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THRESHOLD SELECTION FOR CLUSTER INFERENCE BASED ON LARGE DEVIATION PRINCIPLES

GLORIA BURITICÁ, THOMAS MIKOSCH, AND OLIVIER WINTENBERGER

ABSTRACT. In the setting of regularly varying time series, a cluster of exceedances is a short period for which the supremum norm exceeds a high threshold. We propose to study a generalization of this notion considering short periods, or blocks, with ℓ^p -norm above a high threshold. We derive large deviation principles of blocks and apply these results to improve cluster inference. We focus on blocks estimators and show they are consistent when we use large empirical quantiles from the ℓ^p -norm of blocks as threshold levels. We derive an adaptive threshold selection method for cluster inference in ℓ^p . Our approach focuses on the case $p < \infty$ rather than the classical one for $p = \infty$ where the bias is more difficult to control.

1. INTRODUCTION

We consider an \mathbb{R}^d -valued stationary regularly varying time series $(\mathbf{X}_t)_{t \in \mathbb{Z}}$ with generic element \mathbf{X} ; see Section 2.1 for a definition, cf. Basrak and Segers [4]. It is natural to wonder about the impact of an extreme event on the future behavior of the sequence. Often an exceedance of a high threshold by the norm $|\mathbf{X}_t|$ for some t triggers consecutive exceedances in some neighborhood of t . We refer to them as a *cluster (of exceedances)*. This notion was implicitly introduced in the seminal paper by Davis and Hsing [10] and reviewed in Basrak and Segers [4], Basrak *et al.* [3].

For inference purposes we divide a sample $(\mathbf{X}_t)_{1 \leq t \leq n}$ into disjoint blocks $(\mathbf{X}_{(s-1)b_n + [1, b_n]})_{1 \leq s \leq \lfloor n/b_n \rfloor}$ for a sequence of block lengths (b_n) such that $b_n \rightarrow \infty$ and $b_n/n \rightarrow 0$ as $n \rightarrow \infty$. Typically, inference on clusters starts with selecting blocks whose supremum norm exceeds a high threshold x_{b_n} for a sequence (x_n) such that $n\mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$ as $n \rightarrow \infty$; see Kulik and Soulier [22] for a detailed recent treatment. We generalize this approach by considering blocks whose ℓ^p -norm exceed a high threshold. We define *cluster processes* in the space $\ell^p = \ell^p(\mathbb{R}^d)$ in such a way that we recover the classical clusters letting $p = \infty$. Our main goal is to improve inference procedures for functionals acting on cluster processes in ℓ^p by proposing an

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adaptive threshold selection method for ℓ^p -blocks, and to understand the role which p plays in this context. Our approach heavily relies on new large deviation results.

For inference on cluster functionals in ℓ^p we show that extreme thresholds are more suitable for block selection than moderate ones. Indeed, the latter lead to asymptotically biased estimates. This finding points at the delicate choice of the threshold related to bias and the need for a rigorous definition of extremal blocks for cluster inference. As a matter of fact, thresholds should adapt to the block length and take into account the value of p . In the existing literature for $p = \infty$ no detailed advice is given as to how (b_n) and (x_{b_n}) must be chosen; see for example Drees, Rootzén [15], Drees, Neblung [14], Cissokho, Kulik [8], Drees *et al.* [13] who assume bias conditions. It is common practice to replace x_{b_n} by an upper order statistic of $(|\mathbf{X}_t|)_{1 \leq t \leq n}$, turning the choice of b_n into a delicate problem which has not been studied carefully. This problem occurs for example in the context of the blocks estimator of the *extremal index* proposed by Hsing [19].

Motivated by results on large deviations, we design consistent disjoint blocks methods with thresholds chosen as order statistics of ℓ^p -norms that adapt to the length of the block. We show that the choice of p is crucial: the smaller p the larger the proportion of selected extreme blocks we might consider. This approach reduces the bias compared with the existing methods for $p = \infty$. However, for $p \leq \alpha$ we need a so-called vanishing-small-values condition. Then the large deviation principles sustain that the proposed estimators perform nicely as regards bias; this is supported by a Monte Carlo study for inference on the extremal index in Buriticá *et al.* [7]. In this paper, we focus on the consistency of the blocks estimators. Their asymptotic normality can be derived by combining arguments from Theorem 4.3 in Cissokho and Kulik [8] and the large deviation arguments developed below; this topic is the subject of ongoing work.

We apply our inference procedure to estimate the extremal index taking $p = \alpha > 0$, the index of regular variation of (\mathbf{X}_t) . We also consider inference of *cluster indices* as defined by Mikosch and Wintenberger [27] on partial sum functionals by letting $p = 1$. These functionals are shift-invariant with respect to the backward shift in sequence spaces; see Kulik and Soulier [22] for details. For $p \leq \alpha$, using a simple continuity argument, our approach allows one to consider α -power functionals acting on ℓ^p . Coupled with the random shift analysis of Janssen [21] based on the α th moment of the cluster process, we also extend cluster inference to functionals acting on ℓ^p rather than on shift-invariant spaces.

1.1. Outline of the paper. Section 2 contains an overview on sequential regular variation, notation and the main assumptions for the large deviation results to hold. In Section 3 we define the cluster processes in the space ℓ^p . Section 4 includes the main large deviation principles and discussion of advantages of assessing cluster inference with extreme rather than moderate

threshold levels. In Section 5 we study cluster inference for shift-invariant functionals through Theorem 5.1 fixing thresholds as empirical quantiles of the ℓ^p -norms of blocks. We also highlight advantages of choosing $p < \infty$, and apply our inference procedure for estimating the extremal index as well as the limit parameters in the central limit theorem for regularly varying time series. In Section 6 we show inference for non-shift-invariant functionals in Theorem 6.2. Finally, we defer all the proofs to section 7.

2. PRELIMINARIES

2.1. About regular variation of time series. We consider an \mathbb{R}^d -valued stationary process $\mathbf{X} := (\mathbf{X}_t)$. Following Davis and Hsing [10], we call it regularly varying if the finite-dimensional distributions of the process are regularly varying. This notion involves the vague convergence of certain tail measures; see Resnick [31]. To avoid the concept of vague convergence and infinite limit measures, Basrak and Segers [4] showed that regular variation of (\mathbf{X}_t) is equivalent to the weak convergence relations: for any $h \geq 0$,

$$\mathbb{P}(x^{-1}(\mathbf{X}_t)_{|t| \leq h} \in \cdot \mid |\mathbf{X}_0| > x) \xrightarrow{w} \mathbb{P}(Y(\Theta_t)_{|t| \leq h} \in \cdot), \quad x \rightarrow \infty, \quad (2.1)$$

where Y is Pareto(α)-distributed, i.e., it has tail $\mathbb{P}(Y > y) = y^{-\alpha}$, $y > 1$, independent of the vector $(\Theta_t)_{|t| \leq h}$ and $|\Theta_0| = 1$. According to Kolmogorov's consistency theorem, one can extend the latter finite-dimensional vectors to a sequence $\Theta = (\Theta_t)_{t \in \mathbb{Z}}$ in $(\mathbb{R}^d)^{\mathbb{Z}}$ called the *spectral tail process* of (\mathbf{X}_t) . The regular variation property, say \mathbf{RV}_α , of (\mathbf{X}_t) is determined by the (tail) index $\alpha > 0$ and the spectral tail process.

Extending vague convergence to M_0 -convergence, Hult and Lindskog [20] introduced regular variation for random elements assuming values in a general complete separable metric space; see also Lindskog *et al.* [25]. Segers *et al.* [32] proved regular variation of random elements with values in star-shaped metric spaces. Their results are based on weak convergence in the spirit of (2.1). Our focus will be on a special star-shaped space: the sequence space ℓ^p , $p \in (0, \infty]$ equipped with the metric

$$d_p(\mathbf{x}, \mathbf{y}) := \begin{cases} \|\mathbf{x} - \mathbf{y}\|_p = \left(\sum_{t \in \mathbb{Z}} |\mathbf{x}_t - \mathbf{y}_t|^p \right)^{1/p}, & p \geq 1, \\ \|\mathbf{x} - \mathbf{y}\|_p^p, & p \in (0, 1), \end{cases} \quad \mathbf{x}, \mathbf{y} \in \ell^p,$$

with the usual convention in the case $p = \infty$. We know that d_p makes ℓ^p a separable Banach space for $p \geq 1$, and a separable complete metric space for $p \in (0, 1)$. Using the p -modulus function $\|\cdot\|_p$, the ℓ^p -valued stationary process (\mathbf{X}_t) has the property \mathbf{RV}_α if and only if relation (2.1) holds with $|\mathbf{X}_0|$ replaced by $\|\mathbf{X}_{[0,h]}\|_p$. Equivalently, (Theorem 4.1 in Segers *et al.* [32]), for any $h \geq 0$,

$$\mathbb{P}(x^{-1}\mathbf{X}_{[0,h]} \in \cdot \mid \|\mathbf{X}_{[0,h]}\|_p > x) \xrightarrow{w} \mathbb{P}(Y \mathbf{Q}^{(p)}(h) \in \cdot), \quad x \rightarrow \infty, \quad (2.2)$$

where for $a \leq b$, $\mathbf{X}_{[a,b]} = (\mathbf{X}_a, \dots, \mathbf{X}_b)$, and the Pareto(α) variable Y is independent of $\mathbf{Q}^{(p)}(h) \in \mathbb{R}^{d(h+1)}$, and $\|\mathbf{Q}^{(p)}(h)\|_p = 1$ a.s. We call $\mathbf{Q}^{(p)}(h)$ the *spectral component* of $\mathbf{X}_{[0,h]}$.

2.2. Notation. For integers i and $a < b$ we write $i+[a, b] = \{i+a, \dots, i+b\}$. It is convenient to embed the vectors $\mathbf{x}_{[a,b]} \in \mathbb{R}^{d(b-a+1)}$ in $(\mathbb{R}^d)^\mathbb{Z}$ by assigning zeros to indices $i \notin [a, b]$, and we then also write $\mathbf{x}_{[a,b]} \in (\mathbb{R}^d)^\mathbb{Z}$. We denote $\mathbf{x} := (\mathbf{x}_t) = (\mathbf{x}_t)_{t \in \mathbb{Z}}$, and define truncation at level $\varepsilon > 0$ from above and below by $\underline{\mathbf{x}}_\varepsilon = (\underline{\mathbf{x}}_{t_\varepsilon})_{t \in \mathbb{Z}}$, $\bar{\mathbf{x}}^\varepsilon = (\bar{\mathbf{x}}_t^\varepsilon)_{t \in \mathbb{Z}}$, where $\underline{\mathbf{x}}_{t_\varepsilon} = x_t \mathbb{1}(|x_t| > \varepsilon)$, $\bar{\mathbf{x}}_t^\varepsilon = x_t \mathbb{1}(|x_t| \leq \varepsilon)$.

Recall the *backshift operator* acting on $\mathbf{x} \in (\mathbb{R}^d)^\mathbb{Z}$: $B^k \mathbf{x} = (x_{t-k})_{t \in \mathbb{Z}}$, $k \in \mathbb{Z}$. Let $\tilde{\ell}^p = \ell^p / \sim$ be the quotient space with respect to the equivalence relation \sim in ℓ^p : $\mathbf{x} \sim \mathbf{y}$ holds if there exists $k \in \mathbb{Z}$ such that $B^k \mathbf{x} = \mathbf{y}$. An element of $\tilde{\ell}^p$ is denoted by $[\mathbf{x}] = \{B^k \mathbf{x} : k \in \mathbb{Z}\}$. For ease of notation, we often write \mathbf{x} instead of $[\mathbf{x}]$, and we notice that any element in ℓ^p can be embedded in $\tilde{\ell}^p$ by using the equivalence relation. We define for $[\mathbf{x}], [\mathbf{y}] \in \tilde{\ell}^p$,

$$\tilde{d}_p([\mathbf{x}], [\mathbf{y}]) := \inf_{k \in \mathbb{Z}} \{d_p(B^k \mathbf{a}, \mathbf{b}) : \mathbf{a} \in [\mathbf{x}], \mathbf{b} \in [\mathbf{y}]\}.$$

For $p > 0$, \tilde{d}_p is a metric on $\tilde{\ell}^p$ and turns it into a complete metric space; see Basrak *et al.* [3].

2.3. Assumptions. Consider a stationary sequence (\mathbf{X}_t) satisfying \mathbf{RV}_α and let (x_n) be a thresholding sequences such that $n \mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$ and $x_n \rightarrow \infty$ as $n \rightarrow \infty$. If the conditions **AC** and **CS_p** below are required simultaneously we assume that the sequence (x_n) appearing in them is the same.

Anti-clustering condition AC. For any $\delta > 0$,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(\|\mathbf{X}_{[k,n]}\|_\infty > \delta x_n \mid |\mathbf{X}_0| > \delta x_n) = 0.$$

Condition **AC** ensures that a large value at present time does not persist indefinitely in the extreme future of the time series. This anti-clustering is weaker than the more classical two-sided one:

$$(2.3) \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\max_{k \leq |t| \leq n} |\mathbf{X}_t| > \delta x_n \mid |\mathbf{X}_0| > \delta x_n\right) = 0.$$

A simple sufficient condition, which breaks block-wise extremal dependence into pair-wise, is given by

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{t=k}^n \mathbb{P}(|\mathbf{X}_t| > \delta x_n \mid |\mathbf{X}_0| > \delta x_n).$$

For m -dependent (\mathbf{X}_t) the latter condition turns into $n \mathbb{P}(|\mathbf{X}_0| > \delta x_n) \rightarrow 0$ which is always satisfied.

If $p \leq \alpha$ an extra assumption is required for controlling the accumulation of moderate extremes within a block.

Vanishing-small-values condition \mathbf{CS}_p . For $p \in (0, \alpha]$ we assume that for a sequence (x_n) satisfying $n\mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$ and for any $\delta > 0$, we have

$$(2.4) \quad \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\left| \left\| \overline{x_n^{-1} \mathbf{X}_{[1,n]}}^\epsilon \right\|_p^p - \mathbb{E} \left[\left\| \overline{x_n^{-1} \mathbf{X}_{[1,n]}}^\epsilon \right\|_p^p \right] \right| > \delta)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} = 0.$$

Conditions of a similar type as \mathbf{CS}_p are standard when dealing with sum functionals acting on (\mathbf{X}_t) (see for example Davis and Hsing [10], Bartkiewicz *et al.* [1], Mikosch and Wintenberger [26, 27, 28]), and are also discussed in Kulik and Soulier [22].

Remark 2.1. Assume $\alpha < p < \infty$. Then by Karamata's theorem (see Bingham *et al.* [5]) and since $n\mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$,

$$\mathbb{E} \left[\left\| \overline{x_n^{-1} \mathbf{X}_{[1,n]}}^\epsilon \right\|_p^p \right] = n \mathbb{E} \left[\overline{x_n^{-1} \mathbf{X}}^\epsilon \right]^p = o(1), \quad n \rightarrow \infty.$$

Moreover, applications of Markov's inequality of order 1 and Karamata's theorem yield for $\delta > 0$, as $n \rightarrow \infty$,

$$\begin{aligned} \frac{\mathbb{P} \left(\left\| \overline{x_n^{-1} \mathbf{X}_{[1,n]}}^\epsilon \right\|_p^p > \delta \right)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} &= \frac{\mathbb{P} \left(\sum_{t=1}^n \overline{x_n^{-1} \mathbf{X}_t}^\epsilon > \delta \right)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ &\leq \frac{\mathbb{E} \left[\overline{x_n^{-1} \mathbf{X}}^\epsilon \right]^p}{\delta \mathbb{P}(|\mathbf{X}_0| > \epsilon x_n)} \frac{\mathbb{P}(|\mathbf{X}_0| > \epsilon x_n)}{\mathbb{P}(|\mathbf{X}_0| > x_n)} \rightarrow c e^{p-\alpha}. \end{aligned}$$

The right-hand side converges to zero as $\epsilon \rightarrow 0$. Here and in what follows, c denotes any positive constant whose value is not of interest. We conclude that (2.4) is automatic for $p > \alpha$.

If $p < \alpha$ then $\mathbb{E}[|\mathbf{X}|^p] < \infty$. If we also have $n/x_n^p \rightarrow 0$ then

$$\mathbb{E} \left[\left\| \overline{x_n^{-1} \mathbf{X}_{[1,n]}}^\epsilon \right\|_p^p \right] \leq n x_n^{-p} \mathbb{E}[|\mathbf{X}|^p] \rightarrow 0, \quad n \rightarrow \infty.$$

If $p = \alpha$, $\mathbb{E}[|\mathbf{X}|^\alpha] < \infty$ and $n/x_n^\alpha \rightarrow 0$ then the latter relation remains valid.

If $\mathbb{E}[|\mathbf{X}|^\alpha] = \infty$ then $\mathbb{E}[\overline{x_n^{-1} \mathbf{X}}^\epsilon]^\alpha = \ell(x_n)$ for some slowly varying function ℓ , hence for any small $\delta > 0$ and large n , $\ell(x_n) \leq x_n^\delta$. Then the condition $n x_n^{-\alpha+\delta} \rightarrow 0$ implies that $\mathbb{E}[\left\| \overline{x_n^{-1} \mathbf{X}_{[1,n]}}^\epsilon \right\|_p^p] = o(1)$.

In sum, for $p > \alpha$, (2.4) always holds without the centering term and, for $p \leq \alpha$, under the aforementioned additional growth conditions on (x_n) , if (2.4) holds then it holds without the centering term holds too.

Remark 2.2. Condition \mathbf{CS}_p is challenging to check for $p \leq \alpha$. For $p/\alpha \in (1/2, 1]$, by Čebyšev's inequality,

$$\begin{aligned} &\mathbb{P} \left(\left| \left\| \overline{x_n^{-1} \mathbf{X}_{[1,n]}}^\epsilon \right\|_p^p - \mathbb{E} \left[\left\| \overline{x_n^{-1} \mathbf{X}_{[1,n]}}^\epsilon \right\|_p^p \right] \right| > \delta \right) / [n \mathbb{P}(|\mathbf{X}_0| > x_n)] \\ &\leq \delta^{-2} \text{var} \left(\left\| \overline{x_n^{-1} \mathbf{X}_{[1,n]}}^\epsilon \right\|_p^p \right) / [n \mathbb{P}(|\mathbf{X}_0| > x_n)] \\ &\leq \delta^{-2} \frac{\mathbb{E} \left[\left\| \overline{x_n^{-1} \mathbf{X}}^\epsilon \right\|^{2p} \right]}{\mathbb{P}(|\mathbf{X}_0| > x_n)} \left[1 + 2 \sum_{h=1}^{n-1} \left| \text{corr} \left(\overline{x_n^{-1} \mathbf{X}_0}^\epsilon \right)^p, \overline{x_n^{-1} \mathbf{X}_h}^\epsilon \right|^p \right]. \end{aligned}$$

Now assume that (\mathbf{X}_t) is ρ -mixing with summable rate function (ρ_h) ; cf. Bradley [6]. Then the right-hand side is bounded by

$$\delta^{-2} \frac{\mathbb{E}[\overline{|x_n^{-1} \mathbf{X}^\epsilon|}^{2p}]}{\mathbb{P}(|\mathbf{X}_0| > x_n)} \left[1 + 2 \sum_{h=1}^{\infty} \rho_h \right] \sim \delta^{-2} \epsilon^{2p-\alpha} \left[1 + 2 \sum_{h=1}^{\infty} \rho_h \right], \quad \epsilon \rightarrow 0,$$

where we applied Karamata's theorem in the last step, and \mathbf{CS}_p follows. For Markov chains weaker assumptions such as the drift condition (**DC**) in Mikosch and Wintenberger [27, 28] can be used for checking \mathbf{CS}_p .

Remark 2.3. Condition \mathbf{CS}_p not only restricts the serial dependence of the time series (\mathbf{X}_t) but also the level of thresholds (x_n) . Indeed for $p/\alpha < 1/2$ and (\mathbf{X}_t) iid, since $(\|n^{-1/2} \mathbf{X}'_{[1,n]}\|_p^p - \mathbb{E}[\|n^{-1/2} \mathbf{X}'_{[1,n]}\|_p^p])$ converges in distribution to a Gaussian limit by virtue of the central limit theorem, \mathbf{CS}_p implies necessarily that $x_n/\sqrt{n} \rightarrow \infty$ as $n \mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$.

3. SPECTRAL CLUSTER PROCESS

3.1. The spectral cluster process in ℓ^p . From (2.1) recall the spectral tail process (Θ_t) of a stationary sequence (\mathbf{X}_t) satisfying \mathbf{RV}_α . For $p, \alpha > 0$ assume $\|\Theta\|_p + \|\Theta\|_\alpha < \infty$ a.s. Define

$$(3.1) \quad c(p) := \mathbb{E}[\|\Theta\|_p^\alpha / \|\Theta\|_\alpha^\alpha],$$

and further assume that $c(p) \in (0, \infty)$. This allows one to define a spectral cluster process in the sequence space ℓ^p .

Definition 3.1. *The p -spectral cluster process $\mathbf{Q}^{(p)}$ with values in ℓ^p is given by its distribution:*

$$\mathbb{P}(\mathbf{Q}^{(p)} \in \cdot) := c(p)^{-1} \mathbb{E}[\|\Theta\|_p^\alpha / \|\Theta\|_\alpha^\alpha \mathbf{1}(\Theta / \|\Theta\|_p \in \cdot)].$$

Since $\|\mathbf{Q}^{(p)}\|_p = 1$ a.s. we can interpret $\mathbf{Q}^{(p)}$ as a spectral component of a sequence of random variables in ℓ^p . By definition of $\mathbf{Q}^{(p)}$ we have $\mathbf{Q}^{(\alpha)} \stackrel{d}{=} \Theta / \|\Theta\|_\alpha$ and

$$\mathbb{P}(\mathbf{Q}^{(\alpha)} \in \cdot) = c(p)^{-1} \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_p^\alpha \mathbf{1}(\mathbf{Q}^{(\alpha)} / \|\mathbf{Q}^{(\alpha)}\|_p \in \cdot)].$$

3.1.1. The spectral cluster process in ℓ^α . The following result shows that the α -spectral cluster process $\mathbf{Q}^{(\alpha)}$ is well defined under **AC**.

Proposition 3.2. *Let (\mathbf{X}_t) be a stationary sequence satisfying \mathbf{RV}_α with spectral tail process (Θ_t) . Then the following statements are equivalent:*

- i) $\|\Theta\|_\alpha < \infty$ a.s. and $\mathbf{Q}^{(\alpha)}$ is well defined.
- ii) $|\Theta_t| \rightarrow 0$ a.s. as $t \rightarrow \infty$.
- iii) *The time of the largest record $T^* := \inf\{s : s \in \mathbb{Z} \text{ such that } |\Theta_s| = \sup_{t \in \mathbb{Z}} |\Theta_t|\}$ is finite a.s.*

Moreover, these statements hold under **AC**.

A proof of Proposition 3.2 is given in Lemma 3.6 of Buriticá *et al.* [7] appealing to results by Janssen [21]. We conclude that we also have the a.s. representations $\mathbf{Q}^{(\alpha)} = \Theta / \|\Theta\|_\alpha$ and $\Theta = \mathbf{Q}^{(\alpha)} / |\mathbf{Q}_0^{(\alpha)}|$.

Recall the definition of an ℓ^α -valued stationary sequence (\mathbf{X}_t) with the \mathbf{RV}_α property in ℓ^α given in (2.2) for $p = \alpha$, and recall from (2.2) the sequence of spectral components $(\mathbf{Q}^{(\alpha)}(h))_{h \geq 0}$ of the vectors $(\mathbf{X}_{[0,h]})_{h \geq 0}$ which are characterized by $\|\mathbf{Q}^{(\alpha)}(h)\|_\alpha = 1$ a.s. Our next result relates $\mathbf{Q}^{(\alpha)}$ to $(\mathbf{Q}^{(\alpha)}(h))_{h \geq 0}$.

Proposition 3.3. *Let (\mathbf{X}_t) be a stationary time series satisfying \mathbf{RV}_α and $\lim_{t \rightarrow \infty} |\Theta_t| = 0$ a.s. Then $\mathbf{Q}^{(\alpha)}(h) \xrightarrow{d} \mathbf{Q}^{(\alpha)}$ as $h \rightarrow \infty$ in $(\tilde{\ell}^\alpha, \tilde{d}_\alpha)$.*

The proof will be given in Section 7.1. This result gives raise to the interpretation of $\mathbf{Q}^{(\alpha)}$ as *the spectral component of (\mathbf{X}_t) in ℓ^α* .

3.1.2. *The spectral cluster process in ℓ^∞ .* Assuming **AC**, Basrak and Segers [4] proved that $|\Theta_t| \rightarrow 0$ as $|t| \rightarrow \infty$ and the extremal index $\theta_{|\mathbf{X}|}$ of $(|\mathbf{X}_t|)$ is given by

$$(3.2) \quad \theta_{|\mathbf{X}|} = \mathbb{E}[\|(\Theta_t)_{t \geq 0}\|_\infty^\alpha - \|(\Theta_t)_{t \geq 1}\|_\infty^\alpha].$$

Following Planinić and Soulier [30], the spectral tail process (Θ_t) satisfies the *time-change formula*: for any measurable bounded function $f : (\tilde{\ell}^p, \tilde{d}_p) \rightarrow \mathbb{R}$ such that $f(\lambda \mathbf{x}) = f(\mathbf{x})$ for all $\lambda > 0$, then $\forall t, s \in \mathbb{Z}$,

$$(3.3) \quad \mathbb{E}[f(B^s(\Theta_t)) \mathbb{1}(\Theta_{-s} \neq \mathbf{0})] = \mathbb{E}[|\Theta_s|^\alpha f((\Theta_t))].$$

An application of this formula and a telescoping sum argument yield

$$\begin{aligned} c(p) &= \mathbb{E}[\|\Theta\|_p^\alpha / \|\Theta\|_\alpha^\alpha] \\ &= \sum_{s \in \mathbb{Z}} \mathbb{E}[\|(\Theta_t)_{t \geq -s}\|_p^\alpha / \|\Theta\|_\alpha^\alpha - \|(\Theta_t)_{t \geq -s+1}\|_p^\alpha / \|\Theta\|_\alpha^\alpha] \\ &= \sum_{s \in \mathbb{Z}} \mathbb{E}[|\Theta_s|^\alpha (\|(\Theta_t)_{t \geq 0}\|_p^\alpha / \|\Theta\|_\alpha^\alpha - \|(\Theta_t)_{t \geq 1}\|_p^\alpha / \|\Theta\|_\alpha^\alpha)] \\ &= \mathbb{E}[\|\Theta\|_\alpha^\alpha (\|(\Theta_t)_{t \geq 0}\|_p^\alpha / \|\Theta\|_\alpha^\alpha - \|(\Theta_t)_{t \geq 1}\|_p^\alpha / \|\Theta\|_\alpha^\alpha)] \\ (3.4) \quad &= \mathbb{E}[\|(\Theta_t)_{t \geq 0}\|_p^\alpha - \|(\Theta_t)_{t \geq 1}\|_p^\alpha]. \end{aligned}$$

Thus the representation (3.2) of $\theta_{|\mathbf{X}|}$ extends from $p = \infty$ to $0 < p \leq \infty$ provided $c(p) < \infty$.

It follows from Proposition 4.2 below that we can retrieve the classical definition of the *spectral cluster process* by embedding $\mathbf{Q}^{(\infty)}$ in $(\tilde{\ell}^p, \tilde{d}_p)$ where

$$(3.5) \quad \mathbb{P}(\mathbf{Q}^{(\infty)} \in \cdot) := \theta_{|\mathbf{X}|}^{-1} \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_\infty^\alpha \mathbb{1}(\mathbf{Q}^{(\alpha)} / \|\mathbf{Q}^{(\alpha)}\|_\infty \in \cdot)],$$

in particular $\|\mathbf{Q}^{(\infty)}\|_\infty = 1$ a.s.

If $p \in [\alpha, \infty]$ and $\lim_{t \rightarrow \infty} |\Theta_t| = 0$ a.s. then $\|\Theta\|_p \leq \|\Theta\|_\alpha < \infty$ a.s. hence $c(p) \in [\theta_{|\mathbf{X}|}, 1]$, $\mathbf{Q}^{(p)}$ is well defined and Proposition 3.3 holds with p instead of α . Moreover, $c(\alpha) = 1$ and $c(\infty) = \theta_{|\mathbf{X}|}$.

If $p \in (0, \alpha)$ it is in general not obvious whether $c(p)$ is finite or not. If $\mathbb{E}[\|(\Theta_t)_{t \geq 0}\|_p^{\alpha-p}] < \infty$ a Taylor expansion shows that (3.4) is finite, and $\mathbf{Q}^{(p)}$ is also well defined in this case.

4. LARGE DEVIATION PRINCIPLES

We derive large deviation principles in terms of $\mathbf{Q}^{(\alpha)}$ and show that, for the purposes of statistical inference, $\mathbf{Q}^{(\alpha)}$ might be a competitor to $\mathbf{Q}^{(\infty)}$.

4.1. Extreme thresholds. Recall the properties of the sequence (x_n) from Section 2.3, in particular $n \mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$, and the definition of the constant $c(p)$ from (3.1).

Lemma 4.1. *Consider an \mathbb{R}^d -valued stationary time series (\mathbf{X}_t) satisfying the conditions \mathbf{RV}_α , \mathbf{AC} , \mathbf{CS}_p . Assume $c(p) < \infty$ for some $p > 0$ and, in addition, if $p < \alpha$ also $n/x_n^p \rightarrow 0$ and if $p = \alpha$ also $n/x_n^{\alpha+\delta} \rightarrow 0$ for some $\delta > 0$. Then the following relation holds:*

$$(4.1) \quad \lim_{n \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{X}_{[0,n]}\|_p > x_n)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} = c(p).$$

The proof is given in Section 7.2.

We recall from Remark 2.1 that (2.4) in \mathbf{CS}_p is always satisfied for $p > \alpha$ and, for $p \leq \alpha$ under the growth conditions on (x_n) in Lemma 4.1, centering with the expectation in (2.4) is not necessary.

We refer to a relation of the type (4.1) as *large deviation probabilities* motivated by the following observation. Write $S_k^{(p)} = \sum_{t=1}^k |\mathbf{X}_t|^p$ for $k \geq 1$. Then $|\mathbf{X}|^p$ is regularly varying with index α/p . Relation (4.1) implies that

$$\begin{aligned} \mathbb{P}(\|\mathbf{X}_{[0,n]}\|_p > x_n) &= \mathbb{P}(S_n^{(p)} > x_n^p) \\ &\sim c(p) n \mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Thus the left-hand probability describes the rare event that the sum process $S_n^{(p)}$ exceeds the extreme threshold x_n^p .

Proposition 4.2 below extends the large deviation result for $\|\mathbf{X}_{[0,n]}\|_p$ in (4.1) to a large deviation result for the process $\mathbf{X}_{[0,n]}$ in the sequence space ℓ^p . The proof is given in Section 7.3.

Proposition 4.2. *Assume the conditions of Lemma 4.1. Then,*

$$(4.2) \quad \mathbb{P}(x_n^{-1} \mathbf{X}_{[0,n]} \in \cdot \mid \|\mathbf{X}_{[0,n]}\|_p > x_n) \xrightarrow{w} \mathbb{P}(Y \mathbf{Q}^{(p)} \in \cdot), \quad n \rightarrow \infty,$$

in the space $(\tilde{\ell}^p, \tilde{d}_p)$ where the Pareto(α) random variable Y and $\mathbf{Q}^{(p)}$ are independent.

If $p > \alpha$, or $p < \alpha$ and $n/x_n^p \rightarrow 0$, or $p = \alpha$ and $n/x_n^{\alpha+\delta} \rightarrow 0$ for some $\delta > 0$, Proposition 4.2 provides a family of Borel sets in $\tilde{\ell}^p$ for which the weak limit of the self-normalized blocks $\mathbf{X}_{[0,n]}/\|\mathbf{X}_{[0,n]}\|_p$ exists. This result implies that the sequence of measures

$$\begin{aligned} \mu_n(\cdot) &:= \mathbb{P}(x_n^{-1}\mathbf{X}_{[0,n]} \in \cdot) / \mathbb{P}(\|\mathbf{X}_{[0,n]}\|_p > x_n) \\ \rightarrow \mu(\cdot) &:= \int_0^\infty \mathbb{P}(y\mathbf{Q}^{(p)} \in \cdot) d(-y^{-\alpha}), \quad n \rightarrow \infty, \end{aligned}$$

in the M_0 -sense in $\tilde{\ell}^p$. By the portmanteau theorem for measures (Theorem 2.4. in Hult and Lindskog [20])

$$\mu_n(A) = \mathbb{P}(x_n^{-1}\mathbf{X}_{[0,n]} \in A) / \mathbb{P}(\|\mathbf{X}_{[0,n]}\|_p > x_n) \rightarrow \mu(A),$$

for all Borel sets A in $(\tilde{\ell}^p, \tilde{d}_p)$ satisfying $\mu(\partial A) = 0$ and $\mathbf{0} \notin \bar{A}$. This approach is discussed in Kulik and Soulier [22] where similar conditions are stated for obtaining limit results for $\tilde{\ell}^1$ -functionals. Motivated by inference for the spectral cluster process $\mathbf{Q}^{(p)}$, we establish (4.2) employing weak convergence in the spirit of the polar decomposition in Segers *et al.* [32]. Note that we work under a one-sided anti-clustering condition **AC** together with a telescoping sum argument to compensate for the classical two-sided condition (2.3) used in Kulik and Soulier [22].

4.2. Moderate thresholds. To motivate the results of this section we start by considering an iid sequence (\mathbf{X}_t) satisfying \mathbf{RV}_α for some $\alpha > 0$. Then, for $p > \alpha$, (4.1) holds with limit $c(p) = 1$ and $S_n^{(p)} = \sum_{i=1}^n |\mathbf{X}_i|^p$ has infinite expectation. If $p < \alpha$ the process $(S_n^{(p)})$ has finite expectation and by the law of large numbers, for $n/x_n^p \rightarrow 0$,

$$\begin{aligned} &\mathbb{P}(\|\mathbf{X}_{[0,n]}\|_p > x_n (n x_n^{-p} \mathbb{E}[|\mathbf{X}|^p] + 1)^{1/p}) \\ (4.3) \quad &= \mathbb{P}(S_n^{(p)} - \mathbb{E}[S_n^{(p)}] > x_n^p(1 + o(1))) \rightarrow 0. \end{aligned}$$

Following Nagaev [29], a large deviation result for the centered process holds:

$$\mathbb{P}\left(S_n^{(p)} - \mathbb{E}[S_n^{(p)}] > x_n^p\right) \sim n\mathbb{P}(|\mathbf{X}_0| > x_n), \quad n \rightarrow \infty,$$

provided $n^{\alpha+\delta}/x_n \rightarrow 0$ for $p/\alpha \in (1/2, 1)$ and some $\delta > 0$, and $\sqrt{n \log n}/x_n^p \rightarrow 0$ for $p/\alpha < 1/2$. These conditions are verified for extreme thresholds satisfying $n/x_n^p \rightarrow 0$: in this case the centering term $\mathbb{E}[S_n^{(p)}]$ in (4.3) is always negligible which allows us to derive (4.1).

For inference purposes it is tempting to decrease the threshold level x_n to include more exceedances justified by results such as Nagaev's large deviation principle [29]. This can be achieved by carefully dealing with the centering $\mathbb{E}[S_n^{(p)}]$. This is the content of the next lemma.

Lemma 4.3. *Consider an \mathbb{R}^d -valued stationary process (\mathbf{X}_t) satisfying the conditions \mathbf{RV}_α , \mathbf{AC} , \mathbf{CS}_p and $c(p) < \infty$ for some $p > 0$. If $p < \alpha$ then*

$$(4.4) \quad \lim_{n \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{X}_{[0,n]}\|_p > x_n (n x_n^{-p} \mathbb{E}[|\mathbf{X}|^p] + 1)^{1/p})}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} = c(p).$$

If $p = \alpha$ then

$$(4.5) \quad \lim_{n \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{X}_{[0,n]}\|_\alpha > x_n (n \mathbb{E}[|\overline{\mathbf{X}}/x_n^{-1}|^\alpha] + 1)^{1/\alpha})}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} = c(\alpha) = 1.$$

Moreover, if also $\mathbb{E}[|\mathbf{X}|^\alpha] < \infty$ then equation (4.4) holds for $p = \alpha$.

The proof is given in Section 7.4. Note that restrictions on the level of the thresholds (x_n) are implicitly implied by Condition \mathbf{CS}_p ; see Remark 2.3.

For moderate thresholds x_n we define an auxiliary sequence of levels:

$$z_n := z_n(p) = \begin{cases} x_n (n x_n^{-p} \mathbb{E}[|\mathbf{X}|^p] + 1)^{1/p} & \text{if } p < \alpha, \\ x_n (n \mathbb{E}[|\overline{\mathbf{X}}/x_n^{-1}|^\alpha] + 1)^{1/\alpha} & \text{if } p = \alpha, \\ x_n & \text{if } p > \alpha. \end{cases}$$

For extreme thresholds satisfying $n/x_n^p \rightarrow 0$ we have $z_n \sim x_n$, while for moderate thresholds with $n/x_n^p \rightarrow \infty$ this is no longer the case.

For the purposes of inference this result is not as satisfactory as Lemma 4.1. Indeed, the level z_n in the selection of the exceedances is not the original threshold x_n . For any moderate threshold x_n where $z_n/x_n \rightarrow \infty$ the use of x_n instead of z_n might include a bias. As a toy example, consider the problem of inferring $c(q)/c(p)$ for $p \leq \alpha$, $q > p$. Then an application of Lemma 4.3 ensures that

$$\begin{aligned} & \mathbb{P}(\|\mathbf{X}_{[0,n]}\|_q > z_n(q) \mid \|\mathbf{X}_{[0,n]}\|_p > z_n(p)) \\ & \rightarrow \mathbb{P}(\|Y\mathbf{Q}^{(p)}\|_q > 1) = \mathbb{E}[\|\mathbf{Q}^{(p)}\|_q^\alpha] = c(q)/c(p), \quad n \rightarrow \infty. \end{aligned}$$

However, choosing moderate thresholds (x_n) , we would have

$$\mathbb{P}(\|\mathbf{X}_{[0,n]}\|_q > x_n \mid \|\mathbf{X}_{[0,n]}\|_p > x_n) \sim \frac{\mathbb{P}(n^{1/p} \mathbb{E}[|\mathbf{X}|^p]^{1/p} > x_n)}{\mathbb{P}(n^{1/p} \mathbb{E}[|\mathbf{X}|^p]^{1/p} > x_n)} \rightarrow 1, \quad n \rightarrow \infty.$$

5. CONSISTENT CLUSTER INFERENCE BASED ON SPECTRAL CLUSTER PROCESSES

Let $\mathbf{X}_1, \dots, \mathbf{X}_n$ be a sample from a stationary sequence (\mathbf{X}_t) satisfying \mathbf{RV}_α for some $\alpha > 0$ and choose $p > 0$. We split the sample into disjoint blocks $\mathbf{B}_t := \mathbf{X}_{(t-1)b+1:b}$, $t = 1, \dots, m_n$, where $b = b_n \rightarrow \infty$ and $m = m_n = \lfloor n/b_n \rfloor \rightarrow \infty$. Under the conditions of Lemma 4.1 we also have $\mathbb{P}(\|\mathbf{B}_1\|_p > x_b) \rightarrow 0$.

Our goal is to apply the results of Section 4 for inference on functionals acting on $\mathbf{Q}^{(p)}$. To avoid problems related to moderate threshold levels (see the discussion of Lemma 4.3), we will rather focus on extreme threshold

levels (x_n) satisfying $n \mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$ and the additional conditions for $p \leq \alpha$ described in Lemma 4.1. The objective of this section is to promote cluster inference for $p < \infty$, as well as the use of order statistics of $\|\mathbf{B}_t\|_p$, $t = 1, \dots, m_n$:

$$\|\mathbf{B}\|_{p,(1)} \geq \|\mathbf{B}\|_{p,(2)} \geq \dots \geq \|\mathbf{B}\|_{p,(m)},$$

for inference on $\mathbf{Q}^{(p)}$.

5.1. Cluster functionals and mixing. The real-valued function g on $\tilde{\ell}^p$ is a *cluster functional* for $\mathbf{Q}^{(p)}$ if it vanishes in some neighborhood of the origin and $\mathbb{P}(Y\mathbf{Q}^{(p)} \in D(g)) = 0$ where $D(g)$ denotes the set of discontinuity points of g .

In what follows, it will be convenient to write $\mathcal{G}_+(\tilde{\ell}^p)$ for the non-negative functions on $\tilde{\ell}^p$ which vanish in some neighborhood of the origin.

For asymptotic theory we will need the following *mixing condition*.

Condition \mathbf{MX}_p . There exists an integer sequence (b_n) , $b_n \rightarrow \infty$, $n/b_n \rightarrow \infty$, such that for any Lipschitz-continuous $f \in \mathcal{G}_+(\tilde{\ell}^p)$ the sequence (x_n) satisfies

$$\mathbb{E}\left[e^{-\frac{1}{k} \sum_{t=1}^m f(x_b^{-1} \mathbf{B}_t)}\right] = \left(\mathbb{E}\left[e^{-\frac{1}{k} \sum_{t=1}^{\lfloor m/k \rfloor} f(x_b^{-1} \mathbf{B}_t)}\right]\right)^k + o(1), \quad n \rightarrow \infty, \tag{5.1}$$

and $(m_n), (k_n)$ are defined as $m_n := \lfloor n/b_n \rfloor$, $k_n := \lfloor m_n \mathbb{P}(\|\mathbf{B}\|_p > x_{b_n}) \rfloor$.

In what follows, if \mathbf{MX}_p is required, then the sequences (b_n) , (m_n) and (k_n) appearing are those chosen here.

Condition \mathbf{MX}_p is similar to the mixing conditions \mathcal{A} , \mathcal{A}' considered in Davis and Hsing [10], and Basrak *et al.* [2], respectively, tailored for functionals $f \in \mathcal{G}_+(\tilde{\ell}^p)$ evaluated component-wise. In contrast, we focus on functionals applied to the entire block and extend the mixing-condition in Basrak *et al.* [3] to this situation. Then, borrowing the triangular-array arguments from the proof of Lemma 6.2. in [3], we can establish sufficient conditions for (5.1) for classical models.

5.2. Adaptive extreme threshold. We propose an empirical procedure for spectral cluster process inference using disjoint blocks through Theorem 5.1; see Section 7.5 for a proof.

Theorem 5.1. *Assume the conditions of Lemma 4.1 together with \mathbf{MX}_p . Then $\|\mathbf{B}\|_{p,(k)}/x_b \xrightarrow{\mathbb{P}} 1$ and for all $g \in \mathcal{G}_+(\tilde{\ell}^p)$,*

$$\frac{1}{k} \sum_{t=1}^m g(\|\mathbf{B}\|_{p,(k)}^{-1} \mathbf{B}_t) \xrightarrow{\mathbb{P}} \int_0^\infty \mathbb{E}[g(y\mathbf{Q}^{(p)})] d(-y^{-\alpha}), \quad n \rightarrow \infty. \tag{5.2}$$

Note that the number of order statistics k involved in Theorem 5.1 depends on p . The smaller p the larger the number k_n of extreme ℓ^p -blocks

to consider. Thus disjoint blocks estimation with $p < \infty$ should improve extreme cluster inference compared to $p = \infty$, notably in terms of bias.

5.3. Applications. Theorem 5.1 motivates the adaptive threshold selection method based on ℓ^p -norm order statistics for $p < \infty$. We apply this idea to inference on some indices related to the extremes in a sample and focus on the choices $p = \alpha$ and $p = 1$.

5.3.1. The extremal index. The extremal index of a regularly varying stationary time series has interpretation as a measure of clustering of serial exceedances, and was originally introduced in Leadbetter [23] and Leadbetter *et al.* [24]. If (\mathbf{X}'_t) is iid with the same marginal distribution as (\mathbf{X}_t) then the extremal index relates the expected number of serial exceedances of $(|\mathbf{X}_t|)$ with the serial exceedances of $(|\mathbf{X}'_t|)$.

From (3.1) and the discussion in Section 3.1.2 we recall the identities

$$\theta_{|\mathbf{X}|} = c(\infty) = \mathbb{E} \left[\frac{\|\Theta\|_\infty^\alpha}{\|\Theta\|_\alpha^\alpha} \right] = \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_\infty^\alpha].$$

We consider $g(\mathbf{x}) = (\|\mathbf{x}\|_\infty^\alpha / \|\mathbf{x}\|_\alpha^\alpha) \mathbb{1}(\|\mathbf{x}\|_\alpha > 1)$ and select $p = \alpha$ so that

$$\begin{aligned} \int_0^\infty \mathbb{E}[g(y\mathbf{Q}^{(\alpha)})] d(-y^{-\alpha}) &= \int_0^\infty \mathbb{E} \left[\frac{\|\mathbf{Q}^{(\alpha)}\|_\infty^\alpha}{\|\mathbf{Q}^{(\alpha)}\|_\alpha^\alpha} \mathbb{1}(\|\mathbf{Q}^{(\alpha)}\|_\alpha > y^{-\alpha}) \right] d(-y^{-\alpha}) \\ &= \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_\infty^\alpha] = \theta_{|\mathbf{X}|}. \end{aligned}$$

Now we introduce a new consistent disjoint blocks estimator of the extremal index defined from exceedances of ℓ^α -norm blocks. By an application of Theorem 5.1 with $p = \alpha$ we derive the following corollary.

Corollary 5.2. *Assume the conditions of Theorem 5.1 for $p = \alpha$. Then*

$$(5.3) \quad \hat{\theta}_{|\mathbf{X}|} := \frac{1}{k} \sum_{t=1}^m \frac{\|\mathbf{B}_t\|_\infty^\alpha}{\|\mathbf{B}_t\|_\alpha^\alpha} \mathbb{1}(\|\mathbf{B}_t\|_\alpha > \|\mathbf{B}\|_{\alpha,(k)}) \xrightarrow{\mathbb{P}} \theta_{|\mathbf{X}|}, \quad n \rightarrow \infty.$$

To motivate this estimator we compare it to a more classical cluster-based estimator of $\theta_{|\mathbf{X}|}$. Let $g(\mathbf{x}) := \mathbb{1}(\|\mathbf{x}\|_\infty > 1)$ (this functional describes the blocks estimator in Hsing [19]), then an application of Theorem 5.1 with $p = \infty$ entails that there exists an integer sequence $k' = k'_n \rightarrow \infty$ such that

$$(5.4) \quad \frac{1}{k'} \sum_{t=1}^m \mathbb{1}(\|\mathbf{B}_t\|_\infty \geq \|\mathbf{B}\|_{\infty,(k')}) \xrightarrow{\mathbb{P}} \theta_{|\mathbf{X}|}, \quad n \rightarrow \infty,$$

with $k'_n \sim m_n \mathbb{P}(\|\mathbf{B}\|_\infty > x_b) \sim \theta_{|\mathbf{X}|} n \mathbb{P}(|\mathbf{X}_0| > x_b) \sim \theta_{|\mathbf{X}|} m_n \mathbb{P}(\|\mathbf{B}\|_\alpha > x_b) \sim \theta_{|\mathbf{X}|} k_n$. Thus the proportion of extreme blocks we shall consider for estimating the extremal index from (5.4) is as small as the extremal index itself compared with the number of extreme blocks we can consider by using (5.3).

This example highlights the delicate choice of threshold for cluster inference, which supports the adaptive threshold method and justifies that the choice $p = \alpha$ should perform better in terms of bias. Indeed, the Monte Carlo

experiments in Buriticá *et al.* [7] comparing $\widehat{\theta}_{|\mathbf{X}|}$ in (5.3) with the blocks estimator in Hsing [19] support this idea, and show $\widehat{\theta}_{|\mathbf{X}|}$ is also competitive compared to other estimators of the extremal index regarding bias.

5.3.2. *A cluster index for sums.* In this section we assume $\alpha \in (0, 2)$ and $\mathbb{E}[\mathbf{X}] = \mathbf{0}$ for $\alpha \in (1, 2)$. We study the partial sums $\mathbf{S}_n := \sum_{t=1}^n \mathbf{X}_t$, $n \geq 1$, and introduce a normalizing sequence (a_n) such that $n\mathbb{P}(|\mathbf{X}_0| > a_n) \rightarrow 1$. Starting with Davis and Hsing [10], α -stable central limit theory for (\mathbf{S}_n/a_n) was proven under suitable anti-clustering and mixing conditions.

In this setting, the quantity $c(1)$ appears naturally and was coined cluster index in Mikosch and Wintenberger [27]. For $d = 1$ it can be interpreted as an equivalent of the extremal index for partial sums rather than maxima. Indeed, for $d = 1$, if ξ_α denotes the stable limit,

$$\mathbb{E}[e^{iu\xi_\alpha}] = (\mathbb{E}[e^{iu\xi'_\alpha}])^{c(1)},$$

where $S'_n/a_n \xrightarrow{d} \xi'_\alpha$ for the partial sums (S'_n) of an iid sequence (X'_t) with the same marginal distribution as the sequence (X_t) such as $\mathbb{P}(X \leq -x) = o(\mathbb{P}(X > x))$.

From (3.1) with $g(\mathbf{x}) = (\|\mathbf{x}\|_\alpha^\alpha / \|\mathbf{x}\|_1^\alpha) \mathbb{1}(\|\mathbf{x}\|_1 > 1)$ and $p = 1$, we obtain

$$\begin{aligned} \int_0^\infty \mathbb{E}[g(y\mathbf{Q}^{(1)})]d(-y^{-\alpha}) &= \int_0^\infty \mathbb{E}\left[\frac{\|\mathbf{Q}^{(1)}\|_\alpha^\alpha}{\|\mathbf{Q}^{(1)}\|_1^\alpha} \mathbb{1}(\|\mathbf{Q}^{(1)}\|_1^\alpha > y^{-\alpha})\right]d(-y^{-\alpha}) \\ &= \mathbb{E}[\|\mathbf{Q}^{(1)}\|_\alpha^\alpha] = (c(1))^{-1}. \end{aligned}$$

Theorem 5.1 with $p = 1$ yields a consistent estimator of $c(1)$.

Corollary 5.3. *Assume the conditions of Theorem 5.1. Then*

$$\widehat{c}(1) := \left(\frac{1}{k} \sum_{t=1}^m \frac{\|\mathbf{B}_t\|_\alpha^\alpha}{\|\mathbf{B}_t\|_1^\alpha} \mathbb{1}(\|\mathbf{B}_t\|_1 > \|\mathbf{B}\|_{1,(k)})\right)^{-1} \xrightarrow{\mathbb{P}} c(1), \quad n \rightarrow \infty.$$

Similarly as before, this estimator is appealing since it is based on large ℓ^1 -norms of blocks instead of ℓ^∞ -norms. Indeed, arguing as in Cissokho and Kulik [8], Kulik and Soulier [22], we can extend Theorem 5.1 for $p = \infty$ to hold for bounded ℓ^1 -functionals. Then, for $g(\mathbf{x}) := \mathbb{1}(\|\mathbf{x}\|_1 > 1)$ we deduce that there exists $k' = k'_n \rightarrow \infty$ such that

$$(5.5) \quad \frac{1}{k'} \sum_{t=1}^m \mathbb{1}(\|\mathbf{B}_t\|_1 \geq \|\mathbf{B}\|_{\infty,(k')}) \xrightarrow{\mathbb{P}} c(1), \quad n \rightarrow \infty,$$

with $k'_n \sim m_n \mathbb{P}(\|\mathbf{B}\|_\infty > x_b) \sim \theta_{|\mathbf{X}|} n \mathbb{P}(|\mathbf{X}_0| > x_b) \sim \theta_{|\mathbf{X}|} (c(1))^{-1} m_n \mathbb{P}(\|\mathbf{B}\|_1 > x_b) \sim \theta_{|\mathbf{X}|} (c(1))^{-1} k_n$. In particular, for $\alpha \in (1, 2)$, $c(1) \geq 1$, thus a small proportion (even smaller than $\theta_{|\mathbf{X}|}$) of blocks should be used to infer on $c(1)$. For $\alpha \in (0, 1)$ we can argue similarly and improve inference by proposing an estimator based on ℓ^α -norm order statistics. Then, as in the extremal index example, we highlight the importance of adaptive threshold selection for cluster inference, and we motivate the choice $p < \infty$ for a better control

of bias compared to $p = \infty$. Furthermore, Theorem 5.1 yields estimates for the parameters of the stable limit ξ_α . Indeed, following the theory in Bartkiewicz *et al.* [1], we derive an α -stable limit for (\mathbf{S}_n/a_n) in terms of $\mathbf{Q}^{(1)}$; the proof is given Section 7.6.

Proposition 5.4. *Assume that (\mathbf{X}_t) is a regularly varying stationary sequence with $\alpha \in (0, 1) \cup (1, 2)$ together with the mixing condition:*

$$\mathbb{E}[e^{i\mathbf{u}^\top \mathbf{S}_{b_n/a_n}}] = (\mathbb{E}[e^{i\mathbf{u}^\top \mathbf{S}_{b_n/a_n}}])^{m_n} + o(1), \quad n \rightarrow \infty, \quad \mathbf{u} \in \mathbb{R}^d,$$

and the anti-clustering condition: for every $\delta > 0$,

$$(5.6) \quad \lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} n \sum_{t=l}^{b_n} \mathbb{E}[(|\mathbf{X}_t/a_n| \wedge \delta) (|\mathbf{X}_0/a_n| \wedge \delta)] = 0,$$

Then $\mathbf{S}_n/a_n \xrightarrow{d} \xi_\alpha$ for an α -stable random vector ξ_α with characteristic function $\mathbb{E}[\exp(i\mathbf{u}^\top \xi_\alpha)] = \exp(-c_\alpha \sigma_\alpha(\mathbf{u}) (1 - i\beta(\mathbf{u}) \tan(\alpha\pi/2)))$, $\mathbf{u} \in \mathbb{R}^d$, where $c_\alpha := (\Gamma(2-\alpha)/|1-\alpha|)(1 \wedge \alpha) \cos(\alpha\pi/2)$, and the skewness and scale parameters have representation

$$\begin{aligned} \sigma_\alpha(\mathbf{u}) &:= c(1) \mathbb{E}[|\mathbf{u}^\top \sum_{t \in \mathbb{Z}} \mathbf{Q}_t^{(1)}|^\alpha], \\ \beta(\mathbf{u}) &:= (\mathbb{E}[(\mathbf{u}^\top \sum_{t \in \mathbb{Z}} \mathbf{Q}_t^{(1)})_+^\alpha - (\mathbf{u}^\top \sum_{t \in \mathbb{Z}} \mathbf{Q}_t^{(1)})_-^\alpha]) / \mathbb{E}[|\mathbf{u}^\top \sum_{t \in \mathbb{Z}} \mathbf{Q}_t^{(1)}|^\alpha]. \end{aligned}$$

As for $c(1)$, an application of Theorem 5.1 with $p = 1$ for $\alpha \in (1, 2)$ and $p = \alpha$ for $\alpha \in (0, 1)$ yields natural estimators of the parameters $(\sigma_\alpha(\mathbf{u}), \beta(\mathbf{u}))$ in the central limit theorem of Proposition 5.4.

5.3.3. An example: a regularly varying linear process. We illustrate the index estimators of Corollaries 5.2 and 5.3 for a regularly varying linear process $X_t := \sum_{j \in \mathbb{Z}} \varphi_j Z_{t-j}$, $t \in \mathbb{Z}$, where (Z_t) is an iid real-valued regularly varying sequence (Z_t) with (tail)-index $\alpha > 0$, and (φ_j) are real coefficients such that $\sum_{j \in \mathbb{Z}} |\varphi_j|^{1 \wedge (\alpha - \varepsilon)} < \infty$ for some $\varepsilon > 0$.

Then (X_t) is regularly varying with the same (tail)-index $\alpha > 0$, and the distributions of Z_t and X_t are tail-equivalent; see Davis and Resnick [11]. The cluster process of (X_t) is given by $Q_t^{(\alpha)} = (\varphi_{t+J} / \|\varphi_t\|_\alpha) \Theta_0^Z$, $t \in \mathbb{Z}$, where $\lim_{x \rightarrow \infty} \mathbb{P}(\pm Z_0 > x) / \mathbb{P}(|Z_0| > x) = \mathbb{P}(\Theta_0^Z = \pm 1)$, Θ_0^Z is independent of a random shift J with distribution $\mathbb{P}(J = j) = |\varphi_j|^\alpha / \|\varphi_t\|_\alpha^\alpha$; see Kulik and Soulier [22], (15.3.9). Then

$$\theta_{|X|} = \max_{t \in \mathbb{Z}} |\varphi_t|^\alpha / \|\varphi\|_\alpha^\alpha, \quad c(1) = \left(\sum_{t \in \mathbb{Z}} |\varphi_t| \right)^\alpha / \|\varphi\|_\alpha^\alpha.$$

Noticing that $k_n \sim m_n \mathbb{P}(\|\mathbf{B}\|_p > x_b) \sim n c(p) \mathbb{P}(|\mathbf{X}_0| > x_b) = o(n/b)$, we propose to choose k_n as $k_n := \max\{2, \lfloor n/b_n^{(1+\kappa)} \rfloor\}$ for some tuning parameter $\kappa > 0$. Then fixing k_n this way with $\kappa = 1$, we obtain estimators as a function of b_n . Figures 5.5 and 5.6 show boxplots of the estimators $\hat{\theta}_{|X|}$ and $1/\hat{c}(1)$, as a function of b_n , respectively, for different sample sizes from the causal AR(1) model given by $X_t = \varphi X_{t-1} + Z_t$, $t \in \mathbb{Z}$, $|\varphi| < 1$, thus $\theta_{|X|} = 1 - |\varphi|^\alpha$ and $c(1) = (1 - |\varphi|^\alpha) / (1 - |\varphi|)^\alpha$.

We see in Figures 5.5, 5.6 that the choice $b_n = 32$ is reasonable for all models in our experiment for the tuning parameter $\kappa = 1$, though improvement can be achieved by fine-tuning the parameter $\kappa > 0$. Also notice that the bias for large block lengths is reduced as n increases. However, fixing n we have $\lfloor n/b^{(1+\kappa)} \rfloor \rightarrow 0$ as $b \rightarrow \infty$ which in fact suggests a restriction to the block length for small sample sizes. Here we use the known value of $\alpha > 0$ for computing $\hat{\theta}_{|X|}$, $1/\hat{c}(1)$. For implementing our approach with real data, α must be estimated e.g. via the bias-correction procedure in de Haan *et al.* [17].

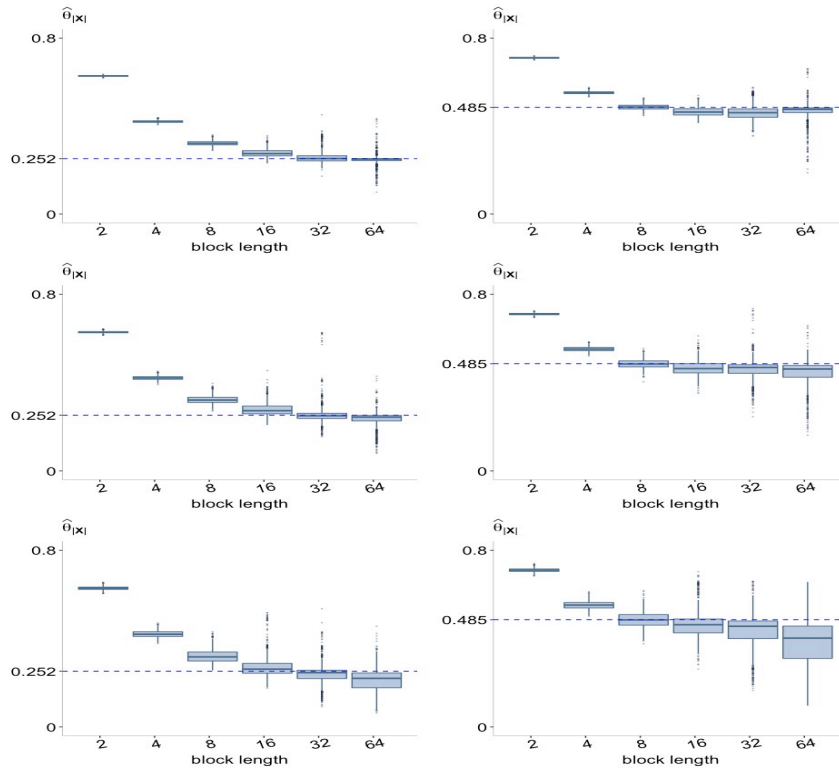


Figure 5.5. Boxplot of estimates $\hat{\theta}_{|X|}$ as a function of b_n for 1000 simulated samples $(X_t)_{t=1,\dots,n}$ from a causal AR(1) model with student(α) noise with $\alpha = 1.3$ and $\varphi = 0.8$ (left column), $\varphi = 0.6$ (right column). Rows correspond to results for $n = 8000, 3000, 1000$ from top to bottom.

6. INFERENCE BEYOND SHIFT-INVARIANT FUNCTIONALS

So far we only considered inference for shift-invariant functionals acting on $(\tilde{\ell}^p, \tilde{d}_p)$ such as maxima and sums. Following the shift-projection ideas in Janssen [21], jointly with continuous mapping arguments, we extend inference to functionals on (ℓ^p, d_p) .

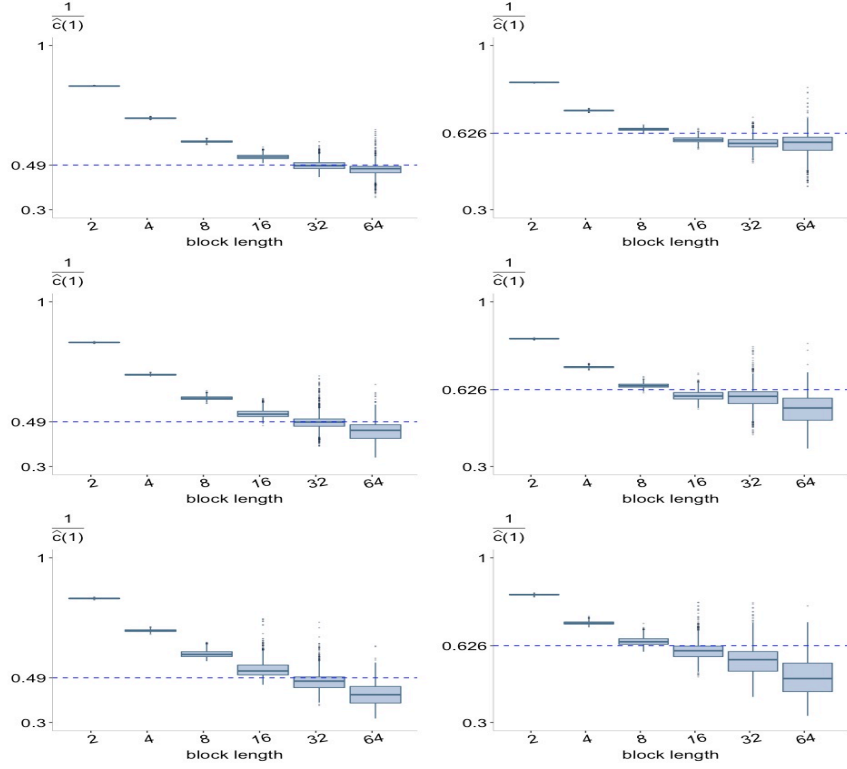


Figure 5.6. Boxplot of estimates $1/\widehat{c}(1)$ as a function of b_n for 1000 simulated samples $(X_t)_{t=1,\dots,n}$ from a causal AR(1) model with student(α) noise with $\alpha = 1.3$ and $\varphi = 0.8$ (left column), $\varphi = 0.6$ (right column). Rows correspond to results for $n = 8000, 3000, 1000$ from top to bottom.

6.1. Inference for cluster functionals in (ℓ^p, d_p) . Let $g : (\ell^p, d_p) \rightarrow \mathbb{R}$ be a bounded measurable function. We define the functional $\psi_g : (\widetilde{\ell}^p, \widetilde{d}_p) \rightarrow \mathbb{R}$ by

$$(6.1) \quad [\mathbf{z}] \mapsto \psi_g([\mathbf{z}]) := \sum_{j \in \mathbb{Z}} |\mathbf{z}_{-j}^*|^\alpha g((B^j \mathbf{z}_t^*)_{t \in \mathbb{Z}}),$$

where $\mathbf{z}_t^* := \mathbf{z}_{t-T^*(\mathbf{z})}$, for $t \in \mathbb{Z}$, such that $T^*(\mathbf{z}) := \inf\{s \in \mathbb{Z} : |\mathbf{z}_s| = \|\mathbf{z}\|_\infty\}$ and $B : \ell^p \rightarrow \ell^p$ is the backward-shift map.

We link the distribution of the cluster process $\mathbf{Q}^{(\alpha)}$ and the distribution of the class $[\mathbf{Q}^{(\alpha)}]$ through the mappings (6.1) in the next proposition whose proof is given in Section 7.7.1.

Proposition 6.1. *The following relation holds for any real-valued bounded measurable function g on ℓ^α*

$$\mathbb{E}[g(\mathbf{Q}^{(\alpha)})] = \mathbb{E}[\psi_g([\mathbf{Q}^{(\alpha)}])].$$

This relation remains valid if α is replaced by p , whenever the p -cluster process is well defined.

The mappings in (6.1) are continuous functionals on $(\tilde{\ell}^p, \tilde{d}_p)$ if $p \leq \alpha$. Therefore we can extend Theorem 5.1 to continuous functionals on (ℓ^p, d_p) evaluated at the cluster process $\mathbf{Q}^{(p)}$.

Theorem 6.2. *Assume the conditions of Theorem 5.1 together with $p \leq \alpha$. Then for any continuous bounded function $g : \ell^p \cap \{\mathbf{x} : \|\mathbf{x}\|_p = 1, |\mathbf{x}_0| > 0\} \rightarrow \mathbb{R}$,*

$$(6.2) \quad \widehat{g}^{(p)} := \frac{1}{k} \sum_{t=1}^m \underbrace{\sum_{j=1}^b W_{j,t}(p) g\left(\frac{B^{j-1}\mathbf{B}_t}{\|\mathbf{B}_t\|_p}\right)}_{=: \psi_g(\mathbf{B}_t)} \mathbb{1}(\|\mathbf{B}_t\|_p > \|\mathbf{B}\|_{p,(k)})$$

$$\xrightarrow{\mathbb{P}} \mathbb{E}[g(\mathbf{Q}^{(p)})], \quad n \rightarrow \infty,$$

where $W_{j,t}(p) = |\mathbf{X}_{(t-1)b+j}|^\alpha / \|\mathbf{B}_t\|_p^\alpha$ for all $1 \leq j \leq b$.

The proof is given in Section 7.7.2.

6.2. Applications. Examples of non-shift-invariant functionals on (ℓ^p, d_p) are measures of serial dependence, probabilities of large deviations as the supremum of a random walk and ruin probabilities; and functionals of the spectral tail process $\Theta \stackrel{d}{=} \mathbf{Q}^{(\alpha)} / |\mathbf{Q}_0^{(\alpha)}|$.

6.2.1. *Measures of serial dependence.* Define $g_h(\mathbf{x}_t) = |\mathbf{x}_h|^\alpha \frac{\mathbf{x}_0^\top \mathbf{x}_h}{|\mathbf{x}_0| |\mathbf{x}_h|}$. Then the following result is straightforward from Theorem 6.2.

Corollary 6.3. *Assume the conditions of Theorem 6.2 for $p = \alpha$. Then*

$$\widehat{g}_h^{(\alpha)} := \frac{1}{k} \sum_{t=1}^m \underbrace{\sum_{j=1}^{b-h} W_{j,t} W_{j+h,t} \frac{\mathbf{X}_{j,t}^\top \mathbf{X}_{j+h,t}}{|\mathbf{X}_{j,t}| |\mathbf{X}_{j+h,t}|}}_{=: \psi_{g_h}(\mathbf{B}_t)} \mathbb{1}(\|\mathbf{B}_t\|_\alpha > \|\mathbf{B}\|_{\alpha,(k)}).$$

$$\xrightarrow{\mathbb{P}} \mathbb{E}[g_h(\mathbf{Q}^{(\alpha)})], \quad n \rightarrow +\infty,$$

where the weights $W_{j,t} = W_{j,t}(\alpha)$ are defined in Theorem 6.2, satisfying $\sum_{j=1}^b W_{j,t} = 1$, and $\mathbf{X}_{j,t} := \mathbf{X}_{(t-1)b+j}$ for $1 \leq j \leq b$.

The function g_h gives a summary of the magnitude and direction of the time series h lags after recording a high-level exceedance of the norm, and satisfies the relation $\sum_{h \in \mathbb{Z}} \mathbb{E}[g_h(\mathbf{Q}^{(\alpha)})] = 1$. In particular, for the linear model we obtain a link with the autocovariance functions as we show in the example below.

Example 6.4. Let (X_t) be a linear process satisfying the assumptions in Example 5.3.3, then

$$(6.3) \quad \mathbb{E}[g_h(Q^{(\alpha)})] = \frac{\sum_{t \in \mathbb{Z}} |\varphi_t|^\alpha |\varphi_{t+h}|^\alpha \text{sign}(\varphi_t) \text{sign}(\varphi_{t+h})}{(\|\varphi\|_\alpha^\alpha)^2}, \quad h \in \mathbb{Z}.$$

The function defined in (6.3) is proportional to the autocovariance function of a finite variance linear process with coefficients $(|\varphi_t|^\alpha \text{sign}(\varphi_t))$. In particular, for $\alpha = 1$ it is proportional to the autocovariance function of a finite variance linear process with coefficients (φ_t) .

6.2.2. *Large deviations for the supremum of a random walk.* We start by reviewing Theorem 4.5 in Mikosch and Wintenberger [28]; see proof is given in Section 7.7.3.

Proposition 6.5. *Consider a univariate stationary sequence (X_t) satisfying \mathbf{RV}_α for some $\alpha \geq 1$, \mathbf{AC} , \mathbf{CS}_1 , and $c(1) < \infty$. Then for all $p \geq 1$,*

$$(6.4) \quad \left| \frac{\mathbb{P}(\sup_{1 \leq t \leq n} S_t > x_n)}{n\mathbb{P}(|X| > x_n)} - c(p) \mathbb{E} \left[\lim_{s \rightarrow \infty} \left(\sup_{t \geq -s} \sum_{i=-s}^t Q_i^{(p)} \right)_+^\alpha \right] \right| \rightarrow 0, \quad n \rightarrow \infty.$$

If $\alpha \geq 1$, then $\|Q^{(1)}\|_\alpha^\alpha \leq \|Q^{(1)}\|_1^\alpha = 1$ and a consistent estimator of $c(1) = 1/\mathbb{E}[\|Q^{(1)}\|_\alpha^\alpha]$ was suggested in Corollary 5.3. A consistent estimator of the term in (6.4) is given next.

Corollary 6.6. *Assume the conditions of Theorem 6.2 for $p = 1$. Then*

$$\left| \frac{\sum_{t=1}^m \left(\sup_{1 \leq j \leq b} \frac{X_{t,j}}{\|\mathbf{B}_t\|_1} \right)_+^\alpha \mathbf{1}(\|\mathbf{B}_t\|_1 > \|\mathbf{B}\|_{1,(k)})}{\sum_{t=1}^m \frac{\|\mathbf{B}_t\|_\alpha^\alpha}{\|\mathbf{B}_t\|_1^\alpha} \mathbf{1}(\|\mathbf{B}_t\|_1 > \|\mathbf{B}\|_{1,(k)})} - c(1) \mathbb{E} \left[\lim_{s \rightarrow \infty} \left(\sup_{t \geq -s} \sum_{i=-s}^t Q_i^{(1)} \right)_+^\alpha \right] \right| \xrightarrow{\mathbb{P}} 0, \quad n \rightarrow \infty,$$

where $X_{t,j} := X_{(t-1)b+j}$, for $1 \leq j \leq b$, $1 \leq t \leq m$.

Following the same ideas and using Theorem 4.9 in [28], one can also derive a consistent estimator for the constant in the related ruin problem.

6.2.3. *Application: a cluster-based method for inferring on (Θ_t) .* Exploiting the relation $(\mathbf{Q}_t^{(\alpha)})/|\mathbf{Q}_0^{(\alpha)}| \stackrel{d}{=} (\Theta_t)$, we propose cluster-based estimation methods for the spectral tail process.

Cluster-based approaches with the goal to improve inference on Θ_1 for Markov chains were considered in Drees *et al.* [16]; see also Davis *et al.* [9], Drees *et al.* [13] for related cluster-based procedures on $(\Theta_t)_{|t| \leq h}$ for fixed $h \geq 0$. Our approach can be seen as an extension for inference on the ℓ^α -valued sequence (Θ_t) .

Consider the continuous re-normalization function $\zeta(\mathbf{x}) = \mathbf{x}/|\mathbf{x}_0|$ on $\{\mathbf{x} \in \ell^\alpha : |\mathbf{x}_0| > 0\}$. We derive the following result from Theorem 6.2; the proof is given in Section 7.7.4.

Proposition 6.7. *Assume the conditions of Theorem 6.2 for $p = \alpha$. Let $\rho : (\ell^\alpha, d_\alpha) \rightarrow \mathbb{R}$ be a homogeneous continuous function and $\rho_\zeta(\mathbf{x}) := (\rho^\alpha \wedge 1) \circ \zeta(\mathbf{x})$. Then*

$$\widehat{\rho}_\zeta^{(\alpha)} := \frac{1}{k} \sum_{t=1}^m \psi_{\rho_\zeta}(\mathbf{B}_t) \mathbb{1}(\|\mathbf{B}_t\|_p > \|\mathbf{B}\|_{p,(k)}) \xrightarrow{\mathbb{P}} \mathbb{P}(\rho(Y \Theta) > 1), \quad n \rightarrow \infty,$$

where $\psi_{\rho_\zeta}(\mathbf{B}_t)$ is defined in (6.2) and the Pareto(α) random variable Y is independent of Θ .

Classical examples of such functionals ρ are $\rho(\mathbf{x}) = \max_{i \geq 0, j \geq i} (x_i - x_j)_+$, functionals related to large deviations such as $\rho(\mathbf{x}) = \sup_{t \geq 0} (\sum_{i=0}^t x_i)_+$, or measures of serial dependence as $\rho(\mathbf{x}) = |\mathbf{x}_h|$.

7. PROOFS

7.1. Proof of Proposition 3.3. Our goal is first to relate the sequence of spectral components $(\mathbf{Q}^{(p)}(h))_{h \geq 0}$ to (Θ_t) , and second to deduce a relation of $(\mathbf{Q}^{(p)}(h))_{h \geq 0}$ with $\mathbf{Q}^{(\alpha)}$ under the assumption $|\Theta_t| \rightarrow 0$ a.s. as $t \rightarrow \infty$, which ensures that $\mathbf{Q}^{(\alpha)}$ is well defined.

We start assuming two auxiliary results whose proofs are given at the end of this section.

Lemma 7.1. *Let (\mathbf{X}_t) be a stationary time series satisfying \mathbf{RV}_α . Then for $h \geq 0$,*

$$\mathbb{P}(\mathbf{Q}^{(p)}(h) \in \cdot) = \frac{1}{c(p, h)} \sum_{k=0}^h \mathbb{E} \left[\frac{\|\Theta_{-k+[0, h]}\|_p^\alpha}{\|\Theta_{-k+[0, h]}\|_\alpha^\alpha} \mathbb{1} \left(\frac{\Theta_{-k+[0, h]}}{\|\Theta_{-k+[0, h]}\|_p} \in \cdot \right) \right], \quad (7.1)$$

where

$$c(p, h) := \sum_{k=0}^h \mathbb{E} \left[\frac{\|\Theta_{-k+[0, h]}\|_p^\alpha}{\|\Theta_{-k+[0, h]}\|_\alpha^\alpha} \right].$$

In particular, $c(\alpha, h) = h + 1$ and

$$\mathbb{P}(\mathbf{Q}^{(\alpha)}(h) \in \cdot) = \mathbb{P}(\Theta_{-U^{(h)}+[0, h]} / \|\Theta_{-U^{(h)}+[0, h]}\|_\alpha \in \cdot), \quad (7.2)$$

where $U^{(h)}$ is uniformly distributed on $\{0, \dots, h\}$ and independent of Θ .

Lemma 7.2. *Assume $|\Theta_t| \rightarrow 0$ as $t \rightarrow \infty$ and let $f : \tilde{\ell}^\alpha \cap \{\mathbf{x} : \|\mathbf{x}\|_p = 1\} \rightarrow (0, \infty)$ be any bounded Lipschitz-continuous function in $(\tilde{\ell}^\alpha, \tilde{d}_\alpha)$. Then, for all $p \geq \alpha$*

$$\frac{c(p, h)}{h + 1} \mathbb{E}[f(\mathbf{Q}^{(p)}(h))] \rightarrow c(p) \mathbb{E}[f(\mathbf{Q}^{(p)})], \quad h \rightarrow \infty. \quad (7.3)$$

We conclude from (7.3) for $f(\mathbf{x}) \equiv 1$ that $\lim_{h \rightarrow \infty} c(p, h)/(h + 1) = c(p)$. If $0 < c(p) < \infty$ then $\lim_{h \rightarrow \infty} \mathbb{E}[f(\mathbf{Q}^{(p)}(h))] = \mathbb{E}[f(\mathbf{Q}^{(p)})]$. Finally, the

portmanteau theorem yields $\mathbf{Q}^{(p)}(h) \xrightarrow{d} \mathbf{Q}^{(p)}$ in $(\tilde{\ell}^p \cap \{\mathbf{x} : \|\mathbf{x}\|_p = 1\}, \tilde{d}_p)$. This finishes the proof of the proposition. \square

Proof of Lemma 7.1. If $\|\mathbf{X}_{[0,h]}/x\|_p > 1$ then for sufficiently small $\epsilon > 0$, $\|\mathbf{X}_{[0,h]}/x\|_\infty > \epsilon$. Therefore, on $\{\|\mathbf{X}_{[0,h]}/x\|_p > 1\}$,

$$\sum_{i=0}^h |\mathbf{X}_i/x|^\alpha \mathbb{1}(|\mathbf{X}_i/x| > \epsilon) > 0.$$

Using stationarity, we obtain

$$\begin{aligned} & \mathbb{P}(\|\mathbf{X}_{[0,h]}/x\|_p > 1) \\ &= \sum_{i=0}^h \mathbb{E} \left[\frac{|\mathbf{X}_i/x|^\alpha \mathbb{1}(|\mathbf{X}_i/x| > \epsilon)}{\sum_{t=0}^h |\mathbf{X}_t/x|^\alpha \mathbb{1}(|\mathbf{X}_t/x| > \epsilon)} \mathbb{1}(\|\mathbf{X}_{[0,h]}/x\|_p > 1) \right] \\ &= \sum_{i=0}^h \mathbb{E} \left[\frac{|\mathbf{X}_0/x|^\alpha \mathbb{1}(|\mathbf{X}_0/x| > \epsilon)}{\sum_{t=-i}^{h-i} |\mathbf{X}_t/x|^\alpha \mathbb{1}(|\mathbf{X}_t/x| > \epsilon)} \mathbb{1}(\|\mathbf{X}_{[-i,h-i]}/x\|_p > 1) \right] \\ &= \mathbb{P}(|\mathbf{X}_0| > x\epsilon) \sum_{i=0}^h \mathbb{E} \left[\frac{|\mathbf{X}_0/x|^\alpha \mathbb{1}(\|\mathbf{X}_{[-i,h-i]}/x\|_p > 1)}{\sum_{t=-i}^{h-i} |\mathbf{X}_t/x|^\alpha \mathbb{1}(|\mathbf{X}_t/x| > \epsilon)} \mid |\mathbf{X}_0| > x\epsilon \right]. \end{aligned}$$

Applying the definition (2.1) of regular variation and dominated convergence, we obtain as $x \rightarrow \infty$,

$$\begin{aligned} \frac{\mathbb{P}(\|\mathbf{X}_{[0,h]}/x\|_p > 1)}{\mathbb{P}(|\mathbf{X}_0| > x)} &\rightarrow \epsilon^{-\alpha} \sum_{i=0}^h \mathbb{E} \left[\frac{|\epsilon Y \Theta_0|^\alpha \mathbb{1}(\|\epsilon Y \Theta_{[-i,h-i]}\|_p > 1)}{\sum_{t=-i}^{h-i} |\epsilon Y \Theta_t|^\alpha \mathbb{1}(|Y \Theta_t| > 1)} \right] \\ &= \sum_{i=0}^h \int_\epsilon^\infty \mathbb{E} \left[\frac{\mathbb{1}(y \|\Theta_{[-i,h-i]}\|_p > 1)}{\sum_{t=-i}^{h-i} |\Theta_t|^\alpha \mathbb{1}(y |\Theta_t| > \epsilon)} \right] d(-y^{-\alpha}). \end{aligned}$$

The left-hand side does not depend on ϵ . Therefore, letting $\epsilon \downarrow 0$, we arrive at

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{X}_{[0,h]}/x\|_p > 1)}{\mathbb{P}(|\mathbf{X}_0| > x)} \\ &= \sum_{i=0}^h \int_0^\infty \mathbb{E} \left[\frac{\mathbb{1}(y \|\Theta_{[-i,h-i]}\|_p > 1)}{\|\Theta_{[-i,h-i]}\|_\alpha^\alpha} \right] d(-y^{-\alpha}) \\ (7.4) \quad &= \sum_{i=0}^h \mathbb{E} \left[\frac{\|\Theta_{-i+[0,h]}\|_p^\alpha}{\|\Theta_{-i+[0,h]}\|_\alpha^\alpha} \right] = c(p, h). \end{aligned}$$

This constant is finite since $\|\Theta_{i+[0,h]}\|_p \leq (h+1)\|\Theta_{i+[0,h]}\|_\infty$.

Next we prove (7.1). For this reason, let A be a continuity set with respect to the limit law in (7.1). An appeal to (7.4) yields

$$\begin{aligned} & c(p, h) \mathbb{P}(x^{-1} \mathbf{X}_{[0, h]} \in A \mid \|\mathbf{X}_{[0, h]}\|_p > x) \\ & \sim \frac{\mathbb{P}(x^{-1} \mathbf{X}_{[0, h]} \in A, \|\mathbf{X}_{[0, h]}\|_p > x)}{\mathbb{P}(\|\mathbf{X}_0\| > x)} =: I(x). \end{aligned}$$

Proceeding as for the derivation of (7.4), we obtain

$$\begin{aligned} I(x) & \sim \int_0^\infty \sum_{i=0}^h \mathbb{E} \left[\frac{\mathbb{1}(y \|\Theta_{[-i, h-i]}\|_p > 1)}{\|\Theta_{[-i, h-i]}\|_\alpha^\alpha} \mathbb{1}(y \Theta_{[-i, h-i]} \in A) \right] d(-y^{-\alpha}) \\ & = \int_1^\infty \sum_{i=0}^h \mathbb{E} \left[\frac{\|\Theta_{[-i, h-i]}\|_p^\alpha}{\|\Theta_{[-i, h-i]}\|_\alpha^\alpha} \mathbb{1}\left(y \frac{\Theta_{[-i, h-i]}}{\|\Theta_{[-i, h-i]}\|_p} \in A\right) \right] d(-y^{-\alpha}). \end{aligned}$$

In the last step we changed the variable, $u = y \|\Theta_{[-i, h-i]}\|_p > 0$ a.s., observing that $\|\Theta_{[-i, h-i]}\|_p \geq \|\Theta_0\| = 1$. This proves (7.1) and the lemma. \square

Proof of Lemma 7.2. Assume $f : \tilde{\ell}^\alpha \cap \{\mathbf{x} : \|\mathbf{x}\|_p = 1\} \rightarrow (0, \infty)$ is any bounded Lipschitz-continuous function in $(\tilde{\ell}^\alpha, \tilde{d}_\alpha)$. By Lemma 7.1 we have for all $p \geq \alpha$,

$$\begin{aligned} & \frac{c(p, h)}{h+1} \mathbb{E}[f(\mathbf{Q}^{(p)}(h))] - c(p) \mathbb{E}[f(\mathbf{Q}^{(p)})] = \\ & = \frac{1}{h+1} \sum_{k=0}^h \mathbb{E} \left[\left(\frac{\|\Theta_{-k+[0, h]}\|_p^\alpha}{\|\Theta_{-k+[0, h]}\|_\alpha^\alpha} - \frac{\|\Theta\|_p^\alpha}{\|\Theta\|_\alpha^\alpha} \right) f(\Theta_{-k+[0, h]} / \|\Theta_{-k+[0, h]}\|_p) \right] + \\ & \quad + \mathbb{E} \left[\frac{\|\Theta\|_p^\alpha}{\|\Theta\|_\alpha^\alpha} \left(f(\Theta_{-k+[0, h]} / \|\Theta_{-k+[0, h]}\|_p) - f(\Theta / \|\Theta\|_p) \right) \right] \\ & =: I + II. \end{aligned}$$

We will prove that I and II vanish as $h \rightarrow \infty$. Since $p \geq \alpha$ subadditivity yields for $k \in [0, h]$,

$$\begin{aligned} & \left| \frac{\|\Theta_{-k+[0, h]}\|_p^\alpha}{\|\Theta_{-k+[0, h]}\|_\alpha^\alpha} - \frac{\|\Theta\|_p^\alpha}{\|\Theta\|_\alpha^\alpha} \right| = \frac{\left| \|\|\Theta\|_\alpha \Theta_{-k+[0, h]}\|_p^\alpha - \|\|\Theta_{-k+[0, h]}\|_\alpha \Theta\|_p^\alpha \right|}{\|\Theta_{-k+[0, h]}\|_\alpha^\alpha \|\Theta\|_\alpha^\alpha} \\ & \leq \frac{\left| \|\|\Theta\|_\alpha \Theta_{-k+[0, h]}\|_p^p - \|\|\Theta_{-k+[0, h]}\|_\alpha \Theta\|_p^p \right|^{\alpha/p}}{\|\Theta_{-k+[0, h]}\|_\alpha^\alpha \|\Theta\|_\alpha^\alpha}. \end{aligned}$$

Moreover,

$$\begin{aligned} & \left| \|\|\Theta\|_\alpha \Theta_{-k+[0, h]}\|_p^p - \|\|\Theta_{-k+[0, h]}\|_\alpha \Theta\|_p^p \right| \\ & \leq \|\Theta_{-k+[0, h]}\|_\alpha^p \left(\sum_{t=-\infty}^{-k-1} |\Theta_t|^p + \sum_{t=-k+h+1}^{+\infty} |\Theta_t|^p \right) \\ & \quad + \left| \|\Theta\|_\alpha - \|\Theta_{-k+[0, h]}\|_\alpha \right|^p \sum_{t=-k}^{-k+h} |\Theta_t|^p. \end{aligned}$$

Thus, $|I|$ is bounded from above by

$$\begin{aligned} & \frac{1}{h+1} \|f\|_\infty \sum_{k=0}^h \left(\mathbb{E} \left[\frac{\|\Theta_{[-\infty, -k-1]}\|_p^\alpha}{\|\Theta\|_\alpha^\alpha} \right] + \mathbb{E} \left[\frac{\|\Theta_{[-k+h+1, \infty]}\|_p^\alpha}{\|\Theta\|_\alpha^\alpha} \right] \right. \\ & \quad \left. + \mathbb{E} \left[\frac{\|\Theta_{[-\infty, -k-1]\|_\alpha^\alpha + \|\Theta_{[-k+h+1, \infty]}\|_\alpha^\alpha}{\|\Theta\|_\alpha^\alpha} \frac{\|\Theta_{-k+[0, h]}\|_p^\alpha}{\|\Theta_{-k+[0, h]}\|_\alpha^\alpha} \right] \right) \\ & \leq \frac{1}{h+1} \|f\|_\infty \sum_{k=0}^h \left(\mathbb{E} \left[\frac{\|\Theta_{[-\infty, -(k+1)]}\|_p^\alpha + \|\Theta_{[-\infty, -(k+1)]}\|_\alpha^\alpha}{\|\Theta\|_\alpha^\alpha} \right] \right. \\ & \quad \left. + \mathbb{E} \left[\frac{\|\Theta_{[k+1, \infty]}\|_p^\alpha + \|\Theta_{[k+1, \infty]}\|_\alpha^\alpha}{\|\Theta\|_\alpha^\alpha} \right] \right). \end{aligned}$$

Taking the limit as $h \rightarrow \infty$, the Cèsaro limit on the right-hand side converges to zero.

We use the Lipschitz-continuity of f to obtain an upper bound of $|II|$:

$$|II| \leq \frac{1}{h+1} c \sum_{k=0}^h \mathbb{E} \left[\frac{\|\Theta_{-k+[0, h]}\|_p^\alpha}{\|\Theta_{-k+[0, h]}\|_\alpha^\alpha} \tilde{d}_\alpha \left(\frac{\Theta_{-k+[0, h]}}{\|\Theta_{-k+[0, h]}\|_p}, \frac{\Theta}{\|\Theta\|_p} \right) \right].$$

Similar arguments as for $|I| \rightarrow 0$ show that $|II| \rightarrow 0$. \square

7.2. Proof of Lemma 4.1. Choose some $\epsilon > 0$, $\delta \in (0, 1)$. Since $\|a_n^{-1} \mathbf{X}_{[0, n]}\|_p^p$ is a sum of non-negative random variables we have the following bounds via truncation

$$\begin{aligned} & \mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0, n]_\epsilon}\|_p^p > 1) \leq \mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0, n]}\|_p^p > 1) \\ (7.5) \quad & \leq \mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0, n]_\epsilon}\|_p^p > (1 - \delta^p)) + \mathbb{P}(\overline{\|x_n^{-1} \mathbf{X}_{[0, n]}\|_p^\epsilon} > \delta^p). \end{aligned}$$

By \mathbf{CS}_p and in view of Remark 2.1 we have

$$(7.6) \quad \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(\overline{\|x_n^{-1} \mathbf{X}_{[0, n]}\|_p^\epsilon} > \delta^p) / (n \mathbb{P}(|\mathbf{X}_0| > x_n)) = 0.$$

Now, for any choice of $u > 0$, it remains to determine the limits of the terms $\mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0, n]_\epsilon}\|_p^p > u) / (n \mathbb{P}(|\mathbf{X}_0| > x_n))$. We start with a telescoping sum representation

$$\begin{aligned} & \mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0, n]_\epsilon}\|_p^p > u) - \mathbb{P}(\|x_n^{-1} \mathbf{X}_{0_\epsilon}\|_p^p > u) \\ & = \sum_{i=1}^n (\mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0, i]_\epsilon}\|_p^p > u) - \mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0, i-1]_\epsilon}\|_p^p > u)) \\ & = \sum_{i=1}^n \mathbb{E} [(\mathbf{1}(\|x_n^{-1} \mathbf{X}_{[0, i]_\epsilon}\|_p^p > u) - \mathbf{1}(\|x_n^{-1} \mathbf{X}_{[1, i]_\epsilon}\|_p^p > u)) \mathbf{1}(|\mathbf{X}_0| > \epsilon x_n)], \end{aligned}$$

where we used stationarity in the last step and the fact that the difference of the indicator functions vanishes on $\{|\mathbf{X}_0| \leq \epsilon x_n\}$. We also observe that the second term on the left-hand side is of the order $o(n \mathbb{P}(|\mathbf{X}_0| > x_n))$. For

any fixed k write $A_k = \{\max_{k \leq t \leq n} |\mathbf{X}_t| > \epsilon x_n\}$. Regular variation of (\mathbf{X}_t) ensures that, as $n \rightarrow \infty$,

$$\begin{aligned} & \frac{\mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p^p > u)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ & \sim \epsilon^{-\alpha} \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\mathbb{1}(\|x_n^{-1} \mathbf{X}_{[0,i]_\epsilon}\|_p^p > u) - \mathbb{1}(\|x_n^{-1} \mathbf{X}_{[1,i]_\epsilon}\|_p^p > u) \mid |\mathbf{X}_0| > \epsilon x_n] \\ & \sim \epsilon^{-\alpha} \mathbb{E}[(\mathbb{1}(\|x_n^{-1} \mathbf{X}_{[0,k-1]_\epsilon}\|_p^p > u) - \mathbb{1}(\|x_n^{-1} \mathbf{X}_{[1,k-1]_\epsilon}\|_p^p > u)) \\ & \quad \times \mathbb{1}(A_k^c) \mid |\mathbf{X}_0| > \epsilon x_n] + \epsilon^{-\alpha} O(\mathbb{P}(A_k \mid |\mathbf{X}_0| > \epsilon x_n)), \end{aligned}$$

where the second term vanishes, first letting $n \rightarrow \infty$ and then $k \rightarrow \infty$, by virtue of **AC**. Now the regular variation property of (\mathbf{X}_t) implies that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{\mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p^p > u)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ & = \lim_{k \rightarrow \infty} \epsilon^{-\alpha} \left(\mathbb{P}\left(\sum_{t=0}^{k-1} |\epsilon Y_{\Theta_t}|^p \mathbb{1}(|Y_{\Theta_t}| > 1) > u\right) \right. \\ & \quad \left. - \mathbb{P}\left(\sum_{t=1}^{k-1} |\epsilon Y_{\Theta_t}|^p \mathbb{1}(|Y_{\Theta_t}| > 1) > u\right) \right), \end{aligned}$$

by a change of variable this term equals

$$\begin{aligned} & = \lim_{k \rightarrow \infty} \mathbb{E} \left[\int_{\epsilon}^{\infty} \left(\mathbb{1}\left(\sum_{t=0}^{k-1} |y_{\Theta_t}|^p \mathbb{1}(|y_{\Theta_t}| > \epsilon) > u\right) \right. \right. \\ & \quad \left. \left. - \mathbb{1}\left(\sum_{t=1}^{k-1} |y_{\Theta_t}|^p \mathbb{1}(|y_{\Theta_t}| > \epsilon) > u\right) \right) d(-y^{-\alpha}) \right] \\ & = \lim_{k \rightarrow \infty} \mathbb{E} \left[\int_0^{\infty} \left(\mathbb{1}(\|y_{\Theta_{[0,k-1]_\epsilon}}\|_p^p > u) - \mathbb{1}(\|y_{\Theta_{[1,k-1]_\epsilon}}\|_p^p > u) \right) d(-y^{-\alpha}) \right]. \end{aligned}$$

In the last step we used the fact that the integrand vanishes for $0 \leq y \leq \epsilon$. The integrand is non-negative and bounded by $\mathbb{1}(y > \epsilon)$ which is integrable. Thus we may take the limit as $k \rightarrow \infty$ inside the integral to derive the quantity

$$\mathbb{E} \left[\int_0^{\infty} \left(\mathbb{1}(\|y_{\Theta_{[0,\infty]_\epsilon}}\|_p^p > u) - \mathbb{1}(\|y_{\Theta_{[1,\infty]_\epsilon}}\|_p^p > u) \right) d(-y^{-\alpha}) \right].$$

By monotone convergence as $\epsilon \downarrow 0$ we get the limit

$$(7.7) \quad u^{-\alpha/p} \mathbb{E} [\|\Theta_{[0,\infty]}\|_p^\alpha - \|\Theta_{[1,\infty]}\|_p^\alpha] = u^{-\alpha/p} c(p);$$

see (3.4). Now an appeal to (7.5) with $u = 1$ and $u = 1 - \delta^p$ yields

$$\begin{aligned} c(p) &\leq \liminf_{n \rightarrow \infty} \frac{\mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0,n]}\|_p^p > 1)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ &\leq \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\|x_n^{-1} \mathbf{X}_{[0,n]}\|_p^p > 1)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \leq (1 - \delta^p)^{-\alpha/p} c(p). \end{aligned}$$

The limit relation (4.1) follows as $\delta \downarrow 0$.

7.3. Proof of Proposition 4.2. Consider any bounded Lipschitz-continuous function $f : (\tilde{\ell}^p, \tilde{d}_p) \rightarrow \mathbb{R}$. The statement is proved if we can show that

$$\begin{aligned} &\lim_{n \rightarrow \infty} \mathbb{E}[f(x_n^{-1} \mathbf{X}_{[0,n]}) \mid \|\mathbf{X}_{[0,n]}\|_p > x_n] \\ &= \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_p^\alpha f(Y \mathbf{Q}^{(\alpha)} / \|\mathbf{Q}^{(\alpha)}\|_p)]. \end{aligned}$$

In view of Lemma 4.1 it suffices to show

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{\mathbb{E}[f(x_n^{-1} \mathbf{X}_{[0,n]}) \mathbf{1}(\|\mathbf{X}_{[0,n]}\|_p > x_n)]}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ &= c(p) \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_p^\alpha f(Y \mathbf{Q}^{(\alpha)} / \|\mathbf{Q}^{(\alpha)}\|_p)]. \end{aligned}$$

In these limit relations we may replace $f(x_n^{-1} \mathbf{X}_{[0,n]})$ by $f(\underline{x_n^{-1} \mathbf{X}_{[0,n]_\epsilon})$ since by (7.6) for any $\delta > 0$, some $K_f > 0$,

$$\begin{aligned} &\lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(|f(x_n^{-1} \mathbf{X}_{[0,n]}) - f(\underline{x_n^{-1} \mathbf{X}_{[0,n]_\epsilon})| > \delta, \|\mathbf{X}_{[0,n]}\|_p > x_n)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ &\leq \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(K_f d_p(\underline{x_n^{-1} \mathbf{X}_{[0,n]_\epsilon}, \mathbf{0}) > \delta)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ &\leq \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(K_f \|\underline{x_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p^p > \delta(p))}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} = 0, \end{aligned}$$

where $\delta(p) = \delta^p$ for $p \geq 1$ and $= \delta$ for $p \in (0, 1)$. We also have for $\delta \in (0, 1)$,

$$\begin{aligned} G_{n,\epsilon} &= \frac{\mathbb{E}[f(\underline{x_n^{-1} \mathbf{X}_{[0,n]_\epsilon}) \mid \mathbf{1}(\|\mathbf{X}_{[0,n]}\|_p > 1) - \mathbf{1}(\|\underline{x_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p > 1)]}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ &\leq c \frac{\mathbb{P}(\|\mathbf{X}_{[0,n]}\|_p > 1 \geq \|\underline{x_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ &\leq c \frac{\mathbb{P}(\|\underline{x_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p > \delta)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} + c \frac{\mathbb{P}(1 \geq \|\underline{x_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p > 1 - \delta)}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} \\ &= G_{n,\epsilon,\delta}^{(1)} + G_{n,\epsilon,\delta}^{(2)}. \end{aligned}$$

Applying (7.6) to $G_{n,\epsilon,\delta}^{(1)}$ and using the calculations in the proof of Lemma 4.1 leading to (7.7) for $G_{n,\epsilon,\delta}^{(2)}$, we conclude that

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} G_{n,\epsilon} &\leq \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} G_{n,\epsilon,\delta}^{(1)} + \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} G_{n,\epsilon,\delta}^{(2)} \\ &= 0 + c \left((1 - \delta)^{-\alpha/p} - 1 \right) \downarrow 0, \quad \delta \downarrow 0. \end{aligned}$$

Thus it suffices to show

$$(7.8) \quad \lim_{\epsilon \downarrow 0} \lim_{n \rightarrow \infty} \frac{\mathbb{E} \left[f(\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon}) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p > 1) \right]}{n \mathbb{P}(|\mathbf{X}_0| > x_n)} = \mathbb{E} \left[\|\mathbf{Q}^{(\alpha)}\|_p f(Y \mathbf{Q}^{(\alpha)} / \|\mathbf{Q}^{(\alpha)}\|_p) \right].$$

This is the goal of the remaining proof.

Choose any $\epsilon > 0$. Noticing that $\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon} = \underline{x}_n^{-1} \mathbf{X}_{[1,n]_\epsilon}$ on $\{|\mathbf{X}_0| \leq \epsilon\}$, we have

$$\begin{aligned} I &:= \mathbb{E} \left[f(\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon}) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p > 1) \right] \\ &= \mathbb{E} \left[\left(f(\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon}) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p > 1) \right) \right. \\ &\quad \left. - f((0, \underline{x}_n^{-1} \mathbf{X}_{[1,n]_\epsilon})) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[1,n]_\epsilon}\|_p > 1) \right) \mathbb{1}(|\mathbf{X}_0| > \epsilon x_n) \Big] \\ &\quad + \mathbb{E} \left[f((0, \underline{x}_n^{-1} \mathbf{X}_{[1,n]_\epsilon})) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[1,n]_\epsilon}\|_p > 1) \right] \\ &= \mathbb{E} \left[\left(f(\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon}) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[0,n]_\epsilon}\|_p > 1) \right) \right. \\ &\quad \left. - f((0, \underline{x}_n^{-1} \mathbf{X}_{[1,n]_\epsilon})) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[1,n]_\epsilon}\|_p > 1) \right) \mathbb{1}(|\mathbf{X}_0| > \epsilon x_n) \Big] \\ &\quad + \mathbb{E} \left[f((0, \underline{x}_n^{-1} \mathbf{X}_{[0,n-1]_\epsilon})) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[0,n-1]_\epsilon}\|_p > 1) \right]. \end{aligned}$$

where we used the stationarity in the last step. Using the same idea recursively, we obtain

$$\begin{aligned} I &= \sum_{j=1}^n \mathbb{E} \left[\left(f((\mathbf{0}^{n-j}, \underline{x}_n^{-1} \mathbf{X}_{[0,j]_\epsilon})) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[0,j]_\epsilon}\|_p > 1) \right) \right. \\ &\quad \left. - f((\mathbf{0}^{n-j+1}, \underline{x}_n^{-1} \mathbf{X}_{[1,j]_\epsilon})) \mathbb{1}(\|\underline{x}_n^{-1} \mathbf{X}_{[1,j]_\epsilon}\|_p > 1) \right) \mathbb{1}(|\mathbf{X}_0| > \epsilon x_n) \Big] \\ &\quad + \mathbb{E} \left[f((\mathbf{0}^n, \underline{x}_n^{-1} \mathbf{X}_0)) \mathbb{1}(|\mathbf{X}_0| > \epsilon x_n) \right], \end{aligned}$$

where $\mathbf{0}^k := \{0\}^k$ for $k \geq 1$. By regular variation of \mathbf{X}_0 the last right-hand term is $o(n \mathbb{P}(|\mathbf{X}_0| > x_n))$. Therefore by regular variation of (\mathbf{X}_t) we obtain

as $n \rightarrow \infty$,

$$\begin{aligned}
& I / (n\mathbb{P}(|\mathbf{X}_0| > x_n)) \\
& \sim \frac{\epsilon^{-\alpha}}{n} \sum_{j=1}^n \mathbb{E} \left[f((\mathbf{0}^{n-j}, \underline{x_n^{-1} \mathbf{X}_{[0,j]_\epsilon}})) \mathbb{1}(\|\underline{x_n^{-1} \mathbf{X}_{[0,j]_\epsilon}\|_p > 1) \right. \\
& \quad \left. - f((\mathbf{0}^{n-j+1}, \underline{x_n^{-1} \mathbf{X}_{[1,j]_\epsilon})) \mathbb{1}(\|\underline{x_n^{-1} \mathbf{X}_{[1,j]_\epsilon}\|_p > 1) \mid |\mathbf{X}_0| > \epsilon x_n \right] \\
& =: II.
\end{aligned}$$

Write $A_k = \{\|\mathbf{X}_{[k,n]}\|_\infty > \epsilon x_n\}$ for fixed $k \geq 1$. By **AC**, $\mathbb{P}(A_k \mid |\mathbf{X}_0| > \epsilon x_n)$ vanishes by first letting $n \rightarrow \infty$ then $k \rightarrow \infty$. Since each of the summands in II is uniformly bounded in absolute value we may restrict the summation to $j \in \{k-1, \dots, n\}$ for any fixed $k \geq 1$. Therefore we have as $n \rightarrow \infty$,

$$\begin{aligned}
& II - O(\mathbb{P}(A_k \mid |\mathbf{X}_0| > \epsilon x_n)) \\
& \sim \frac{\epsilon^{-\alpha}}{n} \sum_{j=k-1}^n \mathbb{E} \left[\left(f((\mathbf{0}^{n-j}, \underline{x_n^{-1} \mathbf{X}_{[0,j]_\epsilon})) \mathbb{1}(\{\|\underline{x_n^{-1} \mathbf{X}_{[0,j]_\epsilon}\|_p > 1\}) \right. \right. \\
& \quad \left. \left. - f((\mathbf{0}^{n-j+1}, \underline{x_n^{-1} \mathbf{X}_{[1,j]_\epsilon})) \mathbb{1}(\{\|\underline{x_n^{-1} \mathbf{X}_{[1,j]_\epsilon}\|_p > 1\}) \right) \mathbb{1}(A_k^c) \mid |\mathbf{X}_0| > \epsilon x_n \right] \\
& = \frac{\epsilon^{-\alpha}}{n} \sum_{j=k-1}^n \mathbb{E} \left[f((\mathbf{0}^{n-j}, \underline{x_n^{-1} \mathbf{X}_{[0,k-1]_\epsilon}, \mathbf{0}^{j-k})) \mathbb{1}(\|\underline{x_n^{-1} \mathbf{X}_{[0,k-1]_\epsilon}\|_p > 1) \right. \\
& \quad \left. - f((\mathbf{0}^{n-j+1}, \underline{x_n^{-1} \mathbf{X}_{[1,k-1]_\epsilon}, \mathbf{0}^{j-k})) \mathbb{1}(\|\underline{x_n^{-1} \mathbf{X}_{[1,k-1]_\epsilon}\|_p > 1) \mid |\mathbf{X}_0| > \epsilon x_n \right].
\end{aligned}$$

Next we apply shift-invariance and regular variation in $\tilde{\ell}^p$:

$$\begin{aligned}
& \sim \epsilon^{-\alpha} \mathbb{E} \left[f(\underline{x_n^{-1} \mathbf{X}_{[0,k-1]_\epsilon}) \mathbb{1}(\|\underline{x_n^{-1} \mathbf{X}_{[0,k-1]_\epsilon}\|_p > 1) \right. \\
& \quad \left. - f(\underline{x_n^{-1} \mathbf{X}_{[1,k-1]_\epsilon}) \mathbb{1}(\|\underline{x_n^{-1} \mathbf{X}_{[1,k-1]_\epsilon}\|_p > 1) \mid |\mathbf{X}_0| > \epsilon x_n \right] \\
& \rightarrow \epsilon^{-\alpha} \mathbb{E} \left[f(\underline{\epsilon Y \Theta_{[0,k-1]_\epsilon}}) \mathbb{1}(\|\underline{\epsilon Y \Theta_{[0,k-1]_\epsilon}\|_p > 1) \right. \\
& \quad \left. - f(\underline{\epsilon Y \Theta_{[1,k-1]_\epsilon}}) \mathbb{1}(\|\underline{\epsilon Y \Theta_{[1,k-1]_\epsilon}\|_p > 1) \right] \\
& = \mathbb{E} \left[f(\underline{\epsilon Y \Theta_{[0,k-1]_1}}) \mathbb{1}(\|\underline{\epsilon Y \Theta_{[0,k-1]_1}\|_p > 1) \right. \\
& \quad \left. - f(\underline{\epsilon Y \Theta_{[1,k-1]_1}}) \mathbb{1}(\|\underline{\epsilon Y \Theta_{[1,k-1]_1}\|_p > 1) \right] =: J_{k,\epsilon}.
\end{aligned}$$

By Proposition 3.2 we have $\|\Theta\|_\alpha < \infty$ a.s., $|\Theta_t| \xrightarrow{\text{a.s.}} 0$ as $|t| \rightarrow \infty$, hence $T := \inf_{t \geq 0} \{t : Y|\Theta_t| < 1\} < \infty$ a.s. Then by monotone convergence as

$k \rightarrow \infty$,

$$\begin{aligned}
J_{k,\epsilon} &= \epsilon^{-\alpha} \mathbb{E} \left[\left(f(\epsilon \underline{Y}_{\Theta_{[0,\infty]}_1}) \mathbb{1}(\|\epsilon \underline{Y}_{\Theta_{[0,\infty]}_1}\|_p > 1) \right. \right. \\
&\quad \left. \left. - f(\epsilon \underline{Y}_{\Theta_{[1,\infty]}_1}) \mathbb{1}(\|\epsilon \underline{Y}_{\Theta_{[1,\infty]}_1}\|_p > 1) \right) \mathbb{1}(T < k) \right] \\
&\quad + O(\mathbb{P}(T \geq k)) \\
&\rightarrow \epsilon^{-\alpha} \mathbb{E} \left[f(\epsilon \underline{Y}_{\Theta_{[0,\infty]}_1}) \mathbb{1}(\|\epsilon \underline{Y}_{\Theta_{[0,\infty]}_1}\|_p > 1) \right. \\
&\quad \left. - f(\epsilon \underline{Y}_{\Theta_{[1,\infty]}_1}) \mathbb{1}(\|\epsilon \underline{Y}_{\Theta_{[1,\infty]}_1}\|_p > 1) \right] \\
&= \int_0^\infty \mathbb{E} [f(\underline{y}_{\Theta_{[0,\infty]}_\epsilon}) \mathbb{1}(\|\underline{y}_{\Theta_{[0,\infty]}_\epsilon}\|_p > 1) \\
&\quad - f(\underline{y}_{\Theta_{[1,\infty]}_\epsilon}) \mathbb{1}(\|\underline{y}_{\Theta_{[1,\infty]}_\epsilon}\|_p > 1)] d(-y^{-\alpha}) =: J_\epsilon.
\end{aligned}$$

In the last step we changed variables, $u = \epsilon y$, and observed that the integrand vanishes for $y < \epsilon$.

Finally, we want to let $\epsilon \downarrow 0$. We start by inter-changing expectation and integral in J_ϵ , and change variables, $u = y \|\Theta_{[0,\infty]}\|_\alpha$, in the first term of the integrand and then proceed similarly for the second term with the convention that it is zero on $\{\|\Theta_{[1,\infty]}\|_\alpha = 0\}$:

$$\begin{aligned}
J_\epsilon &= \mathbb{E} \left[\int_0^\infty \left(\|\Theta_{[0,\infty]}\|_\alpha^\alpha f\left(\frac{y \Theta_{[0,\infty]}}{\|\Theta_{[0,\infty]}\|_\alpha}\right) \mathbb{1}\left(\left\|\frac{y \Theta_{[0,\infty]}}{\|\Theta_{[0,\infty]}\|_\alpha}\right\|_p > 1\right) \right. \right. \\
&\quad \left. \left. - \|\Theta_{[1,\infty]}\|_\alpha^\alpha f\left(\frac{y \Theta_{[1,\infty]}}{\|\Theta_{[1,\infty]}\|_\alpha}\right) \mathbb{1}\left(\left\|\frac{y \Theta_{[1,\infty]}}{\|\Theta_{[1,\infty]}\|_\alpha}\right\|_p > 1\right) \right) d(-y^{-\alpha}) \right] \\
&= \sum_{t=1}^\infty \int_0^\infty \mathbb{E} \left[|\Theta_t|^\alpha \left(f\left(\frac{y \Theta_{[0,\infty]}}{\|\Theta_{[0,\infty]}\|_\alpha}\right) \mathbb{1}\left(\left\|\frac{y \Theta_{[0,\infty]}}{\|\Theta_{[0,\infty]}\|_\alpha}\right\|_p > 1\right) \right. \right. \\
&\quad \left. \left. - f\left(\frac{y \Theta_{[1,\infty]}}{\|\Theta_{[1,\infty]}\|_\alpha}\right) \mathbb{1}\left(\left\|\frac{y \Theta_{[1,\infty]}}{\|\Theta_{[1,\infty]}\|_\alpha}\right\|_p > 1\right) \right) \right] d(-y^{-\alpha}) \\
&\quad + \mathbb{E} \left[\int_0^\infty f\left(\frac{y \Theta_{[0,\infty]}}{\|\Theta_{[0,\infty]}\|_\alpha}\right) \mathbb{1}\left(\left\|\frac{y \Theta_{[0,\infty]}}{\|\Theta_{[0,\infty]}\|_\alpha}\right\|_p > 1\right) d(-y^{-\alpha}) \right].
\end{aligned}$$

Next we apply the time-change formula (3.3) to each summand.

$$\begin{aligned}
J_\epsilon &= \sum_{t=1}^\infty \int_0^\infty \mathbb{E} \left[f\left(\frac{y \Theta_{[-t,\infty]}}{\|\Theta_{[-t,\infty]}\|_\alpha}\right) \mathbb{1}\left(\left\|\frac{y \Theta_{[-t,\infty]}}{\|\Theta_{[-t,\infty]}\|_\alpha}\right\|_p > 1\right) \right. \\
&\quad \left. - f\left(\frac{y \Theta_{[1-t,\infty]}}{\|\Theta_{[1-t,\infty]}\|_\alpha}\right) \mathbb{1}\left(\left\|\frac{y \Theta_{[1-t,\infty]}}{\|\Theta_{[1-t,\infty]}\|_\alpha}\right\|_p > 1\right) \right] d(-y^{-\alpha}) \\
&\quad + \mathbb{E} \left[\int_0^\infty f\left(\frac{y \Theta_{[0,\infty]}}{\|\Theta_{[0,\infty]}\|_\alpha}\right) \mathbb{1}\left(\left\|\frac{y \Theta_{[0,\infty]}}{\|\Theta_{[0,\infty]}\|_\alpha}\right\|_p > 1\right) d(-y^{-\alpha}) \right].
\end{aligned}$$

This is a telescoping sum in t with value

$$J_\epsilon = \mathbb{E} \left[\int_0^\infty f\left(\frac{y \Theta}{\|\Theta\|_\alpha}\right) \mathbb{1}\left(\left\|\frac{y \Theta}{\|\Theta\|_\alpha}\right\|_p > 1\right) d(-y^{-\alpha}) \right].$$

By monotone convergence we have

$$\lim_{\epsilon \downarrow 0} J_\epsilon = \mathbb{E} \left[\int_0^\infty f(y \mathbf{Q}^{(\alpha)}) \mathbf{1} \left(\|y \mathbf{Q}^{(\alpha)}\|_p > 1 \right) d(-y^{-\alpha}) \right].$$

Combining the arguments above, we proved (7.8) as desired. \square

7.4. Proof of Lemma 4.3. The case $p < \alpha$. Choose some $\epsilon > 0$, $\delta \in (0, 1)$. We have the following bounds via truncation

$$\begin{aligned} I_1 - I_2 &:= \mathbb{P} \left(\|\underline{x}_n^{-1} \mathbf{X}_{[0, n]_\epsilon}\|_p^p - \mathbb{E} [\|\underline{x}_n^{-1} \mathbf{X}_{[0, n]_\epsilon}\|_p^p] > 1 + \delta^p \right) \\ &\quad - \mathbb{P} \left(\|\overline{x}_n^{-1} \mathbf{X}_{[0, n]_\epsilon}\|_p^p - \mathbb{E} [\|\overline{x}_n^{-1} \mathbf{X}_{[0, n]_\epsilon}\|_p^p] \leq -\delta^p \right) \\ &\leq \mathbb{P} \left(\|\underline{x}_n^{-1} \mathbf{X}_{[0, n]_\epsilon}\|_p^p - \mathbb{E} [\|\underline{x}_n^{-1} \mathbf{X}_{[0, n]_\epsilon}\|_p^p] > 1 \right) \\ &\leq \mathbb{P} \left(\|\underline{x}_n^{-1} \mathbf{X}_{[0, n]_\epsilon}\|_p^p - \mathbb{E} [\|\underline{x}_n^{-1} \mathbf{X}_{[0, n]_\epsilon}\|_p^p] > 1 - \delta^p \right) \\ &\quad + \mathbb{P} \left(\|\overline{x}_n^{-1} \mathbf{X}_{[0, n]_\epsilon}\|_p^p - \mathbb{E} [\|\overline{x}_n^{-1} \mathbf{X}_{[0, n]_\epsilon}\|_p^p] > \delta^p \right) =: I_3 + I_4. \end{aligned}$$

Taking into account \mathbf{CS}_p for $p < \alpha$, we have

$$\lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} (I_2 + I_4) / (n \mathbb{P}(|\mathbf{X}_0| > x_n)) = 0.$$

Moreover, we observe that by Karamata's theorem for $p < \alpha$

$$\begin{aligned} \mathbb{E} [\|\underline{x}_n^{-1} \mathbf{X}_{[0, n]_\epsilon}\|_p^p] &= n \mathbb{E} [|\mathbf{X}/x_n|^p \mathbf{1}(|\mathbf{X}| > \epsilon x_n)] \\ &= O(n \mathbb{P}(|\mathbf{X}_0| > \epsilon x_n)) = o(1). \end{aligned}$$

Thus centering in I_1 and I_3 is not needed, and one can follow the lines of the proof of Lemma 4.1 to conclude.

The case $p = \alpha$. It requires only slight changes; we omit details. \square

7.5. Proof of Theorem 5.1. We start with a version of Theorem 5.1 for deterministic thresholds (x_b) .

Lemma 7.3. *Assume the conditions of Theorem 5.1. Then for every $g \in \mathcal{G}_+(\tilde{\ell}^p)$,*

$$(7.9) \quad \frac{1}{k} \sum_{t=1}^m g(x_b^{-1} \mathbf{B}_t) \xrightarrow{\mathbb{P}} \int_0^\infty \mathbb{E} [g(y \mathbf{Q}^{(p)})] d(-y^{-\alpha}), \quad n \rightarrow \infty,$$

holds for sequences $k_n \rightarrow \infty$ and $m_n := \lfloor n/b_n \rfloor \rightarrow \infty$ as in \mathbf{MX}_p .

Proof. It suffices to prove that under \mathbf{MX}_p for Lipschitz-continuous $f \in \mathcal{G}_+(\tilde{\ell}^p)$ as $n \rightarrow \infty$,

$$(7.10) \quad \left(\mathbb{E} \left[e^{-\frac{1}{k} \sum_{t=1}^{\lfloor m/k \rfloor} f(x_b^{-1} \mathbf{B}_t)} \right] \right)^k \rightarrow e^{-\mathbb{E} \left[\int_0^\infty f(y \mathbf{Q}^{(p)}) d(-y^{-\alpha}) \right]}.$$

By stationarity,

$$(7.11) \quad \mathbb{E} \left[1 - e^{-\frac{1}{k} \sum_{t=1}^{\lfloor m/k \rfloor} f(x_b^{-1} \mathbf{B}_t)} \right] = O(k^{-2} m \mathbb{E} [f(x_b^{-1} \mathbf{B}_1)]).$$

Since f vanishes in some neighborhood of the origin there exists $c_f > 0$ such that $f(\mathbf{x}) = f(\mathbf{x}) \mathbf{1}(\|\mathbf{x}\|_p > c_f)$. Therefore and by virtue of Proposition 4.2

the right-hand side of (7.11) vanishes as $n \rightarrow \infty$. Now a Taylor expansion argument shows that the left-hand side of (7.10) is of the asymptotic order $\sim \exp\{-(m/k)\mathbb{E}[f(x_b^{-1}\mathbf{B}_t)]\}$, and another application of Proposition 4.2 yields (7.10). \square

We continue with the proof of Theorem 5.1. Lemma 7.3 implies convergence of the empirical measures in $M_0(\tilde{\ell}^p)$:

$$P_n(\cdot) := \frac{1}{k} \sum_{t=1}^m \mathbb{1}(x_b^{-1}\mathbf{B}_t \in \cdot) \xrightarrow{\mathbb{P}} P(\cdot) := \int_0^\infty \mathbb{P}(y\mathbf{Q}^{(p)} \in \cdot) d(-y^{-\alpha}).$$

Using the argument in Resnick [31], p. 81, we may conclude $\|\mathbf{B}\|_{p,(k)}/x_b \xrightarrow{\mathbb{P}} 1$, and thus the joint convergence in $(P_n, \|\mathbf{B}\|_{p,(k)}/x_b) \xrightarrow{\mathbb{P}} (P, 1)$ in $M_0(\tilde{\ell}^p) \times \mathbb{R}_+$ follows. Now (5.2) follows by an application of the continuous mapping theorem to the scaling function $s(P(\cdot), t) = P(\cdot/t)$. To prove continuity of s we use the portmanteau theorem for $M_0(\tilde{\ell}^p)$ -convergence in Hult and Lindskog [20], Theorem 2.4. Thus it suffices to check whether the limit $P_n(f/t) \xrightarrow{\mathbb{P}} P(f)$ holds as $(n, t) \rightarrow (\infty, 1)$ for Lipschitz-continuous $f \in \mathcal{G}_+(\tilde{\ell}^p)$. But we have with Lemma 7.3

$$\begin{aligned} |P_n(f/t) - P(f)| &\leq |P_n(f/t) - P_n(f)| + |P_n(f) - P(f)| \\ &= |P_n(f/t) - P_n(f)| + o_{\mathbb{P}}(1), \quad n \rightarrow \infty. \end{aligned}$$

Then, for all $0 < t_0 \leq t < 2$, for $t_0 \leq 1$, denoting $g(\mathbf{x}) = \|\mathbf{x}\|_p \wedge \|f\|_\infty \mathbb{1}(\{\mathbf{x} : \|\mathbf{x}\|_p > c_f/t_0\})$ we have

$$\begin{aligned} |P_n(f/t) - P(f)| &\leq |t^{-1} - 1| P_n(g) + o_{\mathbb{P}}(1) \\ &\leq |t^{-1} - 1| (c + o_{\mathbb{P}}(1)) + o_{\mathbb{P}}(1), \end{aligned}$$

for some $c > 0$, $c_f > 0$ as above. Letting $t \rightarrow 1$, continuity of s follows. \square

7.6. Proof of Proposition 5.4. The result follows by a direct application of Theorem 3.1 in Bartkiewicz *et al.* [1] on $\mathbf{u}^\top \mathbf{S}_n$ for any $\mathbf{u} \in \mathbb{R}^d$ such that $|\mathbf{u}| = 1$ by checking their conditions **(AC)**, **(TB)**. Condition (5.6) implies that for all $\delta > 0$,

$$\lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} n \sum_{t=l}^{b_n} \mathbb{P}(|\mathbf{X}_t| > \delta a_n, |\mathbf{X}_0| > \delta a_n),$$

from which **(AC)** is immediate. This condition also implies **(TB)**. We show this in two steps. First, we identify the coefficients $b(d)$ in **(TB)** in terms of the spectral tail process. Mikosch and Wintenberger [26] showed that

$$b_\pm(d) - b_\pm(d-1) = \mathbb{E} \left[\left(\sum_{j=0}^d \mathbf{u}^\top \Theta_j \right)_\pm^\alpha \right] - \mathbb{E} \left[\left(\sum_{j=1}^d \mathbf{u}^\top \Theta_j \right)_\pm^\alpha \right],$$

where we suppress in the notation the dependence of the left-hand side on \mathbf{u} in what follows. **(TB)** amounts to verifying that $b_\pm(d) - b_\pm(d-1)$ converges

as $d \rightarrow \infty$. For $\alpha \in (0, 1)$ this follows by concavity since $\|\Theta\|_\alpha < \infty$ a.s. For $1 < \alpha < 2$ this will follow by a convexity argument if $\mathbb{E}[(\sum_{j \geq 0} |\Theta_j|)^{\alpha-1}] < \infty$. By subadditivity and Jensen's inequality, it is enough to check

$$(7.12) \quad \sum_{j=0}^{\infty} (\mathbb{E}[|\Theta_j|^{\alpha-1} \mathbf{1}(|\Theta_j| > 1)] + \mathbb{E}[|\Theta_j| \wedge 1]) < +\infty.$$

We start by showing

$$(7.13) \quad \sum_{j=0}^{\infty} \mathbb{E}[|\Theta_j| \wedge 1] < \infty.$$

Condition (5.6) implies

$$\lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} n \sum_{j=l}^{b_n} \mathbb{E}[(|a_n^{-1} \mathbf{X}_j| \wedge 1) \mathbf{1}(|\mathbf{X}_0| > a_n)] = 0,$$

which yields the following Cauchy criterion: for any $\varepsilon > 0$ there exists K sufficiently large such that for $l \geq K, h \geq 0$,

$$\limsup_{n \rightarrow \infty} n \sum_{j=l}^{l+h} \mathbb{E}[(|a_n^{-1} \mathbf{X}_j| \wedge 1) \mathbf{1}(|\mathbf{X}_0| > a_n)] = \sum_{j=l}^{l+h} \mathbb{E}[|Y \Theta_j| \wedge 1] \leq \varepsilon,$$

where we used regular variation of (\mathbf{X}_t) in the last step. Then, we conclude (7.13) holds. By stationarity we can show similarly

$$(7.14) \quad \sum_{j=0}^{\infty} \mathbb{E}[|\Theta_{-j}| \wedge 1] < +\infty.$$

Then, by the time-change formula in (3.3) we deduce

$$\begin{aligned} +\infty &> \sum_{j=0}^{\infty} \mathbb{E}[|\Theta_{-j}| \wedge 1] \\ &= \sum_{j=0}^{\infty} \mathbb{E}[|\Theta_j|^\alpha (|\Theta_j|^{-1} \wedge 1)] \\ &= \sum_{j=0}^{\infty} \mathbb{E}[|\Theta_j|^{\alpha-1} \wedge |\Theta_j|^\alpha] \\ &> \sum_{j=0}^{\infty} \mathbb{E}[|\Theta_j|^{\alpha-1} \mathbf{1}(|\Theta_j| > 1)], \end{aligned}$$

and (7.12) holds. This finishes the proof of the fact that $\mathbb{E}[(\sum_{j \geq 0} |\Theta_t|)^{\alpha-1}] < +\infty$, in particular $c(1) < \infty$. Applying the mean value theorem and dominated convergence we arrive at the relation

$$b_\pm(d) - b_\pm(d-1) \rightarrow \mathbb{E}\left[\left(\sum_{j=0}^{\infty} \mathbf{u}^\top \Theta_j\right)_\pm^\alpha - \left(\sum_{j=1}^{\infty} \mathbf{u}^\top \Theta_j\right)_\pm^\alpha\right], \quad d \rightarrow \infty.$$

Reasoning for the limit as for (3.4) and recalling that $c(1) < \infty$, we identify

$$\begin{aligned} & \mathbb{E} \left[\left(\sum_{j=0}^{\infty} \mathbf{u}^\top \boldsymbol{\Theta}_j \right)_\pm^\alpha - \left(\sum_{j=1}^{\infty} \mathbf{u}^\top \boldsymbol{\Theta}_j \right)_\pm^\alpha \right] \\ &= \mathbb{E} \left[\left(\sum_{j=-\infty}^{\infty} \mathbf{u}^\top \boldsymbol{\Theta}_j \right)_\pm^\alpha / \|\boldsymbol{\Theta}\|_\alpha^\alpha \right] = \mathbb{E} \left[\left(\sum_{j=-\infty}^{\infty} \mathbf{u}^\top \mathbf{Q}_j^{(\alpha)} \right)_\pm^\alpha \right] \\ &= c(1) \mathbb{E} \left[\left(\sum_{j=-\infty}^{\infty} \mathbf{u}^\top \mathbf{Q}_j^{(1)} \right)_\pm^\alpha \right]. \end{aligned}$$

7.7. Proofs of the results of Section 6.

7.7.1. *Proof of Proposition 6.1.* Notice that ψ_g is bounded and measurable. For $p = \alpha$ we have $\mathbf{Q}^{(\alpha)} \stackrel{d}{=} \boldsymbol{\Theta} / \|\boldsymbol{\Theta}\|_\alpha$. Then the result follows from Proposition 3.6 in Janssen [21]. For $p > 0$, assuming the p-cluster process is well defined we have $\|\boldsymbol{\Theta}\|_p < \infty$ a.s. and $c(p) < \infty$. Then, we introduce the Radon-Nikodym derivative of $\mathcal{L}(\mathbf{Q}^{(p)})$ with respect to $\mathcal{L}(\boldsymbol{\Theta} / \|\boldsymbol{\Theta}\|_p)$ which by (3.2) is the function $h : \ell^p \cap \{\mathbf{x} : \|\mathbf{x}\|_p = 1\} \rightarrow \mathbb{R}_{\geq 0}$ defined by $h(\mathbf{y} / \|\mathbf{y}\|_p) := \|\mathbf{y}\|_\alpha / \|\mathbf{y}\|_p$. Finally, the result follows by another application of Proposition 3.6 in Janssen [21]. \square

7.7.2. *Proof of Theorem 6.2.* The proof is given for $p = \alpha$ only; the case $p \leq \alpha$ extends in a natural way. Let $g : \ell^\alpha \rightarrow \mathbb{R}$ be a continuous bounded function. We start by proving that ψ_g defined in (6.1) is a continuous bounded function on $\tilde{\ell}^\alpha$. Fix $\epsilon > 0$ and $[\mathbf{z}] \in \tilde{\ell}^\alpha \cap \{[\mathbf{y}] : \|\mathbf{y}\|_\alpha = 1\}$. Then for all $[\mathbf{x}] \in \tilde{\ell}^\alpha \cap \{[\mathbf{y}] : \|\mathbf{y}\|_\alpha = 1\}$, $k \in \mathbb{Z}$ and $N \in \mathbb{N}$ we have

$$\begin{aligned} |\psi_g(\mathbf{x}) - \psi_g(\mathbf{z})| &= \left| \sum_{j \in \mathbb{Z}} |\mathbf{x}_j^*|^\alpha g((\mathbf{x}_{j+t}^*)_t) - \sum_{j \in \mathbb{Z}} |\mathbf{z}_j^*|^\alpha g((\mathbf{z}_{j+t}^*)_t) \right| \\ &= \left| \sum_{j \in \mathbb{Z}} |\mathbf{x}_j^* - \mathbf{z}_{j-k}^*|^\alpha g((\mathbf{x}_{j+t}^*)_t) \right. \\ &\quad \left. - \sum_{j \in \mathbb{Z}} |\mathbf{z}_j^*|^\alpha (g((\mathbf{z}_{j+t}^*)_t) - g((\mathbf{x}_{j+t+k}^*)_t)) \right| \\ &\leq \|g\|_\infty d_\alpha^\alpha(B^{-k} \mathbf{z}^*, \mathbf{x}^*) \\ &\quad + 2\|g\|_\infty d_\alpha^\alpha(\mathbf{z}^*, \mathbf{z}_{[-N, N]}^*) \\ &\quad + \sum_{|j| < N} |\mathbf{z}_j^*|^\alpha |g((\mathbf{z}_{j+t}^*)_t) - g((\mathbf{x}_{j+t+k}^*)_t)|. \end{aligned}$$

If $[\mathbf{x}]$ satisfies $\tilde{d}_\alpha^\alpha(\mathbf{z}, \mathbf{x}) < \epsilon(3\|g\|_\infty)^{-1}$ then there exists $k_0 \in \mathbb{Z}$ such that

$$\tilde{d}_\alpha^\alpha(\mathbf{z}, \mathbf{x}) < d_\alpha^\alpha(B^{-k_0} \mathbf{z}^*, \mathbf{x}^*) < \epsilon(3\|g\|_\infty)^{-1}.$$

Furthermore, choose $N_0 \geq 0$ such that $d_\alpha^\alpha(\mathbf{z}^*, \mathbf{z}_{[-N_0, N_0]}^*) < \epsilon(2 \times 3\|g\|_\infty)^{-1}$ and consider the finite set $C_{[\mathbf{z}]} \subset \ell^\alpha \cap \{[\mathbf{y}] : \|\mathbf{y}\|_\alpha = 1\}$, defined by $C_{[\mathbf{z}]} :=$

$\{(\mathbf{z}_{j+t}^*)_t \in \ell^\alpha : |j| < N_0, |\mathbf{z}_j^*| > 0\}$. Notice that for every $\tilde{\mathbf{z}} \in C_{[z]}$ there exists $\delta(\tilde{\mathbf{z}})$ such that if $d_\alpha^\alpha(\tilde{\mathbf{z}}, \mathbf{x}) < \delta(\tilde{\mathbf{z}})$ implies $|g(\tilde{\mathbf{z}}) - g(\mathbf{x})| < \epsilon/3$. Finally, define $\eta(\mathbf{z}) := \min\{\delta(\tilde{\mathbf{z}}) : \tilde{\mathbf{z}} \in C\} \wedge \epsilon(3\|g\|_\infty)^{-1}$. Then, noticing that $\sum_{|j| < N_0} |\mathbf{z}_j^*|^\alpha \leq \|\mathbf{z}\|_\alpha^\alpha = 1$, we obtain also a bound for the last term. Hence, for every $[\mathbf{x}] \in \tilde{\ell}^\alpha$ satisfying $d_\alpha^\alpha(\mathbf{z}, \mathbf{x}) < \eta(\mathbf{z})$ we have $|\psi_g(\mathbf{x}) - \psi_g(\mathbf{z})| < \epsilon$.

This finishes the proof of the continuity of the function ψ_g on $\tilde{\ell}^\alpha \cap \{\mathbf{y} : \|\mathbf{y}\|_\alpha = 1\}$. We conclude with applications of Lemma 7.3 and Proposition 6.1. \square

7.7.3. *Proof of Proposition 6.5.* Theorem 4.5 in Mikosch and Wintenberger [28] yields immediately

$$\left| \frac{\mathbb{P}(\sup_{1 \leq t \leq n} S_t > x_n)}{n\mathbb{P}(|X_1| > x_n)} - \mathbb{E} \left[\left(\sup_{t \geq 0} \sum_{i=0}^t \Theta_i \right)_+^\alpha - \left(\sup_{t \geq 1} \sum_{i=1}^t \Theta_i \right)_+^\alpha \right] \right| \rightarrow 0, \quad n \rightarrow \infty, \quad (7.15)$$

and $n\mathbb{P}(|X_1| > x_n) \rightarrow 0$. We multiply the function inside the limiting expected value by the constant $1 = \|\Theta\|_\alpha^\alpha / \|\Theta\|_\alpha^\alpha$. Moreover, since $c(1) < \infty$, then $\mathbb{E}[(\sum_{t=1}^\infty |\Theta_t|)^{\alpha-1}] < \infty$; see Lemma 3.11 in Planinić and Soulier [30]. Then, by Fubini's theorem,

$$\begin{aligned} & \mathbb{E} \left[\left(\sup_{t \geq 0} \sum_{i=0}^t \Theta_i \right)_+^\alpha - \left(\sup_{t \geq 1} \sum_{i=1}^t \Theta_i \right)_+^\alpha \right] \\ &= \sum_{j \in \mathbb{Z}} \mathbb{E} \left[|\Theta_j|^\alpha \left(\left(\sup_{t \geq 0} \sum_{i=0}^t \frac{\Theta_i}{\|\Theta\|_\alpha} \right)_+^\alpha - \left(\sup_{t \geq 1} \sum_{i=1}^t \frac{\Theta_i}{\|\Theta\|_\alpha} \right)_+^\alpha \right) \right]. \end{aligned}$$

At this point we apply the time-change formula for positive measurable functions of Θ at every term of the sum in $j \in \mathbb{Z}$; see Corollary 2.8. in Dombry *et al.* [12]. By the same argument as in the proof of Proposition 4.2 we obtain the representation of the expectation in (7.15) in terms of the univariate cluster process $Q^{(p)}$.

Now we apply Theorem 6.2 to $f(\mathbf{x}) := \lim_{k \rightarrow \infty} (\sup_{t \geq -k} \sum_{i=-k}^t x_i)_+^\alpha$ on ℓ^1 . It is uniformly continuous and bounded by one on the sphere of ℓ^p , hence (6.2) holds for f . Similarly, the constant $c(1)^{-1}$ can be estimated by employing the function $g(\mathbf{x}) := \|\mathbf{x}\|_\alpha$ on ℓ^1 which is bounded by one on the unitary ℓ^1 -sphere for $\alpha \geq 1$. \square

7.7.4. *Proof of Proposition 6.7.* The re-normalization function ζ is continuous on the unit sphere of (ℓ^α, d_α) , except for sequences with $\mathbf{x}_0 = 0$. Then

$$\begin{aligned} \mathbb{P}(\rho(Y \Theta) > 1) &= \mathbb{E}[\rho(\Theta_t)^\alpha \wedge 1] \\ &= \mathbb{E}[\rho(\mathbf{Q}_t^{(\alpha)} / |\mathbf{Q}_0^{(\alpha)}|)^\alpha \wedge 1] \\ &= \mathbb{E}[(\rho^\alpha \wedge 1) \circ \zeta(\mathbf{Q}^{(\alpha)})]. \end{aligned}$$

The proof is finished by an application of Theorem 6.2. \square

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