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APPROXIMATION OF NULL CONTROLS FOR SEMILINEAR HEAT EQUATIONS USING A LEAST-SQUARES APPROACH

Jérôme Lemoine[1], Irene Marín-Gayte[2] and Arnaud Münch[3,*]

Abstract. The null distributed controllability of the semilinear heat equation $\partial_t y - \Delta y + g(y) = f\,1_\omega$ assuming that $g \in C^1(\mathbb{R})$ satisfies the growth condition $\limsup_{|r|\to\infty} g(r)/(|r|\ln^{3/2}|r|) = 0$ has been obtained by Fernández-Cara and Zuazua (2000). The proof based on a non constructive fixed point theorem makes use of precise estimates of the observability constant for a linearized heat equation. Assuming that g' is bounded and uniformly Hölder continuous on $\mathbb{R}$ with exponent $p \in (0,1]$, we design a constructive proof yielding an explicit sequence converging strongly to a controlled solution for the semilinear equation, at least with order $1+p$ after a finite number of iterations. The method is based on a least-squares approach and coincides with a globally convergent damped Newton method: it guarantees the convergence whatever be the initial element of the sequence. Numerical experiments in the one dimensional setting illustrate our analysis.

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, be a bounded connected open set whose boundary $\partial\Omega$ is Lipschitz and let $\omega \subset\subset \Omega$ be any non-empty open set. Let $T > 0$ and $Q_T := \Omega \times (0,T)$, $q_T := \omega \times (0,T)$ and $\Sigma_T := \partial\Omega \times (0,T)$. We consider the semilinear heat equation

$$\begin{cases} \partial_t y - \Delta y + g(y) = f 1_\omega & \text{in} \quad Q_T, \\ y = 0 \quad \text{on} \quad \Sigma_T, \quad y(\cdot,0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

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where $u_0 \in L^2(\Omega)$ is the initial state of y and $f \in L^2(q_T)$ is a *control* function. Here and throughout the paper, we assume that $g : \mathbb{R} \mapsto \mathbb{R}$ is, at least, locally Lipschitz-continuous and following [16] that g satisfies

$$|g'(r)| \leq C(1 + |r|^m) \quad \text{a.e. with } 1 \leq m \leq 1 + 4/d \tag{1.2}$$

which implies that (1.1) possesses exactly one local in time solution. We recall that under the additional growth condition

$$|g(r)| \leq C(1 + |r| \log(1 + |r|)) \quad \forall r \in \mathbb{R}, \tag{1.3}$$

(1.1) has a unique globally weak solution defined in $[0, T]$ and one has

$$y \in C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)), \tag{1.4}$$

see [6]. Without a growth condition of the kind (1.3), the solutions to (1.1) can blow up before $t = T$; in general, the blow-up time depends on g and the size of $\|u_0\|_{L^2(\Omega)}$.

The system (1.1) is said to be *controllable* at time T if, for any $u_0 \in L^2(\Omega)$ and any globally defined bounded trajectory $y^\star \in C^0([0, T]; L^2(\Omega))$ (corresponding to the data $u_0^\star \in L^2(\Omega)$ and $f^\star \in L^2(q_T)$), there exist controls $f \in L^2(q_T)$ and associated states y that are again globally defined in $[0, T]$ and satisfy (1.4) and

$$y(x, T) = y^\star(x, T), \quad x \in \Omega. \tag{1.5}$$

We refer to [8] for an overview of control problems in nonlinear situations. The uniform controllability strongly depends on the nonlinearity g. Fernández-Cara and Zuazua proved in [16] that if g is too "super-linear" at infinity, then, for some initial data, the control cannot compensate the blow-up phenomenon occurring in $\Omega \backslash \overline{\omega}$.

Theorem 1.1 ([16]). *There exist locally Lipschitz-continuous functions g with $g(0) = 0$ and*

$$|g(r)| \sim |r| \log^p(1 + |r|) \quad \text{as} \quad |r| \to \infty, \quad p > 2,$$

such that (1.1) fails to be controllable for all $T > 0$.

On the other hand, Fernández-Cara and Zuazua also proved that if p is small enough, then the controllability holds true uniformly.

Theorem 1.2 ([16]). *Let $T > 0$ be given. Assume that (1.1) admits at least one solution $y^\star$, globally defined in $[0, T]$ and bounded in Q_T. Assume that $g : \mathbb{R} \mapsto \mathbb{R}$ is locally Lipschitz-continuous and satisfies (1.2). If g satisfies*

$$\frac{g(r)}{|r| \ln^{3/2}(1 + |r|)} \to 0 \quad \text{as} \quad |r| \to \infty. \tag{1.6}$$

Then (1.1) is controllable at time T.

Therefore, if $|g(r)|$ does not grow at infinity faster than $|r| \ln^p(1 + |r|)$ for any $p < 3/2$, then (1.1) is controllable. We also mention [1] which gives the same result assuming additional sign condition on g, namely $g(r)r \geq -C(1 + r^2)$ for every $r \in \mathbb{R}$ and some $C > 0$. The problem remains open when g behaves at infinity like $|r| \ln^p(1 + |r|)$ with $3/2 \leq p \leq 2$. We mention however the recent work of Le Balc'h [20] where uniform controllability results are obtained for $p \leq 2$ assuming additional sign conditions on g, notably that $g(r) > 0$ for $r > 0$ or $g(r) < 0$ for $r < 0$ and T large enough. This condition is not satisfied for $g(r) = -r \ln^p(1 + |r|)$. Let us also mention [9] in the context of Theorem 1.1 where a positive boundary controllability result is proved for a specific class of initial and final data and T large enough.

In the sequel, for simplicity, we shall assume that $g(0) = 0$, $f^\star \equiv 0, u_0^\star \equiv 0$ so that $y^\star$ is the null trajectory. The proof given in [16] is based on a fixed point argument that reduced the controllability problem to the obtention of a suitable *a priori* estimate for the linearized heat equation with a potential. More precisely, it is shown that the operator $\Lambda : L^\infty(Q_T) \to L^\infty(Q_T)$, where $y_z := \Lambda(z)$ is a null controlled solution through the control function f_z of the linear boundary value problem

$$\begin{cases} \partial_t y_z - \Delta y_z + y_z\, \widetilde{g}(z) = f_z 1_\omega & \text{in} \quad Q_T \\ y_z = 0 \quad \text{on} \quad \Sigma_T, \quad y_z(\cdot,0) = u_0 & \text{in} \quad \Omega \end{cases}, \qquad \widetilde{g}(r) := \begin{cases} \dfrac{g(r)}{r} & r \neq 0, \\ g'(0) & r = 0, \end{cases} \tag{1.7}$$

maps a closed ball $B(0,M) \subset L^\infty(Q_T)$ into itself, for some $M > 0$. Under the condition (1.6), the Kakutani's theorem provides the existence of at least one fixed point for the operator Λ, which is also a controlled solution for (1.1).

The main goal of this work is to determine an approximation of the controllability problem associated to (1.1), that is to construct an explicit sequence $(f_k)_{k\in\mathbb{N}}$ converging strongly towards a null control for (1.1). A natural strategy is to take advantage of the method used in [16, 20] and consider, for any element $y_0 \in L^\infty(Q_T)$, the Picard iterations $(y_k)_{k\in\mathbb{N}}$ defined by $y_{k+1} := \Lambda(y_k)$, $k \geq 0$ associated with the operator Λ. The resulting sequence of controls $(f_k)_{k\in\mathbb{N}}$ is then so that $f_{k+1} \in L^2(q_T)$ is a null control for y_{k+1} solution of

$$\begin{cases} \partial_t y_k - \Delta y_k + y_k\, \widetilde{g}(y_{k-1}) = f_k 1_\omega & \text{in} \quad Q_T, \\ y_k = 0 \quad \text{on} \quad \Sigma_T, \quad y_k(\cdot,0) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{1.8}$$

Numerical experiments for the one dimensional case reported in [13] exhibit the non convergence of the sequences $(y_k)_{k\in\mathbb{N}}$ and $(f_k)_{k\in\mathbb{N}}$ for some initial conditions large enough. This phenomenon is related to the fact that the operator Λ is in general not contracting, even if $\widetilde{g}$ is globally Lipschitz (we refer to Rem. 3.17). We also mention [3, 4] where this strategy is implemented. Still in the one dimensional case, a least-squares type approach, based on the minimization over $L^2(Q_T)$ of the functional $\mathcal{R} : L^2(Q_T) \to \mathbb{R}^+$ defined by $\mathcal{R}(z) := \|z - \Lambda(z)\|^2_{L^2(Q_T)}$ has been introduced and analyzed in [13]. Assuming that $\widetilde{g} \in C^1(\mathbb{R})$ and $g' \in L^\infty(\mathbb{R})$, it is proved that $\mathcal{R} \in C^1(L^2(Q_T);\mathbb{R}^+)$ and that, for some constant $C > 0$

$$\|\mathcal{R}'(z)\|_{L^2(Q_T)} \geq (1 - C\|g'\|_\infty \|u_0\|_\infty)\sqrt{2\mathcal{R}(z)}, \quad \forall z \in L^2(Q_T)$$

implying that if $\|g'\|_\infty \|u_0\|_\infty$ is small enough, then any critical point for $\mathcal{R}$ is a fixed point for Λ (see [13], Prop. 3.2). Under this smallness assumption on the data, numerical experiments reported in [13] display the convergence of gradient based minimizing sequences for $\mathcal{R}$ and a better behavior than the Picard iterates associated to Λ. The analysis of convergence is however not performed. As is usual for nonlinear problems and also considered in [13], we may employ a Newton type method to find a zero of the mapping $\widetilde{F} : Y \mapsto W$ defined by

$$\widetilde{F}(y,f) = (\partial_t y - \Delta y + g(y) - f1_\omega, y(\cdot\,,0) - u_0) \quad \forall (y,f) \in Y \tag{1.9}$$

where the Hilbert space Y and W are defined as follows

$$Y := \left\{ (y,f) : \rho y \in L^2(Q_T),\ \rho_1 \nabla y \in L^2(Q_T)^d,\ \rho_0(\partial_t y - \Delta y) \in L^2(Q_T), y = 0 \text{ on } \Sigma_T,\ \rho_0 f \in L^2(q_T) \right\}$$

and $W := L^2(\rho_0; Q_T) \times L^2(\Omega)$ for some appropriates weights ρ, ρ_0 and ρ_1. Here $L^2(\rho_0; Q_T)$ stands for $\{z : \rho_0 z \in L^2(Q_T)\}$. It is shown for $d = 1$ in [13] that, if $g \in C^1(\mathbb{R})$ and $g' \in L^\infty(\mathbb{R})$, then $\widetilde{F} \in C^1(Y; W)$ allowing to derive

the Newton iterative sequence: given (y_0, f_0) in Y, define the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ in $Y^{\mathbb{N}}$ iteratively as follows $(y_{k+1}, f_{k+1}) = (y_k, f_k) - (Y_k, F_k)$ where F_k is a control for Y_k solution of

$$\begin{cases} \partial_t Y_k - \Delta Y_k + g'(y_k)\, Y_k = F_k\, 1_\omega + \partial_t y_k - \Delta y_k + g(y_k) - f_k 1_\omega \quad \text{in} \quad Q_T, \\ Y_k = 0 \quad \text{on} \quad \Sigma_T, \quad Y_k(\cdot, 0) = u_0 - y_k(\cdot, 0) \quad \text{in} \quad \Omega. \end{cases} \tag{1.10}$$

Once again, numerical experiments for $d = 1$ in [13] exhibits the lack of convergence of the Newton method for large enough initial condition, for which the solution y is not close enough to the zero trajectory.

The controllability of nonlinear partial differential equations has attracted a large number of works in the last decades (see the monography [8] and references therein). However, as far as we know, few are concerned with the approximation of exact controls for nonlinear partial differential equations, and the construction of convergent control approximations for controllable nonlinear equations remains a challenge.

Assuming that $g' \in L^\infty(\mathbb{R})$ and in addition that there exists one p in $(0, 1]$ for which g' is uniformly Hölder continuous on $\mathbb{R}$ with exponent $p \in (0, 1]$, we construct, for any initial data $u_0 \in L^2(\Omega)$, a strongly convergent sequence $(f_k)_{k\in\mathbb{N}}$ towards a control for (1.1). Moreover, after a finite number of iterates related to the norm $\|g'\|_{L^\infty(\mathbb{R})}$, the convergence is super linear with a rate equal to $1 + p$. This is done (following and improving [26] devoted to a linear case) by introducing a quadratic functional which measures how a pair $(y, f) \in Y$ is close to a controlled solution for (1.1) and then by determining a particular minimizing sequence enjoying the announced property. A natural example of so-called error (or least-squares) functional is given by $\widetilde{E}(y, f) := \frac{1}{2}\|\widetilde{F}(y, f)\|_W^2$ to be minimized over Y. In view of controllability results for (1.1), the non-negative functional $\widetilde{E}$ achieves its global minimum equal to zero for any control pair $(y, f) \in Y$ of (1.1).

The paper is organized as follows. In Section 2, we derive a controllability result for a linearized heat equation with potential in $L^\infty(Q_T)$ and source term in $L^2(0, T; H^{-1}(\Omega))$. Then, in Section 3, we define the least-squares functional E and the corresponding (nonconvex) optimization problem (3.2) over the Hilbert space $\mathcal{A}$ defined in (3.1). We show that E is Gateaux-differentiable over $\mathcal{A}$ and that any critical point (y, f) for E for which $g'(y)$ belongs to $L^\infty(Q_T)$ is also a zero of E (see Prop. 3.4). This is done by introducing a descent direction (Y^1, F^1) for $E(y, f)$ for which $E'(y, f) \cdot (Y^1, F^1)$ is proportional to $\sqrt{E(y, f)}$. Then, assuming that the function g' is uniformly Hölder continuous on $\mathbb{R}$ with exponent p for some $p \in [0, 1]$, we determine a minimizing sequence based on (Y^1, F^1) converging strongly to a controlled pair for the semilinear heat equation (1.1). Moreover, we prove that after a finite number of iterates, the convergence enjoys a rate equal to $1 + p$ (see Thm. 3.10 for $p = 1$ and Thm. 3.12 for $p \in (0, 1)$). We also emphasize that this least-squares approach coincides with the damped Newton method one may use to find a zero of a mapping similar to $\widetilde{F}$ mentioned above; we refer to Remark 3.15. This explains the convergence of our approach with a super-linear rate. Section 4 gives some numerical illustrations of our result in the one dimensional case and a nonlinear function g uniformly continuous on $\mathbb{R}$ with exponent $p = 1$. We conclude in Section 5 with some perspectives.

As far as we know, the analysis of convergence presented in this work, though some restrictive hypotheses on the nonlinear function g, is the first one in the context of controllability for partial differential equations.

Along the text, we shall denote by $\|\cdot\|_\infty$ the usual norm in $L^\infty(\mathbb{R})$, $(\cdot, \cdot)_X$ the scalar product of X (if X is a Hilbert space) and by $\langle \cdot, \cdot \rangle_{X,Y}$ the duality product between the spaces X and Y.

2. A controllability result for a linearized heat equation with $L^2(H^{-1})$ right hand side

We give in this section a controllability result for a linear heat equation with potential in $L^\infty(Q_T)$ and right hand side in $L^2(0, T; H^{-1}(\Omega))$. As this work concerns the null controllability of parabolic equations, we shall make use of Carleman type weights introduced in this context notably in [17] (we also refer to [12] for a review).

Here, we assume that such weights ρ, ρ_0, ρ_1 and ρ_2 blow up as $t \to T^-$ and satisfy:

$$\begin{cases} \rho = \rho(x,t), \rho_0 = \rho_0(x,t), \rho_1 = \rho_1(x,t) \text{ and } \rho_2 = \rho_2(x,t) \text{ are continuous and } \geq \rho_* > 0 \text{ in } Q_T \\ \rho, \rho_0, \rho_1, \rho_2 \in L^\infty(Q_{T-\delta}) \quad \forall \delta > 0. \end{cases} \tag{2.1}$$

Precisely, we will take $\rho_0 = (T-t)^{3/2}\rho$, $\rho_1 = (T-t)\rho$, $\rho_2 = (T-t)^{1/2}\rho$ and $\rho_3 = (T-t)^{-1/2}\rho$ where ρ is defined as follow

$$\rho(x,t) = \exp\Big(\frac{s\beta(x)}{\ell(t)}\Big), \quad s \geq C(\Omega, \omega, T, \|g'\|_\infty) \tag{2.2}$$

with

$$\ell(t) = \begin{cases} 3T^2/16 & \text{if } 0 \leq t < T/4 \\ t(T-t) & \text{if } T/4 \leq t \leq T. \end{cases}$$

Here $\beta(x) = \exp(2\lambda m\|\eta^0\|_\infty) - \exp(\lambda(m\|\eta^0\|_\infty + \eta^0(x)))$, $m > 1$, $\eta^0 \in \mathcal{C}(\overline{\Omega})$ satisfies $\eta^0 > 0$ in Ω, $\eta^0 = 0$ on $\partial\Omega$ and $|\nabla\eta^0| > 0$ in $\overline{\Omega\backslash\omega}$ (see [12], Lem. 1.2, p. 1401).

In the next section, we shall make use of the following controllability result where $L^2(\rho, A) := \{f : \rho f \in L^2(A)\}$ for any set $A \subset Q_T$ and function ρ.

Proposition 2.1. *Assume $A \in L^\infty(Q_T)$, $\rho_2 B \in L^2(0,T;H^{-1}(\Omega))$ and $z_0 \in L^2(\Omega)$. Then there exists a control $v \in L^2(\rho_0, q_T)$ such that the weak solution z of*

$$\begin{cases} \partial_t z - \Delta z + Az = v1_\omega + B \quad in \quad Q_T, \\ z = 0 \quad on \quad \Sigma_T, \quad z(\cdot,0) = z_0 \quad in \quad \Omega \end{cases} \tag{2.3}$$

satisfies

$$z(\cdot,T) = 0 \text{ in } \Omega. \tag{2.4}$$

Moreover, the unique control u which minimizes together with the corresponding solution z the functional $J : L^2(\rho, Q_T) \times L^2(\rho_0, q_T) \to \mathbb{R}^+$ defined by $J(z,v) := \frac{1}{2}\|\rho\, z\|^2_{L^2(Q_T)} + \frac{1}{2}\|\rho_0\, v\|^2_{L^2(q_T)}$ satisfy the following estimate

$$\|\rho\, z\|_{L^2(Q_T)} + \|\rho_0\, v\|_{L^2(q_T)} \leq C\Big(\|\rho_2 B\|_{L^2(0,T;H^{-1}(\Omega))} + \|z_0\|_{L^2(\Omega)}\Big) \tag{2.5}$$

for some constant $C = C(s, \Omega, \omega, T, \|A\|_\infty)$.

The controlled solution also satisfies, for some constant $C = C(s, \Omega, \omega, T, \|A\|_\infty)$, the estimate

$$\|\rho_1 z\|_{L^\infty(0,T;L^2(\Omega))} + \|\rho_1 \nabla z\|_{L^2(Q_T)^d} \leq C\Big(\|\rho_2\, B\|_{L^2(0,T;H^{-1}(\Omega))} + \|z_0\|_{L^2(\Omega)}\Big). \tag{2.6}$$

Proposition 2.1 follows from several lemmas. Let us first set $P_0 := \{q \in C^2(\overline{Q_T}) \ : \ q = 0 \text{ on } \Sigma_T\}$. The bilinear form

$$(p,q)_P := \int_{Q_T} \rho^{-2} L^\star_A p\, L^\star_A q + \int_{q_T} \rho_0^{-2} p\, q$$

where $L_A^\star q := -\partial_t q - \Delta q + Aq$, is a scalar product on P_0 (see [14]). The completion P of P_0 for the norm $\|\cdot\|_P$ associated to this scalar product is a Hilbert space and the following result proved in [17] holds.

Lemma 2.2. *There exists $C = C(s, \Omega, \omega, T, \|A\|_\infty) > 0$ such that one has the following Carleman estimate:*

$$\int_{Q_T} \left(\rho_3^{-1}|p_t|^2 + \rho_2^{-2}|\nabla p|^2 + \rho_0^{-2}|p|^2\right) \leq C\|p\|_P^2, \quad \forall p \in P. \tag{2.7}$$

Remark 2.3. We denote by P (instead of P_A) the completion of P_0 for the norm $\|\cdot\|_P$ since P does not depend on A (see [13], Lem. 3.1).

Lemma 2.4. *There exists $C = C(s, \Omega, \omega, T, \|A\|_\infty) > 0$ such that one has the following observability inequality:*

$$\|p(\cdot, 0)\|_{L^2(\Omega)} \leq C\|p\|_P, \quad \forall p \in P. \tag{2.8}$$

Proof. From the definition of ρ_0, ρ_1 and ρ_2, $P \hookrightarrow H^1(0, \frac{T}{2}; L^2(\Omega)) \hookrightarrow C([0, \frac{T}{2}]; L^2(\Omega))$ where each embedding is continuous. The result follows from Lemma 2.2. □

Lemma 2.5. *There exists $p \in P$ unique solution of*

$$(p, q)_P = \int_\Omega z_0 q(0) + \int_0^T \langle \rho_2 B, \rho_2^{-1} q\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}, \quad \forall q \in P. \tag{2.9}$$

This solution satisfies the following estimate:

$$\|p\|_P \leq C\big(\|\rho_2 B\|_{L^2(0,T;H^{-1}(\Omega))} + \|z_0\|_{L^2(\Omega)}\big)$$

where $C = C(s, \Omega, \omega, T, \|A\|_\infty) > 0$.

Proof. The linear map $L_1 : P \to \mathbb{R}$, $q \mapsto \int_0^T \langle \rho_2 B, \rho_2^{-1} q\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}$ is continuous. Indeed, for all $q \in P$

$$\Big|\int_0^T \langle \rho_2 B, \rho_2^{-1} q\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\Big| \leq \Big(\int_0^T \|\rho_2 B\|_{H^{-1}(\Omega)}^2\Big)^{1/2}\Big(\int_0^T \|\rho_2^{-1} q\|_{H_0^1(\Omega)}^2\Big)^{1/2}$$

and a.e. in $(0, T)$ $\|\rho_2^{-1} q\|_{H_0^1(\Omega)}^2 = \|\rho_2^{-1} q\|_{L^2(\Omega)}^2 + \|\nabla(\rho_2^{-1} q)\|_{L^2(\Omega)^d}^2$. But since $\rho_0 \leq T\rho_2$ a.e. t in $(0, T)$

$$\|\rho_2^{-1} q\|_{L^2(\Omega)}^2 \leq \frac{1}{T^2}\|\rho_0^{-1} q\|_{L^2(\Omega)}^2, \text{ a.e. } t \in (0, T).$$

Moreover

$$\nabla(\rho_2^{-1} q) = \nabla(\rho_2^{-1}) q + \rho_2^{-1}\nabla q = -\frac{s\nabla\beta(x)}{\ell(t)(T-t)^{1/2}}\rho^{-1} q + \rho_2^{-1}\nabla q$$

and thus, since $\rho_1 \leq T^{1/2}\rho_2$ a.e. t in $(0, T)$:

$$\begin{aligned}\|\nabla(\rho_2^{-1} q)\|_{L^2(\Omega)^d}^2 &\leq \left\|\frac{s\nabla\beta(x)}{\ell(t)(T-t)^{1/2}}\rho^{-1} q\right\|_{L^2(\Omega)^d}^2 + \|\rho_2^{-1}\nabla q\|_{L^2(\Omega)^d}^2 \\ &\leq C(s, \Omega, \omega, T, \|A\|_\infty)\big(\|\rho_0^{-1} q\|_{L^2(\Omega)}^2 + \|\rho_2^{-1}\nabla q\|_{L^2(\Omega)^d}^2\big).\end{aligned}$$

We then deduce that, a.e. in $(0,T)$

$$\|\rho_2^{-1}q\|^2_{H_0^1(\Omega)} \leq C(s,\Omega,\omega,T,\|A\|_\infty)\big(\|\rho_0^{-1}q\|^2_{L^2(\Omega)} + \|\rho_1^{-1}\nabla q\|^2_{L^2(\Omega)^d}\big)$$

and from the Carleman estimate (2.7) that

$$\Big(\int_0^T \|\rho_2^{-1}q\|^2_{H_0^1(\Omega)}\Big)^{1/2} \leq C(s,\Omega,\omega,T,\|A\|_\infty)\|q\|_P$$

and therefore

$$\Big|\int_0^T \langle \rho_2 B, \rho_2^{-1}q\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\Big| \leq C(s,\Omega,\omega,T,\|A\|_\infty)\Big(\int_0^T \|\rho_2 B\|^2_{H^{-1}(\Omega)}\Big)^{1/2}\|q\|_P.$$

Thus L_1 is continuous.

From (2.8) we easily deduce that the linear map $L_2 : P \to \mathbb{R}$, $q \mapsto \int_\Omega z_0 q(0)$ is continuous. Using Riesz's theorem, we conclude that there exists exactly one solution $p \in P$ of (2.9). □

Let us now introduce the convex set

$$C(z_0,T) = \big\{(z,v) : \rho z \in L^2(Q_T),\ \rho_0 v \in L^2(q_T),\ (z,v) \text{ solves (2.3)}{-}\text{(2.4) in the transposition sense}\big\}$$

that is (z,v) is solution of

$$\int_{Q_T} z L_A^\star q = \int_{q_T} vq + \int_\Omega z_0 q(0) + \int_0^T \langle B, q\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}, \quad \forall q \in P.$$

Let us remark that if $(z,v) \in C(z_0,T)$, then since $z_0 \in L^2(\Omega)$, $v \in L^2(q_T)$ and $B \in L^2(0,T;H^{-1}(\Omega))$, z must coincide with the unique weak solution of (2.3) associated with v.

We can now claim that $C(z_0,T)$ is a non empty. Indeed we have:

Lemma 2.6. *Let $p \in P$ defined in Lemma 2.5 and (z,v) defined by*

$$z = \rho^{-2}L_A^\star p \quad \text{and} \quad v = -\rho_0^{-2}p|_{q_T}. \tag{2.10}$$

Then $(z,v) \in C(z_0,T)$ and satisfies the following estimate

$$\|\rho\, z\|_{L^2(Q_T)} + \|\rho_0\, v\|_{L^2(q_T)} \leq C\Big(\|\rho_2\, B\|_{L^2(0,T;H^{-1}(\Omega))} + \|z_0\|_{L^2(\Omega)}\Big) \tag{2.11}$$

where $C = C(s,\Omega,\omega,T,\|A\|_\infty) > 0$.

Proof. Let us prove that (z,v) belongs to $C(z_0,T)$. From the definition of P, $\rho z \in L^2(Q_T)$ and $\rho_0 v \in L^2(q_T)$ and from the definition of ρ, ρ_0, ρ_2, $z \in L^2(Q_T)$ and $v \in L^2(q_T)$. In view of (2.9), (z,v) is solution of

$$\int_{Q_T} z L_A^\star q = \int_{q_T} v\, q + \int_\Omega z_0 q(0) + \int_0^T \langle \rho_2 B, \rho_2^{-1}q\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}, \quad \forall q \in P$$

that is, since from the definition of ρ_2, $B \in L^2(0,T;H^{-1}(\Omega))$ and

$$\int_0^T \langle \rho_2 B, \rho_2^{-1} q\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = \int_0^T \langle B, q\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)},$$

z is the solution of (2.3) associated with v in the transposition sense. Thus $C(z_0,T)\neq\emptyset$. □

Let us now consider the following extremal problem, introduced by Fursikov and Imanuvilov [17]

$$\begin{cases} \text{Minimize } J(z,v) = \dfrac{1}{2}\|(z,v)\|^2_{L^2(\rho,Q_T)\times L^2(\rho_0,q_T)} = \dfrac{1}{2}\displaystyle\int_{Q_T}\rho^2|z|^2 + \dfrac{1}{2}\int_{q_T}\rho_0^2|v|^2 \\ \text{subject to } (z,v)\in C(z_0,T). \end{cases} \tag{2.12}$$

Then $(z,v)\mapsto J(z,v)$ is clearly strictly convex and continuous on $L^2(\rho,Q_T)\times L^2(\rho_0,q_T)$. Therefore (2.12) possesses at most a unique solution in $C(z_0,T)$. More precisely we have:

Proposition 2.7. *$(z,v)\in C(z_0,T)$ defined in Lemma 2.6 is the unique solution of (2.12).*

Proof. Let $(y,w)\in C(z_0,T)$. Since J is convex and differentiable on $L^2(\rho,Q_T)\times L^2(\rho_0,q_T)$ we have:

$$\begin{aligned} J(y,w) &\geq J(z,v) + \int_{Q_T}\rho^2 z(y-z) + \int_{q_T}\rho_0^2 v(w-v) \\ &= J(z,v) + \int_{Q_T} L^\star p(y-z) - \int_{q_T} p(w-v) = J(z,v) \end{aligned}$$

y being the solution of (2.3) associated with w in the transposition sense. Hence (z,v) solves (2.12). □

Proof. (of Prop. 2.1) It suffices to prove that (z,v) satisfies the estimate (2.6). Since z is a weak solution of (2.3) associated with v, $z\in L^2(0,T;H_0^1(\Omega))$ and $z_t\in L^2(0,T;H^{-1}(\Omega))$. Multiplying (2.3) by $\rho_1^2 z$ and integrating by part we obtain, a.e. t in $(0,T)$

$$\begin{aligned} \frac{1}{2}\partial_t\int_\Omega |z|^2\rho_1^2 - \int_\Omega |z|^2\rho_1\partial_t\rho_1 + \int_\Omega \rho_1^2|\nabla z|^2 + 2\int_\Omega \rho_1 z\nabla\rho_1\cdot\nabla z + \int_\Omega \rho_1^2 Azz \\ = \int_\omega v\rho_1^2 z + \langle B, \rho_1^2 z\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}. \end{aligned}$$

But $\partial_t\rho_1 = -\rho - (T-t)\frac{s\beta\ell'(t)}{\ell(t)^2}\rho$, so that

$$\Big|\int_\Omega |z|^2\rho_1\partial_t\rho_1\Big| \leq C(s,\Omega,\omega,T,\|A\|_\infty)\int_\Omega |\rho z|^2.$$

Since $\nabla\rho_1 = (T-t)\nabla\rho = (T-t)\frac{s\nabla\beta}{\ell(t)}\rho$ and $\frac{T-t}{l(t)}\leq C(T)$, we have

$$\Big|\int_\Omega \rho_1 z\nabla\rho_1\cdot\nabla z\Big| \leq C(s,\Omega,\omega,T,\|A\|_\infty)\Big(\int_\Omega |\rho_1\nabla z|^2\Big)^{1/2}\Big(\int_\Omega |\rho z|^2\Big)^{1/2}.$$

The following estimates also hold

$$\Big|\int_\Omega \rho_1^2 Azz\Big| \leq C(T,\|A\|_\infty)\int_\Omega |\rho z|^2,$$

$$\Big|\int_\omega v\rho_1^2 z\Big| \le T^{1/2}\Big|\int_\omega \rho_0 v\rho z\Big| \le T^{1/2}\Big(\int_\omega |\rho_0 v|^2\Big)^{1/2}\Big(\int_\Omega |\rho z|^2\Big)^{1/2}$$

and

$$\begin{aligned}|\langle B,\rho_1^2 z\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}| &= |\langle \rho_1 B,\rho_1 z\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}| \le \|\rho_1 B\|_{H^{-1}(\Omega)}\|\rho_1 z\|_{H_0^1(\Omega)}\\ &\le C(s,\Omega,\omega,T,\|A\|_\infty)\|\rho_2 B\|_{H^{-1}(\Omega)}\big(\|\rho z\|_{L^2(\Omega)}+\|\rho_1\nabla z\|_{L^2(\Omega)^d}\big).\end{aligned}$$

Thus we easily obtain that

$$\partial_t\int_\Omega \rho_1^2|z|^2+\int_\Omega \rho_1^2|\nabla z|^2 \le C(s,\Omega,\omega,T,\|A\|_\infty)\Big(\|\rho_2 B\|^2_{H^{-1}(\Omega)}+\int_\Omega \rho^2|z|^2+\int_\omega |\rho_0 v|^2\Big)$$

and therefore, using (2.11), for all $t\in[0,T]$:

$$\Big(\int_\Omega \rho_1^2|z|^2\Big)(t)+\int_{Q_t}\rho_1^2|\nabla z|^2 \le C(s,\Omega,\omega,T,\|A\|_\infty)\Big(\|\rho_2 B\|^2_{L^2(0,T;H^{-1}(\Omega))}+\|z_0\|^2_{L^2(\Omega)}\Big)$$

which gives (2.6) and concludes the proof of Proposition 2.1. □

3. The least-squares method and its analysis

For any $p\in[0,1]$, we define the space

$$W_p=\Big\{g\in\mathcal{C}(\mathbb{R}),\ g(0)=0,\ g'\in L^\infty(\mathbb{R}),[g']_p:=\sup_{a,b\in\mathbb{R},a\neq b}\frac{|g'(a)-g'(b)|}{|a-b|^p}<\infty\Big\}.$$

The case $p=0$ reduces to $W_0=\{g\in\mathcal{C}(\mathbb{R}),\ g(0)=0,\ g'\in L^\infty(\mathbb{R})\}$ with $[g']_0=2\|g'\|_\infty$. The case $p=1$ corresponds to $W_1=\{g\in\mathcal{C}(\mathbb{R}),\ g(0)=0,\ g'\in L^\infty(\mathbb{R}),g''\in L^\infty(\mathbb{R})\}$ with $[g']_1=\|g''\|_\infty$. The property $[g']_p<\infty$ means that g' is uniformly Hölder continuous with exponent p.

In the sequel, we shall assume that there exists some $p\in(0,1]$ for which the nonlinear function g belongs to W_p. Remark that $g\in W_p$ for some $p\in[0,1]$ satisfies hypotheses (1.2) and (1.6). We shall also assume that $u_0\in L^2(\Omega)$.

3.1. The least-squares method

We introduce the vectorial space $\mathcal{A}_0$ as follows

$$\begin{aligned}\mathcal{A}_0=\Big\{&(y,f):\rho\, y\in L^2(Q_T),\ \rho_1\nabla y\in L^2(Q_T)^d,\ \rho_0 f\in L^2(q_T),\\ &\rho_2(\partial_t y-\Delta y-f\,1_\omega)\in L^2(0,T;H^{-1}(\Omega)),\ y(\cdot,0)=0\text{ in }\Omega,\ y=0\text{ on }\Sigma_T\Big\}\end{aligned}$$

where ρ, ρ_2, ρ_1 and ρ_0 are defined in (2.2). Since $L^2(0,T;H^{-1}(\Omega))$ is also a Hilbert space, $\mathcal{A}_0$ endowed with the following scalar product

$$\begin{aligned}\big((y,f),(\overline{y},\overline{f})\big)_{\mathcal{A}_0}=&\big(\rho y,\rho\overline{y}\big)_2+\big(\rho_1\nabla y,\rho_1\nabla\overline{y}\big)_2+\big(\rho_0 f,\rho_0\overline{f}\big)_2\\ &+\big(\rho_2(\partial_t y-\Delta y-f\,1_\omega),\rho_2(\partial_t\overline{y}-\Delta\overline{y}-\overline{f}\,1_\omega)\big)_{L^2(0,T;H^{-1}(\Omega))}\end{aligned}$$

is a Hilbert space. The corresponding norm is $\|(y,f)\|_{\mathcal{A}_0} = \sqrt{((y,f),(y,f))_{\mathcal{A}_0}}$. We also consider the convex space

$$\mathcal{A} = \Big\{ (y,f) : \rho\, y \in L^2(Q_T),\ \rho_1 \nabla y \in L^2(Q_T)^d,\ \rho_0 f \in L^2(q_T), \\ \rho_2(\partial_t y - \Delta y - f\, 1_\omega) \in L^2(0,T;H^{-1}(\Omega)),\ y(\cdot,0) = u_0 \text{ in } \Omega,\ y = 0 \text{ on } \Sigma_T \Big\} \tag{3.1}$$

so that we can write $\mathcal{A} = (\overline{y},\overline{f}) + \mathcal{A}_0$ for any element $(\overline{y},\overline{f}) \in \mathcal{A}$. We endow $\mathcal{A}$ with the same norm. Clearly, if $(y,f) \in \mathcal{A}$, then $y \in C([0,T];L^2(\Omega))$ and since $\rho\, y \in L^2(Q_T)$, then $y(\cdot,T) = 0$. The null controllability requirement is therefore incorporated in the spaces $\mathcal{A}_0$ and $\mathcal{A}$.

For any fixed $(\overline{y},\overline{f}) \in \mathcal{A}$, we can now consider the following non convex extremal problem:

$$\inf_{(y,f)\in\mathcal{A}_0} E(\overline{y}+y, f+\overline{f}) \tag{3.2}$$

where $E : \mathcal{A} \to \mathbb{R}$ is defined as follows

$$E(y,f) := \frac{1}{2} \left\| \rho_2 \Big(\partial_t y - \Delta y + g(y) - f\, 1_\omega \Big) \right\|^2_{L^2(0,T;H^{-1}(\Omega))}$$

justifying the least-squares terminology we have used.

Let us remark that, if $g \in W_p$ for one $p \geq 0$, then g is Lipschitz and thus, since $g(0) = 0$, there exists $K > 0$ such that $|g(\xi)| \leq K|\xi|$ for all $\xi \in \mathbb{R}$. Consequently, $\rho_2 g(y) \in L^2(Q_T)$ (and then $\rho_2 g(y) \in L^2(0,T;H^{-1}(\Omega))$) since

$$\|\rho_2 g(y)\|_{L^2(Q_T)} = \|(\rho_2\rho^{-1})\rho g(y)\|_{L^2(Q_T)} = \|(T-t)^{1/2}\rho g(y)\|_{L^2(Q_T)} \leq T^{1/2} K \|\rho y\|_{L^2(Q_T)}.$$

Any pair $(y,f) \in \mathcal{A}$ for which $E(y,f)$ vanishes is a controlled pair of (1.1), and conversely. In this sense, the functional E is a so-called error functional which measures the deviation of (y,f) from being a solution of the underlying nonlinear equation. Although any $g \in W_p$ for some $p \geq 0$ satisfies hypotheses (1.2) and (1.6), the controllability result of Theorem 1.2 given in [16] does not imply the existence of zero of E in $\mathcal{A}$, since controls of minimal $L^\infty(q_T)$ norm are considered there. Our constructive approach will show that the extremal problem (3.2) admits solutions.

We also emphasize that the $L^2(0,T;H^{-1}(\Omega))$ norm in E indicates that we are looking for weak solutions of the parabolic equation (1.1). We refer to [22, 23] where a similar so-called weak least-squares method is employed to approximate the solutions of the unsteady Navier-Stokes equation.

A practical way of taking a functional to its minimum is through some clever use of descent directions, i.e the use of its derivative. In doing so, the presence of local minima is always something that may dramatically spoil the whole scheme. The unique structural property that discards this possibility is the strict convexity of the functional E. However, for nonlinear equation like (1.1), one cannot expect this property to hold for the functional E. Nevertheless, we insist in that one may construct a particular minimizing sequence which cannot converge except to a global minimizer leading E down to zero.

In order to construct such minimizing sequence, we look, for any $(y,f) \in \mathcal{A}$, for a pair $(Y^1,F^1) \in \mathcal{A}_0$ solution of the following formulation

$$\begin{cases} \partial_t Y^1 - \Delta Y^1 + g'(y)Y^1 = F^1 1_\omega + \partial_t y - \Delta y + g(y) - f\, 1_\omega \quad \text{in} \quad Q_T, \\ Y^1 = 0 \quad \text{on} \quad \Sigma_T, \quad Y^1(\cdot,0) = 0 \quad \text{in} \quad \Omega. \end{cases} \tag{3.3}$$

Since $(Y^1, F^1) \in \mathcal{A}_0$, F^1 is a null control for Y^1. We have the following property.

Proposition 3.1. *Let any $(y, f) \in \mathcal{A}$. There exists a pair $(Y^1, F^1) \in \mathcal{A}_0$ solution of (3.3) which satisfies the following estimate:*

$$\|(Y^1, F^1)\|_{\mathcal{A}_0} \leq C\sqrt{E(y, f)} \tag{3.4}$$

for some $C = C(s, \Omega, \omega, T, \|g'\|_\infty) > 0$.

Proof. For all $(y, f) \in \mathcal{A}$ we have $\rho_2(\partial_t y - \Delta y + g(y) - f1_\omega) \in L^2(0, T; H^{-1}(\Omega))$. The existence of a null control F^1 is therefore given by Proposition 2.1. Choosing the control F^1 which minimizes together with the corresponding solution Y^1 the functional J defined in Proposition 2.1, we get the following estimate (since $Y^1(\cdot, 0) = 0$)

$$\begin{aligned} \|\rho\, Y^1\|_{L^2(Q_T)} + \|\rho_0 F^1\|_{L^2(q_T)} &\leq C\|\rho_2(\partial_t y - \Delta y + g(y) - f1_\omega)\|_{L^2(0,T;H^{-1}(\Omega))} \\ &\leq C\sqrt{E(y, f)} \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \|\rho_1\, Y^1\|_{L^\infty(0,T;L^2(\Omega))} + \|\rho_1 \nabla Y^1\|_{L^2(Q_T)^d} &\leq C\|\rho_2(\partial_t y - \Delta y + g(y) - f1_\omega)\|_{L^2(0,T;H^{-1}(\Omega))} \\ &\leq C\sqrt{E(y, f)} \end{aligned} \tag{3.6}$$

for some $C = C(s, \Omega, \omega, T, \|g\|_\infty)$ independent of Y^1, F^1 and y. Eventually, from the equation solved by Y^1,

$$\begin{aligned} \|\rho_2(\partial_t Y^1 - \Delta Y^1 - F^1\, 1_\omega)\|_{L^2(0,T;H^{-1}(\Omega))} &\leq \|\rho_2 g'(y) Y^1\|_{L^2(Q_T)} + \|\rho_2(\partial_t y - \Delta y + g(y) - f\, 1_\omega)\|_{L^2(0,T;H^{-1}(\Omega))} \\ &\leq \|(T-t)^{1/2} g'(y)\|_\infty \|\rho Y^1\|_{L^2(Q_T)} + \sqrt{2E(y, f)} \\ &\leq \max\left(1, \|(T-t)^{1/2} g'\|_\infty\right) C\sqrt{E(y, f)} \end{aligned} \tag{3.7}$$

which proves that (Y^1, F^1) belongs to $\mathcal{A}_0$. □

Remark 3.2. From (3.3), we check that $z = y - Y^1$ is a null controlled solution satisfying

$$\begin{cases} \partial_t z - \Delta z + g'(y) z = (f - F^1)1_\omega + g'(y) y - g(y) \quad \text{in} \quad Q_T, \\ z = 0 \quad \text{on} \quad \Sigma_T, \quad z(\cdot, 0) = u_0 \quad \text{in} \quad \Omega \end{cases} \tag{3.8}$$

by the control $(f - F^1) \in L^2(\rho_0, q_T)$.

Remark 3.3. We emphasize that the presence of a right hand side in (3.3), namely $y_t - \Delta y + g(y) - f\, 1_\omega$, forces us to introduce the weights ρ_0, ρ_1, ρ_2 and ρ in the spaces $\mathcal{A}_0$ and $\mathcal{A}$. This can be seen from the equality (2.9): since $\rho_2^{-1} q$ belongs to $L^2(0, T; H^1(\Omega))$ for all $q \in P$, we need to impose that $\rho_2 B \in L^2(0, T; H^{-1}(\Omega))$ with here $B = \partial_t y - \Delta y + g(y) - f\, 1_\omega$. Working with the linearized equation (1.7) (introduced in [16]) which does not make appear an additional right hand side, we may avoid the introduction of Carleman type weights. Actually, the authors in [16] consider controls of minimal $L^\infty(q_T)$ norm. Introduction of weights allows however the characterization (2.9), which is very convenient at the practical level. We refer to [14] where this is discussed at length.

The interest of the pair $(Y^1, F^1) \in \mathcal{A}_0$ lies in the following result.

Proposition 3.4. *Let $(y,f)\in\mathcal{A}$ and let $(Y^1,F^1)\in\mathcal{A}_0$ be a solution of (3.3). Then the derivative of E at the point $(y,f)\in\mathcal{A}$ along the direction (Y^1,F^1) given by $E'(y,f)\cdot(Y^1,F^1):=\lim_{\eta\to0,\eta\neq0}\frac{E((y,f)+\eta(Y^1,F^1))-E(y,f)}{\eta}$ satisfies*

$$E'(y,f)\cdot(Y^1,F^1)=2E(y,f). \tag{3.9}$$

Proof. We preliminary check that for all $(Y,F)\in\mathcal{A}_0$, E is differentiable at the point $(y,f)\in\mathcal{A}$ along the direction $(Y,F)\in\mathcal{A}_0$. For all $\lambda\in\mathbb{R}$, simple computations lead to the equality

$$E(y+\lambda Y,f+\lambda F)=E(y,f)+\lambda E'(y,f)\cdot(Y,F)+h((y,f),\lambda(Y,F))$$

with

$$E'(y,f)\cdot(Y,F):=\Big(\rho_2(\partial_t y-\Delta y+g(y)-f\,1_\omega),\rho_2(\partial_t Y-\Delta Y+g'(y)Y-F\,1_\omega)\Big)_{L^2(0,T;H^{-1}(\Omega))} \tag{3.10}$$

and

$$\begin{aligned}h((y,f),\lambda(Y,F))=&\lambda\Big(\rho_2(\partial_t Y-\Delta Y+g'(y)Y-F\,1_\omega),\rho_2 l(y,\lambda Y)\Big)_{L^2(0,T;H^{-1}(\Omega))}\\&+\frac{\lambda^2}{2}\|\rho_2(\partial_t Y-\Delta Y+g'(y)Y-F\,1_\omega)\|^2_{L^2(0,T;H^{-1}(\Omega))}\\&+\Big(\rho_2(\partial_t y-\Delta y+g(y)-f\,1_\omega),\rho_2 l(y,\lambda Y)\Big)_{L^2(0,T;H^{-1}(\Omega))}\\&+\frac{1}{2}\|\rho_2 l(y,\lambda Y)\|^2_{L^2(0,T;H^{-1}(\Omega))}\end{aligned}$$

where $l(y,\lambda Y)=g(y+\lambda Y)-g(y)-\lambda g'(y)Y$.

The application $(Y,F)\to E'(y,f)\cdot(Y,F)$ is linear and continuous from $\mathcal{A}_0$ to $\mathbb{R}$ as it satisfies

$$\begin{aligned}&|E'(y,f)\cdot(Y,F)|\\&\quad\le\|\rho_2(\partial_t y-\Delta y+g(y)-f\,1_\omega)\|_{L^2(0,T;H^{-1}(\Omega))}\|\rho_2(\partial_t Y-\Delta Y+g'(y)Y-F\,1_\omega)\|_{L^2(0,T;H^{-1}(\Omega))}\\&\quad\le\sqrt{2E(y,f)}\Big(\|\rho_2(\partial_t Y-\Delta Y-F\,1_\omega)\|_{L^2(0,T;H^{-1}(\Omega))}+\|\rho_2 g'(y)Y\|_{L^2(Q_T)}\Big)\\&\quad\le\sqrt{2E(y,f)}\Big(\|\rho_2(\partial_t Y-\Delta Y-F\,1_\omega)\|_{L^2(0,T;H^{-1}(\Omega))}+\|(T-t)^{1/2}g'(y)\|_{L^\infty(Q_T)}\|\rho Y\|_{L^2(Q_T)}\Big)\\&\quad\le\sqrt{2E(y,f)}\max\Big(1,\|(T-t)^{1/2}g'\|_\infty\Big)\|(Y,F)\|_{\mathcal{A}_0}.\end{aligned}$$

Similarly, for all $\lambda\in\mathbb{R}^\star$

$$\begin{aligned}\left|\frac{1}{\lambda}h((y,f),\lambda(Y,F))\right|\le&\Big(\lambda\|\rho_2(\partial_t Y-\Delta Y+g'(y)Y-F\,1_\omega)\|_{L^2(0,T;H^{-1}(\Omega))}+\sqrt{2E(y,f)}\\&+\frac{1}{2}\|\rho_2 l(y,\lambda Y)\|_{L^2(0,T;H^{-1}(\Omega))}\Big)\frac{1}{\lambda}\|\rho_2 l(y,\lambda Y)\|_{L^2(0,T;H^{-1}(\Omega))}\\&+\frac{\lambda}{2}\|\rho_2(\partial_t Y-\Delta Y+g'(y)Y-F\,1_\omega)\|^2_{L^2(0,T;H^{-1}(\Omega))}.\end{aligned}$$

Since $g' \in L^\infty(\mathbb{R})$ we have for a.e. $(x,t) \in Q_T$:

$$\rho_2 \left| \frac{1}{\lambda} l(y, \lambda Y) \right| = \rho_2 \left| \frac{g(y+\lambda Y) - g(y)}{\lambda} - g'(y) Y \right| \leq 2 \|g'\|_\infty |\rho_2 Y|$$

and $\rho_2 Y \in L^2(Q_T)$. Moreover, for a.e. $(x,t) \in Q_T$, $\rho_2 |\frac{1}{\lambda} l(y, \lambda Y)| = \rho_2 |\frac{g(y+\lambda Y)-g(y)}{\lambda} - g'(y)Y| \to 0$ as $\lambda \to 0$; it follows from the Lebesgue's dominated convergence theorem that $\frac{1}{\lambda} \|\rho_2 l(y, \lambda Y)\|_{L^2(Q_T)} \to 0$ as $\lambda \to 0$. It is now easy to see that $h((y,f), \lambda(Y,F)) = o(\lambda)$ and that the functional E is differentiable at the point $(y,f) \in \mathcal{A}$ along the direction $(Y,F) \in \mathcal{A}_0$. Eventually, the equality (3.9) follows from the definition of the pair (Y^1, F^1) given in (3.3). □

Remark that from the equality (3.10), the derivative $E'(y,f)$ is independent of (Y,F). We can then define the norm $\|E'(y,f)\|_{(\mathcal{A}_0)'} := \sup_{(Y,F) \in \mathcal{A}_0 \setminus \{0\}} \frac{E'(y,f)\cdot(Y,F)}{\|(Y,F)\|_{\mathcal{A}_0}}$ associated with $\mathcal{A}_0'$, the topological dual of $\mathcal{A}_0$.

Combining the equality (3.9) and the inequality (3.4), we deduce the following estimates of $E(y,f)$ in term of the norm of $E'(y,f)$.

Proposition 3.5. *For any $(y,f) \in \mathcal{A}$, the inequalities hold true*

$$C_1(s, \Omega, \omega, T, \|g'\|_\infty) \|E'(y,f)\|_{\mathcal{A}_0'} \leq \sqrt{E(y,f)} \leq C_2(s, \Omega, \omega, T, \|g'\|_\infty) \|E'(y,f)\|_{\mathcal{A}_0'}$$

for some constants $C_1, C_2 > 0$.

Proof. (3.9) rewrites $E(y,f) = \frac{1}{2} E'(y,f) \cdot (Y^1, F^1)$ where $(Y^1, F^1) \in \mathcal{A}_0$ is solution of (3.3) and therefore, with (3.4)

$$E(y,f) \leq \frac{1}{2} \|E'(y,f)\|_{\mathcal{A}_0'} \|(Y^1, F^1)\|_{\mathcal{A}_0} \leq C(s, \Omega, \omega, T, \|g'\|_\infty) \|E'(y,f)\|_{\mathcal{A}_0'} \sqrt{E(y,f)}.$$

On the other hand, for all $(Y,F) \in \mathcal{A}_0$ (see the proof of Prop. 3.4):

$$|E'(y,f) \cdot (Y,F)| \leq \sqrt{2E(y,f)} \max\left(1, \|(T-t)^{1/2} g'\|_\infty\right) \|(Y,F)\|_{\mathcal{A}_0}$$

and thus $C_1(s, \Omega, \omega, T, \|g'\|_\infty) \|E'(y,f)\|_{\mathcal{A}_0'} \leq \sqrt{E(y,f)}$. □

In particular, any *critical* point $(y,f) \in \mathcal{A}$ for E (*i.e.* for which $E'(y,f)$ vanishes) is a zero for E, a pair solution of the controllability problem. In other words, any sequence $(y_k, f_k)_{k>0}$ satisfying $\|E'(y_k, f_k)\|_{\mathcal{A}_0'} \to 0$ as $k \to \infty$ is such that $E(y_k, f_k) \to 0$ as $k \to \infty$. We insist that this property does not imply the convexity of the functional E (and *a fortiori* the strict convexity of E, which actually does not hold here in view of the multiple zeros for E) but show that a minimizing sequence for E can not be stuck in a local minimum. Far from the zeros of E, in particular, when $\|(y,f)\|_{\mathcal{A}} \to \infty$, the right hand side inequality indicates that E tends to be convex. On the other hand, the left inequality indicates that the functional E is flat around its zero set. As a consequence, gradient based minimizing sequences may achieve a very low rate of convergence (we refer to [26] and also [24] devoted to the Navier-Stokes equation where this phenomenon is observed).

3.2. A strongly converging minimizing sequence for E

We now examine the convergence of an appropriate sequence $(y_k, f_k)_{k \in \mathbb{N}} \in \mathcal{A}$. In this respect, we observe that the equality (3.9) shows that $-(Y^1, F^1)$ given by the solution of (3.3) is a descent direction for the functional

E. Therefore, we can define at least formally, for any $m \geq 1$, a minimizing sequence $(y_k, f_k)_{k\in\mathbb{N}}$ as follows:

$$\begin{cases} (y_0, f_0) \in \mathcal{A}, \\ (y_{k+1}, f_{k+1}) = (y_k, f_k) - \lambda_k(Y_k^1, F_k^1), \quad k \geq 0, \\ \lambda_k = argmin_{\lambda\in[0,m]} E\big((y_k, f_k) - \lambda(Y_k^1, F_k^1)\big) \end{cases} \tag{3.11}$$

where $(Y_k^1, F_k^1) \in \mathcal{A}_0$ is such that F_k^1 is a null control for Y_k^1, solution of

$$\begin{cases} \partial_t Y_k^1 - \Delta Y_k^1 + g'(y_k)Y_k^1 = F_k^1 1_\omega + (\partial_t y_k - \Delta y_k + g(y_k) - f_k 1_\omega) \quad \text{in} \quad Q_T, \\ Y_k^1 = 0 \quad \text{on} \quad \Sigma_T, \quad Y_k^1(\cdot, 0) = 0 \quad \text{in} \quad \Omega \end{cases} \tag{3.12}$$

and minimizes the functional J defined in Proposition 2.1. The direction Y_k^1 vanishes when E vanishes. The real number $m \geq 1$ is arbitrarily fixed. It is used in the proof of convergence to bound the sequence of optimal descent steps λ_k.

We first perform the analysis assuming the non linear function g in W_1, notably that $g'' \in L^\infty(\mathbb{R})$ (the derivatives here are in the sense of distribution). We first prove the following lemma.

Lemma 3.6. *Assume $g \in W_1$. Let $(y, f) \in \mathcal{A}$ and $(Y^1, F^1) \in \mathcal{A}_0$ defined by (3.3). For any $\lambda \in \mathbb{R}$ and $k \in \mathbb{N}$, the following estimate holds*

$$\sqrt{E\big((y, f) - \lambda(Y^1, F^1)\big)} \leq \sqrt{E(y, f)}\left(|1 - \lambda| + \lambda^2 C(s, \Omega, \omega, T, \|g'\|_\infty)\|g''\|_\infty \sqrt{E(y, f)}\right). \tag{3.13}$$

Proof. With $g \in W_1$, we write that

$$|l(y, -\lambda Y^1)| = |g(y - \lambda Y^1) - g(y) + \lambda g'(y)Y^1| \leq \frac{\lambda^2}{2}\|g''\|_\infty (Y^1)^2 \tag{3.14}$$

and obtain that

$$\begin{aligned} &2E\big((y, f) - \lambda(Y^1, F^1)\big) \\ &= \left\|\rho_2\big(\partial_t y - \Delta y + g(y) - f\,1_\omega\big) - \lambda\rho_2\big(\partial_t Y^1 - \Delta Y^1 + g'(y)Y^1 - F\,1_\omega\big) + \rho_2 l(y, -\lambda Y^1)\right\|^2_{L^2(0,T;H^{-1}(\Omega))} \\ &= \left\|\rho_2(1 - \lambda)\big(\partial_t y - \Delta y + g(y) - f\,1_\omega\big) + \rho_2 l(y, -\lambda Y^1)\right\|^2_{L^2(0,T;H^{-1}(\Omega))} \\ &\leq \left(\big\|\rho_2(1 - \lambda)\big(\partial_t y - \Delta y + g(y) - f\,1_\omega\big)\big\|_{L^2(0,T;H^{-1}(\Omega))} + \big\|\rho_2 l(y, -\lambda Y^1)\big\|_{L^2(0,T;H^{-1}(\Omega))}\right)^2 \\ &\leq 2\left(|1 - \lambda|\sqrt{E(y, f)} + \frac{\lambda^2}{2\sqrt{2}}\|g''\|_\infty \|\rho_2 (Y^1)^2\|_{L^2(0,T;H^{-1}(\Omega))}\right)^2. \end{aligned} \tag{3.15}$$

For $d = 3$ (similar estimates hold for $d = 1$ and $d = 2$), using the continuous embedding of $L^{6/5}(\Omega)$ into $H^{-1}(\Omega)$, we have:

$$\begin{aligned}\|\rho_2(Y^1)^2\|^2_{L^2(0,T;H^{-1}(\Omega))} &\le C(\Omega)\|\rho_2(Y^1)^2\|^2_{L^2(0,T;L^{6/5}(\Omega))}\\ &\le C(\Omega)\int_0^T \|\rho_2 Y^1\|^2_{L^3(\Omega)}\|Y^1\|^2_{L^2(\Omega)}\\ &\le C(\Omega)\int_0^T \|\rho Y^1\|_{L^2(\Omega)}\|\rho_1 Y^1\|_{L^6(\Omega)}\|Y^1\|^2_{L^2(\Omega)}\\ &\le C(\Omega)\int_0^T \|\rho Y^1\|_{L^2(\Omega)}\|\nabla(\rho_1 Y^1)\|_{L^2(\Omega)^d}\|Y^1\|^2_{L^2(\Omega)}.\end{aligned}$$

From the definition of ρ and ρ_1 we have $\nabla\rho_1 = \frac{s\nabla\beta}{\ell(t)(T-t)}\rho_1 = \frac{s\nabla\beta}{\ell(t)}\rho$ and therefore a.e. t in $(0,T)$

$$\begin{aligned}\|\nabla(\rho_1 Y^1)\|_{L^2(\Omega)^d} &\le \|\nabla(\rho_1)Y^1\|_{L^2(\Omega)^d} + \|\rho_1\nabla Y^1\|_{L^2(\Omega)^d}\\ &\le C(s,\Omega,\omega,T,\|g'\|_\infty)\|\rho Y^1\|_{L^2(\Omega)} + \|\rho_1\nabla Y^1\|_{L^2(\Omega)^d}\end{aligned}$$

and thus

$$\begin{aligned}\|\rho_2(Y^1)^2\|^2_{L^2(0,T;H^{-1}(\Omega))} \le C(s,\Omega,\omega,T,\|g'\|_\infty)&\|\rho_1 Y^1\|^2_{L^\infty(0,T;L^2(\Omega))}\|\rho Y^1\|_{L^2(Q_T)}\\ &\times\big(\|\rho Y^1\|_{L^2(Q_T)} + \|\rho_1\nabla Y^1\|_{L^2(Q_T)^d}\big).\end{aligned}$$

Using (3.5) and (3.6), we obtain

$$\|\rho_0(Y^1)^2\|^2_{L^2(0,T;H^{-1}(\Omega))} \le C(s,\Omega,\omega,T,\|g'\|_\infty)E(y,f)^2 \tag{3.16}$$

from which we get (3.13). □

Proceeding as in [23], we are now in position to prove the following convergence result for the sequence $(E(y_k,f_k))_{(k\in\mathbb{N})}$.

Proposition 3.7. *Assume $g \in W_1$. Let $(y_k,f_k)_{k\in\mathbb{N}}$ be the sequence defined by (3.11). Then $E(y_k,f_k)\to 0$ as $k\to\infty$. Moreover, there exists $k_0\in\mathbb{N}$ such that the sequence $(E(y_k,f_k))_{k\ge k_0}$ decays quadratically.*

Proof. We define the function p_k as follows

$$p_k(\lambda) = |1-\lambda| + \lambda^2 c_1\sqrt{E(y_k,f_k)} \quad \text{with} \quad c_1 := C(s,\Omega,\omega,T,\|g'\|_\infty)\|g''\|_\infty.$$

Lemma 3.6 with $(y,f) = (y_k,f_k)$ allows to write that

$$c_1\sqrt{E(y_{k+1},f_{k+1})} \le c_1\sqrt{E(y_k,f_k)}p_k(\widetilde{\lambda_k}), \quad \forall k\ge 0 \tag{3.17}$$

with $p_k(\widetilde{\lambda_k}) := \min_{\lambda\in[0,m]} p_k(\lambda)$.

If $c_1\sqrt{E(y_0,f_0)} < 1$ (and thus $c_1\sqrt{E(y_k,f_k)} < 1$ for all $k\in\mathbb{N}$) then

$$p_k(\widetilde{\lambda_k}) = \min_{\lambda\in[0,m]} p_k(\lambda) \le p_k(1) = c_1\sqrt{E(y_k,f_k)}$$

and thus

$$c_1\sqrt{E(y_{k+1},f_{k+1})} \leq \big(c_1\sqrt{E(y_k,f_k)}\big)^2 \tag{3.18}$$

implying that $c_1\sqrt{E(y_k,f_k)} \to 0$ as $k \to \infty$ with a quadratic rate.

If now $c_1\sqrt{E(y_0,f_0)} \geq 1$, we check that $I := \{k \in \mathbb{N},\ c_1\sqrt{E(y_k,f_k)} \geq 1\}$ is a finite subset of $\mathbb{N}$. For all $k \in I$, since $c_1\sqrt{E(y_k,f_k)} \geq 1$,

$$\min_{\lambda\in[0,m]} p_k(\lambda) = \min_{\lambda\in[0,1]} p_k(\lambda) = p_k\Big(\frac{1}{2c_1\sqrt{E(y_k,f_k)}}\Big) = 1 - \frac{1}{4c_1\sqrt{E(y_k,f_k)}}$$

and thus, for all $k \in I$,

$$c_1\sqrt{E(y_{k+1},f_{k+1})} \leq \Big(1 - \frac{1}{4c_1\sqrt{E(y_k,f_k)}}\Big)c_1\sqrt{E(y_k,f_k)} = c_1\sqrt{E(y_k,f_k)} - \frac{1}{4}. \tag{3.19}$$

This inequality implies that the sequence $(c_1\sqrt{E(y_k,f_k)})_{k\in\mathbb{N}}$ strictly decreases and then that the sequence $(p_k(\widetilde{\lambda_k}))_{k\in\mathbb{N}}$ decreases as well. Thus the sequence $(c_1\sqrt{E(y_k,f_k)})_{k\in\mathbb{N}}$ decreases to 0 at least linearly and there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $c_1\sqrt{E(y_k,f_k)} < 1$, that is I is a finite subset of $\mathbb{N}$. Arguing as in the first case, it follows that $c_1\sqrt{E(y_k,f_k)} \to 0$ as $k \to \infty$. In both cases, remark that $p_k(\widetilde{\lambda_k})$ decreases with respect to k. □

Remark 3.8. Writing from (3.19) that $c_1\sqrt{E(y_k,f_k)} \leq c_1\sqrt{E(y_0,f_0)} - \frac{k}{4}$ for all k such that $c_1\sqrt{E(y_k,f_k)} \geq 1$, we obtain that

$$k_0 \leq \Big\lfloor 4\big(c_1\sqrt{E(y_0,f_0)} - 1\big) + 1 \Big\rfloor$$

where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}^+$.

We also have the following convergence of the optimal sequence $(\lambda_k)_{k\in\mathbb{N}}$.

Lemma 3.9. *The sequence $(\lambda_k)_{k\in\mathbb{N}}$ defined in (3.11) converges to 1 as $k \to \infty$.*

Proof. In view of (3.15), we have, as long as $E(y_k,f_k) > 0$, since $\lambda_k \in [0,m]$

$$\begin{aligned}
(1-\lambda_k)^2 &= \frac{E(y_{k+1},f_{k+1})}{E(y_k,f_k)} - 2(1-\lambda_k)\frac{\langle \rho_2\big(\partial_t y_k + \Delta y_k + g(y_k) - f_k\,1_\omega\big), \rho_2 l(y_k,\lambda_k Y_k^1)\rangle_{L^2(0,T;H^{-1}(\Omega))}}{E(y_k,f_k)} \\
&\quad - \frac{\big\|\rho_2 l(y_k,\lambda_k Y_k^1)\big\|^2_{L^2(0,T;H^{-1}(\Omega))}}{2E(y_k)} \\
&\leq \frac{E(y_{k+1},f_{k+1})}{E(y_k,f_k)} - 2(1-\lambda_k)\frac{\langle \rho_2\big(\partial_t y_k + \Delta y_k + g(y_k) - f_k\,1_\omega\big), \rho_2 l(y_k,\lambda_k Y_k^1)\rangle_{L^2(0,T;H^{-1}(\Omega))}}{E(y_k,f_k)} \\
&\leq \frac{E(y_{k+1},f_{k+1})}{E(y_k,f_k)} + 2\sqrt{2}m\frac{\sqrt{E(y_k,f_k)}\|\rho_2 l(y_k,\lambda_k Y_k^1)\|_{L^2(0,T;H^{-1}(\Omega))}}{E(y_k,f_k)} \\
&\leq \frac{E(y_{k+1},f_{k+1})}{E(y_k,f_k)} + 2\sqrt{2}m\frac{\|\rho_2 l(y_k,\lambda_k Y_k^1)\|_{L^2(0,T;H^{-1}(\Omega))}}{\sqrt{E(y_k,f_k)}}.
\end{aligned}$$

But, from (3.14) and (3.16),

$$\begin{aligned}\|\rho_2 l(y_k,\lambda_k Y_k^1)\|_{L^2(0,T;H^{-1}(\Omega))} &\leq \frac{\lambda_k^2}{2\sqrt{2}}\|g''\|_\infty \|\rho_2 (Y_k^1)^2\|_{L^2(0,T;H^{-1}(\Omega))}\\ &\leq m^2\|g''\|_\infty C(s,T,\Omega,\omega,\|g'\|_\infty)E(y_k,f_k)\end{aligned}$$

and thus

$$(1-\lambda_k)^2 \leq \frac{E(y_{k+1},f_{k+1})}{E(y_k,f_k)} + m^2\|g''\|_\infty C(s,\Omega,\omega,T,\|g'\|_\infty)\sqrt{E(y_k,f_k)}.$$

Consequently, since $E(y_k,f_k)\to 0$ and $\frac{E(y_{k+1},f_{k+1})}{E(y_k,f_k)}\to 0$, we deduce that $(1-\lambda_k)^2\to 0$. □

We are now in position to prove the following convergence result.

Theorem 3.10. *Assume $g\in W_1$. Let $(y_k,f_k)_{k\in\mathbb{N}}$ be the sequence defined by (3.11). Then, $(y_k,f_k)_{k\in\mathbb{N}}\to(y,f)$ in $\mathcal{A}$ where f is a null control for y solution of (1.1). Moreover, the convergence is quadratic after a finite number of iterates.*

Proof. For all $k\in\mathbb{N}$, let $F_k=-\sum_{n=0}^k\lambda_n F_n^1$ and $Y_k=\sum_{n=0}^k\lambda_n Y_n^1$. Let us prove that $(Y_k,F_k)_{k\in\mathbb{N}}$ converges in $\mathcal{A}_0$, *i.e.* that the series $\sum\lambda_n(F_n^1,Y_n^1)$ converges in $\mathcal{A}_0$. Using that $\|(Y_k^1,F_k^1)\|_{\mathcal{A}_0}\leq C\sqrt{E(y_k,f_k)}$ for all $k\in\mathbb{N}$ (see (3.4)), we write

$$\sum_{n=0}^k\lambda_n\|(Y_n^1,F_n^1)\|_{\mathcal{A}_0}\leq m\sum_{n=0}^k\|(Y_n^1,F_n^1)\|_{\mathcal{A}_0}\leq C\sum_{n=0}^k\sqrt{E(y_n,f_n)}.$$

But $\left(\sqrt{E(y_n,f_n)}\right)_{k\in\mathbb{N}}$ and $\left(p_k(\widetilde{\lambda_k})\right)_{k\in\mathbb{N}}$ are decreasing sequences so that

$$\sqrt{E(y_n,f_n)}\leq p_n(\widetilde{\lambda_n})\sqrt{E(y_{n-1},f_{n-1})}\leq p_0(\widetilde{\lambda_0})\sqrt{E(y_{n-1},f_{n-1})}\leq p_0(\widetilde{\lambda_0})^n\sqrt{E(y_0,f_0)}.$$

Since $p_0(0)=1$ and $p_0'(0)=-1$, we infer that $p_0(\widetilde{\lambda_0})<1$ leading to

$$\sum_{n=0}^k\sqrt{E(y_n,f_n)}\leq\sqrt{E(y_0,f_0)}\frac{1-p_0(\widetilde{\lambda_0})^{k+1}}{1-p_0(\widetilde{\lambda_0})}\leq\frac{\sqrt{E(y_0,f_0)}}{1-p_0(\widetilde{\lambda_0})}.$$

We deduce that the series $\sum_n\lambda_n(Y_n^1,F_n^1)$ is normally convergent and so convergent. Consequently, there exists $(Y,F)\in\mathcal{A}_0$ such that $(Y_k,F_k)_{k\in\mathbb{N}}$ converges to (Y,F) in $\mathcal{A}_0$.

Denoting $y=y_0+Y$ and $f=f_0+F$, we then have that $(y_k,f_k)_{k\in\mathbb{N}}=(y_0+Y_k,f_0+F_k)_{k\in\mathbb{N}}$ converges to (y,f) in $\mathcal{A}$.

It suffices now to verify that the limit (y,f) satisfies $E(y,f)=0$. We write that $(Y_k^1,F_k^1)\in\mathcal{A}_0$ and $(y_k,f_k)\in\mathcal{A}$ solve

$$\begin{cases}\partial_t Y_k^1-\Delta Y_k^1+g'(y_k)\cdot Y_k^1=F_k^1 1_\omega+\partial_t y_k-\Delta y_k+g(y_k)-f_k 1_\omega \quad\text{in}\quad Q_T,\\ Y_k^1=0\text{ on }\Sigma_T,\quad Y_k^1(\cdot,0)=0\text{ in }\Omega.\end{cases}\tag{3.20}$$

Using that (Y_k^1,F_k^1) goes to zero in $\mathcal{A}_0$ as $k\to\infty$, we pass to the limit in (3.20) and get, since $g\in W_1$, that $(y,f)\in\mathcal{A}$ solves (1.1), that is $E(y,f)=0$. □

In particular, along the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ defined by (3.11), we have the following coercivity property for E, which confirms the strong convergence of the sequence $(y_k, f_k)_{k\in\mathbb{N}}$.

Proposition 3.11. *Let $(y_k, f_k)_{k\in\mathbb{N}}$ defined by (3.11) and (y, f) its limit. Then, there exists a positive constant $C = C(s, T, \Omega, \omega, \|g'\|_\infty)$ such that*

$$\|(y, f) - (y_k, f_k)\|_{\mathcal{A}_0} \leq C\sqrt{E(y_k, f_k)}, \quad \forall k > 0. \tag{3.21}$$

Proof. We write that

$$\begin{aligned}
\|(y, f) - (y_k, f_k)\|_{\mathcal{A}_0} = \left\| \sum_{p=k+1}^{\infty} \lambda_p(Y_p^1, F_p^1) \right\|_{\mathcal{A}_0} &\leq m \sum_{p=k+1}^{\infty} \|(Y_p^1, F_p^1)\|_{\mathcal{A}_0} \\
&\leq m\, C \sum_{p=k+1}^{\infty} \sqrt{E(y_p, f_p)} \\
&\leq m\, C \sum_{p=k+1}^{\infty} p_0(\widetilde{\lambda}_0)^{p-k} \sqrt{E(y_k, f_k)} \\
&\leq m\, C \frac{p_0(\widetilde{\lambda}_0)}{1 - p_0(\widetilde{\lambda}_0)} \sqrt{E(y_k, f_k)}.
\end{aligned}$$

□

We emphasize, in view of the non uniqueness of the zeros of E, that an estimate (similar to (3.21)) of the form $\|(\overline{y}, \overline{f}) - (y, f)\|_{\mathcal{A}_0} \leq C\sqrt{E(\overline{y}, \overline{f})}$ does not hold for all $(\overline{y}, \overline{f}) \in \mathcal{A}$. We also mention the fact that the sequence $(y_k, f_k)_{k>0}$ and its limits (y, f) are uniquely determined from the initial guess (y_0, f_0) and from our criterion of selection of the control F^1. In other words, the solution (y, f) is unique up to the element (y_0, f_0) and the functional J defined in Proposition 2.1.

3.3. The case $g \in W_p$, $0 \leq p < 1$ and additional remarks

The results of the previous subsection devoted to the case $p = 1$ still hold if we assume only that $g \in W_p$ for some $p \in (0, 1)$. For any $g \in W_p$, we recall the notation $[g']_p := \sup_{a,b\in\mathbb{R}, a\neq b} \frac{|g'(a)-g'(b)|}{|a-b|^p}$. We have the following result.

Theorem 3.12. *Assume that there exists $p \in (0, 1)$ such that $g \in W_p$. Let $(y_k, f_k)_{k\in\mathbb{N}}$ be the sequence defined by (3.11). Then, $(y_k, f_k)_{k\in\mathbb{N}} \to (y, f)$ in $\mathcal{A}$ where f is a null control for y solution of (1.1). Moreover, after a finite number of iterates, the rate of convergence is equal to $1 + p$.*

Proof. We briefly sketch the proof, close to the proof of Theorem 3.10 for the case $p = 1$.

–We first prove for any $(y, f) \in \mathcal{A}$ and $\lambda \in \mathbb{R}$ the following inequality (similar to the inequality (3.13))

$$E\big((y, f) - \lambda(Y^1, F^1)\big) \leq E(y, f)\Big(|1 - \lambda| + \lambda^{1+p} c_1 E(y, f)^{p/2}\Big)^2 \tag{3.22}$$

with $c_1 = C(s,T,\Omega,\omega,\|g'\|_\infty)[g']_p$ and $(Y^1,F^1)\in\mathcal{A}_0$ the solution of (3.12) which minimizes J. For any $(x,y)\in\mathbb{R}^2$ and $\lambda\in\mathbb{R}$, we write $g(x+\lambda y)-g(x)=\int_0^\lambda yg'(x+\xi y)\mathrm{d}\xi$ leading to

$$\begin{aligned}|g(x+\lambda y)-g(x)-\lambda g'(x)y| &\le \int_0^\lambda |y||g'(x+\xi y)-g'(x)|\mathrm{d}\xi\\ &\le \int_0^\lambda |y|^{1+p}|\xi|^p\frac{|g'(x+\xi y)-g'(x)|}{|\xi y|^p}\mathrm{d}\xi\\ &\le [g']_s|y|^{1+p}\frac{\lambda^{1+p}}{1+p}.\end{aligned}$$

It follows that

$$|l(y,-\lambda Y^1)| = |g(y-\lambda Y^1)-g(y)+\lambda g'(y)Y^1| \le [g']_p\frac{\lambda^{1+p}}{1+p}|Y^1|^{1+p}$$

and

$$\begin{aligned}\big\|\rho_2 l(y,\lambda Y^1)\big\|_{L^2(0,T;H^{-1}(\Omega))} &\le \big\|\rho_2 l(y,\lambda Y^1)\big\|_{L^2(0,T;L^{6/5}(\Omega))}\\ &\le [g']_p\frac{\lambda^{1+p}}{1+p}\big\|\rho_2|Y^1|^{1+p}\big\|_{L^2(0,T;L^{6/5}(\Omega))}.\end{aligned}$$

But

$$\begin{aligned}\big\|\rho_2|Y^1|^{1+p}\big\|^2_{L^2(0,T;L^{6/5}(\Omega))} &= \int_0^T\big\|\rho_2|Y^1|^{1+p}\big\|^2_{L^{6/5}(\Omega)} \le \int_0^T\big\|\rho_2 Y^1\big\|^2_{L^3(\Omega)}\big\||Y^1|^p\big\|^2_{L^2(\Omega)}\\ &\le \int_0^T\big\|\rho Y^1\big\|_{L^2(\Omega)}\big\|\rho_1 Y^1\big\|_{L^6(\Omega)}\big\|Y^1\big\|^{2p}_{L^{2p}(\Omega)}\\ &\le C(\Omega)\int_0^T\big\|\rho Y^1\big\|_{L^2(\Omega)}\big\|\nabla(\rho_1 Y^1)\big\|_{L^2(\Omega)^d}\big\|Y^1\big\|^{2p}_{L^{2p}(\Omega)}\\ &\le C(\Omega)\big\|\rho Y^1\big\|_{L^2(Q_T)}\big\|\nabla(\rho_1 Y^1)\big\|_{L^2(Q_T)^d}\big\|Y^1\big\|^{2p}_{L^\infty(0,T;L^{2s}(\Omega))}\\ &\le C(\Omega)\big\|\rho Y^1\big\|_{L^2(Q_T)}\big\|\nabla(\rho_1 Y^1)\big\|_{L^2(Q_T)^d}\big\|Y^1\big\|^{2p}_{L^\infty(0,T;L^2(\Omega))}.\end{aligned}$$

Since $\|\nabla(\rho_1 Y^1)\|_{L^2(\Omega)^d}\le C(s,\Omega,\omega,T,\|g'\|_\infty)\|\rho Y^1\|_{L^2(\Omega)}+\|\rho_1\nabla Y^1\|_{L^2(\Omega)^d}$, we finally get

$$\begin{aligned}\big\|\rho_2|Y^1|^{1+s}\big\|^2_{L^2(0,T;L^{6/5}(\Omega))} &\le C(s,\Omega,\omega,T,\|g'\|_\infty)\|\rho Y^1\|_{L^2(Q_T)}\\ &\quad\times\big(\|\rho Y^1\|_{L^2(Q_T)}+\|\rho_1\nabla Y^1\|_{L^2(Q_T)^d}\big)\|\rho_1 Y\|^{2p}_{L^\infty(0,T;L^2(\Omega))}.\end{aligned}$$

The first inequality of (3.15) then leads to (3.22).

– We then check that the sequence $(E(y_k,f_k))_{k\in\mathbb{N}}$ goes to zero as $k\to\infty$. We define p_k as follows

$$p_k(\lambda) = |1-\lambda|+\lambda^{1+p}c_1E(y_k,f_k)^{p/2}$$

so that

$$\sqrt{E(y_{k+1},f_{k+1})}\le p_k(\widetilde{\lambda_k})\sqrt{E(y_k,f_k)},\quad\forall k\ge 0$$

with $p_k(\widetilde{\lambda_k}) = \min_{\lambda\in[0,m]} p_k(\lambda)$. We have $p_k(\widetilde{\lambda}_k) := \min_{\lambda\in[0,m]} p_k(\lambda) \leq p_k(1) = c_1 E(y_k, f_k)^{p/2}$ and thus

$$c_2\sqrt{E(y_{k+1}, f_{k+1})} \leq \big(c_2\sqrt{E(y_k, f_k)}\big)^{1+p}, \qquad c_2 := c_1^{1/p}.$$

If $c_2\sqrt{E(y_0, f_0)} < 1$ (and thus $c_2\sqrt{E(y_k, f_k)} < 1$ for all $k \in \mathbb{N}$) then the above inequality implies that $c_2\sqrt{E(y_k, f_k)} \to 0$ as $k \to \infty$. If $c_2\sqrt{E(y_0, f_0)} \geq 1$ then let $I = \{k \in \mathbb{N},\ c_2\sqrt{E(y_k, f_k)} \geq 1\}$. I is a finite subset of $\mathbb{N}$; for all $k \in I$, since $c_2\sqrt{E(y_k, f_k)} \geq 1$

$$\min_{\lambda\in[0,m]} p_k(\lambda) = \min_{\lambda\in[0,1]} p_k(\lambda) = p_k\Big(\frac{1}{(1+p)^{1/p} c_2\sqrt{E(y_k, f_k)}}\Big) = 1 - \frac{p}{(1+p)^{\frac{1}{p}+1}} \frac{1}{c_2\sqrt{E(y_k, f_k)}}$$

and thus, for all $k \in I$,

$$c_2\sqrt{E(y_{k+1}, f_{k+1})} \leq \Big(1 - \frac{p}{(1+p)^{\frac{1}{p}+1}} \frac{1}{c_2\sqrt{E(y_k, f_k)}}\Big) c_2\sqrt{E(y_k, f_k)} = c_2\sqrt{E(y_k, f_k)} - \frac{p}{(1+p)^{\frac{1}{p}+1}}.$$

This inequality implies that the sequence $(c_2\sqrt{E(y_k, f_k)})_{k\in\mathbb{N}}$ strictly decreases and then that the sequence $(p_k(\widetilde{\lambda_k}))_{k\in\mathbb{N}}$ decreases as well. Thus the sequence $(c_2\sqrt{E(y_k, f_k)})_{k\in\mathbb{N}}$ decreases to 0 at least linearly and there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $c_2\sqrt{E(y_k, f_k)} < 1$, that is I is a finite subset of $\mathbb{N}$. Similarly, the optimal parameter λ_k goes to one as $k \to \infty$.

– Using that the sequence $(E(y_k, f_k))_{k\in\mathbb{N}}$ goes to zero, we conclude exactly as in the proof of Theorem 3.10. □

On the other hand, if we assume only that g belongs to W_0, then we can not expect the convergence of the sequence $(y_k, f_k)_{k>0}$ if $\|g'\|_\infty$ is too large.

Remark 3.13. Assume that $g \in W_0$. Let any $(y, f) \in \mathcal{A}$ and (Y^1, F^1) the solution of (3.3) which minimizes J. The following inequality holds:

$$\sqrt{E\big((y, f) - \lambda(Y^1, F^1)\big)} \leq \sqrt{E(y, f)}\Big(|1-\lambda| + \lambda C(s, \Omega, \omega, T, \|g'\|_\infty)\|g'\|_\infty\Big)$$

for all $\lambda \in \mathbb{R}$ where $C(s, \Omega, \omega, T, \|g'\|_\infty) \geq 0$ increases with $\|g'\|_\infty$. Indeed, this is a consequence of the following inequality, for all $(y, f) \in \mathcal{A}$, $(Y, F) \in \mathcal{A}_0$:

$$\begin{aligned} 2E\big((y, f) - \lambda(Y^1, F^1)\big) &\leq \Big(\big\|\rho_2(1-\lambda)\big(\partial_t y - \Delta y + g(y) - f\,1_\omega\big)\big\|_{L^2(0,T;H^{-1}(\Omega))} + \big\|\rho_2 l(y, \lambda Y^1)\big\|_{L^2(0,T;H^{-1}(\Omega))}\Big)^2 \\ &\leq \Big(|1-\lambda|\sqrt{2E(y, f)} + 2\lambda\|(T-t)^{1/2} g'(y)\|_{L^\infty(Q_T)}\|\rho Y\|_{L^2(Q_T)}\Big)^2. \end{aligned}$$

Therefore, the sequence $(E(y_k, f_k))_{k\in\mathbb{N}}$ decreases to 0 if g satisfies

$$C(s, \Omega, \omega, T, \|g'\|_\infty)\|g'\|_\infty < 1.$$

Remark 3.14. The estimate (3.4) is a key point in the convergence analysis and is independent of the choice of the functional J defined by $J(Y^1, F^1) = \frac{1}{2}\|\rho_0 F^1\|^2_{L^2(q_T)} + \frac{1}{2}\|\rho Y\|^2_{L^2(Q_T)}$ (see Prop. 2.1) in order to select a pair (Y^1, F^1) in $\mathcal{A}_0$. Thus, we may consider other weighted functionals, for instance $J(Y^1, F^1) = \frac{1}{2}\|\rho_0 F^1\|^2_{L^2(q_T)}$ as discussed in [27].

Remark 3.15. If we introduce $F : \mathcal{A} \to L^2(0,T;H^{-1}(\Omega))$ by $F(y,f) := \rho_2(\partial_t y - \Delta y + g(y) - f\,1_\omega)$, we get that $E(y,f) = \frac{1}{2}\|F(y,f)\|^2_{L^2(0,T;H^{-1}(\Omega))}$ and observe that, for $\lambda_k = 1$, the algorithm (3.11) coincides with the Newton algorithm associated with the mapping F. This explains notably the quadratic convergence of Theorem 3.10 in the case $g \in W_1$ for which we have a control of g'' in $L^\infty(Q_T)$. The optimization of the parameter λ_k allows to get a global convergence of the algorithm and leads to the so-called damped Newton method (for F). Under general hypothesis, global convergence for this kind of method is achieved, with a linear rate (for instance, we refer to [10], Thm. 8.7). As far as we know, the analysis of damped type Newton methods for partial differential equations has deserved very few attention in the literature. We mention [22, 30] in the context of fluid mechanics.

Remark 3.16. Suppose to simplify that λ_k equals one (corresponding to the standard Newton method). Then, for each k, the optimal pair $(Y_k^1, F_k^1) \in \mathcal{A}_0$ is such that the element (y_{k+1}, f_{k+1}) minimizes over $\mathcal{A}$ the functional $(z,v) \to J(z - y_k, v - f_k)$. Instead, we may also select the pair (Y_k^1, F_k^1) such that the element (y_{k+1}, f_{k+1}) minimizes the functional $(z,v) \to J(z,v)$. This leads to the following sequence $(y_k, f_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ defined by

$$\begin{cases} \partial_t y_{k+1} - \Delta y_{k+1} + g'(y_k)y_{k+1} = f_{k+1}1_\omega + g'(y_k)y_k - g(y_k) \quad \text{in } Q_T, \\ y_k = 0 \text{ on } \Sigma_T, \quad (y_{k+1}(\cdot,0), y_{k+1,t}(\cdot,0)) = (u_0, u_1) \text{ in } \Omega. \end{cases} \tag{3.23}$$

This is actually the formulation used in [13]. This formulation is different and the analysis of convergence (at least in the framework of our least-squares setting) is less direct because it is necessary to have a control of the right hand side term $g'(y_k)y_k - g(y_k)$.

Remark 3.17. We emphasize that the explicit construction used here allows to recover the null controllability property of (1.1) for nonlinearities g in W_p for some $p \in (0,1]$. We do not use a fixed point argument as in [16]. On the other hand, the conditions we make on g are more restrictives that in [16].

Eventually, it is also important to remark that these additional conditions does not imply *a priori* a contraction property of the operator Λ introduced in [16]. If we consider the controlled pair associated with the weighted functional J, we can show as in [16] (where the control of minimal $L^\infty(Q_T)$ norm is used) that there exists a real M large enough such that $\Lambda(B_\infty(0,M)) \subset B_\infty(0,M)$ (see [21]). Therefore, let z_i in $L^\infty(Q_T)$, $i = 1,2$ and assume that $\widetilde{g}' \in L^\infty(0,M)$. If $(y_{z_i}, f_{z_i}) = (\Lambda(z_i), f_{z_i})$, $i = 1,2$ are controlled pairs for the system (1.7) minimizing the functional J, then the following inequality holds:

$$\|\rho\big(\Lambda(z_1) - \Lambda(z_2)\big)\|_{L^2(Q_T)} \le C(s,\Omega,\omega,T,\|\tilde{g}\|_{L^\infty(0,M)})\|\widetilde{g}'\|_{L^\infty(0,M)}\|u_0\|_{L^2(\Omega)}\|\rho(z_1 - z_2)\|_{L^2(Q_T)} \tag{3.24}$$

where $C(s,\Omega,\omega,T,\|\tilde{g}\|_\infty)$ is the constant appearing in (2.5). In view of Lemma 2.6 the pair $(y_i, f_i) = (\Lambda(z_i), f_i)$, $i = 1,2$ satisfies $y_i = \rho^{-2}L^\star_{\widetilde{g}(z_i)}p_i$ and $f_i = -\rho_0^{-2}p_i|_{q_T}$ for $p_i \in P_{\widetilde{g}(z_i)}$ solution of

$$(p_i, q)_{P_{\widetilde{g}(z_i)}} = \int_\Omega u_0 q(0), \quad \forall q \in P_{\widetilde{g}(z_i)}.$$

i.e.

$$\int_{Q_T} \rho^{-2} L^\star_{\widetilde{g}(z_i)}p_i\, L^\star_{\widetilde{g}(z_i)}q + \int_{q_T} \rho_0^{-2} p_i\, q = \int_\Omega u_0 q(0), \quad \forall q \in P_{\widetilde{g}(z_i)}$$

leading to

$$\begin{aligned} &\int_{Q_T} \rho^{-2} L^\star_{\widetilde{g}(z_2)}(p_2 - p_1)\, L^\star_{\widetilde{g}(z_2)}q + \int_{q_T} \rho_0^{-2}(p_2 - p_1)\, q \\ &= -\int_{Q_T} \rho^{-2}\Big((L^\star_{\widetilde{g}(z_2)} - L^\star_{\widetilde{g}(z_1)})p_1 L^\star_{\widetilde{g}(z_2)}q + L^\star_{\widetilde{g}(z_1)}p_1 (L^\star_{\widetilde{g}(z_2)} - L^\star_{\widetilde{g}(z_1)})q \Big) \quad \forall q \in P_{\widetilde{g}(z_i)}. \end{aligned}$$

Using that $(L^\star_{\widetilde{g}(z_2)} - L^\star_{\widetilde{g}(z_1)})q = (\widetilde{g}(z_2) - \widetilde{g}(z_1))q$ for all $q \in P_{\widetilde{g}(z_i)}$ and taking $q = p_2 - p_1$, we obtain

$$\begin{aligned}
\|p_2 - p_1\|^2_{P_{\widetilde{g}(z_2)}} &= \|\rho^{-1} L^\star_{\widetilde{g}(z_2)}(p_2 - p_1)\|^2_{L^2(Q_T)} + \|\rho_0^{-1}(p_2 - p_1)\|^2_{L^2(q_T)} \\
&\leq \|\rho^{-1}\big(\widetilde{g}(z_2) - \widetilde{g}(z_1)\big)p_1\|_{L^2(Q_T)} \|\rho^{-1} L^\star_{\widetilde{g}(z_2)}(p_2 - p_1)\|_{L^2(Q_T)} \\
&\quad + \|\rho^{-1} L^\star_{\widetilde{g}(z_1)} p_1\|_{L^2(Q_T)} \|\rho^{-1}\big(\widetilde{g}(z_2) - \widetilde{g}(z_1)\big)(p_2 - p_1)\|_{L^2(Q_T)} \\
&\leq \|\rho^{-1}\big(\widetilde{g}(z_2) - \widetilde{g}(z_1)\big)p_1\|_{L^2(Q_T)} \|p_2 - p_1\|_{P_{\widetilde{g}(z_2)}} + \|\rho\Lambda(z_1)\|_{L^2(Q_T)} \|\rho^{-1}\big(\widetilde{g}(z_2) - \widetilde{g}(z_1)\big)(p_2 - p_1)\|_{L^2(Q_T)}.
\end{aligned} \tag{3.25}$$

Moreover, from Lemma 3.2 of [13], there exists $C > 0$ such that $\|(\xi\rho)^{-1} q\|_{L^\infty(Q_T)} \leq C\|q\|_{P_{\widetilde{g}(z_i)}}$ for all $q \in P_{\widetilde{g}(z_i)}$ and $\xi = (T-t)^{-1/2}$. It follows that

$$\begin{aligned}
\|\rho^{-1}\big(\widetilde{g}(z_2) - \widetilde{g}(z_1)\big)p_1\|_{L^2(Q_T)} &\leq \|\rho\big(\widetilde{g}(z_2) - \widetilde{g}(z_1)\big)\|_{L^2(Q_T)} \|\rho^{-2} p_1\|_{L^\infty(Q_T)} \\
&\leq \|\widetilde{g}'\|_{L^\infty(0,M)} \|\rho(z_1 - z_2)\|_{L^2(Q_T)} \|\zeta\rho^{-1}\|_{L^\infty(Q_T)} \|(\zeta\rho)^{-1} p_1\|_{L^\infty(Q_T)} \\
&\leq C\|\widetilde{g}'\|_{L^\infty(0,M)} \|\zeta\rho^{-1}\|_{L^\infty(Q_T)} \|p_1\|_{P_{\widetilde{g}(z_i)}} \|\rho(z_1 - z_2)\|_{L^2(Q_T)}
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
\|\rho^{-1}\big(\widetilde{g}(z_2) - \widetilde{g}(z_1)\big)(p_2 - p_1)\|_{L^2(Q_T)} &\leq \|\widetilde{g}'\|_{L^\infty(0,M)} \|\rho(z_1 - z_2)\|_{L^2(Q_T)} \|\zeta\rho^{-1}\|_{L^\infty(Q_T)} \|(\zeta\rho)^{-1}(p_2 - p_1)\|_{L^\infty(Q_T)} \\
&\leq C\|\widetilde{g}'\|_{L^\infty(0,M)} \|\zeta\rho^{-1}\|_{L^\infty(Q_T)} \|p_2 - p_1\|_{P_{\widetilde{g}(z_2)}} \|\rho(z_1 - z_2)\|_{L^2(Q_T)}.
\end{aligned}$$

Moreover, from (2.5), $\|\rho\Lambda(z_1)\|_2 \leq C\|u_0\|_2$ while from Lemma 2.5 $\|p_1\|_{P_{\widetilde{g}(z_1)}} \leq C\|u_0\|_2$. Inequality (3.25) then leads to

$$\|\rho^{-1} L^\star_{\widetilde{g}(z_2)}(p_2 - p_1)\|_{L^2(Q_T)} \leq C\|\widetilde{g}'\|_{L^\infty_{loc}} \|\zeta\rho^{-1}\|_{L^\infty(Q_T)} \|\rho(z_1 - z_2)\|_{L^2(Q_T)} \|u_0\|_{L^2(\Omega)}. \tag{3.27}$$

Eventually,

$$\begin{aligned}
\|\rho(\Lambda(z_2) - \Lambda(z_1))\|_2 &= \|\rho^{-1} L^\star_{\widetilde{g}(z_2)}(p_2) - \rho^{-1} L^\star_{\widetilde{g}(z_1)}(p_1)\|_{L^2(Q_T)} \\
&= \|\rho^{-1} L^\star_{\widetilde{g}(z_2)}(p_2 - p_1) - \rho^{-1}(L^\star_{\widetilde{g}(z_1)} - L^\star_{\widetilde{g}(z_2)})(p_1)\|_{L^2(Q_T)} \\
&= \|\rho^{-1} L^\star_{\widetilde{g}(z_2)}(p_2 - p_1) - \rho^{-1}(\widetilde{g}(z_1) - \widetilde{g}(z_2))(p_1)\|_{L^2(Q_T)} \\
&\leq \|\rho^{-1} L^\star_{\widetilde{g}(z_2)}(p_2 - p_1)\|_{L^2(Q_T)} + \|\rho^{-1}(\widetilde{g}(z_1) - \widetilde{g}(z_2))(p_1)\|_{L^2(Q_T)}.
\end{aligned}$$

Combining (3.26) and (3.27), we finally get the estimate (3.24). It follows that a smallness assumption on $\|\widetilde{g}'\|_{L^\infty(0,M)}$ leads to a contraction property for Λ. We observe that $\widetilde{g}'(r) = (g'(r)r - g(r))/r^2$, $r \neq 0$ so that $\widetilde{g}'$ belongs to $L^\infty(0,M)$ if g'' belongs to $L^\infty(0,M)$.

4. Numerical illustrations

We illustrate in this section our results of convergence. We provide some practical details of the algorithm (3.11) then discuss some experiments in the one dimensional case. Approximations of null controls for the heat equation is a delicate issue: we mention the seminal work [5] dealing with the control of minimal L^2-norm which is very oscillatory near the final time $t = T$ and therefore difficult to construct. We also mention [19, 29] where this is discussed at length. On the other hand, introduction of very specific weights in the cost functional J allows to avoid oscillatory phenomena and leads to a robust method and strong convergent approximations with respect to the discretization parameter. We refer to [14, 15].

4.1. Approximation – Algorithm

Each iterate of the algorithm (3.11) requires the determination of the null control of F_k^1 for Y_k^1 solution of

$$\begin{cases} \partial_t Y_k^1 - \Delta Y_k^1 + g'(y_k) Y_k^1 = F_k^1 1_\omega + B_k \quad \text{in } Q_T, \\ Y_k^1 = 0 \quad \text{on} \quad \Sigma_T, \quad Y_k^1(\cdot,0) = 0 \quad \text{in } \Omega \end{cases} \tag{4.1}$$

with $B_k := \partial_t y_k - \Delta y_k + g(y_k) - f_k 1_\omega$. From Lemma 2.6, the pair (F_k^1, Y_k^1) which minimizes the functional J is given by $Y_k^1 = \rho^{-2} L^\star_{g'(y_k)} p_k$ and $F_k^1 = -\rho_0^{-2} p_k \, 1_{q_T}$ where $p_k \in P$ solves the formulation

$$\int_{Q_T} \rho^{-1} L^\star_{g'(y_k)} p_k \, \rho^{-1} L^\star_{g'(y_k)} \overline{p} + \int_{q_T} \rho_0^{-1} p_k \, \rho_0^{-1} \overline{p} = \int_0^T < \rho_2 B_k, \rho_2^{-1} \overline{p} >_{H^{-1}(\Omega), H_0^1(\Omega)} \, \mathrm{d}t \quad \forall \overline{p} \in P. \tag{4.2}$$

The numerical approximation of this variational formulation (of second order in time and fourth order in space) has been discussed at length in [14]. In order, first to avoid numerical instabilities (due to the presence of exponential functions in the formulation), and second to make appear explicitly the controlled solution, we introduce the variables $m_k := \rho_0^{-1} p_k$, $z_k := \rho^{-1} L^\star_{g'(y_k)} p_k$. Since $\rho_2^{-1} p \in L^2(0,T; H_0^1(\Omega))$, we obtain notably that $\rho_2^{-1} p = \rho_2^{-1} \rho_0 m = (T-t) m \in L^2(0,T; H_0^1(\Omega))$. From (4.2), the pair $(m_k, z_k) \in \mathcal{M} \times L^2(Q_T)$ with $\mathcal{M} := \rho_0^{-1} P$ solves

$$\int_{Q_T} z_k \, \overline{z} + \int_{q_T} m_k \, \overline{m} = \int_0^T < \rho_2 B_k, (T-t) \overline{m} >_{H^{-1}(\Omega), H_0^1(\Omega)} \, \mathrm{d}t \quad \forall (\overline{m}, \overline{z}) \in \mathcal{M} \times L^2(Q_T)$$

subjected to the constraint $z_k = \rho^{-1} L^\star_{g'(y_k)} (\rho_0 m_k)$. This constraint leads to the following well-posed mixed formulation: find $(m_k, z_k, \eta_k) \in \mathcal{M} \times L^2(Q_T) \times L^2(Q_T)$ solution of

$$\begin{cases} \displaystyle\int_{Q_T} z_k \, \overline{z} + \int_{q_T} m_k \, \overline{m} + \int_{Q_T} \eta_k \Big(\overline{z} - \rho^{-1} L^\star_{g'(y)} (\rho_0 \, \overline{m}) \Big) \\ \qquad\qquad = \displaystyle\int_0^T < \rho_2 B_k, (T-t) \overline{m} >_{H^{-1}(\Omega), H_0^1(\Omega)} \, \mathrm{d}t, \quad \forall (\overline{m}, \overline{z}) \in \mathcal{M} \times L^2(Q_T), \\ \displaystyle\int_{Q_T} \overline{\eta} \Big(z_k - \rho^{-1} L^\star_{g'(y_k)} (\rho_0 \, m) \Big) = 0, \quad \forall \overline{\eta} \in L^2(Q_T). \end{cases} \tag{4.3}$$

The variable $\eta_k \in L^2(Q_T)$ acts as a Lagrange multiplier. Moreover, from the unique solution (m_k, z_k), we get the explicit form of the controlled pair (Y_k^1, F_k^1) as follows: $Y_k^1 = \rho^{-1} z_k$, $F_k^1 = -\rho_0^{-1} m_k \, 1_{q_T}$.

The algorithm associated with the sequence $(y_k, f_k)_{k \in \mathbb{N}}$ (see (3.11)) may be developed as follows: given $\epsilon > 0$ and $m \geq 1$,

1. We determine the controlled pair $(y_0, f_0) \in \mathcal{A}$ which minimizes the functional J associated with the linear case (for which $g \equiv 0$ in (1.1)). (y_0, f_0) is given by $(y_0, f_0) = (\rho^{-1} z_0, -\rho_0^{-1} m_0 \, 1_{q_T})$ where $(m_0, z_0) \in \mathcal{M} \times L^2(Q_T)$ together with $\eta \in L^2(Q_T)$ solve the formulation:

$$\begin{cases} \displaystyle\int_{Q_T} z \, \overline{z} + \int_{q_T} m \, \overline{m} + \int_{Q_T} \eta \Big(\overline{z} - \rho^{-1} L_0^\star (\rho_0 \, \overline{m}) \Big) = \int_\Omega \rho_0(\cdot,0) \, u_0 \, \overline{m}(\cdot,0), \quad \forall (\overline{m}, \overline{z}) \in \mathcal{M} \times L^2(Q_T), \\ \displaystyle\int_{Q_T} \overline{\eta} \Big(z - \rho^{-1} L_0^\star (\rho_0 \, m) \Big) = 0, \quad \forall \overline{\eta} \in L^2(Q_T). \end{cases}$$

   In view of Proposition 2.1, we check that (y_0, f_0) belongs to $\mathcal{A}$.

2. Assume now that (y_k, f_k) is computed for some $k \geq 0$. We then compute $c_k \in L^2(0,T;H_0^1(\Omega))$, unique solution of

$$\int_{Q_T} \nabla c_k \cdot \nabla \overline{c} = \int_0^T < \rho_2(\partial_t y_k - \Delta y_k + g(y_k) - f_k\, 1_\omega), \overline{c} >_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall \overline{c} \in L^2(0,T;H_0^1(\Omega))$$

and then $E(y_k, f_k) = \frac{1}{2}\|\rho_2(\partial_t y_k - \Delta y_k + g(y_k) - f_k\, 1_\omega)\|^2_{L^2(0,T;H^{-1}(\Omega))} = \frac{1}{2}\|\nabla c_k\|^2_{L^2(Q_T)}$.
3. If $E(y_k, f_k) < \epsilon$, the approximate controlled pair is given by $(\overline{y}, \overline{f}) = (y_k, f_k)$ and the algorithm stops. Otherwise, we determine the solution $(Y_k^1, F_k^1) = (\rho^{-1} z_k, -\rho_0^{-1} m_k\, 1_{q_T})$ where (m_k, z_k) together with $\eta_k \in L^2(Q_T)$ solve (4.3).
4. Set $(y_{k+1}, f_{k+1}) = (y_k, f_k) - \lambda_k(Y_k^1, F_k^1)$ where λ_k minimizes over $[0, m]$ the scalar functional $\lambda \to E((y_k, f_k) - \lambda(Y_k^1, F_k^1))$ defined by (see (3.15))

$$E\big((y_k, f_k) - \lambda(Y_k^1, F_k^1)\big) = \frac{1}{2}\Big\|\rho_2(1-\lambda)\big(\partial_t y_k - \Delta y_k + g(y_k) - f_k\, 1_\omega\big) + \rho_2 l(y_k, -\lambda Y_k^1)\Big\|^2_{L^2(0,T;H^{-1}(\Omega))}$$

with $l(y_k, -\lambda Y_k^1) = g(y_k - \lambda Y_k^1) - g(y_k) + \lambda g'(y_k) Y_k^1$. The minimization is performed using a line search method. Return to step 2.

We use the conformal space-time finite element method described in [14]. We consider a regular family $\mathcal{T} = \{\mathcal{T}_h; h > 0\}$ of triangulation of Q_T such that $\overline{Q_T} = \cup_{K \in \mathcal{T}_h} K$. The family $\mathcal{T}$ is indexed by $h = \max_{K \in \mathcal{T}_h} diam(K)$. The variable z_k and η_k are approximated with the space $P_h = \{p_h \in C(\overline{Q_T}); p_h|_K \in \mathbb{P}_1(K), \forall K \in \mathcal{T}_h\} \subset L^2(Q_T)$ where $\mathbb{P}_1(K)$ denotes the space of affine functions both in x and t. The variable m_k is approximated with the space $V_h = \{v_h \in C^1(\overline{Q_T}); v_h|_K \in \mathbb{P}(K), \forall K \in \mathcal{T}_h\} \subset \mathcal{M}$ where $\mathbb{P}(K)$ denotes the Hsieh-Clough-Tocher C^1 element (we refer to [7], p. 356). These conformal approximation leads to a strong convergent approximation of the controlled pairs (y_0, f_0), (Y_k^1, F_k^1) (and then (y_k, f_k)) for every $k \in \mathbb{N}$ with respect to the parameter h for the norm associated with P: precisely

$$\|\rho_0(f_{k,h} - f_k)\|_{L^2(q_T)} + \|\rho(y_{k,h} - y_k)\|_{L^2(Q_T)} \to 0 \text{ as } h \to 0^+, \quad \forall k \in \mathbb{N}.$$

Remark that this strong convergent approximation of the linear control f_k together with the convergent result of Theorem 3.10 allows to get, for k large enough and h small enough, an approximation $f_{k,h}$ of a control f for the nonlinear equation (1.1), writing that

$$\|\rho_0(f - f_{k,h})\|_{L^2(q_T)} \leq \|\rho_0(f - f_k)\|_{L^2(q_T)} + \|\rho_0(f_k - f_{k,h})\|_{L^2(q_T)}, \quad \forall k \in \mathbb{N}, \forall h > 0.$$

4.2. Experiments

We present some numerical experiments in the one dimensional setting and $\Omega = (0,1)$. The control is located on $\omega = (0.1, 0.3)$. We consider $T = 1/2$; moreover, in order to reduce the dissipation of the solution of (1.1) when $g \equiv 0$, we replace the term $-\Delta y$ in (1.1) by $-\nu \Delta y$ with $\nu = 10^{-1}$. We consider the nonlinear even function g as follows

$$g(r) = \begin{cases} l(r), & r \in [-a, a], \\ -|r|^\alpha \ln^{3/2}(1 + |r|), & |r| \geq a \end{cases}$$

with $a, \alpha \in (0,1)$. l denotes the (even) polynomial of order two so that $g(0) = 0$ and $g \in C^1([-a,a])$: this requires $l(0) = 0$, $l(a) = -|a|^\alpha \ln^{3/2}(1 + |a|)$ and $(-|r|^\alpha \ln^{3/2}(1 + |r|))'(r = a) = l'(a)$. We use in the sequel the values

TABLE 1. Results for the algorithm (3.11); $\beta = 10$.

♯iteration k	$\frac{\|y_k - y_{k-1}\|_{L^2(Q_T)}}{\|y_{k-1}\|_{L^2(Q_T)}}$	$\frac{\|f_k - f_{k-1}\|_{L^2(q_T)}}{\|f_{k-1}\|_{L^2(q_T)}}$	$\|y_k\|_{L^2(Q_T)}$	$\|f_k\|_{L^2(q_T)}$	$\sqrt{2E(y_k, f_k)}$	λ_k
0	–	–	4.528	4.391	5.58×10^{-1}	0.961
1	1.83×10^{-2}	1.28×10^{-3}	4.651	4.402	1.81×10^{-3}	0.996
2	4.45×10^{-4}	9.07×10^{-5}	4.661	4.403	2.72×10^{-6}	1.
3	1.12×10^{-6}	3.74×10^{-7}	4.662	4.404	4.88×10^{-8}	–

TABLE 2. Results for the algorithm (3.11); $\beta = 10^2$.

♯iteration k	$\frac{\|y_k - y_{k-1}\|_{L^2(Q_T)}}{\|y_{k-1}\|_{L^2(Q_T)}}$	$\frac{\|f_k - f_{k-1}\|_{L^2(q_T)}}{\|f_{k-1}\|_{L^2(q_T)}}$	$\|y_k\|_{L^2(Q_T)}$	$\|f_k\|_{L^2(q_T)}$	$\sqrt{2E(y_k, f_k)}$	λ_k
0	–	–	45.28	43.91	9.31×10^{-1}	0.534
1	8.41×10^{-1}	1.23×10^{-2}	35.8908	38.76	1.12×10^{-1}	0.591
2	1.93×10^{-1}	2.91×10^{-3}	36.7302	38.92	3.40×10^{-2}	0.701
3	3.65×10^{-2}	1.01×10^{-3}	37.0919	39.12	6.12×10^{-3}	0.812
4	1.12×10^{-2}	2.69×10^{-4}	37.2124	40.01	1.12×10^{-3}	0.881
5	3.23×10^{-4}	4.23×10^{-5}	37.2426	40.04	2.13×10^{-4}	0.912
6	1.27×10^{-5}	6.23×10^{-6}	37.2518	40.05	3.05×10^{-5}	0.999
7	5.09×10^{-6}	8.12×10^{-7}	37.2520	40.05	2.10×10^{-6}	0.999
8	7.40×10^{-8}	8.21×10^{-9}	37.2520	40.05	5.10×10^{-9}	–

$a = 10^{-1}$ and $\alpha = 0.95$. We check that g belongs to W_1, in particular $g'' \in L^\infty(\mathbb{R})$ in the sense of distribution. Remark as well that g is sublinear.

As for the initial condition to be controlled, we consider simply $u_0(x) = \beta \sin(\pi x)$ parametrized by $\beta > 0$.

The experiments are performed with the Freefem++ package developed at the Sorbonne university (see [18]), very well-adapted to the space-time formulation we employ. The algorithm is stopped when the value $E(y_k, f_k)$ is smaller than $\epsilon = 10^{-6}$. We take the value $m = 1$ in (3.11), meaning that the optimal steps λ_k are searched in the interval $[0, 1]$.

Tables 1–3 collect some norms for the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ defined by the algorithm (3.11), initialized with the linear controlled solution, for $\beta = 10$, $\beta = 10^2$ and $\beta = 10^3$ respectively. We use a structured mesh composed of 20 000 triangles, 10 201 vertices and for which $h \approx 1.11 \times 10^{-2}$. For $\beta = 10$, we observe the convergence after 4 iterations. The optimal steps λ_k are very close to one since $\max_k |\lambda_k - 1| < 0.05$; consequently, the algorithm (3.11) provides similar results than the Newton algorithm (for which $\lambda_k = 1$ for all k). For $\beta = 10^2$, the convergence remains fast and is reached after 8 iterates. We can observe that some optimal steps differ from one since $\max_k |\lambda_k - 1| > 0.4$. Nevertheless, the Newton algorithm still converges after 17 iterations. More interestingly, the value $\beta = 10^3$ illustrates the features and robustness of the algorithm: the convergence is achieved after 19 iterations. Far away from a zero of E, the variations of the error functional $k \to E(y_k, f_k)$ are first quite slow, then increase to become very fast after 16 iterations, when λ_k is close to one. In contrast, for $\beta = 10^3$, the Newton algorithm still initialized with the linear solution diverges (see Tab. 4). As discussed in [22], in that case, a continuation method with respect to the parameter β may be combined with the Newton algorithm.

On the contrary, we mention that with these data, the sequences obtained from the algorithm (1.8) based on the linearization introduced in [16], remain bounded but do not converge, including for the value $\beta = 10$. The convergence is observed notably with a larger size of the domain ω, for instance $\omega = (0.2, 0.8)$ (see [13], Sect. 4.2).

TABLE 3. Results for the algorithm (3.11); $\beta = 10^3$.

♯iteration k	$\frac{\|y_k - y_{k-1}\|_{L^2(Q_T)}}{\|y_{k-1}\|_{L^2(Q_T)}}$	$\frac{\|f_k - f_{k-1}\|_{L^2(q_T)}}{\|f_{k-1}\|_{L^2(q_T)}}$	$\|y_k\|_{L^2(Q_T)}$	$\|f_k\|_{L^2(q_T)}$	$\sqrt{2E(y_k, f_k)}$	λ_k
0	–	–	452.80	439.18	9.809×10^{-1}	0.4215
1	8.21×10^{-1}	6.00×10^{-1}	320.12	330.15	8.536×10^{-1}	0.3919
2	6.19×10^{-1}	3.29×10^{-2}	324.02	334.12	8.012×10^{-1}	0.1566
3	4.18×10^{-1}	1.37×10^{-2}	325.65	338.21	7.953×10^{-1}	0.1767
4	3.11×10^{-2}	1.34×10^{-2}	326.11	340.12	7.851×10^{-1}	0.0937
5	2.98×10^{-2}	5.85×10^{-3}	326.35	342.24	7.688×10^{-2}	0.0491
6	3.37×10^{-2}	7.00×10^{-3}	326.91	344.65	7.417×10^{-2}	0.1296
7	4.17×10^{-2}	9.69×10^{-3}	327.23	346.12	6.864×10^{-2}	0.1077
8	2.89×10^{-2}	8.09×10^{-3}	327.42	347.19	6.465×10^{-2}	0.0859
9	1.09×10^{-2}	6.40×10^{-3}	327.49	347.29	6.182×10^{-2}	0.0968
10	1.02×10^{-2}	6.72×10^{-3}	327.92	347.38	5.805×10^{-2}	0.1184
11	6.32×10^{-3}	6.91×10^{-3}	328.13	347.41	5.371×10^{-2}	0.1730
12	5.53×10^{-3}	7.41×10^{-3}	328.16	347.43	4.825×10^{-2}	0.2579
13	4.32×10^{-3}	8.22×10^{-3}	328.19	347.45	4.083×10^{-2}	0.3817
14	2.13×10^{-3}	8.14×10^{-3}	328.21	347.48	3.164×10^{-2}	0.4946
15	3.57×10^{-3}	7.34×10^{-3}	328.22	347.50	2.207×10^{-2}	0.8294
16	1.01×10^{-3}	6.68×10^{-3}	328.25	347.51	1.174×10^{-2}	0.9845
17	5.68×10^{-4}	3.84×10^{-4}	328.26	347.51	2.191×10^{-3}	0.9999
18	2.14×10^{-4}	5.85×10^{-5}	328.26	347.52	4.674×10^{-5}	1.
19	3.21×10^{-6}	1.57×10^{-7}	328.27	347.52	5.843×10^{-7}	–

TABLE 4. Results for the algorithm (3.11) with $\lambda_k = 1$ for all k; $\beta = 10^3$.

♯iterate k	$\frac{\|y_k - y_{k-1}\|_{L^2(Q_T)}}{\|y_{k-1}\|_{L^2(Q_T)}}$	$\frac{\|f_k - f_{k-1}\|_{L^2(q_T)}}{\|f_{k-1}\|_{L^2(q_T)}}$	$\|y_k\|_{L^2(Q_T)}$	$\|f_k\|_{L^2(q_T)}$	$\sqrt{2E(y_k, f_k)}$
0	–	–	452.80	439.18	9.809×10^{-1}
1	9.76×10^{-1}	1.05	330.21	334.15	9.812×10^{-1}
2	1.02	1.11	344.37	336.12	1.356
3	1.27	1.13	366.92	338.23	4.319
4	1.18	1.25	406.06	343.12	4.799
5	1.01	1.14	481.53	405.03	13.131

5. CONCLUSIONS

We have constructed an explicit sequence of functions $(f_k)_{k\in\mathbb{N}}$ converging strongly for the $L^2(q_T)$ norm towards a null control for the semilinear heat equation $\partial_t y - \Delta y + g(y) = f\,1_\omega$. The construction of the sequence is based on the minimization of a $L^2(0,T;H^{-1}(\Omega))$ least-squares functional. The use of a specific descent direction allows to achieve a global convergence (uniform with respect to the data and to the initial guess) with a super-linear rate related to the regularity of the nonlinear function g. Numerical experiments confirm the robustness of the approach. In this analysis, we have assumed in particular that the derivative of g is uniformly bounded in $\mathbb{R}$. This allows to get a uniform bound of the observability constant $C(s,\Omega,\omega,T,\|g'(y)\|_\infty)$ appearing in the estimate (2.5). In order to remove this assumption and be able to consider super-linear function g (as in the seminal work [16] by Fernández-Cara and Zuazua, assuming that g is locally Lipschitz-continuous and the asymptotic behavior (1.6)), the analysis needs to be refined in order to keep as explicit as possible the observability constant $C(s,\Omega,\omega,T,\|g'(y)\|_{L^\infty})$ in term of the norm $\|g'(y)\|_{L^\infty}$ and the parameter s (which appears in the Carleman weights). We refer to [11] for precise estimates of the observability constant. In the one dimensional setting, this

analysis has been recently made in [21] assuming that the asymptotic behavior $\limsup_{|r|\to\infty} \frac{|g'(r)|}{\ln^{3/2}(1+|r|)} \leq \beta$ for some $\beta > 0$ small enough.

We also emphasize that this least-squares approach is very general and may be used to address other PDEs. We also mention the explicit construction of controls for nonlinear wave equation (initially studied in [31] making use of a fixed-point argument) recently performed in [2, 28]. Eventually, as a challenge, it would be interesting to apply the method to approximate controls for the Navier-Stokes system. In this direction, we mention [25] devoted to the Stokes system and [22, 23] devoted to the direct problem for the Navier-Stokes system in the incompressible regime.

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