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# UNRESTRICTED QUANTUM MODULI ALGEBRAS, II: NOETHERIANITY AND SIMPLE FRACTION RINGS AT ROOTS OF 1

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**ABSTRACT.** We prove that the quantum graph algebra and the quantum moduli algebra associated to a punctured sphere and complex semisimple Lie algebra  $\mathfrak{g}$  are Noetherian rings and finitely generated rings over  $\mathbb{C}(q)$ . Moreover, we show that these two properties still hold on  $\mathbb{C}[q, q^{-1}]$  for the integral version of the quantum graph algebra. We also study the specializations  $\mathcal{L}_{0,n}^\epsilon$  of the quantum graph algebra at a root of unity  $\epsilon$  of odd order, and show that  $\mathcal{L}_{0,n}^\epsilon$  and its invariant algebra under the quantum group  $U_\epsilon(\mathfrak{g})$  have classical fraction algebras which are central simple algebras of PI degrees that we compute.

*Keywords:* quantum groups, invariant theory, TQFT

*AMS subject classification 2020:* 16R30, 17B37, 20G42, 57R56

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## 1. INTRODUCTION

This paper is the second part of our work, initiated in [31], on the quantum graph algebra  $\mathcal{L}_{g,n}(\mathfrak{g})$  and the quantum moduli algebra  $\mathcal{M}_{g,n}(\mathfrak{g})$ , which are associated to a surface  $\Sigma_{g,n+1}$  of genus  $g$  with  $n+1$  punctures and a complex semisimple Lie algebra  $\mathfrak{g}$ . As in [31] we focus in this paper on punctured spheres ( $g=0$ ,  $n \geq 1$ ). From now on we fix  $\mathfrak{g}$ , and when no confusion may arise we omit it from the notations of the various algebras.

The algebras  $\mathcal{L}_{g,n}$  and  $\mathcal{M}_{g,n}$  are defined over the field  $\mathbb{C}(q)$ . They were introduced in the mid '90s by Alekseev-Grosse-Schomerus [2, 3] and Buffenoir-Roche [32, 33] by a method called *combinatorial quantization*. By this method, the pair formed by  $\mathcal{L}_{g,n}$  and  $\mathcal{M}_{g,n}$  appear naturally as a  $q$ -deformation of the Fock-Rosly [56] lattice model of the algebra of functions on the “classical” moduli space  $\mathcal{M}_{g,n}^{cl}$  of flat  $\mathfrak{g}$ -connections on the surface  $\Sigma_{g,n+1}$ .

In [31] we showed that both  $\mathcal{L}_{0,n}$  and  $\mathcal{M}_{0,n}$  have integral forms  $\mathcal{L}_{0,n}^A$  and  $\mathcal{M}_{0,n}^A$  defined over the ring  $A = \mathbb{C}[q, q^{-1}]$ ; one can thus consider the specializations of these algebras at  $q = \epsilon \in \mathbb{C}^*$ , which we denote by  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{M}_{0,n}^{A,\epsilon}$  respectively. The algebra  $\mathcal{L}_{0,n}^A$  is endowed with an action of the Lusztig integral form  $U_A^{res} = U_A^{res}(\mathfrak{g})$  of the quantum group  $U_q = U_q(\mathfrak{g})$ , and  $\mathcal{M}_{0,n}^A$  is the subalgebra of invariant elements under this action. Therefore

$$\mathcal{M}_{0,n}^A := (\mathcal{L}_{0,n}^A)^{U_A^{res}}, \quad \mathcal{M}_{0,n} := \mathcal{L}_{0,n}^{U_q} = \mathcal{M}_{0,n}^A \otimes_A \mathbb{C}(q).$$

The definition of  $\mathcal{L}_{0,n}^A$  is based on the original combinatorial quantization method, together with twists of module-algebras and Lusztig’s theory of canonical basis of quantum groups [76]. This allows us to address the structure and representation theory of  $\mathcal{L}_{0,n}^A$  and  $\mathcal{M}_{0,n}^A$  by means of quantum groups, following ideas of classical invariant theory. In particular, we obtained that  $\mathcal{L}_{0,n}$  and  $\mathcal{L}_{0,n}^\epsilon$  have no non trivial zero divisors (and therefore do as well the subalgebras  $\mathcal{M}_{0,n}$ ,  $\mathcal{L}_{0,n}^A$ ,  $\mathcal{M}_{0,n}^A$  and  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon^{res}}$ , where  $U_\epsilon^{res}$  is the specialization of  $U_A^{res}$  at  $q = \epsilon$ ). Also, by extending the quantum coadjoint action of De Concini-Kac-Procaci [41, 42, 43], we described in the  $sl(2)$  case an action by derivations of the center  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  of  $\mathcal{L}_{0,n}^\epsilon$  on  $\mathcal{L}_{0,n}^\epsilon$ , and we defined a subalgebra  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}} \subset \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  which is a finite extension of the ring of regular functions on the character variety of the sphere with  $(n+1)$  punctures (see Corollary 7.20 and Theorem 8.8 of [31]). Moreover, from these results we derived an action by derivation of  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}}$  on  $\mathcal{M}_{0,n}^{A,\epsilon}(sl(2))$ .

Representations of a quotient (the semisimplification) of  $\mathcal{M}_{g,n}^{A,\epsilon}$  were already constructed and classified in [4]; they involve only the irreducible representations of the finite dimensional “small” quantum group  $u_\epsilon(\mathfrak{g})$ . Moreover, [4] deduced from these representations of  $\mathcal{M}_{g,n}^{A,\epsilon}$  a family of representations of the mapping class group of surfaces, that is equivalent to the one associated to the Witten-Reshetikin-Turaev TQFT [97, 89]. Recently, representations of another, larger quotient of  $\mathcal{M}_{g,n}^{A,\epsilon}$ , and the corresponding representations of the mapping class groups of surfaces, were constructed in [52, 53]. These representations are equivalent to those previously obtained by Lyubashenko-Majid [78], and are associated to the so called non-semisimple TQFT defined by Geer, Patureau-Mirand and their collaborators (see eg. [48, 49]). In the  $sl(2)$  case they involve the irreducible and also the principal indecomposable representations of the small quantum group  $u_\epsilon(sl(2))$ . The related link and 3-manifold invariants coincide with those of [80] and [20].

In general, the representation theory of  $\mathcal{M}_{g,n}^{A,\epsilon}$  is by now far from being understood. Because  $\mathcal{M}_{g,n}^{A,\epsilon}$  deforms the classical moduli space  $\mathcal{M}_{g,n}^{cl}$ , it is natural to expect that its representation theory provides  $(2+1)$ -dimensional TQFTs for 3-manifolds endowed with general flat  $\mathfrak{g}$ -connections, extending the known TQFTs based on quantum groups (where purely topological ones correspond to the trivial connection). A family of such invariants, called *quantum hyperbolic invariants*, has already been defined for  $\mathfrak{g} = sl(2)$  by means of certain  $6j$ -symbols, *Deus ex machina*; they are closely connected to classical Chern-Simons theory, provide generalized Volume Conjectures, and contain quantum Teichmüller theory (see [13]–[19]). It is part of our present program, initiated in [9], to shed light on these invariants and to generalize them to arbitrary  $\mathfrak{g}$  by developing the representation theory of  $\mathcal{M}_{g,n}^{A,\epsilon}$ .

The quantum moduli algebras have also been recognized as central objects from the viewpoints of factorization homology [21], multiplicative quiver varieties [58] and (stated) skein theory [23, 54, 37, 29]. In another direction, one may expect that the equivalence proved in [79] between combinatorial quantisation for the Drinfeld double  $D(H)$  of a finite-dimensional semisimple Hopf algebra  $H$ , and Kitaev's lattice model in topological quantum computation, can be extended to the setup of quantum moduli algebras.

In the present paper we study  $\mathcal{L}_{0,n}$ , its integral form  $\mathcal{L}_{0,n}^A$ , and the specialization  $\mathcal{L}_{0,n}^\epsilon$  of  $\mathcal{L}_{0,n}^A$  at  $q = \epsilon$  a primitive root of unity of odd order. We study also the subalgebras of invariant elements  $\mathcal{M}_{0,n} = \mathcal{L}_{0,n}^{U_q}$  and  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon^{res}}$ . Our results hold for every complex semisimple Lie algebra  $\mathfrak{g}$ . The main ones are proofs that  $\mathcal{L}_{0,n}$ ,  $\mathcal{L}_{0,n}^A$  and  $\mathcal{M}_{0,n}$  are Noetherian and finitely generated rings (Theorem 1.1), and that the classical fraction algebras of  $\mathcal{L}_{0,n}^\epsilon$  and  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon^{res}}$  are central simple algebras of PI degrees  $l^{nN}$  and  $l^{N(n-1)-m}$  respectively (Theorem 1.3). Here,  $m$  and  $N$  are the rank and the number of positive roots of  $\mathfrak{g}$ .

In the sequel [29] to this paper, in collaboration with M. Faitg, we extend Theorem 1.1 to the algebras  $\mathcal{L}_{g,n}$  and  $\mathcal{M}_{g,n}$ , associated to arbitrary finite type surfaces (arbitrary genus and number of punctures). Also, we show that  $\mathcal{M}_{g,n}$  is isomorphic to the  $\mathfrak{g}$ -skein algebra of  $\Sigma_{g,n+1}$ , and  $\mathcal{L}_{g,n}$  to the stated skein algebra of the compact surface  $\overline{\Sigma}_{g,n+1}$  with one boundary component and one marked point on the boundary component. This was proved for  $\mathfrak{g} = sl(2)$  in [54].

By using the analysis developed in the present paper for  $\mathcal{L}_{0,n}^A$ , one can define the integral form  $\mathcal{L}_{g,n}^A$  as well, and show that it is a Noetherian and finitely generated ring. We do not have a proof yet of these properties for the algebra  $\mathcal{M}_{0,n}^A$ , which seems to be much more difficult to handle. We note that there is a strict inclusion

$$\mathcal{M}_{0,n}^{A,\epsilon} \subset (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon^{res}}.$$

This is discussed after Theorem 1.2. In [30] we study further properties of  $(\mathcal{L}_{g,n}^\epsilon)^{U_\epsilon^{res}}$ , and we consider also the subalgebra  $\mathcal{M}_{g,n}^{A,\epsilon}$ .

**1.1. Statement of results.** Let us recall a few notations and facts from [31]. Let  $U_q$  be the simply-connected quantum group of  $\mathfrak{g}$ , defined over the field  $\mathbb{C}(q)$ . From  $U_q$  one can define a  $U_q$ -module algebra  $\mathcal{L}_{0,n}$ , called (quantum, daisy) *graph algebra*, where  $U_q$  acts by means of a right coadjoint action. The set of invariant elements of  $\mathcal{L}_{0,n}$  for this action is an algebra; we denote it

$$\mathcal{M}_{0,n} := \mathcal{L}_{0,n}^{U_q}$$

and call it *quantum moduli algebra*. As a  $\mathbb{C}(q)$ -module  $\mathcal{L}_{0,n}$  is just  $\mathcal{O}_q^{\otimes n}$ , where  $\mathcal{O}_q = \mathcal{O}_q(G)$  is the standard quantum function algebra of the connected and simply-connected Lie group  $G$  with Lie algebra  $\mathfrak{g}$ . The product of  $\mathcal{L}_{0,n}$  is obtained by twisting both the product of each factor  $\mathcal{O}_q$  and the product between them. It is equivariant with respect to a (right) coadjoint action of  $U_q$ , which defines the structure of  $U_q$ -module of  $\mathcal{L}_{0,n}$ .

The module algebra  $\mathcal{L}_{0,n}$  has an integral form  $\mathcal{L}_{0,n}^A$ , which is defined over  $A = \mathbb{C}[q, q^{-1}]$ , and endowed with an (coadjoint) action of the Lusztig [75] integral form  $U_A^{res}$  of  $U_q$ . It is obtained by replacing  $\mathcal{O}_q$  in the construction of  $\mathcal{L}_{0,n}$  with the restricted dual  $\mathcal{O}_A$  of the integral form  $U_A^{res}$ , or equivalently with the restricted dual of the integral form  $\Gamma$  of  $U_q$  defined by De Concini-Lyubashenko [45]. Since  $U_A^{res}$  contains the De Concini-Kac [41] integral form  $U_A$ , and  $U_A$  has the same set of invariant elements in  $\mathcal{L}_{0,n}^A$ , we systematically denote the latter

$$\mathcal{M}_{0,n}^A := (\mathcal{L}_{0,n}^A)^{U_A} (= (\mathcal{L}_{0,n}^A)^{U_A^{res}}).$$

We call  $\mathcal{M}_{0,n}^A$  the *integral* quantum moduli algebra.

A cornerstone of the theory of  $\mathcal{M}_{0,n}$  is a map  $\Phi_n$  originally due to Alekseev [1], building on works of Drinfeld [39] and Reshetikhin and Semenov-Tian-Shansky [86]. In [31] we showed that  $\Phi_n$  eventually provides isomorphisms of module algebras and algebras respectively,

$$\Phi_n: \mathcal{L}_{0,n} \rightarrow (U_q^{\otimes n})^{lf}, \Phi_n: \mathcal{M}_{0,n} \rightarrow (U_q^{\otimes n})^{U_q}$$

where  $U_q^{\otimes n}$  is endowed with a right adjoint action of  $U_q$ , and  $(U_q^{\otimes n})^{lf}$  is the subalgebra of locally finite elements with respect to this action. When  $n = 1$  the algebra  $U_q^{lf}$  has been studied in great detail by Joseph-Letzter [62, 63, 61]; we will use simplified proofs of their results, obtained in [96].

All the material we need about the results discussed above is described in [31], and recalled in Section 2.1-2.2.

Our first result, proved in Section 3, is:

**Theorem 1.1.**  $\mathcal{L}_{0,n}$ ,  $\mathcal{M}_{0,n}$  and the integral form  $\mathcal{L}_{0,n}^A$  are Noetherian rings, and finitely generated rings.

It follows immediately from the theorem that the specializations  $\mathcal{L}_{0,n}^\epsilon$ ,  $\epsilon \in \mathbb{C} \setminus \{0, 1\}$ , are Noetherian and finitely generated rings as well. In [31] we proved that all these algebras (and therefore  $\mathcal{M}_{0,n}^A$  and  $\mathcal{M}_{0,n}^{A,\epsilon}$ ) have no non-trivial zero divisors.

After we finished this work we discovered that [46] already proved that  $\mathcal{L}_{0,1}(gl(n))$  and  $\mathcal{L}_{0,n}(gl(2))$  are Noetherian and finitely generated rings. Our approach here is completely different. For  $\mathcal{L}_{0,n}$  we adapt the proof given by Voigt-Yuncken [96] of a result of Joseph [61], which asserts that  $U_q^{lf}$  is a Noetherian ring (Theorem 3.1). For  $\mathcal{M}_{0,n}$  we deduce the result from the one for  $\mathcal{L}_{0,n}$ , by following a line of proof of the Hilbert-Nagata theorem in classical invariant theory (Theorem 3.2).

The result for  $\mathcal{L}_{0,n}^A$  follows the same line, but uses also in a crucial way the Kashiwara-Lusztig theory of canonical bases of quantum groups. We describe the background material on this theory in Section 2.2.2; we have tried to make the exposition pedestrian and self-contained, so as to be accessible to non experts.

At present we do not have a proof that  $\mathcal{M}_{0,n}^A$  is a Noetherian and finitely generated ring for arbitrary  $\mathfrak{g}$  and  $n \geq 1$ , though it is natural to expect it is the case. Indeed, in the case  $\mathfrak{g} = sl(2)$ ,  $\mathcal{M}_{0,n}^A(sl(2))$  is isomorphic to the skein algebra of a sphere with  $n+1$  punctures (see Theorem 8.6 in [31]), which is finitely generated and Noetherian by results of [12] and [82], or by the particular case  $\mathfrak{g} = sl(2)$  of the results in [29]. In our general situation, key arguments in the proof of Theorem 1.1 for  $\mathcal{M}_{0,n}$  depend on the existence of a Reynolds operator on the  $U_q$ -module  $\mathcal{L}_{0,n}$ , and one can easily show there is no Reynolds operator on  $\mathcal{L}_{0,n}^A$ . This follows from the corresponding fact for the integral quantum coordinate ring  $\mathcal{O}_A$  (see Remark 2.14).

From Section 4 we consider the specializations  $\mathcal{L}_{0,n}^\epsilon$  of  $\mathcal{L}_{0,n}^A$  at  $q = \epsilon$ , a primitive root of unity of odd order  $l$  (and coprime to 3 if  $\mathfrak{g}$  has  $G_2$  components). In [45], De Concini-Lyubashenko introduced a central subalgebra  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  of  $\mathcal{O}_\epsilon$  isomorphic to the coordinate ring  $\mathcal{O}(G)$ , and proved that the  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ -module  $\mathcal{O}_\epsilon$  is projective of rank  $l^{dim \mathfrak{g}}$ . As observed by Brown-Gordon-Stafford [27], Bass' Cancellation theorem in  $K$ -theory and the fact that  $K_0(\mathcal{O}(G)) \cong \mathbb{Z}$ , proved by Marlin [84], imply that this module is free. Alternatively, this follows also from the fact that  $\mathcal{O}_\epsilon$  is a cleft extension of  $\mathcal{O}(G)$  by the dual of the Hopf algebra  $u_\epsilon(\mathfrak{g})$ , as proved by Andruskiewitsch-Garcia (see [6], Remark 2.18(b), and also Section 3.2 of [22]; this argument was explained to us by K. A. Brown).

The section 4 proves the analogous property for  $\mathcal{L}_{0,n}^\epsilon$ . Namely:

**Theorem 1.2.**  $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$  is a central subalgebra of  $\mathcal{L}_{0,n}^\epsilon$ , and  $\mathcal{L}_{0,n}^\epsilon$  is a free  $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$ -module of rank  $l^{n \cdot \dim \mathfrak{g}}$ , isomorphic to the  $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$ -module  $\mathcal{O}_\epsilon^{\otimes n}$ .

In the sequel we systematically denote

$$\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) := \mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}.$$

We prove the first and third claims of Theorem 1.2 in Proposition 4.1. The arguments use results of De Concini-Kac [41], De Concini-Procesi [42, 43], and De Concini-Lyubashenko [45], that we recall in Section 2.3-2.5. Let us stress that the algebra structures of  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{O}_\epsilon^{\otimes n}$  are completely different.

Since  $\mathcal{Z}_0(\mathcal{O}_\epsilon) \cong \mathcal{O}(G)$ , we may deduce the second claim of Theorem 1.2 from the first and third claims together with the results of [45] and [84], or [6], recalled above. Nevertheless we give a self-contained proof that  $\mathcal{L}_{0,1}^\epsilon$  is finite projective of rank  $l^{\dim \mathfrak{g}}$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , by adapting the original arguments of Theorem 7.2 of De Concini-Lyubashenko [45]. In particular we study the coregular action of the braid group of  $\mathfrak{g}$  on  $\mathcal{L}_{0,1}^\epsilon$ ; by the way, in the Appendix we provide different proofs of some technical facts shown in [45]. Of course, it remains an exciting problem to describe the centralizing extension  $\mathcal{O}(G)^{\otimes n} \subset \mathcal{L}_{0,n}^\epsilon$  (and similarly  $\mathcal{O}(G)^{\otimes n} \subset (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  below), aiming at generalizing the results of [6] and finding a direct, more structural proof of freeness in Theorem 1.2.

Also, we note that basis of  $\mathcal{L}_{0,n}^\epsilon$  over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  are complicated. The only case we know is for  $\mathcal{O}_\epsilon(sl(2))$ , described in [47], and it is far from being obvious (see (103)).

In Section 5 we turn to fraction rings. As mentioned above  $\mathcal{L}_{0,n}^\epsilon$  has no non-trivial zero divisors. Therefore its center  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is an integral domain. Denote by  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  its fraction field. Denote by  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  the subring of  $\mathcal{L}_{0,n}^\epsilon$  formed by the invariant elements of  $\mathcal{L}_{0,n}^\epsilon$  with respect to the right coadjoint action of  $U_\epsilon$ . The center  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  of  $\mathcal{L}_{0,n}^\epsilon$  is contained in  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  (this follows from [31], Proposition 6.19). Note also that we trivially have an inclusion  $\mathcal{M}_{0,n}^{A,\epsilon} \subset (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ , and these two algebras are distinct in general. For instance, when  $n = 1$  we have  $(\mathcal{L}_{0,1}^\epsilon)^{U_\epsilon} = \mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)$ , which is a finite extension of  $\mathcal{Z}_0(\mathcal{O}_\epsilon) \cong \mathcal{O}(G)$  (see Lemma 5.1). On another hand,  $\mathcal{M}_{0,1}^{A,\epsilon}$  is the specialization at  $q = \epsilon$  of  $\mathcal{Z}(\mathcal{L}_{0,1}^A)$ , a polynomial algebra in  $\text{rk}(\mathfrak{g})$  variables, which may be identified via  $\Phi_1$  with the center  $\mathcal{Z}(U_A)$  of the integral form  $U_A$ .

Consider the rings

$$Q(\mathcal{L}_{0,n}^\epsilon) = Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} \mathcal{L}_{0,n}^\epsilon$$

and

$$Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) = Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}.$$

In general, given a ring  $A$  with center  $\mathcal{Z}(A)$  an integral domain we reserve the notation  $Q(A)$  to the central localization of  $A$ , ie.  $Q(A) := Q(\mathcal{Z}(A)) \otimes_{\mathcal{Z}(A)} A$ . Though the center  $\mathcal{Z}((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  of  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is larger than  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ , the notation  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is valid, for  $\mathcal{Z}((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is an integral domain finite over  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ , and hence the central localization of  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  coincides with  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  as defined above. Throughout the paper, unless we mention it explicitly we follow the conventions of Mc Connell-Robson [85] as regards the terminology of ring theory; in particular, for the notions of central simple algebras and PI degrees, see in [85] the sections 5.3 and 13.3.6-13.6.7.

Denote by  $m$  the rank of  $\mathfrak{g}$ , and by  $N$  the number of its positive roots. In section 5 we prove:



**Theorem 1.3.** (1)  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a division algebra and a central simple algebra of PI degree  $l^{nN}$ .

(2)  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ ,  $n \geq 2$ , is a division algebra and a central simple algebra of PI degree  $l^{N(n-1)-m}$ .

The second claim of (1) means that  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a complex subalgebra of a full matrix algebra  $Mat_d(\mathbb{F})$ , where  $d = l^{nN}$  and  $\mathbb{F}$  is a finite extension of  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  such that

$$\mathbb{F} \otimes_{Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} Q(\mathcal{L}_{0,n}^\epsilon) = Mat_d(\mathbb{F}).$$

That  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a division algebra and a central simple algebra follows from Theorem 1.2 and the fact that  $\mathcal{L}_{0,n}^\epsilon$  has no non trivial zero divisors (proved in [31]). The computation of  $d = l^{nN}$  uses a lower bound coming from the representation theory of  $U_\epsilon$ , and a lower bound for the degree of  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  as a field extension of  $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ , obtained by using specializations to  $q = \epsilon$  of certain central elements in  $\mathcal{L}_{0,n}$  (for  $q$  generic). In this computation a main role is played by results of De Concini-Kac [41].

We deduce (2) from (1), the double centralizer theorem for central simple algebras, a few results of [31] and [45], and Theorem 1.2 again.

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**1.2. Basic notations.** Given a ring  $R$ , we denote by  $\mathcal{Z}(R)$  its center. When  $R$  is commutative and has no non-trivial zero divisors,  $Q(R)$  denotes its fraction field.

Given a Hopf algebra  $H$  with product  $m$  and and coproduct  $\Delta$ , we denote by  $H^{cop}$  (resp.  $H_{op}$ ) the Hopf algebra with the same algebra (resp. coalgebra) structure as  $H$  but the opposite coproduct  $\Delta^{cop} := \sigma \circ \Delta$  (resp. opposite product  $m \circ \sigma$ ), where  $\sigma(x \otimes y) = y \otimes x$ , and the antipode  $S^{-1}$ . We use Sweedler's coproduct notation,  $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ ,  $x \in H$ , and we set  $\Delta^{(1)} := id$ ,  $\Delta^{(2)} := \Delta$ , and  $\Delta^{(n)} := (\Delta \otimes id) \Delta^{(n-1)}$  for  $n \geq 3$  (this is not the convention used in [31]).

The results of this paper hold true for any finite dimensional complex semisimple Lie algebra  $\mathfrak{g}$ , but unless we state it differently, we will assume  $\mathfrak{g}$  is simple. We will denote its rank by  $m$ , and its Cartan matrix by  $(a_{ij})$ . We fix a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  and a basis of simple roots  $\alpha_i \in \mathfrak{h}_{\mathbb{R}}^*$ , and denote by  $\mathfrak{b}_{\pm}$  the Borel subalgebras associated to it. We denote by  $N$  the number of positive roots of  $\mathfrak{g}$ , and by  $\rho$  half the sum of the positive roots.

We denote by  $d_1, \dots, d_m$  the unique coprime positive integers such that the matrix  $(d_i a_{ij})$  is symmetric, and  $(, )$  the unique inner product on  $\mathfrak{h}_{\mathbb{R}}^*$  such that  $d_i a_{ij} = (\alpha_i, \alpha_j)$ . For any root  $\alpha$  the coroot is  $\check{\alpha} = 2\alpha/(\alpha, \alpha)$ ; in particular  $\check{\alpha}_i = d_i^{-1} \alpha_i$ . The root lattice  $Q$  is the  $\mathbb{Z}$ -lattice in  $\mathfrak{h}_{\mathbb{R}}^*$  defined by  $Q = \sum_{i=1}^m \mathbb{Z} \alpha_i$ . The weight lattice  $P$  is the  $\mathbb{Z}$ -lattice formed by all  $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$  such that  $(\lambda, \check{\alpha}_i) \in \mathbb{Z}$  for every  $i = 1, \dots, m$ . So  $P = \sum_{i=1}^m \mathbb{Z} \varpi_i$ , where  $\varpi_i$  is the fundamental weight dual to the simple coroot  $\check{\alpha}_i$ , which satisfies  $(\varpi_i, \check{\alpha}_j) = \delta_{i,j}$ . Note that  $(\lambda, \alpha) \in \mathbb{Z}$  for every  $\lambda \in P$ ,  $\alpha \in Q$ . We denote by  $D$  the cardinal of the quotient lattice  $P/Q$ . Then  $D$  is the smallest positive integer such that  $D(\lambda, \mu) \in \mathbb{Z}$  for every  $\lambda, \mu \in P$ , that is, such that  $DP \subset Q$ .

We denote by  $P_+ := \sum_{i=1}^m \mathbb{Z}_{\geq 0} \varpi_i$  the cone of dominant integral weights, and we put  $Q_+ := \sum_{i=1}^m \mathbb{Z}_{\geq 0} \alpha_i$ . Though  $Q \subset P$ , it is not true that  $Q_+ \subset P_+$ , but we have  $DP_+ \subset Q_+$ . This last property is not trivial, and follows from the classical fact that the inverse of the Cartan matrix  $(a_{ij})$  has coefficients in  $D^{-1}\mathbb{N}$ .

We will use the standard partial order relation  $\leq$  on  $P$ , defined by:  $\lambda, \mu \in P$  satisfy  $\lambda \leq \mu$  if  $\mu - \lambda \in Q_+$ . In section 3, we will also use the partial order relation  $\preceq$  on  $P$  defined by:  $\lambda \preceq \mu$  if  $\mu - \lambda \in D^{-1}Q_+$ .

We denote by  $\mathcal{B}(\mathfrak{g})$  the braid group of  $\mathfrak{g}$ ; we recall its standard defining relations in the Appendix (Section 6.1).

We denote by  $G$  the connected and simply-connected algebraic group with Lie algebra  $\mathfrak{g}$ , and by  $T_G$  the maximal torus of  $G$  with Lie algebra  $\mathfrak{h}$ ;  $N(T_G)$  is the normalizer of  $T_G$ ,  $W = N(T_G)/T_G$  is the Weyl group,  $B_\pm$  are the Borel subgroups of  $G$  with Lie algebra  $\mathfrak{b}_\pm$ , and  $U_\pm \subset B_\pm$  are their unipotent subgroups.

We denote by  $\mathcal{O}(G)$  the coordinate ring of  $G$ . It is a commutative Hopf algebra, which can be identified with the restricted dual of the universal enveloping algebra  $U(\mathfrak{g})$  (see [69], and also [77]). Similarly we denote by  $\mathcal{O}(B_\pm)$  the coordinate ring of  $B_\pm$ .

We let  $q$  be an indeterminate, set  $A = \mathbb{C}[q, q^{-1}]$ ,  $q_i = q^{d_i}$ ,  $q_\beta = q^{(\beta, \beta)/2}$  for  $\beta \in Q$ , and given integers  $p, k$  with  $0 \leq k \leq p$  we put

$$\begin{aligned} [p]_q &= \frac{q^p - q^{-p}}{q - q^{-1}}, \quad [0]_q! = 1, \quad [p]_q! = [1]_q [2]_q \cdots [p]_q, \quad \begin{bmatrix} p \\ k \end{bmatrix}_q = \frac{[p]_q!}{[p-k]_q! [k]_q!} \\ (p)_q &= \frac{q^p - 1}{q - 1}, \quad (0)_q! = 1, \quad (p)_q! = (1)_q (2)_q \cdots (p)_q, \quad \binom{p}{k}_q = \frac{(p)_q!}{(p-k)_q! (k)_q!}. \end{aligned}$$

We denote by  $\mathcal{A}_0 \subset \mathbb{C}(q)$  the ring of functions regular at  $q = 0$ ; this ring is used only in Section 2.2.2.

We denote by  $\epsilon$  a primitive  $l$ -th root of unity such that  $\epsilon^{2d_i} \neq 1$  is also a primitive  $l$ -th root of unity for all  $i \in \{1, \dots, m\}$ . This means that  $l$  is odd, and coprime to 3 if  $\mathfrak{g}$  is  $G_2$ .

In this paper we use the definition of the unrestricted integral form  $U_A(\mathfrak{g})$  given in [43], [45]; in [31] we used the one of [41], [42]. The two are (trivially) isomorphic, and have the same specialization at  $q = \epsilon$ . Also, we denote here by  $L_i$  the generators of  $U_q(\mathfrak{g})$  we denoted by  $\ell_i$  in [31].

In order to facilitate the comparison with the results of [45] we note that their generators denoted  $K_i, E_i$  and  $F_i$ , that we will denote by  $\tilde{K}_i, \tilde{E}_i$  and  $\tilde{F}_i$ , can be written as  $K_i, K_i^{-1}E_i$  and  $F_i K_i$  in our notations. They satisfy the same algebra relations.

## 2. BACKGROUND RESULTS

**2.1. On  $U_q, \mathcal{O}_q, \mathcal{L}_{0,n}, \mathcal{M}_{0,n}$ , and  $\Phi_n$ .** Except when stated differently, we refer to [31], Sections 2-4 and 6, and the references therein for details about the material of this section. We stress that the simply-connected quantum group, that we denote  $U_q$  below, was denoted  $\tilde{U}_q$  in [31]. Also, the adjoint quantum group  $U_q^{ad}$  was denoted  $U_q$ .

The *simply-connected* quantum group  $U_q = U_q(\mathfrak{g})$  is the Hopf algebra over  $\mathbb{C}(q)$  with generators  $E_i, F_i, L_i, L_i^{-1}$ ,  $1 \leq i \leq m$ , and defining relations

$$\begin{aligned} L_i L_j &= L_j L_i, \quad L_i L_i^{-1} = L_i^{-1} L_i = 1, \quad L_i E_j L_i^{-1} = q_i^{\delta_{i,j}} E_j, \quad L_i F_j L_i^{-1} = q_i^{-\delta_{i,j}} F_j \\ E_i F_j - F_j E_i &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r &= 0 \quad \text{if } i \neq j \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{ij}-r} F_j F_i^r &= 0 \quad \text{if } i \neq j \end{aligned}$$



where for  $\lambda = \sum_{i=1}^m m_i \varpi_i \in P$  we set  $K_\lambda = \prod_{i=1}^m L_i^{m_i}$ , and  $K_i = K_{\alpha_i} = \prod_{j=1}^m L_j^{a_{ji}}$ . The coproduct  $\Delta$ , antipode  $S$ , and counit  $\varepsilon$  of  $U_q$  are given by

$$\begin{aligned} \Delta(L_i) &= L_i \otimes L_i, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i \\ S(E_i) &= -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i, \quad S(L_i) = L_i^{-1} \\ \varepsilon(E_i) &= \varepsilon(F_i) = 0, \quad \varepsilon(L_i) = 1. \end{aligned}$$

We fix a reduced expression  $s_{i_1} \dots s_{i_N}$  of the longest element  $w_0$  of the Weyl group of  $\mathfrak{g}$ . It induces a total ordering of the positive roots,

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \dots, \beta_N = s_{i_1} \dots s_{i_{N-1}}(\alpha_{i_N}).$$

The root vectors of  $U_q$  with respect to such an ordering are defined by

$$(1) \quad E_{\beta_k} = T_{i_1} \dots T_{i_{k-1}}(E_{i_k}), \quad F_{\beta_k} = T_{i_1} \dots T_{i_{k-1}}(F_{i_k})$$

where  $T_i$  is the Lusztig algebra automorphism of  $U_q$  associated to the simple root  $\alpha_i$  ([76, 75], see also [38], Ch. 8). The braid group  $\mathcal{B}(\mathfrak{g})$  acts on  $U_q$  by means of the Lusztig automorphisms. In the Appendix we recall the relation between  $T_i$  and the generator  $\hat{w}_i$  of the quantum Weyl group, which we will mostly use. Let us just recall here that the monomials  $F_{\beta_1}^{r_1} \dots F_{\beta_N}^{r_N} K_\lambda E_{\beta_N}^{t_N} \dots E_{\beta_1}^{t_1}$  ( $r_i, t_i \in \mathbb{N}$ ,  $\lambda \in P$ ) form a basis of  $U_q$ , the *PBW basis*.

$U_q$  is a *pivotal* Hopf algebra, with pivotal element

$$\ell := K_{2\rho} = \prod_{j=1}^m L_j^2.$$

So  $\ell$  is group-like, and  $S^2(x) = \ell x \ell^{-1}$  for every  $x \in U_q$ .

The *adjoint* quantum group  $U_q^{ad} = U_q^{ad}(\mathfrak{g})$  is the Hopf subalgebra of  $U_q$  generated by the elements  $E_i, F_i$  ( $i = 1, \dots, m$ ) and  $K_\alpha$  with  $\alpha \in Q$ ; so  $\ell \in U_q^{ad}$ . When  $\mathfrak{g} = \mathfrak{sl}(2)$ , we simply write the above generators  $E = E_1, F = F_1, L = L_1, K = K_1$ .

We denote by  $U_q(\mathfrak{n}_+)$ ,  $U_q(\mathfrak{n}_-)$  and  $U_q(\mathfrak{h})$  the subalgebras of  $U_q$  generated respectively by the  $E_i$ , the  $F_i$ , and the  $K_\lambda$  ( $\lambda \in P$ ), and by  $U_q(\mathfrak{b}_+)$  and  $U_q(\mathfrak{b}_-)$  the subalgebras generated by the  $E_i$  and the  $K_\lambda$ , and by the  $F_i$  and the  $K_\lambda$ , respectively (they are the two-sided ideals generated by  $U_q(\mathfrak{n}_\pm)$ ). We do similarly with  $U_q^{ad}$ , where now  $U_q^{ad}(\mathfrak{h})$  is generated by the  $K_\lambda$  with  $\lambda \in Q$ .

The Hopf algebra  $U_q^{ad}$  is not braided in a strict sense, but it has braided categorical completions. Let us recall briefly what this means and implies. For details we refer to the sections 2 and 3 of [31] (see also [96], Section 3.10, where  $\mathbb{U}_q$  below is formulated in terms of multiplier Hopf algebras).

A  $U_q^{ad}$ -module  $V$  is said of *type 1* if it has finite dimension and the generators  $K_i$  are diagonalizable on  $V$  with eigenvalues in  $q_i^{\mathbb{Z}}$ . We denote by  $\mathcal{C}$  the category of  $U_q^{ad}$ -modules of type 1, by  $Vect$  the category of finite dimensional  $\mathbb{C}(q)$ -vector spaces, and by  $F_{\mathcal{C}} : \mathcal{C} \rightarrow Vect$  the forgetful functor. The category  $\mathcal{C}$  is semisimple. The simple objects are highest weight  $U_q^{ad}$ -modules; we denote by  $V_\mu$  the simple module with highest weight  $\mu \in P_+$ . Note that  $V_\mu$  is canonically endowed with a structure of  $U_q$ -module of type 1, the generators  $L_i$  being diagonalizable with eigenvalues in  $q_i^{\mathbb{Z}/D}$ . The *categorical completion*  $\mathbb{U}_q^{ad}$  of  $U_q^{ad}$  is the set of natural transformations  $F_{\mathcal{C}} \rightarrow F_{\mathcal{C}}$ . An element of  $\mathbb{U}_q^{ad}$  is a collection  $(a_V)_{V \in Ob(\mathcal{C})}$ , where  $a_V \in \text{End}_{\mathbb{C}(q)}(V)$  satisfies  $F_{\mathcal{C}}(f) \circ a_V = a_W \circ F_{\mathcal{C}}(f)$  for any objects  $V, W$  of  $\mathcal{C}$  and any arrow  $f \in \text{Hom}_{U_q^{ad}}(V, W)$ . It is not hard to see that  $\mathbb{U}_q^{ad}$  inherits from  $\mathcal{C}$  a natural structure of (completed) Hopf algebra such that the map

$$(2) \quad \begin{aligned} \iota : U_q^{ad} &\longrightarrow \mathbb{U}_q^{ad} \\ x &\longmapsto (\pi_V(x))_{V \in Ob(\mathcal{C})} \end{aligned}$$

is a morphism of Hopf algebras, where  $\pi_V : U_q^{ad} \rightarrow \text{End}(V)$  is the representation associated to a module  $V$  in  $\mathcal{C}$ . It is a theorem that this map is injective. From now on, let us extend the coefficient ring of the modules and morphisms in  $\mathcal{C}$  to  $\mathbb{C}(q^{1/D})$ . Put

$$\mathbb{U}_q = \mathbb{U}_q^{ad} \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D}).$$

The map  $\iota$  above extends to an embedding of  $U_q$  in  $\mathbb{U}_q$ . The category  $\mathcal{C}$ , with coefficients extended to  $\mathbb{C}(q^{1/D})$ , is braided and ribbon; we postpone a discussion of that fact to Section 2.3, where it will be developed. As a consequence, we can regard  $\mathbb{U}_q$  as a quasitriangular and ribbon Hopf algebra in a generalized sense (see [31]). The R-matrix of  $\mathbb{U}_q$  is the family of morphisms

$$R = (R_{V,W})_{V,W \in \text{Ob}(\mathcal{C})}$$

where  $R_{V,W} \in \text{End}(V \otimes W)$  is the endomorphism defined by the action of Drinfeld's universal R-matrix on  $V \otimes W$ . The ribbon element of  $\mathbb{U}_q$  is defined similarly by Drinfeld's universal ribbon element. One defines the *categorical tensor product*  $\mathbb{U}_q^{\hat{\otimes} 2}$  similarly as  $\mathbb{U}_q$ ; in particular it contains all the infinite series of elements of  $\mathbb{U}_q^{\hat{\otimes} 2}$  having only a finite number of non-zero terms when evaluated on a given module  $V \otimes W$  of  $\mathcal{C}$ . There is an expansion of  $R$  as an infinite series in  $\mathbb{U}_q^{\hat{\otimes} 2}$ . Adapting Sweedler's coproduct notation  $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$  we find convenient to write this series as

$$(3) \quad R = \sum_{(R)} R_{(1)} \otimes R_{(2)}.$$

We put  $R^+ := R$ ,  $R^- := (\sigma \circ R)^{-1}$  where  $\sigma$  is the flip map  $x \otimes y \mapsto y \otimes x$ . We will not use any explicit formula of  $R$ , but the following factorization formula

$$(4) \quad R = \Theta \hat{R}$$

where

$$\Theta = q^{\sum_{i,j=1}^m (B^{-1})_{ij} H_i \otimes H_j} \in \mathbb{U}_q^{\hat{\otimes} 2},$$

with  $B \in M_m(\mathbb{Q})$  the matrix with entries  $B_{ij} := d_j^{-1} a_{ij}$ , and

$$\hat{R} = \sum_{(\hat{R})} \hat{R}_{(1)} \otimes \hat{R}_{(2)} \in \mathbb{U}_q(\mathfrak{n}_+) \hat{\otimes} \mathbb{U}_q(\mathfrak{n}_-)$$

(see [31], Section 3.2, and for details eg. [38], Theorem 8.3.9, or [96], Theorem 3.108). If  $x, y$  are weight vectors of weights  $\mu, \nu$  respectively, then  $\Theta(x \otimes y) = q^{(\mu, \nu)} x \otimes y$ . Moreover,  $\hat{R}$  has weight 0 for the adjoint action of  $U_q(\mathfrak{h})$ ; that is, complementary components  $\hat{R}_{(1)}$  and  $\hat{R}_{(2)}$  have opposite weights.

Recall that we denote by  $G$  the connected and simply-connected algebraic group with Lie algebra  $\mathfrak{g}$ . The *quantum function Hopf algebra*  $\mathcal{O}_q = \mathcal{O}_q(G)$  is defined as the restricted dual of  $U_q^{ad}$  with respect to the category  $\mathcal{C}$ , that is, the set of  $\mathbb{C}(q)$ -linear maps  $f : U_q^{ad} \rightarrow \mathbb{C}(q)$  such that  $\text{Ker}(f)$  contains a cofinite two sided ideal  $I$  (ie. such that  $I \oplus M = U_q$  for some finite dimensional vector space  $M$ ), and  $\prod_{s=-r}^r (K_i - q_i^s) \in I$  for some  $r \in \mathbb{N}$  and every  $i$  (see eg. [24], Chapter I.7).

The space  $\mathcal{O}_q$  is a Hopf algebra, with structure maps defined dually to those of  $U_q^{ad}$ . We denote by  $\star$  its product. The algebras  $\mathcal{O}_q(T_G)$ ,  $\mathcal{O}_q(U_{\pm})$ ,  $\mathcal{O}_q(B_{\pm})$  are defined similarly, by replacing  $U_q^{ad}$  with  $U_q^{ad}(\mathfrak{h})$ ,  $U_q^{ad}(\mathfrak{n}_{\pm})$ ,  $U_q^{ad}(\mathfrak{b}_{\pm})$  respectively. As a vector space,  $\mathcal{O}_q$  is generated by the functionals  $x \mapsto w(\pi_V(x)v)$ ,  $x \in U_q^{ad}$ , for every object  $V \in \text{Ob}(\mathcal{C})$  and vectors  $v \in V$ ,  $w \in V^*$ . Such functionals are called *matrix coefficients*. Because the morphism  $\iota : U_q^{ad} \rightarrow \mathbb{U}_q$  is injective (see (2)), the Hopf duality pairing  $\langle \cdot, \cdot \rangle : \mathcal{O}_q \times U_q^{ad} \rightarrow \mathbb{C}(q)$  is non degenerate. By

extending the coefficient ring from  $\mathbb{C}(q)$  to  $\mathbb{C}(q^{1/D})$ , we can uniquely extend it to a bilinear pairing

$$\langle \cdot, \cdot \rangle: (\mathcal{O}_q \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})) \times \mathbb{U}_q \rightarrow \mathbb{C}(q^{1/D})$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}_q \otimes U_q^{ad} & \xrightarrow{\langle \cdot, \cdot \rangle} & \mathbb{C}(q) \\ id \otimes \iota \downarrow & & \downarrow \\ (\mathcal{O}_q \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})) \otimes \mathbb{U}_q & \xrightarrow{\langle \cdot, \cdot \rangle} & \mathbb{C}(q^{1/D}) \end{array}$$

This pairing is defined by

$$\langle {}_Y \phi_v^w, (a_X) \rangle = w(a_Y v)$$

for every  $(a_X) \in \mathbb{U}_q$  and  ${}_Y \phi_v^w \in \mathcal{O}_q$ . It is non degenerate.

The maps

$$(5) \quad \begin{aligned} \Phi^\pm : \mathcal{O}_q &\longrightarrow U_q^{cop} \\ \alpha &\longmapsto (\alpha \otimes id)(R^\pm) = \sum_{(R^\pm)} \langle \alpha, R_{(1)}^\pm \rangle R_{(2)}^\pm \end{aligned}$$

are well-defined morphisms of Hopf algebras. Here we stress that it is the simply-connected quantum group  $U_q^{cop}$  that is the range of  $\Phi^\pm$ . This will be explained with more details in Section 2.3.

Let us make two simple observations, for future reference. Firstly, because  $\mathcal{O}_q$  is spanned by the matrix coefficients of the objects of  $\mathcal{C}$ , and  $\mathcal{C}$  is semisimple with simple objects the  $U_q^{ad}$ -modules  $V_\mu$ ,  $\mu \in P_+$ , there is a decomposition of  $U_q$ -bimodule

$$(6) \quad \mathcal{O}_q = \bigoplus_{\mu \in P_+} C(\mu),$$

where  $C(\mu) = V_\mu^* \otimes V_\mu$ , the space of matrix coefficients of  $V_\mu$ , is endowed with the left action on the factor  $V_\mu$  and the right action on  $V_\mu^*$ , and  $\mathcal{O}_q$  has the left and right coregular actions  $\triangleleft$  and  $\triangleleft$ , defined by

$$x \triangleright \alpha := \sum_{(\alpha)} \alpha_{(1)} \langle \alpha_{(2)}, x \rangle, \quad \alpha \triangleleft x := \sum_{(\alpha)} \langle \alpha_{(1)}, x \rangle \alpha_{(2)}$$

for all  $x \in U_q$  and  $\alpha \in \mathcal{O}_q$ . Here we recall that each  $U_q^{ad}$ -module  $V_\mu$  can be regarded as a  $U_q$ -module, so the above expressions make sense. The decomposition (6) is the *Peter-Weyl* decomposition of  $\mathcal{O}_q$ . It will be refined in Section 2.2.2.

Moreover, the algebra  $\mathcal{O}_q$  is generated by the matrix coefficients of the simple  $U_q^{ad}$ -modules  $V_{\varpi_k}$  with highest weights the fundamental weights  $\varpi_k$ ,  $k = 1, \dots, m$ ; see eg. Proposition I.7.8 of [24] for a proof. This relies on the standard fact that, for any  $\mu, \nu \in P_+$  we have a direct sum decomposition of modules (where  $m(\lambda) \in \mathbb{N}$ )

$$(7) \quad V_\mu \otimes V_\nu = V_{\mu+\nu} \oplus \bigoplus_{\lambda < \mu+\nu} V_\lambda^{\oplus m(\lambda)}.$$

In particular, this decomposition implies that, up to scalar multiples, there is a unique non zero morphism  $V_{\mu+\nu} \rightarrow V_\mu \otimes V_\nu$ , which is injective and splits. Dually, this means that, applying the product in  $\mathcal{O}_q$  followed by the projection onto the subspace  $C(\mu + \nu)$  we get a canonical projection map

$$(8) \quad C(\mu) \otimes C(\nu) \rightarrow C(\mu + \nu).$$

The *loop algebra*  $\mathcal{L}_{0,1} = \mathcal{L}_{0,1}(\mathfrak{g})$  is defined by twisting the product  $\star$  of  $\mathcal{O}_q$ , keeping the same underlying linear space. The new product is equivariant with respect to the right coadjoint action  $coad^r$  of  $U_q$ , defined by

$$coad^r(x)(\alpha) = \sum_{(x)} S(x_{(2)}) \triangleright \alpha \triangleleft x_{(1)}$$

for all  $x \in U_q$  and  $\alpha \in \mathcal{O}_q$ . By equivariant we mean that  $\mathcal{L}_{0,1}$  is a  $U_q$ -module algebra. Let us spell out its product and equivariance property. Using the fact that  $U_q$  can be regarded as a subspace of  $\mathbb{U}_q$ , the actions  $\triangleleft$  and  $\triangleright$  extend naturally to actions of  $\mathbb{U}_q$ , and the product of  $\mathcal{L}_{0,1}$  is expressed in terms of  $\star$  by the formula ([31], Proposition 4.1):

$$(9) \quad \alpha\beta = \sum_{(R),(R')} (R_{(2')} S(R_{(2)}) \triangleright \alpha) \star (R_{(1')} \triangleright \beta \triangleleft R_{(1)}),$$

where  $\sum_{(R)} R_{(1)} \otimes R_{(2)}$  and  $\sum_{(R)} R_{(1')} \otimes R_{(2')}$  are expansions of two copies of  $R \in \mathbb{U}_q^{\otimes 2}$ . Note that the sum in (9) has only a finite number of non zero terms. By using that  $R\Delta = \Delta^{cop}R$ , this product can equivalently be expressed as

$$(10) \quad \alpha\beta = \sum_{(R),(R')} (\beta \triangleleft R_{(1)} R_{(1')}) \star (S(R_{(2)}) \triangleright \alpha \triangleleft R_{(2')}).$$

This product gives  $\mathcal{L}_{0,1}$  (like  $\mathcal{O}_q$ ) a structure of  $U_q$ -module algebra for the actions  $\triangleright$ ,  $\triangleleft$ , but also for  $coad^r$  (which is not the case of  $\mathcal{O}_q$ ). Spelling this out for  $coad^r$ , this means

$$coad^r(x)(\alpha\beta) = \sum_{(x)} coad^r(x_{(1)})(\alpha) coad^r(x_{(2)})(\beta).$$

The relations between  $\mathcal{O}_q$ ,  $\mathcal{L}_{0,1}$  and  $U_q$  are encoded by the map

$$(11) \quad \begin{aligned} \Phi_1: \mathcal{O}_q &\longrightarrow \mathbb{U}_q \\ \alpha &\longmapsto (\alpha \otimes id)(RR') \end{aligned}$$

where  $R' = \sigma \circ R$ , and as usual  $\sigma: x \otimes y \mapsto y \otimes x$ . Note that

$$(12) \quad \Phi_1 = m \circ (\Phi^+ \otimes (S^{-1} \circ \Phi^-)) \circ \Delta.$$

We call  $\Phi_1$  the *RSD* map, for Drinfeld, Reshetikhin and Semenov-Tian-Shansky introduced it first (see [39, 86],[83]). It is a fundamental result of the theory (see [35, 61, 11]) that  $\Phi_1$  affords an isomorphism of  $U_q$ -modules

$$\Phi_1: \mathcal{O}_q \rightarrow U_q^{lf}.$$

For full details on that result we refer to Section 3.12 of [96]. Here,  $U_q^{lf}$  is the set of *locally finite* elements of  $U_q$ , endowed with the right adjoint action  $ad^r$  of  $U_q$ . It is defined by

$$U_q^{lf} := \{x \in U_q \mid rk_{\mathbb{C}(q)}(ad^r(U_q)(x)) < \infty\}$$

and

$$ad^r(y)(x) = \sum_{(y)} S(y_{(1)})xy_{(2)}$$

for every  $x, y \in U_q$ . The action  $ad^r$  gives in fact  $U_q^{lf}$  a structure of right  $U_q$ -module algebra. It is also a right coideal, that is  $\Delta(U_q^{lf}) \subset U_q^{lf} \otimes U_q$ . Moreover,  $\Phi_1$  affords an isomorphism of  $U_q$ -module algebras

$$(13) \quad \Phi_1: \mathcal{L}_{0,1} \rightarrow U_q^{lf}.$$

It is a fact that  $\Phi_1$  affords an isomorphism between the centers  $\mathcal{Z}(\mathcal{L}_{0,1})$  of  $\mathcal{L}_{0,1}$  and  $\mathcal{Z}(U_q)$  of  $U_q$  ([31], Proposition 6.24). Since  $\Phi_1$  is an isomorphism of  $U_q$ -modules and  $\mathcal{Z}(U_q) = U_q^{U_q}$ , it follows that  $\mathcal{Z}(\mathcal{L}_{0,1}) = \mathcal{L}_{0,1}^{U_q}$ .

Let us recall a few fundamental results about  $U_q^{lf}$  that we will meet again later. Denote by  $T \subset U_q$  the multiplicative Abelian group formed by the elements  $K_\lambda$ ,  $\lambda \in P$ , and by  $T_2 \subset T$  the subgroup formed by the elements  $K_\lambda$ ,  $\lambda \in 2P$ . Consider the subset  $T_{2-} \subset T_2$  formed by the elements  $K_{-\lambda}$ ,  $\lambda \in 2P_+$ . Clearly  $T_2 = T_{2-}^{-1}T_{2-}$  and  $\text{Card}(T/T_2) = 2^m$ .

- Theorem 2.1.** (1)  $U_q^{lf} = \bigoplus_{\lambda \in 2P_+} ad^r(U_q)(K_{-\lambda})$ .  
 (2)  $U_q = T_{2-}^{-1}U_q^{lf}[T/T_2]$ , so  $U_q$  is a free  $T_{2-}^{-1}U_q^{lf}$ -module of rank  $2^m$ .  
 (3) The ring  $U_q^{lf}$  is (left and right) Noetherian.

These results were proved by Joseph-Letzter in Theorem 4.10 of [63], Theorem 6.4 of [62], and Theorem 7.4.8 of [61], respectively (see also the sections 7.1.6, 7.1.13 and 7.1.25 in [61]). For (1) and (3) we refer also to Theorem 3.113 and 3.137 in [96], which provides simpler proofs. For instance, in the  $sl(2)$  case we have

$$U_q(sl(2)) = U_q(sl(2))^{lf}[K] \oplus U_q(sl(2))^{lf}[K].L.$$

The actual values of  $\Phi_1$  are complicated in general (see Theorem 2.15 and the comments thereafter). However, there is a simple important one, that we describe now. Let  $V_{-\lambda}$  be the type 1 simple  $U_q^{ad}$ -module of lowest weight  $-\lambda \in -P_+$  (ie. the highest weight  $U_q^{ad}$ -module  $V_{-w_0(\lambda)}$  of highest weight  $-w_0(\lambda)$ , where  $w_0$  is the longest element of the Weyl group; note that  $-w_0$  permutes the simple roots). Let  $v \in V_{-\lambda}$  be a lowest weight vector, and  $v^* \in V_{-\lambda}^*$  be such that  $v^*(v) = 1$  and  $v^*$  vanishes on a  $U_q^{ad}(\mathfrak{h})$ -invariant complement of  $v$ . Define  $\psi_{-\lambda} \in \mathcal{O}_q$  by  $\langle \psi_{-\lambda}, x \rangle = v^*(xv)$ ,  $x \in U_q$ . From the definition (11) it is quite easy to see that

$$(14) \quad \Phi_1(\psi_{-\lambda}) = K_{-2\lambda}.$$

In particular  $\Phi_1(\psi_{-\rho}) = \ell^{-1}$ , where as usual  $\ell$  is the pivotal element of  $U_q$ .

**Remark 2.2.** Since  $\mathcal{L}_{0,1} = \mathcal{O}_q$  as a vector space, we still denote by  $C(\mu)$ ,  $\mu \in P^+$ , the linear subspace generated by the matrix coefficients of  $V_\mu$ , the  $U_q^{ad}$ -module of type 1 and highest weight  $\mu$ . It can be proved (see Section 7.1.22 in [61], or page 156 of [96], where different conventions are used) that  $\Phi_1$  yields an isomorphism of  $U_q$ -modules

$$(15) \quad \Phi_1: C(-w_0(\mu)) \rightarrow ad^r(U_q)(K_{-2\mu}).$$

Therefore, the summands in (1) are finite-dimensional  $U_q$ -modules, and the action  $ad^r$  is completely reducible on  $U_q^{lf}$ . In fact,  $U_q^{lf}$  is the socle of  $ad^r$  on  $U_q$ .

**Remark 2.3.** Because  $\ell = \prod_{j=1}^m L_j^2$  and  $\Phi_1(\psi_{-\rho}) = \ell^{-1}$ , a natural question is the factorization of  $\psi_{-\rho}$  in  $\mathcal{L}_{0,1}$  (see Corollary 2.18). This question is considered in [64], where  $\mathcal{L}_{0,1}(\mathfrak{g})$  for  $\mathfrak{g} = gl(r+1)$  is analysed and quantum minors are extensively studied. Let us review here some of their results in relation with  $\psi_{-\rho}$ .

First note that for  $\mathfrak{g} = sl(r+1)$  the irreducible representation  $V_{-\rho}$  of lowest weight  $-\rho$  is isomorphic to the representation of highest weight  $V_\rho$  because  $-w_0(\rho) = \rho$ . By the Weyl formula the dimension of this representation is  $\prod_{\alpha > 0} \frac{(2\rho, \alpha)}{(\rho, \alpha)} = 2^N$ . In [73] a presentation of  $U_q(gl(r+1))$  is given, which differs from our presentation of  $U_q(sl(r+1))$  only by its subalgebra  $U_q(\mathfrak{h})$ , generated by  $r+1$  elements  $\mathbb{K}_1, \dots, \mathbb{K}_{r+1}$ . The inclusion  $U_q(sl(r+1)) \subset U_q(gl(r+1))$  is such that  $K_i = \mathbb{K}_i^2 \mathbb{K}_{i+1}^{-2}$ ,  $i = 1, \dots, r$ . The quantum minors, properly defined in [64], of the matrix of matrix elements of the natural representation of  $U_q(gl(r+1))$  are denoted  $det_q(A_{\geq k})$

for  $k = 1, \dots, r+1$ . We have  $\det_q(A_{\geq 1}) = 1$  in the case of  $sl(r+1)$ . Then [64] proves that  $\det_q(A_{\geq k}) = (\mathbb{K}_k \dots \mathbb{K}_{r+1})^2$ , and there exists an element  $\mathbb{K} \in U_q(gl(r+1))$  such that

$$\mathbb{K}^{-2\rho} = \det_q(A_{\geq 1})^{-r} \det_q(A_{\geq 2}) \dots \det_q(A_{\geq r+1}).$$

This has to be interpreted in the  $sl(r+1)$  case as  $K_{-2\rho} = \Phi_1(\det_q(A_{\geq 2}) \dots \det_q(A_{\geq r+1}))$ . As a result this gives the equality

$$\psi_{-\rho} = \det_q(A_{\geq 2}) \dots \det_q(A_{\geq r+1}).$$

The (*quantum*) *graph algebra*  $\mathcal{L}_{0,n} = \mathcal{L}_{0,n}(\mathfrak{g})$  is the braided tensor product of  $n$  copies of  $\mathcal{L}_{0,1}$  (considered as a  $U_q$ -module algebra). As a linear space and  $U_q$ -bimodule with actions  $\triangleleft$  and  $\triangleright$ , it coincides with  $\mathcal{L}_{0,1}^{\otimes n}$ , and thus with  $\mathcal{O}_q^{\otimes n}$ . It is also a right  $U_q$ -module algebra, with the following action of  $U_q$  (extending  $\text{coad}^r$  on  $\mathcal{L}_{0,1}$ ):

$$(16) \quad \text{coad}_n^r(y)(\alpha^{(1)} \otimes \dots \otimes \alpha^{(n)}) = \sum_{(y)} \text{coad}^r(y_{(1)})(\alpha^{(1)}) \otimes \dots \otimes \text{coad}^r(y_{(n)})(\alpha^{(n)})$$

for all  $y \in U_q$  and  $\alpha^{(1)} \otimes \dots \otimes \alpha^{(n)} \in \mathcal{L}_{0,n}$ . The product of  $\mathcal{L}_{0,n}$  can be explicited as follows. For every  $1 \leq a \leq n$  define  $\mathbf{i}_a: \mathcal{L}_{0,1} \rightarrow \mathcal{L}_{0,n}$  by  $\mathbf{i}_a(x) = 1^{\otimes(a-1)} \otimes x \otimes 1^{\otimes(n-a)}$ ;  $\mathbf{i}_a$  is an embedding of  $U_q$ -module algebras. We will use the notations

$$(17) \quad \mathcal{L}_{0,n}^{(a)} := \text{Im}(\mathbf{i}_a), \quad (\alpha)^{(a)} := \mathbf{i}_a(\alpha).$$

Take  $(\alpha)^{(a)}, (\alpha')^{(a)} \in \mathcal{L}_{0,n}^{(a)}$  and  $(\beta)^{(b)}, (\beta')^{(b)} \in \mathcal{L}_{0,n}^{(b)}$  with  $a < b$ . Then the product of  $\mathcal{L}_{0,n}$  is given by the following formula (see [31], section 6):

$$(18) \quad \begin{aligned} & \left( (\alpha)^{(a)} \otimes (\beta)^{(b)} \right) \left( (\alpha')^{(a)} \otimes (\beta')^{(b)} \right) \\ &= \sum_{(R^1), \dots, (R^4)} \left( \alpha \left( S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha' \triangleleft R_{(1)}^1 R_{(1)}^2 \right) \right)^{(a)} \\ & \quad \otimes \left( \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \right) \beta' \right)^{(b)} \end{aligned}$$

where  $R^i = \sum_{(R^i)} R_{(1)}^i \otimes R_{(2)}^i$ ,  $i \in \{1, 2, 3, 4\}$ , are expansions of four copies of  $R \in \mathbb{U}_q^{\hat{\otimes} 2}$ , and on the right-hand side the product is componentwise that of  $\mathcal{L}_{0,1}$ . Later we will use the fact that the product of  $\mathcal{L}_{0,n}$  is obtained from the standard (componentwise) product of  $\mathcal{L}_{0,1}^{\otimes n}$  by a process that may be inverted. Indeed, (18) can be rewritten as

$$(19) \quad \left( (\alpha)^{(a)} \otimes (\beta)^{(b)} \right) \left( (\alpha')^{(a)} \otimes (\beta')^{(b)} \right) = \sum_{(F)} (\alpha)^{(a)} \left( (\alpha')^{(a)} \cdot F_{(2)} \right) \otimes \left( (\beta)^{(b)} \cdot F_{(1)} \right) (\beta')^{(b)}$$

where  $F = \sum_{(F)} F_{(1)} \otimes F_{(2)} := (\Delta \otimes \Delta)(R')$ , and the symbol “ $\cdot$ ” stands for the right action of  $\mathbb{U}_q^{\hat{\otimes} 2}$  on  $\mathcal{L}_{0,1}$  that may be read from (18). The tensor  $F$  is known as a twist. Then, by replacing  $F$  with its inverse  $\bar{F} = (\Delta \otimes \Delta)(R'^{-1})$ , one can express the product of  $\mathcal{L}_{0,1}^{\otimes n}$  in terms of the product of  $\mathcal{L}_{0,n}$  by

$$(20) \quad (\alpha)^{(a)} (\alpha')^{(a)} \otimes (\beta)^{(b)} (\beta')^{(b)} = \sum_{(\bar{F})} \left( (\alpha)^{(a)} \otimes \left( (\beta)^{(b)} \cdot \bar{F}_{(1)} \right) \right) \left( \left( (\alpha')^{(a)} \cdot \bar{F}_{(2)} \right) \otimes (\beta')^{(b)} \right).$$

We call *quantum moduli algebra* and denote by

$$\mathcal{M}_{0,n} = \mathcal{L}_{0,n}^{U_q}$$

the subalgebra of  $\mathcal{L}_{0,n}$  formed by the  $U_q$ -invariant elements.



The map  $\Phi_1$  can be extended to  $\mathcal{L}_{0,n}$  as follows. Consider the following action of  $U_q$  on the tensor product algebra  $U_q^{\otimes n}$ , which extends  $ad^r$  on  $U_q$ :

$$(21) \quad ad_n^r(y)(x) = \sum_{(y)} \Delta^{(n)}(S(y_{(1)}))x\Delta^{(n)}(y_{(2)})$$

for all  $y \in U_q$ ,  $x \in U_q^{\otimes n}$ . This action gives  $U_q^{\otimes n}$  a structure of right  $U_q$ -module algebra. In [1] Alekseev introduced a morphism of  $U_q$ -module algebras  $\Phi_n: \mathcal{L}_{0,n} \rightarrow U_q^{\otimes n}$  which extends  $\Phi_1$ . In Proposition 6.7 of [31] we showed that  $\Phi_n$  affords isomorphisms

$$(22) \quad \Phi_n: \mathcal{L}_{0,n} \rightarrow (U_q^{\otimes n})^{lf}, \quad \Phi_n: \mathcal{M}_{0,n} \rightarrow (U_q^{\otimes n})^{U_q}$$

where  $(U_q^{\otimes n})^{lf}$  is the set of  $ad_n^r$ -locally finite elements of  $U_q^{\otimes n}$ . We call  $\Phi_n$  the *Alekseev map*; we do not recall here the definition of  $\Phi_n$ , for we will not use it. It is a key argument of the proof of (22) that the set of locally finite elements of  $U_q^{\otimes n}$  for  $(ad^r)^{\otimes n} \circ \Delta^{(n)}$  coincides with  $(U_q^{lf})^{\otimes n}$ ; this follows from the main result of [72]. Using that the map

$$(23) \quad \psi_n = \Phi_n \circ (\Phi_1^{-1})^{\otimes n}: (U_q^{lf})^{\otimes n} \rightarrow (U_q^{\otimes n})^{lf}$$

is surjective and intertwines the actions  $(ad^r)^{\otimes n} \circ \Delta^{(n-1)}$  and  $ad_n^r$ , we deduced that  $\text{Im}(\Phi_n) = (U_q^{\otimes n})^{lf}$ .

**Remark 2.4.** We have  $(U_q^{lf})^{\otimes n} \neq (U_q^{\otimes n})^{lf}$ , and in fact there is not even an inclusion. Indeed let  $\Omega = (q - q^{-1})^2 FE + qK + q^{-1}K^{-1}$  be the Casimir element of  $U_q(sl(2))$ . We trivially have  $\Delta(\Omega) \in (U_q^{\otimes 2})^{lf}$  but

$$\Delta(\Omega) = (q - q^{-1})^2(K^{-1}E \otimes FK + F \otimes E) + \Omega \otimes K + K^{-1} \otimes \Omega - (q + q^{-1})K^{-1} \otimes K$$

and therefore  $\Delta(\Omega) \notin (U_q^{lf})^{\otimes 2}$ , since  $K \notin U_q^{lf}$  (see eg. Theorem 2.1 (2)). This reflects the fact that  $U_q^{lf}$  is only a right coideal (and not a subcoalgebra).

As in Remark 2.2, denote by  $C(\mu)$ ,  $\mu \in P^+$ , the linear subspace of  $\mathcal{L}_{0,1}$  generated by the matrix coefficients of  $V_\mu$ . For every tuple  $[\mu] = (\mu_1, \dots, \mu_n) \in P_+^n$  put

$$(24) \quad C([\mu]) = C(\mu_1) \otimes \dots \otimes C(\mu_n)$$

Then  $\mathcal{L}_{0,n} = \bigoplus_{[\mu] \in P_+^n} C([\mu])$ . Each space  $C([\mu])$  is a finite dimensional  $U_q$ -module under the action  $coad_n^r$ , whence it is completely reducible. Therefore

$$(25) \quad \mathcal{L}_{0,n} = \mathcal{M}_{0,n} \oplus I$$

as  $U_q$ -modules, where  $I$  the sum of non trivial isotypical components of  $\mathcal{L}_{0,n}$ . The  $\mathbb{C}(q)$ -linear projection map

$$(26) \quad \mathcal{R}: \mathcal{L}_{0,n} \rightarrow \mathcal{M}_{0,n}, \quad \text{Ker}(\mathcal{R}) = I$$

is called the *Reynolds operator*. For all  $\alpha \in \mathcal{M}_{0,n}$ ,  $\beta \in \mathcal{L}_{0,n}$  it satisfies

$$(27) \quad \mathcal{R}(\alpha\beta) = \alpha\mathcal{R}(\beta).$$

This property will be crucial in the sequel, so let us recall a (classical) proof of it. We can write  $\beta = \mathcal{R}(\beta) + \gamma$  with  $\gamma \in I$ , and then we have to show  $\alpha\gamma \in I$ . We can reduce to the case where  $\gamma$  is contained in a simple summand  $V$  of  $I$ . Multiplication by the invariant element  $\alpha$  yields a surjective map  $V \rightarrow \alpha V$ , which is a morphism of  $U_q$ -modules. Since  $V$  is simple, it is either the 0 map, or an isomorphism. In either cases it follows  $\alpha V \subset I$  (in fact the first case cannot happen, for  $\mathcal{L}_{0,n}$  has no non-trivial zero divisors, as explained after (31)).

We can formulate the Reynolds operator in the following way. Recall that  $\mathcal{O}_q$  has a unique left (or right, or 2-sided) Haar integral, that is a linear map  $h: \mathcal{O}_q \rightarrow \mathbb{C}(q)$  such that

$$(28) \quad h(1) = 1 \text{ and } (id \otimes h)\Delta(\alpha) = h(\alpha)1, \quad \forall \alpha \in \mathcal{O}_q.$$

(See eg. [38], Proposition 13.3.6). It vanishes on all matrix coefficients except the one of the trivial representation, to which it gives the value 1. Denote by  $\Delta_{\mathcal{L}}: \mathcal{L}_{0,n} \rightarrow \mathcal{L}_{0,n} \otimes \mathcal{O}_q$  the right coaction dual to the action  $coad_n^r$  of  $U_q$  on  $\mathcal{L}_{0,n}$ . Then, in analogy with the formula of the averaging operator  $\mathcal{C}^\infty(G) \rightarrow \mathcal{C}^\infty(G)^G$ ,  $f \rightarrow [f] = \int_G f(g^{-1} \cdot g) d\mu(g)$ , for a locally compact group  $G$  with Haar measure  $d\mu(g)$ , it is straightforward that

$$(29) \quad \mathcal{R} = (id \otimes h)\Delta_{\mathcal{L}}.$$

Note that the complete reducibility of  $\mathcal{L}_{0,n}$  discussed after (24) follows also from Theorem 2.1 (1), since by (23) we have an isomorphism of  $U_q$ -modules

$$(30) \quad \mathcal{L}_{0,n} \xrightarrow{\Phi_n} (U_q(\mathfrak{g})^{\otimes n})^{lf} \xrightarrow{\psi_n^{-1}} U_q^{lf}(\mathfrak{g})^{\otimes n},$$

where  $lf$  means respectively locally finite for the action  $ad_n^r$  of  $U_q(\mathfrak{g})$  on  $U_q(\mathfrak{g})^{\otimes n}$ , and locally finite for the action  $ad^r$  of  $U_q(\mathfrak{g})$  on  $U_q(\mathfrak{g})$ . An explicit basis of  $\mathcal{M}_{0,n}$  is described in Proposition 6.22 of [31].

Finally, let us point out here two important consequences of (22). First,  $\Phi_n$  yields isomorphisms between centers,  $\mathcal{Z}(\mathcal{L}_{0,n}) \cong \mathcal{Z}(U_q)^{\otimes n}$  and  $\mathcal{Z}(\mathcal{L}_{0,n}^{U_q}) \cong \mathcal{Z}((U_q^{\otimes n})^{U_q})$ , where one can show that ([31], Lemma 6.29)

$$(31) \quad \mathcal{Z}((U_q^{\otimes n})^{U_q}) \cong \Delta^{(n)}(\mathcal{Z}(U_q)) \otimes_{\mathbb{C}(q)} \mathcal{Z}(U_q)^{\otimes n}.$$

Second,  $\mathcal{L}_{0,n}$  (and therefore  $\mathcal{M}_{0,n}$ ) has no non-trivial zero divisors because of the isomorphisms  $\Phi_n: \mathcal{L}_{0,n} \rightarrow (U_q^{\otimes n})^{lf} \subset U_q^{\otimes n}$  and  $U_q^{\otimes n} \cong U_q(\mathfrak{g}^{\oplus n})$ , and the fact that  $U_q(\mathfrak{g}^{\oplus n})$  has no non-trivial zero divisors (proved eg. in [41]).

**2.2. Integral forms and specializations.** Let  $A = \mathbb{C}[q, q^{-1}]$ . We call integral form of a (Hopf)  $\mathbb{C}(q)$ -algebra  $H$  a (Hopf)  $A$ -subalgebra  ${}_A H$  such that the canonical map  ${}_A H \otimes_A \mathbb{C}(q) \rightarrow H$  is an isomorphism. Note that the standard notion of integral form of  $\mathbb{C}(q)$ -algebra uses  $\mathbb{Z}[q, q^{-1}]$  instead of  $\mathbb{C}[q, q^{-1}]$ ; our choice is made for simplicity ( $\mathbb{C}[q, q^{-1}]$  is a principal ideal domain, whereas  $\mathbb{Z}[q, q^{-1}]$  is not).

**2.2.1. Definitions.** The *unrestricted* integral form of  $U_q$  is the  $A$ -subalgebra  $U_A = U_A(\mathfrak{g})$  introduced by De Concini–Kac–Procesi in [43], Section 12 (and in a differently normalized form in [41] and [42]). It is the smallest  $A$ -subalgebra of  $U_q$  which contains the elements ( $i = 1, \dots, m$ )

$$(32) \quad \bar{E}_i = (q_i - q_i^{-1})E_i, \quad \bar{F}_i = (q_i - q_i^{-1})F_i, \quad L_i, \quad L_i^{-1}$$

and is stable under the action of  $\mathcal{B}(\mathfrak{g})$  given by the Lusztig automorphisms (see (1)). Recall the root vectors  $E_{\beta_k}, F_{\beta_k}$  defined in (1). Let us put  $q_{\beta} := q^{(\beta, \beta)/2}$ . The algebra  $U_A$  is a free  $A$ -module with basis the monomials  $\bar{E}_{\beta_1}^{p_1} \dots \bar{E}_{\beta_N}^{p_N} K_{\lambda} \bar{F}_{\beta_N}^{n_N} \dots \bar{F}_{\beta_1}^{n_1}$ , where  $\lambda \in P$  and we set

$$\bar{E}_{\beta_k} = (q_{\beta_k} - q_{\beta_k}^{-1})E_{\beta_k}, \quad \bar{F}_{\beta_k} = (q_{\beta_k} - q_{\beta_k}^{-1})F_{\beta_k}.$$

We denote

$$U_A^{lf} := U_A \cap U_q^{lf}.$$

The unrestricted integral form of  $U_q^{ad}$  is defined similarly, as the smallest  $A$ -subalgebra  $U_A^{ad} \subset U_A$  which contains the elements  $\bar{E}_i, \bar{F}_i$  and  $K_i^{\pm 1}$ , for  $i = 1, \dots, m$ , and is stable under the Lusztig action of  $\mathcal{B}(\mathfrak{g})$ .

For  $\beta$  a positive root, we define the divided powers

$$E_\beta^{(k)} = \frac{E_\beta^k}{[k]_{q_\beta}!}, \quad F_\beta^{(k)} = \frac{F_\beta^k}{[k]_{q_\beta}!}, \quad k \in \mathbb{N}.$$

The Lusztig *restricted* integral form of  $U_q^{ad}$  ([75, 76], see also [38], Chapter 9.3) is the  $A$ -subalgebra  $U_A^{res}$  generated by the elements  $(i = 1, \dots, m, k \in \mathbb{N}^*)$

$$E_i^{(k)} = \frac{E_i^k}{[k]_{q_i}!}, \quad F_i^{(k)} = \frac{F_i^k}{[k]_{q_i}!}, \quad K_i, \quad K_i^{-1}.$$

The algebra  $U_A^{res}$  is a free  $A$ -module with Poincaré-Birkhoff-Witt (PBW) basis

$$(33) \quad E_{\beta_1}^{(p_1)} \dots E_{\beta_N}^{(p_N)} \prod_{i=1}^m K_i^{\sigma_i} [K_i; t_i]_{q_i} F_{\beta_N}^{(n_N)} \dots F_{\beta_1}^{(n_1)}$$

where  $\sigma_i \in \{0, 1\}$ ,  $n_i, p_i, t_i \in \mathbb{N}$ , and we set  $[K_i; 0]_{q_i} := 1$  and

$$[K_i; t]_{q_i} = \prod_{s=1}^t \frac{K_i q_i^{-s+1} - K_i^{-1} q_i^{s-1}}{q_i^s - q_i^{-s}}.$$

The integral forms  $U_A(\mathfrak{h})$ ,  $U_A(\mathfrak{b}_\pm)$  and  $U_A^{res}(\mathfrak{h})$ ,  $U_A^{res}(\mathfrak{b}_\pm)$  associated to the subalgebras  $\mathfrak{h}$ ,  $\mathfrak{b}_\pm \subset \mathfrak{g}$  are the subalgebras of  $U_A$  and  $U_A^{res}$ , respectively, defined in the obvious way. For instance the ‘‘Cartan’’ subalgebra  $U_A^{res}(\mathfrak{h}) = U_q(\mathfrak{h}) \cap U_A^{res}$  is generated as a  $A$ -module by the elements  $\prod_{i=1}^m K_i^{\sigma_i} [K_i; t_i]_{q_i}$ .

Denote by  $\mathcal{C}_A$  the category of  $U_A^{res}$ -modules of *type 1*, ie. free  $A$ -modules of finite rank which have a basis where the elements  $K_i$  act diagonally with eigenvalues of the form  $q_i^k$ ,  $k \in \mathbb{Z}$  (in general, finiteness of the rank imposes eigenvalues of the form  $\pm q_i^k$ ,  $k \in \mathbb{Z}$ ). The category  $\mathcal{C}_A$  is a rigid and tensor category. It is not semisimple, and this makes the study of  $\mathcal{C}_A$  a complicated task; for this, see [31], and section 2.2.2 below. Every type 1 finite dimensional simple  $U_q$ -module  $V_\mu$ ,  $\mu \in P_+$ , has a  $U_A^{res}$ -invariant full  $A$ -sublattice, that we denote by  ${}_A V_\mu$ . These  $U_A^{res}$ -modules form the simple objects of  $\mathcal{C}_A$ . Moreover  $\mathcal{C}_A \otimes \mathbb{C}[q^{1/D}, q^{-1/D}]$  is a ribbon category (see Section 2.3).

The *integral* quantum function Hopf algebra  $\mathcal{O}_A = \mathcal{O}_A(G)$  is the (type 1) restricted dual of  $U_A^{res}$ , that is, the  $A$ -span of the matrix coefficients  $x \mapsto v^i(\pi_V(x)v_i)$ ,  $x \in U_A^{res}$ , for every module  $V$  in  $\mathcal{C}_A$ , where  $(v_i)$  is an  $A$ -basis of  $V$  and  $(v^i)$  the dual  $A$ -basis of the dual module  $V^*$  (compare with the definition of  $\mathcal{O}_q$ ). We can also regard  $\mathcal{O}_A$  as the set of  $A$ -linear maps  $f: U_A^{res} \rightarrow A$  such that  $\text{Ker}(f)$  contains a cofinite two sided ideal  $I$ , and  $\prod_{s=-r}^r (K_i - q_i^s) \in I$  for some  $r \in \mathbb{N}$  and every  $i$ . Because of the inclusions of  $U_A^{res}(\mathfrak{h})$ ,  $U_A^{res}(\mathfrak{n}_\pm)$ ,  $U_A^{res}(\mathfrak{b}_\pm)$  in  $U_A^{res}$ , there are Hopf epimorphisms from  $\mathcal{O}_A$  to the  $A$ -duals of these subalgebras, that we denote by  $\mathcal{O}_A(T_G)$ ,  $\mathcal{O}_A(U_\pm)$  and  $\mathcal{O}_A(B_\pm)$  respectively.

The algebra  $\mathcal{O}_A$  has been introduced by Lusztig in [75, 76]. It is an integral form of  $\mathcal{O}_q$ , so

$$\mathcal{O}_q = \mathcal{O}_A \otimes_A \mathbb{C}(q).$$

$\mathcal{O}_A$  is also the restricted dual of the integral form  $\Gamma = \Gamma(\mathfrak{g})$  of  $U_q^{ad}$  introduced by De Concini-Lyubashenko in [45], Sections 2-3;  $\Gamma$  is the  $A$ -subalgebra of  $U_q^{ad}$  generated by the elements  $(i = 1, \dots, m)$

$$E_i^{(k)} = \frac{E_i^k}{[k]_{q_i}!}, \quad F_i^{(k)} = \frac{F_i^k}{[k]_{q_i}!}, \quad (K_i; t)_{q_i} = \prod_{s=1}^t \frac{K_i q_i^{-s+1} - 1}{q_i^s - 1}, \quad K_i^{-1}$$

where  $k \in \mathbb{N}$ ,  $t \in \mathbb{N}$  (setting  $(K_i; 0)_{q_i} = 1$  by convention). Note that the definition of  $\Gamma$  is less symmetric than that of  $U_A^{res}$ . However,  $\Gamma$  contains the elements  $K_i$ , and the commutation

relations between the generators  $E_i^{(k)}, F_i^{(k)}$  imply that the symmetrized elements  $[K_i; t]_{q_i}$  belong to  $\Gamma$ . In fact, let us denote  $\Gamma(\mathfrak{h}) = U_q(\mathfrak{h}) \cap \Gamma$  and  $\Gamma(\mathfrak{b}_\pm) = U_q(\mathfrak{b}_\pm) \cap \Gamma$ . It is proved in [45], Theorem 3.1, that  $\Gamma(\mathfrak{h})$  contains  $U_A^{res}(\mathfrak{h})$ , and that the elements  $\prod_{i=1}^m K_i^{-\sigma(t_i)}(K_i; t_i)_{q_i}$ ,  $t_i \in \mathbb{N}$ , where  $\sigma(t) \in \{0, 1\}$  and  $\sigma(t) = t \bmod(2)$ , is an  $A$ -basis of  $\Gamma(\mathfrak{h})$ . A PBW basis of  $\Gamma$  is formed by the monomials

$$(34) \quad E_{\beta_1}^{(p_1)} \dots E_{\beta_N}^{(p_N)} \prod_{i=1}^m K_i^{-\sigma(t_i)}(K_i; t_i)_{q_i} F_{\beta_N}^{(n_N)} \dots F_{\beta_1}^{(n_1)}.$$

The inclusion  $U_A^{res} \subset \Gamma$  is strict, for the elements  $(K_i; t)_{q_i}$ ,  $t \neq 0$ , do not belong to  $U_A^{res}$ . However the restriction functor  $\mathcal{C}_\Gamma \rightarrow \mathcal{C}_A$  is obviously an equivalence, where  $\mathcal{C}_\Gamma$  is the category of  $\Gamma$ -modules of *type 1*, ie. free  $A$ -modules of finite rank which have a basis where the elements  $K_i$  act diagonally with eigenvalues of the form  $q_i^k$ ,  $k \in \mathbb{Z}$ . Therefore we can identify the two categories, and  $\mathcal{O}_A$  with the (type 1) restricted dual of  $\Gamma$ . We will thus consider the  $U_A^{res}$ -modules  ${}_A V_\mu$ ,  $\mu \in P_+$ , equally as  $\Gamma$ -modules. We will sometimes use  $\Gamma$  instead of  $U_A^{res}$  in order to make direct the connection with results of De Concini-Lyubashenko about the integral pairings  $\pi_A^\pm$  considered in Section 2.3.

The *integral* form  $\mathcal{L}_{0,1}^A$  of  $\mathcal{L}_{0,1}$  is defined as the  $U_A^{res}$ -module  $\mathcal{O}_A$  endowed with the product of  $\mathcal{L}_{0,1}$ . The *integral* form  $\mathcal{L}_{0,n}^A$  of  $\mathcal{L}_{0,n}$  is the braided tensor product of  $n$  copies of  $\mathcal{L}_{0,1}^A$ ; in particular  $\mathcal{L}_{0,n}^A = \mathcal{O}_A^{\otimes n}$  as  $U_A^{res}$ -modules. That the products of  $\mathcal{L}_{0,1}$  and  $\mathcal{L}_{0,n}$  are well-defined over  $A$  was shown in Proposition 6.9 of [31].

The *integral* quantum moduli algebra is

$$\mathcal{M}_{0,n}^A := (\mathcal{L}_{0,n}^A)^{U_A^{res}} = (\mathcal{L}_{0,n}^A)^{U_A}.$$

Finally, given  $q = \epsilon \in \mathbb{C}^\times$  we define the *specializations*  $U_\epsilon, \Gamma_\epsilon, \mathcal{O}_\epsilon, \mathcal{L}_{0,n}^{\epsilon}$  and  $\mathcal{M}_{0,n}^{A,\epsilon}$  as the  $\mathbb{C}$ -algebras obtained by tensoring  $U_A, \Gamma, \mathcal{O}_A, \mathcal{L}_{0,n}^A$  and  $\mathcal{M}_{0,n}^A$  respectively with  $\mathbb{C}_\epsilon$ , the  $A$ -module  $\mathbb{C}$  where  $q$  acts by multiplication by  $\epsilon$ . Each one can also be defined as the quotient by the ideal generated by  $q - \epsilon$ . We find convenient to use the notations

$$(35) \quad (U_A^{\otimes n})_\epsilon^{U_A} := (U_A^{\otimes n})^{U_A} \otimes_A \mathbb{C}_\epsilon, \quad (U_A^{\otimes n})_\epsilon^{lf} := (U_A^{\otimes n})^{lf} \otimes_A \mathbb{C}_\epsilon.$$

Let us stress here that when  $\epsilon$  is a root of unity, taking the locally finite part and taking the specialization at  $\epsilon$  are non commuting operations. Indeed, as shown by Theorem 2.22 below  $U_\epsilon$  is finite over  $\mathcal{Z}_0(U_\epsilon)$  and therefore all its elements are locally finite for  $ad^r$ ; on another hand  $U_\epsilon^{lf} = U_A^{lf} \otimes_A \mathbb{C}_\epsilon$  does not contain the elements  $L_i$ .

Similarly, taking invariants and taking the specialization at  $\epsilon$  are non commuting operations when  $\epsilon$  is a root of unity: indeed, it is easily checked that in this case  $(U_A^{\otimes n})_\epsilon^{U_A}$  and  $(U_\epsilon^{\otimes n})^{U_\epsilon}$ , or  $\mathcal{M}_{0,n}^{A,\epsilon} = \mathcal{M}_{0,n}^A \otimes_A \mathbb{C}_\epsilon$  and  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ , are distinct spaces. When  $\epsilon$  is a root of unity, we will not consider the algebras  $\mathcal{M}_{0,n}^{A,\epsilon}$  in this paper.

Arguments similar to those mentioned at the end of Section 2.1 imply that the algebras  $\mathcal{L}_{0,n}^A, \mathcal{M}_{0,n}^A$  and  $\mathcal{L}_{0,n}^{\epsilon'}, \mathcal{M}_{0,n}^{A,\epsilon'}$ ,  $\epsilon' \in \mathbb{C}^\times$ , have no non-trivial zero divisors (see [31], Proposition 6.11 and 6.30).

**2.2.2. Canonical basis and modified quantum groups.** Because the category  $\mathcal{C}_A$  is not semisimple, it is not clear from the above definition of  $\mathcal{O}_A$  whether or not it is a finitely generated algebra, or if it satisfies a Peter-Weyl  $A$ -module direct sum decomposition similar to (6), if  $\mathcal{M}_{0,n}^A$  is a direct summand of the  $A$ -module  $\mathcal{L}_{0,n}^A$ , or if the projection map (8) may be refined to a morphism between underlying  $A$ -modules.

Such properties indeed hold true, and will play a key role later. We state them precisely in Proposition 2.7, Theorem 2.6, Theorem 2.10 and Proposition 2.8. These results are consequences of the existence of an  $A$ -basis of  $\mathcal{O}_A$  with favourable properties, which implies in particular that  $\mathcal{O}_A$  is a free  $A$ -module. In order to introduce this  $A$ -basis it is necessary to consider a variant of  $U_q^{ad}$  introduced by Lusztig [76], called *modified quantum group*, and use the Kashiwara-Lusztig theory of canonical bases ([65, 66, 67], [76]). We are going to recall the background material step by step.

The Lusztig *modified quantum group* is the  $\mathbb{C}(q)$ -algebra  $\dot{\mathbf{U}}$  obtained by replacing  $U_q^{ad}(\mathfrak{h})$  with the direct sum of infinitely many one-dimensional algebras, generated by orthogonal idempotents  $1_\lambda$  indexed by the elements  $\lambda$  of the weight lattice  $P$  ([76], Chapter 23). Namely, as a vector space  $\dot{\mathbf{U}} = \bigoplus_{\lambda', \lambda'' \in P} \lambda' \dot{\mathbf{U}}_{\lambda''}$ , where

$$\lambda' \dot{\mathbf{U}}_{\lambda''} = U_q^{ad} / \left( \sum_{\alpha \in Q} (K_\alpha - q^{(\alpha, \lambda')}) U_q^{ad} + \sum_{\alpha \in Q} U_q^{ad} (K_\alpha - q^{(\alpha, \lambda'')}) \right).$$

Denote by  $\pi_{\lambda', \lambda''}: U_q^{ad} \rightarrow \lambda' \dot{\mathbf{U}}_{\lambda''}$  the canonical projection. The product of  $\dot{\mathbf{U}}$  is given by  $\pi_{\lambda'_1, \lambda''_1}(s) \pi_{\lambda'_2, \lambda''_2}(t) = \pi_{\lambda'_1, \lambda''_2}(st)$  if  $\lambda'_1 = \lambda'_2$  and zero otherwise. Set  $1_\lambda := \pi_{\lambda, \lambda}(1)$ . The algebra  $\dot{\mathbf{U}}$  has not unit, but the family  $(1_\lambda)_{\lambda \in P}$  can be regarded as a substitute of it. Denote by  $\Delta$  the collection of maps  $\Delta_{\lambda'_1, \lambda'_2, \lambda''_1, \lambda''_2}: \lambda'_1 + \lambda'_2 \dot{\mathbf{U}}_{\lambda''_1 + \lambda''_2} \rightarrow \lambda'_1 \dot{\mathbf{U}}_{\lambda''_1} \otimes \lambda'_2 \dot{\mathbf{U}}_{\lambda''_2}$  such that

$$(36) \quad \Delta_{\lambda'_1, \lambda'_2, \lambda''_1, \lambda''_2} \pi_{\lambda'_1 + \lambda'_2, \lambda''_1 + \lambda''_2} = (\pi_{\lambda'_1, \lambda''_1} \otimes \pi_{\lambda'_2, \lambda''_2}) \Delta_{U_q^{ad}},$$

where  $\Delta_{U_q^{ad}}$  is the coproduct of  $U_q^{ad}$ . We can regard  $\Delta$  as a (categorically completed) coproduct  $\Delta: \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}^{\hat{\otimes} 2}$ . There is a natural structure of  $U_q^{ad}$ -bimodule on  $\dot{\mathbf{U}}$ , defined by

$$(37) \quad t' \pi_{\lambda', \lambda''}(s) t'' = \pi_{\lambda' + \nu', \lambda'' - \nu''}(t' s t'')$$

for all  $s \in U_q^{ad}$  and all elements  $t', t'' \in U_q^{ad}$  of respective weights  $\nu', \nu''$ . This structure affords a triangular decomposition of  $\dot{\mathbf{U}}$ : given basis  $\{b^\pm\}$  of  $U_q^{ad}(\mathfrak{n}_\pm)$ , the set of elements  $b^+ 1_\lambda b^-$  (or  $b^- 1_\lambda b^+$ , or  $b^+ b^- 1_\lambda$ ), where  $\lambda \in P$ , is a basis of  $\dot{\mathbf{U}}$ .

Given any  $U_q^{ad}$ -module  $X$  of type 1, and any weight subspace  $X^\lambda \subset X$  of weight  $\lambda \in P$ , one can define the action of an element  $u 1_\lambda \in \dot{\mathbf{U}}$ ,  $u \in U_q^{ad}$ , on  $X$  as the projection onto  $X^\lambda$  followed by the action of  $u$ . This way, one can identify the category  $\mathcal{C}$  with the one of finite dimensional *unital*  $\dot{\mathbf{U}}$ -modules, where unital means that all elements  $1_\lambda$  act as 0 but a finite number of them, and  $\sum_{\lambda \in P} 1_\lambda$  acts as the identity. It is proved in [76], Section 29.5.1, that

$$(38) \quad \mathcal{O}_q = \left\{ f: \dot{\mathbf{U}} \rightarrow \mathbb{C}(q) \mid \begin{array}{l} f \text{ is } \mathbb{C}(q)\text{-linear and vanishes on some} \\ \text{two - sided ideal of finite codimension of } \dot{\mathbf{U}} \end{array} \right\}.$$

There is an analogous realization of  $\mathcal{O}_A$ , of the form (see [76], sections 23.2 and 29.5.2, and [77])

$$(39) \quad \mathcal{O}_A = \left\{ f: \dot{\mathbf{U}}_A \rightarrow A \mid \begin{array}{l} f \text{ is } A\text{-linear and vanishes on some} \\ \text{two - sided ideal of finite corank of } \dot{\mathbf{U}}_A \end{array} \right\},$$

where  $\dot{\mathbf{U}}_A$  is the  $A$ -subalgebra of  $\dot{\mathbf{U}}$  generated by the elements  $E_i^{(k)} 1_\lambda$  and  $F_i^{(k)} 1_\lambda$ , for all  $i \in \{1, \dots, m\}$ ,  $k \in \mathbb{N}$  and  $\lambda \in P$ . It is a  $U_A^{res}$ -subbimodule of  $\dot{\mathbf{U}}$ , and the coproduct restricts to a map  $\Delta: \dot{\mathbf{U}}_A \rightarrow \dot{\mathbf{U}}_A^{\hat{\otimes} 2}$ . The above identification of the category  $\mathcal{C}$  with the one of finite dimensional unital  $\dot{\mathbf{U}}$ -modules yields an identification of the category  $\mathcal{C}_A$  of  $U_A^{res}$ -modules of type 1 with the category of  $\dot{\mathbf{U}}_A$ -modules of finite rank.

The key advantage of this realization of  $\mathcal{O}_A$  is that  $\dot{\mathbf{U}}_A$  can be equipped with a canonical  $A$ -basis  $\dot{\mathbf{B}}$ . The construction of  $\dot{\mathbf{B}}$  is described in [76], Chapter 25. It relies on the Kashiwara-Lusztig *canonical basis* of  $U_A^{res}(\mathfrak{n}_-)$ . This last basis, denoted by  $\mathbf{B}^-$ , is defined in [76], Chapter 14, and [65] (a review can be found in [38], Chapter 14). It enjoys the following nice properties. Denote by  $\bar{\cdot} : \mathbb{C}(q) \rightarrow \mathbb{C}(q)$  the field involution such that  $\bar{q} = q^{-1}$ , and by  $\bar{\cdot} : U_q^{ad} \rightarrow U_q^{ad}$  the homomorphism of  $\mathbb{C}$ -algebras such that

$$\bar{E}_i = E_i \quad \bar{F}_i = F_i \quad , \quad \bar{K}_\lambda = K_{-\lambda} \quad , \quad \bar{f}x = \bar{f}\bar{x}$$

for all  $f \in \mathbb{C}(q)$ ,  $x \in U_q^{ad}$  ( $\bar{E}_i$  and  $\bar{F}_i$  above, which will not appear elsewhere, should not be confused with the normalized elements in (32)). The map  $\bar{\cdot}$  yields a  $\mathbb{C}$ -algebra homomorphism  $\bar{\cdot} : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$ . Then we have:

- (1) the elements of  $\mathbf{B}^-$  are weight vectors under the adjoint action of  $U_q^{ad}(\mathfrak{h})$ ;
- (2) for every  $b \in \mathbf{B}^-$ ,  $\bar{b} = b$ ;
- (3) for every  $b, b' \in \mathbf{B}^-$ ,  $bb' = \sum_{b'' \in \mathbf{B}^-} N_{b''}^{bb'} b''$  where  $N_{b''}^{bb'} \in \mathbb{N}[q, q^{-1}]$ ;
- (4) for every  $b, b' \in \mathbf{B}^-$ ,  $\Delta(b) = \sum_{b', b'' \in \mathbf{B}^-} C_{b', b''}^b b' \otimes b''$  where  $C_{b', b''}^b \in \mathbb{N}[q, q^{-1}]$ ;
- (5) For every  $\mu \in P^+$ , denoting by  $v_\mu$  the highest weight vector of the  $U_A^{res}$ -module  ${}_A V_\mu$ , the elements  $bv_\mu$  which are non zero, where  $b \in \mathbf{B}^-$ , form an  $A$ -basis of  ${}_A V_\mu$ .

In the case of  $\mathfrak{g} = sl(2)$  the elements of  $\mathbf{B}^-$  are just the divided powers  $F^{(k)}$ ,  $k \in \mathbb{N}$ . Formulas in terms of PBW basis elements are known also for  $\mathfrak{g} = sl(3)$  and  $sl(4)$ , and an algorithm exists in the general case (see [57] and the references therein).

Correspondingly to  $\mathbf{B}^-$ , the set  $\mathbf{B}^+ = \omega(\mathbf{B}^-)$  is a basis of  $U_A^{res}(\mathfrak{n}_+)$ , where  $\omega : U_q^{ad} \rightarrow U_q^{ad}$  is the ( $\mathbb{C}(q)$ -linear) *Cartan automorphism*, defined by

$$\omega(E_i) = F_i \quad , \quad \omega(F_i) = E_i \quad , \quad \omega(K_i) = K_i^{-1}$$

for  $i = 1, \dots, m$ . The triangular decomposition of  $\dot{\mathbf{U}}$  implies that the elements  $b^+ 1_\lambda b'^-$ , where  $b^+ \in \mathbf{B}^+$ ,  $b'^- \in \mathbf{B}^-$  and  $\lambda \in P$ , form a basis of  $\dot{\mathbf{U}}$ . They form in fact an  $A$ -basis of  $\dot{\mathbf{U}}_A$ , and its elements are fixed by the involution  $\bar{\cdot} : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$ .

Lusztig has constructed another  $A$ -basis of  $\dot{\mathbf{U}}_A$ , denoted  $\dot{\mathbf{B}}$ , and called the *canonical basis* of  $\dot{\mathbf{U}}_A$ . It satisfies numerous properties that we now review. Its elements are denoted by  $b \diamond_\lambda b'$ , where  $b, b' \in \mathbf{B}^-$  and  $\lambda \in P$ , and are related to the elements  $b^+ b'^- 1_\lambda$ , where  $b^+ := \omega(b)$  and  $b'^- := b'$ , by a specific trigonal change of basis with coefficients in  $A$ . Although we always have  $b^+ 1_\lambda, b'^- 1_\lambda \in \dot{\mathbf{B}}$ , to our knowledge explicit formulas of the elements of  $\dot{\mathbf{B}}$  as linear combinations of elements  $b^+ 1_\lambda b'^-$  or  $b'^- 1_\lambda b^+$  are known only for  $\mathfrak{g} = sl(2)$  or  $sl(3)$  (see [76], 25.3, and [36]). In the former case, identifying  $P$  with  $\mathbb{Z}$  and  $Q$  with  $2\mathbb{Z}$  the canonical basis  $\dot{\mathbf{B}}$  is formed by the elements

$$(40) \quad E^{(k)} 1_{-n} F^{(l)} \quad \text{and} \quad F^{(l)} 1_n E^{(k)} \quad , \quad k, l, n \in \mathbb{N}, n \geq k + l$$

where  $E^{(k)} 1_{-n} F^{(l)} = F^{(l)} 1_n E^{(k)}$  for  $n = k + l$ .

We are going to review Lusztig's construction of  $\dot{\mathbf{B}}$ , its canonical partition  $\dot{\mathbf{B}} = \cup_{\lambda \in P_+} \dot{\mathbf{B}}[\lambda]$ , the Kashiwara's dual basis  $\dot{\mathbf{B}}^*$ , and its dual partition ([66, 67]). The latter is stated in Theorem 2.6 below. At first we need to recall the notions of based module and balanced triple; for details we refer to [76], Chapter 27, and [66] (see also [68], the sections 3.15 and 3.16 in [96], or Chapter 14 in [38] for overviews).

Denote by  $\mathcal{A}_0 \subset \mathbb{C}(q)$  the ring of rational functions regular at  $q = 0$ . By applying the involution  $\bar{\cdot}$ , put

$$\mathcal{A}_\infty = \overline{\mathcal{A}_0}.$$



Since  $\mathcal{A}_0$  is the localization of  $\mathbb{C}[q]$  at  $q = 0$ , we may regard  $\mathcal{A}_\infty$  as the localization of  $\mathbb{C}[q^{-1}]$  at  $q = \infty$ .

Let us recall briefly the definition of crystal basis (see [65]). Denote by  $U_q^{ad}(\mathfrak{g})_i$  the subalgebra of  $U_q^{ad}(\mathfrak{g})$  generated by  $E_i, F_i$  and  $K_i^{\pm 1}$ ; thus  $U_q^{ad}(\mathfrak{g})_i$  is isomorphic to  $U_{q_i}(sl(2))$ . Let  $M$  be a  $U_q^{ad}$ -module of type 1. Denote  $M^\zeta$  the subspace of  $M$  of weight  $\zeta \in P$ . For every  $i = 1, \dots, m$  we can regard  $M$  as a  $U_q^{ad}(\mathfrak{g})_i$ -module, so  $M \cong \bigoplus_j V_{\lambda_j}$  for some simple  $U_q^{ad}(\mathfrak{g})_i$ -modules  $V_{\lambda_j}$ . These being generated by primitive weight vectors, the PBW basis of  $U_q^{ad}(\mathfrak{g})_i$  yields

$$M = \bigoplus_{\zeta \in P} \bigoplus_{0 \leq n \leq (\check{\alpha}_i, \zeta)} F_i^{(n)}(Ker(E_i) \cap M^\zeta).$$

The *Kashiwara operators*  $\tilde{e}_i, \tilde{f}_i$  are the endomorphisms of  $M$  defined by, for every  $v \in Ker(E_i) \cap M^\zeta$  and  $0 \leq n \leq (\check{\alpha}_i, \zeta)$ ,

$$(41) \quad \tilde{f}_i(F_i^{(n)}v) = F_i^{(n+1)}v, \quad \tilde{e}_i(F_i^{(n)}v) = F_i^{(n-1)}v.$$

A *crystal basis* of  $M$  at  $q = 0$  consists of a pair  $(\mathcal{L}, \mathcal{B})$ , where:

- $\mathcal{L}$  is a free  $\mathcal{A}_0$ -sublattice of  $M$  such that the canonical map  $\mathcal{L} \otimes_{\mathcal{A}_0} \mathbb{C}(q) \rightarrow M$  is an isomorphism;
- $\mathcal{B}$  is a basis of the  $\mathbb{C}$ -vector space  $\mathcal{L}/q\mathcal{L}$ ;
- $\mathcal{L} = \bigoplus_{\zeta \in P} \mathcal{L}^\zeta$  and  $\mathcal{B} = \coprod_{\zeta \in P} (\mathcal{B} \cap \mathcal{L}^\zeta / q\mathcal{L}^\zeta)$ , where  $\mathcal{L}^\zeta = \mathcal{L} \cap M^\zeta$ ;
- for every  $i = 1, \dots, m$  the Kashiwara operators  $\tilde{e}_i, \tilde{f}_i$  preserve  $\mathcal{L}$ , and the induced maps on  $\mathcal{L}/q\mathcal{L}$  send  $\mathcal{B}$  into  $\mathcal{B} \cup \{0\}$ , and satisfy  $b' = \tilde{f}_i(b)$  if and only if  $b = \tilde{e}_i(b')$  for every  $b, b' \in \mathcal{B}$ .

Crystal basis at  $q = \infty$  are defined similarly, by replacing  $\mathcal{A}_0$  with  $\mathcal{A}_\infty$ .

A  $U_q^{ad}$ -module  $M$  of type 1 endowed with a  $\mathbb{C}(q)$ -basis  $B$  is called a *based module* if the following conditions hold:

- (i) For every weight  $\zeta \in P$ , the set  $B \cap M^\zeta$  is a basis of the weight subspace  $M^\zeta \subset M$ .
- (ii) The  $A$ -module  ${}_A M$  generated by  $B$  is stable under  $U_A^{res}$ .

We will denote by  $\mathcal{L}_M$  the  $\mathcal{A}_0$ -submodule of  $M$  generated by  $B$ , and by  $\bar{\mathcal{L}}_M$  the  $\mathcal{A}_\infty$ -submodule of  $M$  generated by  $B$ .

- (iii) The  $\mathbb{C}$ -linear involution  $\bar{\cdot} : M \rightarrow M$  defined by  $\overline{fb} = \bar{f}\bar{b}$  for all  $f \in \mathbb{C}(q)$  and  $b \in B$  is compatible with the action of  $U_q^{ad}$  in the sense that  $\overline{x\bar{m}} = \bar{x}\bar{m}$  for all  $U_q^{ad}$ ,  $m \in M$ .
- (iv) The  $\mathcal{A}_\infty$ -submodule  $\bar{\mathcal{L}}_M$  of  $M$  together with the image of  $B$  in  $\bar{\mathcal{L}}_M/q^{-1}\bar{\mathcal{L}}_M$  forms a crystal basis of  $M$  at  $q = \infty$ .

If  $(M, B)$  is a based module, we will denote by  $\bar{\mathcal{B}}$  the image of  $B$  in  $\bar{\mathcal{L}}_M/q^{-1}\bar{\mathcal{L}}_M$ . From the notion of balanced triple that we recall now, denoting by  $\mathcal{B}$  the image of  $B$  in  $\mathcal{L}_M/q\mathcal{L}_M$ , we see that  $(\mathcal{L}_M, \mathcal{B})$  is a crystal basis at  $q = 0$ .

Indeed, consider more generally a  $\mathbb{C}(q)$ -vector space  $V$ , finite dimensional or not, a sub- $A$ -module  ${}_A V$ , a sub- $\mathcal{A}_0$ -module  ${}_{\mathcal{A}_0} V$  and a sub- $\mathcal{A}_\infty$ -module  ${}_{\mathcal{A}_\infty} V$  satisfying the conditions (all isomorphisms being the canonical maps)

$$V \cong \mathbb{C}(q) \otimes_A {}_A V, \quad V \cong \mathbb{C}(q) \otimes_{\mathcal{A}_0} {}_{\mathcal{A}_0} V, \quad V \cong \mathbb{C}(q) \otimes_{\mathcal{A}_\infty} {}_{\mathcal{A}_\infty} V.$$

Denote the  $\mathbb{C}$ -vector space  $E := {}_A V \cap {}_{\mathcal{A}_0} V \cap {}_{\mathcal{A}_\infty} V$ . Then  $({}_A V, {}_{\mathcal{A}_0} V, {}_{\mathcal{A}_\infty} V)$  is a *balanced triple* ([65, 66]) if the canonical maps

$$(42) \quad A \otimes_{\mathbb{C}} E \rightarrow {}_A V, \quad \mathcal{A}_0 \otimes_{\mathbb{C}} E \rightarrow {}_{\mathcal{A}_0} V, \quad \mathcal{A}_\infty \otimes_{\mathbb{C}} E \rightarrow {}_{\mathcal{A}_\infty} V$$

are isomorphisms. Equivalently,  $({}_A V, {}_{\mathcal{A}_0} V, {}_{\mathcal{A}_\infty} V)$  is balanced if and only if the canonical map  $E \rightarrow {}_{\mathcal{A}_0} V/q{}_{\mathcal{A}_0} V$  is an isomorphism, if and only if the canonical map  $E \rightarrow {}_{\mathcal{A}_\infty} V/q^{-1}{}_{\mathcal{A}_\infty} V$  is an isomorphism ([66], Lemma 2.1.1).

Given a based module  $(M, B)$ , the elements of  $B$  are weight vectors and  $\bar{b} = b$  for every  $b \in B$ . Also, if an element  $m \in {}_A M$  satisfies  $\bar{m} = m$  and  $m \in B + q^{-1}\bar{\mathcal{L}}_M$ , then  $m \in B$  (see 27.1.5 in [76] for details on this fact). It follows that the canonical quotient map

$$(43) \quad {}_A M \cap \mathcal{L}_M \cap \bar{\mathcal{L}}_M \rightarrow \bar{\mathcal{L}}_M/q^{-1}\bar{\mathcal{L}}_M$$

is an isomorphism of  $\mathbb{C}$ -vector spaces. This provides another way of viewing based modules: by (43),  $({}_A M, \mathcal{L}_M, \bar{\mathcal{L}}_M)$  is a balanced triple, and by (42) the  $A$ -lattice  ${}_A M$  is completely determined by the crystal base  $(\bar{\mathcal{L}}_M, \bar{\mathcal{B}})$ . We will say that  $(\bar{\mathcal{L}}_M, \bar{\mathcal{B}})$  (or the corresponding crystal base  $(\mathcal{L}_M, \mathcal{B})$  at  $q = 0$ ) is *melted into* the based module  $(M, B)$ .

We will indifferently apply the notion of based module to finite-dimensional unital  $\dot{\mathbf{U}}$ -modules, since they are equivalent to  $U_q^{ad}$ -modules of type 1.

Every module  $V_\mu$ ,  $\mu \in P^+$ , supports a structure of based module ([76], Section 14.4.10, [65]); the corresponding basis, called *canonical basis* and that we will denote by  $\underline{\mathbf{B}}_\mu$ , is formed by the elements  $bv_\mu \in {}_A V_\mu$  which are non zero, where  $b \in \mathbf{B}^-$  and  $v_\mu$  is the canonical highest weight vector of  $V_\mu$ , corresponding to the coset of  $1 \in U_q^{ad}(\mathfrak{n}_-)$  in the Verma module construction of  $V_\mu$ . Note that the involution  $\bar{\cdot} : V_\mu \rightarrow V_\mu$  defined by (iii) above is indeed an automorphism, for the space  $V_\mu$  with action of  $U_q^{ad}$  defined by  $x \cdot v := \bar{x}v$ , for all  $x \in U_q^{ad}$ ,  $v \in V_\mu$ , has highest weight  $\mu$ , and is thus isomorphic to  $V_\mu$  via the map  $\bar{\cdot}$ . The crystal base  $(\mathcal{L}_\mu^{low}, \mathcal{B}_\mu^{low})$  at  $q = 0$  is formed by the  $\mathcal{A}_0$ -sublattice  $\mathcal{L}_\mu^{low}$  of  $V_\mu$  generated by  $\underline{\mathbf{B}}_\mu$  (which is eventually the same as the  $\mathcal{A}_0$ -sublattice generated by the vectors of the form  $\tilde{f}_{i_1} \circ \dots \circ \tilde{f}_{i_k}(v_\mu)$ , where  $i_1, \dots, i_k \in \{1, \dots, m\}$ ), and  $\mathcal{B}_\mu^{low}$  is the set of non-zero images of these vectors in  $\mathcal{L}_\mu^{low}/q\mathcal{L}_\mu^{low}$ .

There is the following uniqueness result ([65], Theorem 3):

**Theorem 2.5.** *Let  $M$  be a  $U_q^{ad}$ -module of type 1, and  $(\mathcal{L}, \mathcal{B})$  a crystal base at  $q = 0$  of  $M$ . Then there exists a  $\mathbb{C}(q)$ -isomorphism  $M \rightarrow \bigoplus_j V_{\lambda_j}$  by which  $(\mathcal{L}, \mathcal{B})$  is  $\mathcal{A}_0$ -isomorphic to  $\bigoplus_j (\mathcal{L}_{\lambda_j}^{low}, \mathcal{B}_{\lambda_j}^{low})$ .*

The based modules form a category. Given based modules  $(M, B)$  and  $(M', B')$ , a morphism of  $U_q^{ad}$ -modules  $f : M \rightarrow M'$  is a morphism of based modules if:

- (a)  $f(b) \in B' \cup \{0\}$  for any  $b \in B$ ;
- (b)  $B \cap \text{Ker}(f)$  is a basis of  $\text{Ker}(f)$ .

The direct sum of based modules  $(M, B)$  and  $(M', B')$  is a based module  $(M \oplus M', B \cup B')$ ; and a submodule  $M'$  of a based module  $(M, B)$  spanned over  $\mathbb{C}(q)$  by a subset  $B'$  of  $B$  forms a based module  $(M', B')$ . The quotient module  $M/M'$  together with the image of  $B \setminus B'$  is then a based module.

The tensor product of based modules  $(M, B)$ ,  $(M', B')$  is also defined. Namely, consider the  $\mathbb{C}$ -linear map  $\Psi : M \otimes M' \rightarrow M \otimes M'$  defined by  $\Psi(m \otimes m') = \hat{R}^{-1}(\bar{m} \otimes \bar{m}')$ , where  $\hat{R} = \Theta^{-1}R$ , see (4) (note that, as we use the coproduct opposite to [76] our quasi-R-matrix is  $\hat{R}^{-1}$ ). It can be checked that  $\Psi$  is an involution compatible with the action of  $\dot{\mathbf{U}}$  in the sense of (iii) above. Moreover, denote by  $\mathcal{L}_{M, M'}$  the  $\mathbb{C}[q^{-1}]$ -submodule of  $M \otimes M'$  spanned by the basis elements  $b \otimes b'$ , where  $b \in B$ ,  $b' \in B'$ . It is shown in [76], Section 27.3, that for every couple  $(b, b') \in B \times B'$  there is a unique element  $b \diamond b' \in \mathcal{L}_{M, M'}$  such that

- (a)  $\Psi(b \diamond b') = b \diamond b'$ ,

$$(b) \ b \diamond b' - b \otimes b' \in q^{-1} \mathcal{L}_{M, M'}.$$

Moreover,  $B_\diamond = \{b \diamond b', b \in B, b' \in B'\}$  is a basis of  $M \otimes M'$ , a  $\mathbb{C}[q^{-1}]$ -basis of  $\mathcal{L}_{M, M'}$ , a  $\mathbb{C}[q, q^{-1}]$ -basis of the  $\mathbb{C}[q, q^{-1}]$ -module  ${}_A \mathcal{L}_{M, M'}$  of  $M \otimes M'$  generated by the elements  $b \otimes b'$ , where  $b \in B, b' \in B'$ , and  $(M \otimes M', B_\diamond)$  is a based module.

This construction of  $B_\diamond$  is associative. Since  $(V_\mu, \underline{\mathbf{B}}_\mu)$  is for every  $\mu \in P_+$  a based module, it follows that any tensor product  $M$  of a finite number of the simple modules  $V_\mu$  is naturally a based module. The corresponding basis is called *the canonical basis* of  $M$ . These canonical basis have been computed explicitly in [55] in the case  $\mathfrak{g} = sl(2)$ .

Consider now the  $U_q^{ad}$ -module  ${}^\omega V_\mu$  with underlying space  $V_\mu$ ,  $\mu \in P_+$ , and action defined by  $x \cdot {}^\omega v := \omega(x)v$ , for every  $x \in U_q^{ad}$  and  $v \in V_\mu$  (as usual  $\omega: U_q^{ad} \rightarrow U_q^{ad}$  is the Cartan automorphism). Note that there are isomorphisms  ${}^\omega V_\mu \cong V_{-w_0(\mu)} \cong V_\mu^*$  (endowed with the standard left action of  $U_q^{ad}$ ). Let us denote by  ${}^\omega v_\mu$  the vector  $v_\mu$  regarded in  ${}^\omega V_\mu$  (ie. its canonical lowest weight vector), and by  ${}^\omega \underline{\mathbf{B}}_\mu := \{b \cdot {}^\omega v_\mu \neq 0, b \in \mathbf{B}^+\}$  its canonical basis. Then  ${}^\omega V_{\mu'} \otimes V_{\mu''}$  has the canonical basis  $\underline{\mathbf{B}}_{\mu', \mu''} := \{\underline{b}' \diamond \underline{b}'', \underline{b}' \in {}^\omega \underline{\mathbf{B}}_{\mu'}, \underline{b}'' \in \underline{\mathbf{B}}_{\mu''}\}$ . Since  $\underline{b}' \diamond \underline{b}''$  is canonically determined by the elements  $\underline{b}', \underline{b}'' \in \mathbf{B}^-$  such that  $\underline{b}' = \omega(b') \cdot {}^\omega v_{\mu'}, \underline{b}'' = b'' v_{\mu''}$ , following Lusztig we denote it  $(b' \diamond b'')_{\mu', \mu''}$ .

Denote by  $v_{w_0(\mu)}$  the canonical lowest weight vector of  $V_\mu$ , and by  ${}^\omega v_{w_0(\mu)}$  the vector  $v_{w_0(\mu)}$  regarded in  ${}^\omega V_\mu$ . It is a crucial observation that  ${}^\omega v_{w_0(\mu')} \otimes v_{w_0(\mu'')}$  is a cyclic vector of  ${}^\omega V_{\mu'} \otimes V_{\mu''}$  (see eg. [76], 23.3.6; note that  ${}^\omega v_{w_0(\mu')} \otimes v_{w_0(\mu'')}$  plays the role of  $\xi_{-\mu'} \otimes \eta_{\mu''} := {}^\omega v_{\mu'} \otimes v_{\mu''}$  in [76], because we use opposite coproducts on  $U_q^{ad}$  but the factors  ${}^\omega V_{\mu'}$  and  $V_{\mu''}$  are ordered in the same way).

We can now state the definition of the canonical basis  $\dot{\mathbf{B}}$  of  $\dot{\mathbf{U}}$ : each element  $u$  of  $\dot{\mathbf{B}}$  belongs to  $\dot{\mathbf{U}}_A 1_\zeta$  for some (unique)  $\zeta \in P$ , and it is then uniquely determined by the property that, for every  $\mu', \mu'' \in P^+$  such that  $\mu'' - \mu' = \zeta$ , we have

$$(44) \quad u({}^\omega v_{w_0(\mu')} \otimes v_{w_0(\mu'')}) = (b' \diamond b'')_{\mu', \mu''}$$

for some  $(b' \diamond b'')_{\mu', \mu''} \in \underline{\mathbf{B}}_{\mu', \mu''}$  ([76], Chapter 25.2). We write  $u = b' \diamond_\zeta b''$ , and as in [77] we denote by  $\dot{\mathbf{B}}_{\mu', \mu''}$  the finite subset of  $\dot{\mathbf{B}}$  which is in bijection with  $\underline{\mathbf{B}}_{\mu', \mu''}$  under the map  $u \mapsto u({}^\omega v_{w_0(\mu')} \otimes v_{w_0(\mu'')})$ . So

$$(45) \quad \dot{\mathbf{B}} = \cup_{\mu', \mu'' \in P_+} \dot{\mathbf{B}}_{\mu', \mu''}.$$

In a sense, one can view  $\dot{\mathbf{U}}$  as the projective limit of an inverse system formed by the  $(U_q^{ad} \otimes U_q^{ad})$ -modules  ${}^\omega V_{\mu'} \otimes V_{\mu''}$ , where  $\mu', \mu'' \in P^+$ ; then  $\dot{\mathbf{B}}$  is the basis resulting from the corresponding inverse system of basis  $\{\dot{\mathbf{B}}_{\mu', \mu''}\}_{\mu', \mu''}$ .

Lusztig has produced a partition of  $\dot{\mathbf{B}}$  as follows. First, consider the situation of a based module  $(M, B)$ . For every  $\lambda \in P_+$  denote by  $M[\lambda]$  the sum of the simple submodules of  $M$  isomorphic to  $V_\lambda$  (ie. its isotypical component). Set

$$(46) \quad M[\geq \lambda] = \oplus_{\lambda' \geq \lambda} M[\lambda'].$$

Then, for every base element  $b \in B$  there is a unique  $\lambda \in P_+$  such that  $b \in M[\geq \lambda]$  and  $\lambda$  is maximal with this property ([76], 27.2). Denote by  $B[\lambda]$  the set of all  $b \in B$  that give rise to  $\lambda \in P_+$  in this way. Clearly the sets  $B[\lambda], \lambda \in P_+$ , form a partition of  $B$ .

Now, given  $b \in \dot{\mathbf{B}}$ , let  $\zeta \in P$  be the unique weight such that  $b \in \dot{\mathbf{U}}_A 1_\zeta$ , and let  $\mu', \mu'' \in P^+$  be such that  $\mu'' - \mu' = \zeta$ , and  $(\check{\alpha}_i, \mu')$  is large enough for all  $i = 1, \dots, m$  so that  $u({}^\omega v_{w_0(\mu')} \otimes v_{w_0(\mu'')})$  is non zero. This element belongs to the canonical basis  $\underline{\mathbf{B}}_{\mu', \mu''}$  of  ${}^\omega V_{\mu'} \otimes V_{\mu''}$ , and therefore to one of the subsets  $\underline{\mathbf{B}}_{\mu', \mu''}[\lambda]$ , for a unique  $\lambda \in P_+$ . It is a result

that  $\lambda$  does not depend on the choice of  $\mu', \mu''$  (see [76], 29.1.1). Hence there is a well-defined map  $\dot{\mathbf{B}} \rightarrow P_+$ ,  $b \mapsto \lambda$ . Denoting by  $\dot{\mathbf{B}}[\lambda]$  the fiber of this map, we thus obtain a partition

$$(47) \quad \dot{\mathbf{B}} = \coprod_{\lambda \in P_+} \dot{\mathbf{B}}[\lambda].$$

The sets  $\dot{\mathbf{B}}[\lambda]$  are called *2-sided cells*. They are finite sets and have the following remarkable properties. For every  $\lambda \in P_+$  denote by  $\dot{\mathbf{U}}[\geq \lambda]$  and  $\dot{\mathbf{U}}[> \lambda]$  the subspaces of  $\dot{\mathbf{U}}$  spanned by  $\coprod_{\lambda' \geq \lambda} \dot{\mathbf{B}}[\lambda']$  and  $\coprod_{\lambda' > \lambda} \dot{\mathbf{B}}[\lambda']$  respectively. Then  $\dot{\mathbf{U}}[\geq \lambda]$  (respectively  $\dot{\mathbf{U}}[> \lambda]$ ) consists of the elements  $u \in \dot{\mathbf{U}}$  such that if  $u$  acts on  $V_\mu$  by a non zero linear map, then  $\mu \geq \lambda$  (respectively  $\mu > \lambda$ ) ([76], 29.1.3-29.1.4). Both  $\dot{\mathbf{U}}[\geq \lambda]$  and  $\dot{\mathbf{U}}[> \lambda]$  are two-sided ideals of  $\dot{\mathbf{U}}$ . Moreover the algebra homomorphism  $\pi_\lambda: \dot{\mathbf{U}}[\geq \lambda] \rightarrow \text{End}(V_\lambda)$  given by the  $\dot{\mathbf{U}}$ -module structure on  $V_\lambda$  descends to an algebra and  $U_q^{ad}$ -bimodule isomorphism (keeping the same notation)

$$(48) \quad \bar{\pi}_\lambda: \dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda] \rightarrow \text{End}(V_\lambda)$$

([76], Proposition 29.2.2). Therefore  $\text{End}(V_\lambda)$  has a natural full  $A$ -sublattice  ${}_A \text{End}(V_\lambda)$ , namely the  $A$ -module generated by  $\pi_\lambda(\dot{\mathbf{B}}[\lambda])$ , and this lattice is isomorphic to the  $A$ -submodule of  $\dot{\mathbf{U}}_A$  generated by  $\dot{\mathbf{B}}[\lambda]$ .

For instance, when  $\mathfrak{g} = \mathfrak{sl}(2)$  the 2-sided cell  $\dot{\mathbf{B}}[n]$  associated to the simple  $U_q^{ad}(\mathfrak{sl}(2))$ -module of type 1 and dimension  $n + 1$  is the set ([76], 29.4.3)

$$(49) \quad \dot{\mathbf{B}}[n] = \{E^{(k)}1_{-n}F^{(l)}, n \geq k + l\} \cup \{F^{(l)}1_nE^{(k)}, n \geq k + l\}.$$

As we are mainly interested in  $\mathcal{O}_A$  we are going to describe the dual partition of  $\dot{\mathbf{B}}^*$ ; see Theorem 2.6. The duality with (47) is discussed after that theorem, see (59). We follow the approach of Kashiwara [66, 67].

For every  $\lambda \in P_+$  denote by  $V_\lambda^r$  the dual space of  $V_\lambda$  endowed with its natural structure of right  $U_q^{ad}$ -module, defined by

$$(fx)(v) = f(xv)$$

for every  $f \in V_\lambda^r$ ,  $x \in U_q^{ad}$ ,  $v \in V_\lambda$ . Clearly  $V_\lambda^r$  is a simple module of highest weight  $\lambda$ . Let  $\varphi: U_q^{ad} \rightarrow U_q^{ad}$  be the anti-automorphism of  $\mathbb{C}(q)$ -algebra given by  $\varphi(E_i) = F_i$ ,  $\varphi(F_i) = E_i$ ,  $\varphi(K_\lambda) = K_\lambda$ . By using  $\varphi$  any right  $U_q^{ad}$ -module can be considered as a left  $U_q^{ad}$ -module. In particular, by the Verma module construction of  $V_\lambda$  it follows

$$V_\lambda^r \cong U_q^{ad} / \left( \sum_{\mu \in P_+} (K_\mu - q^{(\lambda, \mu)}) U_q^{ad} + \sum_{i=1}^m E_i^{1+(\alpha_i, \lambda)} U_q^{ad} \right),$$

and  $\varphi$  affords an isomorphism of the right module  $V_\lambda^r$  with the *left* module  $V_\lambda$ . We will denote by  $f_\lambda$  the highest weight vector of  $V_\lambda^r$ .

The space  $V_\lambda^r \otimes V_\lambda$  can be identified with  $\text{End}(V_\lambda)^*$ , and thus acquires by duality a natural structure of  $U_q^{ad}$ -bimodule (or equivalently left  $U_q^{ad} \otimes (U_q^{ad})^{op}$ -module); the left and right actions are given by

$$(50) \quad x(f \otimes v)y = fy \otimes xv$$

for every  $x, y \in U_q^{ad}$ ,  $f \in V_\lambda^r$ ,  $v \in V_\lambda$ . The space  $V_\lambda^r \otimes V_\lambda$  also acquires by duality a natural ‘‘upper’’ crystal structure over  $U_q^{ad} \otimes (U_q^{ad})^{op}$ , as we explain now. Denote by  $\langle \cdot, \cdot \rangle_\lambda: V_\lambda \times V_\lambda \rightarrow \mathbb{C}(q)$  the unique symmetric bilinear form such that

$$(51) \quad \langle v_\lambda, v_\lambda \rangle_\lambda = 1 \quad \text{and} \quad \langle \varphi(x)u, v \rangle_\lambda = \langle u, xv \rangle_\lambda$$

for every  $u, v \in V_\lambda$  and  $x \in U_q^{ad}$ . Recall the crystal base  $(\mathcal{L}_\mu^{low}, \mathcal{B}_\mu^{low})$  at  $q = 0$  introduced before Theorem 2.5. In Kashiwara’s terminology ([65, 66]) the pair  $(\mathcal{L}_\lambda^{low}, \mathcal{B}_\lambda^{low})$  is the *lower*

crystal base of  $V_\lambda$  at  $q = 0$ . Applying the involution  $\bar{\cdot} : V_\lambda \rightarrow V_\lambda$  one obtains the lower crystal base  $(\overline{\mathcal{L}_\lambda^{low}}, \overline{\mathcal{B}_\lambda^{low}})$  at  $q = \infty$ . Because the canonical bases are determined by the crystal bases (see the discussion about (43)), we call  $(V_\lambda, \underline{\mathbf{B}}_\lambda)$  the *lower based module* of  $V_\lambda$ , and  $\underline{\mathbf{B}}_\lambda$  the *lower canonical basis* of  $V_\lambda$ .

Put

$$(52) \quad \begin{aligned} {}_A V_\lambda^{up} &:= \{v \in V_\lambda, \langle v, {}_A V_\lambda \rangle_\lambda \subset A\}, \\ \mathcal{L}_\lambda^{up} &:= \{v \in V_\lambda, \langle v, \mathcal{L}_\lambda^{low} \rangle_\lambda \subset \mathcal{A}_0\}, \\ \overline{\mathcal{L}_\lambda^{up}} &:= \{v \in V_\lambda, \langle v, \overline{\mathcal{L}_\lambda^{low}} \rangle_\lambda \subset \mathcal{A}_\infty\}. \end{aligned}$$

Then  $({}_A V_\lambda^{up}, \mathcal{L}_\lambda^{up}, \overline{\mathcal{L}_\lambda^{up}})$  is a balanced triple ([66], Lemma 4.2.1). Denote by  $\mathcal{B}_\lambda^{up}$  the basis of  $\mathcal{L}_\lambda^{up}/q\mathcal{L}_\lambda^{up}$  dual to  $\mathcal{B}_\lambda^{low}$  by the induced pairing  $\langle \cdot, \cdot \rangle_\lambda : \mathcal{L}_\lambda^{low}/q\mathcal{L}_\lambda^{up} \times \mathcal{L}_\lambda^{low}/q\mathcal{L}_\lambda^{low} \rightarrow \mathbb{C}$ . The pair  $(\mathcal{L}_\lambda^{up}, \mathcal{B}_\lambda^{up})$  is the *upper crystal base* of  $V_\lambda$  at  $q = 0$ . The weight spaces of the  $\mathcal{A}_0$ -modules  $\mathcal{L}_\lambda^{low}$  and  $\mathcal{L}_\lambda^{up}$  are related by

$$(53) \quad (\mathcal{L}_\lambda^{up})^\mu = q^{\frac{(\lambda, \lambda)}{2} - \frac{(\mu, \mu)}{2}} (\mathcal{L}_\lambda^{low})^\mu \quad , \quad \mu \in P.$$

Correspondingly, denoting  $(\mathcal{B}_\lambda^{up})^\mu := \mathcal{B}_\lambda^{up} \cap (\mathcal{L}_\lambda^{up})^\mu$  and  $(\mathcal{B}_\lambda^{low})^\mu := \mathcal{B}_\lambda^{low} \cap (\mathcal{L}_\lambda^{low})^\mu$ , we have ([65], [66], eq. (4.2.9))

$$(\mathcal{B}_\lambda^{up})^\mu = q^{\frac{(\lambda, \lambda)}{2} - \frac{(\mu, \mu)}{2}} (\mathcal{B}_\lambda^{low})^\mu.$$

The  $A$ -module  ${}_A V_\lambda^{up}$  is characterized by the following two properties ([66], eq. (4.2.10)-(4.2.12)):

$$\begin{aligned} ({}_A V_\lambda^{up})^\lambda &= \mathbb{C}[q, q^{-1}]v_\lambda \\ ({}_A V_\lambda^{up})^\mu &= \{v \in V_\lambda \mid U_A^{res}(\mathfrak{n}^+)^{\lambda-\mu}v \in \mathbb{C}[q, q^{-1}]v_\lambda\} \end{aligned}$$

where  $U_A^{res}(\mathfrak{n}^+)^\gamma = \{u \in U_A^{res}(\mathfrak{n}^+) \mid \forall \nu \in P, K_\nu u K_\nu^{-1} = q^{(\nu, \gamma)}u\}$ . Denote by  $\underline{\mathbf{B}}_\lambda^{up}$  the inverse image of  $\mathcal{B}_\lambda^{up}$  by the isomorphism  ${}_A V_\lambda^{up} \cap \mathcal{L}_\lambda^{up} \cap \overline{\mathcal{L}_\lambda^{up}} \rightarrow \mathcal{L}_\lambda^{up}/q\mathcal{L}_\lambda^{up}$ . By (42) the set  $\underline{\mathbf{B}}_\lambda^{up}$  is a basis of  ${}_A V_\lambda^{up}$ ; we call it the *upper canonical basis* of  $V_\lambda$ .

Similarly, the right module  $V_\lambda^r$  with its canonical basis  $\underline{\mathbf{B}}_\lambda^r = \{f_\lambda b, b \in \mathbf{B}^+\} \setminus \{0\}$  has the lower crystal base  $(\mathcal{L}_\lambda^{r, low}, \mathcal{B}_\lambda^{r, low})$ , and it supports a balanced triple  $({}_A V_\lambda^{r, up}, \mathcal{L}_\lambda^{r, up}, \overline{\mathcal{L}_\lambda^{r, up}})$  defined again by duality. We denote by  $(\mathcal{L}_\lambda^{r, up}, \mathcal{B}_\lambda^{r, up})$  and  $\underline{\mathbf{B}}_\lambda^{r, up}$  the corresponding crystal base and upper canonical basis of  $V_\lambda^r$  respectively.

It follows that  $({}_A V_\lambda^{r, up} \otimes_A {}_A V_\lambda^{up}, \mathcal{L}_\lambda^{r, up} \otimes_{\mathcal{A}_0} \mathcal{L}_\lambda^{up}, \overline{\mathcal{L}_\lambda^{r, up}} \otimes_{\mathcal{A}_\infty} \overline{\mathcal{L}_\lambda^{up}})$  is a balanced triple; equivalently  $V_\lambda^r \otimes V_\lambda$  with the bimodule structure (50) and the basis  $\underline{\mathbf{B}}_\lambda^{r, up} \otimes \underline{\mathbf{B}}_\lambda^{up}$  is a based  $(U_q^{ad} \otimes (U_q^{ad})^{op})$ -module.

Denote again by  $\langle \cdot, \cdot \rangle : \mathcal{O}_q \times \dot{\mathbf{U}} \rightarrow \mathbb{C}(q)$  the pairing of  $U_q^{ad}$ -bimodules induced by the canonical pairing  $\langle \cdot, \cdot \rangle : \mathcal{O}_q \times U_q^{ad} \rightarrow \mathbb{C}(q)$ , and let  $\Phi_\lambda : V_\lambda^r \otimes V_\lambda \rightarrow \mathcal{O}_q$ ,  $\lambda \in P_+$ , be the ‘‘matrix coefficient’’ map, ie.

$$(54) \quad \langle \Phi_\lambda(f \otimes v), x \rangle = \langle fx, v \rangle_\lambda$$

for every  $f \in V_\lambda^r$ ,  $x \in U_q^{ad}$ ,  $v \in V_\lambda$ . The map  $\Phi := \bigoplus_{\lambda \in P_+} \Phi_\lambda$  is an isomorphism of  $U_q^{ad}$ -bimodules, so let us use it to identify  $\mathcal{O}_q$  with  $\bigoplus_{\lambda \in P_+} V_\lambda^r \otimes V_\lambda$  (which is the content of the Peter-Weyl decomposition (6)). Define

$$\begin{aligned} \mathcal{L}(\mathcal{O}_q) &= \bigoplus_{\lambda \in P_+} \mathcal{L}_\lambda^{r, up} \otimes_{\mathcal{A}_0} \mathcal{L}_\lambda^{up}, \quad \mathcal{B}(\mathcal{O}_q) := \prod_{\lambda \in P_+} \mathcal{B}_\lambda^{r, up} \otimes \mathcal{B}_\lambda^{up} \\ \overline{\mathcal{L}}(\mathcal{O}_q) &= \bigoplus_{\lambda \in P_+} \overline{\mathcal{L}_\lambda^{r, up}} \otimes_{\mathcal{A}_\infty} \overline{\mathcal{L}_\lambda^{up}}, \quad \overline{\mathcal{B}}(\mathcal{O}_q) := \prod_{\lambda \in P_+} \overline{\mathcal{B}_\lambda^{r, up}} \otimes \overline{\mathcal{B}_\lambda^{up}}. \end{aligned}$$

**Theorem 2.6.** (i) The triple  $(\mathcal{O}_A, \mathcal{L}(\mathcal{O}_q), \overline{\mathcal{L}}(\mathcal{O}_q))$  is balanced. Therefore, denoting by  $G$  the inverse of the canonical map  $\mathcal{O}_A \cap \mathcal{L}(\mathcal{O}_q) \cap \overline{\mathcal{L}}(\mathcal{O}_q) \rightarrow \mathcal{L}(\mathcal{O}_q)/q\mathcal{L}(\mathcal{O}_q)$ , we have

$$\mathcal{O}_A = \bigoplus_{b \in \mathcal{B}(\mathcal{O}_q)} A.G(b).$$

(ii) The basis  $G(\mathcal{B}(\mathcal{O}_q)) := \{G(b), b \in \mathcal{B}(\mathcal{O}_q)\}$  coincides with the dual canonical basis  $\dot{\mathbf{B}}^*$ .

The statement (i) is Theorem 1 of [66], and (ii) is Theorem 10.1 and Proposition 10.2.2 of [67]. The basis  $G(\mathcal{B}(\mathcal{O}_q))$  is called the *global basis*, or *canonical basis*, of  $\mathcal{O}_q$ . As an example, in the case  $\mathfrak{g} = sl(2)$  the basis  $G(\mathcal{B}(\mathcal{O}_q))$  is formed by the monomials  $b^n d^m c^l$ , where  $n, m, l \in \mathbb{N}$ , and  $b^n a^m c^l$ , where  $n, l \in \mathbb{N}$  and  $m > 0$  ([66], Proposition 9.1.1, [45], Proposition 1.3). Here  $a, b, c, d$  are the matrix coefficients in the canonical basis  $(v_+, v_- := Fv_+)$  of  $V_1$ , the simple  $U_q^{ad}(sl(2))$ -module of type 1 and dimension two, read from the top left to the bottom right. In that case the upper canonical basis  $\underline{\mathbf{B}}_1^{rup}$  and  $\underline{\mathbf{B}}_1^{up}$  coincide with the lower ones, but this is not true in general, see Example 2.11.

Put  $\underline{\mathbf{B}}_{\lambda^r, \lambda}^{up} = G(\mathcal{B}(\mathcal{O}_q)) \cap (V_{\lambda^r} \otimes V_{\lambda})$ , and consider the partition

$$(55) \quad G(\mathcal{B}(\mathcal{O}_q)) = \coprod_{\lambda \in P_+} \underline{\mathbf{B}}_{\lambda^r, \lambda}^{up}.$$

We have

$$(56) \quad \underline{\mathbf{B}}_{\lambda^r, \lambda}^{up} = \underline{\mathbf{B}}_{\lambda}^{rup} \otimes \underline{\mathbf{B}}_{\lambda}^{up}.$$

Indeed, since the canonical basis are determined by their crystal bases (see the discussion about (43)), and the one associated to  $\underline{\mathbf{B}}_{\lambda}^{rup} \otimes \underline{\mathbf{B}}_{\lambda}^{up}$  is  $\mathcal{B}_{\lambda}^{rup} \otimes \mathcal{B}_{\lambda}^{up}$ , by Theorem 2.6 (ii) it is enough to check that  $\mathcal{B}(\mathcal{O}_q)$  is dual to the crystal base of  $\dot{\mathbf{U}}$  associated to the canonical basis  $\dot{\mathbf{B}}$ . The latter is described in [67] (see also [68]); let us denote it  $(\mathcal{L}(\dot{\mathbf{U}}), \mathcal{B}(\dot{\mathbf{U}}))$ . The proof of Theorem 2.6 (ii) in [67] exhibits an isomorphism of crystals over  $U_q^{ad} \otimes (U_q^{ad})^{op}$ ,

$$(57) \quad \psi: \mathcal{B}(\mathcal{O}_q) \rightarrow \mathcal{B}(\dot{\mathbf{U}}),$$

such that  $\langle G(b), G(b') \rangle = \delta_{\psi(b), b'}$  for every  $b \in \mathcal{B}(\mathcal{O}_q)$ ,  $b' \in \mathcal{B}(\dot{\mathbf{U}})$ . Therefore (56) follows at once.

For every  $\lambda \in P_+$  we will denote

$$(58) \quad {}_A C(\lambda) := \bigoplus_{b \in \underline{\mathbf{B}}_{\lambda^r, \lambda}^{up}} Ab.$$

Thus, Theorem 2.6 (i) identifies the  $U_A^{res}$ -bimodule  $\mathcal{O}_A$  with  $\bigoplus_{\lambda \in P_+} {}_A C(\lambda)$ . We stress that  ${}_A C(\lambda)$  is not spanned over  $A$  by the matrix coefficients of  ${}_A V_{\lambda}$  (see Remark 2.14 (b)).

Let us discuss the dual structure. Consider the pairing  $(V_{\lambda^r} \otimes V_{\lambda}) \times \text{End}(V_{\lambda}) \rightarrow \mathbb{C}(q)$ ,  $(f \otimes v, x) \mapsto f(xv)$ , for every  $f \in V_{\lambda^r}$ ,  $v \in V_{\lambda}$  and  $x \in \text{End}(V_{\lambda})$ . It is non degenerate and compatible with the  $U_q^{ad}$ -bimodule structures, so  $\text{End}(V_{\lambda})$  inherits a lower crystal basis over  $U_q^{ad} \otimes (U_q^{ad})^{op}$  dual to  $(\mathcal{L}_{\lambda}^{rup} \otimes_{A_0} \mathcal{L}_{\lambda}^{up}, \mathcal{B}_{\lambda}^{rup} \otimes \mathcal{B}_{\lambda}^{up})$  ([66], Proposition 3.2.2), which melts into a balanced triple ([66], Lemma 2.2.3). In other words,  $(V_{\lambda^r} \otimes V_{\lambda}, \underline{\mathbf{B}}_{\lambda^r, \lambda}^{up})$  is in duality with a structure of lower based  $(U_q^{ad} \otimes (U_q^{ad})^{op})$ -module on  $\text{End}(V_{\lambda})$ . By the isomorphism (48) and Theorem 2.6 (ii), the basis of this based module is  $\pi_{\lambda}(\dot{\mathbf{B}}[\lambda])$ . On another hand, it is clear that under the standard identification of  $U_q^{ad}$ -bimodules  $\text{End}(V_{\lambda}) = V_{\lambda^r} \otimes V_{\lambda}$  the basis dual



to  $\underline{\mathbf{B}}_{\lambda^r, \lambda}^{up}$  is  $\underline{\mathbf{B}}_{\lambda^r, \lambda} := \underline{\mathbf{B}}_{\lambda}^r \otimes \underline{\mathbf{B}}_{\lambda}$  (ie. the tensor product of lower canonical basis). Hence the  $A$ -sublattice of  $End(V_{\lambda})$  generated by  $\pi_{\lambda}(\dot{\mathbf{B}}[\lambda])$  is just

$$(59) \quad {}_A End(V_{\lambda}) = {}_A V_{\lambda}^r \otimes {}_A V_{\lambda}.$$

Now recall the property (45). Consider in particular the finite subsets  $\dot{\mathbf{B}}_{0, \varpi_i}$  and  $\dot{\mathbf{B}}_{\varpi_i, 0}$  associated to the fundamental weights  $\varpi_i$ ,  $i = 1, \dots, m$ . The map  $u \mapsto u({}^{\omega}v_0 \otimes v_{w_0(\varpi_i)})$ ,  $u \in \dot{\mathbf{U}}$ , allows one to identify  $\dot{\mathbf{B}}_{0, \varpi_i}$  with the canonical basis  $\underline{\mathbf{B}}_{\varpi_i}$  of  ${}^{\omega}V_0 \otimes V_{\varpi_i} \cong V_{\varpi_i}$ , and therefore with a uniquely determined finite subset  $\mathbf{B}_{\varpi_i}$  of the canonical basis  $\mathbf{B}^-$  of  $U_q^{ad}(\mathfrak{n}_-)$ ; similarly, one can identify  $\dot{\mathbf{B}}_{\varpi_i, 0}$  with a uniquely determined finite subset  ${}^{\omega}\mathbf{B}_{\varpi_i}$  of the canonical basis  $\mathbf{B}^+$  of  $U_q^{ad}(\mathfrak{n}_+)$ . The elements of  $\dot{\mathbf{B}}_{0, \varpi_i}$  and  $\dot{\mathbf{B}}_{\varpi_i, 0}$  are respectively of the form  $b^- 1_{\varpi_i}$  and  $b^+ 1_{-\varpi_i}$ , where  $b^- \in \mathbf{B}_{\varpi_i}$  and  $b^+ \in {}^{\omega}\mathbf{B}_{\varpi_i}$ , and we have (see [77], Proposition 3.3 and section 3.4):

**Proposition 2.7.** *The algebra  $\mathcal{O}_A$  is finitely generated. A system of generators is provided by the elements  $a^* \in \dot{\mathbf{B}}^*$ ,  $a \in \cup_{i=1}^m (\dot{\mathbf{B}}_{0, \varpi_i} \cup \dot{\mathbf{B}}_{\varpi_i, 0})$ , defined by  $a^*(a') = \delta_{a, a'}$  for every  $a' \in \dot{\mathbf{B}}$ .*

Note that the above system of generators of  $\mathcal{O}_A$  has  $2 \sum_{i=1}^m \dim(V_{\varpi_i})$  elements. By Theorem 2.6 (ii) and (55) these generators can be identified with elements of some sets  $\underline{\mathbf{B}}_{\lambda^r, \lambda}^{up}$ . In fact, recall that  $\varphi: U_q^{ad} \rightarrow U_q^{ad}$  is the anti-automorphism given by  $\varphi(E_i) = F_i$ ,  $\varphi(F_i) = E_i$ ,  $\varphi(K_{\lambda}) = K_{\lambda}$ , and denote by  $v_{-\varpi_i}$  and  $f_{-\varpi_i}$  the canonical lowest-weight vectors of the highest weight modules  $V_{-w_0(\varpi_i)}$  and  $V_{-w_0(\varpi_i)}^r$  respectively.

For every  $b^- \in \mathbf{B}_{\varpi_i}$  and  $b^+ \in {}^{\omega}\mathbf{B}_{\varpi_i}$  we have

$$(60) \quad (b^- 1_{\varpi_i})^* = \Phi_{\varpi_i}(f_{\varpi_i} \varphi(b^-) \otimes v_{\varpi_i})$$

$$(61) \quad (b^+ 1_{-\varpi_i})^* = \Phi_{-w_0(\varpi_i)}(f_{-\varpi_i} \varphi(b^+) \otimes v_{-\varpi_i})$$

In other words,  $(b^- 1_{\varpi_i})^*$  and  $(b^+ 1_{-\varpi_i})^*$  are the matrix coefficients lying on the first and last columns of the matrix representations in the canonical basis of the spaces  $V_{\varpi_i}$ ,  $i = 1, \dots, m$ . This can be checked by using the isomorphism (57). The key observation is that

$$\langle \Phi_{\lambda}(f_{\lambda} \otimes v_{\lambda}), 1_{\mu} \rangle = \langle f_{\lambda}, 1_{\mu} v_{\lambda} \rangle_{\lambda} = \delta_{\lambda, \mu}$$

for every  $\lambda \in P_+$ ,  $\mu \in P$ , and therefore  $\Phi_{\lambda}(f_{\lambda} \otimes v_{\lambda}) = 1_{\lambda}^*$ . Then the computation proceeds by using the equivariance of  $\Phi$  under the action of  $U_q^{ad} \otimes (U_q^{ad})^{op}$ , the fact that  $\langle \cdot, \cdot \rangle$  dualizes the bimodules structures on  $\mathcal{O}_q$  and  $\dot{\mathbf{U}}$ , and the description of the associated Kashiwara operators on  $V_{\lambda}^r \otimes V_{\lambda}$  and  $\mathcal{B}(\dot{\mathbf{U}})$ .

Though the  $A$ -module  ${}_A V_{\mu} \otimes_A {}_A V_{\nu}$  has no decomposition like (7), we can refine the map  $C(\mu) \otimes C(\nu) \rightarrow C(\mu + \nu)$  in (8) to an  $A$ -linear map defined on  ${}_A C(\mu) \otimes_A {}_A C(\nu)$ . Indeed, there is a unique injective morphism of  $U_A^{res}$ -modules  $\mathfrak{T}_{\mu, \nu}: {}_A V_{\mu + \nu} \rightarrow {}_A V_{\mu} \otimes_A {}_A V_{\nu}$ , which is given by  $\mathfrak{T}_{\mu, \nu}(v_{\mu + \nu}) = v_{\mu} \otimes v_{\nu}$  ([76], Proposition 25.1.2 (a)-(b)). It defines a morphism of based modules

$$(V_{\mu + \nu}, \underline{\mathbf{B}}_{\mu + \nu}) \rightarrow (V_{\mu} \otimes V_{\nu}, \underline{\mathbf{B}}_{\mu} \diamond \underline{\mathbf{B}}_{\nu})$$

where  $\underline{\mathbf{B}}_{\mu} \diamond \underline{\mathbf{B}}_{\nu} := \{b \diamond b', b \in \underline{\mathbf{B}}_{\mu}, b' \in \underline{\mathbf{B}}_{\nu}\}$  ([76], 27.1.7). Therefore  $\mathfrak{T}_{\mu, \nu}$  is a split  $A$ -linear map. Similarly, the unique morphism of  $U_A^{res}$ -modules  ${}^{\omega}\mathfrak{T}_{\mu, \nu}: {}^{\omega}V_{\mu + \nu} \rightarrow {}^{\omega}V_{\mu} \otimes_A {}^{\omega}V_{\nu}$  is a split injection. Define  $\rho_{\mu', \mu''}: \dot{\mathbf{U}}_A \rightarrow {}^{\omega}V_{\mu'} \otimes_A {}_A V_{\mu''}$  by

$$\rho_{\mu', \mu''}(u) = u({}^{\omega}v_{w_0(\mu')} \otimes_A v_{w_0(\mu'')}),$$

and  $\rho_{\mu', \mu'', \nu', \nu''}: \dot{\mathbf{U}}_A^{\otimes 2} \rightarrow {}^{\omega}V_{\mu'} \otimes_A {}_A V_{\mu''} \otimes_A {}^{\omega}V_{\nu'} \otimes_A {}_A V_{\nu''}$  by

$$\rho_{\mu', \mu'', \nu', \nu''}(u) = u({}^{\omega}v_{w_0(\mu')} \otimes_A v_{w_0(\mu'')} \otimes_A {}^{\omega}v_{w_0(\nu')} \otimes_A v_{w_0(\nu'')}).$$

Define  $\tau_{\mu', \mu'', \nu', \nu''} : {}^\omega V_{\mu'+\nu'} \otimes_A {}^\omega V_{\mu''+\nu''} \rightarrow {}^\omega V_{\mu'} \otimes_A {}^\omega V_{\mu''} \otimes_A {}^\omega V_{\nu'} \otimes_A {}^\omega V_{\nu''}$  by

$$\tau_{\mu', \mu'', \nu', \nu''} = (1 \otimes \hat{R}^{-1} \otimes 1)({}^\omega \mathfrak{T}_{\mu', \nu'} \otimes \mathfrak{T}_{\mu'', \nu''}).$$

It is an injective morphism of  $U_A^{res}$ -modules. Denote  $\mu := \mu'', \nu = \nu'' \in P_+$ , and take  $\mu' = -w_0(\mu), \nu' = -w_0(\nu)$ , so that  ${}^\omega V_{\mu'}, {}^\omega V_{\nu'}$  are the left modules associated to  $V_\mu^r, V_\nu^r$  by applying the antiautomorphism  $\varphi$ . By (58) and (59),  ${}_A C(\lambda)$  is the  $U_A^{res}$ -bimodule dual to  ${}_A V_\lambda^r \otimes_A {}_A V_\lambda$ , so the transpose of  $\tau_{\mu, \nu} := \tau_{\mu', \mu', \nu', \nu'}$  is a surjective morphism of  $U_A^{res}$ -modules

$$\tau_{\mu, \nu}^t : {}_A C(\mu) \otimes_A {}_A C(\nu) \rightarrow {}_A C(\mu + \nu).$$

In Section 1.13 of [77], Lusztig proved that  $\tau_{\mu', \mu'', \nu', \nu''}$  is a split  $A$ -linear map ([77] uses  $\hat{R}$  instead of  $\hat{R}^{-1}$ , since our coproducts on  $U_q^{ad}$  are opposite), and that it satisfies

$$(62) \quad \tau_{\mu', \mu'', \nu', \nu''} \rho_{\mu'+\mu'', \nu'+\nu''} = \rho_{\mu', \mu'', \nu', \nu''} \Delta$$

where  $\Delta$  is the coproduct of  $\dot{U}_A$ , see (36). This implies the following statement.

**Proposition 2.8.** *The morphism  $\tau_{\mu, \nu}^t : {}_A C(\mu) \otimes_A {}_A C(\nu) \rightarrow {}_A C(\mu + \nu)$  is split as an  $A$ -linear map, and it coincides with the product in  $\mathcal{O}_A$  followed by the projection onto the component  ${}_A C(\mu + \nu)$ .*

Finally, we consider for any  $n \geq 1$  the invariant elements of  $\mathcal{O}_A^{\otimes n}$  endowed with the action  $coad_n^r$  of  $U_A$ , see (16) (recall that  $\mathcal{L}_{0, n} = \mathcal{O}_q^{\otimes n}$  as  $U_q^{ad}$ -module).

First note that, by definition,  $\mathcal{O}_A(G^n)$  is the restricted dual of the Hopf algebra  $U_A^{res}(\mathfrak{g}^{\oplus n})$ , associated to its category of type 1 modules. By ordering the summands of  $\mathfrak{g}^{\oplus n}$  we get an isomorphism  $U_A^{res}(\mathfrak{g}^{\oplus n}) \cong U_A^{res}(\mathfrak{g})^{\otimes n}$ , and any type 1 simple  $U_A^{res}(\mathfrak{g})^{\otimes n}$ -module is isomorphic to  $V_{[\lambda]} := \otimes_{i=1}^n V_{\lambda_i}$  endowed with the componentwise action, for some  $[\lambda] := (\lambda_1, \dots, \lambda_n) \in P_+^n$  (this is a classical fact; see eg. Theorem 3.10.2 in [51]). Therefore we have an isomorphism  $\mathcal{O}_A(G^n) \cong \mathcal{O}_A^{\otimes n}$ .

Moreover,  $\otimes_{i=1}^n (V_{\lambda_i}^r \otimes V_{\lambda_i}, \underline{\mathbf{B}}_{\lambda_i}^{r \text{ up}} \otimes \underline{\mathbf{B}}_{\lambda_i}^{up})$  is a based  $(U_q^{ad} \otimes (U_q^{ad})^{op})^{\otimes n}$ -module. Let us put

$$(63) \quad \begin{aligned} \underline{\mathbf{B}}_{[\lambda^r, \lambda]}^{up} &:= \otimes_{i=1}^n (\underline{\mathbf{B}}_{\lambda_i}^{r \text{ up}} \otimes \underline{\mathbf{B}}_{\lambda_i}^{up}) \\ {}_A C([\lambda]) &:= \otimes_{i=1}^n {}_A C(\lambda_i) = \bigoplus_{b \in \underline{\mathbf{B}}_{[\lambda^r, \lambda]}^{up}} Ab. \end{aligned}$$

Note that  ${}_A C([\lambda]) \otimes_A \mathbb{C}(q) = C([\lambda])$ , the space defined in (24). We thus obtain a decomposition into based  $(U_q^{ad} \otimes (U_q^{ad})^{op})^{\otimes n}$ -modules

$$(64) \quad (\mathcal{O}_q^{\otimes n}, G(\mathcal{B}(\mathcal{O}_q^{\otimes n}))) = \bigoplus_{[\lambda] \in P_+^n} (C([\lambda]), \underline{\mathbf{B}}_{[\lambda^r, \lambda]}^{up}).$$

Now  $coad_n^r = (coad^r)^{\otimes n} \circ \Delta^{(n-1)}$  and  $coad^r$  give structures of  $U_q^{ad}$ -modules to  $\mathcal{O}_q^{\otimes n}$  and  $C(\lambda)$  respectively. Therefore we give  $C(\lambda)$  the basis  $\underline{\mathbf{B}}_{\lambda_i}^{r \text{ up}} \diamond \underline{\mathbf{B}}_{\lambda_i}^{up}$ , and  $C([\lambda])$  the basis

$$(65) \quad \underline{\mathbf{B}}_{[\lambda]}^{up} := (\underline{\mathbf{B}}_{\lambda_1}^{r \text{ up}} \diamond \underline{\mathbf{B}}_{\lambda_1}^{up}) \diamond \dots \diamond (\underline{\mathbf{B}}_{\lambda_n}^{r \text{ up}} \diamond \underline{\mathbf{B}}_{\lambda_n}^{up}).$$

We thus obtain a decomposition into based  $U_q^{ad}$ -modules

$$(66) \quad \mathcal{O}_q^{\otimes n} = \bigoplus_{[\lambda] \in P_+^n} (C([\lambda]), \underline{\mathbf{B}}_{[\lambda]}^{up}).$$

with underlying  $U_A^{res}$ -module

$$(67) \quad \mathcal{O}_A^{\otimes n} = \bigoplus_{[\lambda] \in P_+^n} AC([\lambda]).$$

Finally we state the last property of based modules we need. Let  $(M, B)$  be a based module. Recall the notations introduced around (46). It is proved in Proposition 27.1.8 of [76] that for every  $\lambda \in P_+$  the submodule  $M[\geq \lambda]$  is a sub-based module of  $M$ , and that it has the basis

$$(68) \quad B \cap M[\geq \lambda] = \cup_{\lambda' \geq \lambda} B[\lambda'].$$

Consider  $M[\neq 0] := \bigoplus_{\lambda \neq 0} M[\lambda]$ , the largest proper submodule of  $M$  that contains no non-zero invariant element. Recall that the space of *coinvariants* of  $M$  is

$$\begin{aligned} M_{U_q^{ad}} &= M/M[\neq 0] \\ &= M/\mathbb{C}(q)\{um - \varepsilon(u)m, m \in M, u \in U_q^{ad}\} \end{aligned}$$

that is, the largest quotient of  $M$  with trivial action, where  $\varepsilon: U_q^{ad} \rightarrow \mathbb{C}(q)$  is the counit. It follows from (68) that  $M[\neq 0]$  is a sub-based module of  $M$ , with the basis  $\cup_{\lambda \neq 0} B[\lambda]$ , and we have (this is [76], Proposition 27.2.6):

**Proposition 2.9.** *The quotient map  $\pi: M \rightarrow M_{U_q^{ad}}$  is a morphism of based modules, where  $M_{U_q^{ad}}$  is endowed with the basis  $B_{U_q^{ad}} := \pi(B[0])$ .*

As a consequence, the transpose map  $\pi^t: M_{U_q^{ad}}^* \rightarrow M^*$  is a monomorphism mapping the dual basis  $B_{U_q^{ad}}^*$  to the subset  $B[0]^*$  of  $B^*$ . The image of  $\pi^t$  is  $(M^*)_{U_q^{ad}}^{U_q^{ad}}$ , the subspace of  $U_q^{ad}$ -invariant elements of  $M^*$ , regarded as a right module in the natural way. Denote by  ${}_A M \subset M$  the  $A$ -module generated by  $B$ , by  ${}_A M^* \subset M^*$  the  $A$ -module generated by  $B^*$ , and set  ${}_A M_{U_A^{res}} = \pi({}_A M)$ . Then  $\pi^t(({}_A M_{U_A^{res}})^*) = ({}_A M^*)_{U_A^{res}}^{U_A^{res}}$  is the  $A$ -submodule of  ${}_A M^*$  generated by  $B[0]^*$ , and therefore

$$(69) \quad {}_A M^* = ({}_A M^*)_{U_A^{res}}^{U_A^{res}} \oplus {}_A N$$

where  ${}_A N \subset {}_A M^*$  is the  $A$ -submodule generated by  $\cup_{\lambda \neq 0} B[\lambda]^*$ . Note that, since  $B[0]$  is in general not invariant under the action of  $U_A^{res}$ ,  ${}_A N$  need not be stable under this action.

We are now ready to draw consequences of this discussion and the previous results. As usual denote by  $(\mathcal{O}_A^{\otimes n})_{U_A^{res}}^{U_A^{res}}$  the subspace of invariant elements of  $\mathcal{O}_A^{\otimes n}$  for the action  $coad_n^r$ . In the case  $n = 1$  it is just the center  $\mathcal{Z}(\mathcal{O}_A)$ .

**Theorem 2.10.**  *$(\mathcal{O}_A^{\otimes n})_{U_A^{res}}^{U_A^{res}}$  is a direct summand of the  $A$ -module  $\mathcal{O}_A^{\otimes n}$  for any  $n \geq 1$ .*

*Proof.* By (67) it is enough to show that for every  $\lambda \in P_+$  the invariant elements of  ${}_A C(\lambda)$  form a direct summand. But this follows by applying Proposition 2.9 and the subsequent comments to the based module  $(M, B) = (\otimes_{i=1}^n End(V_{\lambda_i}), \underline{\mathbf{B}}_{[\lambda]})$  dual to  $(C([\lambda]), \underline{\mathbf{B}}_{[\lambda]}^{up})$ .  $\square$

**Example 2.11.** The simplest case is already instructive. Namely, consider  $V_1$  and  $V_2$ , the simple  $U_q^{ad}(sl(2))$ -modules of type 1 and dimension two and three (we refer to [60] for the computation of many other examples of crystal and global basis).

On  $V_1$  we have the lower canonical basis vectors  $v_+$  and  $v_-$ , such that  $Kv_+ = qv_+$ ,  $Ev_+ = 0$ ,  $v_- = Fv_+$ . The canonical lower and upper basis of  $V_1$  are both  $\{v_+, v_-\}$ . Using the relation (44) we see that the elements of  $\dot{\mathbf{B}}_{0,1}$  and  $\dot{\mathbf{B}}_{1,0}$  are  $1_1$ ,  $F1_1$  and  $1_{-1}$ ,  $E1_{-1}$  respectively; the dual linear forms generate  $\mathcal{O}_A(SL_2)$ , they are the matrix coefficients  $a$ ,  $c$ ,  $d$  and  $b$  respectively. By (49) we have  $\dot{\mathbf{B}}[1] = \dot{\mathbf{B}}_{0,1} \amalg \dot{\mathbf{B}}_{1,0}$ .

Next consider  $V_2$ . On  $V_2$  we have the canonical highest weight vector  $v_+$  of weight 2, and lower canonical basis  $\underline{\mathbf{B}}_2 = \{v_+, v_0, v_-\}$ , where  $v_0 = Fv_+$  and  $v_- = F^{(2)}v_+$ . A direct computation shows that  $\langle v_+, v_+ \rangle_2 = \langle v_-, v_- \rangle_2 = 1$  and  $\langle v_0, v_0 \rangle_2 = [2]_q$ , so the upper canonical basis of  $V_2$  is  $\underline{\mathbf{B}}_2^{up} = \{v_+, [2]_q^{-1}v_0, v_-\}$ . We can identify the ambient space of the right module  $V_2^r$  with that of  $V_2$ ; its highest weight vector is then  $v_+$ , and its canonical lower and upper basis are  $\underline{\mathbf{B}}_2^r = \{v_+, v_0, v_-\}$  and  $\underline{\mathbf{B}}_2^{rup} = \{v_+, [2]_q^{-1}v_0, v_-\}$ . Now recall the pairing (54), the morphism  $\pi_2$  in (48), and the 2-side cell  $\dot{\mathbf{B}}[2]$  in (49). Consider the pairing  $(V_2^r \otimes V_2) \times \text{End}(V_2) \rightarrow \mathbb{C}(q)$ ,  $(f \otimes v, x) \mapsto f(xv)$ . It is immediate that  $\pi_2(\dot{\mathbf{B}}[2])$  is the basis of  $\text{End}(V_2)$  dual by this pairing to the upper canonical basis  $\underline{\mathbf{B}}_{2^r, 2}^{up} = \underline{\mathbf{B}}_2^{rup} \otimes \underline{\mathbf{B}}_2^{up}$  (this equality is (56)). Under the identification  $\text{End}(V_2) = V_2^r \otimes V_2$ , for an element  $x \in \text{End}(V_2)$  of the form  $g \otimes w$  we have  $f(xv) = \langle f, g \rangle_2 \langle w, v \rangle_2$ , hence that dual basis  $\text{End}(V_2)$  is  $\underline{\mathbf{B}}_2^r \otimes \underline{\mathbf{B}}_2$ , as expected from (59).

Consider now the module  ${}^\omega V_1 \otimes V_1$ . We have  $\hat{R} = \sum_{n=0}^{\infty} \frac{(q-q^{-1})^n}{[n]_q!} q^{n(n-1)/2} E^n \otimes F^n$ , so the matrix of the involution  $\Psi = \hat{R}^{-1} \circ^-$  in the basis  $v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-$  is

$$(\hat{R}^{-1} \circ^-)_{\omega V_1, V_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q^{-1} - q & 0 & 0 & 1 \end{pmatrix}.$$

Therefore the canonical basis  $\underline{\mathbf{B}}_{1,1}$  is formed by the vectors  $v_+ \diamond v_+ = v_+ \otimes v_+ + q^{-1}v_- \otimes v_-$  and  $v_+ \diamond v_- = v_+ \otimes v_-, v_- \diamond v_+ = v_- \otimes v_+, v_- \diamond v_- = v_- \otimes v_-$ . Consider the partition  $\underline{\mathbf{B}}_{1,1} = \underline{\mathbf{B}}_{1,1}[2] \cup \underline{\mathbf{B}}_{1,1}[0]$ . We have  $\underline{\mathbf{B}}_{1,1}[2] = \{v_- \diamond v_+, v_+ \diamond v_+, v_+ \diamond v_-\}$ , which is a basis of the three-dimensional submodule of  $V_1 \otimes V_1$ . Since  $\underline{\mathbf{B}}_{1,1}$  is an  $A$ -basis of  ${}^\omega V_1 \otimes_A V_1$ , it follows that the epimorphism  $\tau_{1,1}^t: {}_A C(1) \otimes_A {}_A C(1) \rightarrow {}_A C(2)$  splits (see Proposition 2.8). The vector  $v_- \diamond v_-$  is cyclic, so  $\underline{\mathbf{B}}_{1,1}[0] = \{v_- \diamond v_-\}$ . By the definitions we have  $v_+ \diamond v_+ = (1 \diamond_0 1)_{1,1}$ ,  $v_+ \diamond v_- = (1 \diamond_0 F)_{1,1}$ ,  $v_- \diamond v_+ = (F \diamond_0 1)_{1,1}$ ,  $v_- \diamond v_- = (F \diamond_0 F)_{1,1}$ , so the corresponding elements of  $\dot{\mathbf{B}}_{1,1} \subset \dot{\mathbf{B}}$  are respectively  $1_0, 1_{-2}F, 1_2E$ , and  $F1_2E = E1_{-2}F$ .

The invariant submodule of  ${}^\omega V_1 \otimes V_1$  is generated by  $v' = v_- \otimes v_- - q^{-1}v_+ \otimes v_+$ ; in particular  $v_+ \otimes v_+ = [2]_q^{-1}(qv_+ \diamond v_+ - v')$   $\notin {}_A V_2 \oplus {}_A V_1$ , and therefore  ${}^\omega V_1 \otimes_A V_1 \neq {}_A V_2 \oplus {}_A V_1$  though both sides are equal over  $\mathbb{C}(q)$ . The module of coinvariants is

$$({}^\omega V_1 \otimes V_1)_{U_q^{ad}} = \mathbb{C}(q)\{\pi(v_- \otimes v_-)\},$$

where as usual  $\pi: {}^\omega V_1 \otimes V_1 \rightarrow ({}^\omega V_1 \otimes V_1)_{U_q^{ad}}$  is the quotient map. The transpose map  $\pi^t: ({}^\omega V_1 \otimes V_1)_{U_q^{ad}}^* \rightarrow ({}^\omega V_1 \otimes V_1)^*$  sends  $(v_- \diamond v_-)^*$  to the unique  $U_q^{ad}$ -invariant linear map  $ev_1: {}^\omega V_1 \otimes V_1 \rightarrow \mathbb{C}(q)$  such that  $ev_1(v_- \otimes v_-) = 1$ .

**Remark 2.12.** Let  $(M, B), (M', B')$  be based modules, with tensor product  $(M \otimes M', B_\diamond)$ . By Proposition 2.9 the canonical basis of the space of coinvariants  $(M \otimes M')^*$  is in bijection with a subset  $B_\diamond[0] \subset B_\diamond$ . This subset is described in Proposition 27.3.8 of [76] in terms of  $B$  and  $B'$ . This result can be applied recursively to  $\otimes_{i=1}^n \text{End}(V_{\lambda_i})$ , endowed with the basis  $B_\diamond := \underline{\mathbf{B}}_{[\lambda]}$  dual to  $\underline{\mathbf{B}}_{[\lambda]}^{up}$ , and the action dual to  $\text{coad}_n^r$ . As a result there is a complete (though highly non trivial) characterization of the basis  $B_\diamond[0]^*$  of  $(\mathcal{O}_A^{\otimes n})_{U_A^{res}}^{Ures}$ . Examples can be found in [76], 27.3.10. In the case  $\mathfrak{g} = sl(2)$ , the canonical basis of the dual space  $\text{End}(V_1^{\otimes n})^*$  has been identified in [55] with the canonical basis of the Temperley-Lieb algebra  $TL_n(q)$ .

2.2.3. *Some consequences on  $\mathcal{L}_{0,n}^A$  and  $\mathcal{M}_{0,n}^A$ .* Recall from Section 2.2.1 the definition of the integral forms  $\mathcal{L}_{0,n}^A$  and  $\mathcal{M}_{0,n}^A$ .

**Proposition 2.13.**  $\mathcal{L}_{0,n}^A$  and  $\mathcal{M}_{0,n}^A$  are free  $A$ -modules, and  $\mathcal{M}_{0,n}^A$  is a direct summand of the  $A$ -module  $\mathcal{L}_{0,n}^A$ . Moreover,  $\mathcal{L}_{0,n}^A$  is a finitely generated ring.

*Proof.* Since  $\mathcal{L}_{0,n}^A = \mathcal{O}_A^{\otimes n}$  as  $U_A^{res}$ -modules, by (67) it has the basis  $\cup_{[\lambda] \in P_+^n} \mathbf{B}_{[\lambda]}^{up}$ . Therefore  $\mathcal{L}_{0,n}^A$  is a free  $A$ -module. Since  $A$  is a principal ideal domain,  $\mathcal{M}_{0,n}^A$  is a free  $A$ -submodule. By Theorem 2.10 there is a direct sum decomposition as  $A$ -module

$$(70) \quad \mathcal{L}_{0,n}^A = \mathcal{M}_{0,n}^A \oplus {}_A N,$$

and the proof identifies a basis of  $\mathcal{M}_{0,n}^A$  as a subset of  $\cup_{[\lambda] \in P_+^n} \mathbf{B}_{[\lambda]}^{up}$ .

Next, consider the question of finite generation. By the formula (18) it is enough to verify this for  $\mathcal{L}_{0,1}^A$ , but  $\mathcal{L}_{0,1}^A = \mathcal{O}_A$  as an  $A$ -module, and  $\mathcal{O}_A$  is finitely generated by the matrix coefficients of the fundamental  $U_A^{res}$ -modules  ${}_A V_{\varpi_k}$ ,  $k \in \{1, \dots, m\}$  (see Proposition 2.7). Then our claim follows from the formula inverse to (9), expressing the product  $\star$  in terms of the product of  $\mathcal{L}_{0,1}$ , and the integrality properties of the R-matrix (see in [31] the formula (4.8) and Section 3.3).  $\square$

**Remark 2.14.** (a) As noted in (69), the  $A$ -module  ${}_A N$  in the decomposition (70) is in general not a  $U_A^{res}$ -module. Therefore the  $A$ -linear projection map  $\mathcal{R}_A: \mathcal{L}_{0,n}^A \rightarrow \mathcal{M}_{0,n}^A$  such that  $\text{Ker}(\mathcal{R}_A) = {}_A N$  is not a Reynolds operator, for it does not satisfy the identity  $\mathcal{R}_A(\alpha\beta) = \alpha\mathcal{R}_A(\beta)$  for all  $\alpha \in \mathcal{M}_{0,n}^A$ ,  $\beta \in \mathcal{L}_{0,n}^A$ .

(b) Recall (29). In coherence with (a) above, there is no Haar measure on  $\mathcal{O}_A$ . Indeed, by extending scalars over  $\mathbb{C}(q)$  it should otherwise coincide with the Haar measure  $h: \mathcal{O}_q \rightarrow \mathbb{C}(q)$ , but in the notations of Example 2.11, a straightforward computation shows that the matrix coefficient  $v_0^* \otimes v_0$  of the module  $V_2$  is equal to  $[2]_q bc + 1$  (in particular  $bc$  cannot be expressed as a linear combination over  $A$  of matrix coefficients in the canonical basis, which shows that these cannot form a basis of  $\mathcal{O}_A$ ). Since  $h(v_0^* \otimes v_0) = 0$  we have  $h(bc) = -1/(q + q^{-1})$ , whence  $h$  cannot be defined on  $\mathcal{O}_A$ .

(c) The Haar measure yields a well-defined  $\mathcal{A}_0$ -linear map  $h: \mathcal{L}(\mathcal{O}_q) \rightarrow \mathcal{A}_0$  (and analogously  $\mathcal{A}_0$ -linear and  $\mathcal{A}_\infty$ -linear maps  $h: \mathcal{L}_\diamond(\mathcal{O}_q^{\otimes n}) \rightarrow \mathcal{A}_0$  and  $\bar{h}: \bar{\mathcal{L}}_\diamond(\mathcal{O}_q^{\otimes n}) \rightarrow \mathcal{A}_\infty$  for any  $n \geq 1$ , where  $(\mathcal{L}_\diamond(\mathcal{O}_q^{\otimes n}), \mathcal{B}_{[\lambda]}^{up})$  is the crystal base at  $q = 0$  underlying the based  $U_q^{ad}$ -module (66)). Indeed, by (53) the lattice  $\mathcal{L}_\lambda^{rup} \otimes_{\mathcal{A}_0} \mathcal{L}_\lambda^{up}$  is generated by the matrix coefficients in the canonical basis of  $V_\lambda^r$  and  $V_\lambda$ . Since the normalisation by powers of  $q$  is vacuous on the trivial module  $V_0^* \otimes V_0$ , and  $h$  vanishes on  $V_\lambda^* \otimes V_\lambda$  for  $\lambda \in P_+ \setminus \{0\}$ , the claim follows.

**2.3. Perfect pairings.** We will need to restrict the morphisms  $\Phi^+$ ,  $\Phi^-$  in (5) on the integral forms  $\mathcal{O}_A(B_+)$ ,  $\mathcal{O}_A(B_-)$ . We collect their properties in Theorem 2.15 and the discussion thereafter. In order to state it, we recall first a few facts about  $R$ -matrices and related pairings.

Recall that  $\mathcal{C}_A$  is the category of  $U_A^{res}$ -modules of type 1. In [75, 76] Lusztig proved that  $\mathcal{C}_A \otimes_A \mathbb{C}[q^{\pm 1/D}]$  is braided and ribbon, with braiding given by the collection of endomorphisms

$$R = (R_{V,W})_{V,W \in \text{Ob}(\mathcal{C}_A)}.$$

Actually,  $R_{V,W}$  is represented by a matrix with coefficients in  $q^{\mathbb{Z}/D} \mathbb{C}[q^{\pm 1}]$  on the tensor product of the lower canonical basis of  $V$  and  $W$  (see Corollary 24.1.5 of [76]).

This can be rephrased as follows in Hopf algebra terms. Denote by  $\mathbb{U}_\Gamma$  the categorical completion of  $\Gamma$ , ie. the Hopf algebra of natural transformations  $F_{\mathcal{C}_A} \rightarrow F_{\mathcal{C}_A}$ , where  $F_{\mathcal{C}_A}: \mathcal{C}_A \rightarrow A\text{-Mod}_f$  is the forgetful functor towards the category  $A\text{-Mod}_f$  of finite rank  $A$ -modules. Then  $\mathbb{U}_\Gamma \otimes_A \mathbb{C}[q^{\pm 1/D}]$  is quasi-triangular and ribbon with R-matrix

$$R \in \mathbb{U}_\Gamma^{\hat{\otimes} 2} \otimes_A \mathbb{C}[q^{\pm 1/D}].$$

As in (3), we can write

$$R^\pm = \sum_{(R)} R_{(1)}^\pm \otimes R_{(2)}^\pm.$$

There are pairings of Hopf algebras naturally related to the R-matrix  $R$ , considered as an element of  $\mathbb{U}_q^{\otimes 2}$ . What follows is standard (see eg. [70, 71, 74]), for details we refer to the results 3.73, 3.75, 3.92, 3.106 and 3.107 in [96]:

- There is a unique pairing of Hopf algebras  $\rho: U_q(\mathfrak{b}_-)^{cop} \otimes U_q(\mathfrak{b}_+) \rightarrow \mathbb{C}(q^{1/D})$  such that, for every  $\alpha, \lambda \in P$  and  $l, k \in U_q(\mathfrak{h})$ ,

$$(71) \quad \rho(K_\lambda, K_\alpha) = q^{(\lambda, \alpha)}, \quad \rho(F_i, E_j) = \delta_{i,j}(q_i - q_i^{-1})^{-1}, \quad \rho(l, E_j) = \rho(F_i, k) = 0.$$

- The *Drinfeld pairing*  $\tau: U_q(\mathfrak{b}_+)^{cop} \otimes U_q(\mathfrak{b}_-) \rightarrow \mathbb{C}(q^{1/D})$  is the bilinear map defined by  $\tau(X, Y) = \rho(S(Y), X)$ ; it satisfies

$$(72) \quad \tau(K_\lambda, K_\alpha) = q^{-(\lambda, \alpha)}, \quad \tau(E_j, F_i) = -\delta_{i,j}(q_i - q_i^{-1})^{-1}, \quad \tau(l, F_i) = \tau(E_j, k) = 0.$$

- $\rho$  and  $\tau$  are perfect pairings; this means that they yield *isomorphisms* of Hopf algebras  $i_\pm: U_q(\mathfrak{b}_\pm) \rightarrow \mathcal{O}_q(B_{\mp})_{op}$  (with coefficients *a priori* extended to  $\mathbb{C}(q^{1/D})$ , but see below) defined by, for every  $X \in U_q(\mathfrak{b}_+)$ ,  $Y \in U_q(\mathfrak{b}_-)$ ,

$$\langle i_+(X), Y \rangle = \tau(S(X), Y), \quad \langle i_-(Y), X \rangle = \tau(X, Y).$$

Since  $\mathcal{O}_q(B_{\mp})_{op}$  is equipped with the *inverse* of the antipode of  $\mathcal{O}_q(B_{\mp})$ , which is induced by the antipode  $S_{\mathcal{O}_q}$  of  $\mathcal{O}_q$ , it follows that  $i_\pm \circ S = S_{\mathcal{O}_q}^{-1} \circ i_\pm$ .

- Denote by  $p_\pm: \mathcal{O}_q(G) \rightarrow \mathcal{O}_q(B_\pm)$  the canonical projection map, ie. the Hopf algebra homomorphism dual to the inclusion map  $U_q(\mathfrak{b}_\pm) \hookrightarrow U_q(\mathfrak{g})$ . For every  $\alpha, \beta \in \mathcal{O}_q(G)$  we have

$$(73) \quad \langle \alpha \otimes \beta, R \rangle = \tau(i_+^{-1}(p_-(\beta)), i_-^{-1}(p_+(\alpha))).$$

Note that it is the use of weights  $\alpha, \lambda \in P$  that forces the pairings  $\rho, \tau$  to be defined over  $\mathbb{C}(q^{1/D})$ , instead of  $\mathbb{C}(q)$ . Then, let us consider the restrictions  $\pi_q^+$  of  $\rho$ , and  $\pi_q^-$  of  $\tau$  defined by the formulas (71) and (72), where now  $\alpha \in Q$  and  $k \in U_q^{ad}(\mathfrak{h})$ . They take values in  $\mathbb{C}(q)$ , and define pairings

$$\pi_q^+: U_q(\mathfrak{b}_-)^{cop} \otimes U_q^{ad}(\mathfrak{b}_+) \rightarrow \mathbb{C}(q), \quad \pi_q^-: U_q(\mathfrak{b}_+)^{cop} \otimes U_q^{ad}(\mathfrak{b}_-) \rightarrow \mathbb{C}(q).$$

By the same arguments as for  $\rho$  and  $\tau$  (eg. in [96], Proposition 3.92), it follows that  $\pi_q^\pm$  are perfect pairings. Note also that  $\pi_q^- = \kappa \circ \pi_q^+ \circ (\kappa \otimes \kappa)$ , where  $\kappa: U_q \rightarrow U_q$  is the  $\mathbb{C}$ -linear automorphism extending  $\bar{\cdot}: U_q^{ad} \rightarrow U_q^{ad}$  in Section 2.2.2, so defined by

$$(74) \quad \kappa(E_i) = F_i, \quad \kappa(F_i) = E_i, \quad \kappa(K_\lambda) = K_{-\lambda}, \quad \kappa(q) = q^{-1}.$$

In [45], De Concini-Lyubashenko described integral forms of  $\pi_q^\pm$  as follows. Denote by  $m^*: \mathcal{O}_A \rightarrow \mathcal{O}_A(B_+) \otimes \mathcal{O}_A(B_-)$  the map dual to the multiplication map  $\Gamma(\mathfrak{b}_+) \otimes \Gamma(\mathfrak{b}_-) \rightarrow \Gamma$ , so  $m^* = (p_+ \otimes p_-) \circ \Delta_{\mathcal{O}_A}$ . Let  $U_A(G^*)$  be the smallest  $A$ -subalgebra of  $U_A(\mathfrak{b}_-)^{cop} \otimes U_A(\mathfrak{b}_+)^{cop}$  which contains the elements  $(i = 1, \dots, m)$

$$1 \otimes K_i^{-1} \bar{E}_i, \quad \bar{F}_i K_i \otimes 1, \quad L_i^{\pm 1} \otimes L_i^{\mp 1}.$$

and is stable under the diagonal action of  $\mathcal{B}(\mathfrak{g})$ . The reason for the notation  $U_A(G^*)$  will be explained at the beginning of Section 2.5. Note that  $U_A(G^*)$  is free over  $A$ , a Hopf subalgebra, and that a basis is given by the elements

$$(75) \quad \bar{F}_{\beta_1}^{n_1} \dots \bar{F}_{\beta_N}^{n_N} K_{n_1 \beta_1 + \dots + n_N \beta_N} K_\lambda \otimes K_{-\lambda} K_{-p_1 \beta_1 \dots - p_N \beta_N} \bar{E}_{\beta_1}^{p_1} \dots \bar{E}_{\beta_N}^{p_N}$$

where  $\lambda \in P$  and  $n_1, \dots, n_N, p_1, \dots, p_N \in \mathbb{N}$ .



Now, let  $v$  be a lowest weight vector of the lowest weight  $\Gamma$ -module  ${}_A V_{-\lambda}$ ,  $\lambda \in P_+$ . As after Theorem 2.1, denote by  $v^* \in {}_A V_{-\lambda}^*$  the dual vector, and by  $\psi_{-\lambda} \in \mathcal{O}_A$  the matrix coefficient defined by  $\langle \psi_{-\lambda}, x \rangle = v^*(xv)$  for every  $x \in \Gamma$ . Consider the maps  $j_q^\pm : \mathcal{O}_q(B_\pm) \rightarrow U_q(\mathfrak{b}_\mp)^{cop}$  defined by

$$\langle \alpha_+, X \rangle = \pi_q^+(j_q^+(\alpha_+), X), \quad \langle \alpha_-, Y \rangle = \pi_q^-(j_q^-(\alpha_-), Y)$$

where  $\alpha_\pm \in \mathcal{O}_q(B_\pm)$ ,  $X \in U_q^{ad}(\mathfrak{b}_+)$ , and  $Y \in U_q^{ad}(\mathfrak{b}_-)$ .

The following theorem summarizes results proved in the sections 3 and 4 of [45]. Denote by  $\mathcal{O}_A[\psi_{-\rho}^{-1}]$  the localization of  $\mathcal{O}_A$  by the element  $\psi_{-\rho}$ ; this localization is well-defined, for the set  $\{\psi_{-\rho}^n\}_{n \in \mathbb{N}}$  is a left and right multiplicative Ore subset of  $\mathcal{O}_A$  (see Corollary 2.18 below for an analogous statement for  $\mathcal{L}_{0,1}^A$ ). For the sake of clarity let us spell out the correspondence of notations between statements:  $\pi_q^+$ ,  $\pi_q^-$ ,  $U_q(\mathfrak{b}_\mp)^{cop}$ ,  $U_A(\mathfrak{b}_\mp)^{cop}$ ,  $\mathcal{O}_A(B_\pm)$ ,  $U_A(G^*)$  and  $\Phi$  are denoted in [45] respectively by  $\pi''$ ,  $\bar{\pi}''$ ,  $U_q(\mathfrak{b}_\mp)_{op}$ ,  $R_q[B_\pm]''$ ,  $R_q[B_\pm]$ ,  $A''$  and  $\mu''$  (the definition of  $j_A^\pm$  is implicit in the section 4.2 of [45]).

**Theorem 2.15.** (1)  $\pi_q^\pm$  restricts to a perfect Hopf pairing between the unrestricted and restricted integral forms,  $\pi_A^\pm : U_A(\mathfrak{b}_\mp)^{cop} \otimes \Gamma(\mathfrak{b}_\pm) \rightarrow A$ .

(2)  $j_q^\pm$  yields an isomorphism of Hopf algebras  $j_A^\pm : \mathcal{O}_A(B_\pm) \rightarrow U_A(\mathfrak{b}_\mp)^{cop}$ , satisfying  $\langle \alpha_\pm, x_\pm \rangle = \pi_A^\pm(j_A^\pm(\alpha_\pm), x_\pm)$  for every  $\alpha_\pm \in \mathcal{O}_A(B_\pm)$ ,  $x_\pm \in \Gamma(\mathfrak{b}_\pm)$ .

(3) The map  $\Phi := (j_A^+ \otimes j_A^-) \circ m^* : \mathcal{O}_A \rightarrow U_A(G^*) \subset U_A(\mathfrak{b}_-)^{cop} \otimes U_A(\mathfrak{b}_+)^{cop}$  is an embedding of Hopf algebras, and it extends to an isomorphism  $\Phi : \mathcal{O}_A[\psi_{-\rho}^{-1}] \rightarrow U_A(G^*)$ .

For our purposes it is necessary to reformulate this result. Consider the morphisms of Hopf algebras  $\Phi^\pm : \mathcal{O}_A(B_\pm) \rightarrow U_A(\mathfrak{b}_\mp)^{cop}$ ,  $\alpha \mapsto (\alpha \otimes id)(R^\pm)$ .

**Lemma 2.16.** We have  $\Phi^\pm = j_A^\pm$ .

*Proof.* By definitions, for every  $X \in U_q(\mathfrak{b}_+)^{cop}$ ,  $Y \in U_q^{ad}(\mathfrak{b}_-)$  we have  $\langle i_+(S^{-1}(X)), Y \rangle = \pi_q^-(X, Y)$ , and similarly for every  $X \in U_q^{ad}(\mathfrak{b}_+)$ ,  $Y \in U_q(\mathfrak{b}_-)^{cop}$  we have  $\langle i_-(S^{-1}(Y)), X \rangle = \pi_q^+(Y, X)$ . By keeping these notations for  $X$  and  $Y$ , we deduce  $j_q^-(i_+(S^{-1}(X))) = X$  and  $j_q^+(i_-(S^{-1}(Y))) = Y$ , ie.

$$(76) \quad j_q^\pm = S \circ i_\mp^{-1}.$$

Because  $S_{\mathcal{O}_q}^{-1} \circ i_\pm = i_\pm \circ S$ , it follows that

$$(77) \quad j_q^\pm \circ S_{\mathcal{O}_q} = S^{-1} \circ j_q^\pm.$$

Also, for every  $\alpha_- \in \mathcal{O}_q(B_-)$  we have

$$\langle \alpha_-, \Phi^+(i_-(Y)) \rangle = \langle i_-(Y) \otimes \alpha_-, R \rangle = \tau(i_+^{-1}(\alpha_-), Y) = \pi_q^-(j_q^-(S_{\mathcal{O}_q}(\alpha_-)), Y) = \langle \alpha_-, S(Y) \rangle$$

where the first equality is by definition of  $\Phi^+$  (see (5)), the second is (73), the third follows from (77), and the last from the definition of  $j_q^-$ . Similarly, for every  $\alpha_+ \in \mathcal{O}_q(B_+)$  we have

$$\begin{aligned} \langle \alpha_+, \Phi^-(i_+(X)) \rangle &= \langle i_+(X) \otimes \alpha_+, R^- \rangle \\ &= \langle \alpha_+ \otimes S_{\mathcal{O}_q}^{-1} \circ i_+(X), R \rangle \\ &= \langle \alpha_+ \otimes i_+(S(X)), R \rangle \\ &= \tau(S(X), i_+^{-1}(\alpha_+)) \\ &= \pi_q^+(S(i_+^{-1}(\alpha_+)), S(X)) = \pi_q^+(j_q^+(\alpha_+), S(X)) = \langle \alpha_+, S(X) \rangle. \end{aligned}$$

These computations imply  $\Phi^\pm = S \circ i_\mp^{-1} = j_q^\pm$ , and the result follows by taking integral forms.

□

**2.4. Integral form, specialization and localization of  $\Phi_n$ .** Recall the isomorphism of  $U_q$ -module algebras  $\Phi_1: \mathcal{L}_{0,1} \rightarrow U_q^{lf}$ , and that  $U_A^{lf} = U_A \cap U_q^{lf}$ . We have:

**Corollary 2.17.** *The map  $\Phi_1$  affords an embedding of  $U_A^{res}$ -module algebras  $\Phi_1: \mathcal{L}_{0,1}^A \rightarrow U_A^{lf}$ .*

*Proof.* The only thing to be proved is that  $\Phi_1(\mathcal{O}_A) \subset U_A^{lf}$ , since  $\mathcal{L}_{0,1}^A = \mathcal{O}_A$  as  $A$ -module. But Lemma 2.16 and (12) imply  $\Phi_1 = m \circ (id \otimes S^{-1}) \circ \Phi$ , and  $\Phi$  maps  $\mathcal{O}_A$  into  $U_A(\mathfrak{b}_-)^{cop} \otimes U_A(\mathfrak{b}_+)^{cop}$  by Theorem 2.15. The conclusion follows.  $\square$

Let us denote

$$d = \psi_{-\rho} \in \mathcal{L}_{0,1}^A.$$

(The linear forms  $\psi_{-\lambda}$  have been introduced before Theorem 2.15.) When  $\mathfrak{g} = sl(2)$  the element  $d$  is one of the ‘‘standard’’ generators of  $\mathcal{L}_{0,1}(sl(2))$  (see (104) below). In this case we have shown in Lemma 5.7 of [31] that  $\mathcal{L}_{0,1}^A$  has a well-defined localization  $\mathcal{L}_{0,1}^A[d^{-1}]$ , and that  $\Phi_1: \mathcal{L}_{0,1}^A[d^{-1}] \rightarrow U_A^{ad} = T_{2-}^{-1}U_A^{lf}$  is an isomorphism of algebras. A generalization of these facts to any  $\mathfrak{g}$  is provided by the following statement. As usual  $\ell = K_{2\rho}$ , the pivotal element.

**Corollary 2.18.** (1) *The set  $\{d^n\}_{n \in \mathbb{N}}$  is a left and right multiplicative Ore set in  $\mathcal{L}_{0,1}^A$ . We can therefore define the localization  $\mathcal{L}_{0,1}^A[d^{-1}]$ .*

(2)  *$\Phi_1$  extends to an embedding of  $U_A^{res}$ -module algebras  $\Phi_1: \mathcal{L}_{0,1}^A[d^{-1}] \rightarrow U_A^{lf}[\ell]$ , and  $U_A^{lf}[\ell] = T_{2-}^{-1}U_A^{lf}$ .*

*Proof.* (1) Because  $\mathcal{L}_{0,1}^A$  has no non-trivial zero divisors,  $d$  is a regular element. We have to show that for all  $x \in \mathcal{L}_{0,1}^A$  there exists elements  $y, y' \in \mathcal{L}_{0,1}^A$  and  $d', d'' \in \{d^n\}_{n \in \mathbb{N}}$  such that  $xd' = dy$  and  $d''x = y'd$ . In fact  $d' = d'' = d$  in the present situation. Indeed by (14) we have  $\Phi_1(x)\Phi_1(d) = \Phi_1(x)K_{-2\rho} = K_{-2\rho}ad^r(K_{2\rho})(\Phi_1(x))$ , and  $ad^r(K_{2\rho})(\Phi_1(x)) = \Phi_1(\text{coad}^r(K_{2\rho})(x))$ . Therefore the left Ore condition is satisfied with  $y = \text{coad}^r(K_{2\rho})(x)$ . Similarly one finds  $y'$ .

(2) The first claim follows immediately from Corollary 2.17 and  $\Phi_1(d) = \ell^{-1}$ , which is a regular element of  $U_A$ . For the second claim, since  $K_{-2\rho} = \prod_{j=1}^m L_j^{-2}$ , localizing in  $d$  we obtain  $L_j^2 = \prod_{k \neq j} L_k^{-2} \Phi_1(d^{-1}) = \Phi_1(\prod_{k \neq j} \psi_{-\varpi_k} d^{-1}) \in \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$ . Therefore  $T_{2-}^{-1} \subset \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$ , which implies the assertion (2).  $\square$

We expect that the inclusion  $\Phi_1(\mathcal{O}_A) \subset U_A^{lf}$  is an equality, but have no proof yet. However, recall Joseph-Letzter’s Theorem 2.1 (1) and (2).

**Proposition 2.19.** *We have  $U_A = T_{2-}^{-1}U_A^{lf}[T/T_2] = \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2]$ , and therefore  $\Phi_1: \mathcal{L}_{0,1}^A[d^{-1}] \rightarrow T_{2-}^{-1}U_A^{lf}$  is an isomorphism. Moreover  $\Phi_1(\mathcal{O}_A) = \bigoplus_{\lambda \in 2P_+} ad^r(U_A^{res})(K_{-\lambda})$ .*

*Proof.* The inclusions  $T \subset U_A$ ,  $U_A^{lf} \subset U_A$  and  $\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}]) \subset T_{2-}^{-1}U_A^{lf}$  imply

$$\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2] \subset T_{2-}^{-1}U_A^{lf}[T/T_2] \subset U_A.$$

For the inverse inclusion, it is enough to show that any PBW basis vector of  $U_A$  lies in  $\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2]$ . This will follow at once if this is true of all root vectors  $\bar{E}_{\beta_k}, \bar{F}_{\beta_k}$ . Let us show this explicitly for the simple root vectors  $\bar{E}_i$  and  $\bar{F}_i$ . For every positive root  $\alpha$  define elements  $\psi_{-\lambda}^\alpha, \psi_{-\lambda}^{-\alpha} \in \mathcal{O}_A$  by the formulas, where  $x \in \Gamma$ :

$$\langle \psi_{-\lambda}^\alpha, x \rangle = v^*(xE_\alpha v), \quad \langle \psi_{-\lambda}^{-\alpha}, x \rangle = v^*(F_\alpha x v).$$

It is shown in [45], Lemma 4.5, that

$$\Phi(\psi_{-\lambda}) = K_{-\lambda} \otimes K_\lambda, \quad \Phi(\psi_{-\varpi_j}^{\alpha_i}) = -\delta_{i,j} q_i L_i^{-1} \otimes L_i K_i^{-1} \bar{E}_i, \quad \Phi(\psi_{-\varpi_j}^{-\alpha_i}) = \delta_{i,j} q_i^{-1} \bar{F}_i K_i L_i^{-1} \otimes L_i.$$

(Note that the generators denoted by  $E_i$  and  $F_i$  in [45] are respectively  $K_i^{-1}E_i$  and  $F_iK_i$  in our notations, which explains the factors  $q_i, q_i^{-1}$  in the formulas below; also  $\kappa$  in (74) maps  $\bar{E}_i, \bar{F}_i$  to  $-\bar{F}_i, -\bar{E}_i$ , whence the sign for the expression of  $\Phi(\psi_{-\varpi_j}^{\alpha_i})$ ). Since  $\Phi_1 = m \circ (id \otimes S^{-1}) \circ \Phi$ , we have

$$(78) \quad \Phi_1(\psi_{-\lambda}) = K_{-2\lambda}, \quad \Phi_1(\psi_{-\varpi_j}^{\alpha_i}) = \delta_{i,j} L_i^{-2} \bar{E}_i, \quad \Phi_1(\psi_{-\varpi_j}^{-\alpha_i}) = \delta_{i,j} q_i^{-1} \bar{F}_i K_i L_i^{-2}.$$

Therefore  $\bar{E}_i, \bar{F}_i, L_i^{\pm 1} \in T_{2-}^{-1} \Phi_1(\mathcal{L}_{0,1}^A)[T/T_2] = \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2]$ . These elements do not generate  $U_A$ ; it is necessary to consider general root vectors. By the stability of  $U_A(G^*)$  under  $\mathcal{B}(\mathfrak{g})$  and the isomorphism  $\mathcal{O}_A[\psi_{-\rho}^{-1}] \rightarrow U_A(G^*)$  of Theorem 2.15 (3), for every positive root  $\beta_k$  we have  $1 \otimes K_{\beta_k}^{-1} \bar{E}_{\beta_k}, \bar{F}_{\beta_k} K_{\beta_k} \otimes 1 \in \Phi(\mathcal{O}_A[\psi_{-\rho}^{-1}]) = \Phi(\mathcal{L}_{0,1}^A[d^{-1}])$ . Therefore  $\bar{F}_{\beta_k} K_{\beta_k}, S^{-1}(\bar{E}_{\beta_k}) K_{\beta_k} \in \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$ , and  $\bar{F}_{\beta_k}, S^{-1}(\bar{E}_{\beta_k}) \in \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2]$ . The sets  $S^{-1}(\bar{E}_{\beta_k}) U_A(\mathfrak{h})$  generate the subalgebra  $U_A(\mathfrak{b}_+)$  of  $U_A$  (in fact, let us quote that a formula of  $S^{-1}(\bar{E}_{\beta_k})$  is given in [98]). From the triangular decomposition  $U_A = U_A(\mathfrak{n}_-) U_A(\mathfrak{h}) U_A(\mathfrak{n}_+)$ , the inclusion  $U_A \subset \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2]$  follows, whence the equality too. In particular,  $U_A$  is a free  $\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$ -module with a basis formed by representatives of the cosets in  $T/T_2$ . By the uniqueness of this free decomposition, we find  $\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}]) = T_{2-}^{-1} U_A^{lf}$ . Therefore  $\Phi_1$  in Corollary 2.18 (2) is surjective.

For the third claim, recall the isomorphism  $\Phi_1: C(-w_0(\mu)) \rightarrow ad^r(U_q)(K_{-2\mu})$  (see (15)), and that  $\psi_{-\mu}$  is the matrix coefficient dual to the vector  ${}^\omega v_{-\mu} \otimes v_{-\mu} \in \text{End}_A(V_{-w_0(\mu)})$ . This vector is cyclic by (44), so by equivariance  $\Phi_1: {}_A C(-w_0(\mu)) \rightarrow ad^r(U_A^{res})(K_{-2\mu})$  is an isomorphism of  $U_A^{res}$ -modules. The second claim follows from this and (67) for  $n = 1$ .  $\square$

Recall from (22) the isomorphisms of  $U_q$ -module algebras  $\Phi_n: \mathcal{L}_{0,n} \rightarrow (U_q^{\otimes n})^{lf}$  and of algebras  $\Phi_n: \mathcal{M}_{0,n} \rightarrow (U_q^{\otimes n})^{U_q}$ , and from (35) the notations for specializations. Corollary 2.17 can be extended to  $\Phi_n$  as follows:

**Corollary 2.20.** *The map  $\Phi_n$  affords embeddings of module algebras  $\Phi_n: \mathcal{L}_{0,n}^A \rightarrow (U_A^{\otimes n})^{lf}$  and  $\Phi_n: \mathcal{L}_{0,n}^{\epsilon'} \rightarrow (U^{\otimes n})_{\epsilon'}^{lf}$ ,  $q = \epsilon' \in \mathbb{C}^\times$ .*

*Proof.* For the first claim the only thing to prove is the inclusion  $\Phi_n(\mathcal{L}_{0,n}^A) \subset U_A^{\otimes n}$ . It follows from Corollary 2.17 and the expression of  $\Phi_n$  in terms of  $\Phi_1$  and  $R$ -matrices (in particular, the fact that they preserve integrality, see [31], Lemma 6.10). For the specialization at  $q = \epsilon' \in \mathbb{C}^\times$ , we have to justify that  $\Phi_n$  is injective. One uses the fact, to be developed in Theorem 2.24 below, that  $\Phi: \mathcal{O}_\epsilon \rightarrow U_\epsilon(G^*)$  is an embedding. The algebra  $U_\epsilon(G^*)$  has the basis elements (75), and the map  $m \circ (id \otimes S^{-1})$  sends this basis to a free family of  $U_\epsilon$ . Therefore  $\Phi_1: \mathcal{L}_{0,1}^\epsilon \rightarrow U_\epsilon$  is injective. Since  $\Phi_n$  differs from  $\Phi_1^{\otimes n}$  by a linear isomorphism (induced by the conjugation action of  $R$ -matrices on the components  ${}_A C([\lambda])$  of  $\mathcal{L}_{0,n}^A$  in (67), see equation (6.10) in [31]),  $\Phi_n: \mathcal{L}_{0,n}^\epsilon \rightarrow U_\epsilon^{\otimes n}$  is an embedding as well.  $\square$

**Remark 2.21.** (1) It is a natural problem to determine the image of  $\Phi_n$ . One may expect that it would be  $(T_{2-}^{-1} U_A^{lf})^{\otimes n}$ , because this is true for  $n = 1$ , as well as for any  $n$  in the  $sl(2)$  case, as shown in [31]. Unfortunately this is not so. This comes from the fact, eg. for  $n = 2$ , that the matrix elements of  $R_{02} R_{01} R'_{01} R_{02}^{-1}$  do not belong to  $(T_{2-}^{-1} U_A^{lf})^{\otimes 2}$  as can be shown by an explicit computation in the  $sl(3)$  case.

(2) In the case of  $\mathfrak{g} = sl(2)$  we defined in [31] an algebra  $loc \mathcal{L}_{0,n}^A$  generalizing  $\mathcal{L}_{0,1}^A[d^{-1}]$  above, containing  $\mathcal{L}_{0,n}^A$  as a subalgebra, and such that  $\Phi_n$  extends to  $loc \mathcal{L}_{0,n}^A$  and yields an isomorphism  $\Phi_n: loc \mathcal{L}_{0,n}^A \rightarrow U_A^{ad}(sl(2))^{\otimes n}$ . The definition of  $loc \mathcal{L}_{0,n}^A$  involves elements  $\xi^{(i)} \in$

$\mathcal{L}_{0,n}^A$  ( $i = 1, \dots, n$ ) such that  $\Phi_n(\xi^{(i)}) = (K^{-1})^{(i)} \dots (K^{-1})^{(n)}$ . It may be of interest to study a similar extension of  $\Phi_n$  for general  $\mathfrak{g}$ .

**2.5. Structure theorems for  $U_\epsilon$  and  $\mathcal{O}_\epsilon$ .** As usual we denote by  $\epsilon$  a primitive  $l$ -th root of unity, where  $l$  is odd, and coprime to 3 if  $\mathfrak{g}$  has  $G_2$ -components.

Recall the subgroups  $T_G$ ,  $U_\pm$  and  $B_\pm$  of  $G$ . Let  $G^0 = B_+B_-$  (the *big cell* of  $G$ ), and define the subgroup

$$G^* = \{(u_+t, u_-t^{-1}), t \in T_G, u_\pm \in U_\pm\} \subset B_+^{op} \times B_-^{op}$$

where  $B_\pm^{op}$  is the group  $B_\pm$  with opposite multiplication. The group  $G^*$  can be naturally identified with the Poisson-Lie dual of  $G$  with its standard structure.

Recall also that there is an injective homomorphism  $\gamma_q^{-1} \circ h_q: \mathcal{Z}(U_q) \rightarrow U_q(\mathfrak{h})$ , defined by means of the quantum Harish-Chandra homomorphism (see eg. [38], section 9.1.C, or [96], Section 3.13). The image of  $\gamma_q^{-1} \circ h_q$  is the set  $U_q(\mathfrak{h})^{\tilde{W}}$  of invariant elements under  $\tilde{W}$ , the subgroup of  $W \times P_2^*$  generated by the conjugates  $\sigma W \sigma$  of  $W$  by elements  $\sigma \in P_2^*$ . Here,  $P_2^*$  is the group of homomorphisms  $P \rightarrow \mathbb{Z}/2\mathbb{Z}$ , and the semidirect product  $W \times P_2^*$  acts on  $U_q(\mathfrak{h})$  by the standard action of the Weyl group  $W$ , and by the action of  $P_2^*$  given by  $\sigma \cdot K_\lambda := \sigma(\lambda)K_\lambda$ .

It was shown in [43], section 21.1, that the inverse map  $h_q^{-1} \circ \gamma_q: U_q(\mathfrak{h})^{\tilde{W}} \rightarrow \mathcal{Z}(U_q)$  induces by specializing  $q$  to  $\epsilon$  a well-defined injection

$$U_\epsilon(\mathfrak{h})^{\tilde{W}} \rightarrow \mathcal{Z}(U_\epsilon).$$

We denote its image by  $\mathcal{Z}_1(U_\epsilon)$ . For instance, when  $U_\epsilon = U_\epsilon(sl(2))$ ,  $\mathcal{Z}_1(U_\epsilon)$  is the polynomial algebra generated by the Casimir element  $\Omega = (\epsilon - \epsilon^{-1})^2 FE + \epsilon K + \epsilon^{-1} K^{-1}$ .

Denote by  $\mathcal{Z}_0(U_\epsilon) \subset U_\epsilon$  the smallest subalgebra containing the elements  $E_i^l, F_i^l, K_\alpha^l$ , for  $i \in \{1, \dots, m\}$ ,  $\alpha \in P$ , and stable under  $\mathcal{B}(\mathfrak{g})$ ; it is also the subalgebra generated by  $E_{\beta_k}^l, F_{\beta_k}^l, L_i^{\pm l}$ , for  $k \in \{1, \dots, N\}$  and  $i \in \{1, \dots, m\}$  ([43], Section 18). We will denote by  $\mathcal{Z}_0(U_\epsilon(\mathfrak{n}_-))$ ,  $\mathcal{Z}_0(U_\epsilon(\mathfrak{h}))$  and  $\mathcal{Z}_0(U_\epsilon(\mathfrak{n}_+))$  the subalgebras of  $\mathcal{Z}_0(U_\epsilon)$  generated by the elements  $F_{\beta_k}^l, K_\lambda^l$  ( $\lambda \in P$ ), and  $E_{\beta_k}^l$  respectively. In [41], Section 1.8-3.3-3.8, and [43], Theorem 14.1 and Section 20-21, the following results are proved:

**Theorem 2.22.** (1)  $U_\epsilon$  has no non-trivial zero divisors,  $\mathcal{Z}_0(U_\epsilon)$  is a central Hopf subalgebra of  $U_\epsilon$ , and  $U_\epsilon$  is a free  $\mathcal{Z}_0(U_\epsilon)$ -module of rank  $l^{dim \mathfrak{g}}$ . Moreover the classical fraction algebra  $Q(U_\epsilon) = Q(\mathcal{Z}(U_\epsilon)) \otimes_{\mathcal{Z}(U_\epsilon)} U_\epsilon$  is a central simple algebra of PI degree  $l^N$ , and  $U_\epsilon$  is a maximal order of  $Q(U_\epsilon)$ .

(2)  $\text{Maxspec}(\mathcal{Z}_0(U_\epsilon))$  is a group isomorphic to  $G^*$  above, and the multiplication map yields an isomorphism  $\mathcal{Z}_0(U_\epsilon) \otimes_{\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon)} \mathcal{Z}_1(U_\epsilon) \rightarrow \mathcal{Z}(U_\epsilon)$ .

By this theorem the dimension of  $Q(U_\epsilon)$  over its center  $Q(\mathcal{Z}(U_\epsilon))$  is  $l^{2N}$ , and its dimension over  $Q(\mathcal{Z}_0(U_\epsilon))$  is  $l^{dim \mathfrak{g}} = l^{m+2N}$ . Therefore the field  $Q(\mathcal{Z}(U_\epsilon))$  is an extension of  $Q(\mathcal{Z}_0(U_\epsilon))$  of degree  $l^m$ .

Note that, because  $\mathcal{Z}_0(U_\epsilon)$  is an affine and commutative algebra, the maximal spectrum  $\text{Maxspec}(\mathcal{Z}_0(U_\epsilon))$ , viewed as the set of characters of  $\mathcal{Z}_0(U_\epsilon)$ , acquires by duality a structure of affine algebraic group. Thus, the first claim of (2) in the theorem means precisely that this group can be identified with  $G^*$ . See for instance Section 7.2.1 of [31] for an explicit description in the  $sl(2)$  case.

In addition,  $\text{Maxspec}(\mathcal{Z}_0(U_\epsilon))$  and  $G^*$  have natural Poisson structures which correspond one to the other under the isomorphism of (2), and we have the following identifications (see [43], Section 21.2). The dual isomorphism  $\mathcal{O}(G^*) \rightarrow \mathcal{Z}_0(U_\epsilon)$  identifies  $\mathcal{O}(T_G)$  with  $\mathcal{Z}_0(U_\epsilon) \cap U_\epsilon(\mathfrak{h}) = \mathbb{C}[lP]$ , where as usual  $U_\epsilon(\mathfrak{h}) = U_A(\mathfrak{h}) \otimes_A \mathbb{C}_\epsilon$ . Therefore we can identify  $\mathbb{C}[P]$

with  $\mathcal{O}(\tilde{T}_G)$ , the coordinate ring of the  $l^m$ -fold covering space  $\tilde{T}_G \rightarrow T_G$ . The quantum Harish-Chandra isomorphism identifies  $\mathcal{Z}_1(U_\epsilon)$  with  $\mathbb{C}[2P]^W \cong \mathcal{O}(\tilde{T}_G/(2))^W$ , where we denote by (2) the subgroup of 2-torsion elements in  $\tilde{T}_G$ . Consider the map

$$\begin{aligned} \sigma : B_+ \times B_- &\longrightarrow G^0 \\ (b_+, b_-) &\longmapsto b_+ b_-^{-1}. \end{aligned}$$

The restriction of  $\sigma$  to  $G^*$  is an unramified covering map of degree  $2^m$ . Composing  $\sigma: G^* \rightarrow G^0$  with the quotient map under conjugation,  $G^0 \hookrightarrow G \rightarrow G//G$ , we get dually an embedding of  $\mathcal{O}(G//G) = \mathcal{O}(G)^G$  in  $\mathcal{O}(G^*)$ . Collecting these observations, we see that the isomorphism of Theorem 2.22 (2) affords identifications

$$\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon) \cong \mathcal{O}(G)^G$$

as a subalgebra of  $\mathcal{Z}_0(U_\epsilon) \cong \mathcal{O}(G^*)$ , and

$$\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon) = \mathbb{C}[2lP]^W \cong \mathcal{O}(\tilde{T}_G/(2l))^W \cong \mathcal{O}(T_G/(2))^W$$

as a subalgebra of  $\mathcal{Z}_1(U_\epsilon) \cong \mathcal{O}(\tilde{T}_G/(2))^W$ .

We will use the following obvious though crucial fact.

**Lemma 2.23.** *For every  $U_A^{res}$ -module  $V$  of type 1, the action of  $\mathcal{Z}_0(U_\epsilon)$  on the specialization  $V_\epsilon := V \otimes_A \mathbb{C}_\epsilon$  is trivial.*

*Proof.* This comes from  $E_i^l = [l]!E_i^{(l)}$ ,  $F_i^l = [l]!F_i^{(l)}$  and the fact that  $K_i$  acts on  $V$  by powers of  $q_i$ . Specializing to  $q = \epsilon$  ends the proof.  $\square$

A result similar to Theorem 2.22 holds true for  $\mathcal{O}_\epsilon$ . Namely, take the specializations at  $q = \epsilon$  in Theorem 2.15. Denote by  $\mathcal{Z}_0(U_\epsilon(G^*))$  the subalgebra of  $U_\epsilon(G^*)$  generated by the elements  $(k \in \{1, \dots, N\}, i \in \{1, \dots, m\})$

$$1 \otimes K_{-l\beta_k} E_{\beta_k}^l, F_{\beta_k}^l K_{l\beta_k} \otimes 1, L_i^{\pm l} \otimes L_i^{\mp l}.$$

It is a central Hopf subalgebra. Recall that the coordinate ring  $\mathcal{O}(G)$  can be identified as a Hopf subalgebra with  $U(\mathfrak{g})^\circ$ , the restricted dual of the envelopping algebra  $U(\mathfrak{g})$  over  $\mathbb{C}$ . In [45], Section 6, De Concini-Lyubashenko introduced an epimorphism of Hopf algebras  $\eta: \Gamma_\epsilon \rightarrow U(\mathfrak{g})$  (essentially a version of Lusztig's ‘‘Frobenius’’ epimorphism in [75]), defined by

$$(79) \quad \eta(E_i^{(p)}) = \begin{cases} \frac{e_i^{p/l}}{(p/l)!} & \text{if } l \text{ divides } p \\ 0 & \text{otherwise} \end{cases}, \quad \eta(F_i^{(p)}) = \begin{cases} \frac{f_i^{p/l}}{(p/l)!} & \text{if } l \text{ divides } p \\ 0 & \text{otherwise} \end{cases}$$

$$\eta(K_i) = 1, \quad \eta((K_i; p)_{q_i}) = \begin{cases} \frac{h_i(h_i - 1) \dots (h_i - (p/l) + 1)}{(p/l)!} & \text{if } l \text{ divides } p \\ 0 & \text{otherwise} \end{cases}$$

where  $p \in \mathbb{N}$ , and  $e_i, f_i$  and  $h_i, i \in \{1, \dots, m\}$ , denote the standard generators of  $U(\mathfrak{g})$ . The kernel of  $\eta$  is generated by the elements  $E_i, F_i, K_i - 1$ , and  $(K_i; p)_{q_i}$  where  $l$  does not divide  $p$ . Put

$$(80) \quad \mathcal{Z}_0(\mathcal{O}_\epsilon) := \eta^*(\mathcal{O}(G))$$

where  $\eta^*: U(\mathfrak{g})^\circ \rightarrow \Gamma_\epsilon^\circ$  is the monomorphism dual to  $\eta$ . Let us define special matrix coefficients, analogous to those introduced in Theorem 2.15. Denote by  $v_{\varpi_i}$  and  $v_{w_0(\varpi_i)}$  a highest weight vector and a lowest weight vector of the  $\Gamma$ -module  ${}_A V_{\varpi_i}$ . Denote also by  $v_{w_0(\varpi_i)}^*$  and



$v_{\varpi_i}^*$  a highest and lowest weight vector of the dual module  $\Gamma$ -module  ${}_A V_{\varpi_i}^* \cong {}_A V_{-w_0(\varpi_i)}$ . Define the matrix coefficients  $b_{\varpi_i}, c_{\varpi_i} \in \mathcal{O}_A$  by

$$b_{\varpi_i}(x) = v_{\varpi_i}^*(xv_{w_0(\varpi_i)}) , \quad c_{\varpi_i}(x) = v_{w_0(\varpi_i)}^*(xv_{\varpi_i})$$

for all  $x \in \Gamma$ . We consider them as elements of  $\mathcal{O}_\epsilon$ . Denote by  $\mathcal{Z}_1(\mathcal{O}_\epsilon)$  the subalgebra of  $\mathcal{O}_\epsilon$  generated by the elements  $b_{\varpi_i}^k c_{\varpi_i}^{l-k}$  for  $1 \leq i \leq m$  and  $0 \leq k \leq l$ .

**Theorem 2.24.** (1)  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  is a central Hopf subalgebra of  $\mathcal{O}_\epsilon \subset \Gamma_\epsilon^\circ$ , and  $Q(\mathcal{Z}(\mathcal{O}_\epsilon))$  is an extension of  $Q(\mathcal{Z}_0(\mathcal{O}_\epsilon))$  of degree  $l^m$ .

(2)  $\psi_{-l\rho} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ , and  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  is generated by matrix coefficients of irreducible  $\Gamma$ -modules of highest weight  $l\lambda$ ,  $\lambda \in P_+$ . Moreover the multiplication map yields an isomorphism  $\mathcal{Z}_0(\mathcal{O}_\epsilon) \otimes_{\mathcal{Z}_0(\mathcal{O}_\epsilon) \cap \mathcal{Z}_1(\mathcal{O}_\epsilon)} \mathcal{Z}_1(\mathcal{O}_\epsilon) \rightarrow \mathcal{Z}(\mathcal{O}_\epsilon)$ , and the map  $\Phi$  in Theorem 2.15 affords an algebra embedding  $\mathcal{Z}_0(\mathcal{O}_\epsilon) \rightarrow \mathcal{Z}_0(U_\epsilon(G^*))$  and algebra isomorphisms  $\mathcal{Z}_0(\mathcal{O}_\epsilon)[\psi_{-l\rho}^{-1}] \rightarrow \mathcal{Z}_0(U_\epsilon(G^*))$  and  $\mathcal{O}_\epsilon[\psi_{-l\rho}^{-1}] \rightarrow U_\epsilon(G^*)$ .

(3)  $\mathcal{O}_\epsilon$  has no non-trivial zero divisors, and it is a free  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ -module of rank  $l^{\dim \mathfrak{g}}$ . Moreover the classical fraction algebra  $Q(\mathcal{O}_\epsilon) = Q(\mathcal{Z}(\mathcal{O}_\epsilon)) \otimes_{\mathcal{Z}(\mathcal{O}_\epsilon)} \mathcal{O}_\epsilon$  is a central simple algebra of PI degree  $l^N$ , and  $\mathcal{O}_\epsilon$  is a maximal order of  $Q(\mathcal{O}_\epsilon)$ .

For the proof, see in [45]: the proposition 6.4 for the first claim of (1) (where  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  and  $\mathcal{Z}_0(U_\epsilon(G^*))$  are denoted  $F_0$  and  $A_0$  respectively), the appendix of Enriquez and [50] for the second claim of (1) and (2), the propositions 6.4-6.5 for the other claims of (2), the theorem 7.2 (where  $\mathcal{O}_\epsilon$  is shown to be projective over  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ ) and [27] (which provides the additional K-theoretic arguments to deduce that  $\mathcal{O}_\epsilon$  is free), or Remark 2.18(b) of [6], for the second claim of (3), and the corollary 7.3 and theorem 7.4 for the third claim. The fact that  $\mathcal{O}_\epsilon$  has no non-trivial zero divisors follows from the embedding  $\mathcal{O}_\epsilon \rightarrow U_\epsilon(G^*)$  via  $\Phi$ .

As above for  $U_\epsilon$ , it follows directly from (3) that  $Q(\mathcal{Z}(\mathcal{O}_\epsilon))$  has degree  $l^m$  over  $Q(\mathcal{Z}_0(\mathcal{O}_\epsilon))$ . For a more complete description of  $\mathcal{Z}(\mathcal{O}_\epsilon)$  we refer to [50] and Enriquez' Appendix in [45], as well as [26].

We do not know a basis of  $\mathcal{O}_\epsilon$  over  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  for general  $G$ , but see [47] for the case of  $SL_2$ . We will recall the known results in this case of  $SL_2$  before Lemma 4.5.

Finally, there is a natural action of the braid group  $\mathcal{B}(\mathfrak{g})$  on  $\mathcal{O}_\epsilon$ , that we will use. Namely, let  $n_i \in N(T_G)$  be a representative of the reflection  $s_i \in W = N(T_G)/T_G$  associated to the simple root  $\alpha_i$ . In [93, 92] Soibelman-Vaksman introduced functionals  $t_i : \mathcal{O}_q \rightarrow \mathbb{C}(q)$  which quantize the elements  $n_i$ . They correspond dually to generators of the quantum Weyl group of  $\mathfrak{g}$ ; in the Appendix we recall their main properties, in particular they map  $\mathcal{O}_A$  to  $A$  (see also [38], Section 8.2, and [70, 93, 74, 71, 45]). Denote by  $\triangleleft$  the natural right action of functionals on  $\mathcal{O}_A$ , namely (using Sweedler's notation)

$$\alpha \triangleleft h = \sum_{(\alpha)} h(\alpha_{(1)})\alpha_{(2)}$$

for every  $\alpha \in \mathcal{O}_A$  and  $h \in \mathcal{O}_A \rightarrow A$ . Let us identify  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  with  $\mathcal{O}(G)$  by means of (80). We have ([45], Proposition 7.1):

**Proposition 2.25.** *The maps  $\triangleleft t_i$  on  $\mathcal{O}_\epsilon$  preserve  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ , and satisfy  $(f \triangleleft t_i)(a) = f(n_i a)$  and  $(f \star \alpha) \triangleleft t_i = (f \triangleleft t_i)(\alpha \triangleleft t_i)$  for every  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ ,  $a \in G$ ,  $\alpha \in \mathcal{O}_\epsilon$ .*

We provide an alternative, non computational, proof of this result in the Appendix (Section 6.2).



## 3. NOETHERIANITY AND FINITENESS

In this section we prove Theorem 1.1. Recall that by Noetherian we mean right and left Noetherian. We begin with:

**Theorem 3.1.** *The algebras  $\mathcal{L}_{0,n}$ ,  $\mathcal{L}_{0,n}^A$  and  $\mathcal{L}_{0,n}^{\epsilon'}$ ,  $\epsilon' \in \mathbb{C}^\times$ , are Noetherian.*

By Proposition 2.13 each of the algebras in this theorem is finitely generated.

The theorem 3.1 for  $\mathcal{L}_{0,1}$  and any  $\mathfrak{g}$  follows immediately from Joseph-Letzter's Theorem 2.1, claim (3), by identifying  $\mathcal{L}_{0,1}$  with  $U_q^{lf}$  via  $\Phi_1$ . The method of proof uses filtration arguments. An alternative proof in the case of  $sl(n)$ , which works also for  $\mathcal{L}_{0,1}^A$ , was obtained by Domokos-Lenagan in [46], by exhibiting special sequences of generators of  $\mathcal{L}_{0,1}^A$  satisfying *polynormal* relations. Here, given a Noetherian Abelian ring  $R$  and a finitely generated  $R$ -algebra  $B$  with product  $\circ$ , we call polynormal a set of relations between generators  $u_1, \dots, u_M$  of  $B$ , of the form (see [96], Proposition 3.133)

$$(81) \quad u_i \circ u_j - q_{ij} u_j \circ u_i = \sum_{s=1}^{j-1} \sum_{t=1}^M (\alpha_{ij}^{st} u_s \circ u_t + \beta_{ij}^{st} u_t \circ u_s)$$

for all  $1 \leq j < i \leq M$ , where  $\alpha_{ij}^{st}, \beta_{ij}^{st} \in R$ , and the elements  $q_{ij} \in R$  are invertible (note that this definition is more restrictive than the more standard one, in e.g. [24], II.4.1). If such a set of relations exists in  $B$ , then  $B$  can be endowed with an algebra filtration such that the associated graded algebra is a quotient of a skew-polynomial algebra ([24], Proposition I.8.17). By classical results, if the algebra filtration is well-founded, then  $B$  is a Noetherian ring (see eg. [85], 1.2.9-1.6.9-1.6.11, or [96], Lemma 3.130-3.131). In [46], the theorem 3.1 is also proved for any  $n \geq 1$  in the case of  $\mathfrak{g} = sl(2)$  by considering  $\mathcal{L}_{0,n}^A(sl(2))$  as an iterated overring of  $\mathcal{L}_{0,1}(sl(2))$ .

The proof of Theorem 3.1 that we develop for any  $\mathfrak{g}$  and  $n \geq 1$  is also based on polynormal relations. In our proof, the generating set of  $\mathcal{L}_{0,n}$  that we will consider is evident, as they are matrix coefficients in the modules  $V_{\varpi_k}$ ,  $k \in \{1, \dots, m\}$ ; the task is then to exhibit a set of polynormal relations between them, that hold in a certain graded algebra associated to  $\mathcal{L}_{0,n}$ . Indeed, as explained above this will imply that the graded algebra is Noetherian, and that  $\mathcal{L}_{0,n}$  is Noetherian as well. In the case of  $\mathcal{L}_{0,n}^A$ , the proof is formally similar, but it needs the use of canonical basis discussed in Section 2.2.2.

*Proof of Theorem 3.1.* First we develop the proof for  $\mathcal{L}_{0,n}$ , and then for  $\mathcal{L}_{0,n}^A$ ; the result for  $\mathcal{L}_{0,n}^{\epsilon'} = \mathcal{L}_{0,n}^A/(q-\epsilon')\mathcal{L}_{0,n}^A$  follows immediately by lifting ideals by the quotient map  $\mathcal{L}_{0,n}^A \rightarrow \mathcal{L}_{0,n}^{\epsilon'}$ .

We adapt the proof of Theorem 2.1 (3) given in Theorem 3.137 of [96]. Let us begin by recalling these arguments. In doing this, let us stress that [96] takes on  $\mathcal{O}_q$  and  $\mathcal{L}_{0,1}$  the product opposite to ours, and below in (88) and (89) we respect their convention.

As usual let  $C(\mu)$  be the vector space generated by the matrix coefficients of  $V_\mu$ , the simple  $U_q^{ad}$ -module of highest weight  $\mu \in P_+$ . Denote by  $C(\mu)_\lambda \subset C(\mu)$  the subspace of weight  $\lambda$  for the left coregular action of  $U_q(\mathfrak{h})$ ; so  $\alpha \in C(\mu)_\lambda$  if  $K_\nu \triangleright \alpha = q^{(\nu, \lambda)} \alpha$ ,  $\nu \in P$ . Consider the semigroup

$$\Lambda = \{(\mu, \lambda) \in P_+ \times P, \lambda \text{ is a weight of } V_\mu\}.$$

Recall that the partial order  $\preceq$  on  $P$  is defined by:  $\mu \preceq \mu'$  if and only if  $\mu' - \mu \in D^{-1}Q_+$ . Define  $\preceq$  on  $\Lambda$  by:  $(\mu, \lambda) \preceq (\mu', \lambda')$  if and only if  $\mu' - \mu \in D^{-1}Q_+$  and  $\lambda' - \lambda \in D^{-1}Q_+$ . If  $(\mu, \lambda) \preceq (\mu', \lambda')$  and  $(\mu, \lambda) \neq (\mu', \lambda')$ , we write  $(\mu, \lambda) \prec (\mu', \lambda')$ . Since  $\mathcal{L}_{0,1}$  and  $\mathcal{O}_q$  are

isomorphic vector spaces, we have  $\mathcal{L}_{0,1} = \bigoplus_{\mu \in P_+} C(\mu) = \bigoplus_{(\mu,\lambda) \in \Lambda} C(\mu)_\lambda$ . Consider the family of subspaces

$$\begin{aligned} \mathcal{F}_2^{\mu,\lambda} &:= \bigoplus_{(\mu',\lambda') \preceq (\mu,\lambda)} C(\mu')_{\lambda'}, (\mu,\lambda) \in \Lambda, \\ \mathcal{F}_2^{\prec\mu,\lambda} &:= \bigoplus_{(\mu',\lambda') \prec (\mu,\lambda)} C(\mu')_{\lambda'}, (\mu,\lambda) \in \Lambda. \end{aligned}$$

We have

$$(82) \quad \mathcal{L}_{0,1} = \bigcup_{(\mu,\lambda) \in \Lambda} \mathcal{F}_2^{\mu,\lambda}.$$

Indeed, clearly  $\mathcal{L}_{0,1} = \sum_{(\mu,\lambda) \in \Lambda} \mathcal{F}_2^{\mu,\lambda}$ , so (82) follows from the following fact: for every  $(\mu,\lambda), (\mu',\lambda') \in \Lambda$ , the element  $(\mu'',\lambda'') := (\mu + \mu', \lambda + \lambda')$  is such that

$$\mathcal{F}_2^{\mu,\lambda} + \mathcal{F}_2^{\mu',\lambda'} \subset \mathcal{F}_2^{\mu'',\lambda''}.$$

Note that in general, since  $Q_+ \not\subset P_+$  (but  $P_+ \subset D^{-1}Q_+$ ), it is not true that there exists an element  $(\mu'',\lambda'')$  satisfying such an inclusion if one replaces  $\preceq$  with the standard ‘‘product’’ partial order  $\leq$  on  $\Lambda$ , defined by:  $(\mu,\lambda) \leq (\mu',\lambda')$  if and only if  $\mu' - \mu \in Q_+$  and  $\lambda' - \lambda \in Q_+$ . Note also that  $\preceq$  is finer than  $\leq$ , in the sense that if  $\mu \leq \mu'$ , then  $\mu \preceq \mu'$ . Again, this would not be true if we had replaced  $D^{-1}Q_+$  by  $P_+$  in the definition of  $\preceq$ .

The family  $\mathcal{F}_2 := \{\mathcal{F}_2^{\mu,\lambda}\}_{(\mu,\lambda) \in \Lambda}$  is a filtration of the vector space  $\mathcal{L}_{0,1}$ , which is clearly well-founded (ie. every subset of  $\Lambda$  contains a minimal element, or equivalently any decreasing infinite sequence of elements in  $\Lambda$  is eventually constant).

Consider the associated graded vector space  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1}) := \bigoplus_{(\mu,\lambda)} \mathcal{F}_2^{\mu,\lambda} / \mathcal{F}_2^{\prec\mu,\lambda}$ . By identifying an element  $x \in C(\mu)_\lambda$  with its coset  $\bar{x} \in \mathcal{F}_2^{\mu,\lambda} / \mathcal{F}_2^{\prec\mu,\lambda}$  we get an equality of vector spaces  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1}) = \bigoplus_{(\mu,\lambda) \in \Lambda} C(\mu)_\lambda$ . Now, one has the following facts:

(i) Taking the product in  $\mathcal{L}_{0,1}$  we have

$$(83) \quad \alpha\beta \in \mathcal{F}_2^{\mu_1+\mu_2, \lambda_1+\lambda_2} \quad \text{for } \alpha \in C(\mu_1)_{\lambda_1}, \beta \in C(\mu_2)_{\lambda_2}.$$

This follows from (7) and the fact that, for every  $\nu \in P_+$  and every summand of the formula (9), denoting by  $-r \in -Q_+$  the weight of the  $R$ -matrix component  $R_{(2)}$  we have

$$(84) \quad \begin{aligned} K_\nu \triangleright ((R_{(2')})S(R_{(2)}) \triangleright \alpha) \star (R_{(1')} \triangleright \beta \triangleleft R_{(1)}) \\ = q^{(\nu, \lambda_1 + \lambda_2 - r)} (R_{(2')})S(R_{(2)}) \triangleright \alpha \star (R_{(1')} \triangleright \beta \triangleleft R_{(1)}). \end{aligned}$$

(Details of a similar computation are given below (94)). It follows from (83) that  $\mathcal{F}_2$  is an algebra filtration of  $\mathcal{L}_{0,1}$ , and then  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  is a graded algebra.

(ii) Denote by  $\alpha \circ \beta$  the product in  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  of  $\alpha, \beta \in \mathcal{L}_{0,1}$ . The space  $C(\mu_1 + \mu_2)$  has multiplicity one in  $C(\mu_1) \otimes C(\mu_2)$  (again by (7)), therefore if  $\alpha \in C(\mu_1)_{\lambda_1}$  and  $\beta \in C(\mu_2)_{\lambda_2}$ , then  $\alpha \circ \beta$  is the projection of  $\alpha\beta$  onto  $C(\mu_1 + \mu_2)_{\lambda_1 + \lambda_2}$ . Denote by  $\bar{\star}$  the product  $\star$  of  $\mathcal{O}_q$  followed by the projection onto the component  $C(\mu + \nu)$ . Then we have

$$(85) \quad C(\mu) \circ C(\nu) = C(\mu) \bar{\star} C(\nu) = C(\mu + \nu).$$

This follows from the formula (9), and the fact that it is given by an invertible twist of the product  $\star$ .

(iii) For every  $\mu \in P_+$  fix a basis of weight vectors  $e_1^\mu, \dots, e_{d(\mu)}^\mu$  of  $V_\mu$ . Denote by  $e_\mu^1, \dots, e_\mu^{d(\mu)} \in V_\mu^*$  the dual basis, and by  $w(e_i^\mu)$  the weight of  $e_i^\mu$ . Consider the matrix

coefficients  ${}_{\mu}\phi_j^i(x) := e_{\mu}^i(\pi_V(x)(e_j^{\mu}))$ ,  $x \in U_q$ . By using the formula (9) and the explicit form of the  $R$ -matrix one can check that

$$(86) \quad \begin{aligned} {}_{\mu}\phi_j^i \circ {}_{\nu}\phi_l^k &= \sum_{j',l'}' c_{j',l'}^{ikjl} {}_{\mu}\phi_{j'}^i \bar{\star} {}_{\nu}\phi_{l'}^k \\ &= q^{(w(e_j^{\mu}), w(e_l^{\nu}) - w(e_k^{\nu}))} {}_{\mu}\phi_j^i \bar{\star} {}_{\nu}\phi_l^k + \sum_{\substack{j',l' \\ j' \neq j, l' \neq l}}' d_{j',l'}^{ikjl} {}_{\mu}\phi_{j'}^i \circ {}_{\nu}\phi_{l'}^k \end{aligned}$$

where  $\sum_{j',l'}'$  is the sum over indices with weights satisfying  $w(e_j^{\mu}) + w(e_l^{\nu}) = w(e_{j'}^{\mu}) + w(e_{l'}^{\nu})$ ,  $w(e_{j'}^{\mu}) \leq w(e_j^{\mu})$  and  $w(e_{l'}^{\nu}) \geq w(e_l^{\nu})$ , and the coefficient  $c_{j',l'}^{ikjl}$ , equal to  $q^{(w(e_j^{\mu}), w(e_l^{\nu}) - w(e_k^{\nu}))}$ , is computed from the term  $\Theta$  in the  $R$ -matrix factorization (4). In general, all the coefficients  $c_{j',l'}^{ikjl}$  and  $d_{j',l'}^{ikjl}$  belong to  $\mathbb{C}(q)$  (see Proposition 4.1 of [31]); in particular  $q^{(w(e_j^{\mu}), w(e_l^{\nu}) - w(e_k^{\nu}))} \in q^{\mathbb{Z}}$  since  $w(e_l^{\nu}) - w(e_k^{\nu}) \in Q$ . The second equality follows by repeated use of the first and (85). Similarly, by using (10) one gets

$$\begin{aligned} {}_{\nu}\phi_l^k \circ {}_{\mu}\phi_j^i &= \sum_{i',k'}' e_{i',k'}^{kilj} {}_{\mu}\phi_{j'}^{i'} \bar{\star} {}_{\nu}\phi_l^{k'} \\ &= q^{(w(e_i^{\mu}), w(e_k^{\nu}) - w(e_l^{\nu}))} {}_{\mu}\phi_j^i \bar{\star} {}_{\nu}\phi_l^k + \sum_{\substack{i',k' \\ i' \neq i, k' \neq k}}' e_{i',k'}^{kilj} {}_{\mu}\phi_{j'}^{i'} \bar{\star} {}_{\nu}\phi_l^{k'} \\ &= q^{(w(e_i^{\mu}), w(e_k^{\nu}) - w(e_l^{\nu}))} {}_{\mu}\phi_j^i \bar{\star} {}_{\nu}\phi_l^k + \sum_{\substack{i',k',j',l' \\ i' \neq i, k' \neq k}}' f_{i',k',j',l'}^{kilj} {}_{\mu}\phi_{j'}^{i'} \circ {}_{\nu}\phi_l^{k'} \end{aligned}$$

where  $e_{i',k'}^{kilj}, f_{i',k',j',l'}^{kilj} \in \mathbb{C}(q)$ , and  $\sum_{i',k'}'$  is the sum over indices with weights satisfying  $w(e_i^{\mu}) + w(e_k^{\nu}) = w(e_{i'}^{\mu}) + w(e_{k'}^{\nu})$ ,  $w(e_{i'}^{\mu}) \leq w(e_i^{\mu})$  and  $w(e_{k'}^{\nu}) \geq w(e_k^{\nu})$ , and  $e_{i',k'}^{kilj} = q^{(w(e_i^{\mu}), w(e_k^{\nu}) - w(e_l^{\nu}))}$ . The third equality comes from the second and (86); the sum is over indices with weights satisfying  $w(e_i^{\mu}) + w(e_k^{\nu}) = w(e_{i'}^{\mu}) + w(e_{k'}^{\nu})$ ,  $w(e_{i'}^{\mu}) < w(e_i^{\mu})$ ,  $w(e_{k'}^{\nu}) > w(e_k^{\nu})$ ,  $w(e_{j'}^{\mu}) \leq w(e_j^{\mu})$  and  $w(e_{l'}^{\nu}) \geq w(e_l^{\nu})$ . By eliminating the leading term  ${}_{\mu}\phi_j^i \bar{\star} {}_{\nu}\phi_l^k$  one deduces

$$(87) \quad \begin{aligned} {}_{\nu}\phi_l^k \circ {}_{\mu}\phi_j^i - q_{ijkl} {}_{\mu}\phi_j^i \circ {}_{\nu}\phi_l^k &= \\ &= \sum_{\substack{i',k',j',l' \\ i' \neq i, k' \neq k}}' f_{i',k',j',l'}^{kilj} {}_{\mu}\phi_{j'}^{i'} \circ {}_{\nu}\phi_{l'}^{k'} - \sum_{\substack{j',l' \\ j' \neq j, l' \neq l}}' q_{ijkl} d_{j',l'}^{ikjl} {}_{\mu}\phi_{j'}^i \circ {}_{\nu}\phi_{l'}^k \end{aligned}$$

where  $q_{ijkl} = q^{(w(e_j^{\mu}) + w(e_i^{\mu}), w(e_k^{\nu}) - w(e_l^{\nu}))}$ .

(iv) We can always reorder the weight vectors  $e_1^{\mu}, \dots, e_{d(\mu)}^{\mu}$  so that  $w(e_i^{\mu}) > w(e_j^{\mu})$  implies  $i < j$ ; then (87) reads

$$(88) \quad \begin{aligned} {}_{\nu}\phi_l^k \circ {}_{\mu}\phi_j^i - q_{ijkl} {}_{\mu}\phi_j^i \circ {}_{\nu}\phi_l^k &= \sum_{r=i}^{d(\mu)} \sum_{s=1}^k \sum_{u=1}^{l-1} \sum_{v=j+1}^{d(\mu)} \delta_{rsuv}^{ijkl} {}_{\mu}\phi_v^r \circ {}_{\nu}\phi_u^s \\ &\quad - \sum_{r=i+1}^{d(\mu)} \sum_{s=1}^{k-1} q_{ijkl} \gamma_{rs}^{ijkl} {}_{\mu}\phi_j^r \circ {}_{\nu}\phi_l^s \end{aligned}$$

where  $\gamma_{rs}^{ijkl}, \delta_{rsuv}^{ijkl} \in \mathbb{C}(q)$  are such that  $\gamma_{rs}^{ijkl} = 0$  unless  $w(e_r^{\mu}) < w(e_i^{\mu})$  and  $w(e_s^{\nu}) > w(e_k^{\nu})$ , and  $\delta_{rsuv}^{ijkl} = 0$  unless  $w(e_u^{\nu}) > w(e_l^{\nu})$ ,  $w(e_v^{\mu}) < w(e_j^{\mu})$ ,  $w(e_r^{\mu}) \leq w(e_i^{\mu})$  and  $w(e_s^{\nu}) \geq w(e_k^{\nu})$ . Now,

from (88) one can extract a defining set of polynormal relations for  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$ , as in (81). Indeed, like  $\mathcal{L}_{0,1}$  the algebra  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  is generated by the matrix coefficients  $\varpi_k \phi_i^j$  of the fundamental representations  $V_{\varpi_k}$ . One can list these matrix coefficients, say  $M$  in number, in an ordered sequence  $u_1, \dots, u_M$  such that the following condition holds: if  $w(e_k^{\varpi_s}) < w(e_i^{\varpi_r})$ , or  $w(e_k^{\varpi_s}) = w(e_i^{\varpi_r})$  and  $w(e_l^{\varpi_s}) < w(e_j^{\varpi_r})$ , then  $u_a := \varpi_r \phi_j^i$  and  $u_b := \varpi_s \phi_l^k$  satisfy  $b < a$ . Then denoting  $\mu \phi_j^i, \nu \phi_l^k$  in (88) by  $u_j, u_i$  respectively, and assuming  $u_j < u_i$ , one finds that all terms  $u_s := \mu \phi_v^r, \mu \phi_j^r$  in the sums are  $< u_j$ . Therefore, for all  $1 \leq j < i \leq M$  it takes the form:

$$(89) \quad u_i \circ u_j - q_{ij} u_j \circ u_i = \sum_{s=1}^{j-1} \sum_{t=1}^M \alpha_{ij}^{st} u_s \circ u_t$$

for some  $q_{ij} \in q^{\mathbb{Z}}$  and  $\alpha_{ij}^{st} \in \mathbb{C}(q)$ . As explained after (81) it follows that  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  is a Noetherian ring, and since the filtration  $\mathcal{F}_2$  is well-founded, it implies that  $\mathcal{L}_{0,1}$  is Noetherian too.

We are going to extend all these facts to  $\mathcal{L}_{0,n}$ ,  $n > 1$ . First we need to refine the filtration  $\mathcal{F}_2$  on  $\mathcal{L}_{0,1}$ . Consider the action of  $U_q(\mathfrak{h})$  on  $C(\mu)_\lambda$  given by

$$(90) \quad K_\nu \cdot \alpha := \text{coad}(K_\nu^{-1})(\alpha), \quad \nu \in P, \quad \alpha \in C(\mu)_\lambda.$$

Denote by  $C(\mu)_{\lambda,\gamma} \subset C(\mu)_\lambda$  the subspace of weight  $\gamma$  for this action; so  $\alpha \in C(\mu)_{\lambda,\gamma}$  if  $K_\nu \cdot \alpha = q^{(\nu,\gamma)} \alpha$ . Consider the semigroup

$$\Lambda_P = \{(\mu, \lambda, \gamma) \in P_+ \times P^2, \lambda \text{ is a weight of } V_\mu \text{ for } \triangleright, \gamma \text{ is a weight of } V_\mu \text{ for } \cdot\}$$

with the partial order  $(\mu, \lambda, \gamma) \preceq (\mu', \lambda', \gamma')$  if and only if  $\mu' - \mu, \lambda' - \lambda, \gamma' - \gamma \in D^{-1}Q_+$ . Define

$$(91) \quad [\Lambda_P] = \{([\mu], [\lambda], [\gamma]) \in P_+^n \times P^n \times P^n \\ | (\mu_i, \lambda_i, \gamma_i) \in \Lambda_P, [\mu] = (\mu_i)_{i=1}^n, [\lambda] = (\lambda_i)_{i=1}^n, [\gamma] = (\gamma_i)_{i=1}^n\}.$$

Let us put the following lexicographic order on  $[\Lambda_P]$ , starting from the tail:  $([\mu'], [\lambda'], [\gamma']) \preceq ([\mu], [\lambda], [\gamma])$  if  $(\mu'_n, \lambda'_n, \gamma'_n) \prec (\mu_n, \lambda_n, \gamma_n)$ , or  $(\mu_n, \lambda_n, \gamma_n) = (\mu'_n, \lambda'_n, \gamma'_n)$  and  $(\mu'_{n-1}, \lambda'_{n-1}, \gamma'_{n-1}) \prec (\mu_{n-1}, \lambda_{n-1}, \gamma_{n-1}), \dots$ , or  $(\mu_k, \lambda_k, \gamma_k) = (\mu'_k, \lambda'_k, \gamma'_k)$  for all  $1 < k \leq n$  and  $(\mu'_1, \lambda'_1, \gamma'_1) \preceq (\mu_1, \lambda_1, \gamma_1)$ . (As usual we write  $([\mu'], [\lambda'], [\gamma']) \prec ([\mu], [\lambda], [\gamma])$  for  $([\mu'], [\lambda'], [\gamma']) \preceq ([\mu], [\lambda], [\gamma])$  and  $([\mu'], [\lambda'], [\gamma']) \neq ([\mu], [\lambda], [\gamma])$ .)

Now recall that  $\mathcal{L}_{0,n} = \mathcal{L}_{0,1}^{\otimes n} = \mathcal{O}_q^{\otimes n}$  as vector spaces. For every  $([\mu], [\lambda], [\gamma]) \in [\Lambda_P]$  consider the subspace  $C([\mu])_{[\lambda],[\gamma]} \subset \mathcal{L}_{0,n}$  defined by

$$C([\mu]) = C(\mu_1) \otimes \dots \otimes C(\mu_n) \\ C([\mu])_{[\lambda],[\gamma]} = C(\mu_1)_{\lambda_1, \gamma_1} \otimes \dots \otimes C(\mu_n)_{\lambda_n, \gamma_n}.$$

Then  $\mathcal{L}_{0,n} = \bigoplus_{[\mu] \in P_+^n} C([\mu])$  and  $C([\mu]) = \bigoplus_{([\lambda],[\gamma])} C([\mu])_{[\lambda],[\gamma]}$ . For every  $([\mu], [\lambda], [\gamma]) \in [\Lambda_P]$  define

$$(92) \quad \mathcal{F}_3^{[\mu],[\lambda],[\gamma]} = \bigoplus_{([\mu'], [\lambda'], [\gamma']) \preceq ([\mu], [\lambda], [\gamma])} C([\mu'])_{[\lambda'], [\gamma']} \\ \mathcal{F}_3^{\prec([\mu],[\lambda],[\gamma])} = \bigoplus_{([\mu'], [\lambda'], [\gamma']) \prec ([\mu], [\lambda], [\gamma])} C([\mu'])_{[\lambda'], [\gamma']}.$$

Clearly  $\mathcal{L}_{0,n}$  is the union of the subspaces  $\mathcal{F}_3^{[\mu],[\lambda],[\gamma]}$  over all  $([\mu],[\lambda],[\gamma]) \in [\Lambda_P]$ , so these form a vector space filtration of  $\mathcal{L}_{0,n}$ . Let us denote it  $\mathcal{F}_3$ , and define

$$Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu],[\lambda],[\gamma]} = \mathcal{F}_3^{[\mu],[\lambda],[\gamma]} / \mathcal{F}_3^{\prec[\mu],[\lambda],[\gamma]}.$$

This space is canonically identified with  $C([\mu]_{[\lambda],[\gamma]})$ , so the graded vector space associated to  $\mathcal{F}_3$  is

$$(93) \quad Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n}) = \bigoplus_{([\mu],[\lambda],[\gamma]) \in [\Lambda_P]} Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu],[\lambda],[\gamma]} = \bigoplus_{([\mu],[\lambda],[\gamma]) \in [\Lambda_P]} C([\mu]_{[\lambda],[\gamma]}).$$

We claim that  $\mathcal{F}_3$  is an algebra filtration with respect to the product of  $\mathcal{L}_{0,n}$ , and therefore  $Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})$  is a graded algebra.

For notational simplicity let us prove it for  $n = 2$ , the general case being strictly similar. Recall the  $R$ -matrix factorization (4). Take tuples  $([\mu],[\lambda],[\gamma]) = ((\mu_1, \mu_2), (\lambda_1, \lambda_2), (\gamma_1, \gamma_2))$  and  $([\mu'], [\lambda'], [\gamma']) = ((\mu'_1, \mu'_2), (\lambda'_1, \lambda'_2), (\gamma'_1, \gamma'_2))$  in  $[\Lambda_P]$ , and elements  $\alpha \otimes \beta \in C([\mu]_{[\lambda],[\gamma]})$  and  $\alpha' \otimes \beta' \in C([\mu']_{[\lambda'],[\gamma']})$ . Recall from (18) that the product of  $\mathcal{L}_{0,2}$  is given by the formula

$$(94) \quad (\alpha \otimes \beta)(\alpha' \otimes \beta') = \sum_{(R^1, \dots, R^4)} \alpha \left( S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha' \triangleleft R_{(1)}^1 R_{(1)}^2 \right) \otimes \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \right) \beta'.$$

For every  $\nu \in P$  and any of the components  $R_{(2)}^1, \dots, R_{(2)}^4$ , denoting by  $-r_j \in -Q_+$  the weight of  $R_{(2)}^j$  we have

$$\begin{aligned} K_\nu \triangleright \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \right) &= \sum_{(\beta), (\beta)} \beta_{(1)}(R_{(2)}^2 R_{(2)}^4) \left( K_\nu S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta_{(2)} \right) \\ &= q^{-(\nu, r_1 + r_3)} \sum_{(\beta), (\beta)} \beta_{(1)}(R_{(2)}^2 R_{(2)}^4) \left( S(R_{(2)}^1 R_{(2)}^3) K_\nu \triangleright \beta_{(2)} \right) \\ &= q^{(\nu, \lambda_2 - r_1 - r_3)} \sum_{(\beta), (\beta)} \beta_{(1)}(R_{(2)}^2 R_{(2)}^4) \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta_{(2)} \right) \\ &= q^{(\nu, \lambda_2 - r_1 - r_3)} \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \right). \end{aligned}$$

By similar computations for the action  $coad(K_\nu^{-1})$ , and for all terms in the right-hand side of (94), and using (83) componentwisely, we find that

$$\alpha \left( S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha' \triangleleft R_{(1)}^1 R_{(1)}^2 \right) \otimes \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \right) \beta' \in \mathcal{F}_3^{[\mu]+[\mu'], [\lambda]+[\lambda'], [\gamma]+[\gamma']},$$

where

$$\begin{aligned} \lambda'' &= (\lambda_1 + \lambda'_1 + r_3 + r_4, \lambda_2 + \lambda'_2 - r_1 - r_3) \\ \gamma'' &= (\gamma_1 + \gamma'_1 + r_1 + r_2 + r_3 + r_4, \gamma_2 + \gamma'_2 - r_1 - r_2 - r_3 - r_4). \end{aligned}$$

Since  $r_1 + r_2 + r_3 + r_4 = 0$  implies  $r_1 = r_2 = r_3 = r_4 = 0$ , by the order we have put on  $[\Lambda_P]$  we deduce

$$(\alpha \otimes \beta)(\alpha' \otimes \beta') \in \mathcal{F}_3^{[\mu]+[\mu'], [\lambda]+[\lambda'], [\gamma]+[\gamma']}.$$

Note that the filtration  $\mathcal{F}_3$ , taking the action (90) into account, is crucial for this argument to work. Similar arguments work for any  $n \geq 2$ . This proves that  $Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})$  is a graded algebra. We denote its product by  $\circ_n$ .

Next we show that (85) implies the analogous property for the product  $\circ_n$ . For simplicity of notations let us again assume that  $n = 2$ . Recall that the product  $\circ_2$  is defined on homogeneous elements  $\overline{\alpha \otimes \beta} \in Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu],[\lambda]}$  and  $\overline{\alpha' \otimes \beta'} \in Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu'],[\lambda']}$  by

$$\overline{\alpha \otimes \beta} \circ_n \overline{\alpha' \otimes \beta'} = (\alpha \otimes \beta)(\alpha' \otimes \beta') + \mathcal{F}_3^{\prec[\mu+\mu'],[\lambda+\lambda']}$$

Clearly (85) gives  $(C(\mu_1) \circ C(\mu'_1)) \otimes (C(\mu_2) \circ C(\mu'_2)) = C([\mu + \mu'])$ , and (94) gives

$$C([\mu]) \circ_n C([\mu']) \subset (C(\mu_1) \circ C(\mu'_1)) \otimes (C(\mu_2) \circ C(\mu'_2)).$$

The converse inclusion holds true as well, as one can see by expressing, reciprocally, the (componentwise) product of  $\mathcal{L}_{0,1}^{\otimes n}$  in terms of the product of  $\mathcal{L}_{0,n}$  via the formula (20). In conclusion

$$(95) \quad C([\mu]) \circ_n C([\mu']) = C([\mu + \mu']).$$

We are left to show that (88) generalizes to  $\mathcal{L}_{0,n}$ . First note that for every  $1 \leq a \leq n$  the embedding  $i_a: \mathcal{L}_{0,1} \rightarrow \mathcal{L}_{0,n}$  in (17) is a morphism of the filtered algebras  $(\mathcal{L}_{0,1}, \mathcal{F}_2)$  and  $(\mathcal{L}_{0,n}, \mathcal{F}_3)$ , in the sense that  $i_a(\mathcal{F}_2^{\mu,\lambda}) \subset \sum_{\gamma \in P} \mathcal{F}_3^{[\mu_a],[\lambda_a],[\gamma_a]}$ , where by definition  $[\mu_a] = (0, \dots, 0, \mu, 0, \dots, 0)$  with  $\mu$  on the  $a$ -th entry, and similarly  $[\lambda_a] = (0, \dots, 0, \lambda, 0, \dots, 0)$  and  $[\gamma_a] = (0, \dots, 0, \gamma, 0, \dots, 0)$ . Therefore the relation (88) yields in  $Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})$  similar relations between elements of the form (matrix coefficient) $\otimes 1$ , or  $1 \otimes$ (matrix coefficient).

We now consider the case of “mixed” products. We give the details when  $n = 2$ , the general case being similar. Let us write the twist  $F$  in (19) as

$$F = \sum_{(F)} F_{(1)} \otimes F_{(2)} = \sum_{(F)} F_{(1)1} \otimes F_{(1)2} \otimes F_{(2)1} \otimes F_{(2)2}$$

that is, we set  $F_{(1)1} := R_{(2)}^2 R_{(2)}^4$ ,  $F_{(1)2} := R_{(2)}^1 R_{(2)}^3$ ,  $F_{(2)1} := R_{(1)}^1 R_{(1)}^2$ ,  $F_{(2)2} := R_{(1)}^3 R_{(1)}^4$ . Put  $d(\mu) := \dim(V_\mu)$ ,  $\mu \in P_+$ , and

$$\Delta^{(2)}(\mu_2 \phi_{l_2}^{k_2}) = \sum_{p,s=1}^{d(\mu_2)} \mu_2 \phi_p^{k_2} \otimes \mu_2 \phi_s^p \otimes \mu_2 \phi_{l_2}^s, \quad \Delta^{(2)}(\mu'_1 \phi_{l'_1}^{k'_1}) = \sum_{p',s'=1}^{d(\mu'_1)} \mu'_1 \phi_{p'}^{k'_1} \otimes \mu'_1 \phi_{s'}^{p'} \otimes \mu'_1 \phi_{l'_1}^{s'}.$$

From (94) one obtains

$$(96) \quad \left(1 \otimes \mu_2 \phi_{l_2}^{k_2}\right) \left(\mu'_1 \phi_{l'_1}^{k'_1} \otimes 1\right) \\ = \sum_{(F)} \sum_{p,s=1}^{d(\mu_2)} \sum_{p',s'=1}^{d(\mu'_1)} \left(\mu'_1 \phi_{s'}^{p'} \left(\mu'_1 \phi_{p'}^{k'_1}(F_{(2)1}) \mu'_1 \phi_{l'_1}^{s'}(S(F_{(2)2}))\right)\right) \\ \otimes \left(\mu_2 \phi_s^p \left(\mu_2 \phi_p^{k_2}(F_{(1)1}) \mu_2 \phi_{l_2}^s(S(F_{(1)2}))\right)\right).$$

It is immediate that

$$\mu'_1 \phi_{s'}^{p'} \otimes \mu_2 \phi_s^p \in C(\mu'_1)_{w(e_{s'}^{\mu'_1}), w(e_{s'}^{\mu'_1}) - w(e_{p'}^{\mu'_1})} \otimes C(\mu_2)_{w(e_s^{\mu_2}), w(e_s^{\mu_2}) - w(e_p^{\mu_2})}.$$

As in (iv) above, for every  $\mu \in P_+$  we order the weight vectors  $e_1^\mu, \dots, e_m^\mu$  so that  $w(e_i^\mu) > w(e_j^\mu)$  implies  $i < j$ . With such an ordering the factorization  $R = \Theta \hat{R}$  (see (4)) implies  $\mu_2 \phi_p^{k_2}(F_{(1)1}) \mu_2 \phi_{l_2}^s(S(F_{(1)2})) = 0$  unless  $k_2 \geq p$  and  $s \geq l_2$ , and  $\mu'_1 \phi_{p'}^{k'_1}(F_{(2)1}) \mu'_1 \phi_{l'_1}^{s'}(S(F_{(2)2})) = 0$  unless  $k'_1 \leq p'$  and  $s' \leq l'_1$ . Since  $s > l_2$  we have  $w(e_s^{\mu_2}) \leq w(e_{l_2}^{\mu_2})$ , and if  $w(e_s^{\mu_2}) < w(e_{l_2}^{\mu_2})$  then  $\mu_2 \phi_s^p \in \mathcal{F}_2^{\leq \mu_2, w(e_{l_2}^{\mu_2})}$ . In this last situation the summands  $\mu'_1 \phi_{s'}^{p'} \otimes \mu_2 \phi_s^p$  in the sum above vanish in  $Gr_{\mathcal{F}_3}(\mathcal{L}_{0,2})$ . In order to find all the non zero summands we have to consider also



the weights with respect to the action (90). Since  $k_2 \geq p$  implies  $w(e_{k_2}^{\mu_2}) \leq w(e_p^{\mu_2})$ , we have  $w(e_s^{\mu_2}) - w(e_p^{\mu_2}) \leq w(e_{l_2}^{\mu_2}) - w(e_{k_2}^{\mu_2})$ . Therefore the summands which are non zero in  $Gr_{\mathcal{F}_3}(\mathcal{L}_{0,2})$  have both weights  $w(e_s^{\mu_2}) = w(e_{l_2}^{\mu_2})$  and  $w(e_p^{\mu_2}) = w(e_{k_2}^{\mu_2})$ . Doing similarly with the weights of  $\mu'_1 \phi_{s'}^{p'}$ , we find that also  $w(e_{s'}^{\mu'_1}) = w(e_{l'_1}^{\mu'_1})$  and  $w(e_{p'}^{\mu'_1}) = w(e_{k'_1}^{\mu'_1})$ . When all these conditions on weights are satisfied, the corresponding components  $F_{(1)1}$ ,  $F_{(1)2}$ ,  $F_{(2)1}$ ,  $F_{(2)2}$  have zero weight. Therefore the sum reduces to

$$\begin{aligned} \sum_{(F)} \mu_2 \phi_{k_2}^{k_2}(F_{(1)1}) \mu_2 \phi_{l_2}^{l_2}(S(F_{(1)2})) \mu'_1 \phi_{k'_1}^{k'_1}(F_{(2)1}) \mu'_1 \phi_{l'_1}^{l'_1}(S(F_{(2)2})) \\ = \left\langle \mu_2 \phi_{k_2}^{k_2} \otimes \mu_2 \phi_{l_2}^{l_2} \otimes \mu'_1 \phi_{k'_1}^{k'_1} \otimes \mu'_1 \phi_{l'_1}^{l'_1}, \Theta_{13} \Theta_{14}^{-1} \Theta_{24} \Theta_{23}^{-1} \right\rangle \\ = q \left( w(e_{k_2}^{\mu_2}) - w(e_{l_2}^{\mu_2}), w(e_{k'_1}^{\mu'_1}) - w(e_{l'_1}^{\mu'_1}) \right). \end{aligned}$$

Denoting by  $q'_{k_2 l_2 k'_1 l'_1}$  this scalar, it follows

$$\begin{aligned} \left( 1 \otimes \mu_2 \phi_{l_2}^{k_2} \right) \circ_2 \left( \mu'_1 \phi_{l'_1}^{k'_1} \otimes 1 \right) &= q'_{k_2 l_2 k'_1 l'_1} \mu'_1 \phi_{l'_1}^{k'_1} \otimes \mu_2 \phi_{l_2}^{k_2} \\ &= q'_{k_2 l_2 k'_1 l'_1} \left( \mu'_1 \phi_{l'_1}^{k'_1} \otimes 1 \right) \circ_2 \left( 1 \otimes \mu_2 \phi_{l_2}^{k_2} \right). \end{aligned}$$

This is the relation analogous to (88) for mixed products in  $Gr_{\mathcal{F}_3}(\mathcal{L}_{0,2})$ .

Recall that in (89) we denoted by  $u_1, \dots, u_M$  the ordered list of matrix coefficients  $\varpi_k \phi_i^j$ . Let us order in a lexicographic way the elements  $u_i \otimes u_j$ , ie. as a sequence  $u_1^{(2)}, \dots, u_{M^2}^{(2)}$  such that the following condition holds: if  $\varpi_{i'} \phi_{s'}^{t'} < \varpi_{k'} \phi_{i'}^{j'}$ , or  $\varpi_{i'} \phi_{s'}^{t'} = \varpi_{k'} \phi_{i'}^{j'}$  and  $\varpi_i \phi_s^t < \varpi_k \phi_i^j$ , then  $u_a^{(2)} := \varpi_k \phi_i^j \otimes \varpi_{k'} \phi_{i'}^{j'}$  and  $u_b^{(2)} := \varpi_i \phi_s^t \otimes \varpi_{i'} \phi_{s'}^{t'}$  satisfy  $u_b^{(2)} < u_a^{(2)}$ . Then, for this ordering the polynomial relations (89) hold true for all  $1 \leq u_j^{(2)} < u_i^{(2)} \leq M^2$ . As described after (81), it follows that  $Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})$  is Noetherian. The filtration  $\mathcal{F}_3$  being well-founded, it implies that  $\mathcal{L}_{0,n}$  is Noetherian too.

Finally we consider the  $A$ -algebra  $\mathcal{L}_{0,n}^A$ , and prove it is Noetherian. We proceed in exactly the same way as for  $\mathcal{L}_{0,n}$ , changing the generators and replacing key arguments of the steps **(i)**–**(iv)** by the corresponding results over  $A$ . Let us describe these modifications step by step.

First consider the case  $n = 1$ . Recall the full  $A$ -sublattices  ${}_A C(\lambda) \subset C(\lambda)$ , see (58). Consider the  $A$ -modules  ${}_A C(\mu)_\lambda = C(\mu)_\lambda \cap \mathcal{O}_A$ , and the filtration  ${}_A \mathcal{F}_2$  of  $\mathcal{O}_A$  by the submodules  ${}_A \mathcal{F}_2^{\mu, \lambda} = \mathcal{F}_2^{\mu, \lambda} \cap \mathcal{O}_A$ . By Theorem 2.6 and the fact that  $\mathcal{O}_A = \mathcal{L}_{0,1}^A$  as an  $A$ -module we have the  $A$ -module decomposition

$$Gr_{{}_A \mathcal{F}_2}(\mathcal{L}_{0,1}^A) = \bigoplus_{(\mu, \lambda) \in \Lambda} {}_A C(\mu)_\lambda.$$

Exactly as in step **(i)** one proves that  ${}_A \mathcal{F}_2$  is an algebra filtration of  $\mathcal{L}_{0,1}^A$ . By Proposition 2.8 the  $A$ -module  ${}_A C(\mu_1 + \mu_2)$  has multiplicity one in  ${}_A C(\mu_1) \otimes {}_A C(\mu_2)$ , so as in step **(ii)** one obtains that

$$(97) \quad {}_A C(\mu) \circ {}_A C(\nu) = {}_A C(\mu + \nu).$$

In step **(iii)** we fixed a basis of each space  $C(\mu)$ , consisting of a set of matrix coefficients  $\{\mu \phi_j^i\}$  with respect to dual basis of weight vectors of the modules  $V_\mu$  and  $V_\mu^*$ . In step **(iv)** the basis elements of  $V_\mu$  and  $V_\mu^*$  were ordered by means of the weights, and we used the fact that the matrix coefficients in the spaces  $C(\varpi_1), \dots, C(\varpi_m)$  form a generating set of the algebra  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$ . The only property of the matrix coefficients used in the computations was that

they are weight vectors for the left coregular action (and later, in the case  $n > 1$ , for the action (90)).

Now recall that any set of generators of  $\mathcal{O}_A$  generates  $\mathcal{L}_{0,1}^A$  as well (see the comment after Theorem 3.1). Then, by working over  $A$  it follows from Theorem 2.6 and Proposition 2.7 that one can replace the basis  $\{\mu\phi_j^i\}$  in each space  $C(\mu)$  with the canonical basis  $\underline{\mathbf{B}}_{\mu^r,\mu}^{up}$ ; in particular its elements are also weight vectors, and form a basis in each weight space. By the integrality properties satisfied by the  $R$ -matrix and the twists, all the computations in the proof using such basis elements eventually take place over  $A$  (see the propositions 4.10 and 6.9 in [31]). Therefore we obtain a relation like (89) with coefficients  $\alpha_{ij}^{st} \in A$ . Since  $A$  is a Noetherian ring, again this proves  $Gr_{A\mathcal{F}_2}(\mathcal{L}_{0,1}^A)$ , whence  $\mathcal{L}_{0,1}^A$ , are Noetherian rings.

This being done, the adaptation of the proof when  $n > 1$  is immediate. The filtration  $\mathcal{F}_3$  has to be replaced with  ${}_A\mathcal{F}_3 := \{{}_A\mathcal{F}_3^{[\mu],[\lambda],[\gamma]}\}_{([\mu],[\lambda],[\gamma])}$ , where  ${}_A\mathcal{F}_3^{[\mu],[\lambda],[\gamma]}$  is the  $A$ -module defined by

$${}_A\mathcal{F}_3^{[\mu],[\lambda],[\gamma]} = \bigoplus_{([\mu'],[\lambda'],[\gamma']) \preceq ([\mu],[\lambda],[\gamma])} AC([\mu'])_{[\lambda'],[\gamma']}$$

where  $AC([\mu])_{[\lambda],[\gamma]} = AC(\mu_1)_{\lambda_1,\gamma_1} \otimes \dots \otimes AC(\mu_n)_{\lambda_n,\gamma_n}$ , and  $AC(\mu_1)_{\lambda,\gamma} = C(\mu_1)_{\lambda,\gamma} \cap \mathcal{O}_A$ . Then the proof proceeds in exactly the same way, replacing in (96) and all subsequent computations the matrix coefficients by the generators of  $\mathcal{O}_A$  provided by Proposition 2.7. This concludes the proof.  $\square$

**Theorem 3.2.** *The algebra  $\mathcal{M}_{0,n} = \mathcal{L}_{0,n}^{U_q}$  is Noetherian and generated over  $\mathbb{C}(q)$  by a finite number of elements.*

Our method of proof follows closely that of the Hilbert-Nagata theorem (see [40]). Let us recall one version of this theorem. Let  $K$  be an arbitrary field,  $\mathfrak{A}$  a commutative algebra over  $K$  finitely generated by elements  $a_1, \dots, a_n$ , and  $G$  a group of algebra automorphisms of  $\mathfrak{A}$ .

**Theorem 3.3.** *If the action of  $G$  on  $\mathfrak{A}$  is completely reducible on finite dimensional representations, then the ring  $\mathfrak{A}^G$  of invariants of  $\mathfrak{A}$  with respect to  $G$  is Noetherian and a finitely generated algebra over  $K$ .*

We recall here the main steps of the proof that we will adapt in order to prove Theorem 3.2:

- (a) From the complete reducibility of the action of  $G$  on  $\mathfrak{A}$  one can define a linear map

$$R : \mathfrak{A} \rightarrow \mathfrak{A}^G$$

namely the projection onto the space of invariant elements along the sum of non-trivial isotypical components of  $\mathfrak{A}$ . This linear map is the Reynolds operator; we already discussed it in (26) in the case of  $U_q$  acting on  $\mathcal{L}_{0,n}$ . By the same arguments we developed there, it satisfies

$$(98) \quad R(hf) = hR(f)$$

for every  $f \in \mathfrak{A}$ ,  $h \in \mathfrak{A}^G$ .

(b) Let  $I$  be an ideal of  $\mathfrak{A}^G$ . Then  $I = R(\mathfrak{A}I) = \mathfrak{A}I \cap \mathfrak{A}^G$ . Because  $\mathfrak{A}I$  is an ideal of  $\mathfrak{A}$ , and  $\mathfrak{A}$  is Noetherian, there exist elements  $b_1, \dots, b_s$ , that can be chosen in  $I \subset \mathfrak{A}^G$ , such that  $\mathfrak{A}I = \mathfrak{A}b_1 + \dots + \mathfrak{A}b_s$ . Since  $I = R(\mathfrak{A}I) = R(\mathfrak{A}b_1 + \dots + \mathfrak{A}b_s) = \mathfrak{A}^G b_1 + \dots + \mathfrak{A}^G b_s$ ,  $I$  is finitely generated over  $\mathfrak{A}^G$ . Therefore  $\mathfrak{A}^G$  is Noetherian.

(c) Let  $\mathfrak{B}$  be an algebra graded over  $\mathbb{N}$  (for simplicity of notations):  $\mathfrak{B} = \bigoplus_{n=0}^{+\infty} \mathfrak{B}_n$ , with  $\mathfrak{B}_m \cdot \mathfrak{B}_n \subset \mathfrak{B}_{m+n}$ . The augmentation ideal of  $\mathfrak{B}$  is  $\mathfrak{B}^+ := \bigoplus_{n=1}^{+\infty} \mathfrak{B}_n$ . If  $\mathfrak{B}^+$  is a Noetherian

ideal of  $\mathfrak{B}$ , then  $\mathfrak{B}$  is a finitely generated algebra over  $\mathfrak{B}_0$ . This is Lemma 2.4.5 of [91] (in that statement  $\mathfrak{B}$  is commutative, but this hypothesis is not necessary for the proof).

(d) Assume that  $\mathfrak{A}^G$  is graded over  $\mathbb{N}$  (for simplicity of notations):  $\mathfrak{A}^G = \bigoplus_{n=0}^{+\infty} \mathfrak{A}_n^G$  with  $\mathfrak{A}_0^G = K$ . Then  $\mathfrak{A}^{G+} = \bigoplus_{n=1}^{+\infty} \mathfrak{A}_n^G$  is an ideal of  $\mathfrak{A}^G$ , which is Noetherian by (b) above. Applying (c) we deduce that  $\mathfrak{A}^G$  is a finitely generated algebra over  $K$ .

*Proof of Theorem 3.2.* Consider the filtration  $\mathcal{F}$  of  $\mathcal{L}_{0,n}$  by the subspaces

$$\mathcal{F}^{[\mu]} = \bigoplus_{[\mu'] \preceq [\mu]} C([\mu']), \mu \in P_+^n$$

where  $P_+^n$  is given the lexicographic partial order induced from  $[\Lambda]$ . It is easily seen that  $\mathcal{F}$  is an algebra filtration: the coregular actions  $\triangleright, \triangleleft$  fix globally each component  $C(\mu)$  of  $\mathcal{L}_{0,1}$ , so the claim follows from (9), (18) and the fact that  $C(\mu) \star C(\nu) \subset C(\mu + \nu)$  for all  $\mu, \nu \in P_+$ . Denote by  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})$  the corresponding graded algebra. As a vector space we have:

$$(99) \quad Gr_{\mathcal{F}}(\mathcal{L}_{0,n}) = \bigoplus_{[\mu] \in P_+^n} C([\mu]).$$

Because each space  $C([\mu])$  is stabilized by the coadjoint action of  $U_q$ , (99) has a key advantage on the refined decomposition (93). Indeed, since  $\mathcal{L}_{0,n}$  is a  $U_q$ -module algebra, the action of  $U_q$  is well-defined on  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})$  and gives it a structure of  $U_q$ -module algebra. As vector spaces we have

$$(100) \quad Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q} = \bigoplus_{[\mu] \in P_+^n} C([\mu])^{U_q}.$$

Now we can adapt the different steps (a)–(d) recalled above:

(a') The action of  $U_q$  on  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})$  is completely reducible. This follows from (99) and the fact that the spaces  $C(\mu)$  are finite-dimensional and thus completely reducible  $U_q$ -modules. We can therefore define the Reynolds operator as in (a),

$$R: Gr_{\mathcal{F}}(\mathcal{L}_{0,n}) \rightarrow Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}.$$

(b')  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})$  is Noetherian, because (99) shows it is filtered by  $\mathcal{F}_3$ , and the associated graded algebra  $Gr_{\mathcal{F}_3}(Gr_{\mathcal{F}}(\mathcal{L}_{0,n})) = Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})$  is Noetherian by Theorem 3.1. As in (b) we deduce that  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$  is Noetherian. But  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q} = Gr_{\mathcal{F}}(\mathcal{L}_{0,n}^{U_q})$ , which implies that  $\mathcal{L}_{0,n}^{U_q}$  is Noetherian.

(c'-d') Then we can apply the steps (c)–(d). As a result  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$  is finitely generated, say by  $k$  non zero elements  $\bar{x}_1, \dots, \bar{x}_k$ , which we may assume homogeneous.

(e') We can now deduce that  $\mathcal{L}_{0,n}^{U_q}$  is generated by elements  $x_i$  with leading terms the  $\bar{x}_i$ 's. Indeed, let  $x \in \mathcal{L}_{0,n}^{U_q}$ , and  $[\mu] \in P_+^n$  such that  $x \in \mathcal{F}^{[\mu]} \setminus \mathcal{F}^{<[\mu]}$ , where  $\mathcal{F}^{<[\mu]} := \bigoplus_{[\mu'] \prec [\mu]} C([\mu'])$ . In  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})_{[\mu]}^{U_q} = \mathcal{F}^{[\mu]} / \mathcal{F}^{<[\mu]}$  we have  $\bar{x} = \sum_{(i_1, \dots, i_k) \in I} \lambda_{(i_1, \dots, i_k)} \bar{x}_1^{i_1} \dots \bar{x}_k^{i_k}$  for some finite set  $I \subset \mathbb{N}^k$ , scalars  $\lambda_{(i_1, \dots, i_k)} \in \mathbb{C}(q)$ , and monomials  $\bar{x}_1^{i_1} \dots \bar{x}_k^{i_k}$  of degree  $[\mu]$ . By definition of the product in  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$ ,  $\bar{x}_1^{i_1} \dots \bar{x}_k^{i_k} = x_1^{i_1} \dots x_k^{i_k} + \mathcal{F}^{<[\mu]}$ , so  $x_1^{i_1} \dots x_k^{i_k} \in \mathcal{F}^{[\mu]} \setminus \mathcal{F}^{<[\mu]}$ , whence  $\bar{x}_1^{i_1} \dots \bar{x}_k^{i_k} = \overline{x_1^{i_1} \dots x_k^{i_k}}$  and  $x - \sum_{(i_1, \dots, i_k) \in I} \lambda_{(i_1, \dots, i_k)} x_1^{i_1} \dots x_k^{i_k} \in \mathcal{F}^{<[\mu]}$ . The conclusion follows by decreasing induction on  $[\mu]$ , since at last we terminate at  $\mathcal{F}^{[0]} \cong \mathbb{C}(q)$ .

By combining the steps (a') to (e'), we get that  $\mathcal{M}_{0,n}$  is a Noetherian and finitely generated ring.  $\square$

**Remark 3.4.** (1) Because  $\mathcal{L}_{0,1}^{U_q}$  is the center of  $\mathcal{L}_{0,1}$ , (e') proves it is finitely generated. Of course this follows also from the isomorphism  $\mathcal{L}_{0,1} \cong U_q^{lf}$  and the fact that the center of  $U_q^{lf}$  is the center of  $U_q$  (by Theorem 2.1), plus the well-known description of the latter.

(2) In the  $sl(2)$  case the filtration  $\mathcal{F}$  on  $\mathcal{L}_{0,n}^{U_q}$  should be related via the Wilson loop isomorphism (defined in [31], Section 8.2) to the filtration of skein algebras of spheres with  $n+1$  punctures used in [82].

#### 4. PROOF OF THEOREM 1.2

As usual we let  $\epsilon$  be a primitive  $l$ -th root of unity with  $l$  odd and  $l > d_i$  for all  $i \in \{1, \dots, m\}$ . We now consider the specialization  $\mathcal{L}_{0,n}^\epsilon$  of  $\mathcal{L}_{0,n}$  at  $q = \epsilon$ , defined in Section 2.2.1. Recall the isomorphism of algebras  $\eta^*: \mathcal{O}(G) \rightarrow \mathcal{Z}_0(\mathcal{O}_\epsilon)$  (see (79)), and that  $\mathcal{L}_{0,n}^\epsilon = \mathcal{O}_\epsilon^{\otimes n}$  as a vector space. Consider the linear subspace of  $\mathcal{L}_{0,n}^\epsilon$  defined by

$$\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) := \mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}.$$

This space is naturally a subalgebra of  $\mathcal{O}_\epsilon^{\otimes n}$  (endowed with the componentwise product  $\star$ ). In fact we also have:

**Proposition 4.1.** (1)  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a central subalgebra of the algebra  $\mathcal{L}_{0,n}^\epsilon$ , and the  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -modules  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{O}_\epsilon^{\otimes n}$ , with actions defined by the respective products of these algebras, do coincide.

(2)  $\mathcal{L}_{0,n}^\epsilon$  is a free  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module of rank  $l^{n \cdot \dim \mathfrak{g}}$ .

(3)  $(\eta^{*-1})^{\otimes n}: \mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) \rightarrow \mathcal{O}(G)^{\otimes n}$  is an isomorphism of algebras, and  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a Noetherian ring.

(4) The  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module  $\mathcal{L}_{0,n}^\epsilon$  is finite and Noetherian. Therefore  $\mathcal{L}_{0,n}^\epsilon$  is a Noetherian ring.

Note that the proof we give in (4) of the fact that  $\mathcal{L}_{0,n}^\epsilon$  is Noetherian is independent from the proof of Theorem 3.1.

*Proof.* (1) Let us show that  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a central subalgebra of  $\mathcal{L}_{0,n}^\epsilon$ . In the case  $n = 1$  the formula (9) implies that  $\alpha\beta = \alpha \star \beta$  for all  $\alpha \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$  and  $\beta \in \mathcal{L}_{0,1}^\epsilon$ . Indeed, by (9) we have

$$\begin{aligned} \alpha\beta &= \sum_{(R),(R)} (R_{(2')})S(R_{(2)}) \triangleright \alpha \star (R_{(1')}) \triangleright \beta \triangleleft R_{(1)} \\ &= \sum_{(R),(R),(\alpha),(\beta)} \alpha_{(1)} \star \left( \beta_{(1)} (R_{(1)} \alpha_{(3)} (S(R_{(2)}))) \beta_{(3)} (R_{(1')} \alpha_{(2)} (R_{(2')})) \beta_{(2)} \right) \end{aligned}$$

where all components  $\alpha_{(1)}, \alpha_{(2)}, \alpha_{(3)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ , since  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  is a Hopf subalgebra of  $\mathcal{O}_\epsilon$ . But  $\sum_{(R)} R_{(1)} \alpha_{(3)} (S(R_{(2)})) = S^{-1}(\Phi^-(S_{\mathcal{O}_\epsilon}(\alpha_{(3)}))) \in \mathcal{Z}_0(U_\epsilon)$ , since  $\Phi^-(S_{\mathcal{O}_\epsilon}(\alpha_{(3)})) \in \mathcal{Z}_0(U_\epsilon)$  by Theorem 2.24 (2). Similarly,  $\sum_{(R)} R_{(1')} \alpha_{(2)} (R_{(2')}) \in \mathcal{Z}_0(U_\epsilon)$ . By Lemma 2.23,  $\mathcal{Z}_0(U_\epsilon)$  acts by the trivial character (the counit  $\varepsilon$ ) on  $\Gamma_\epsilon$ -modules. Therefore

$$\alpha\beta = \sum_{(R),(R),(\alpha),(\beta)} \alpha_{(1)} \star \left( \varepsilon(\beta_{(1)}) \varepsilon(\alpha_{(3)}) \varepsilon(\beta_{(3)}) \varepsilon(\alpha_{(2)}) \beta_{(2)} \right) = \alpha \star \beta.$$

This shows  $\mathcal{L}_{0,1}^\epsilon$  and  $\mathcal{O}_\epsilon$  coincide as modules over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$ . Next we show that the subalgebras  $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$  are central in  $\mathcal{L}_{0,n}^\epsilon$  for all  $a = 1, \dots, n$ . This fact will conclude the proof that  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{O}_\epsilon^{\otimes n}$  coincide as  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -modules, because the subalgebras  $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$  generate the space  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  in  $(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$ , and hence in  $\mathcal{L}_{0,n}^\epsilon$  (this follows from the comment before (19)).

In order to show that  $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$  is central in  $\mathcal{L}_{0,n}^\epsilon$  for all  $a = 1, \dots, n$ , it is enough to show  $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$  commutes with the elements of  $\mathcal{L}_{0,n}^\epsilon$  supported by the tensor factors  $(\mathcal{L}_{0,1}^\epsilon)^{(b)}$  with  $b \neq a$ . Since  $(\alpha)^{(a)} \otimes (\beta)^{(b)} = ((\alpha)^{(a)} \otimes 1)(1 \otimes (\beta)^{(b)})$  by the definition, we have to show that  $(1 \otimes (\beta)^{(b)})((\alpha)^{(a)} \otimes 1) = (\alpha)^{(a)} \otimes (\beta)^{(b)}$  whenever  $\alpha \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ . We have (denoting  $\sum_{(\alpha),(\alpha),(\alpha),(\alpha)}$  by  $\sum_{(\alpha)^4}$ ,  $\Delta(\alpha_{(1)}) = \sum_{(\alpha)} \alpha_{(1)(1)} \otimes \alpha_{(1)(2)}$  etc.):

$$\begin{aligned} (1 \otimes (\beta)^{(b)})((\alpha)^{(a)} \otimes 1) &= \sum_{(R^i)} \left( S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha \triangleleft R_{(1)}^1 R_{(1)}^2 \right)^{(a)} \\ &\quad \otimes \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \right)^{(b)} \\ &= \sum_{(R^i), (\alpha)^4, (\beta)^2} (\alpha_{(2)})^{(a)} \otimes (\beta_{(2)})^{(b)} \\ &\quad \times \beta_{(1)} \left( \alpha_{(1)(2)}(R_{(1)}^2) R_{(2)}^2 \alpha_{(3)(1)}(S(R_{(1)}^4)) R_{(2)}^4 \right) \\ &\quad \times \beta_{(3)} \left( \alpha_{(3)(2)}(R_{(1)}^3) R_{(2)}^3 \alpha_{(1)(1)}(R_{(1)}^1) S(R_{(2)}^1) \right). \end{aligned}$$

By Theorem 2.24 (2) it follows that  $\alpha_{(1)(2)}(R_{(1)}^2) R_{(2)}^2 = \Phi^+(\alpha_{(1)(2)}) \in \mathcal{Z}_0(U_\epsilon)$ , and similarly  $\alpha_{(3)(1)}(S(R_{(1)}^4)) R_{(2)}^4$ ,  $\alpha_{(3)(2)}(R_{(1)}^3) R_{(2)}^3$ ,  $\alpha_{(1)(1)}(R_{(1)}^1) S(R_{(2)}^1) \in \mathcal{Z}_0(U_\epsilon)$ . Denote by  $z$  any such element;  $\mathcal{Z}_0(U_\epsilon)$  acts by the trivial character (the counit  $\varepsilon$ ) on  $\Gamma_\epsilon$ -modules, hence the expression of  $z$  in terms of the corresponding  $\alpha_{(i)(j)}$  implies  $\varepsilon(z) = \varepsilon(\alpha_{(i)(j)})$ . It follows

$$\begin{aligned} \beta_{(1)} \left( \alpha_{(1)(2)}(R_{(1)}^2) R_{(2)}^2 \alpha_{(3)(1)}(S(R_{(1)}^4)) R_{(2)}^4 \right) &= \varepsilon(\alpha_{(1)(2)}) \varepsilon(\alpha_{(3)(1)}) \varepsilon(\beta_{(1)}) \\ &= \varepsilon(\alpha_{(1)(2)}) \varepsilon(\alpha_{(3)(1)}) \varepsilon(\beta_{(1)}) \\ \beta_{(3)} \left( \alpha_{(3)(2)}(R_{(1)}^3) R_{(2)}^3 \alpha_{(1)(1)}(R_{(1)}^1) S(R_{(2)}^1) \right) &= \varepsilon(\alpha_{(3)(2)}) \varepsilon(\alpha_{(1)(1)}) \varepsilon(\beta_{(3)}). \end{aligned}$$

Therefore  $(1 \otimes (\beta)^{(b)})((\alpha)^{(a)} \otimes 1) = (\alpha)^{(a)} \otimes (\beta)^{(b)}$ . It follows that  $\mathcal{L}_{0,n}^\epsilon = \mathcal{O}_\epsilon^{\otimes n}$  as modules over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ ; for instance when  $n = 2$ , given  $\alpha', \beta' \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  we have  $(\alpha' \otimes \beta')(\alpha \otimes \beta) = (\alpha' \otimes 1)(1 \otimes \beta')(\alpha \otimes 1)(1 \otimes \beta)$  immediately by (18), and  $(1 \otimes \beta')(\alpha \otimes 1) = \alpha \otimes \beta' = (\alpha \otimes 1)(1 \otimes \beta')$  as above. Then  $(\alpha' \otimes \beta')(\alpha \otimes \beta) = \alpha' \alpha \otimes \beta' \beta$ . In particular  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a central subalgebra of  $\mathcal{L}_{0,n}^\epsilon$ .

(2) Since  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{O}_\epsilon^{\otimes n}$  coincide as modules over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon^{\otimes n})$ , the claim follows from Theorem 2.24, that is: from Theorem 7.2 of [45], which shows that  $\mathcal{O}_\epsilon$  is a finitely generated projective module of rank  $l^{dim \mathfrak{g}}$  over  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ , and from the arguments of [27] (using that  $K_0(\mathcal{O}(G)) = \mathbb{Z}$  by [84]), which imply that this module is free. Alternatively, it follows from the fact that  $\mathcal{O}_\epsilon$  is a cleft extension of  $\mathcal{O}(G)$  (see [6], Remark 2.18 (b), and Section 3.2 of [22]).

(3) The linear isomorphism  $(\eta^{*-1})^{\otimes n}: \mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) \rightarrow \mathcal{O}(G)^{\otimes n}$  is an isomorphism of algebras because  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is central in  $\mathcal{L}_{0,n}^\epsilon$ . It implies that  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a Noetherian ring, since  $\mathcal{O}(G)^{\otimes n} = \mathcal{O}(G^n)$  and  $G^n$  is an affine algebraic variety.

(4) The fact that  $\mathcal{L}_{0,n}^\epsilon$  is a finitely generated  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module follows from (2); an alternative proof of this fact will be provided at the end of the proof of Theorem 4.9. Since  $\mathcal{L}_{0,n}^\epsilon$  is finite over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  and  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is Noetherian,  $\mathcal{L}_{0,n}^\epsilon$  is a Noetherian  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module (eg. by [7] Proposition 6.5). It follows that  $\mathcal{L}_{0,n}^\epsilon$  is a Noetherian ring (by eg. [85], Section 1.3 of Chapter 1).  $\square$

Recall that we denote  $U_\epsilon^{lf} = U_A^{lf} \otimes_A \mathbb{C}_\epsilon$  (see (35)), and  $\mathcal{Z}_0(U_\epsilon) \subset U_\epsilon$  is the central polynomial subalgebra generated by  $E_{\beta_k}^l, F_{\beta_k}^l, L_i^{\pm l}$ , for  $k \in \{1, \dots, N\}$  and  $i \in \{1, \dots, m\}$ . Since  $\Phi_1: \mathcal{L}_{0,1}^\epsilon \rightarrow U_\epsilon^{lf}$  is an embedding of algebras (see Corollary 2.20), it identifies  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  with a central subalgebra of  $U_\epsilon^{lf}$ . Put

$$\mathcal{Z}_0(U_\epsilon^{lf}) := \Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)).$$

Recall Theorem 2.1, Proposition 2.19, and let  $T^{(l)}, T_{2-}^{(l)}$  and  $T_2^{(l)}$  be the subsets of  $T, T_{2-}$  and  $T_2$  formed by the elements  $K_{\lambda l}$  with  $\lambda \in P, \lambda \in -2P_+$  and  $\lambda \in 2P$  respectively.

**Proposition 4.2.** *We have  $U_\epsilon = T_{2-}^{-1} U_\epsilon^{lf} [T/T_2] = \Phi_1(\mathcal{L}_{0,1}^\epsilon [d^{-1}]) [T/T_2]$ , and therefore the map  $\Phi_1: \mathcal{L}_{0,1}^\epsilon [d^{-1}] \rightarrow T_{2-}^{-1} U_\epsilon^{lf}$  is an isomorphism. Moreover  $\mathcal{Z}(U_\epsilon^{lf}) = U_\epsilon^{lf} \cap \mathcal{Z}(U_\epsilon)$ , and*

$$\mathcal{Z}_0(U_\epsilon) = T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{lf}) [T^{(l)}/T_2^{(l)}], \quad \mathcal{Z}(U_\epsilon) = T_{2-}^{(l)-1} \mathcal{Z}(U_\epsilon^{lf}) [T^{(l)}/T_2^{(l)}].$$

*Proof.* The first claim follows immediately from Proposition 2.19 by specialization at  $q = \epsilon$ . For the second claim, the inclusion  $U_\epsilon^{lf} \cap \mathcal{Z}(U_\epsilon) \subset \mathcal{Z}(U_\epsilon^{lf})$  is clear, and for the converse inclusion we only have to show that the elements of  $\mathcal{Z}(U_\epsilon^{lf})$  commute with  $T$ . They commute with  $T_2 \subset U_\epsilon^{lf}$ , so the conjugation action by elements of  $T$  on  $\mathcal{Z}(U_\epsilon^{lf})$  has order at most 2. Let  $x \in \mathcal{Z}(U_\epsilon^{lf})$  with decomposition  $x = \sum_i c_i x_i$  with all  $c_i \in \mathbb{C}$  and  $x_i$  PBW basis vectors, and let  $\lambda \in P$ . We have  $K_\lambda x K_{-\lambda} = \sum_i c_i q(x_i) x_i$ , where  $q(x_i) \in \epsilon^{\mathbb{Z}}$  satisfies  $q(x_i)^2 = 1$  for all  $i$ . Because  $\epsilon$  has odd order the only possibility is  $q(x_i) = 1$ , whence  $K_\lambda x K_{-\lambda} = x$ . The conclusion follows.

The inclusion  $\mathcal{Z}_0(U_\epsilon^{lf}) \subset \mathcal{Z}_0(U_\epsilon)$  follows from the definition  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$ , the formula  $\Phi_1 = m \circ (id \otimes S^{-1}) \circ \Phi$ , and the fact that  $\Phi$  affords an embedding  $\mathcal{Z}_0(\mathcal{O}_\epsilon) \rightarrow \mathcal{Z}_0(U_\epsilon(G^*))$  (see Theorem 2.24 (2)). Since  $T^{(l)} \subset \mathcal{Z}_0(U_\epsilon)$ , we obtain  $T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{lf}) [T^{(l)}/T_2^{(l)}] \subset \mathcal{Z}_0(U_\epsilon)$ . The proof of the converse inclusion is similar to that in Proposition 2.19. The isomorphism  $\mathcal{Z}_0(\mathcal{O}_\epsilon) [\psi_{-l\rho}^{-1}] \rightarrow \mathcal{Z}_0(U_\epsilon(G^*))$  of Theorem 2.24 (2) implies  $F_{\beta_k}^l K_{\beta_k}^l \otimes 1, 1 \otimes K_{\beta_k}^{-l} E_{\beta_k}^l \in \Phi(\mathcal{Z}_0(\mathcal{O}_\epsilon) [\psi_{-l\rho}^{-1}])$  for every positive root  $\beta_k$ . Since  $\psi_{-l\rho} = \Phi_1^{-1}(K_{-2l\rho}) = \psi_{-\rho}^l$  (the  $l$ -th power of  $\psi_{-\rho}$  in  $\mathcal{L}_{0,1}^\epsilon$ ), and  $\Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) [\psi_{-\rho}^{-l}]) = T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{lf})$ , it follows that  $F_{\beta_k}^l K_{\beta_k}^l, S^{-1}(E_{\beta_k}^l) K_{\beta_k}^l \in T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{lf})$ , hence  $F_{\beta_k}^l, S^{-1}(E_{\beta_k}^l) \in T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{lf}) [T^{(l)}/T_2^{(l)}]$ . The sets  $S^{-1}(E_{\beta_k}^l) \mathcal{Z}_0(U_\epsilon(\mathfrak{h}))$ ,  $k = 1, \dots, N$ , generate the subalgebra  $\mathcal{Z}_0(U_\epsilon(\mathfrak{b}_+))$  of  $\mathcal{Z}_0(U_\epsilon)$ , so from the triangular decomposition  $\mathcal{Z}_0(U_\epsilon) = \mathcal{Z}_0(U_\epsilon(\mathfrak{n}_-)) \mathcal{Z}_0(U_\epsilon(\mathfrak{h})) \mathcal{Z}_0(U_\epsilon(\mathfrak{n}_+))$  this proves the inclusion  $\mathcal{Z}_0(U_\epsilon) \subset T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{lf}) [T^{(l)}/T_2^{(l)}]$ . By the isomorphism  $\mathcal{Z}_0(U_\epsilon) \otimes_{\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon)} \mathcal{Z}_1(U_\epsilon) \rightarrow \mathcal{Z}(U_\epsilon)$  (see Theorem 2.22), and the fact that  $\mathcal{Z}(U_q) \subset U_q^{lf}$  (whence  $\mathcal{Z}_1(U_\epsilon) \subset \mathcal{Z}(U_\epsilon^{lf})$ ), the equality  $\mathcal{Z}(U_\epsilon) = T_{2-}^{(l)-1} \mathcal{Z}(U_\epsilon^{lf}) [T^{(l)}/T_2^{(l)}]$  follows at once.  $\square$

**Remark 4.3.** Let us explain how this can be used to give an interpretation of the isomorphism  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \cong \mathcal{O}(G)$ . Recall the notations introduced around Theorem 2.22. Since  $G^* = U_+ T_G U_-$ , we have  $\mathcal{O}(G^*) = \mathcal{O}(U_+) \mathcal{O}(T_G) \mathcal{O}(U_-)$ , and the map  $\sigma$  yields an identification

$$(101) \quad \mathcal{O}(G^0) = \mathcal{O}(U_+) \mathcal{O}(T_G/(2)) \mathcal{O}(U_-).$$

We can identify  $\mathcal{O}(G^0)$  with the subalgebra  $(\sigma|_{G^*})^*(\mathcal{O}(G^0)) \subset \mathcal{O}(G^*)$ . Consider the exterior power  $V = \wedge^N \mathfrak{g}$  endowed with the action  $\wedge^N Ad$  of  $G$ . Put on  $\mathfrak{g}$  a basis consisting of one element  $e_\alpha$  per root space  $\mathfrak{g}_\alpha$ , along with a basis of  $\mathfrak{h}$ . Let  $v \in V$  be the exterior power of the  $e_\alpha$ 's for  $\alpha$  negative, and  $v^*$  a dual vector such that  $v^*(v) = 1$  and  $v^*$  vanishes on a  $T_G$ -invariant complement of  $v$ . It is classical that  $G \setminus G^0$  has defining equation  $\delta(g) = 0$ , where  $\delta$  is the matrix coefficient  $\delta(g) = v^*(\pi_V(g)v)$  (see eg. [59], page 174). Hence  $\mathcal{O}(G^0) = \mathcal{O}(G) [\delta^{-1}]$ . On  $G^0$  we have  $\delta(u_+ t u_-) = \chi_{-2\rho}(t)$ , where  $\chi_{-2\rho}$  is the character of  $T_G$  associated to the



root  $-2\rho$ . Now we can make the connection with  $U_\epsilon$ . The isomorphism  $\mathcal{Z}_0(U_\epsilon) \cong \mathcal{O}(G^*)$  of Theorem 2.22 (2) identifies  $\mathcal{Z}_0(U_\epsilon(\mathfrak{h})) = \mathbb{C}[T^{(l)}]$  with  $\mathcal{O}(T_G)$  by mapping  $K_{\lambda l}$  to the character of  $T_G$  associated to  $\lambda$ . Therefore it maps  $\mathbb{C}[T_2^{(l)}]$  to  $\mathcal{O}(T_G/(2))$ , and  $T_2^{(l)-1} \mathcal{Z}_0(U_\epsilon^{lf})$  to  $\mathcal{O}(G^0)$  by (101) and Proposition 4.2. Since  $\mathcal{O}(G^0) = \mathcal{O}(G)[\delta^{-1}]$  and  $T_2^{(l)-1} \mathcal{Z}_0(U_\epsilon^{lf}) = \mathcal{Z}_0(U_\epsilon^{lf})[\ell^l]$ , it follows that  $\mathcal{Z}_0(U_\epsilon^{lf})$  and  $\mathcal{O}(G)$  coincide after localization by  $\ell^l$  and  $\delta$  respectively. By using the Bruhat decomposition of  $G$  as in (106) in the proof of Theorem 4.9 below, one can deduce  $\mathcal{Z}_0(U_\epsilon^{lf}) \cong \mathcal{O}(G)$ , whence  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \cong \mathcal{O}(G)$  by injectivity of  $\Phi_1$ .

Let us make the following observation. Since  $\mathcal{L}_{0,n}^\epsilon = \mathcal{L}_{0,n}^A \otimes_A \mathbb{C}_\epsilon$ , with  $\mathcal{L}_{0,n}^A = \mathcal{O}_A^{\otimes n}$  as an  $A$ -module, and a generating system of  $\mathcal{O}_A^{\otimes n}$  is also a generating system of  $\mathcal{L}_{0,n}^\epsilon$ , it follows from Proposition 2.7 and the identities 60-61 that  $\mathcal{L}_{0,n}^\epsilon$  is generated by elements of the form  $\alpha_1 \otimes \dots \otimes \alpha_n$ , where  $\alpha_1, \dots, \alpha_n$  belong to the set  $C_{gen}$  of matrix coefficients lying on the first and last columns of the matrix representations of  $U_A^{res}$  in the canonical basis of the modules  $AV_{\varpi_i}$ ,  $i = 1, \dots, m$ . Denote by  $\alpha^{*k}$ ,  $k \in \mathbb{N}$ , the  $k$ -th power of an element  $\alpha \in \mathcal{O}_A$ .

**Lemma 4.4.** *For all  $\alpha \in C_{gen}$ ,  $\alpha^{*l} \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ .*

*Proof.* Recall that the Frobenius epimorphism  $\eta: U_A^{res} \otimes_A \mathbb{C}_\epsilon \rightarrow U(\mathfrak{g})$  in (79) has kernel the ideal  $I$  generated by the elements  $E_i, F_i, K_i - 1$ , and  $(K_i; p)_{q_i}$  where  $l$  does not divide  $p$ ,  $i = 1, \dots, m$ . It follows that an element of  $\mathcal{O}_\epsilon$  belongs to  $\mathcal{Z}_0(\mathcal{O}_\epsilon) = \eta^*(\mathcal{O}(G))$  if and only if it vanishes on  $I$ . But this is immediate to check for the elements of the form  $\alpha^{*l}$  with  $\alpha \in C_{gen}$ , using that  $K_i$  is grouplike and the pure summands of  $\Delta(E_i)$  and  $\Delta(F_i)$  have one component equal to 1 or  $K_i^{\pm 1}$  and the other component equal to  $E_i$  or  $F_i$ . For instance,  $\psi_{\varpi_i}^{*l}(K_i - 1) = \psi_{\varpi_i}(K_i)^l - 1 = \epsilon^{l(\alpha_i, \varpi_i)} - 1 = 0$ . Similarly, for every  $\alpha \in C_{gen}$  we find  $\alpha^{*l}(E_i) = \alpha^{\otimes l}(\Delta^{(l)}(E_i)) = 0$ , and  $\alpha^{*l}(F_i) = \alpha^{*l}(K_i - 1) = 0$ .  $\square$

We need below explicit descriptions of the centers of  $\mathcal{O}_\epsilon(SL_2)$  and  $\mathcal{L}_{0,1}^\epsilon(sl(2))$  and their  $\mathcal{Z}_0$ -subalgebras. Denote by  $a, b, c, d$  the standard generators of  $\mathcal{O}_q(SL_2)$ , ie. the matrix coefficients in the basis of weight vectors  $v_0, v_1 = F.v_0$  of the 2-dimensional irreducible representation  $V_1$  of  $U_q(sl(2))$ . As above, denote by  $x^{*k}$ ,  $k \in \mathbb{N}$ , the  $k$ -th power of an element  $x \in \mathcal{O}_A(SL_2)$ . The algebra  $\mathcal{O}_A(SL_2)$  is generated by  $a, b, c, d$ ; the monomials  $a^{*i} \star b^{*j} \star d^{*r}$  and  $a^{*i} \star c^{*k} \star d^{*r}$ ,  $i, j, k, r \in \mathbb{N}, k > 0$ , form an  $A$ -basis of  $\mathcal{O}_A(SL_2)$ . The algebra  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$  is generated by  $a^{*l}, b^{*l}, c^{*l}, d^{*l}$ ; the monomials  $a^{*il} \star b^{*jl} \star d^{*rl}$  and  $a^{*il} \star c^{*kl} \star d^{*rl}$  form a basis of  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ , and  $\mathcal{Z}(\mathcal{O}_\epsilon(SL_2))$  is generated by  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$  and the elements  $b^{*(l-k)} \star c^{*k}$ ,  $k = 0, \dots, l$  (see [45], Proposition 1.4 and the Appendix). We have the relation

$$(102) \quad a^{*l} \star d^{*l} - b^{*l} \star c^{*l} = 1$$

and the Frobenius isomorphism of Parshall-Wang (see [81], Chapter 7) coincides with the map

$$Fr_{PW}: \mathcal{O}(SL_2) \rightarrow \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$$

induced by  $\eta^*$ ; it sends the standard generators  $\underline{a}, \underline{b}, \underline{c}, \underline{d}$  of  $\mathcal{O}(SL_2) = \mathcal{O}_1(SL_2)$  respectively to  $a^{*l}, b^{*l}, c^{*l}, d^{*l}$ . Finally, we have seen that  $\mathcal{O}_\epsilon(SL_2)$  is a free  $\mathcal{Z}_0(\mathcal{O}(SL_2))$ -module of rank  $l^3$  (see Theorem 2.24 (3)). In [47] it is shown that a basis of this module is formed by the monomials  $a^m b^n c^{s'} d^r$  and  $b^n c^{s''} d^r$ , with the integers  $m, n, r, s', s''$  in the range

$$(103) \quad 1 \leq m \leq l-1, 0 \leq n, r \leq l-1, m \leq s' \leq l-1, 0 \leq s'' \leq l-r-1.$$

Now consider  $\mathcal{L}_{0,1}^A(sl(2))$ . Recall that  $\mathcal{L}_{0,1}^A = \mathcal{O}_A$  as  $U_A$ -modules. The algebra  $\mathcal{L}_{0,1}^A(sl(2))$  is also generated by  $a, b, c, d$ ; a set of defining relations is (see [31], Section 5):

$$(104) \quad \begin{aligned} ad &= da & , & \quad ab - ba = -(1 - q^{-2})bd \\ db &= q^2bd & , & \quad cb - bc = (1 - q^{-2})(da - d^2) \\ cd &= q^2dc & , & \quad ac - ca = (1 - q^{-2})dc \\ & & & \quad ad - q^2bc = 1. \end{aligned}$$

The element  $\omega := qa + q^{-1}d$  is central. Let  $T_k, k \in \mathbb{N}$ , be such that  $T_k(x)/2$  is the  $k$ -th Chebyshev polynomial of the first type in the variable  $x/2$ . We have (see [31], Proposition 7.2, for the generalization to  $\mathcal{L}_{0,n}^\epsilon(sl(2))$ ):

**Lemma 4.5.**  $\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon(sl(2))) = \mathbb{C}[\omega, b^l, c^l, d^l]/\mathcal{I}$  and  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2))) = \mathbb{C}[(T_l(\omega), b^l, c^l, d^l)]/\mathcal{I}$ , where  $\mathcal{I}$  is the ideal of  $\mathbb{C}[\omega, b^l, c^l, d^l]$  generated by  $(T_l(\omega) - d^l)d^l - b^l c^l - 1$ .

Here  $b^l, c^l, d^l$  are the  $l$ -th powers of  $b, c, d$  computed using the product of  $\mathcal{L}_{0,1}^A(sl(2))$ , not the product  $\star$  of  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ . The above generator of  $\mathcal{I}$  can be interpreted as a determinant, and  $\omega$  as a quantum trace on  $V_1$ .

**Lemma 4.6.** Viewed as elements of  $\mathcal{O}_A(SL_2)$ ,  $T_l(\omega) - d^l = a^{\star l}$  and  $x^l = x^{\star l}$ ,  $x \in \{b, c, d\}$ .

*Proof.* Let  $\alpha$  and  $\varpi$  be the simple root and fundamental weight of  $sl(2)$ . In the notations of (78) we have  $b = \psi_{-\varpi}^{-\alpha}$ ,  $c = \psi_{-\varpi}^{\alpha}$ ,  $d = \psi_{-\varpi}$ ; the formulas give  $\Phi_1(b^{\star l}) = (q - q^{-1})^l F^l$ ,  $\Phi_1(c^{\star l}) = (q - q^{-1})^l E^l K^{-l}$ ,  $\Phi_1(d^{\star l}) = K^{-l}$ . These coincide respectively with  $\Phi_1(b^l)$ ,  $\Phi_1(c^l)$ ,  $\Phi_1(d^l)$  (see (32) in [31]). By passing to the localization  $\mathcal{O}_A(SL_2)[d^{-1}]$ , and using Parshall-Wang's relation (102), one deduces easily  $\Phi_1(a^{\star l}) = K^l + (q - q^{-1})^{2l} F^l E^l = T_l(\Omega) - K^{-l}$ , where  $\Omega = (\epsilon - \epsilon^{-1})^2 FE + \epsilon K + \epsilon^{-1} K^{-1}$  is the Casimir element, and  $T_l(x)/2$  is the  $l$ -th Chebyshev polynomial of the first type in the variable  $x/2$ . We have  $\Phi_1(\omega) = \Omega$ , so  $\Phi_1(a^{\star l}) = T_l(\omega) - d^l$ . The conclusion follows from the injectivity of  $\Phi_1$ .  $\square$

This lemma proves that we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}(SL_2) & \xrightarrow{Fr_{PW}} & \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2)) & \hookrightarrow & \mathcal{O}_\epsilon(SL_2) \\ & \searrow Fr & \downarrow & & \downarrow \\ & & \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2))) & \hookrightarrow & \mathcal{L}_{0,1}^\epsilon(sl(2)) \end{array}$$

where  $Fr_{PW}$  is Parshall-Wang's Frobenius isomorphism recalled above,  $Fr$  is the Frobenius isomorphism introduced in [31], Definition 7.1, and the vertical arrows are the identifications as vector spaces (the middle one proved by Proposition 4.1).

**Remark 4.7.** By Lemma 4.5 the monomials  $T_l(\omega)^i b^{jl} d^{rl}$  and  $T_l(\omega)^i c^{kl} d^{rl}$ , for  $i, j, k, r \in \mathbb{N}$  and  $k > 0$ , form an  $A$ -basis of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$ . It is straightforward (though cumbersome) to express these basis elements in terms of the basis elements  $a^{\star il} \star b^{\star jl} \star d^{\star rl}$  and  $a^{\star il} \star c^{\star kl} \star d^{\star rl}$  of  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ , and conversely; this can be done by using Lemma 4.6, the formula (9) or the inverse one (expressing  $\star$  in terms of the product of  $\mathcal{L}_{0,1}$ , see (18) in [31]), and the formula of the coproduct  $\Delta: \mathcal{L}_{0,1}^\epsilon(sl(2)) \rightarrow \mathcal{L}_{0,2}^\epsilon(sl(2))$  restricted to  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$  (given in Lemma 7.5 of [31]).

Since  $\mathcal{L}_{0,1}^A = \mathcal{O}_A$  as an  $A$ -module, the functionals  $t_i$  in Proposition 2.25 can be seen as maps  $t_i: \mathcal{L}_{0,1}^A \rightarrow A$ . Though the algebra structures of  $\mathcal{O}_\epsilon$  and  $\mathcal{L}_{0,1}^\epsilon$  are very different,  $\mathcal{L}_{0,1}^\epsilon$  satisfies a result analogous to Proposition 2.25:

**Proposition 4.8.** *The maps  $\triangleleft t_i$  preserve  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , and they satisfy  $(f \triangleleft t_i)(a) = f(n_i a)$  and  $(f\alpha) \triangleleft t_i = (f \triangleleft t_i)(\alpha \triangleleft t_i)$  for every  $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ ,  $a \in G$ ,  $\alpha \in \mathcal{L}_{0,1}^\epsilon$ .*

*Proof.* The first two claims follow from Proposition 2.25 and the definition  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$ .

The last claim follows from the case  $\mathfrak{g} = \mathfrak{sl}(2)$ , as in the proof of Proposition 7.1 of [45]. In fact it is enough to show that  $t(fg) = t(f)t(g)$  for every  $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}(2)))$ ,  $g \in \mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}(2))$ ; for completeness we explain this in the Appendix, see (127). A word of caution is needed: the proof of (127) uses that  $\Delta: \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon \otimes \mathcal{O}_\epsilon$  is a morphism of algebras. The analogous property for  $\mathcal{L}_{0,1}^\epsilon$  is that  $\Delta$  yields a morphism of algebras  $\Delta: \mathcal{L}_{0,1}^\epsilon \rightarrow \mathcal{L}_{0,2}^\epsilon$ . Since the algebra structure of  $\mathcal{L}_{0,2}^\epsilon$  is not the product one on  $\mathcal{L}_{0,1}^\epsilon \otimes \mathcal{L}_{0,1}^\epsilon$ , it is not true in general that  $\sum_{(f),(g)} (f_{(1)} \otimes f_{(2)})(g_{(1)} \otimes g_{(2)}) = \sum_{(f),(g)} f_{(1)}g_{(1)} \otimes f_{(2)}g_{(2)}$  for every  $f, g \in \mathcal{L}_{0,1}^\epsilon$ . However it holds whenever  $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , since  $\Delta(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)) \subset \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \otimes \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  and therefore  $f_{(2)} \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$  commutes in  $\mathcal{L}_{0,2}^\epsilon$  with any  $g_{(1)} \in \mathcal{L}_{0,1}^\epsilon = \mathcal{O}_\epsilon$ .

It is enough to prove the identity  $t(fg) = t(f)t(g)$  when  $f$  ranges in a set of generators of the algebra  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}(2)))$ . So one can take  $f$  among, say,  $T_l(\omega) - d^l = a^{*l}$  and  $x^l = x^{*l}$ ,  $x \in \{b, c, d\}$  (using Lemma 4.5). By (9) and Proposition 6.1 in the Appendix we have

$$t(fg) = \sum_{(R),(R)} t(R_{(2')}S(R_{(2)}) \triangleright f) t(R_{(1')} \triangleright g \triangleleft R_{(1)}).$$

Expanding coproducts and using that  $R^{-1} = (S \otimes id)(R)$  we deduce

$$\begin{aligned} t(fg) &= \sum_{(f),(R),(R)} t(f_{(1)}) \langle f_{(2)}, R_{(2')}S(R_{(2)}) \rangle t(R_{(1')} \triangleright g \triangleleft R_{(1)}) \\ &= \sum_{(f),(R),(R)} t(f_{(1)}) t\left(\langle f_{(2)}, R_{(2')} \rangle R_{(1')} \triangleright g \triangleleft \langle f_{(3)}, S(R_{(2)}) \rangle R_{(1)}\right) \\ &= \sum_{(f)} t(f_{(1)}) t\left(S^{-1}(\Phi^-(f_{(2)})) \triangleright g \triangleleft S^{-2}(\Phi^-(f_{(3)}))\right) \\ &= \sum_{(f)} t(f_{(1)}) \left\langle g, S^{-2}(\Phi^-(f_{(3)})) \underline{w} S^{-1}(\Phi^-(f_{(2)})) \right\rangle \\ &= \sum_{(f)} t(f_{(1)}) \varepsilon\left(S^{-2}(\Phi^-(f_{(3)}))\right) \varepsilon\left(S^{-1}(\Phi^-(f_{(2)}))\right) t(g) \end{aligned}$$

where  $\underline{w} \in \mathbb{U}_\Gamma$  is the quantum Weyl group element dual to  $t$  (see Section 6.1), and in the last equality we used that  $\Phi^-$  maps  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  into  $\mathcal{Z}_0(U_\epsilon)$  (see Theorem 2.24 (2)), which acts on  $\Gamma$ -modules by the trivial character (the counit)  $\varepsilon: U_\epsilon \rightarrow \mathbb{C}$ . By (119)-(120) in the Appendix we have  $t(a^{*l}) = t(d^{*l}) = 0$  and  $t(b^{*l}) = 1$ ,  $t(c^{*l}) = -1$ . Now the computation of  $t(fg)$  follows easily. For instance, taking  $f = b^l = b^{*l}$ , by using  $\Delta(b^{*l}) = a^{*l} \otimes b^{*l} + b^{*l} \otimes d^{*l}$  and  $\Delta(d^{*l}) = c^{*l} \otimes b^{*l} + d^{*l} \otimes d^{*l}$  we get

$$t(b^l g) = \varepsilon\left(S^{-2}(\Phi^-(b^{*l}))\right) \varepsilon\left(S^{-1}(\Phi^-(c^{*l}))\right) t(g) + \varepsilon\left(S^{-2}(\Phi^-(d^{*l}))\right) \varepsilon\left(S^{-1}(\Phi^-(d^{*l}))\right) t(g)$$

Since  $b^{*l} \in \mathcal{O}_\epsilon(U_+)$ ,  $\Phi^-(b^{*l}) = 0$ . Also, it is immediate from the definition of  $\Phi^-$  that  $\Phi^-(d^{*l}) = \Phi^-(d)^l = L^l$ ; alternatively, one can bypass this computation by observing that  $\Phi^-$  sets an isomorphism from  $\mathcal{O}_\epsilon(T_G) = \mathcal{O}_\epsilon(B_+) \cap \mathcal{O}_\epsilon(B_-)$  to  $\mathbb{C}[L^{\pm 1}] = U_\epsilon(\mathfrak{b}_+) \cap U_\epsilon(\mathfrak{b}_-)$ , mapping

a generator  $d$  to  $L$  or  $L^{-1}$ . We have  $\varepsilon(L^l) = 1$ , and therefore

$$t(b^l g) = t(g) = t(b^l)t(g).$$

The other cases  $f = T_l(\omega) - d^l, c^l, d^l$  are similar.  $\square$

**Theorem 4.9.**  $\mathcal{L}_{0,n}^\epsilon$  is a free  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module of rank  $l^{n \cdot \dim \mathfrak{g}}$ , and  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is a Noetherian ring and a finite, whence Noetherian,  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module.

*Proof.* We already proved the first claim in Proposition 4.1, and that  $\mathcal{L}_{0,n}^\epsilon$  is a Noetherian  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module. For the second claim, it follows that the  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -submodule  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is necessarily finitely generated. But  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  being Noetherian,  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is then a Noetherian  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module and a Noetherian ring.

For the sake of clarity let us provide a self-contained proof of the first claim, not invoking directly [45, 27] or [6, 22], but applying the same arguments directly on  $\mathcal{L}_{0,n}^\epsilon$ . Since  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{L}_{0,1}^{\otimes n}$  coincide as modules over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) = \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$  by Proposition 4.1, the result will follow from the case  $n = 1$ . Then we argue in four steps. First, using Theorem 2.1 we show that a certain localization of  $\mathcal{L}_{0,1}^\epsilon$  is a free module of rank  $l^{\dim \mathfrak{g}}$ . Then, assuming that  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective, we explain why it has constant rank  $l^{\dim \mathfrak{g}}$  (this is very classical). Thirdly, we prove that  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective as in Theorem 7.2 of [45]. Finally we obtain that it is a free module as in [27].

Recall Proposition 4.2:  $U_\epsilon$  is a free  $\Phi_1(\mathcal{L}_{0,1}^\epsilon[d^{-l}])$ -module of rank  $2^m$  (note that  $\mathcal{L}_{0,1}^\epsilon[d^{-l}] = \mathcal{L}_{0,1}^\epsilon[d^{-1}]$ ), and  $\mathcal{Z}_0(U_\epsilon)$  is free over  $T_{2^-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{lf}) = \Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}])$  of rank  $2^m$ . Since  $U_\epsilon$  is also free of rank  $l^{\dim \mathfrak{g}}$  over  $\mathcal{Z}_0(U_\epsilon)$  (Theorem 2.22 (1)), it is free over  $\Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}])$  of rank  $2^m l^{\dim \mathfrak{g}}$ . The decomposition being unique,  $\Phi_1(\mathcal{L}_{0,1}^\epsilon[d^{-l}])$  is free of rank  $l^{\dim \mathfrak{g}}$  over  $\Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}])$ , and injectivity of  $\Phi_1$  implies that  $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$  is free of rank  $l^{\dim \mathfrak{g}}$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$ .

Assume now that  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective. Let us show that its rank is  $l^{\dim \mathfrak{g}}$ . The localization  $(\mathcal{L}_{0,1}^\epsilon)_P$  of  $\mathcal{L}_{0,1}^\epsilon$  at any prime ideal  $P$  of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  is a free module over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)_P$  ([88], Proposition 2.12.15); the ranks of such modules are finite in number ([88], Proposition 2.12.20). If these ranks are all equal, then, by definition, it is the rank of  $\mathcal{L}_{0,1}^\epsilon$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ . This happens if  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  has no non-trivial (ie.  $\neq 1$ ) idempotent ([88], Corollary 2.12.23), which is the case since it has no non-trivial zero divisors. To compute the rank, suppose  $P$  does not contain  $d^l = \psi^l_{-p}$ . Such ideals  $P$  are in 1-1 correspondence with the prime ideals of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$  by the natural ring monomorphism  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \rightarrow \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$ . The set  $S = \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \setminus P$  is multiplicatively closed, and we have also a ring morphism  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}] \rightarrow S^{-1} \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , which is also an injection (there are no zero divisors in  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , whence in  $S$ ). Then

$$(105) \quad (\mathcal{L}_{0,1}^\epsilon)_P = S^{-1} \mathcal{L}_{0,1}^\epsilon = \mathcal{L}_{0,1}^\epsilon[d^{-l}] \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]} S^{-1} \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$$

shows that  $(\mathcal{L}_{0,1}^\epsilon)_P$  has over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)_P = S^{-1} \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  the same rank  $l^{\dim \mathfrak{g}}$  as  $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$ . This proves our claim.

In order to show that  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  it is enough to show it is finite locally free, ie. there are elements  $d_i \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  such that the localizations  $\mathcal{L}_{0,1}^\epsilon[d_i^{-1}]$  are finite free  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d_i^{-1}]$ -modules, and  $\text{Maxspec}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))$  is covered by the open sets  $U(d_i)$  made of the ideals not containing  $d_i$  (see Lemma 77.2 of [94]).

We have seen above that  $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$  is free of rank  $l^{\dim \mathfrak{g}}$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$ . By Remark 4.3,  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}] \cong \mathcal{Z}_0(U_\epsilon^{lf})[\ell^l]$  is isomorphic to  $\mathcal{O}(G^0)$ , and  $\mathcal{O}(G^0) = \mathcal{O}(G)[\delta^{-1}]$ . Now, given

$w \in W$  with a reduced expression  $s_{i_1} \dots s_{i_k}$ , put  $t_w = t_{i_1} \dots t_{i_k}$ . Let  $w$  be represented by  $n_w = n_{i_1} \dots n_{i_k}$  in  $N(T_G)$ . By Proposition 4.8 we have  $(f \triangleleft t_w)(x) = f(n_w x)$  for every  $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ ,  $x \in G$ . Then

$$(106) \quad \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}] \triangleleft t_w \cong \mathcal{O}(n_w^{-1}G^0) \cong \mathcal{O}(G)[(\delta \triangleleft t_w)^{-1}].$$

If  $b_1, \dots, b_r$  ( $r := l^{\dim \mathfrak{g}}$ ) is a basis of  $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$ , then  $\mathcal{L}_{0,1}^\epsilon[d^{-l}] \triangleleft t_w$  is free over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[(d \triangleleft t_w)^{-l}] \cong \mathcal{O}(n_w^{-1}G^0)$  with basis  $b_1 \triangleleft t_w, \dots, b_r \triangleleft t_w$ . Consider the Bruhat decomposition of  $G$ : any  $g \in G$  can be written in the form  $g = b_1 n b_2$ , where  $b_1, b_2 \in B_-$ ,  $n \in W$ . Hence  $g = n n^{-1} b_1 n b_2 \in n B_+ B_- = n G^0$ , and therefore

$$G = \cup_{w \in W} (B_- n_w B_-) = \cup_{w \in W} (n_w G^0).$$

For every  $w \in W$  put

$$d_w^l := d^l \triangleleft t_w.$$

Under the isomorphism of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  with  $\mathcal{O}(G)$ , we thus get that  $\text{Maxspec}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))$  is covered by the open sets  $U(d_w^l) \cong n_w G^0$ , and  $\mathcal{L}_{0,1}^\epsilon[d_w^l]$  is finite free over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d_w^l]$ . Therefore  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ .

Finally, let us explain why  $\mathcal{L}_{0,1}^\epsilon$  is free over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , following the arguments of [27]. Let  $R$  be a commutative Noetherian ring, put  $X = \text{Maxspec}(R)$ , and let  $P$  be an  $R$ -module. Denote by  $R_I, P_I$  the localizations of  $R, P$  at a maximal ideal  $I \in X$ . Define the  $f$ -rank of  $P$  as  $\text{f-rank}(P) = \inf_{I \in X} \{ \text{f-rank}_{R_I}(P_I) \}$ , where  $\text{f-rank}_{R_I}(P_I) = \sup \{ r \in \mathbb{N}, R_I^{\otimes r} \subset P_I \} \in \mathbb{N} \cup \{+\infty\}$  (ie. the maximal dimension of a free summand of  $P_I$ ). Bass' Cancellation theorem asserts that if  $P$  is projective and  $\text{f-rank}(P) > \dim(X)$ , and  $P \oplus Q \cong M \oplus Q$  for some  $R$ -modules  $Q$  and  $M$  such that  $Q$  is finitely generated and projective, then  $P \cong M$  (see [10], IV.3.5 and pages 167 and 170, taking  $A = R$ , or [85], section 11.7.13). Let us apply this to  $R = \mathcal{O}(G)$  and  $P = \mathcal{L}_{0,1}^\epsilon$ . We proved above that  $\text{f-rank}_{R_I}(P_I) = l^{\dim \mathfrak{g}}$ , a constant, and we have  $l^{\dim \mathfrak{g}} > \dim \mathfrak{g} = \dim(G)$ . By a result of Marlin [84],  $G$  being semisimple and simply connected the Grothendieck ring  $K_0(\mathcal{O}(G))$  is isomorphic to  $\mathbb{Z}$ . Therefore  $\mathcal{L}_{0,1}^\epsilon \oplus Q \cong \mathcal{O}(G)^r$  for some free  $\mathcal{O}(G)$ -module  $Q$  and  $r \in \mathbb{N}$ . Then Bass' Cancellation implies  $\mathcal{L}_{0,1}^\epsilon$  is free over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \cong \mathcal{O}(G)$ .  $\square$

## 5. PROOF OF THEOREM 1.3

We begin with the following lemma, interesting by itself.

**Lemma 5.1.**  *$\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is a finite  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module and a Noetherian ring. Therefore the ring  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is integral over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ .*

*Proof.* We know by Proposition 4.1 that  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a Noetherian ring, and  $\mathcal{L}_{0,n}^\epsilon$  is a finite Noetherian  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module. Therefore the submodule  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is finitely generated. Being finite over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ , it is necessarily a Noetherian ring (by eg. Proposition 7.2 of [7]).

Let  $x \in \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ . The  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -submodule  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)[x]$  of  $\mathcal{L}_{0,n}^\epsilon$  is finitely generated by the same argument. Using the fact that an element  $x$  is integral over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  if and only if  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)[x]$  is a finitely generated  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module (by eg. Proposition 5.1 of [7]), this proves the last claim.  $\square$

We will use the following notations. Let  $A$  be a ring with no non-trivial zero divisors. The center  $Z = Z(A)$  is a commutative integral domain. We denote by  $Q(Z)$  its field of fractions, and put

$$Q(A) := Q(Z) \otimes_Z A.$$

It is an algebra over its center  $Q(Z)$ . Since  $\mathcal{L}_{0,n}^\epsilon$  has no non-trivial zero divisors ([31], Proposition 6.30), we can take  $A = \mathcal{L}_{0,n}^\epsilon$ , or  $A = (\mathcal{L}_{0,n}^\epsilon)^{U^\epsilon}$ .



By the lemma  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is finite over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ , so the ring  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon) \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)} Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$  is a field. Necessarily it coincides with  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , and therefore

$$(107) \quad Q(\mathcal{L}_{0,n}^\epsilon) = Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} \mathcal{L}_{0,n}^\epsilon = Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)} \mathcal{L}_{0,n}^\epsilon.$$

Recall that we denote by  $N$  the number of positive roots of  $\mathfrak{g}$ .

**Theorem 5.2. (Theorem 1.3(1))**  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a division algebra and a central simple algebra of PI degree  $l^{Nn}$ .

*Proof.* It follows from (107) and Theorem 4.9 that  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a vector space of dimension  $l^{n \cdot \dim \mathfrak{g}}$  over  $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ , and therefore has finite dimension over its center  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ . Because  $\mathcal{L}_{0,n}^\epsilon$  has no non-trivial divisors ([31], Proposition 6.30) and  $Q(\mathcal{L}_{0,n}^\epsilon)$  is finite-dimensional over  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ ,  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a division algebra, whence a central simple algebra. By classical theory (see eg. Section 13.3.5 of [85], or [88], Corollary 2.3.25) there is a finite extension  $\mathbb{F}$  of  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , a *splitting field*, such that

$$\mathbb{F} \otimes_{Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} Q(\mathcal{L}_{0,n}^\epsilon) = M_d(\mathbb{F})$$

where  $d \in \mathbb{N}$ , the PI degree of  $Q(\mathcal{L}_{0,n}^\epsilon)$ , satisfies

$$(108) \quad d^2 = [Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))] = \frac{[Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))]}{[Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))]}.$$

We have to show  $d^2 = l^{2nN}$ . We will obtain this equality by proving firstly that  $d^2 \geq l^{2nN}$ , and then  $d^2 \leq l^{2nN}$ .

In order to show that  $d^2 \geq l^{2nN}$ , it is enough to exhibit an irreducible representation  $V$  of  $\mathcal{L}_{0,n}^\epsilon$  of dimension  $k := l^{nN}$ . Indeed, the representation map  $\rho_V : \mathcal{L}_{0,n}^\epsilon \rightarrow \text{End}_{\mathbb{C}}(V)$  being surjective, given basis elements  $v_1, \dots, v_{k^2} \in \text{End}(V)$ , and elements  $\alpha_1, \dots, \alpha_{k^2} \in \mathcal{L}_{0,n}^\epsilon$  such that  $\rho(\alpha_i) = v_i$  for every  $i \in \{1, \dots, k^2\}$ , necessarily  $\alpha_1, \dots, \alpha_{k^2}$  form a free family of  $Q(\mathcal{L}_{0,n}^\epsilon)$ . For, if there was a non trivial relation  $\sum_i z_i \alpha_i = 0$ , with  $z_i \in Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ , by clearing denominators and then applying the representation map  $\rho_V$  we would get a non trivial relation in  $\text{End}_{\mathbb{C}}(V)$  between  $v_1, \dots, v_{k^2}$ .

Now, by Theorem 2.22 (1) (see [43], section 20), the dimension of a generic irreducible representation space of  $U_\epsilon$  is  $l^N$ . Because  $U_\epsilon = T_2^{-1} U_\epsilon^{lf} [T/T_2]$  (Proposition 4.2), an irreducible representation of  $U_\epsilon$  yields an irreducible representation of  $U_\epsilon^{lf}$ . Moreover, the tensor product of  $n$  irreducible representation spaces of  $U_\epsilon^{lf}$  of dimension  $l^N$  is an irreducible representation space of  $(U_\epsilon^{lf})^{\otimes n}$  of dimension  $l^{nN}$  (see eg. Theorem 3.10.2 in [51]). Applying the linear isomorphism  $\psi_n = \Phi_n \circ (\Phi_1^{-1})^{\otimes n}$  in (23) thus provides an irreducible representation of  $\mathcal{L}_{0,n}^\epsilon$  of dimension  $l^{nN}$ .

It remains to show  $d^2 \leq l^{2nN}$ , which by  $[Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = l^{n(2N+m)}$  is equivalent to  $[Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] \geq l^{mn}$ . For this, it is enough to exhibit an extension of  $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$  contained in  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  and of degree  $l^{mn}$ . There is a very natural one, which we denote by  $Q(\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon))$  and is constructed as follows. Consider for every  $\lambda \in P_+$  the matrices

$$M_\lambda := ({}_{AV_\lambda} \phi_{e_k}^{e_l})_{k,l} \in \text{End}({}_A V_\lambda) \otimes \mathcal{L}_{0,n}^A, \quad M_\lambda^{(i)} := (({}_{AV_\lambda} \phi_{e_k}^{e_l})^{(i)})_{k,l} \in \text{End}({}_A V_\lambda) \otimes \mathcal{L}_{0,n}^A,$$

where  $i = 1, \dots, n$ , and as usual  ${}_{AV_\lambda} \phi_{e_k}^{e_l}$  is a matrix coefficient of  ${}_A V_\lambda$ ,  $\{e_k\}$  the canonical basis of  ${}_A V_\lambda$ , and  $({}_{V_\lambda} \phi_{e_k}^{e_l})^{(i)} := 1^{\otimes(i-1)} \otimes {}_{V_\lambda} \phi_{e_k}^{e_l} \otimes 1^{\otimes(n-i)}$ . Set

$$\lambda \omega := \text{Tr}(\pi_{V_\lambda}(\ell) M_\lambda), \quad \lambda \omega^{(i)} := \text{Tr}(\pi_{V_\lambda}(\ell) M_\lambda^{(i)})$$



where  $Tr$  is the standard trace on  $End(V_\lambda)$ . Clearly  $\lambda\omega \in \mathcal{L}_{0,1}^A$ ,  $\lambda\omega^{(i)} \in \mathcal{L}_{0,n}^A$ . By Proposition 4.8 and 6.24 of [31], the family of elements  $\prod_{i=1}^n \lambda_i \omega^{(i)}$ , where  $\lambda_1, \dots, \lambda_n \in P_+$ , is a basis of  $\mathcal{Z}(\mathcal{L}_{0,n})$ ; moreover the Alekseev map  $\Phi_n$  affords an isomorphism from  $\mathcal{Z}(\mathcal{L}_{0,n})$  to  $\mathcal{Z}(U_q)^{\otimes n}$ , and  $\Phi_n(\lambda\omega^{(i)}) = (\Phi_1(\lambda\omega))^{(i)}$ . For  $n = 1$ , specializing  $q$  to  $\epsilon$  it follows

$$(109) \quad \mathcal{Z}_1(U_\epsilon) = Vect \{ \Phi_1(\lambda\omega), \lambda \in P_+ \},$$

where  $\mathcal{Z}_1(U_\epsilon)$  is defined before Theorem 2.22. Then, for every  $i = 1, \dots, n$  define

$$\mathcal{Z}_{0,(i)}(\mathcal{L}_{0,n}^\epsilon) := \mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)[\{\lambda\omega^{(i)}, \lambda \in P_+\}]$$

and let  $\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon) \subset \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  be the algebra generated by  $\mathcal{Z}_{0,(1)}(\mathcal{L}_{0,n}^\epsilon), \dots, \mathcal{Z}_{0,(n)}(\mathcal{L}_{0,n}^\epsilon)$ . The fields  $Q(\hat{\mathcal{Z}}_{0,(i)}(\mathcal{L}_{0,n}^\epsilon))$  are  $n$  linearly disjoint extensions of  $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ , so

$$[Q(\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = \prod_{i=1}^n [Q(\hat{\mathcal{Z}}_{0,(i)}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))].$$

Now, by Proposition 4.2 we know that  $\Phi_1$  affords isomorphisms  $Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)) \cong Q(\mathcal{Z}_0(U_\epsilon^{lf}))$  and  $Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)) \cong Q(\mathcal{Z}(U_\epsilon^{lf}))$ , and moreover

$$(110) \quad Q(\mathcal{Z}_0(U_\epsilon)) = Q(\mathcal{Z}_0(U_\epsilon^{lf}))(T^{(l)}/T_2^{(l)}), \quad Q(\mathcal{Z}(U_\epsilon)) = Q(\mathcal{Z}(U_\epsilon^{lf}))(T^{(l)}/T_2^{(l)}).$$

Computing via the field embedding  $\Phi_1^{\otimes n} : Q(\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon)) \rightarrow Q(\mathcal{Z}(U_\epsilon^{\otimes n}))$ , we deduce

$$\begin{aligned} & [Q(\hat{\mathcal{Z}}_{0,(i)}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] \\ &= [\Phi_1^{\otimes n}(Q(\hat{\mathcal{Z}}_{0,(i)}(\mathcal{L}_{0,n}^\epsilon))) : \Phi_1^{\otimes n}(Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)))] \\ &= \left[ Q(\mathcal{Z}_0(U_\epsilon^{lf})^{\otimes n})[\{(\Phi_1(\lambda\omega))^{(i)}, \lambda \in P_+, i = 1, \dots, n\}] : Q(\mathcal{Z}_0(U_\epsilon^{lf})^{\otimes n}) \right] \\ &= \left[ Q(\mathcal{Z}_0(U_\epsilon)^{\otimes n})[\{(\Phi_1(\lambda\omega))^{(i)}, \lambda \in P_+, i = 1, \dots, n\}] : Q(\mathcal{Z}_0(U_\epsilon)^{\otimes n}) \right] \\ &= l^m. \end{aligned}$$

The second and third equalities follow from (110) and the properties of  $\Phi_1$  recalled before it, and the last equality follows from Theorem 2.24 (2) and (109). As a result we have  $[Q(\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = l^{mn}$ , whence  $[Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] \geq l^{mn}$ . Since  $[Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = l^{n(m+2N)}$ , by (108) we obtain  $d^2 \leq l^{2nN}$ , which concludes the proof.  $\square$

**Remark 5.3.** By the degree computation above, it follows  $[Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = l^{mn}$ , whence  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) = Q(\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon))$ . We expect  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon) = \hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon)$ .

**Theorem 5.4. (Theorem 1.3(2))**  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ ,  $n \geq 2$ , is a division algebra and a central simple algebra of PI degree  $l^{N(n-1)-m}$ .

*Proof.* The center of  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  contains  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ , so the finite dimensionality of  $Q(\mathcal{L}_{0,n}^\epsilon)$  over  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  implies the finite dimensionality of  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  over its center. Since it has no non-zero divisors, this proves  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is a division algebra.

Now denote by  $\Delta^{(n)} : \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon^{\otimes n}$ ,  $n \geq 2$ , the  $n$ -fold coproduct, ie.  $\Delta^{(2)} := \Delta$ , the standard coproduct of  $\mathcal{O}_\epsilon$ , and  $\Delta^{(n)} := (id \otimes \Delta) \circ \Delta^{(n-1)}$  for  $n \geq 3$ . Identifying  $\mathcal{L}_{0,n}^\epsilon$  with  $\mathcal{O}_\epsilon^{\otimes n}$  as a vector space, we consider  $\Delta^{(n)}$  as a map  $\Delta^{(n)} : \mathcal{L}_{0,1}^\epsilon \rightarrow \mathcal{L}_{0,n}^\epsilon$ . It is an algebra morphism ([31], Proposition 6.18), injective because  $(\epsilon^{\otimes(n-1)} \otimes id)\Delta^{(n)} = id$ . Then it extends uniquely to the fraction algebra  $Q(\mathcal{L}_{0,1}^\epsilon)$ . As noted above  $Q(\mathcal{L}_{0,1}^\epsilon) = Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)) \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)} \mathcal{L}_{0,1}^\epsilon$ . Since

$\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$  is a Hopf subalgebra of  $\mathcal{O}_\epsilon$  ([45], Proposition 6.4),  $\Delta^{(n)}$  maps  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  to  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$ . Then, extending the scalars of  $\Delta^{(n)}(Q(\mathcal{L}_{0,1}^\epsilon))$  by the field  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , consider the algebra

$$\begin{aligned} Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) &:= Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))} \Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon) \\ &= Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)))} \Delta^{(n)}(Q(\mathcal{L}_{0,1}^\epsilon)) \\ &= \left( Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)))} \Delta^{(n)}(Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))) \right) \\ &\quad \otimes_{\Delta^{(n)}(Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)))} \Delta^{(n)}(Q(\mathcal{L}_{0,1}^\epsilon)). \end{aligned}$$

By Proposition 5.2,  $\Delta^{(n)}(Q(\mathcal{L}_{0,1}^\epsilon))$  is a  $\Delta^{(n)}(Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)))$ -central simple algebra. The left factor is a field, so  $Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$  is a central simple algebra over it (see eg. [88], Theorem 1.7.27, or [95], Lemma 4.9). Note that the left factor can also be written as

$$\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) := Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))} \Delta^{(n)}(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))$$

for it contains  $\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , it is contained in its fraction field, and  $\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  is a field because  $\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)$  is finite over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  and has no non trivial zero divisors. Note that

$$[\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))] = l^m$$

We proved in Proposition 6.19 of [31] that the ring  $(\mathcal{L}_{0,n}^A)^{U_A}$  is the centralizer of  $\Delta^{(n)}(\mathcal{L}_{0,1}^A)$  in  $\mathcal{L}_{0,n}^A$ ; the same arguments show that  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is the centralizer of  $\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)$  in  $\mathcal{L}_{0,n}^\epsilon$ . So the algebra

$$Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) := Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$$

is the centralizer of  $Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$  in  $Q(\mathcal{L}_{0,n}^\epsilon)$ . Since the latter is simple, we can apply the double centralizer theorem (see eg. [88], Theorem 7.1.9, or [95], Theorem 7.1):  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is a simple algebra, we have

$$[Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))] = \frac{[Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]}{[Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]} = l^{2nN-(2N+m)},$$

and the centralizer of  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is  $Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$ . In particular  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  has center  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) \cap Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$ , which is easily shown to be  $\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ . It then follows

$$\begin{aligned} [Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) : \tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))] &= \frac{[Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]}{[\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]} \\ &= l^{2nN-(2N+m)} l^{-m} = l^{2(N(n-1)-m)}. \end{aligned}$$

Therefore  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is a central simple algebra of PI degree  $l^{N(n-1)-m}$ .  $\square$

## 6. APPENDIX

**6.1. Quantum Weyl group.** We recall some of the formulas of [34]. Let  $e_q(z)$  be the formal power series in  $z$  with coefficients in  $\mathbb{C}(q)$  defined by:

$$(111) \quad e_q(z) = \sum_{n=0}^{+\infty} \frac{z^n}{(n)_q!}.$$

We first consider the case of  $\mathfrak{g} = sl(2)$ . As explained in [31], Section 3, the Cartan element  $H \in \mathfrak{g}$  defines an element of  $\mathbb{U}_q(sl(2))$ . Viewed as elements of  $\mathbb{U}_q$  we have  $L = q^{H/2}$ . The

series  $\Theta = q^{H \otimes H/2}$  defines an element of  $\mathbb{U}_q(\mathfrak{sl}(2))^{\hat{\otimes} 2}$ , its image under multiplication being  $q^{H^2/2}$ . The  $R$ -matrix can be expressed as  $R = \Theta \hat{R}$  where  $\hat{R} = e_{q^{-1}}((q - q^{-1})E \otimes F)$  is a well defined element of  $\mathbb{U}_q^{\hat{\otimes} 2}$ . Consider the Lusztig [75] braid group automorphism of  $U_q(\mathfrak{sl}(2))$ , defined by

$$(112) \quad T(L) = L^{-1}, T(E) = -FK^{-1}, T(F) = -KE.$$

For every  $x \in U_q(\mathfrak{sl}(2))$  it satisfies:

$$(113) \quad \Delta(T(x)) = \hat{R}^{-1}(T \otimes T)(\Delta(x))\hat{R}.$$

Define the quantum Weyl group element  $\hat{w} \in \mathbb{U}_q(\mathfrak{sl}(2))$  by Saito's formula [90]:

$$(114) \quad \hat{w} = e_{q^{-1}}(F)q^{-H^2/4}e_{q^{-1}}(-E)q^{-H^2/4}e_{q^{-1}}(F)q^{-H/2}.$$

For every  $x \in U_q(\mathfrak{sl}(2))$  it satisfies:

$$(115) \quad T(x) = \hat{w}x\hat{w}^{-1},$$

$$(116) \quad \Delta(\hat{w}) = \hat{R}^{-1}(\hat{w} \otimes \hat{w}),$$

$$(117) \quad \hat{w}^2 = q^{H^2/2}\xi\theta,$$

where  $\theta \in \mathbb{U}_q(\mathfrak{sl}(2))$  is the ribbon element, and  $\xi \in \mathbb{U}_q(\mathfrak{sl}(2))$  is the central group element whose value on the type 1 simple module  $V_k$  of  $U_q^{ad}(\mathfrak{sl}(2))$  of dimension  $k + 1$  is the scalar endomorphism  $(-1)^k id_{V_k}$ .

In order to compare our setting to the one of [45] we need an explicit formula of  $\hat{w}$ . Consider the basis vectors  $v_0, \dots, v_k$  of  $V_k$  such that:

$$\begin{aligned} K.v_j &= \epsilon^{k-2j}v_j, \\ F.v_j &= v_{j+1} \text{ if } j < k, \quad F.v_k = 0, \\ E.v_j &= [j]_\epsilon[k-j+1]_\epsilon v_{j-1} \text{ if } j > 0, \quad E.v_0 = 0. \end{aligned}$$

Setting  $v'_j = v_j/[j]!$  and using (112), (115) and (117), we obtain:

$$(118) \quad \hat{w}v'_j = (-1)^j q^{-j(k-j-1)-k} v'_{k-j}.$$

In [45] another quantum Weyl group element  $\underline{w}$  is defined. It is dual to the Vaksman-Soibelman functional  $t: \mathcal{O}_q(SL_2) \rightarrow \mathbb{C}(q)$  of [93, 92], that is,  $t(\alpha) = \langle \alpha, \underline{w} \rangle$  for all  $\alpha \in \mathcal{O}_q(SL_2)$ . By comparing (118) with the formulas defining the action of  $t$  in Section 1.7 of [45], we find

$$\underline{w} = \xi \hat{w} K$$

and the basis vectors  $w_r^p$  of [45], where  $p \in (1/2)\mathbb{N}$  and  $r \in \{-p, -p+1, \dots, p-1, p\}$ , are related to the vectors  $v'_j$  above as follows:

$$v'_j = \lambda_j w_r^p$$

where  $k = 2p$ ,  $j = p - r$ ,  $\lambda_0 = 1$ ,  $\lambda_1 = [k]q^{-k}$ , and

$$\lambda_j = \frac{[k]!}{[j]![k-(j-2)]!} q^{j(j+1)-j(k+2)}, \quad j \geq 2.$$

Explicit formulas of the evaluation of  $t$  on basis vectors of  $\mathcal{O}_q(SL_2)$  can be computed. We get:

$$(119) \quad t(\tilde{a}^{\star m} \star \tilde{b}^{\star n} \star \tilde{d}^{\star p}) = \delta_{m,p} q^{-np} \prod_{i=1}^p (1 - q^{-2i}),$$

$$(120) \quad t(\tilde{a}^{\star m} \star \tilde{c}^{\star n} \star \tilde{d}^{\star p}) = (-1)^n \delta_{m,p} q^{-n(p+1)} \prod_{i=1}^p (1 - q^{-2i})$$

where

$$(121) \quad \tilde{a} = a, \quad \tilde{b} = qb, \quad \tilde{c} = q^{-1}c, \quad \tilde{d} = d$$

and as usual  $a, b, c, d$  are the standard generators of  $\mathcal{O}_q(SL_2)$ , ie. the matrix coefficients in the basis of weight vectors  $v_0, v_1$  of the 2-dimensional irreducible representation  $V_1$  of  $U_q(\mathfrak{sl}(2))$  such that  $K.v_0 = qv_0$  and  $v_1 = F.v_0$ . Here we have introduced the generators  $\tilde{a}, \dots, \tilde{d}$  to facilitate the comparison with the formulas in [45]; these generators come naturally in their setup because they use different generators  $E_i$  and  $F_i$  of  $U_q(\mathfrak{g})$ , which in our notations can be written respectively as  $K_i^{-1}E_i$  and  $F_iK_i$ .

The formulas (119)-(120) can be shown by two independent methods. The first uses a definition of  $t$  as a GNS state associated to an infinite dimensional representation of  $\mathcal{O}_q(SL_2)$ , as recalled in Section 1.6 of [45]. The second is to write eg.

$$(122) \quad t(\tilde{a}^{\star m} \star \tilde{b}^{\star n} \star \tilde{d}^{\star p}) = \left\langle \tilde{a}^{\otimes m} \otimes \tilde{b}^{\otimes n} \otimes \tilde{d}^{\otimes p}, \Delta^{(m+n+p-1)}(\underline{w}) \right\rangle$$

and to use explicit expressions of  $\Delta^{(m+n+p-1)}(\underline{w})$  when represented on  $V_1^{\otimes(m+n+p)}$ . In general one can check that

$$(123) \quad \Delta^{(n)}(\hat{\omega}) = \left( \Delta^{(n-1)} \otimes id \right) \left( \hat{R}^{-1} \right) \left( \left( \Delta^{(n-2)} \otimes id \right) \left( \hat{R}^{-1} \right) \otimes id \right) \\ \dots \left( \left( \Delta \otimes id \right) \left( \hat{R}^{-1} \right) \otimes id^{\otimes(n-3)} \right) \left( \hat{R}^{-1} \otimes id^{\otimes(n-2)} \right) \hat{\omega}^{\otimes n}.$$

By (118) or (119)-(120) we see that  $\hat{w}$  (or  $\underline{w}$ ) and  $t$  are well-defined on the integral forms,

$$\hat{w} \in \mathbb{U}_\Gamma, \quad t: \mathcal{O}_A(SL_2) \rightarrow A.$$

We now consider the case where  $\mathfrak{g}$  is of rank  $m \geq 2$ . To each simple root  $\alpha_i$ ,  $1 \leq i \leq m$ , is associated the subalgebra of  $U_q$  generated by  $E_i, F_i, L_i, L_i^{-1}$ . It is a copy of  $U_{q_i}(\mathfrak{sl}(2))$ , where  $q_i = q^{d_i}$ . Let  $\hat{w}_i$  be the corresponding quantum Weyl group element in  $\mathbb{U}_q = \mathbb{U}_q(\mathfrak{g})$ , defined by Saito's formula (114), replacing  $H, E, F$  by  $H_i, E_i$  and  $F_i$ . Also, denote by  $\nu_i: \mathcal{O}_q \rightarrow \mathcal{O}_{q_i}(SL_2)$  the projection map dual to the inclusion  $U_{q_i}(\mathfrak{sl}(2)) \otimes_{\mathbb{C}(q_i)} \mathbb{C}(q) \hookrightarrow U_q$ , and put  $t_i = t \circ \nu_i$ . Let  $\underline{w}_i$  be the corresponding quantum Weyl group element in  $\mathbb{U}_q$ , ie.  $t_i(\alpha) = \langle \alpha, \underline{w}_i \rangle$  for all  $\alpha \in \mathcal{O}_q$ . On integral forms they yield well-defined elements  $\hat{w}_i, \underline{w}_i \in \mathbb{U}_\Gamma$  and  $t_i: \mathcal{O}_A \rightarrow A$  (see [45], Proposition 5.1, and [77] for a different construction). They satisfy the defining relations of the braid group  $\mathcal{B}(\mathfrak{g})$  of  $\mathfrak{g}$  [70]:

$$\hat{w}_i \hat{w}_j \hat{w}_i = \hat{w}_j \hat{w}_i \hat{w}_j \quad \text{if } a_{ij} a_{ji} = 1 \\ (\hat{w}_i \hat{w}_j)^k = (\hat{w}_j \hat{w}_i)^k \quad \text{for } k = 1, 2, 3 \text{ if } a_{ij} a_{ji} = 0, 2, 3$$

and similarly by replacing  $\hat{w}_i$  with  $\underline{w}_i$ , or with  $t_i$  (see [92] for the latter). The Weyl group  $W = W(\mathfrak{g}) = N(T_G)/T_G$  is generated by the reflexions  $s_i$  associated to the simple roots  $\alpha_i$ . Denote by  $n_i \in N(T_G)$  a representative of  $s_i$ . Let  $w \in W$  and denote by  $w = s_{i_1} \dots s_{i_k}$  a reduced expression. Because of the braid group relations the elements  $\hat{w} = \hat{w}_{i_1} \dots \hat{w}_{i_k}$ ,  $\underline{w} = \underline{w}_{i_1} \dots \underline{w}_{i_k}$  and the functional  $t_w = t_{i_1} \dots t_{i_k}$  do not depend on the choice of reduced

expression. The Lusztig [75] braid group automorphism  $T_w: \Gamma \rightarrow \Gamma$  associated to  $w$  satisfies (see [45]):

$$T_w(x) = \hat{w}x\hat{w}^{-1}, \quad x \in \Gamma.$$

Let  $w_0$  be the longest element in  $W$ . We have

$$(124) \quad \Delta(\hat{w}_0) = \hat{R}^{-1}(\hat{w}_0 \otimes \hat{w}_0)$$

where as usual  $R = \Theta\hat{R}$ .

**6.2. Regular action on  $\mathcal{O}_\epsilon$ .** The following result is proved in Section 1.10 of [45]. For completeness let us give a (different) proof. Recall from (80) that we may identify  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  with  $\mathcal{O}(G)$ .

**Proposition 6.1.** *For every  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ ,  $g \in \mathcal{O}_\epsilon$  we have*

$$(125) \quad t_i(f) = f(n_i)$$

$$(126) \quad t_i(f \star g) = t_i(f)t_i(g).$$

*Proof.* It is sufficient to prove the results for  $SL_2$  because  $\nu_i: \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon(SL_2)$  is a morphism of Hopf algebras and  $\nu_i(\mathcal{Z}_0(\mathcal{O}_\epsilon)) \subset \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ . In this case (125) can be proved by using (119)-(120), evaluating  $t$  on basis elements of  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$  as is done in Lemma 1.5 (a) of [45]. Such a basis is formed by monomials like in (119)-(120), with all exponents divisible by  $l$ ; then for instance

$$t(\tilde{a}^{\star ml} \star \tilde{b}^{\star nl} \star \tilde{d}^{\star pl}) = \delta_{p,0}\delta_{m,0} = \underline{a}^m \underline{b}^n \underline{d}^p(n)$$

where  $\underline{a}, \dots, \underline{d}$  are the generators of  $\mathcal{O}(G) = \mathcal{O}_1(G)$  corresponding to  $a, \dots, d$ , and we take

$$n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

as representative of the reflexion  $s$  generating the Weyl group  $W(sl(2))$ . Here is an alternative proof of (125): (126) shows that  $t$  is a homomorphism on  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ , so by proving (126) at first one is reduced to check (125) on the generators  $a^{\star l}, \dots, d^{\star l}$ , which is easy by means of (122) and (124).

We provide a proof of (126) that we find more conceptual than the one in Lemma 1.5 (b) of [45] (which uses again (119)-(120)). As above let us denote  $\underline{w} = \xi\hat{w}K$ . For any  $f, g \in \mathcal{O}_\epsilon$  we have

$$\begin{aligned} t(f \star g) &= (f \otimes g)(\Delta(\underline{w})) \\ &= (f \otimes g)\left(\hat{R}^{-1}(\underline{w} \otimes \underline{w})\right) \\ &= \sum_{(\hat{R}^{-1})} f\left((\hat{R}^{-1})_{(1)}\underline{w}\right) g\left((\hat{R}^{-1})_{(2)}\underline{w}\right) \\ &= \sum_{(\hat{R}^{-1}), (f)} f_{(1)}\left((\hat{R}^{-1})_{(1)}\right) f_{(2)}(\underline{w}) g\left((\hat{R}^{-1})_{(2)}\underline{w}\right) \\ &= \sum_{(f)} f_{(2)}(\underline{w}) g\left((f_{(1)} \otimes id)(\hat{R}^{-1})\underline{w}\right). \end{aligned}$$

Assume now  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ . Since  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$  is a Hopf subalgebra of  $\mathcal{O}_\epsilon(SL_2)$  we have  $f_{(1)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ . From Theorem 2.24 (2) we deduce

$$(f_{(1)} \otimes id)(\hat{R}^{-1}) \in U_\epsilon(\mathfrak{n}_-) \cap \mathcal{Z}_0(U_\epsilon).$$

Denote by  $z$  this element. Note that from its expression we have  $\epsilon(z) = \epsilon(f_{(1)})$ . Now  $g(z\underline{w}) = \sum_{(g)} g_{(1)}(z)g_{(2)}(\underline{w})$ , but  $g_{(1)}$  is a linear combination of matrix elements of  $\Gamma$ -modules, on which  $\mathcal{Z}_0(U_\epsilon)$  acts by the trivial character. Therefore

$$g(z\underline{w}) = \sum_{(g)} \epsilon(z)g_{(1)}(1)g_{(2)}(\underline{w}) = \epsilon(z)g(\underline{w}) = \epsilon(f_{(1)})g(\underline{w})$$

and eventually

$$t(f \star g) = \sum_{(f)} f_{(2)}(\underline{w})\epsilon(f_{(1)})g(\underline{w}) = t(f)t(g).$$

This concludes the proof.  $\square$

For the sake of completeness, let us show how this result implies:

*Proof of Proposition 2.25 (ie. Proposition 7.1 of [45]).* We have  $f \triangleleft t_i = \sum_{(f)} t_i(f_{(1)})f_{(2)}$ ,  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ . Since  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  is a Hopf subalgebra of  $\mathcal{O}_\epsilon$ ,  $f_{(2)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$  and therefore the maps  $\triangleleft t_i: \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon$  preserve  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ . Moreover,  $(f \triangleleft t_i)(a) = \sum_{(f)} f_{(1)}(n_i)f_{(2)}(a) = f(n_i a)$ ,  $a \in G$ , by (125).

It remains to show that  $(f \star \alpha) \triangleleft t_i = (f \triangleleft t_i)(\alpha \triangleleft t_i)$  for every  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ ,  $\alpha \in \mathcal{O}_\epsilon$ . We have

$$\begin{aligned} (f \star g) \triangleleft t_i &= \sum_{(f \star g)} t_i((f \star g)_{(1)})(f \star g)_{(2)} = \sum_{(f),(g)} t_i(f_{(1)} \star g_{(1)})f_{(2)} \star g_{(2)} \\ &= \sum_{(f),(g)} t(\nu_i(f_{(1)})\nu_i(g_{(1)}))f_{(2)} \star g_{(2)} \\ (127) \qquad &= \sum_{(f),(g)} t(\nu_i(f_{(1)}))t(\nu_i(g_{(1)}))f_{(2)} \star g_{(2)} \end{aligned}$$

using that  $\nu_i$  is a homomorphism in the third equality, and (126) in the last one. The result is just  $(f \triangleleft t_i)(g \triangleleft t_i)$ .  $\square$

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