

Supplementary material to: “A Bayesian approach for the separation of the acoustic and the correlated aerodynamic wall pressure fluctuations”

Alice Dinsenmeyer¹, Quentin Leclère¹, Jérôme Antoni¹, and Emmanuel Julliard²

¹Univ Lyon, INSA Lyon, LVA, EA677, 69621 Villeurbanne, France

²Airbus Operations S.A.S, Acoustics Department, Toulouse, France

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Abstract

This document complement the article *A Bayesian approach for the separation of the acoustic and the correlated aerodynamic wall pressure fluctuations* (from the same authors). It contains the expressions of the posterior distributions required for the implementation of the PFA-Corr approach described in the article, as well as the pseudo-code of the separation algorithm based on these posteriors.

1 Posterior distributions

In this section are given the expressions of the likelihood of the data and the posterior distributions required for the implementation of the CSM-based Gibbs sampler, to solve the PFA-Corr separation problem. The same mathematical notations as in the related article are used.

1.1 Summary of the PFA-Corr model

The Fa-Corr decomposition in the vector form is

$$\mathbf{y}_j = \mathbf{L}\mathbf{c}_j + p\boldsymbol{\nu}_j + \boldsymbol{\epsilon}, \quad j = 1, \dots, N_s, \quad (1)$$

with the following priors on each parameters:

$$[\mathbf{c}] = \mathcal{N}_{\mathbb{C}}(\mathbf{0}, [\boldsymbol{\sigma}_{\mathbf{c}}^2]), \quad (2)$$

$$[\boldsymbol{\sigma}_{\mathbf{c}}^2] = \mathcal{IG}(\mathbf{a}_{\mathbf{c}}, \mathbf{b}_{\mathbf{c}}), \quad (3)$$

$$[\boldsymbol{\nu}] = \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\nu}}^2(\boldsymbol{\theta})), \quad (4)$$

$$[\boldsymbol{\theta}] = \mathcal{N}_{\mathbb{R}}(\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^2), \quad (5)$$

$$[\mathbf{L}] = \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \frac{\mathbf{I}_{\kappa M}}{\kappa}), \quad (6)$$

$$[\boldsymbol{\epsilon}] = \mathcal{N}_{\mathbb{C}}(\mathbf{0}, [\boldsymbol{\sigma}_{\boldsymbol{\epsilon}}^2]), \quad (7)$$

$$[\boldsymbol{\sigma}_{\boldsymbol{\epsilon}}^2] = \mathcal{IG}(\mathbf{a}_{\boldsymbol{\epsilon}}, \mathbf{b}_{\boldsymbol{\epsilon}}), \quad (8)$$

$$[p] = \mathcal{N}_{\mathbb{C}}(\mu_p, \sigma_p^2). \quad (9)$$

1.2 Likelihood of the measurements

From the statement of the separation problem given in (1), one can write the likelihood as follows:

$$[\mathbf{y}_j | \infty] = \mathcal{N}_{\mathbb{C}}(\mathbf{L}\mathbf{c}_j + p\boldsymbol{\nu}_j, [\boldsymbol{\sigma}_{\boldsymbol{\epsilon}}^2]). \quad (10)$$

1.3 Sampling of \mathbf{c}

Using the expression of the likelihood in Eq. (10) and the Gaussian prior on \mathbf{c} (Eq. (2)) leads to:

$$[\mathbf{c}_j | \infty] \propto [\mathbf{c}_j][\mathbf{y}_j | \infty], \quad (11)$$

$$\propto \mathcal{N}_{\mathbb{C}}(\boldsymbol{\mu}_{\mathbf{c}_j}, \boldsymbol{\Omega}_{\mathbf{c}}),$$

$$\text{with } \boldsymbol{\mu}_{\mathbf{c}_j} = \underbrace{\boldsymbol{\Omega}_{\mathbf{c}} \mathbf{L}^H [\boldsymbol{\sigma}_{\boldsymbol{\epsilon}}^{-2}]}_{\mathbf{T}_{\mathbf{c}}^H} (\mathbf{y}_j - p\boldsymbol{\nu}_j), \quad (12)$$

$$\text{and } \boldsymbol{\Omega}_{\mathbf{c}} = (\mathbf{L}^H [\boldsymbol{\sigma}_{\boldsymbol{\epsilon}}^{-2}] \mathbf{L} + [\boldsymbol{\sigma}_{\mathbf{c}}^{-2}])^{-1}. \quad (13)$$

For the implementation of the quadratic Gibbs sampler, the same approach as in Ref. 1 is followed. As $\mathbf{c}_j | \infty$ is Gaussian, it can be written

$$\mathbf{c}_j | \infty = \boldsymbol{\mu}_{\mathbf{c}_j} + \mathbf{x}_j \quad \text{with} \quad [\mathbf{x}_j] = \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{\Omega}_{\mathbf{c}}). \quad (14)$$

Then,

$$\mathbf{S}_{cc} | \infty = \frac{1}{N_s} \sum_{j=1}^{N_s} \boldsymbol{\mu}_{\mathbf{c}_j} \boldsymbol{\mu}_{\mathbf{c}_j}^H + \frac{1}{N_s} \sum_{j=1}^{N_s} \mathbf{x}_j \mathbf{x}_j^H + \frac{2}{N_s} \sum_{j=1}^{N_s} \mathbf{x}_j \boldsymbol{\mu}_{\mathbf{c}_j}^H. \quad (15)$$

Since \mathbf{x}_j and $\boldsymbol{\mu}_{\mathbf{c}_j}$ are independent random variables, the last terms tends to zero

$$\begin{aligned} \mathbf{S}_{cc} | \infty &\approx \mathbb{E}\{\boldsymbol{\mu}_{\mathbf{c}_j} \boldsymbol{\mu}_{\mathbf{c}_j}^H\} + \frac{1}{N_s} \mathbf{W}_{\mathbf{c}} \\ &= \mathbf{T}_{\mathbf{c}}^H (\mathbf{S}_{yy} + p\mathbf{S}_{\boldsymbol{\nu}\boldsymbol{\nu}} p^H - p\mathbf{S}_{\boldsymbol{\nu}y} - \mathbf{S}_{y\boldsymbol{\nu}} p^H) \mathbf{T}_{\mathbf{c}} \\ &\quad + \frac{1}{N_s} \mathbf{W}_{\mathbf{c}}, \end{aligned} \quad (16)$$

where $\mathbb{E}\{\cdot\}$ is the expected value over the snapshots, \mathbf{W}_c is a random matrix that follows a complex Wishart distribution, with N_s degrees of freedom and variance matrix $\mathbf{\Omega}_c$. The expression of the CSMs $\mathbf{S}_{\nu\nu}$ and $\mathbf{S}_{y\nu}$ are given later in Eq. (31).

1.4 Sampling of ν

Similarly, from the Gaussian prior assigned to ν in Eq. (4),

$$\begin{aligned} [\nu_j | \infty] &\propto [\nu_j][y_j | \infty], \\ &\propto \mathcal{N}_{\mathbb{C}}(\mu_{\nu_j}, \mathbf{\Omega}_{\nu}), \end{aligned} \quad (17)$$

$$\text{with } \mu_{\nu_j} = \underbrace{\mathbf{\Omega}_{\nu} p^H [\sigma_{\epsilon}^{-2}]^{-2}}_{\mathbf{T}_{\nu}^H} (\mathbf{y}_j - \mathbf{L} \mathbf{c}_j), \quad (18)$$

$$\text{and } \mathbf{\Omega}_{\nu} = (p^H [\sigma_{\epsilon}^{-2}] p + \mathbf{\Sigma}_{\nu}(\boldsymbol{\theta})^{-2})^{-1}. \quad (19)$$

For the CSM-based Gibbs sampler:

$$\begin{aligned} \mathbf{S}_{\nu\nu} | \infty &\approx \mathbb{E}\{\mu_{\nu_j} \mu_{\nu_j}^H\} + \frac{1}{N_s} \mathbf{W}_{\nu} \quad (20) \\ &= \mathbf{T}_{\nu}^H (\mathbf{S}_{yy} + \mathbf{L} \mathbf{S}_{cc} \mathbf{L}^H - \mathbf{L} \mathbf{S}_{cy} - \mathbf{S}_{yc} \mathbf{L}^H) \mathbf{T}_{\nu} \\ &\quad + \frac{1}{N_s} \mathbf{W}_{\nu}, \end{aligned} \quad (21)$$

where \mathbf{W}_{ν} follows a complex Wishart distribution, with N_s degrees of freedom and variance matrix $\mathbf{\Omega}_{\nu}$. The expression of the CSMs \mathbf{S}_{cy} and \mathbf{S}_{yc} are given later in Eq. (29).

1.5 Sampling of L

The sampling of L is made using a vectorized form of L , written $\boldsymbol{\lambda} = \text{vec}(L)$. From the prior on L given in Eq. (6), the posterior is calculated as follows:

$$\begin{aligned} [\boldsymbol{\lambda} | \infty] &\propto [\boldsymbol{\lambda}] \prod_{j=1}^{N_s} [y_j | \infty], \\ &\propto \mathcal{N}_{\mathbb{C}}(\mu_{\boldsymbol{\lambda}}, \mathbf{\Omega}_{\boldsymbol{\lambda}}). \end{aligned} \quad (22)$$

Using some properties of the Kronecker product \otimes (see Ref.2) gives

$$\begin{aligned} \mathbf{\Omega}_{\boldsymbol{\lambda}} &= \left(\sum_j (\mathbf{c}_j^T \otimes \mathbf{I}_M)^H [\sigma_{\epsilon}^{-2}] (\mathbf{c}_j^T \otimes \mathbf{I}_M) + \kappa \mathbf{I}_{M\kappa} \right)^{-1} \\ &= ((\mathbf{S}_{cc}^*) \otimes [\sigma_{\epsilon}^{-2}] + \kappa \mathbf{I}_{M\kappa})^{-1}, \end{aligned} \quad (23)$$

$$\begin{aligned} \text{and } \mu_{\boldsymbol{\lambda}} &= \mathbf{\Omega}_{\boldsymbol{\lambda}} \sum_j (\mathbf{c}_j^T \otimes \mathbf{I}_M)^H [\sigma_{\epsilon}^{-2}] (\mathbf{y}_j - p \nu_j) \\ &= \mathbf{\Omega}_{\boldsymbol{\lambda}} \text{vec}([\sigma_{\epsilon}^{-2}] (\mathbf{S}_{yc} - p \mathbf{S}_{\nu c})), \end{aligned} \quad (24)$$

where \cdot^* is the conjugate operator. Note that $\mathbf{\Omega}_{\boldsymbol{\lambda}}^{-1}$ is sparse, which can be taken into account to reduce the numerical cost of its inversion. The expression of the CSMs \mathbf{S}_{yc} and $\mathbf{S}_{\nu c}$ is given later in Eqs. (29) and (33) respectively.

1.6 Sampling of p

The complex scalar parameter p is assigned a Gaussian prior (Eq. (9)), which gives the following posterior:

$$\begin{aligned} [p | \infty] &\propto [p] \prod_j [y_j | \infty], \\ &\propto \mathcal{N}_{\mathbb{C}}(\mu_p, \mathbf{\Omega}_p), \end{aligned} \quad (25)$$

$$\begin{aligned} \text{with } \mathbf{\Omega}_p &= (\text{tr}(\sigma_{\epsilon}^{-2} \mathbf{S}_{\nu\nu}) + \sigma_p^{-2})^{-1}, \quad (26) \\ \text{and } \mu_p &= \mathbf{\Omega}_p (\text{tr}(\sigma_{\epsilon}^{-2} (\mathbf{S}_{y\nu} - \mathbf{L} \mathbf{S}_{c\nu})) + \sigma_p^{-2} a_p). \end{aligned} \quad (27)$$

The expression of the CSMs $\mathbf{S}_{y\nu}$ and $\mathbf{S}_{c\nu}$ is given later in Eqs. (31) and (33) respectively.

1.7 Expression of the cross-correlation matrices

The previous calculations require the expression of some cross-correlations, which are detailed in the present section. Using the decomposition of \mathbf{c}_j of Eq. (14),

$$\begin{aligned} \mathbf{S}_{yc} &= \frac{1}{N_s} \sum_{j=1}^{N_s} \mathbf{y}_j \mu_{c_j}^H + \frac{1}{N_s} \sum_{j=1}^{N_s} \mathbf{y}_j \mathbf{x}_j^H, \\ &\approx \sum_{j=1}^{N_s} \mathbf{y}_j \mathbf{c}_j^H, \end{aligned} \quad (28)$$

and replacing \mathbf{c}_j by its posterior mean value (given in Eq. (12)) leads to

$$\mathbf{S}_{yc} = (\mathbf{S}_{yy} - \mathbf{S}_{y\nu}) \mathbf{T}_c = \mathbf{S}_{cy}^H. \quad (29)$$

Similarly,

$$\mathbf{S}_{y\nu} = \sum_{j=1}^{N_s} \mathbf{y}_j \nu_j^H, \quad (30)$$

and replacing ν by its posterior mean value (given in Eq. (18)) leads to

$$\mathbf{S}_{y\nu} = (\mathbf{S}_{yy} - \mathbf{S}_{yc} \mathbf{L}^H) \mathbf{T}_{\nu} = \mathbf{S}_{\nu y}^H. \quad (31)$$

Again the calculations for $\mathbf{S}_{c\nu}$ starts from,

$$\mathbf{S}_{c\nu} = \sum_{j=1}^{N_s} \mathbf{c}_j \nu_j^H, \quad (32)$$

and at this step, either ν or \mathbf{c} can be replaced by its mean posterior. Choosing arbitrarily to replace ν leads to

$$\mathbf{S}_{c\nu} = (\mathbf{S}_{cy} - \mathbf{S}_{cc} \mathbf{L}^H) \mathbf{T}_{\nu} = \mathbf{S}_{\nu c}^H. \quad (33)$$

1.8 Sampling of σ_ϵ^2

The posterior expression of the additive noise is directly obtained from the conjugacy of the inverse-gamma with the Gaussian:

$$[\sigma_{\epsilon_m}^2 | \infty] \propto [\sigma_\epsilon^2] \prod_j [\mathbf{y}_j | \infty],$$

$$\propto \mathcal{IG}(\mathbf{a}_\epsilon + N_s, \mathbf{b}_\epsilon + \mathbf{T}_{mm}), \quad (34)$$

with

$$\mathbf{T} = \sum_j (\mathbf{y}_j - \mathbf{L}\mathbf{c}_j - p\boldsymbol{\nu}_j)(\mathbf{y}_j - \mathbf{L}\mathbf{c}_j - p\boldsymbol{\nu}_j)^H \quad (35)$$

$$\begin{aligned} \mathbf{T} &\approx \sum_j (\mathbf{y}_j - p\boldsymbol{\nu}_j - \mathbf{L}\mathbf{T}_c^H(\mathbf{y}_j - p\boldsymbol{\nu}_j))(\mathbf{y}_j - p\boldsymbol{\nu}_j - \mathbf{L}\mathbf{T}_c^H(\mathbf{y}_j - p\boldsymbol{\nu}_j))^H + \mathbf{L}\mathbf{W}_c\mathbf{L}^H \\ &= (\mathbf{I}_M - \mathbf{L}\mathbf{T}_c^H)(\mathbf{S}_{yy} + p\mathbf{S}_{\nu\nu}p^H - \mathbf{S}_{y\nu}p^H - p\mathbf{S}_{\nu y})(\mathbf{I}_M - \mathbf{T}_c\mathbf{L}^H) + \mathbf{L}\mathbf{W}_c\mathbf{L}^H. \end{aligned}$$

1.9 Sampling of σ_c^2

Still using the Bayes' rule, the posterior for the k^{th} element in the vector σ_c^2 can be written as follows:

$$[\sigma_{c_k}^2 | \infty] \propto [\sigma_{c_k}^2] \prod_{j=1}^{N_s} [c_j | \sigma_{c_k}^2],$$

$$\propto \mathcal{IG}(\mathbf{a}_c, \mathbf{b}_c) \prod_{j=1}^{N_s} \mathcal{N}_{\mathbb{C}}(\mathbf{0}, [\sigma_c^2]). \quad (37)$$

The use of the conjugacy of the inverse-gamma with the Gaussian directly gives the expression of the posterior:

$$[\sigma_{c_k}^2 | \infty] \propto \mathcal{IG}(\mathbf{a}_{c_k} + N_s, \mathbf{b}_{c_k} + \mathbf{S}_{cc_{kk}}). \quad (38)$$

1.10 Sampling of $\boldsymbol{\theta}$

The posterior distribution for $\boldsymbol{\theta}$ can be computed:

$$[\boldsymbol{\theta} | \infty] \propto [\boldsymbol{\theta}] \prod_{j=1}^{N_s} [\boldsymbol{\nu}_j | \infty], \quad (39)$$

$$\propto \frac{e^{-\sum_j \boldsymbol{\nu}_j^H \boldsymbol{\Sigma}_\nu^{-2}(\boldsymbol{\theta}) \boldsymbol{\nu}_j}}{\prod_j |\boldsymbol{\Sigma}_\nu^2(\boldsymbol{\theta})|} \frac{e^{-\frac{1}{2}(\boldsymbol{\theta} - \boldsymbol{\mu}_\theta)^H \boldsymbol{\Sigma}_\theta^{-2}(\boldsymbol{\theta} - \boldsymbol{\mu}_\theta)}}{|\boldsymbol{\Sigma}_\theta^2|}. \quad (40)$$

No closed-form of this expression can be developed for the sampling, therefore, a Metropolis-Hastings (MH) step is implemented within the Gibbs sampler.

$$\begin{aligned} &= \mathbf{S}_{yy} + \mathbf{L}\mathbf{S}_{cc}\mathbf{L}^H + p\mathbf{S}_{\nu\nu}p^H - \mathbf{S}_{yc}\mathbf{L}^H - \mathbf{L}\mathbf{S}_{cy} \\ &\quad - \mathbf{S}_{y\nu}p^H - p\mathbf{S}_{\nu y} + \mathbf{L}\mathbf{S}_{c\nu}p^H + p\mathbf{S}_{\nu c}\mathbf{L}^H. \end{aligned} \quad (36)$$

However, due to the iterative process, this expression can be slightly negative, which can lead to a negative parametrization of the inverse-gamma. Therefore, an alternative expression is proposed, replacing \mathbf{c}_j by $\mathbf{c}_j | \infty = \boldsymbol{\mu}_{c_j} + \mathbf{x}_j$, with $[\mathbf{x}_j] = \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{\Omega}_c)$ in Eq. (35):

Metropolis-Hastings within Gibbs implementation

The principle of the MH algorithm is to iteratively pick a parameter set in a candidate distribution based on the current set, and then accept or reject it as the new set with some probability. In the case of a symmetric proposal distribution, the probability of acceptance is given by the ratio of the posterior distribution computed with the current parameter set and with the proposed ones. The acceptance-rejection procedure is thus performed in three steps. First, a parameter set is sampled in a Gaussian proposal distribution

$$\boldsymbol{\theta}_i = \mathcal{N}_{\mathbb{R}}(\boldsymbol{\theta}_{(i-1)}, \boldsymbol{\Sigma}_{\text{prop}}^2), \quad (41)$$

where $\boldsymbol{\theta}_{(i-1)}$ is the Corcos' parameter set at the current state and $\boldsymbol{\Sigma}_{\text{prop}}^2$ is the covariance of the proposal. Then, the acceptance rate r is computed by

$$r = \min\left(1, \frac{[\boldsymbol{\theta}_{(i-1)} | \infty]}{[\boldsymbol{\theta}_i | \infty]}\right), \quad (42)$$

where \cdot_i indicates the i^{th} iteration. And finally the update is rejected if r is smaller than a sample u drawn in a uniform distribution $\mathcal{U}(0, 1)$, and accepted otherwise.

In order to explore properly the target distribution, the proposal should have the same shape as the target. This implies that the covariance of the proposal has to be adjusted so that the proposed steps are large enough to ensure good convergence, but not too large to avoid having a too high rejection rate. This covariance can be tuned automatically with adaptive MH algorithm, or manually by checking the convergence rate. It has been shown that the

optimal trade-off between the step size and the number of rejections is reached when the acceptance rate is 23.4% for a univariate Gaussian proposal and 44% for a Gaussian whose dimension tends to infinity [3].

2 Pseudo-code of the Gibbs sampler

The pseudo-code for the Gibbs sampler used to perform the separation proposed in the article is given in Alg. 1. In this code, $\delta \mathbf{r}$ is the set of relative coordinates of the microphones, required for the Corcos-like model.

Algorithm 1 CSM-based Gibbs sampler for PFA-Corr.

Initialization: $\mathbf{L}_0, \sigma_{\epsilon_0}^2, \sigma_{c_0}^2, \mathbf{S}_{\nu\nu_0}, \boldsymbol{\theta}_0, p_0$

Require: $\hat{\mathbf{S}}_{yy}, \kappa, \mathbf{a}_c, \mathbf{b}_c, \mathbf{a}_\epsilon, \mathbf{b}_\epsilon, \mu_p, \sigma_p^2, \boldsymbol{\mu}_\theta, \boldsymbol{\Sigma}_\theta^2, N_{\text{run}}, \delta \mathbf{r}, f, \boldsymbol{\Sigma}_{\text{prop}}^2$

for $i = 1, \dots, N_{\text{run}}$ **do**

Sample \mathbf{S}_{cc_i} following Eq. (16)

Sample \mathbf{L}_i in Eq. (22)

Sample $\sigma_{c_i}^2$ in Eq. (38)

Update \mathbf{S}_{yc} following Eq. (29)

Sample $\mathbf{S}_{\nu\nu_i}$ following Eq. (21)

Update $\mathbf{S}_{y\nu}$ following Eq. (31)

Sample p in Eq. (25)

▷ *Metropolis-Hastings step*

Sample $\boldsymbol{\theta}_i$ in Eq. (41)

Calculate the acceptance rate as Eq. (42)

if $r < (u \sim \mathcal{U}(0,1))$ **then**

$\boldsymbol{\theta}_i = \boldsymbol{\theta}_{i-1}$ ▷ *Rejection of the sample*

$\boldsymbol{\Sigma}_{\nu_i}^2 = \boldsymbol{\Sigma}_{\nu_{i-1}}^2$

else

$\boldsymbol{\Sigma}_{\nu_i}^2 = \mathbf{G}(\boldsymbol{\theta}_i)$ ▷ *G given by the Corcos-like model*

end if

Sample $\sigma_{\epsilon_i}^2$ in Eq. (34)

end for

return Posterior PDFs of $\mathbf{S}_{cc}, \mathbf{S}_{\nu\nu}, p, \mathbf{L}, \sigma_{\epsilon}^2, \sigma_c^2, \boldsymbol{\theta}$

References

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