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Classification of first strain-gradient elasticity tensors by symmetry planes

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First strain-gradient elasticity is a generalized continuum theory capable of modelling size effects in materials. This extended capability comes from the inclusion in the mechanical energy density of terms related to the strain-gradient. In its linear formulation, the constitutive law is defined by three elasticity tensors whose orders range from four to six. In the present contribution, the symmetry properties of the sixth-order elasticity tensors involved in this model are investigated. If their classification with respect to the orthogonal symmetry group is known, their classification with respect to symmetry planes is still missing. This last classification is important since it is deeply connected with some identification procedures. The classification of sixth-order elasticity tensors in terms of invariance properties with respect to symmetry planes is given in the present contribution. Precisely, it is demonstrated that there exist 11 reflection symmetry classes. This classification is distinct from the one obtained with respect to the orthogonal group, according to which there exist 17 different symmetry classes. These results for the sixth-order elasticity tensor are very different from those obtained for the classical fourth-order elasticity tensor, since in the latter case the two classifications coincide. A few numerical examples are provided to illustrate how some different orthogonal classes merge into one reflection class.

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1. Introduction

In the classical continuum theory of mechanics of solids (see e.g. [1,2]) only the first displacement gradient is taken into account in the strain-energy. In other words, all the higher-displacement gradients are discarded by assuming their effects to be very small with respect to those of the first displacement gradient. As a consequence, when the material constituting a solid is taken to be linearly elastic, the constitutive law involves only the classical fourth-order elasticity tensor $\mathbf{C}$ [3].

The classical continuum theory is applicable in most situations of practical interest at the usual engineering scale. However, in some situations of mechanical and/or physical interest, due to the effects of microstructure or internal constraints, size effects and/or non-local behaviour become important. These effects are even more pronounced in dynamic situations in which elastic waves propagate in very heterogeneous environments. When the ratio of the wave length to the microstructural characteristic one is small, new phenomena such as dispersivity, directivity, higher-order anisotropy appear [4–8]. These non-standard effects induced by strongly heterogeneous microstructures (or mesostructures) offer fascinating engineering perspectives like cloaking, superlensing, high resolution imaging... These promising applications are studied in detail in the literature devoted to metamaterials [9].

Being invariant by scale change, the equations of the classical theory of elasticity are not adapted to describing the effective behaviour, whether static or dynamic, of a complex architectured continuum. One possibility to solve this problem while remaining in the field of continuum mechanics is to use a generalised continuum theory [10,11]. Within the framework of these theories, approaches that consider higher order gradients of the displacement field are known to be efficient. There is extensive literature on the development and application of these approaches. The pioneer works are the ones reported in [10,12–17].

For the last two decades, higher-order continuum theories have largely been applied to a variety of situations such as polycrystalline materials, biomaterials and nanostructured materials in which size and non-local effects have to be captured (see e.g. [18–22]). Among all the higher-order continuum theories, the most widely used one is the first strain-gradient elasticity (see e.g. [16]). The latter is nowadays a theory that has gained importance for describing the overall elasticity of materials with a coarse mesostructure such as, for instance, architectured materials [7,23].

In the first strain-gradient elasticity, the strain energy density function depends linearly both on the classical second-order strain tensor $\boldsymbol{\varepsilon}$ and the third-order strain-gradient tensor $\boldsymbol{\eta}$. The second-order Cauchy stress tensor $\boldsymbol{\sigma}$, dual to $\boldsymbol{\varepsilon}$, and the third-order hyperstress tensor $\boldsymbol{\tau}$, dual to $\boldsymbol{\eta}$, are linearly related to $\boldsymbol{\varepsilon}$ and $\boldsymbol{\eta}$ by the classical fourth-order elasticity tensor $\mathbf{C}$ and the two additional constitutive tensors $\mathbf{M}$ and $\mathbf{A}$ which are, respectively, fifth- and sixth-order tensors. When the microstructure of a continuum is centro-symmetric, the fifth-order tensor $\mathbf{M}$ is null and the constitutive tensors reduce to the pair ($\mathbf{C}$,$\mathbf{A}$). This centrosymmetry hypothesis will be made throughout the rest of this work.

Using homogenisation techniques, it is now possible to determine the associated effective sixth-order elasticity tensor of a given periodic or random mesostructure [24–27]. This approach has recently been downscaled to molecular scale and atomistic representations for an arbitrary multi-lattice of the sixth-order tensor were derived [28]. If these approaches are mostly numerical, it is important to note that recent efforts have also been made to develop experimental ones to evidence and measure higher-order effects emerging from an actual mesostructure [29–31]. Continued advances in digital image correlation (DIC) techniques and growing interest in architectured materials for engineering applications suggest that robust experimental measurements of higher-order properties would be available soon.

Now, let us assume that the sixth-order elasticity tensor for a mesostructure is obtained by using a homogenization method. Curie-Neumann's principle [32,33] implies that the invariance properties of the obtained sixth-order elasticity tensor includes the symmetries exhibited by the mesostructure. This is a necessary (but not sufficient) condition for the validation of the homogenisation method used. To check if this condition is satisfied, it is indispensable to know the set of symmetry classes of the sixth-order elasticity tensor.

Next, let us discuss the following two points.

a) Classification of tensorial symmetries:

The symmetries of a material and their consequences on its constitutive law were initiated with the work of Pierre Curie[1] [32]. Concerning linear elasticity, the problem of classifying and characterizing the fourth-order elasticity tensors **C**, which was investigated by [34] over 70 years ago, has recently been studied in the fields of mechanics, physics, applied mathematics and engineering (see, for example, [33,35–43]). A definite answer was obtained in 1996 by Forte and Vianello [37] who proved, for the first time, that the space of fourth-order elasticity tensors is partitioned into 8 symmetry classes. Their approach falls within the orthogonal group theory.

Another approach of classifying elasticity tensors is based on their invariances with respect to plane reflections [44]. As demonstrated by Chadwick *et al.* [40], for the classical fourth-order elasticity tensors, the two classification approaches give the same result. This coincidence is by no means trivial. Indeed, since the orthogonal classification is "finer" than that involving planes of symmetry, the number of symmetry classes obtained by the latter approach should be less than that obtained from the orthogonal group. The particular result that occurs for the classical fourth-order elasticity tensor has been the source of conceptual confusion in the literature.

While these issues are resolved in the case of the classical **C** elasticity tensor, they remained, until recently, open for higher order elasticity tensors. More specifically, the classifications of the fifth- and sixth-order elasticity tensors of strain-gradient elasticity were lacking. This situation has recently changed owing to a great number of works dedicated to comprehensibly understanding higher-order elasticity tensors [45–51].

In the 3D context, it has been proved that the number of symmetry classes for fifth-order elasticity tensors **M** is 29 while the one for sixth-order elasticity tensors **A** is 17 [46,52–54]. Compared to the 8 symmetry classes of classical elasticity, these results highlight the complexity and richness of strain-gradient elasticity. It is important to notice that all the symmetry classification results on the fifth- and sixth-order elasticity tensors were obtained by using an orthogonal group approach.

b) Identification of tensorial symmetries:

The question now arises as to how to identify the symmetry properties of a sixth-order tensor from the knowledge of the matrix of its components with respect to a given basis. There are at least four strategies to do this:

(i) **Direct inspection of the matrix form:** As the matrix representations of the sixth-order anisotropic tensors are known, we can compare the reconstructed operator to the *catalogue* of these elementary matrices. As the reference matrices are provided with respect to a basis associated with symmetry elements, the method only works if *a priori* information on the symmetry elements is known, and fails in the general case.

(ii) **Polynomial relations between invariants of A:** To characterize symmetry classes intrinsically, polynomial relations between *invariants* of **A** can be used. If this is true from a theoretical standpoint, some previous works conducted on the 3D fourth-order elasticity [55,56] suggest that its extension to the 3D sixth-order elasticity tensors would be extremely complicated. Further, being encoded by high-order polynomials, the characteristic relations are very sensitive to noise in the data.

(iii) **Spectral properties of the associated matrix:** Following works initially conducted in [42, 44] in the case of classical elasticity, invariance properties of **A** should be read off from the determination of the eigenvalues and eigentensors of its matrix representation. Since involving the computation of eigenvalues of tensors, this approach is also highly sensitive to the inherent noise of the data.

[1]This project is fully contained in the first sentence the cited paper: "I think it would be useful to introduce into the study of physical phenomena the symmetry considerations familiar to crystallographers."

(iv) **Pole figures of the matrix:** Aside these purely algebraical methods, a more *graphical* approach has been proposed by Francois *et al.* in [57]. The method consists in considering a function over a unit sphere, constructed from the tensor matrix representation, the zeros of which indicate the normal vectors to physical symmetry planes. This method has the main advantages of being easy to implement and providing visual information even when the components of the tensors are corrupted by noise.

In the present contribution, we explore the possibility of extending to the sixth-order tensor the pole figure approach initially used by Francois *et al.* in [57] in the context of classical elasticity. It is important to note that, in the case of the fourth-order elasticity tensor **C**, the method relies on the fact that every symmetry class can be characterized by its set of symmetry planes [40]. As already pointed out, if this is true for the fourth-order elasticity tensor, there is no reason for this to hold for higher-order tensors. As such, and with respect to its importance for practical identification applications, the main objective of this work is to classify the symmetry properties of the sixth-order elasticity tensors **A** with respect to mirror symmetries, and to establish a bridge between this classification and the one based on the full orthogonal group.

To achieve this goal, the first part of this paper is devoted to determining, using the plane of symmetry approach, the number and types of all possible classes of symmetry for the space of the sixth-order elasticity tensor $\mathbb{E}\mathrm{la}(6)$. Precisely, we prove that the two classifications lead to quite different results: $\mathbb{E}\mathrm{la}(6)$ is partitioned into 17 orthogonal symmetry classes but only 11 reflection symmetry classes. In a second time, these differences are verified and illustrated by numerical experiments. These experiments are conducted by using the matrix representations obtained in a former contribution [46]. Random tensors belonging to different orthogonal symmetry classes are generated, and the set of unit directions normal to reflection planes are plotted on the unit sphere S^2. This evidences, in a visual manner, that different orthogonal symmetry classes can merge into one reflection symmetry class. This result implies that the numerical and experimental procedures for identifying the reflection symmetry classes of the sixth-order elasticity tensors by polar figures are not applicable to identify all 17 orthogonal symmetry classes of $\mathbb{E}\mathrm{la}(6)$. At the same time, situations in which different orthogonal classes are reduced to the same reflection class are identified.

The paper is organized as follows. Section 2 is devoted to the description of the constitutive laws of the strain-gradient elasticity model. In Section 3, the main geometrical concepts for symmetry classification are introduced. Classification methods in relation to orthogonal transformations and in relation to planes of symmetry are introduced and positioned in relation to each other. The main findings of this study are detailed and discussed in Section 4, while their proofs are postponed, for the interested reader, to section 5. Finally, in Section 6, pole figures are numerically produced to graphically evidence the theoretical results that have been demonstrated in the previous section. The paper is closed, in Section 7, with some concluding remarks.

Notations

Throughout this paper, we consider the three-dimensional Euclidean space $\mathcal{E}^3$. This space is endowed with a system of Cartesian coordinates $\{x_1, x_2, x_3\}$ associated to a right-handed orthonormal basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. For notational convenience, blackboard fonts such as $\mathbb{T}$ denote tensor spaces; uppercase Roman bold fonts or lowercase Greek bold fonts, like $\mathbf{T}$ or $\boldsymbol{\tau}$, are reserved for tensors of order > 1; lowercase Roman bold fonts such as $\mathbf{t}$ are employed to designate vectors.

The orthogonal group in $\mathbb{R}^3$ is defined as

$$\mathrm{O}(3) = \{\mathbf{Q} \in \mathrm{GL}(3) \mid \mathbf{Q}\mathbf{Q}^T = \mathbf{I}\} \tag{1.1}$$

where $\mathrm{GL}(3)$ is the set of invertible transformations acting on $\mathbb{R}^3$, $\mathbf{Q}^T$ denotes the transpose of $\mathbf{Q}$ and $\mathbf{I}$ is the second-order identity tensor. As a subgroup of $\mathrm{O}(3)$, the rotation group is defined by

$$\mathrm{SO}(3) = \{\mathbf{Q} \in \mathrm{O}(3) \mid \det \mathbf{Q} = 1\}. \tag{1.2}$$

In particular, use will be made of the following elements of $\mathrm{O}(3)$:

- $\mathbf{Q} \in \mathrm{O}(3)$ a generic orthogonal transformation;
- $\mathbf{R}(\mathbf{v};\theta) \in \mathrm{SO}(3)$ the rotation about $\mathbf{v} \in \mathbb{R}^3$ through an angle $\theta \in [0; 2\pi[$;
- $-\mathbf{I} \in \mathrm{O}(3)\backslash\mathrm{SO}(3)$ the inversion with respect to the origin;
- $\overline{\mathbf{R}}(\mathbf{v};\theta) \in \mathrm{O}(3)\backslash\mathrm{SO}(3)$ the roto-inversion about $\mathbf{v} \in \mathbb{R}^3$ through an angle $\theta \in [0; 2\pi[$;
- $\mathbf{P}_\mathbf{n} \in \mathrm{O}(3)\backslash\mathrm{SO}(3)$ the reflection through the plane normal to $\mathbf{n}$, namely $\mathbf{P}_\mathbf{n} = \mathbf{I} - 2\mathbf{n} \otimes \mathbf{n}$.

The notation $\mathfrak{I}^\alpha(\mathbb{V})$ means the collection of symmetry classes of the space $\mathbb{V}$, and the exponent α indicates the type of symmetry considered: o for orthogonal, r for rotation and p for reflection. Finally, we denote by S^n the unit n-sphere defined by $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\| = 1\}$.

2. Strain-gradient elasticity: constitutive laws

In this section the basic equations of strain-gradient elasticity are recapitulated. For a deeper introduction to the mathematics and physics of this model, the interested readers can refer to [10,21,58]. By letting $\boldsymbol{\varepsilon}$ be the infinitesimal strain tensor, its components ε_{ij} relative to the orthonormal basis $\mathcal{B}$ are calculated from the displacement vector $\mathbf{u}$ by

$$\varepsilon_{ij} = \varepsilon_{ji} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.1}$$

Above and hereafter, a subscript, say i, following a comma denotes the derivative with respect to x_i. According to the strain-gradient elasticity theory of Mindlin [10,14], the elastic strain energy density, denoted by w, is a quadratic form of both $\boldsymbol{\varepsilon}$ and its first gradient noted as $\boldsymbol{\eta}$. The matrix components of $\boldsymbol{\eta}$ relative to $\mathcal{B}$ are calculated by

$$\eta_{ijk} = \eta_{jik} = \varepsilon_{ij,k}. \tag{2.2}$$

In the most general case, w takes the form

$$\mathrm{w} = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + M_{ijklm}\varepsilon_{ij}\eta_{klm} + \frac{1}{2}A_{ijklmn}\eta_{ijk}\eta_{lmn}, \tag{2.3}$$

where C_{ijkl}, M_{ijklm} and A_{ijklmn} are the components of the fourth-, fifth- and sixth-order elastic stiffness tensors, $\mathbf{C}$, $\mathbf{M}$ and $\mathbf{A}$, respectively.

The Cauchy stress tensor $\boldsymbol{\sigma}$ and the third-order hyperstress tensor $\boldsymbol{\tau}$ are linearly related to the strain tensor $\boldsymbol{\varepsilon}$ and the strain-gradient tensor $\boldsymbol{\eta}$ by following constitutive laws:

$$\begin{cases} \sigma_{ij} = \sigma_{ji} = \frac{\partial \mathrm{w}}{\partial \varepsilon_{ij}} = C_{ijkl}\varepsilon_{kl} + M_{ijmns}\eta_{mns}, \\ \tau_{pqr} = \tau_{qpr} = \frac{\partial \mathrm{w}}{\partial \eta_{pqr}} = M_{klpqr}\varepsilon_{kl} + A_{pqrmns}\eta_{mns}. \end{cases} \tag{2.4}$$

For more details about the physical and graphical interpretations of the third-order hyperstress tensor $\boldsymbol{\tau}$, the reader can refer to the work [59]. In the particular case of a centro-symmetric microstructure, the fifth-order tensor $\mathbf{M}$ vanishes, so that the stress-strain relation reduces to

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \quad \tau_{pqr} = A_{pqrmns}\eta_{mns}. \tag{2.5}$$

Thus, $\boldsymbol{\sigma}$ depends only on $\boldsymbol{\varepsilon}$, and $\boldsymbol{\tau}$ on $\boldsymbol{\eta}$ alone. Due to the index symmetry $\varepsilon_{ij} = \varepsilon_{ji}$ of $\boldsymbol{\varepsilon}$, the index symmetry $\eta_{ijk} = \eta_{jik}$ of $\boldsymbol{\eta}$, and the existence of w as specified by (2.3), the components of $\mathbf{C}$, $\mathbf{M}$ and $\mathbf{A}$ verify the following index permutation properties:

$$C_{ijkl} = C_{jikl} = C_{klij}, \quad M_{ijklm} = M_{jiklm} = M_{ijlkm}, \quad A_{ijklmn} = A_{jiklmn} = A_{lmnijk}. \tag{2.6}$$

This work focuses on the sixth-order elasticity tensor $\mathbf{A}$. Thus, we introduce the space $\mathbb{E}\mathrm{la}(6)$ composed of all sixth-order elasticity tensors $\mathbf{A}$ verifying the relevant index symmetry properties in (2.6):

$$\mathbb{E}\text{la}(6) = \{\mathbf{A} \in \otimes^6\mathbb{R}^3 \mid A_{ijklmn} = A_{jiklmn} = A_{lmnijk}\}. \tag{2.7}$$

The dimension of $\mathbb{E}\text{la}(6)$ is 171, which corresponds to the number of independent components of $\mathbf{A}$ in the triclinic symmetry class.

3. Symmetry properties of a sixth-order elasticity tensor $\mathbf{A}$

(a) The orthogonal group $\mathrm{O}(3)$

The 3D orthogonal group $\mathrm{O}(3)$, defined by (1.1), consists of all the isometries of $\mathbb{R}^3$. Amongst them, the rotations, which are the spatial-orientation preserving isometries, constitute the subgroup $\mathrm{SO}(3)$ as defined by (1.2). Each element of $\mathrm{SO}(3)$ is characterized by an invariant direction $\mathbf{n} \in S^2$ and an angle $\theta \in [0; 2\pi[$. Thus, any element of $\mathrm{SO}(3)$ can be written as $\mathbf{R}(\mathbf{n}; \theta)$. A general expression for $\mathbf{R}(\mathbf{n}; \theta)$ is provided by the well-known Rodrigues formula:

$$\mathbf{R}(\mathbf{n}; \theta) = \cos(\theta)\mathbf{I} - \sin(\theta)\epsilon \cdot \mathbf{n} + [1 - \cos(\theta)]\mathbf{n} \otimes \mathbf{n} \tag{3.1}$$

in which ϵ denotes the Levi-Civita third-order tensor in $\mathbb{R}^3$.

In addition to rotations, $\mathrm{O}(3)$ contains different transformations of determinant equal to -1. In the following list, the first two operations can be considered as elementary improper transformations while the following two are compound transformations involving rotations.

- $-\mathbf{I} \in \mathrm{O}(3)\backslash\mathrm{SO}(3)$ is the inversion with respect to the origin. For subsequent uses, let us designate by Z_2^c the group generated by the inversion, i.e. $\mathrm{Z}_2^c = \{\mathbf{I}, -\mathbf{I}\}$;
- $\mathbf{P}(\mathbf{n}) \in \mathrm{O}(3)\backslash\mathrm{SO}(3)$ is the reflection through the plane $\mathcal{P}(\mathbf{n})$ perpendicular to $\mathbf{n}$[2], which can be formulated by

$$\mathbf{P}(\mathbf{n}) = \mathbf{I} - 2\mathbf{n} \otimes \mathbf{n}. \tag{3.2}$$

  Observe that $\mathbf{P}$ is an even function of $\mathbf{n}$, i.e. $\mathbf{P}(-\mathbf{n}) = \mathbf{P}(\mathbf{n})$. Hence, to each reflection two unit directions can be associated. As a consequence, the cardinal of the set of unit directions associated to a symmetry plane is twice the cardinal of the set of symmetry planes;
- $\overline{\mathbf{R}}(\mathbf{n}; \theta) \in \mathrm{O}(3)\backslash\mathrm{SO}(3)$ is the rotoinversion[3] of unit direction $\mathbf{n}$ and angle θ, which is given by

$$\overline{\mathbf{R}}(\mathbf{n}; \theta) = -\mathbf{I} \cdot \mathbf{R}(\mathbf{n}; \theta) = -\cos(\theta)\mathbf{I} + \sin(\theta)\epsilon \cdot \mathbf{n} - (1 - \cos(\theta))\mathbf{n} \otimes \mathbf{n}. \tag{3.3}$$

  It is immediate from (3.2) that $\mathbf{P}(\mathbf{n})$ can be viewed as a π-angle rotoinversion along $\mathbf{n}$, i.e.,

$$\mathbf{P}(\mathbf{n}) = \overline{\mathbf{R}}(\mathbf{n}; \pi);$$

- $\hat{\mathbf{R}}(\mathbf{n}; \theta) \in \mathrm{O}(3)\backslash\mathrm{SO}(3)$ is the rotoreflection[4] of unit direction $\mathbf{n}$ and angle θ, which is given by

$$\hat{\mathbf{R}}(\mathbf{n}; \theta) = \mathbf{P}(\mathbf{n}) \cdot \mathbf{R}(\mathbf{n}; \theta) = \cos(\theta)\mathbf{I} - \sin(\theta)\epsilon \cdot \mathbf{n} - (1 + \cos(\theta))\mathbf{n} \otimes \mathbf{n}. \tag{3.4}$$

  Rotoreflections and rotoinvertions are related by the following relation:

$$\overline{\mathbf{R}}(\mathbf{n}; \theta + \pi) = \hat{\mathbf{R}}(\mathbf{n}; \theta).$$

The previous list demonstrated the following fact

[2] More precisely, it consists in the linear subspace of $\mathbb{R}^3$ characterized by

$$\mathcal{P}(\mathbf{n}) = \{\mathbf{h} \in \mathbb{R}^3 \mid \mathbf{h} \cdot \mathbf{n} = 0\}$$

.

[3] A rotoinvertion is compound transformation consisting of a rotation followed by the inversion.

[4] A rotoreflection is a compound transformation consisting of a rotation followed by a reflection along the same characteristic direction.

Proposition 3.1. *Any element* $\mathbf{Q} \in \mathrm{O}(3)$ *can be expressed as*

$$\mathbf{Q} = \mathbf{J}\mathbf{R}(\mathbf{n};\theta) = \mathbf{R}(\mathbf{n};\theta)\mathbf{J}$$

with $\mathbf{J} \in \mathrm{Z}_2^c$ *and* $\mathbf{R}(\mathbf{n};\theta) \in \mathrm{SO}(3)$.

(b) Action on $\mathbb{E}\mathrm{la}(6)$

Now let us take a look at how a tensor of order 6 transforms with respect to an orthogonal operation.

General action

A linear representation ϕ of a group G on a vector space $\mathbb{V}$ is defined as a group morphism from G to $\mathrm{GL}(\mathbb{V})$. In the present work, $\mathrm{O}(3)$ acts on $\mathbb{E}\mathrm{la}(6)$ in the following classical way:

$$\phi(\mathbf{Q})_{ijkpqwlmnsrt} = Q_{il}Q_{jm}Q_{kn}Q_{ps}Q_{qr}Q_{wt} \in \mathrm{GL}(\mathbb{E}\mathrm{la}(6)).$$

Hence, for any $\mathbf{Q} \in \mathrm{O}(3)$, $\phi(\mathbf{Q})$ defines a linear map from $\mathbb{E}\mathrm{la}(6)$ to itself. The image $\mathbf{A}^\star$ of $\mathbf{A}$ by $\phi(\mathbf{Q})$ is given in components by

$$(\mathbf{A}^\star)_{ijkpqw} = Q_{il}Q_{jm}Q_{kn}Q_{ps}Q_{qr}Q_{wt}A_{lmnsrt}. \tag{3.5}$$

This action can be compactly formulated by using the so-called Rayleigh product $*$ as follows:

$$\mathbf{A}^\star = \mathbf{Q} * \mathbf{A}. \tag{3.6}$$

From (3.5) or (3.6), we define the set of orthogonal transformations letting $\mathbf{A}$ invariant:

$$\mathrm{G}_{\mathbf{A}} := \{\mathbf{Q} \in \mathrm{O}(3) \mid \mathbf{Q} * \mathbf{A} = \mathbf{A}\}. \tag{3.7}$$

This subgroup of $\mathrm{O}(3)$ is said to be the orthogonal symmetry group of $\mathbf{A}$. Since $\mathbf{A}$ is an even-order tensor, $\mathrm{G}_{\mathbf{A}}$ always includes the inversion $-\mathbf{I}$:

$$\forall \mathbf{A} \in \mathbb{E}\mathrm{la}(6),\ -\mathbf{I} \in \mathrm{G}_{\mathbf{A}}. \tag{3.8}$$

Restricted action

In the definition of the symmetry group $\mathrm{G}_{\mathbf{A}}$ by (3.7), all the orthogonal transformations are considered. This definition can be restricted to specific subsets of $\mathrm{O}(3)$, thereafter the cases of i) rotations and ii) reflections are considered.

Rotation symmetry group. In this case, the $\mathrm{O}(3)$-action on $\mathbb{E}\mathrm{la}(6)$ is restricted to the $\mathrm{SO}(3)$-action. Thus, the rotation symmetry group of a tensor $\mathbf{A} \in \mathbb{E}\mathrm{la}(6)$, denoted $\mathrm{G}_{\mathbf{A}}^R$, is defined as

$$\mathrm{G}_{\mathbf{A}}^R := \{\mathbf{R} \in \mathrm{SO}(3) \mid \mathbf{R} * \mathbf{A} = \mathbf{A}\}.$$

We have the obvious inclusion

$$\mathrm{G}_{\mathbf{A}}^R \subset \mathrm{G}_{\mathbf{A}}.$$

More precisely, since in this special case $\mathrm{G}_{\mathbf{A}}$ includes the inversion (cf. Equation (3.8)), we have,

$$\mathrm{G}_{\mathbf{A}} = \mathrm{G}_{\mathbf{A}}^R \cup (-\mathbf{I})\mathrm{G}_{\mathbf{A}}^R.$$

Proof. Since ϕ is linear representation, $\phi(-\mathbf{IR}) = \phi(-\mathbf{I})\phi(\mathbf{R})$, as such by exploiting the fact that both $-\mathbf{I}$ and $\mathbf{R}$ belong to $\mathrm{G}_{\mathbf{A}}$

$$\phi(-\mathbf{IR}) \star \mathbf{A} = \phi(-\mathbf{I}) \star (\phi(\mathbf{R}) \star \mathbf{A}) = \phi(-\mathbf{I}) \star \mathbf{A} = \mathbf{A}$$

This result shows that $\forall \mathbf{R} \in \mathrm{G}_{\mathbf{A}}^R$, $-\mathbf{IR} \in \mathrm{G}_{\mathbf{A}}$. ■

Hence, to each symmetry element in $\mathrm{G}^R_{\mathbf{A}}$ corresponds two elements in $\mathrm{G}_{\mathbf{A}}$, a rotation and a rotoinversion. Therefore, $\mathrm{G}^R_{\mathbf{A}}$ is the quotient group of $\mathrm{G}_{\mathbf{A}}$ by Z^c_2. For even-order tensors, studying the properties of invariance with respect to $\mathrm{O}(3)$ or $\mathrm{SO}(3)$ leads to essentially the same classification. In other words, there is a one-to-one correspondence between the two classifications for even-order tensors in $\mathbb{R}^3$. For this reason, it is sufficient to make the symmetry classification of even-order tensors in $\mathbb{R}^3$ with respect to $\mathrm{SO}(3)$, the classification with respect to $\mathrm{O}(3)$ being automatically deduced.

Reflection symmetry set. The set of reflections letting a tensor $\mathbf{A} \in \mathbb{E}\mathrm{la}(6)$ invariant is defined as

$$\mathrm{P}_{\mathbf{A}} := \{\mathbf{P}(\mathbf{n}) \in \mathrm{O}(3) \mid \mathbf{P}(\mathbf{n}) * \mathbf{A} = \mathbf{A},\ \ \mathbf{n} \in \mathrm{S}^2\}. \tag{3.9}$$

In addition, let $\mathcal{N}(\mathbf{A})$ be the set defined by

$$\mathcal{N}(\mathbf{A}) = \{\mathbf{n} \in S^2 | \mathbf{P}(\mathbf{n}) * \mathbf{A} = \mathbf{A}\}. \tag{3.10}$$

Since $\mathbf{P}$ is an even function of $\mathbf{n}$, the number of elements in $\mathcal{N}(\mathbf{A})$ is twice the one in $\mathrm{P}_{\mathbf{A}}$. Further, the plane $\mathcal{P}(\mathbf{n})$ associated to an element $\mathbf{n}$ of $\mathcal{N}(\mathbf{A})$ is referred to as a symmetry plane of $\mathbf{A}$, and $\mathbf{n}$ acts as a unit normal to this physical plane[5]. Clearly, we have

$$\mathrm{P}_{\mathbf{A}} \subset (-\mathbf{I})\mathrm{G}^R_{\mathbf{A}} \subset \mathrm{G}_{\mathbf{A}}.$$

It has to be noted that $\mathrm{P}_{\mathbf{A}}$ is not a group since, for instance, $\mathrm{P}_{\mathbf{A}}$ does not include the identity transformation $\mathbf{I}$ which is a proper transformation. To associate a group to this collection of symmetry operations, we can use the following construction. Consider the rotations that exchange the normals of the symmetry planes of $\mathbf{A}$, the associate set of operations constitutes a group:

$$\mathrm{G}^R_{\mathcal{N}(\mathbf{A})} := \{\mathbf{R} \in \mathrm{SO}(3) \mid \forall \mathbf{n} \in \mathcal{N}(\mathbf{A}),\ \overline{\mathbf{n}} = \mathbf{R}\mathbf{n} \in \mathcal{N}(\mathbf{A})\}. \tag{3.11}$$

Proof. First, it is clear from (3.11) that $\mathbf{I}$, called identity or neutral element, belongs to $\mathrm{G}^R_{\mathcal{N}(\mathbf{A})}$. Second, consider $\mathbf{R}_1$ and $\mathbf{R}_2 \in \mathrm{G}^R_{\mathcal{N}(\mathbf{A})}$, we have

$$\forall \mathbf{n} \in \mathcal{N}(\mathbf{A}),\ \overline{\mathbf{n}} = \mathbf{R}_1\mathbf{n} \in \mathcal{N}(\mathbf{A}); \quad \forall \mathbf{n} \in \mathcal{N}(\mathbf{A}),\ \overline{\mathbf{n}} = \mathbf{R}_2\mathbf{n} \in \mathcal{N}(\mathbf{A}). \tag{3.12}$$

Since, for instance, the last relation is true for any $\mathbf{n} \in \mathcal{N}(\mathbf{A})$ it is true for $\overline{\mathbf{n}}_1 = \mathbf{R}_1\mathbf{n}$ which implies that $\overline{\mathbf{n}} = \mathbf{R}_2\mathbf{R}_1\mathbf{n} \in \mathcal{N}(\mathbf{A})$ showing that $\mathbf{R}_2\mathbf{R}_1 \in \mathrm{G}^R_{\mathcal{N}(\mathbf{A})}$. Third, we consider now a rotation $\mathbf{R}(\mathbf{m};\theta) \in \mathrm{G}^R_{\mathcal{N}(\mathbf{A})}$ whose the expression is given by (3.1). Since the definition (3.11), it can be shown that if $\mathbf{n} \in \mathcal{N}(\mathbf{A})$ then $\mathbf{n}^* = \underbrace{\mathbf{R}(\mathbf{m};\theta)\cdot \ldots \cdot \mathbf{R}(\mathbf{m};\theta)}_{p \text{ times}}\,\mathbf{n} = \mathbf{R}(\mathbf{m};p\theta)\mathbf{n} \in \mathcal{N}(\mathbf{A})$ for any value of p. This holds if and only if $\theta = \frac{2\pi}{k}$ with k being an integer. Consequently, by setting $p = k-1$, we have $\underbrace{\mathbf{R}(\mathbf{m};\theta)\cdot \ldots \cdot \mathbf{R}(\mathbf{m};\theta)}_{k-1 \text{ times}} = \mathbf{R}(\mathbf{m};(k-1)\theta) = \mathbf{R}(\mathbf{m};2\pi-\theta) = \mathbf{R}(\mathbf{m};-\theta) = \mathbf{R}^{-1}(\mathbf{m};\theta)$. This means that $\mathbf{R}^{-1}(\mathbf{m};\theta)$ belongs also to $\mathrm{G}^R_{\mathcal{N}(\mathbf{A})}$. As a result, $\mathrm{G}^R_{\mathcal{N}(\mathbf{A})}$ constitutes therefore a group. ∎

(c) Symmetry classes

Elements of symmetry, as defined above, are related to a specific orientation. To speak about the symmetry of an object intrinsically, the reference to a specific orientation has to be removed. This leads naturally to the notion of *symmetry class*.

[5]Here a *physical symmetry plane* means a symmetry plane for the tensor, this has to be distinguished from a *material symmetry plane* which means a symmetry plane for the microstructure. If to each material plane one can associate a physical plane, the converse is not true. A classical example is the effective elasticity of a honeycomb material. The microstructure possesses 7 symmetry planes, while the elasticity tensors is invariant by $\infty + 1$ symmetry planes.

Let us designate by $\mathrm{S}_{\mathbf{A}}$ a set of symmetry elements of $\mathbf{A} \in \mathbb{E}\mathrm{la}(6)$. According to the considered type of classification, the content of this set can be specified as follows:

$$\mathrm{S}_{\mathbf{A}} = \begin{cases} \mathrm{G}_{\mathbf{A}}, & \text{if all elements of } \mathrm{O}(3) \text{ are considered;} \\ \mathrm{G}^{R}_{\mathbf{A}}, & \text{if only rotations are considered } (\mathrm{SO}(3)\text{-restriction}); \\ \mathrm{P}_{\mathbf{A}}, & \text{if only symmetry planes are considered.} \end{cases}$$

The symmetry *type* of $\mathrm{S}_{\mathbf{A}}$ can be defined, in a first time, without resorting to the specific nature of its elements. To that aim, let us introduce the conjugacy class of a symmetry set. The conjugacy class of $\mathrm{S}_{\mathbf{A}}$, denoted $[\mathrm{S}_{\mathbf{A}}]$, is defined as

$$[\mathrm{S}_{\mathbf{A}}] = \{\mathrm{S}'_{\mathbf{A}} = \mathbf{R}\mathrm{S}_{\mathbf{A}}\mathbf{R}^{T}, \mathbf{R} \in \mathrm{SO}(3)\}.$$

Note that $[\mathrm{S}_{\mathbf{A}}]$ indicates the symmetry class and not the symmetry group of the element $\mathbf{A} \in \mathbb{E}\mathrm{la}(6)$. We have the fundamental property:

Proposition 3.2. *Consider a general rotation $\mathbf{R}(\mathbf{n}, \theta)$ and another rotation $\tilde{\mathbf{R}}$ such as $\mathbf{n}^{\star} = \tilde{\mathbf{R}}\mathbf{n}$. The conjugacy of $\mathbf{R}(\mathbf{n}, \theta)$ by $\tilde{\mathbf{R}}$ amount to changing the characteristic axis from $\mathbf{n}$ to $\mathbf{n}^{\star}$:*

$$\tilde{\mathbf{R}}\mathbf{R}(\mathbf{n}, \theta)\tilde{\mathbf{R}}^{T} = \mathbf{R}(\mathbf{n}^{\star}, \theta)$$

Proof. The conjugate of $\mathbf{R}(\mathbf{n}, \theta)$ by $\tilde{\mathbf{R}}$ is defined to be

$$\mathbf{R}^{\star} = \tilde{\mathbf{R}}\mathbf{R}(\mathbf{n}, \theta)\tilde{\mathbf{R}}^{T}$$

In components and using the Rodrigues formula:

$$\begin{aligned} R^{\star}_{ij} &= \tilde{R}_{ip}\tilde{R}_{jq}\left\{\cos(\theta)\delta_{pq} - \sin(\theta)\epsilon_{pqr}n_r + [1-\cos(\theta)]n_p n_q\right\} \\ &= \cos(\theta)\tilde{R}_{ip}\tilde{R}_{jp} - \sin(\theta)\tilde{R}_{ip}\tilde{R}_{jq}\epsilon_{pqr}n_r + [1-\cos(\theta)]\tilde{R}_{ip}n_p\tilde{R}_{jq}n_q \\ &= \cos(\theta)\delta_{ij} - \sin(\theta)\epsilon_{ijk}n^{\star}_r + [1-\cos(\theta)]n^{\star}_i n^{\star}_j \end{aligned}$$

To get from the second line to the third one, observe that:

$$\tilde{R}_{ip}\tilde{R}_{jq}\epsilon_{pqr}n_r = \tilde{R}_{ip}\tilde{R}_{jq}\epsilon_{pqs}\delta_{st}n_t = \tilde{R}_{ip}\tilde{R}_{jq}\epsilon_{pqs}\tilde{R}_{ks}\tilde{R}_{kt}n_t = \tilde{R}_{ip}\tilde{R}_{jq}\tilde{R}_{ks}\epsilon_{pqs}n^{\star}_k$$

and use the identity

$$\tilde{R}_{ip}\tilde{R}_{jq}\tilde{R}_{ks}\epsilon_{pqs} = (\det\tilde{\mathbf{R}})\epsilon_{ijk} = \epsilon_{ijk}.$$

As a result

$$\mathbf{R}^{\star} = \cos(\theta)\mathbf{I} - \sin(\theta)\epsilon \cdot \mathbf{n}^{\star} + [1-\cos(\theta)]\mathbf{n}^{\star} \otimes \mathbf{n}^{\star} \equiv \mathbf{R}(\mathbf{n}^{\star}, \theta).$$

■

Since, as stated in the Proposition 3.1, any element of $\mathrm{O}(3)$ can be obtained from a rotation $\mathbf{R}(\mathbf{n}, \theta)$, the previous result can be extended to $\mathrm{O}(3)$.

Corollary 3.1. *Consider a transformation $\mathbf{Q}(\mathbf{n})$ having a characteristic direction $\mathbf{n}$ and a rotation $\tilde{\mathbf{R}}$. The conjugacy of $\mathbf{Q}(\mathbf{n})$ by $\tilde{\mathbf{R}}$ amounts to changing the characteristic axis from $\mathbf{n}$ to $\mathbf{n}^{\star} = \tilde{\mathbf{R}}\mathbf{n}$:*

$$\tilde{\mathbf{R}}\mathbf{Q}(\mathbf{n})\tilde{\mathbf{R}}^{T} = \mathbf{Q}(\mathbf{n}^{\star}).$$

Two elements of $\mathbb{E}\mathrm{la}(6)$, $\mathbf{A}$ and $\mathbf{A}^{\star}$, are said to have the same type of symmetry if and only if the elements of their symmetry sets $\mathrm{S}_{\mathbf{A}}$ and $\mathrm{S}_{\mathbf{A}^{\star}}$ differ at most to by common rotation of their characteristic axes:

$$\mathbf{A}^\star \sim \mathbf{A} \Leftrightarrow \exists \mathbf{R} \in \mathrm{SO}(3) \text{such that } \mathrm{S}_{\mathbf{A}^\star} = \mathbf{R}\mathrm{S}_{\mathbf{A}}\mathbf{R}^T \tag{3.13}$$

in which

$$\mathbf{R}\mathrm{S}_{\mathbf{A}}\mathbf{R}^T = \{\mathbf{S}' \in \mathrm{O}(3) \mid \mathbf{S}' = \mathbf{R}\mathbf{S}\mathbf{R}^T, \mathbf{S} \in \mathrm{S}_{\mathbf{A}}\}.$$

Equation (3.13) defines an equivalence relation between elements of $\mathbb{E}\mathrm{la}(6)$. As a result, it induces a partition of $\mathbb{E}\mathrm{la}(6)$ into a family of classes $\Sigma_{[S_i]}$, called strata, verifying the following properties:

$$\bigcup_{1 \le i \le N} \Sigma_{[S_i]} = \mathbb{E}\mathrm{la}(6) \quad \text{and} \quad \Sigma_{[S_i]} \cap \Sigma_{[S_j]} = \emptyset \text{ for } i \neq j. \tag{3.14}$$

In other words, $\{\Sigma_{[S_i]}\}_{1 \le i \le N}$ constitutes a partition of $\mathbb{E}\mathrm{la}(6)$, and each stratum $\Sigma_{[S_i]}$ of this partition contains all tensors whose symmetry class corresponds to $[S_i]$. The specific type of a symmetry class depends on the considered classification. Precisely, we can distinguish between:

- Orthogonal symmetry class: all elements of $\mathrm{O}(3)$ are considered;
- Rotation symmetry class: only rotations are concerned ($\mathrm{SO}(3)$-restriction);
- Reflection symmetry class: only symmetry planes are under consideration.

(i) Results concerning symmetry classification

General results concerning the classification with respect to $\mathrm{SO}(3)$ have been obtained for even-order tensor spaces in [52]. The application of these general theorems to the specific situation of $\mathbb{E}\mathrm{la}(6)$ provides the following classification theorem:

Theorem 3.1. *With respect to the classical* $\mathrm{SO}(3)$*-action, the space* $\mathbb{E}\mathrm{la}(6)$ *is divided into the following* 17 *rotation symmetry classes:*

$$\mathfrak{I}^r(\mathbb{E}\mathrm{la}(6)) = \{[\mathbf{I}], [\mathrm{Z}_2], \cdots, [\mathrm{Z}_6], [\mathrm{D}_2], \cdots, [\mathrm{D}_6], [\mathrm{SO}(2)], [\mathrm{O}(2), [\mathcal{T}], [\mathcal{O}], [\mathcal{I}], [\mathrm{SO}(3)]\}.$$

The description of the groups involved in this classification can be found in [54,60]. As previously discussed, for even-order tensors, the classifications with respect to $\mathrm{SO}(3)$ and $\mathrm{O}(3)$ are in one-to-one correspondence. This is due to the fact that the inversion is contained in the symmetry group of any even-tensor in $\mathbb{R}^3$. As a consequence, any orthogonal-symmetry class $[G]$ of an even-tensor in $\mathbb{R}^3$ has the following form $[H \times \mathrm{Z}_2^c]$, with H being a closed subgroup of $\mathrm{SO}(3)$ and Z_2^c the group generated by the inversion. As a consequence, orthogonal symmetry classes of $\mathbb{E}\mathrm{la}(6)$ can be deduced from the rotation classes under $\mathrm{SO}(3)$:

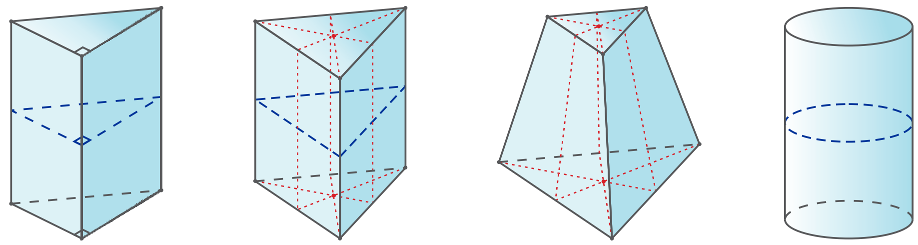
Theorem 3.2. *With respect to the classical* $\mathrm{O}(3)$*-action, the space* $\mathbb{E}\mathrm{la}(6)$ *is divided into the following* 17 *orthogonal symmetry classes:*

$$\begin{aligned}\mathfrak{I}^o(\mathbb{E}\mathrm{la}(6)) = \{&[\mathrm{Z}_2^c], [\mathrm{Z}_2 \times \mathrm{Z}_2^c], \cdots, [\mathrm{Z}_6 \times \mathrm{Z}_2^c], [\mathrm{D}_2 \times \mathrm{Z}_2^c], \cdots, [\mathrm{D}_6 \times \mathrm{Z}_2^c],\\ &[\mathrm{SO}(2) \times \mathrm{Z}_2^c], [\mathrm{O}(2) \times \mathrm{Z}_2^c], [\mathcal{T} \times \mathrm{Z}_2^c], [\mathcal{O} \times \mathrm{Z}_2^c], [\mathcal{I} \times \mathrm{Z}_2^c], [\mathrm{O}(3)]\}.\end{aligned}$$

For any rotational (and hence orthogonal) symmetry class of $\mathbb{E}\mathrm{la}(6)$, the associated matrix representation has been provided in [46]. It remains to find the results relative to the classification by symmetry planes. First, we have the following lemma:

Lemma 3.1. *Reflection symmetry classes cannot be more than orthogonal symmetry classes.*

Proof. Taking $\mathbf{T}_1$ and $\mathbf{T}_2$ as two elements belonging to the same (non-empty) stratum $\Sigma_{[G_i]}$, let $\mathrm{G}_{\mathbf{T}_i}$ be the symmetry group of $\mathbf{T}_1$ and $\mathrm{P}_{\mathbf{T}_i} \subset \mathrm{G}_{\mathbf{T}_i}$ be the, possibly empty, subset of symmetry planes of $\mathrm{G}_{\mathbf{T}_i}$. Since $\mathbf{T}_1$ and $\mathbf{T}_2$ belong to the same stratum $\Sigma_{[G_i]}$, then $\exists \mathbf{R} \in \mathrm{SO}(3)$, $\mathrm{G}_{\mathbf{T}_2} = \mathbf{R}\mathrm{G}_{\mathbf{T}_1}\mathbf{R}^T$. As a consequence, for the same $\mathbf{R} \in \mathrm{SO}(3)$, we have $\mathrm{P}_{\mathbf{T}_2} = \mathbf{R}\mathrm{P}_{\mathbf{T}_1}\mathbf{R}^T$. Hence $\mathbf{T}_1$ and $\mathbf{T}_2$ belong to the same (non-empty) stratum $\Sigma_{[P_i]}$. ∎



(a) (b) (c) (d)

Fig1: Geometrical illustration of the sets of reflection transformations: (a) $\mathcal{P}_h$; (b) $\mathcal{P}_{hv_3}$; (c) $\mathcal{P}_{v_3}$ and (d) $\mathcal{P}_{hv_\infty}$.

Hence, a natural question is the following one: among the orthogonal symmetry classes, how many of them are distinguished by the classification based on symmetry planes ? Conversely, how many different orthogonal symmetry classes possess the same reflection symmetry set ? As pointed in the introduction, the answer to those questions are important for constructing a symmetry identification procedure based on Tensor's Pole Figures.

4. All symmetry classes of an elastic material: results and discussions

Our main results are presented and discussed in this section. For the sake of clarity, their derivations and the relevant technical details are postponed to section 5.

Let us define the unit vector

$$\mathbf{r}_i(\theta) = \sin\theta\mathbf{e}_j + \cos\theta\mathbf{e}_k \tag{4.1}$$

with $\{i, j, k\}$ being a cycle permutation of $\{1, 2, 3\}$. We introduce the following sets of reflection transformations.

- $\mathcal{P}_h$ being the set defined by $\mathbf{P}(\mathbf{e}_3)$: this set contains one reflection. The label h means *horizontal*;
- $\mathcal{P}_{v_k}$ being the set defined by $\mathcal{P}_{v_k} = \{\mathbf{P}(\mathbf{r}_3(\frac{2p\pi}{k})\}_{1\le p\le k}$: this set contains k elements. The label v means *vertical*;
- $\mathcal{P}_{hv_k}$ $(k \ge 1)$ being the set defined by the k elements of $\mathcal{P}_{v_k}$ completed by $\mathbf{P}(\mathbf{e}_3)$: this set comprises $k+1$ elements;
- $\mathcal{P}_{\mathcal{O}}$ denoting the cubic set consisting of 9 reflections with respect to the nine planes of which the normals of 6 pass through the center of each edge of a regular cube and the normals of 3 through the center of each face of the latter;
- $\mathcal{P}_{\mathcal{I}}$ symbolizing the icosahedral set of 15 reflections with respect to the fifteen planes whose normals pass through the center of each edge of a regular icosahedron;
- $\mathcal{P}_{\mathrm{O}(3)}$ representing the set composed of all reflections $\mathbf{P}(\mathbf{n})$ with any $\mathbf{n} \in \mathrm{S}^2$.

Some sets of reflection transformations are illustrated in Figure 1.

Theorem 4.1. *With respect to the classical* $\mathrm{O}(3)$*-action, the space* $\mathbb{E}\mathrm{la}(6)$ *is divided into the following* 11 *reflection symmetry classes:*

$$\mathfrak{I}^{\mathcal{P}}(\mathbb{E}\mathrm{la}(6)) = \{\emptyset, [\mathcal{P}_h], [\mathcal{P}_{hv_2}], [\mathcal{P}_{v_3}], [\mathcal{P}_{hv_4}], [\mathcal{P}_{v_5}], [\mathcal{P}_{hv_6}], [\mathcal{P}_{hv_\infty}], [\mathcal{P}_{\mathcal{O}}], [\mathcal{P}_{\mathcal{I}}], [\mathcal{P}_{\mathrm{O}(3)}]\}.$$

The characteristics of each reflection symmetry class, and especially its link with the orthogonal ones, are detailed in Table 1. In this table are indicated the rotation symmetry class of the normal set, the associated lattice structure allowing to establish the transition from a reflection class to another. The resulting transition diagram is illustrated in Figure 2.

System	Reflection class	Number of symmetry planes	Rotation class of the normal set	Orthogonal class	# independent parameters
Triclinic	$\emptyset$	0	$[\mathbf{I}]$	$[\mathrm{Z}_2^c]$, $[\mathrm{Z}_3 \times \mathrm{Z}_2^c]$, $[\mathrm{Z}_5 \times \mathrm{Z}_2^c]$	171
Monoclinic	$[\mathcal{P}_h]$	1	$[Z_2]$	$[\mathrm{Z}_2 \times \mathrm{Z}_2^c]$, $[\mathrm{Z}_4 \times \mathrm{Z}_2^c]$, $[\mathrm{Z}_6 \times \mathrm{Z}_2^c]$, $[\mathrm{SO}(2) \times \mathrm{Z}_2^c]$	91
Orthotropic	$[\mathcal{P}_{hv_2}]$	3	$[\mathrm{D}_2]$	$[\mathrm{D}_2 \times \mathrm{Z}_2^c]$, $[\mathcal{T} \times \mathrm{Z}_2^c]$	51
Trigonal	$[\mathcal{P}_{v_3}]$	3	$[\mathrm{D}_3]$	$[\mathrm{D}_3 \times \mathrm{Z}_2^c]$	34
Tetragonal	$[\mathcal{P}_{hv_4}]$	5	$[\mathrm{D}_4]$	$[\mathrm{D}_4 \times \mathrm{Z}_2^c]$	28
Pentagonal	$[\mathcal{P}_{v_5}]$	5	$[\mathrm{D}_5]$	$[\mathrm{D}_5 \times \mathrm{Z}_2^c]$	23
Hexagonal	$[\mathcal{P}_{hv_6}]$	7	$[\mathrm{D}_6]$	$[\mathrm{D}_6 \times \mathrm{Z}_2^c]$	22
Trans. isotropic	$[\mathcal{P}_{hv_\infty}]$	$\infty + 1$	$[\mathrm{O}(2)]$	$[\mathrm{O}(2) \times \mathrm{Z}_2^c]$	21
Cubic	$[\mathcal{P}_{\mathcal{O}}]$	9	$[\mathcal{O}]$	$[\mathcal{O} \times \mathrm{Z}_2^c]$	11
Icosahedral	$[\mathcal{P}_{\mathcal{I}}]$	15	$[\mathcal{I}]$	$[\mathcal{I} \times \mathrm{Z}_2^c]$	6
Isotropic	$[\mathcal{P}_{\mathrm{O}(3)}]$	∞^3	$[\mathrm{SO}(3)]$	$[\mathrm{O}(3)]$	5

Table 1: Symmetry plane stratification of $\mathbb{E}\mathrm{la}(6)$.

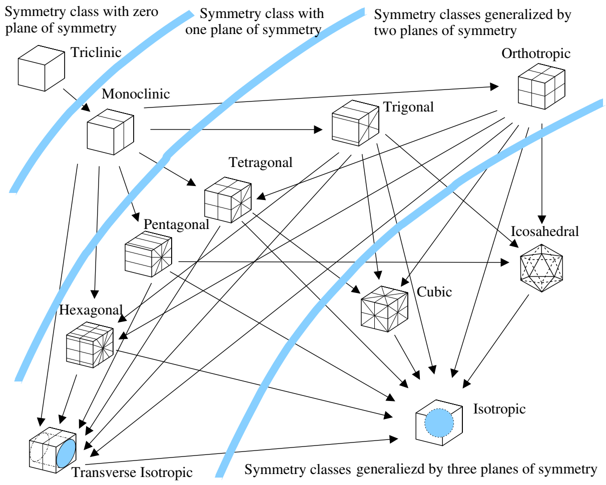


Fig2: Diagrammatic representation of the 11 reflection symmetry classes for the sixth-order elastic tensor. The dotted lines indicate the positions of the planes of symmetry.

The relationships of the reflection symmetry classes are illustrated by the diagram of Figure 3.
In this diagram, an arrow going from $[\mathrm{H}_1]$ to $[\mathrm{H}_2]$ signifies that groups in $[\mathrm{H}_2]$ possess at least a subgroup that belongs to $[\mathrm{H}_1]$. From this lattice, the following diagram can be constructed:

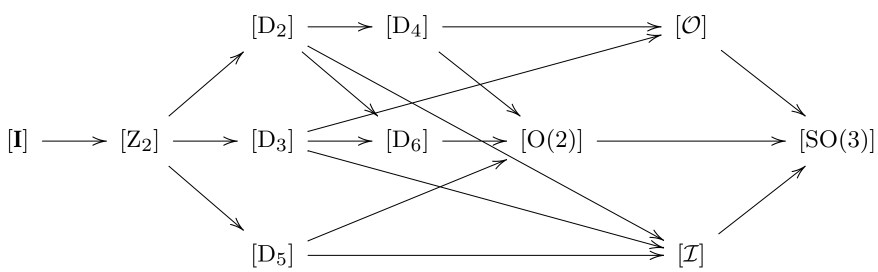


Fig3: Lattice of group inclusions of the rotation class of the normal set

5. Proofs of the main result

Let us start by establishing some basic but useful lemmas.

Lemma 5.1. *Let* $\mathbf{Q} \in \mathrm{O}(3) \backslash \mathrm{SO}(3)$, $\mathbf{Q}$ *is a reflection if and only if* $\mathrm{Tr}\,\mathbf{Q} = 1$.

Proof. Suppose that $\mathbf{Q}$ is a reflection, hence

$$\exists \mathbf{n} \in S^2, \; \mathbf{Q} = \mathbf{I} - 2\mathbf{n} \otimes \mathbf{n} \quad \text{and} \quad \mathrm{Tr}\,\mathbf{Q} = 1.$$

On the other hand, we consider now an element $\mathbf{Q}$ of $\mathrm{O}(3) \backslash \mathrm{SO}(3)$ verifying $\mathrm{Tr}\,\mathbf{Q} = 1$. As a general element of $\mathrm{O}(3) \backslash \mathrm{SO}(3)$, $\mathbf{Q}$ is a rotoinversion, i.e. $\mathbf{Q} = \overline{\mathbf{R}}(\mathbf{n}; \theta)$ with $\overline{\mathbf{R}}(\mathbf{n}; \theta)$ taking the following expression:

$$\overline{\mathbf{R}}(\mathbf{n}; \theta) = -\cos(\theta)\mathbf{I} + \sin(\theta)\epsilon \cdot \mathbf{n} - (1 - \cos(\theta))\mathbf{n} \otimes \mathbf{n}$$

which trace verifies

$$\mathrm{Tr}(\overline{\mathbf{R}}(\mathbf{n}; \theta)) = -1 - 2\cos\theta = 1.$$

Solving the relation in terms of θ within the interval $[0, 2\pi[$ gives $\theta = \pi$ as the unique solution. And, finally,

$$\overline{\mathbf{R}}(\mathbf{n}; \theta) = \mathbf{I} - 2\mathbf{n} \otimes \mathbf{n}$$

which is a reflection across vector plane having $\mathbf{n} \in S^2$ as normal. ∎

Lemma 5.2. *Let G be a closed subgroup of* $\mathrm{O}(3)$ *and have the form* $H \times \mathrm{Z}_2^c$ *with H being a closed subgroup of* $\mathrm{SO}(3)$. *Then, to each* $\mathbf{R}(\mathbf{n}; \pi) \in G$ *corresponds a unique* $\mathbf{P}(\mathbf{n}) \in G$.

Proof. Suppose there exists an $\mathbf{n} \in S^2$ such as $\mathbf{R}(\mathbf{n}; \pi) \in H$, this element verifies $\mathrm{Tr}(\mathbf{R}(\mathbf{n}; \theta)) = -1$. Since G is of the form $H \times \mathrm{Z}_2^c$ with H in $\mathrm{SO}(3)$, there exists a unique element $-\mathbf{R}(\mathbf{n}; \pi)$ which belongs also to G. The trace of this element is 1, hence it is a reflection by Lemma 5.1. ∎

Lemma 5.3. *Consider the cyclic group* $\mathrm{Z}_k^{\mathbf{n}}$, *generated by the rotation* $\mathbf{R}(\mathbf{n}; \frac{2\pi}{k})$. *Then,* $\mathbf{R}(\mathbf{n}; \pi) \in \mathrm{Z}_k^{\mathbf{n}}$ *if and only if* $k = 2p$.

Proof. Consider a rotation $\mathbf{R}^*$ generated by $\mathbf{R}(\mathbf{n}; \frac{2\pi}{k})$ such as $\mathbf{R}^* = \underbrace{\mathbf{R}(\mathbf{n}; \frac{2\pi}{k}) \cdot \ldots \cdot \mathbf{R}(\mathbf{n}; \frac{2\pi}{k})}_{p \text{ times}}$. Since along a given axis of rotation the composition law of rotations is as follows: $\mathbf{R}(\mathbf{n}; \theta_1)\mathbf{R}(\mathbf{n}; \theta_2) = \mathbf{R}(\mathbf{n}; \theta_1 + \theta_2)$, $\mathbf{R}^* = \mathbf{R}(\mathbf{n}; \frac{2p\pi}{k})$ and is equal to $\mathbf{R}(\mathbf{n}; \pi)$ if and only if $k = 2p$. ∎

As a consequence, the identification of the set of symmetry planes for subgroups of $\mathrm{O}(3)$ having the following structure $H \times \mathrm{Z}_2^c$ reduces to the one of the set of rotations within the same group. This identification is direct by using a theorem from [61]. Before enunciating this result, the following group decomposition, introduced by [61,62] and called the disjoint union of subgroups, has to be introduced:

Definition 5.1. *Let H_1, H_2, $\cdots$, H_s be subgroups of G. Then G is the* disjoint union *of the H_i's if and only if*

$$G = \bigcup_{i=1}^{s} H_i \text{ with } H_i \cap H_j = \{\mathbf{I}\} \text{ for } i \neq j.$$

This decomposition is noted as $G = \biguplus_{i=1}^{s} H_i$ *which indicates a* disjoint union *of subgroups. The following result borrowed from [61,62] shows that any closed subgroup of* $\mathrm{SO}(3)$ *acts as a disjoint union of cyclic groups.*

Lemma 5.4. *The closed subgroups of* $\mathrm{SO}(3)$ *admit the following decompositions into disjoint union of cyclic groups.*

$$\mathrm{D}_n = \mathrm{Z}_n^p \biguplus_{l=0}^{n-1} \mathrm{Z}_2^{o_l}, \quad \mathcal{T} = \biguplus_{i=1}^{4} \mathrm{Z}_3^{v_i} \biguplus_{j=1}^{3} \mathrm{Z}_2^{e_j}, \quad \mathcal{O} = \biguplus_{i=1}^{3} \mathrm{Z}_4^{f_i} \biguplus_{j=1}^{4} \mathrm{Z}_3^{v_j} \biguplus_{l=1}^{6} \mathrm{Z}_2^{e_l}, \quad \mathcal{I} = \biguplus_{i=1}^{6} \mathrm{Z}_5^{f_i} \biguplus_{j=1}^{10} \mathrm{Z}_3^{v_j} \biguplus_{l=1}^{15} \mathrm{Z}_2^{e_l}.$$

in which the exponents p*,* o*,* v*,* e *and* f *indicate the position of the principal axis, orthogonal (to* p*) axis, vertex, edge and face, respectively.*

Theorem 5.1. *Consider the application* $\mathfrak{P}$ *which associate to an orthogonal symmetry class of* $\mathrm{O}(3)$ *its reflection class. We have the following result:*

$$\mathfrak{P}([\mathrm{Z}_{2p} \times \mathrm{Z}_2^c]) = [\mathcal{P}_h], \quad \mathfrak{P}([\mathrm{Z}_{2p+1} \times \mathrm{Z}_2^c]) = \emptyset, \quad \mathfrak{P}([\mathrm{D}_{2p} \times \mathrm{Z}_2^c]) = [\mathcal{P}_{hv_{2p}}], \quad \mathfrak{P}([\mathcal{T} \times \mathrm{Z}_2^c]) = [\mathcal{P}_{hv_2}],$$
$$\mathfrak{P}([\mathrm{D}_{2p+1} \times \mathrm{Z}_2^c]) = [\mathcal{P}_{v_{2p+1}}], \quad \mathfrak{P}([\mathcal{O} \times \mathrm{Z}_2^c]) = [\mathcal{P}_{\mathcal{O}}], \quad \mathfrak{P}([\mathcal{I} \times \mathrm{Z}_2^c]) = [\mathcal{P}_{\mathcal{I}}], \quad \mathfrak{P}([\mathrm{O}(3)]) = [\mathcal{P}_{\mathrm{O}(3)}].$$

Proof.

- Consider the class $[\mathrm{Z}_{2p} \times \mathrm{Z}_2^c]$. By using Lemma 5.2 each symmetry plane is associated to a rotation of order 2, and by applying Lemma 5.3 there is a unique rotation of that kind in Z_{2p} along $\mathbf{e}_3$. As consequence, there is a unique symmetry plane $\mathcal{P}(\mathbf{e}_3)$ associated to $\mathrm{Z}_{2p} \times \mathrm{Z}_2^c$. In terms of symmetry class, $\mathfrak{P}([\mathrm{Z}_{2p} \times \mathrm{Z}_2^c]) = [\mathcal{P}_h]$;
- Consider the class $[\mathrm{Z}_{2p+1} \times \mathrm{Z}_2^c]$. By applying Lemmas 5.2 and 5.3, it appears that there is no rotation in Z_{2p+1} which corresponds to a symmetry plane. As consequence, there is no symmetry plane associated to $\mathrm{Z}_{2p+1} \times \mathrm{Z}_2^c$, or equivalently, $\mathfrak{P}([\mathrm{Z}_{2p+1} \times \mathrm{Z}_2^c]) = \emptyset$;
- Consider the class $[\mathrm{D}_{2p+1} \times \mathrm{Z}_2^c]$. By Lemma 5.3 that there is no rotation in Z_{2p+1} along $\mathbf{e}_3$ corresponding to an horizontal symmetry plane. From the Lemma 5.3, $\mathbf{R}(\mathbf{e}_1; \pi)$ generates a vertical symmetry plane, from it the action of $\mathbf{R}\left(\mathbf{e}_3; \frac{2\pi}{2p+1}\right)$ generates the symmetry planes $\mathcal{P}(\mathbf{r}_3(\frac{2k\pi}{2p+1}))$ associated to $\mathbf{P}(\mathbf{r}_3(\frac{2k\pi}{2p+1})$ with $k = 1, 2, ..., 2p+1$. This yields that $\mathfrak{P}([\mathrm{D}_{2p+1} \times \mathrm{Z}_2^c]) = [\mathcal{P}_{v_{2p+1}}]$.
- Consider the class $[\mathrm{D}_{2p} \times \mathrm{Z}_2^c]$. In addition to the previous class $[\mathrm{D}_{2p+1} \times \mathrm{Z}_2^c]$, Lemma 5.3 shows that there is a unique horizontal symmetry plane $\mathcal{P}(\mathbf{e}_3)$ in Z_{2p}. As consequence, we have $\mathfrak{P}([\mathrm{D}_{2p} \times \mathrm{Z}_2^c]) = [\mathcal{P}_{hv_{2p}}]$.
- Consider the class $[\mathcal{T} \times \mathrm{Z}_2^c]$. It can be demonstrated that $\mathcal{P}(\mathbf{e}_1)$, $\mathcal{P}(\mathbf{e}_2)$ and $\mathcal{P}(\mathbf{e}_3)$ are 3 symmetry planes, or equivalently, $\mathfrak{P}([\mathcal{T} \times \mathrm{Z}_2^c]) = [\mathcal{P}_{hv_2}]$.
- Consider the class $[\mathcal{O} \times \mathrm{Z}_2^c]$, a look at lemma 5.4 and the use of the intermediate lemmas 5.1-5.3 indicate that this class possesses 9 symmetry planes. Still from lemma 5.4 it can be read of that the normals to 6 of those 9 planes pass through the center of each edge of a regular cube, while 3 of them go through the center of each face.
- Consider the class $[\mathcal{I} \times \mathrm{Z}_2^c]$, lemma 5.4 with the aid of lemmas 5.1-5.3 implies that this class possesses 15 symmetry planes. Moreover, it can be obtained from lemma 5.4 that the normals to all of them pass through the center of each edge of a regular icosahedron. ∎

6. Higher-order Tensorial Pole Figures

(a) Pole Figure

Following an idea initially proposed by [57], in this last section we will study pole figures that can be associated to the sixth-order elasticity tensor $\mathbf{A} \in \mathbb{E}\mathrm{la}(6)$. Since the explicit matrix representation of $\mathbf{A} \in \mathbb{E}\mathrm{la}(6)$ is explicitly known for each orthogonal symmetry class [46], numerical computations of the pole figures can be done on simulated data for any symmetry classes.

Pole figures consist in identifying the set of unit normals to symmetry planes of a given tensor. To be more specific, consider the reflection through $\mathcal{P}(\mathbf{n})$ with unit normal $\mathbf{n}$:

$$\mathbf{P}(\mathbf{n}) = \mathbf{I} - 2\mathbf{n} \otimes \mathbf{n}.$$

Any element $\mathbf{n} \in S^2$ can be parametrized as follows:

$$\mathbf{n} = \sin\theta\cos\phi\mathbf{e}_1 + \sin\theta\sin\phi\mathbf{e}_2 + \cos\theta\mathbf{e}_3$$

where $(\phi, \theta) \in [0, 2\pi[\times[0, \pi[$ denote, respectively, the longitude and colatitude angles relative to a system of spherical coordinates.

Consider now, for a given $\mathbf{T} \in \mathbb{E}\mathrm{la}(6)$ the function:

$$\mathrm{I}(\theta, \phi) = \|\mathbf{P}(\theta, \phi) * \mathbf{T} - \mathbf{T}\| \tag{6.1}$$

in which $\|\cdot\|$ is the Frobenius norm inherited from the scalar product on $\mathbb{E}\mathrm{la}(6)$, $\mathbf{P}(\theta, \phi)$ is a reflection operator parametrized with the longitude and colatitude angles and $*$ refers to the operation defined in (3.6). The vanishing loci of $\mathrm{I}(\theta, \phi)$ give the unit normals to the symmetry planes of $\mathbf{T}$.

(b) Numerical Procedure

For computations, the function (6.1) is evaluated on discrete values:

$$M_{ij} = I(\theta_i, \phi_j) \quad \text{with} \quad \theta_i = i\frac{2\pi}{N} \quad \text{and} \quad \phi_j = j\frac{2\pi}{N}.$$

In the following computations, we take $N = 160$. Owing to its symmetry with respect to θ and ϕ, $\mathrm{I}(\theta, \phi)$ can be evaluated only on the domain $[0, \pi[\times[0, \frac{\pi}{2}[$.

To evaluate the function $\mathrm{I}(\theta, \phi)$, we use first the matrix representations which have been shown in [46]. Then, by starting with the fully anisotropic case where the corresponding matrix $\mathbf{A}$ is in $\Sigma_{[\mathbf{I}]}$, the numerical component values of $\mathbf{A}$ are determined as random integers picked-up in the range $\{-20, 20\}$. Next, for higher symmetry classes, their matrix components can be directly deduced from the full anisotropic tensor. It should be noted that tensors obtained by this procedure are generically not positive definite. But since symmetry properties are independent of it, this point is by no mean problematic. We show in Figure 4-Figure 13, the loci of the zeros of the function $\mathrm{I}(\theta, \phi)$ plotted on the unit sphere and on the $\theta - \phi$ plane for different rotation symmetry classes and reflection symmetry classes. It can be seen from Figure 4-Figure 13 that the number of symmetry planes for a given rotation symmetry class or reflection symmetry class coincides exactly with the one obtained in Table 1. This confirms the validity of our results.

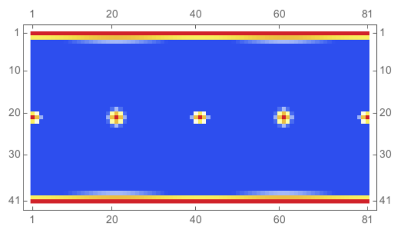
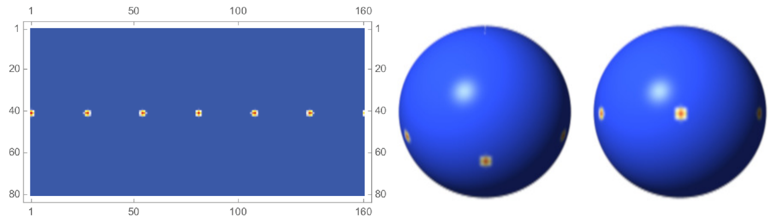
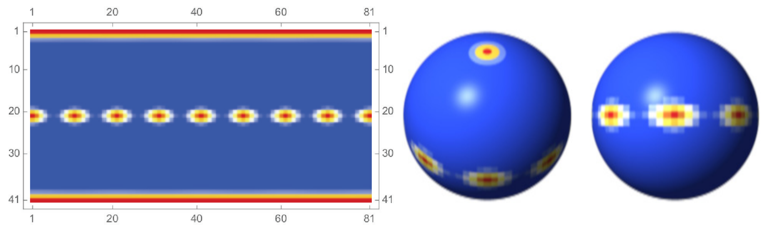


Fig4: Normals to symmetry planes of a random tensor in $\Sigma_{[\mathrm{D}_2]}$ and belonging to reflection class $[\mathcal{P}_{hv_2}]$



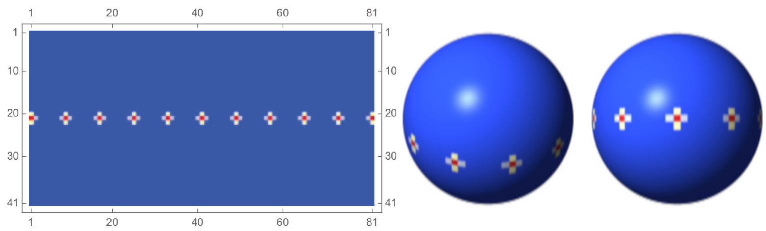
(a) (b) (c)

Fig5: Normals to symmetry planes and loci of the zeros of the function $\mathrm{I}(\theta, \phi)$ of a random tensor in $\varSigma_{[\mathrm{D}_3]}$ and belonging to reflection class $[\mathcal{P}_{v_3}]$ plotted on: (a) the $\theta - \phi$ plane; (b) and (c) the unit sphere.



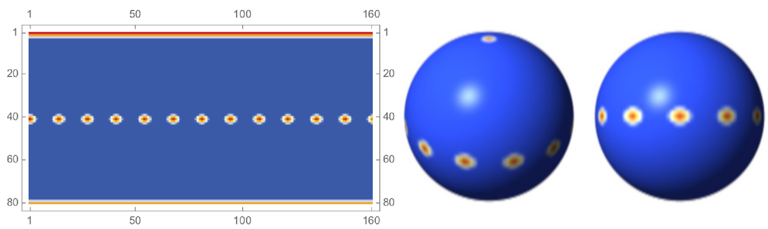
(a) (b) (c)

Fig6: Normals to symmetry planes and loci of the zeros of the function $\mathrm{I}(\theta, \phi)$ of a random tensor in $\varSigma_{[\mathrm{D}_4]}$ and belonging to the reflection class $[\mathcal{P}_{hv_4}]$ plotted on: (a) the $\theta - \phi$ plane; (b) and (c) the unit sphere.



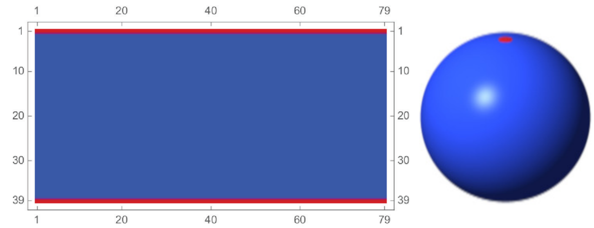
(a) (b) (c)

Fig7: Normals to symmetry planes and loci of the zeros of the function $\mathrm{I}(\theta, \phi)$ of a random tensor in $\varSigma_{[\mathrm{D}_5]}$ and belonging to the reflection class $[\mathcal{P}_{v_5}]$ plotted on: (a) the $\theta - \phi$ plane; (b) and (c) the unit sphere.



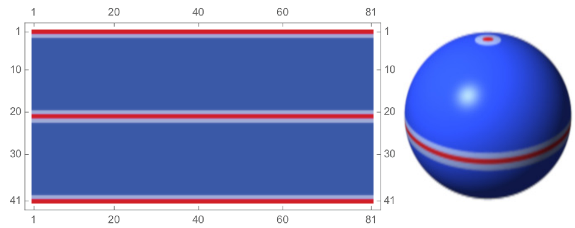
(a) (b) (c)

Fig8: (a) Normals to symmetry planes and loci of the zeros of the function $\mathrm{I}(\theta, \phi)$ of a random tensor in $\varSigma_{[\mathrm{D}_6]}$ and belonging to the reflection class $[\mathcal{P}_{hv_6}]$ plotted on: (a) the $\theta - \phi$ plane; (b) and (c) the unit sphere.



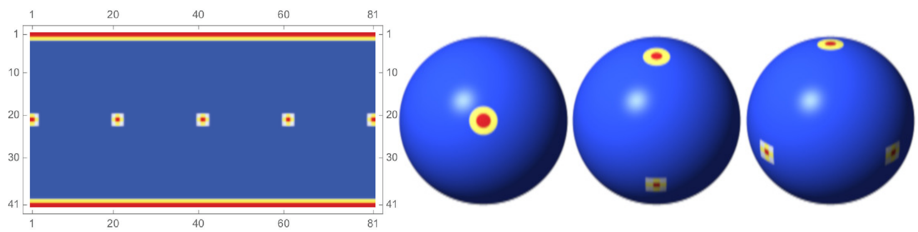
(a) (b)

Fig9: (a) Normals to symmetry planes and loci of the zeros of the function $\mathrm{I}(\theta, \phi)$ of a random tensor in $\Sigma_{[\mathrm{SO}(2)]}$ and belonging to the reflection class $[\mathcal{P}_h]$ plotted on: (a) the $\theta - \phi$ plane; (b) and (c) the unit sphere.



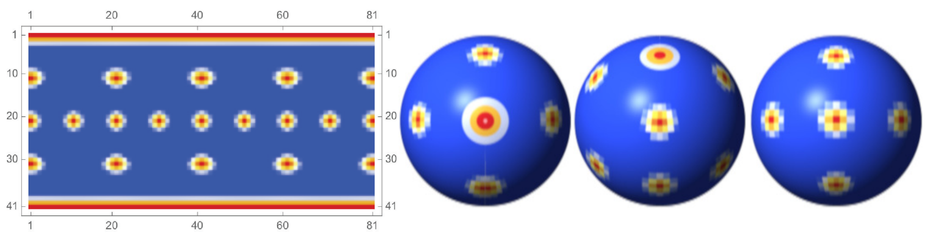
(a) (b)

Fig10: (a) Normals to symmetry planes and loci of the zeros of the function $\mathrm{I}(\theta, \phi)$ of a random tensor in $\Sigma_{[\mathrm{O}(2)]}$ and belonging to the reflection class $[\mathcal{P}_{hv_\infty}]$ plotted on (a) the $\theta - \phi$ plane; (b) the unit sphere.



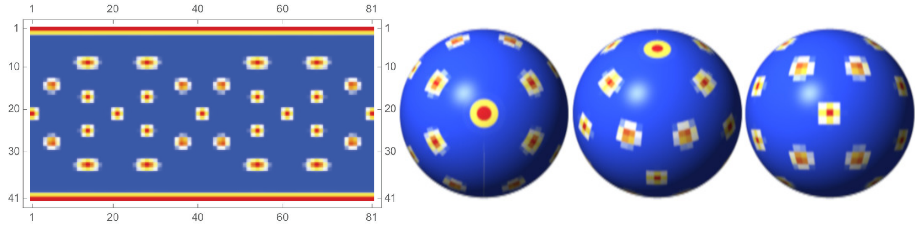
(a) (b) (c) (d)

Fig11: (a) Normals to symmetry planes and loci of the zeros of the function $\mathrm{I}(\theta, \phi)$ of a random tensor in $\Sigma_{[\mathcal{T}]}$ and belonging to reflection class $[\mathcal{P}_{hv_2}]$ plotted on (a) the $\theta - \phi$ plane; (b), (c) and (d) the unit sphere.



(a) (b) (c) (d)

Fig12: (a) Normals to symmetry planes and loci of the zeros of the function $\mathrm{I}(\theta, \phi)$ of a random tensor in $\Sigma_{[\mathcal{O}]}$ and belonging to reflection class $[\mathcal{P}_{\mathcal{O}}]$ plotted on (a) the $\theta - \phi$ plane; (b), (c) and (d) the unit sphere.



(a) (b) (c) (d)

Fig13: (a) Normals to symmetry planes and loci of the zeros of the function $\mathrm{I}(\theta, \phi)$ of a random tensor in $\Sigma_{[\mathcal{I}]}$ and belonging to reflection class $[\mathcal{P}_{\mathcal{I}}]$ plotted on (a) the $\theta - \phi$ plane; (b), (c) and (d) the unit sphere.

7. Concluding remarks

In this work, we have classified the sixth-order elasticity tensors with respect to the reflection symmetry planes they possess. Specifically, it has been shown that the number of reflection symmetry classes for the sixth-order elasticity tensor space is 11, whereas in the case of orthogonal symmetry classes the number is 17. This result is in sharp contrast to that known for classical fourth-order elasticity tensors since, in this latter case, both classifications give the same result. In addition, we have proposed a procedure for identifying planes of symmetry based on the exploitation of tensor pole figures. This method makes it possible to find numerically and graphically the reflection symmetry class to which a given sixth-order elasticity tensor belongs.

The methods elaborated in the present work for sixth-order elasticity tensors are applicable to fifth-order elasticity tensors involved the first strain-gradient elasticity. This study will be presented in a forthcoming work.

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