}\|_\alpha \in \cdot), \quad (7.9)$$

where $U^{(h)}$ is uniformly distributed on $\{0, \dots, h\}$ and independent of Θ .

Lemma 7.4. *Assume $|\Theta_t| \rightarrow 0$ as $t \rightarrow \infty$ and let $f : \tilde{\ell}^\alpha \cap \{\mathbf{x} : \|\mathbf{x}\|_p = 1\} \rightarrow (0, \infty)$ be any bounded Lipschitz-continuous function in $(\tilde{\ell}^\alpha, \tilde{d}_\alpha)$. Then, for every $p \geq \alpha$,*

$$\frac{c(p, h)}{h+1} \mathbb{E}[f(\mathbf{Q}^{(p)}(h))] \rightarrow \mathbb{E}[\|\Theta / \|\Theta\|_\alpha\|_p^\alpha f(\Theta / \|\Theta\|_p)], \quad (7.10)$$

as $h \rightarrow +\infty$

We conclude from (7.10) for $f(\mathbf{x}) \equiv 1$ that $\lim_{h \rightarrow \infty} c(p, h)/(h+1) = c(p)$. If $0 < c(p) < \infty$,

$$\lim_{h \rightarrow \infty} \mathbb{E}[f(\mathbf{Q}^{(p)}(h))] = c(p)^{-1} \mathbb{E}[\|\Theta / \|\Theta\|_\alpha\|_p^\alpha f(\Theta / \|\Theta\|_p)]. \quad (7.11)$$

Finally, the portmanteau theorem yields $\mathbf{Q}^{(p)}(h) \xrightarrow{d} \mathbf{Q}^{(p)}(\infty)$ in $(\tilde{\ell}^p \cap \{\mathbf{x} : \|\mathbf{x}\|_p = 1\}, \tilde{d}_p)$ where $\mathbf{Q}^{(p)}(\infty)$ is well defined in view of the right-hand side of (7.11). This finishes the proof of the proposition. \square

Proof of Lemma 7.3. If $\|\mathbf{X}_{[0,h]}/x\|_p > 1$ then for sufficiently small $\epsilon > 0$, $\|\mathbf{X}_{[0,h]}/x\|_\infty > \epsilon$. Therefore, on $\{\|\mathbf{X}_{[0,h]}/x\|_p > 1\}$,

$$\sum_{i=0}^h |\mathbf{X}_i/x|^\alpha \mathbf{1}(|\mathbf{X}_i/x| > \epsilon) > 0.$$

Using stationarity, we obtain

$$\begin{aligned} & \mathbb{P}(\|\mathbf{X}_{[0,h]}/x\|_p > 1) \\ &= \sum_{i=0}^h \mathbb{E} \left[\frac{|\mathbf{X}_i/x|^\alpha \mathbf{1}(|\mathbf{X}_i/x| > \epsilon)}{\sum_{t=0}^h |\mathbf{X}_t/x|^\alpha \mathbf{1}(|\mathbf{X}_t/x| > \epsilon)} \mathbf{1}(\|\mathbf{X}_{[0,h]}/x\|_p > 1) \right] \\ &= \sum_{i=0}^h \mathbb{E} \left[\frac{|\mathbf{X}_0/x|^\alpha \mathbf{1}(|\mathbf{X}_0/x| > \epsilon)}{\sum_{t=-i}^{h-i} |\mathbf{X}_t/x|^\alpha \mathbf{1}(|\mathbf{X}_t/x| > \epsilon)} \mathbf{1}(\|\mathbf{X}_{[-i,h-i]}/x\|_p > 1) \right] \\ &= \mathbb{P}(|\mathbf{X}_0| > x\epsilon) \sum_{i=0}^h \mathbb{E} \left[\frac{|\mathbf{X}_0/x|^\alpha \mathbf{1}(\|\mathbf{X}_{[-i,h-i]}/x\|_p > 1)}{\sum_{t=-i}^{h-i} |\mathbf{X}_t/x|^\alpha \mathbf{1}(|\mathbf{X}_t/x| > \epsilon)} \mid |\mathbf{X}_0| > x\epsilon \right]. \end{aligned}$$

Applying the definition (2.1) of regular variation and dominated convergence, we obtain as $x \rightarrow \infty$,

$$\begin{aligned} \frac{\mathbb{P}(\|\mathbf{X}_{[0,h]}/x\|_p > 1)}{\mathbb{P}(|\mathbf{X}_0| > x)} &\rightarrow \epsilon^{-\alpha} \sum_{i=0}^h \mathbb{E} \left[\frac{|\epsilon Y \Theta_0|^\alpha \mathbf{1}(\|\epsilon Y \Theta_{[-i,h-i]}\|_p > 1)}{\sum_{t=-i}^{h-i} |\epsilon Y \Theta_t|^\alpha \mathbf{1}(|Y \Theta_t| > 1)} \right] \\ &= \sum_{i=0}^h \int_\epsilon^\infty \mathbb{E} \left[\frac{\mathbf{1}(y \|\Theta_{[-i,h-i]}\|_p > 1)}{\sum_{t=-i}^{h-i} |\Theta_t|^\alpha \mathbf{1}(y |\Theta_t| > \epsilon)} \right] d(-y^{-\alpha}). \end{aligned}$$

The left-hand side does not depend on ϵ . Therefore, letting $\epsilon \downarrow 0$, we arrive at

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{X}_{[0,h]}/x\|_p > 1)}{\mathbb{P}(|\mathbf{X}_0| > x)} &= \sum_{i=0}^h \int_0^\infty \mathbb{E} \left[\frac{\mathbf{1}(y \|\Theta_{[-i,h-i]}\|_p > 1)}{\|\Theta_{[-i,h-i]}\|_\alpha^\alpha} \right] d(-y^{-\alpha}) \\ (7.12) \qquad \qquad \qquad &= \sum_{i=0}^h \mathbb{E} \left[\frac{\|\Theta_{-i+[0,h]}\|_p^\alpha}{\|\Theta_{-i+[0,h]}\|_\alpha^\alpha} \right] = c(p, h). \end{aligned}$$

This constant is finite since $\|\Theta_{i+[0,h]}\|_p \leq (h+1)\|\Theta_{i+[0,h]}\|_\infty$.

Next we prove (7.8). For this reason, let A be a continuity set with respect to the limit law in (7.8). An appeal to (7.12) yields

$$\begin{aligned} & c(p, h) \mathbb{P}(x^{-1} \mathbf{X}_{[0,h]} \in A \mid \|\mathbf{X}_{[0,h]}\|_p > x) \\ & \sim \frac{\mathbb{P}(x^{-1} \mathbf{X}_{[0,h]} \in A, \|\mathbf{X}_{[0,h]}\|_p > x)}{\mathbb{P}(\|\mathbf{X}_{[0,h]}\|_p > x)} =: I(x). \end{aligned}$$

Proceeding as for the derivation of (7.12), we obtain

$$\begin{aligned} I(x) &\sim \int_0^\infty \sum_{i=0}^h \mathbb{E} \left[\frac{\mathbb{1}(y \|\Theta_{[-i, h-i]}\|_p > 1)}{\|\Theta_{[-i, h-i]}\|_\alpha^\alpha} \mathbb{1}(y \Theta_{[-i, h-i]} \in A) \right] d(-y^{-\alpha}) \\ &= \int_1^\infty \sum_{i=0}^h \mathbb{E} \left[\frac{\|\Theta_{[-i, h-i]}\|_p^\alpha}{\|\Theta_{[-i, h-i]}\|_\alpha^\alpha} \mathbb{1}\left(y \frac{\Theta_{[-i, h-i]}}{\|\Theta_{[-i, h-i]}\|_p} \in A\right) \right] d(-y^{-\alpha}). \end{aligned}$$

In the last step we changed the variable, $u = y \|\Theta_{[-i, h-i]}\|_p > 0$ a.s., observing that $\|\Theta_{[-i, h-i]}\|_p \geq |\Theta_0| = 1$. This proves (7.8) and the lemma. \square

Proof of Lemma 7.4. Assume $f : \tilde{\ell}^\alpha \cap \{\mathbf{x} : \|\mathbf{x}\|_p = 1\} \rightarrow (0, \infty)$ is any bounded Lipschitz-continuous function in $(\tilde{\ell}^\alpha, \tilde{d}_\alpha)$. By Lemma 7.3 we have for all $p \geq \alpha$,

$$\begin{aligned} &\frac{c(p, h)}{h+1} \mathbb{E}[f(\mathbf{Q}^{(p)}(h))] - c(p) \mathbb{E}[f(\mathbf{Q}^{(p)})] = \\ &= \frac{1}{h+1} \sum_{k=0}^h \mathbb{E} \left[\left(\frac{\|\Theta_{-k+[0, h]}\|_p^\alpha}{\|\Theta_{-k+[0, h]}\|_\alpha^\alpha} - \frac{\|\Theta\|_p^\alpha}{\|\Theta\|_\alpha^\alpha} \right) f(\Theta_{-k+[0, h]} / \|\Theta_{-k+[0, h]}\|_p) \right] + \\ &\quad + \mathbb{E} \left[\frac{\|\Theta\|_p^\alpha}{\|\Theta\|_\alpha^\alpha} \left(f(\Theta_{-k+[0, h]} / \|\Theta_{-k+[0, h]}\|_p) - f(\Theta / \|\Theta\|_p) \right) \right] \\ &=: I + II. \end{aligned}$$

We will prove that I and II vanish as $h \rightarrow \infty$. Since $p \geq \alpha$ subadditivity yields for $k \in [0, h]$,

$$\begin{aligned} &\left| \frac{\|\Theta_{-k+[0, h]}\|_p^\alpha}{\|\Theta_{-k+[0, h]}\|_\alpha^\alpha} - \frac{\|\Theta\|_p^\alpha}{\|\Theta\|_\alpha^\alpha} \right| = \frac{\left| \|\|\Theta\|_\alpha \Theta_{-k+[0, h]}\|_p^\alpha - \|\|\Theta_{-k+[0, h]}\|_\alpha \Theta\|_p^\alpha \right|}{\|\Theta_{-k+[0, h]}\|_\alpha^\alpha \|\Theta\|_\alpha^\alpha} \\ &\leq \frac{\left| \|\|\Theta\|_\alpha \Theta_{-k+[0, h]}\|_p^p - \|\|\Theta_{-k+[0, h]}\|_\alpha \Theta\|_p^p \right|^{\alpha/p}}{\|\Theta_{-k+[0, h]}\|_\alpha^\alpha \|\Theta\|_\alpha^\alpha}. \end{aligned}$$

Moreover,

$$\begin{aligned} &\left| \|\|\Theta\|_\alpha \Theta_{-k+[0, h]}\|_p^p - \|\|\Theta_{-k+[0, h]}\|_\alpha \Theta\|_p^p \right| \\ &\leq \|\Theta_{-k+[0, h]}\|_\alpha^p \left(\sum_{t=-\infty}^{-k-1} |\Theta_t|^p + \sum_{t=-k+h+1}^{+\infty} |\Theta_t|^p \right) \\ &\quad + \left| \|\Theta\|_\alpha - \|\Theta_{-k+[0, h]}\|_\alpha \right|^p \sum_{t=-k}^{-k+h} |\Theta_t|^p. \end{aligned}$$

Thus, $|I|$ is bounded from above by

$$\begin{aligned} & \frac{1}{h+1} \|f\|_\infty \sum_{k=0}^h \left(\mathbb{E} \left[\frac{\|\Theta_{[-\infty, -k-1]}\|_p^\alpha}{\|\Theta\|_\alpha^\alpha} \right] + \mathbb{E} \left[\frac{\|\Theta_{[-k+h+1, \infty]}\|_p^\alpha}{\|\Theta\|_\alpha^\alpha} \right] \right. \\ & \quad \left. + \mathbb{E} \left[\frac{\|\Theta_{[-\infty, -k-1]\|_\alpha^\alpha + \|\Theta_{[-k+h+1, \infty]}\|_\alpha^\alpha}{\|\Theta\|_\alpha^\alpha} \frac{\|\Theta_{-k+[0, h]}\|_p^\alpha}{\|\Theta_{-k+[0, h]}\|_\alpha^\alpha} \right] \right) \\ & \leq \frac{1}{h+1} \|f\|_\infty \sum_{k=0}^h \left(\mathbb{E} \left[\frac{\|\Theta_{[-\infty, -(k+1)]}\|_p^\alpha + \|\Theta_{[-\infty, -(k+1)]}\|_\alpha^\alpha}{\|\Theta\|_\alpha^\alpha} \right] \right. \\ & \quad \left. + \mathbb{E} \left[\frac{\|\Theta_{[k+1, \infty]}\|_p^\alpha + \|\Theta_{[k+1, \infty]}\|_\alpha^\alpha}{\|\Theta\|_\alpha^\alpha} \right] \right). \end{aligned}$$

Taking the limit as $h \rightarrow \infty$, the Cèsaro limit on the right-hand side converges to zero.

We use the Lipschitz-continuity of f to obtain an upper bound of $|II|$:

$$|II| \leq \frac{1}{h+1} c \sum_{k=0}^h \mathbb{E} \left[\frac{\|\Theta_{-k+[0, h]}\|_p^\alpha}{\|\Theta_{-k+[0, h]}\|_\alpha^\alpha} \tilde{d}_\alpha \left(\frac{\Theta_{-k+[0, h]}}{\|\Theta_{-k+[0, h]}\|_p}, \frac{\Theta}{\|\Theta\|_p} \right) \right].$$

Similar arguments as for $|I| \rightarrow 0$ show that $|II| \rightarrow 0$. \square

7.4. Proofs of the results of Section 4.

7.4.1. *Proof of Theorem 4.1.* We start with a version of Theorem 4.1 for deterministic thresholds (x_b) .

Lemma 7.5. *Assume the conditions of Theorem 4.1. Then for every $g \in \mathcal{G}_+(\ell^p)$,*

$$(7.13) \quad \frac{1}{k} \sum_{t=1}^m g(x_b^{-1} \mathbf{B}_t) \xrightarrow{\mathbb{P}} \int_0^\infty \mathbb{E}[g(y \mathbf{Q}^{(p)})] d(-y^{-\alpha}), \quad n \rightarrow \infty,$$

holds for sequences $k_n \rightarrow \infty$ and $m_n := \lfloor n/b_n \rfloor \rightarrow \infty$ as in \mathbf{MX}_p .

Proof. If \mathbf{MX}_p holds for Lipschitz-continuous $f \in \mathcal{G}_+(\tilde{\ell}^p)$, then it holds for functions $g \in \mathcal{G}_+(\ell^p)$ of the form $g(\mathbf{x}_t) = \mathbb{1}(\mathbf{x}_t \in A)$ where A is a continuity-set of $\tilde{\ell}^p$ and $\mathbf{0} \notin \bar{A}$. It suffices to prove that

$$(7.14) \quad \left(\mathbb{E} \left[e^{-\frac{1}{k} \sum_{t=1}^{\lfloor m/k \rfloor} g(x_b^{-1} \mathbf{B}_t)} \right] \right)^k \rightarrow e^{-\mathbb{E} \left[\int_0^\infty g(y \mathbf{Q}^{(p)}) d(-y^{-\alpha}) \right]}.$$

By stationarity,

$$(7.15) \quad \mathbb{E} \left[1 - e^{-\frac{1}{k} \sum_{t=1}^{\lfloor m/k \rfloor} g(x_b^{-1} \mathbf{B}_t)} \right] = O(k^{-2} m \mathbb{E}[g(x_b^{-1} \mathbf{B}_1)]).$$

Since g vanishes in some neighborhood of the origin there exists $c_g > 0$ such that $g(\mathbf{x}) = g(\mathbf{x}) \mathbb{1}(\|\mathbf{x}\|_p > c_g)$. Therefore and by virtue of Proposition 7.2 the right-hand side of (7.15) vanishes as $n \rightarrow \infty$. Now a Taylor expansion argument shows that the left-hand side of (7.14) is of the asymptotic order $\sim \exp\{-(m/k) \mathbb{E}[g(x_b^{-1} \mathbf{B}_t)]\}$, and another application of Proposition 7.2 yields

(7.14). We conclude by the portmanteau theorem for $M_0(\tilde{\ell}^p)$ -convergence in Hult and Lindskog [21], Theorem 2.4. that (7.13) holds. \square

We continue with the proof of Theorem 4.1. Lemma 7.5 implies convergence of the empirical measures in $M_0(\tilde{\ell}^p)$:

$$P_n(\cdot) := \frac{1}{k} \sum_{t=1}^m \mathbb{1}(x_b^{-1} \mathbf{B}_t \in \cdot) \xrightarrow{\mathbb{P}} P(\cdot) := \int_0^\infty \mathbb{P}(y \mathbf{Q}^{(p)} \in \cdot) d(-y^{-\alpha}).$$

Using the argument in Resnick [33], p. 81, we may conclude $\|\mathbf{B}\|_{p,(k)}/x_b \xrightarrow{\mathbb{P}} 1$, and thus the joint convergence in $(P_n, \|\mathbf{B}\|_{p,(k)}/x_b) \xrightarrow{\mathbb{P}} (P, 1)$ in $M_0(\tilde{\ell}^p) \times \mathbb{R}_+$ follows. Now (4.2) follows by an application of the continuous mapping theorem to the scaling function $s(P(\cdot), t) = P(t \cdot)$. To prove continuity of s we use again the portmanteau theorem for $M_0(\tilde{\ell}^p)$ -convergence in Hult and Lindskog [21], Theorem 2.4. Thus it suffices to check whether the limit $P_n f(\cdot/t) \xrightarrow{\mathbb{P}} P f$ holds as $(n, t) \rightarrow (\infty, 1)$ for Lipschitz-continuous $f \in \mathcal{G}_+(\tilde{\ell}^p)$. But we have with Lemma 7.5

$$\begin{aligned} |P_n f(\cdot/t) - P f| &\leq |P_n f(\cdot/t) - P_n f| + |P_n f - P f| \\ &= |P_n f(\cdot/t) - P_n f| + o_{\mathbb{P}}(1), \quad n \rightarrow \infty. \end{aligned}$$

Then, for all $0 < t_0 \leq t < 2$, for $t_0 \leq 1$, setting $g(\mathbf{x}) = (\|\mathbf{x}\|_p \wedge \|f\|_\infty) \mathbb{1}(\{\mathbf{x} : \|\mathbf{x}\|_p > c_f/t_0\})$, we have

$$\begin{aligned} |P_n f(\cdot/t) - P f| &\leq |t^{-1} - 1| P_n g + o_{\mathbb{P}}(1) \\ &\leq |t^{-1} - 1| (c + o_{\mathbb{P}}(1)) + o_{\mathbb{P}}(1), \end{aligned}$$

for some $c > 0$, $c_f > 0$ as above. Letting $t \rightarrow 1$, continuity of s follows. \square

7.4.2. *Proof of Proposition 4.4.* The result follows by a direct application of Theorem 3.1 in Bartkiewicz *et al.* [1] on $\mathbf{u}^\top \mathbf{S}_n$ for any $\mathbf{u} \in \mathbb{R}^d$ such that $|\mathbf{u}| = 1$ by checking their conditions **(AC)**, **(TB)**. Condition (4.8) implies that for all $\delta > 0$,

$$\lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} n \sum_{t=l}^{b_n} \mathbb{P}(|\mathbf{X}_t| > \delta a_n, |\mathbf{X}_0| > \delta a_n),$$

from which **(AC)** is immediate. This condition also implies **(TB)**. We show this in two steps. First, we identify the coefficients $b(d)$ in **(TB)** in terms of the spectral tail process. Mikosch and Wintenberger [27] showed that

$$b_{\pm}(d) - b_{\pm}(d-1) = \mathbb{E} \left[\left(\sum_{j=0}^d \mathbf{u}^\top \boldsymbol{\Theta}_j \right)_{\pm}^{\alpha} \right] - \mathbb{E} \left[\left(\sum_{j=1}^d \mathbf{u}^\top \boldsymbol{\Theta}_j \right)_{\pm}^{\alpha} \right],$$

where we suppress in the notation the dependence of the left-hand side on \mathbf{u} in what follows. **(TB)** amounts to verifying that $b_{\pm}(d) - b_{\pm}(d-1)$ converges as $d \rightarrow \infty$. For $\alpha \in (0, 1)$ this follows by concavity since $\|\boldsymbol{\Theta}\|_{\alpha} < \infty$ a.s. For

$1 < \alpha < 2$ this will follow by a convexity argument if $\mathbb{E}[(\sum_{j \geq 0} |\Theta_j|)^{\alpha-1}] < \infty$. By subadditivity and Jensen's inequality, it is enough to check

$$(7.16) \quad \sum_{j=0}^{\infty} (\mathbb{E}[|\Theta_j|^{\alpha-1} \mathbf{1}(|\Theta_j| > 1)] + \mathbb{E}[|\Theta_j| \wedge 1]) < +\infty.$$

We start by showing

$$(7.17) \quad \sum_{j=0}^{\infty} \mathbb{E}[|\Theta_j| \wedge 1] < \infty.$$

Condition (4.8) implies

$$\lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} n \sum_{j=l}^{b_n} \mathbb{E}[(|a_n^{-1} \mathbf{X}_j| \wedge 1) \mathbf{1}(|\mathbf{X}_0| > a_n)] = 0,$$

which yields the following Cauchy criterion: for any $\varepsilon > 0$ there exists K sufficiently large such that for $l \geq K, h \geq 0$,

$$\limsup_{n \rightarrow \infty} n \sum_{j=l}^{l+h} \mathbb{E}[(|a_n^{-1} \mathbf{X}_j| \wedge 1) \mathbf{1}(|\mathbf{X}_0| > a_n)] = \sum_{j=l}^{l+h} \mathbb{E}[|Y \Theta_j| \wedge 1] \leq \varepsilon,$$

where we used regular variation of (\mathbf{X}_t) in the last step. Then, we conclude (7.17) holds. By stationarity we can show similarly

$$(7.18) \quad \sum_{j=0}^{\infty} \mathbb{E}[|\Theta_{-j}| \wedge 1] < +\infty.$$

Then, by the time-change formula in (3.5) we deduce

$$\begin{aligned} \infty &> \sum_{j=0}^{\infty} \mathbb{E}[|\Theta_{-j}| \wedge 1] = \sum_{j=0}^{\infty} \mathbb{E}[|\Theta_j|^\alpha (|\Theta_j|^{-1} \wedge 1)] \\ &= \sum_{j=0}^{\infty} \mathbb{E}[|\Theta_j|^{\alpha-1} \wedge |\Theta_j|^\alpha] > \sum_{j=0}^{\infty} \mathbb{E}[|\Theta_j|^{\alpha-1} \mathbf{1}(|\Theta_j| > 1)], \end{aligned}$$

and (7.16) holds. This finishes the proof of the fact that $\mathbb{E}[(\sum_{j \geq 0} |\Theta_t|)^{\alpha-1}] < +\infty$, in particular $c(1) < \infty$. Applying the mean value theorem and dominated convergence we arrive at the relation

$$b_{\pm}(d) - b_{\pm}(d-1) \rightarrow \mathbb{E}\left[\left(\sum_{j=0}^{\infty} \mathbf{u}^\top \Theta_j\right)_{\pm}^{\alpha} - \left(\sum_{j=1}^{\infty} \mathbf{u}^\top \Theta_j\right)_{\pm}^{\alpha}\right], \quad d \rightarrow \infty.$$

Reasoning for the limit as for (3.6) and recalling that $c(1) < \infty$, we identify

$$\begin{aligned} & \mathbb{E} \left[\left(\sum_{j=0}^{\infty} \mathbf{u}^\top \Theta_j \right)_\pm^\alpha - \left(\sum_{j=1}^{\infty} \mathbf{u}^\top \Theta_j \right)_\pm^\alpha \right] \\ &= \mathbb{E} \left[\left(\sum_{j=-\infty}^{\infty} \mathbf{u}^\top \Theta_j \right)_\pm^\alpha / \|\Theta\|_\alpha^\alpha \right] = \mathbb{E} \left[\left(\sum_{j=-\infty}^{\infty} \mathbf{u}^\top \mathbf{Q}_j^{(\alpha)} \right)_\pm^\alpha \right] \\ &= c(1) \mathbb{E} \left[\left(\sum_{j=-\infty}^{\infty} \mathbf{u}^\top \mathbf{Q}_j^{(1)} \right)_\pm^\alpha \right]. \end{aligned}$$

7.5. Proofs of the results of Section 6.

7.5.1. *Proof of Proposition 6.1.* Notice that ψ_g is bounded and measurable. For $p = \alpha$ we have $\mathbf{Q}^{(\alpha)} \stackrel{d}{=} \Theta / \|\Theta\|_\alpha$. Then the result follows from Proposition 3.6 in Janssen [22]. For $p > 0$, assuming the spectral cluster process $\mathbf{Q}^{(p)}$ is well defined we have $\|\Theta\|_p < \infty$ a.s. and $c(p) < \infty$. Then, we introduce the Radon-Nikodym derivative of $\mathcal{L}(\mathbf{Q}^{(p)})$ with respect to $\mathcal{L}(\Theta / \|\Theta\|_p)$ which by (3.1) is the function $h : \ell^p \cap \{\mathbf{x} : \|\mathbf{x}\|_p = 1\} \rightarrow \mathbb{R}_{\geq 0}$ defined by $h(\mathbf{y} / \|\mathbf{y}\|_p) := \|\mathbf{y}\|_\alpha / \|\mathbf{y}\|_p$. Finally, the result follows by another application of Proposition 3.6 in Janssen [22]. \square

7.5.2. *Proof of Theorem 6.2.* The proof is given for $p = \alpha$ only; the case $p \leq \alpha$ extends in a natural way. Let $g : \ell^\alpha \rightarrow \mathbb{R}$ be a continuous bounded function. We start by proving that ψ_g defined in (6.1) is a continuous bounded function on $\tilde{\ell}^\alpha$. Fix $\epsilon > 0$ and $[\mathbf{z}] \in \tilde{\ell}^\alpha \cap \{[\mathbf{y}] : \|\mathbf{y}\|_\alpha = 1\}$. Then for all $[\mathbf{x}] \in \tilde{\ell}^\alpha \cap \{[\mathbf{y}] : \|\mathbf{y}\|_\alpha = 1\}$, $k \in \mathbb{Z}$ and $N \in \mathbb{N}$, we have

$$\begin{aligned} & |\psi_g(\mathbf{x}) - \psi_g(\mathbf{z})| = \left| \sum_{j \in \mathbb{Z}} |\mathbf{x}_j^*|^\alpha g((\mathbf{x}_{j+t}^*)_t) - \sum_{j \in \mathbb{Z}} |\mathbf{z}_j^*|^\alpha g((\mathbf{z}_{j+t}^*)_t) \right| \\ &= \left| \sum_{j \in \mathbb{Z}} |\mathbf{x}_j^* - \mathbf{z}_{j-k}^*|^\alpha g((\mathbf{x}_{j+t}^*)_t) - \sum_{j \in \mathbb{Z}} |\mathbf{z}_j^*|^\alpha (g((\mathbf{z}_{j+t}^*)_t) - g((\mathbf{x}_{j+t+k}^*)_t)) \right| \\ &\leq \|g\|_\infty d_\alpha^\alpha(B^{-k} \mathbf{z}^*, \mathbf{x}^*) + 2\|g\|_\infty d_\alpha^\alpha(\mathbf{z}^*, \mathbf{z}_{[-N, N]}^*) \\ &\quad + \sum_{|j| < N} |\mathbf{z}_j^*|^\alpha |g((\mathbf{z}_{j+t}^*)_t) - g((\mathbf{x}_{j+t+k}^*)_t)|. \end{aligned}$$

If $[\mathbf{x}]$ satisfies $\tilde{d}_\alpha^\alpha(\mathbf{z}, \mathbf{x}) < \epsilon(3\|g\|_\infty)^{-1}$ then there exists $k_0 \in \mathbb{Z}$ such that

$$\tilde{d}_\alpha^\alpha(\mathbf{z}, \mathbf{x}) < d_\alpha^\alpha(B^{-k_0} \mathbf{z}^*, \mathbf{x}^*) < \epsilon(3\|g\|_\infty)^{-1}.$$

Furthermore, choose $N_0 \geq 0$ such that $d_\alpha^\alpha(\mathbf{z}^*, \mathbf{z}_{[-N_0, N_0]}^*) < \epsilon(2 \times 3\|g\|_\infty)^{-1}$ and consider the finite set $C_{[\mathbf{z}]} \subset \ell^\alpha \cap \{[\mathbf{y}] : \|\mathbf{y}\|_\alpha = 1\}$, defined by $C_{[\mathbf{z}]} := \{(\mathbf{z}_{j+t}^*)_t \in \ell^\alpha : |j| < N_0, |\mathbf{z}_j^*| > 0\}$. Notice that for every $\tilde{\mathbf{z}} \in C_{[\mathbf{z}]}$ there exists $\delta(\tilde{\mathbf{z}})$ such that if $d_\alpha^\alpha(\tilde{\mathbf{z}}, \mathbf{x}) < \delta(\tilde{\mathbf{z}})$ implies $|g(\tilde{\mathbf{z}}) - g(\mathbf{x})| < \epsilon/3$. Finally, define $\eta(\mathbf{z}) := \min\{\delta(\tilde{\mathbf{z}}) : \tilde{\mathbf{z}} \in C\} \wedge \epsilon(3\|g\|_\infty)^{-1}$. Then, noticing that

$\sum_{|j| < N_0} |\mathbf{z}_j^*|^\alpha \leq \|\mathbf{z}\|_\alpha^\alpha = 1$, we also obtain a bound for the last term. Hence, for every $[\mathbf{x}] \in \tilde{\ell}^\alpha$ satisfying $d_\alpha^\alpha(\mathbf{z}, \mathbf{x}) < \eta(\mathbf{z})$ we have $|\psi_g(\mathbf{x}) - \psi_g(\mathbf{z})| < \epsilon$.

This finishes the proof of the continuity of the function ψ_g on $\tilde{\ell}^\alpha \cap \{\mathbf{y} : \|\mathbf{y}\|_\alpha = 1\}$. We conclude with applications of Lemma 7.5 and Proposition 6.1. \square

7.5.3. *Proof of Proposition 6.5.* Theorem 4.5 in Mikosch and Wintenberger [29] yields immediately

$$\left| \frac{\mathbb{P}(\sup_{1 \leq t \leq n} S_t > x_n)}{n\mathbb{P}(|X_1| > x_n)} - \mathbb{E} \left[\left(\sup_{t \geq 0} \sum_{i=0}^t \Theta_i \right)_+^\alpha - \left(\sup_{t \geq 1} \sum_{i=1}^t \Theta_i \right)_+^\alpha \right] \right| \rightarrow 0, \quad n \rightarrow \infty, \quad (7.19)$$

and $n\mathbb{P}(|X_1| > x_n) \rightarrow 0$. We multiply the function inside the limiting expected value by the constant $1 = \|\Theta\|_\alpha^\alpha / \|\Theta\|_\alpha^\alpha$. Moreover, since $c(1) < \infty$, then $\mathbb{E}[(\sum_{t=1}^\infty |\Theta_t|)^{\alpha-1}] < \infty$; see Lemma 3.11 in Planinić and Soulier [32]. Then, by Fubini's theorem,

$$\begin{aligned} & \mathbb{E} \left[\left(\sup_{t \geq 0} \sum_{i=0}^t \Theta_i \right)_+^\alpha - \left(\sup_{t \geq 1} \sum_{i=1}^t \Theta_i \right)_+^\alpha \right] \\ &= \sum_{j \in \mathbb{Z}} \mathbb{E} \left[|\Theta_j|^\alpha \left(\left(\sup_{t \geq 0} \sum_{i=0}^t \frac{\Theta_i}{\|\Theta\|_\alpha} \right)_+^\alpha - \left(\sup_{t \geq 1} \sum_{i=1}^t \frac{\Theta_i}{\|\Theta\|_\alpha} \right)_+^\alpha \right) \right]. \end{aligned}$$

At this point we apply the time-change formula for positive measurable functions of Θ at every term of the sum in $j \in \mathbb{Z}$; see Corollary 2.8. in Dombry *et al.* [13]. By the same argument as in the proof of Proposition 7.2 we obtain the representation of the expectation in (7.19) in terms of the univariate spectral cluster process $Q^{(p)}$.

Now we apply Theorem 6.2 to $f(\mathbf{x}) := \lim_{k \rightarrow \infty} (\sup_{t \geq -k} \sum_{i=-k}^t x_i)_+^\alpha$ on ℓ^1 . It is uniformly continuous and bounded by one on the sphere of ℓ^p , hence (6.2) holds for f . Similarly, the constant $c(1)^{-1}$ can be estimated by employing the function $g(\mathbf{x}) := \|\mathbf{x}\|_\alpha$ on ℓ^1 which is bounded by one on the unitary ℓ^1 -sphere for $\alpha \geq 1$. \square

7.5.4. *Proof of Proposition 6.7.* The re-normalization function ζ is continuous on the unit sphere of (ℓ^α, d_α) , except for sequences with $\mathbf{x}_0 = 0$. Then

$$\begin{aligned} \mathbb{P}(\rho(Y \Theta) > 1) &= \mathbb{E}[\rho(\Theta_t)^\alpha \wedge 1] = \mathbb{E}[\rho(\mathbf{Q}_t^{(\alpha)} / |\mathbf{Q}_0^{(\alpha)}|)^\alpha \wedge 1] \\ &= \mathbb{E}[(\rho^\alpha \wedge 1) \circ \zeta(\mathbf{Q}^{(\alpha)})]. \end{aligned}$$

The proof is finished by an application of Theorem 6.2. \square

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