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# On feasible solutions with guaranteed suboptimality for Quadratic Programming

Xiang Dai<sup>1\*</sup>, Romain Bourdais<sup>2</sup> and Hervé Guéguen<sup>2</sup>

<sup>1\*</sup>GIPSA-lab, Université Grenoble Alpes, CNRS, Grenoble INP, Grenoble, 38000, France.

<sup>2</sup>Department of Automatic Control, IETR, CentraleSupélec, Rennes, 35000, France.

\*Corresponding author(s). E-mail(s):

[xiang.dai@gipsa-lab.grenoble-inp.fr](mailto:xiang.dai@gipsa-lab.grenoble-inp.fr);

Contributing authors: [romain.bourdais@centralesupelec.fr](mailto:romain.bourdais@centralesupelec.fr);

[herve.gueguen@centralesupelec.fr](mailto:herve.gueguen@centralesupelec.fr);

## Abstract

Primal-dual gradient methods are tractable approaches to solve quadratic programs, especially for large scale problems with sparse structures. The associated iterative mechanism allows these methods to converge to the optimal solution. However, the convergence may require a considerable number of iterations, and if one decides to terminate the iterative process earlier, the resulting solution may not fulfill the original constraints of the problem. For the first issue, we propose a proactive method based on the Karush–Kuhn–Tucker (KKT) conditions to check whether the active set updated during the iterations is optimal, which can terminate the iterations before its convergence. To address the second issue and faster terminate the iterations, we first introduce a degree of suboptimality for objective value, and then propose a suboptimal method to solve for suboptimal and feasible solutions. Mathematical developments prove both the feasibility and the guarantee of predefined suboptimality for the suboptimal method. Various random simulations illustrate the effectiveness of the proposed methods.

**Keywords:** Quadratic programming, Suboptimality, Primal-dual algorithm, Gradient method, Feasibility

# 1 Introduction

Quadratic programming (QP) has long tracked massive interest in the society of control system, applied mathematics and computer science (Floudas and Visweswaran, 1995), for it not only encompasses a large variety of applications (including computational geometry, finance, process networks, robotics, telecommunications, energy, and data confidentiality, etc. (Furini et al, 2019)) but also serves as preliminaries in many methods for general constrained optimization (Nocedal and Wright, 2006).

Active set methods, interior point methods and the gradient projection method are often-used approaches to solve QP (Nocedal and Wright, 2006). Nonetheless, for large scale problems, active set methods are incapable as the iterations required increases proportionally to the problem size (Forsgren et al, 2016) (Arnrström and Axehill, 2021), and interior point methods (Mehrotra, 1992) (Wright, 1997) are prohibitive by utmost computation burden in computing inverse of Hessian (Chong and Zak, 2004). The gradient projection method (Zhu and Rockafellar, 1993), however, is most efficient only when box constraints are dealt with, as the projection step itself can be costly for general linear inequality constraints (Conn et al, 1988) (Burke and Moré, 1994).

Note that in implementing the primal-dual gradient methods, the optimal solution may take enormous iterations to generate, which leads to undesirable long computation time. Another fact is that the fulfillment of constraints in QP is mandatory in many applications due to security restrictions or physical limits. On the other hand, the suboptimality of objective value can be manipulated as a parameter in response to time requirements in various applications. Consequently, feasible solutions with guaranteed suboptimality can be of great interest in solving QP using primal-dual gradient methods. And the motivation of this paper is to reduce the iteration number needed in obtaining feasible solutions with guaranteed suboptimality.

In this paper, we first propose a proactive method by consolidating the primal-dual gradient method and the KKT criterion (Kuhn and Tucker, 1951) to identify if the dynamically updated active set is the optimal active set, from which the optimal solution can be analytically solved. Further, given a prespecified suboptimality, we conceive a suboptimal method to generate feasible solutions with guaranteed suboptimality using the dynamically updated active set, which proceeds in 2 steps: i) the original QP is transformed into the only equality constrained QP by regarding the active inequality constraints as equality constraints, whose optimal objective value can be solved explicitly; ii) taking advantage of the gap between the above-mentioned optimal objective value and the best dual objective value obtained during iterations, the cone programming (CP) can be initiated to search for feasible solutions with guaranteed suboptimality. It is essential to mention that the updated active set only needs to be close enough, not necessarily identical, to the optimal active set, making it possible to consume fewer iterations than the proactive method.

The main contributions of this paper are threefold. First, the proactive method can deliver the optimal solution using a relatively small number of

iterations compared to conventional primal-dual gradient methods. Second, the suboptimal method, requiring fewer iterations than the proactive method, can generate feasible solutions with guaranteed suboptimality. Third, the lower bound of suboptimality is demonstrated such that any suboptimality greater than the bound can make the suboptimal method terminated with finite iterations.

This paper is organized as follows. Section 2 sets up the QP problem and fundamentals. Section 3 proposes the proactive method combining the KKT criterion and the primal-dual gradient method. Section 4 illustrates transformations from the original QP into equality constrained QP, and proposes the suboptimal method to generate feasible solutions with guaranteed suboptimality. Numerical experiments and results discussions are presented in Section 5. And conclusions are given in Section 6.

*Notation:* For  $n \in \mathbb{N}_{>0}$ ,  $\mathbb{S}_+^n$  and  $\mathbb{S}_{++}^n$  denote semidefinite positive and definite positive matrix of size  $n \times n$  respectively. The norm  $\|\cdot\|$  denotes the Euclidean norm, for  $x \in \mathbb{R}^n$ ,  $R \in \mathbb{S}_+^n$ ,  $\|x\|_R = \sqrt{x^T R x}$ ,  $\min \text{eig}(R)$  denotes the minimal eigenvalue of  $R$ . For matrix  $A \in \mathbb{R}^{m_1 \times n}$  and  $B \in \mathbb{R}^{m_2 \times n}$  ( $m_1, m_2 \in \mathbb{N}_+$ ),  $A \oplus B = (A^T, B^T)^T$ , and  $\text{rank}(A)$  denotes the rank of  $A$ .  $\mathbf{1}$  denotes column vector with elements being 1 of appropriate size. For symmetric matrix  $C, D \in \mathbb{S}^n$ ,  $C \succeq D$  means that  $C - D \in \mathbb{S}_+^n$ .

## 2 Problem Statement and Fundamentals

In this section, we formulate the QP problem in the first place. After, the primal-dual gradient method, and the definitions of  $\epsilon$  primal solution and active set are introduced to build fundamentals of methods studied later.

### 2.1 Problem Statement

In this paper, we consider below a QP with optimizer  $\mathbf{y}^*$ :

$$\mathcal{J}^* = \min_{\mathbf{y}} \mathcal{J}(\mathbf{y}) \quad (1a)$$

$$\text{s.t. } \mathbf{A}\mathbf{y} = \mathbf{b}, \quad (1b)$$

$$\mathbf{C}\mathbf{y} \leq \mathbf{d}, \quad (1c)$$

where  $\mathcal{J}(\mathbf{y}) = \frac{1}{2}\|\mathbf{y}\|_{\mathbf{R}}^2$ ,  $\mathbf{y} \in \mathbb{R}^{ny}$ ,  $\mathbf{R} \in \mathbb{S}_{++}^{ny}$ ,  $\mathbf{A} \in \mathbb{R}^{n_\epsilon \times ny}$ ,  $\text{rank}(\mathbf{A}) = n_r$ ,  $\mathbf{b} \in \mathbb{R}^{n_\epsilon}$ ,  $\mathbf{C} \in \mathbb{R}^{n_{i\epsilon} \times ny}$  and  $\mathbf{d} \in \mathbb{R}^{n_{i\epsilon}}$ .

**Definition 1** (Feasible set) *We define  $Y$  as the feasible set of problem (1):*

$$Y = \{\mathbf{y} \in \mathbb{R}^{ny} \mid \mathbf{A}\mathbf{y} = \mathbf{b}, \mathbf{C}\mathbf{y} \leq \mathbf{d}\}. \quad (2)$$

**Assumption 2.1** *We assume that  $Y$  is compact, closed and not empty, and there is at least one  $\mathbf{y}$  in the interior of  $Y$ . We also assume that  $n_r < ny$ .*

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**Definition 2** ( $\epsilon$  primal solution)  $\mathbf{y}$  is said to be an  $\epsilon$  ( $\epsilon > 0$ ) primal solution of problem (1) if and only if  $\mathbf{y} \in Y$  and

$$\mathcal{J}(\mathbf{y}) - \mathcal{J}^* \leq \epsilon. \quad (3)$$

The main objective of this paper is to find an efficient way to solve  $\epsilon$  primal solutions of problem (1).

## 2.2 The Primal-dual Gradient Method

The dual problem of problem (1) is formulated as:

$$g^* = \max_{\boldsymbol{\theta}, \boldsymbol{\lambda} \geq 0} g(\boldsymbol{\theta}, \boldsymbol{\lambda}) = \max_{\boldsymbol{\theta}, \boldsymbol{\lambda} \geq 0} \min_{\mathbf{y}} \mathcal{L}(\mathbf{y}, \boldsymbol{\theta}, \boldsymbol{\lambda}), \quad (4)$$

$$\mathcal{L}(\mathbf{y}, \boldsymbol{\theta}, \boldsymbol{\lambda}) = \frac{1}{2} \|\mathbf{y}\|_{\mathbf{R}}^2 + \boldsymbol{\theta}^T (\mathbf{A}\mathbf{y} - \mathbf{b}) + \boldsymbol{\lambda}^T (\mathbf{C}\mathbf{y} - \mathbf{d}),$$

where  $\boldsymbol{\theta} \in \mathbb{R}^{n_e}$  and  $\boldsymbol{\lambda} \in \mathbb{R}_+^{n_{ie}}$  are the dual variables associated with constraint (1b) and (1c) respectively.

**Remark 1** As  $\mathcal{J}(\mathbf{y})$  is quadratic with polyhedron constraints and  $\mathbf{R} \in \mathbb{S}_+^{n_y}$ , problem (1) is convex. Next, the Slater's condition is satisfied by Assumption 2.1, thus the strong duality holds by Slater's theorem (Boyd and Vandenberghe, 2004), namely  $\mathcal{J}^* = g^*$ .

Commonly, the dual problem (4) is solved in a iterative manner, and the primal-dual gradient method is adopted in this paper for its accessible implementation and good scalability, whose general general formulation is:

$$\boldsymbol{\theta}^{k+1} = \boldsymbol{\theta}^k + \alpha_{\boldsymbol{\theta}}^k (\mathbf{A}\mathbf{y}^k - \mathbf{b}), \quad (5a)$$

$$\boldsymbol{\lambda}^{k+1} = \max\{0, \boldsymbol{\lambda}^k + \alpha_{\boldsymbol{\lambda}}^k (\mathbf{C}\mathbf{y}^k - \mathbf{d})\}, \quad (5b)$$

$$\mathbf{y}^{k+1} = -\mathbf{R}^{-1} (\mathbf{A}^T \boldsymbol{\theta}^{k+1} + \mathbf{C}^T \boldsymbol{\lambda}^{k+1}), \quad (5c)$$

where  $\boldsymbol{\theta}^k \in \mathbb{R}^{n_e}$  and  $\boldsymbol{\lambda}^k \in \mathbb{R}_{\geq 0}^{n_{ie}}$  are dual variables associated with constraint (1b) and (1c) respectively,  $\alpha_{\boldsymbol{\theta}}^k \in \mathbb{R}_{>0}$  and  $\alpha_{\boldsymbol{\lambda}}^k \in \mathbb{R}_{>0}$  are step size associated with  $\boldsymbol{\theta}^k$  and  $\boldsymbol{\lambda}^k$  respectively.

Note that the iterative process (5) can be amenable to formulate in flexible manner through exploiting the specialized problem structure for higher computation efficiency, such as distributed structure (Giselsson et al, 2013), parallelized structure (Bertsekas and Tsitsiklis, 2015), splitting scheme (Combettes and Pesquet, 2012), and layering structure (Chiang et al, 2007). Since the methods of solving problem (1) studied later in this paper are generally compatible with aforementioned structures, we keep a concise formulation of (5) hereinafter for readability.

**Assumption 2.2** We assume that  $\alpha_{\theta}^k$  and  $\alpha_{\lambda}^k$  satisfy one of the following step size conditions: minimization rule, Armijo rule and diminishing step size (Bertsekas, 1999), such that the sequence  $\{\mathbf{y}^k\}$  converges to  $\mathbf{y}^*$ , e.g.  $\lim_{k \rightarrow \infty} \mathbf{y}^k = \mathbf{y}^*$ .

Let  $\mathcal{P} = \{1, \dots, n_{ie}\}$ , the definitions of active constraint, active set and inactive set are given as follows.

**Definition 3** (active set) During iterative process (5), the  $i$ -th constraint of (1c),  $\mathbf{C}_i \mathbf{y} \leq \mathbf{d}_i$  ( $\mathbf{C} = \bigoplus_{i=1}^{n_{ie}} \mathbf{C}_i$ ,  $\mathbf{d} = \bigoplus_{i=1}^{n_{ie}} \mathbf{d}_i$ ), is said to be active at  $k$ -th iteration if  $\mathbf{C}_i \mathbf{y}^k \geq \mathbf{d}_i$ , i.e., the equality is reached or the inequality constraint is violated; or  $\lambda_i^k > 0$ , its corresponding dual variable is turned positive. Let  $\mathcal{A}^k$  and  $\mathcal{I}^k$  denote the active and the inactive set of constraints (1c) at  $k$ -th iteration, which are defined as:

$$\mathcal{A}^k = \{i \in \mathcal{P} \mid \mathbf{C}_i \mathbf{y}^k \geq \mathbf{d}_i, \text{ or } \lambda_i^k > 0\}, \quad (6)$$

$$\mathcal{I}^k = \mathcal{P} \setminus \mathcal{A}^k. \quad (7)$$

The optimal active and inactive set are defined respectively as:  $\mathcal{A}^* = \{i \in \mathcal{P} \mid \mathbf{C}_i \mathbf{y}^* = \mathbf{d}_i\}$ ,  $\mathcal{I}^* = \mathcal{P} \setminus \mathcal{A}^*$ .

### 2.3 KKT Criterion of The Optimal Active Set

In this paper, it is not assumed that the strict complementarity condition or the linear independence constraint qualification (LICQ) is satisfied<sup>1</sup>, where the former is said to hold if matrix  $\mathbf{A}_{\mathcal{A}^k}$  ( $\mathbf{A}_{\mathcal{A}^k} = \mathbf{A} \oplus \mathbf{C}_{\mathcal{A}^k}$ ) has full row rank and the latter is said to hold if  $\lambda_i > 0$  for each  $\mathbf{C}_i \mathbf{y}^* = \mathbf{d}_i$ . As a result,  $\mathcal{A}^*$  is not necessary to be unique. Nonetheless, since strong duality holds for problem (1) and its objective and constraint functions are differentiable, any  $\mathbf{y}^*$  must satisfy the KKT conditions (Boyd and Vandenberghe, 2004).

As such, a KKT conditions based linear programming (LP) (Gupta et al, 2011) below can generate  $\mathbf{y}^*$ , if the input  $\mathcal{A}^k$  is  $\mathcal{A}^*$ . In other words, it can be used to check whether  $\mathcal{A}^k$  is  $\mathcal{A}^*$ .

$$\min_{\mathbf{y}, \lambda_{\mathcal{A}^k}, \mathbf{s}_{\mathcal{I}^k}, h} -h \quad (8a)$$

$$s.t. \mathbf{R}\mathbf{y} + \mathbf{A}^T \theta + \mathbf{C}_{\mathcal{A}^k}^T \lambda_{\mathcal{A}^k} = \mathbf{0}, \quad (8b)$$

$$\mathbf{A}\mathbf{y} - \mathbf{b} = \mathbf{0}, \quad (8c)$$

$$\mathbf{C}_{\mathcal{A}^k} \mathbf{y} - \mathbf{d}_{\mathcal{A}^k} = \mathbf{0}, \quad (8d)$$

$$\mathbf{C}_{\mathcal{I}^k} \mathbf{y} - \mathbf{d}_{\mathcal{I}^k} + \mathbf{s}_{\mathcal{I}^k} = \mathbf{0}, \quad (8e)$$

$$h \cdot \mathbf{1} \leq \lambda_{\mathcal{A}^k}, \quad (8f)$$

$$h \cdot \mathbf{1} \leq \mathbf{s}_{\mathcal{I}^k}, \quad (8g)$$

$$0 \leq h, \quad (8h)$$

---

<sup>1</sup>The notion of LICQ and strict complementarity condition are borrowed from (Nocedal and Wright, 2006).

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where  $C_{\mathcal{A}^k} = \bigoplus_{i \in \mathcal{A}^k} C_i$ ,  $C_{\mathcal{A}^k} \in \mathbb{R}^{c_k \times ny}$ ,  $d_{\mathcal{A}^k} = \bigoplus_{i \in \mathcal{A}^k} d_i$ ,  $d_{\mathcal{A}^k} \in \mathbb{R}^{c_k}$ ,  $C_{\mathcal{I}^k} = \bigoplus_{i \in \mathcal{I}^k} C_i$ ,  $d_{\mathcal{I}^k} = \bigoplus_{i \in \mathcal{I}^k} d_i$ .

### 3 Proactive Optimal Active Set Identification Method (POASIM)

In this section, combining the iterative process (5) and KKT criterion (8), we propose a proactive method to solve  $\mathbf{y}^*$ . It proceeds by dynamically checking if  $\mathcal{A}^k$  is  $\mathcal{A}^*$ , which can generally terminate process (5) before the convergence of iterative process (5), as merely  $\mathcal{A}^*$  instead of  $\mathbf{y}^*$  is sought after. An illustrative example will show this feature in Subsection 5.1.

This proactive method is summarized in Alg. 1. Note that it is possible to have  $\mathcal{A}^k = \mathcal{A}^{k-i}$  ( $i = 1, \dots, k$ ), thus it is sufficient to only test  $\mathcal{A}^k$  that has not been tested before. We can obtain  $\mathbf{y}^*$  if  $\mathcal{A}^*$  is identified during iterations of (5), but that cannot be guaranteed to happen. Therefore, it is one crucial drawback of POASIM, and we will show that this can be overcome by the suboptimal method proposed in the next section.

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#### Algorithm 1 Proactive Optimal Active Set Identification Method

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- 1: Initialize:  $\boldsymbol{\theta}^{-1}$ ,  $\boldsymbol{\lambda}^{-1}$ ,  $k = 0$  and  $\epsilon > 0$ .  $\mathbf{y}^{-1}$  is obtained by (5c).
  - 2: **repeat**
  - 3:   Update primal and dual variables by (5), update  $\mathcal{A}^k$  by (6)
  - 4:   **if** LP (8) has a solution **then return**  $\mathbf{y}^*$
  - 5:   **end if**
  - 6:    $k \leftarrow k + 1$
  - 7: **until**  $\mathbf{y}^*$  is returned
- 

## 4 Active Set Based $\epsilon$ Suboptimal Method

In this section, based on Definition 3, problem (1) can be dynamically transformed into only equality constrained QP, which can be solved explicitly. During iterative process (5), with the optimal objective value of the equality constrained QP and the best dual objective value of problem (4) found so far, a CP can be formulated to check whether an  $\epsilon$  primal solution of problem (1) is available under  $\mathcal{A}^k$ .

### 4.1 Transforming Active Inequality Constraints Into Equality Constraints

By considering the inequality constraints contained in  $\mathcal{A}^k$  as equality constraints, problem (1) can be dynamically converted into the only equality

constrained QP (denote  $\mathbf{y}_{\mathcal{A}^k}^*$  its optimizer) as:

$$\mathcal{J}_{\mathcal{A}^k}^* = \min_{\mathbf{y}} \mathcal{J}(\mathbf{y}), \quad (9a)$$

$$s.t. \mathbf{A}_{\mathcal{A}^k} \mathbf{y} = \mathbf{b}_{\mathcal{A}^k}, \quad (9b)$$

where  $\mathbf{A}_{\mathcal{A}^k} = \mathbf{A} \oplus \mathbf{C}_{\mathcal{A}^k}$ , and  $\mathbf{b}_{\mathcal{A}^k} = \mathbf{b} \oplus \mathbf{d}_{\mathcal{A}^k}$ .

**Lemma 1**  $\mathbf{y}_{\mathcal{A}^k}^*$  can be solved by the linear equation group below:

$$\begin{cases} \mathbf{A}_{\mathcal{A}^k} \mathbf{y} = \mathbf{b}_{\mathcal{A}^k}, \\ \mathbf{F}_{\mathcal{A}^k}^T \mathbf{R} \mathbf{y} = \mathbf{0}, \end{cases} \quad (10)$$

where  $\mathbf{F}_{\mathcal{A}^k} \in \mathbb{R}^{ny \times (ny - n_r - c_k)}$  is a orthonormal null space matrix of  $\mathbf{A}_{\mathcal{A}^k}$  satisfying  $\mathbf{A}_{\mathcal{A}^k} \mathbf{F}_{\mathcal{A}^k} = \mathbf{0}$ .

*Proof* By Assumption 2.1,  $\text{rank}(\mathbf{F}_{\mathcal{A}^k}^T) = ny - n_r - c_k$ . Since  $\text{rank}(\mathbf{A}_{\mathcal{A}^k}) = c_k + n_r$  and  $\mathbf{R} \in \mathbb{S}_{++}^{ny}$ , the linear equation group (10) has row rank as  $ny$ , and it has the unique solution.

The feasible set  $Y_{\mathcal{A}^k}$  of problem (9) is characterized as:

$$\begin{aligned} Y_{\mathcal{A}^k} &= \{\mathbf{y} \in \mathbb{R}^{ny} \mid \mathbf{A}_{\mathcal{A}^k} \mathbf{y} = \mathbf{b}_{\mathcal{A}^k}\} \\ &= \{\hat{\mathbf{y}}_{\mathcal{A}^k} + \mathbf{F}_{\mathcal{A}^k} \mathbf{t}_{\mathcal{A}^k} \mid \mathbf{t}_{\mathcal{A}^k} \in \mathbb{R}^{ny - n_r - c_k}\}, \end{aligned} \quad (11)$$

where the second equation is based on any  $\hat{\mathbf{y}}_{\mathcal{A}^k} \in Y_{\mathcal{A}^k}$ .

It is trivial that the solution of the linear equation group is a feasible solution of problem (9) since (9b) is satisfied.

Next, the optimality will be proved. By (11), problem (9) is equivalent to:

$$\mathbf{J}_{\mathcal{A}^k}^* = \min_{\mathbf{t}_{\mathcal{A}^k}} \mathbf{J}(\mathbf{t}_{\mathcal{A}^k}), \quad (12)$$

where  $\mathbf{J}_{\mathcal{A}^k}(\mathbf{t}_{\mathcal{A}^k}) = \frac{1}{2} \|\hat{\mathbf{y}}_{\mathcal{A}^k} + \mathbf{F}_{\mathcal{A}^k} \mathbf{t}_{\mathcal{A}^k}\|_{\mathbf{R}}^2$ , and  $\mathbf{J}_{\mathcal{A}^k}^* = \mathcal{J}_{\mathcal{A}^k}^*$ .

By viewing (12), we have  $\nabla \mathbf{J}_{\mathcal{A}^k}(\mathbf{t}_{\mathcal{A}^k}) = \mathbf{F}_{\mathcal{A}^k}^T \mathbf{R}(\hat{\mathbf{y}}_{\mathcal{A}^k} + \mathbf{F}_{\mathcal{A}^k} \mathbf{t}_{\mathcal{A}^k})$ ,  $\nabla^2 \mathbf{J}_{\mathcal{A}^k}(\mathbf{t}_{\mathcal{A}^k}) = \mathbf{F}_{\mathcal{A}^k}^T \mathbf{R} \mathbf{F}_{\mathcal{A}^k}$ . Next, given  $\hat{\mathbf{y}}_{\mathcal{A}^k} \in Y_{\mathcal{A}^k}$ , then for any  $\mathbf{t}_{\mathcal{A}^k} \in \mathbb{R}^{ny - n_r - c_k}$ , there is a corresponding  $\mathbf{y}_{\mathcal{A}^k}$  to satisfy  $\mathbf{y}_{\mathcal{A}^k} = \hat{\mathbf{y}}_{\mathcal{A}^k} + \mathbf{F}_{\mathcal{A}^k} \mathbf{t}_{\mathcal{A}^k}$ . As a result, we have  $\nabla \mathbf{J}_{\mathcal{A}^k}(\mathbf{t}_{\mathcal{A}^k}) = \mathbf{F}_{\mathcal{A}^k}^T \mathbf{R} \mathbf{y}_{\mathcal{A}^k}$ .

Since the necessary and sufficient optimality condition of unconstrained convex optimization (12) is:  $\|\nabla \mathbf{J}_{\mathcal{A}^k}(\mathbf{t}_{\mathcal{A}^k})\| = 0$ , the proof can be concluded by (10).  $\square$

## 4.2 Active Set Based Suboptimal Method (ASBSM)

To emphasis the main contribution, the suboptimal method will be delineated only for case  $c_k + n_r < ny$ <sup>2</sup> henceforth. Let  $g_{best}^k = \sup_{i \leq k, i \in \mathbb{N}_{>0}} g(\boldsymbol{\theta}^i, \boldsymbol{\lambda}^i)$ . By the primal-dual theory (Boyd and Vandenberghe, 2004), we have  $g_{best}^k \leq \mathcal{J}^*$ . Combining with Definition 2, if  $\mathbf{y} \in Y$ , and satisfies

$$\mathcal{J}(\mathbf{y}) - g_{best}^k \leq \epsilon, \quad (13)$$

<sup>2</sup>The case  $c_k + n_r > ny$  is unsolvable, executing  $\mathcal{A}^k \leftarrow \mathcal{A}^k \setminus \{i \in \mathcal{A}^k \mid \lambda_i^k < \lambda'^k\}$  ( $\lambda'^k$  denote the  $ny$ -th largest value of  $\lambda_i^k$ ,  $i \in \mathcal{A}^k$ ), it can be converted into case  $c_k + n_r = ny$ , in which case the only solution can be obtained by solving linear equation (9b).



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then  $\mathbf{y}$  is an  $\epsilon$  primal solution of problem (1). During the iterations of (5), the use of  $g_{best}^k$  in (13) can reduce the gap between  $g^k$  and  $\mathcal{J}^*$  by taking advantage of the sequence  $\{g^k\}$  generated.

During iterative process (5), depending on the discrepancy between  $\mathcal{A}^k$  and  $\mathcal{A}^*$ , and difference between  $g_{best}^k$  and  $\mathcal{J}^*$ , there are 3 possible relations between  $\mathcal{J}_{\mathcal{A}^k}^*$  and  $g_{best}^k$  given an  $\epsilon > 0$ :

$$\mathcal{J}_{\mathcal{A}^k}^* < g_{best}^k, \quad (14)$$

$$0 \leq \mathcal{J}_{\mathcal{A}^k}^* - g_{best}^k \leq \epsilon, \quad (15)$$

$$\epsilon < \mathcal{J}_{\mathcal{A}^k}^* - g_{best}^k. \quad (16)$$

Note that in case (16), no deterministic  $\epsilon$  suboptimality criterion can be derived without knowing  $\mathcal{A}^*$ , since even  $\mathcal{J}_{\mathcal{A}^k}^*$  fails (13).

The following 2 propositions will be used to build criteria for  $\epsilon$  primal solutions when (14) or (15) is satisfied.

**Proposition 1** *Given  $\Delta > 0$ , if  $\mathbf{y}$  satisfies (9b) and  $\|\mathbf{F}_{\mathcal{A}^k}^T \mathbf{R}\mathbf{y}\| \leq (2\beta_{\mathcal{A}^k}\Delta)^{\frac{1}{2}}$ , where  $\beta_{\mathcal{A}^k} = \min \text{eig}(\mathbf{F}_{\mathcal{A}^k}^T \mathbf{R}\mathbf{F}_{\mathcal{A}^k})$ , then it must hold that:*

$$\mathcal{J}(\mathbf{y}) - \mathcal{J}_{\mathcal{A}^k}^* \leq \Delta. \quad (17)$$

*Proof* As  $\mathbf{y}$  satisfies (9b), it is a feasible solution of problem (9). Since problem (12) is convex and unconstrained, and  $\nabla^2 \mathcal{J}_{\mathcal{A}^k}(\mathbf{t}_{\mathcal{A}^k}) = \mathbf{F}_{\mathcal{A}^k}^T \mathbf{R}\mathbf{F}_{\mathcal{A}^k} \succeq \beta_{\mathcal{A}^k} \mathbf{I}$ , (17) holds by applying (9.10) of (Boyd and Vandenberghe, 2004).  $\square$

Now, we consider below a CP with optimizer  $(\mathbf{y}_{sub}, s_{\mathcal{I}^k}^1, h^1)$ :

$$\min_{\mathbf{y}, s_{\mathcal{I}^k}, h} -h \quad (18a)$$

$$\text{s.t. (9b), (8e), (8g), (8h),}$$

$$\|\mathbf{F}_{\mathcal{A}^k}^T \mathbf{R}\mathbf{y}\| \leq (2\beta_{\mathcal{A}^k}(\epsilon + g_{best}^k - \mathcal{J}_{\mathcal{A}^k}^*))^{\frac{1}{2}}. \quad (18b)$$

**Proposition 2** *If CP (18) has a solution and*

$$\mathcal{J}_{\mathcal{A}^k}^* - g_{best}^k \leq \epsilon, \quad (19)$$

*then  $\mathbf{y}_{sub}$  is an  $\epsilon$  primal solution of problem (1).*

*Proof* We first prove for case (14). Let  $\delta^k = g_{best}^k - \mathcal{J}_{\mathcal{A}^k}^*$ . As  $\mathbf{y}_{sub}$  satisfies (9b), (8e), (8g), (8h), we have  $\mathbf{A}_{\mathcal{A}^k} \mathbf{y}_{sub} = \mathbf{b}_{\mathcal{A}^k}$ , and  $\mathbf{C}_{\mathcal{I}^k} \mathbf{y}_{sub} \leq \mathbf{d}_{\mathcal{I}^k}$ . Since at each iteration,  $\mathbf{C}$  consists of  $\mathbf{C}_{\mathcal{I}^k}$  and  $\mathbf{C}_{\mathcal{A}^k}$ , then by (2),  $\mathbf{y}_{sub} \in Y$ . In the continuation, by the primal-dual theory (Boyd and Vandenberghe, 2004), we have

$$0 \leq \mathcal{J}(\mathbf{y}_{sub}) - g_{best}^k. \quad (20)$$

As (18b) is satisfied by  $\mathbf{y}_{sub}$ , using Proposition 1 we have:

$$0 \leq \mathcal{J}(\mathbf{y}_{sub}) - g_{best}^k \leq \epsilon. \quad (21)$$

As a consequence, combining  $\mathbf{y}_{sub} \in Y$ , (20) and (21),  $\mathbf{y}_{sub}$  is an  $\epsilon$  primal solution of problem (1) by (13).

Second, consider case (15), let  $\Delta^k = \mathcal{J}_{\mathcal{A}^k}^* - g_{best}^k$ , by (17) it holds that:  $\mathcal{J}(\mathbf{y}) - \mathcal{J}_{\mathcal{A}^k}^* \leq \epsilon - \Delta^k$ . Namely,  $\mathcal{J}(\mathbf{y}) - g_{best}^k \leq \epsilon$ . The rest of the proof remains the same with that of case (14).  $\square$

In a cost ascending order, ASBSM (detailed in Alg.2) presents how to compute an  $\epsilon$  primal solution<sup>3</sup> of problem (1) with  $\mathcal{A}^k$  and  $\mathbf{y}_{\mathcal{A}^k}^*$  updated at  $k$ -th iteration of (5): firstly, check whether  $\mathbf{y}_{\mathcal{A}^k}^*$  is an  $\epsilon$  primal solution; if not, check whether  $\mathcal{A}^k$  is  $\mathcal{A}^*$  by (8); if not, then check whether an  $\epsilon$  primal solution can be found by CP (18) using  $g_{best}^k$ .

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### Algorithm 2 Active Set Based Suboptimal Method (ASBSM)

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- 1: Initialize:  $\boldsymbol{\theta}^{-1}$ ,  $\boldsymbol{\lambda}^{-1}$ ,  $k = 0$  and  $\epsilon > 0$ .  $\mathbf{y}^{-1}$  is obtained by (5c).
  - 2: **repeat**
  - 3:   Update primal and dual variables by (5), update  $\mathcal{A}^k$  by (6), compute  $\mathbf{y}_{\mathcal{A}^k}^*$  by (10)
  - 4:   **if**  $\mathbf{y}_{\mathcal{A}^k}^* \in Y$  **then**
  - 5:     **if**  $\mathbf{y}_{\mathcal{A}^k}^*$  satisfies (13) **then return**  $\mathbf{y}_{\mathcal{A}^k}^*$
  - 6:     **end if**
  - 7:   **end if**
  - 8:   **if** LP (8) has a solution for  $\mathcal{A}^k$  **then return**  $\mathbf{y}^*$
  - 9:   **end if**
  - 10:   **if**  $\mathbf{y}_{\mathcal{A}^k}^*$  satisfies (19) **then**
  - 11:     **if** CP (18) has a solution **then return**  $\mathbf{y}_{sub}$
  - 12:     **end if**
  - 13:   **end if**
  - 14:    $k \leftarrow k + 1$
  - 15: **until** one of  $\mathbf{y}_{\mathcal{A}^k}^*$ ,  $\mathbf{y}^*$  and  $\mathbf{y}_{sub}$  is returned
- 

## 4.3 Optimization Properties of ASBSM

From here, we will use the following 2 lemmas to derive the lower bound of  $\epsilon$  such that for any  $\epsilon$  above the bound, ASBSM can be terminated with finite iterations.

**Lemma 2** *Under Assumption 2.2, in implementing ASBSM, (19) can be satisfied with finite iterations  $\forall \epsilon > 0$ .*

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<sup>3</sup>The optimal solution of problem (1) is trivially an  $\epsilon$  primal solution of problem (1).

*Proof* First, consider the following problem

$$\delta = \inf_{\mathbf{y} \in \mathcal{I}^*} \{ \min_{\mathbf{y}} \|\mathbf{y} - \mathbf{y}^*\| \mid \mathbf{C}_i \mathbf{y} = \mathbf{d}_i \}.$$

As  $\mathbf{y}^k$  asymptotically converges to  $\mathbf{y}^*$  by Assumption 2.2, there exists a  $k_1$  such that  $\forall k \geq k_1$ ,  $\|\mathbf{y}^k - \mathbf{y}^*\| \leq \delta$ . So, for  $\mathcal{A}^k$  generated by (6),  $\forall k \geq k_1$ , we have  $\mathcal{I}^* \cap \mathcal{A}^k = \emptyset$ . Therefore, we have  $\mathcal{A}^k \subset \mathcal{A}^*$ ,  $\forall k \geq k_1$ , which means  $Y$  is a proper subset of the feasible set of problem (9), thus we have:

$$\mathcal{J}_{\mathcal{A}^k}^* \leq \mathcal{J}^*, \quad \forall k \geq k_1. \quad (22)$$

Given a  $\epsilon > 0$ , there exists a  $k_2$  by Remark 1 such that

$$\mathcal{J}^* - g_{best}^k \leq \epsilon, \quad \forall k \geq k_2. \quad (23)$$

Consequently, combining (22) and (23), we have

$$\mathcal{J}_{\mathcal{A}^k}^* - g_{best}^k \leq \epsilon, \quad \forall k \geq \max\{k_1, k_2\}. \quad (24)$$

And this completes the proof.  $\square$

Here, we consider the following norm minimization:

$$\bar{\delta} = \min_{\mathbf{y}} \|\mathbf{F}_{\mathbf{A}}^T \mathbf{R} \mathbf{y}\|, \quad s.t. \mathbf{y} \in Y,$$

where  $\mathbf{F}_{\mathbf{A}} \in \mathbb{R}^{ny \times (ny - n_r)}$  is an orthonormal null space matrix of  $\mathbf{A}$  satisfying  $\mathbf{A} \mathbf{F}_{\mathbf{A}} = \mathbf{0}$ .

**Lemma 3** *Under Assumption 2.2, an  $\epsilon$  primal solution of problem (1) can be generated with finite iterations in implementing ASBSM  $\forall \epsilon > \bar{\delta}/2\beta_{\mathbf{A}}$ , where  $\beta_{\mathbf{A}} = \min \text{eig}(\mathbf{F}_{\mathbf{A}}^T \mathbf{R} \mathbf{F}_{\mathbf{A}})$ .*

*Proof* By Lemma 2, given an arbitrary  $\epsilon' > 0$ , there exists a  $k$  such that

$$\mathcal{J}_{\mathcal{A}^k}^* - g_{best}^k \leq \epsilon'. \quad (25)$$

Next, consider the problem:

$$\delta^k = \min_{\mathbf{y}} \|\mathbf{F}_{\mathcal{A}^k}^T \mathbf{R} \mathbf{y}\|, \quad s.t. (9b), \mathbf{C}_{\mathcal{I}^k} \mathbf{y} \leq \mathbf{d}_{\mathcal{I}^k}.$$

Observing (18b), for  $\epsilon^k = (\delta^k)^2/2\beta_{\mathcal{A}^k} + \mathcal{J}_{\mathcal{A}^k}^* - g_{best}^k$ , an  $\epsilon^k$  suboptimal solution of problem (1) can be found by solving CP (18).

By (25), a  $((\delta^k)^2/2\beta_{\mathcal{A}^k} + \epsilon')$  suboptimal solution of problem (1) can be generated with at most  $k$  iterations. If it can be shown that

$$(\delta^k)^2/2\beta_{\mathcal{A}^k} \leq \bar{\delta}^2/2\beta_{\mathbf{A}}, \quad (26)$$

then for  $\epsilon = \bar{\delta}/2\beta_{\mathbf{A}} + \epsilon'$ , an  $\epsilon$  primal solution of problem (1) can be generated with at most  $k$  iterations. Since  $\epsilon'$  can be arbitrarily small, then an  $\epsilon$  primal solution of problem (1) can be generated with finite iterations  $\forall \epsilon > \bar{\delta}/2\beta_{\mathbf{A}}$ .

We give the proof of (26) from here. When  $\mathbf{A}_{\mathcal{A}^k} = \mathbf{A}$ , (26) trivially holds. When  $\mathbf{A}_{\mathcal{A}^k} \neq \mathbf{A}$ , which indicates  $\mathbf{A}_{\mathcal{A}^k} = \mathbf{A} \oplus \mathbf{C}_{\mathcal{A}^k}$ . Subsequently, by  $\mathbf{A} \mathbf{F} = \mathbf{0}$ ,  $\mathbf{A} \mathbf{F}_{\mathcal{A}^k} = \mathbf{0}$ , and  $\mathbf{C}_{\mathcal{A}^k} \mathbf{F}_{\mathcal{A}^k} = \mathbf{0}$ , the null space of  $\mathbf{A}_{\mathcal{A}^k}$  is a subspace of the null space of  $\mathbf{A}$ . Since  $\mathbf{F} \in \mathbb{R}^{ny \times (ny - n_r)}$  and  $\mathbf{F}_{\mathcal{A}^k} \in \mathbb{R}^{ny \times (ny - n_r - c_k)}$ , there exists a semi-orthogonal

matrix  $P \in \mathbb{R}^{(ny-n_r) \times (ny-n_r-c_k)}$  with  $P^T P = \mathbf{I}$ , such that  $\mathbf{F}P = \mathbf{F}_{\mathcal{A}^k}$ . It follows that  $\mathbf{F}_{\mathcal{A}^k}^T \mathbf{R} \mathbf{F}_{\mathcal{A}^k} = P^T \mathbf{F}^T \mathbf{R} \mathbf{F} P$ . Then by Poincaré separation theorem (Magnus and Neudecker, 2019), we have

$$\beta_{\mathbf{A}} \leq \beta_{\mathcal{A}^k}. \quad (27)$$

Then we have:

$$\begin{aligned} \|\mathbf{F}_{\mathcal{A}^k}^T \mathbf{R} \mathbf{y}\| &= \|P^T \mathbf{F}^T \mathbf{R} \mathbf{y}\| \\ &\leq \|P^T\| \|\mathbf{F}^T \mathbf{R} \mathbf{y}\| = \|\mathbf{F}^T \mathbf{R} \mathbf{y}\|, \forall \mathbf{y} \in Y \end{aligned} \quad (28)$$

where the inequality uses Cauchy-Schwarz inequality, and the second equality uses the property of semi-orthogonal matrix that  $\|P^T\| = \|P\| = 1$ .

Finally, we can conclude (26) by (27) and (28), and this completes the proof.  $\square$

**Remark 2** Note that  $\epsilon > \bar{\delta}/2\beta_{\mathbf{A}}$  is a sufficient condition for ASBSM to be terminated with finite iterations. In practice, it is possible to take  $\epsilon$  much lower than  $\bar{\delta}/2\beta_{\mathbf{A}}$ , which will be illustrated with a numerical example in Section 5.

**Remark 3** Suppose that POASIM is terminated at  $k$ -th iteration, since POASIM and ASBSM use the same iterative process (5), and the step 8 in ASBSM tests if  $\mathcal{A}^k$  is  $\mathcal{A}^*$ , ASBSM can also be terminated at  $k$ -th iteration with  $\mathbf{y}^*$  returned. Alternatively, if one of  $\mathbf{y}_{\mathcal{A}^k}^*$  or  $\mathbf{y}_{sub}$  is returned prior to  $k$ -th iteration, ASBSM can be terminated with fewer iterations than  $k$ . In conclusion, if POASIM can be terminated with finite iterations, ASBSM can be terminated using the same or fewer iterations.

In terms of ASBSM, economical treatments as follows can further improve the efficiency of computation:

1. for each distinct  $\mathcal{A}^k$ , its corresponding variables  $\mathbf{C}_{\mathcal{A}^k}$ ,  $\mathbf{d}_{\mathcal{A}^k}$ ,  $\mathbf{C}_{\mathcal{I}^k}$ ,  $\mathbf{d}_{\mathcal{I}^k}$ ,  $\mathbf{y}_{\mathcal{A}^k}^*$ ,  $\mathbf{F}_{\mathcal{A}^k}$  and  $\mathcal{J}_{\mathcal{A}^k}^*$  can be stored, then if  $\mathcal{A}^{k+i} = \mathcal{A}^k$ ,  $i = 1, 2, \dots$ , the above mentioned variables can be retrieved from the stored data, instead to compute from the scratch<sup>4</sup>;
2.  $\mathcal{A}^k$  used in Step 11 of ASBSM is required to have not been tested by CP (18) before, which can avoid repeated test of CP (18) under the same  $\mathcal{A}^k$ ;
3. at each iteration, if  $\mathbf{y}_{\mathcal{A}^k}^*$  satisfies (13) and  $\mathbf{y}_{\mathcal{A}^k}^* \in Y$ , then ASBSM can be terminated without excess computation. In addition, to avoid unnecessary solving of CP (18), for each distinct  $\mathbf{y}_{\mathcal{A}^k}^*$  that satisfies  $\mathbf{y}_{\mathcal{A}^k}^* \in Y$  and  $\mathcal{J}(\mathbf{y}) - g_{best}^{\bar{k}} > \epsilon$ , denoting  $D_{\mathcal{A}^k} = \mathcal{J}_{\mathcal{A}^k}^* - \epsilon$ . Then for every  $\bar{k} > k$ , one can claim that  $\mathbf{y}_{\mathcal{A}^k}^*$  is an  $\epsilon$  primal solution of problem (1) if

$$g_{best}^{\bar{k}} \geq D_{\mathcal{A}^k}. \quad (29)$$

## 5 Numerical Experiments

For numerical experiments, the Nesterov gradient descent (Nesterov, 1983) (Giselsson et al, 2013), which has been proved to be the best gradient method

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<sup>4</sup>This treatment is also applied to POASIM.

**Table 1:** Performance comparison among the Nesterov gradient descent, POASIM, and ASBSM of a single test with  $ny=10$ , and relative suboptimality<sup>5</sup> as  $1 \times 10^{-2}$ 

	Nesterov		POASIM	ASBSM
	$\epsilon^*$	$\epsilon + Y^{**}$		
Computation time (s)	$5.8 \times 10^{-3}$	$4.3 \times 10^{-1}$	$8.9 \times 10^{-2}$	$1.7 \times 10^{-2}$
# of iterations or (8) solved	15	74803	20	15
Primal feasible	no	yes	yes	yes

\* solved by (5a)-(5b), a posterior  $\epsilon$  suboptimality criterion is used:  $\mathcal{J}^* - g(\theta^k, \lambda^k) \leq 0.01\mathcal{J}^*$ , where  $\mathcal{J}^*$  is known as a parameter.

\*\* solved by (5a)-(5b),  $\epsilon$  suboptimality criterion:  $\mathcal{J}^* - g(\theta^k, \lambda^k) \leq 0.01\mathcal{J}^*$ , feasibility criteria:  $\|\mathbf{A}\mathbf{y} - \mathbf{b}\| \leq 1 \times 10^{-16}$ ,  $\mathbf{C}\mathbf{y} - \mathbf{d} \leq 1 \times 10^{-16}$ . (Because the numerical results of  $\|\mathbf{A}\mathbf{y}^* - \mathbf{b}\|$  and  $\max\{0, \mathbf{C}\mathbf{y}^* - \mathbf{d}\}$  are of magnitude  $10^{-16}$ .)

in (Nesterov, 2018), is adopted for iteration (5a) (5b) as follows.

$$\begin{aligned}\theta^{k+1} &= \hat{\theta}^k + \frac{1}{L}(\mathbf{A}\hat{\mathbf{y}}^k - \mathbf{b}), \\ \lambda^{k+1} &= \max\{0, \hat{\lambda}^k + \frac{1}{L}(\mathbf{C}\hat{\mathbf{y}}^k - \mathbf{d})\},\end{aligned}$$

where for a vector  $\nu$ ,  $\hat{\nu}^k = \nu^k + \frac{k-1}{k+2}(\nu^k - \nu^{k-1})$ ,  $L = \|\mathbf{E}\mathbf{R}^{-1}\mathbf{E}^T\|_2$ , and  $\mathbf{E} = \mathbf{A} \oplus \mathbf{C}$ .

Specifically, we carry out 2 groups of experiments: (i), single small size problem for a clear-cut comparison of time and iteration number among POASIM, ASBSM, and the Nesterov gradient descent; (ii), 1000 randomly generated problems (of size  $ny = 100$ ) for general performance comparison between POASIM and ASBSM. All numerical experiments are carried out using MATLAB 2020b on a Windows 10 PC with 2.20 GHz Core i7-8750H CPU and 16GB RAM. The complete datasets generated in the numerical experiments are available at <https://github.com/SettingTheWorld/epsilon-Suboptimality-QP>.

In detail, we use  $0 \leq \mathbf{y} \leq \mathbf{1}$  for inequality constraints (1c), and set  $n_e = ny/2$ , and  $\mathbf{R} = \mathbf{I}$ . The sparsity concerning matrix  $\mathbf{A}$  is randomly drawn from uniform distribution (0, 1) of each problem, and each non zero entry of  $\mathbf{A}$  is randomly drawn from uniform distribution (-0.5, 0.5), and the  $i$ -th element of  $\mathbf{b}$  is randomly drawn from uniform distribution (0,  $\mathbf{A}_i \cdot \mathbf{1}$ ) to ensure that  $Y \neq \emptyset$ , where  $\mathbf{A}_i$  denote the  $i$ -th row of  $\mathbf{A}$ .

## 5.1 Single Test among 3 Methods

In this subsection, a small size ( $ny=10$ ) test is carried out to present an intuitive comparison among POASIM, ASBSM, and the Nesterov gradient descent, Table. 1 shows that when  $\epsilon$  suboptimality is only concerned, the Nesterov gradient descent discloses evident superiority in iteration number and

**Table 2:** Average and maximal relative error of ASBSM of 1000 random tests with  $ny = 100$ 

Predefined Rel. Subopt.	$10^{-4}$	$10^{-3}$	$10^{-2}$	$10^{-1}$
Ave. Rel. Error	0.15	0.06	0.21	0.25
Max. Rel. Error	0.81	0.91	0.84	0.78

time. However, if feasibility is required, the Nesterov gradient descent requires 5000 times more iterations, resulting in worse performance than POASIM and ASBSM. By contrast, POASIM and ASBSM reveal more favorable results in iteration number and computation time. They hence will be investigated further with randomly generated problems of larger size in the following subsection.

Specifically, by Lemma 3 we have  $\bar{\delta}/2\beta_{\mathcal{A}} = 0.0348$  in this test. Then let  $\epsilon = \bar{\delta}/2\beta_{\mathcal{A}}$ , the corresponding relative suboptimality is 0.0424, which is a sufficient condition for ASBSM to be terminated with finite iterations. In fact, it is quite reasonable to take much smaller  $\epsilon$  in practice, e.g. in the next subsection, predefined relative suboptimality will be set as small as 0.0001, and all the tests can generate  $\epsilon$  primal solution with in finite iterations.

## 5.2 Random Tests between POASIM and ASBSM

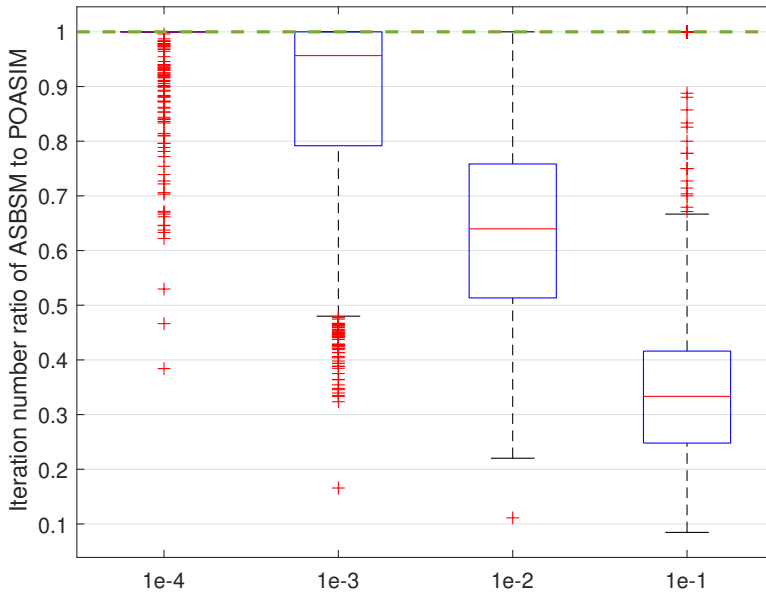
In this subsection, we use 1000 independent randomly generated QP to test POASIM and ASBSM; for each problem, ASBSM is tested under 4 different relative suboptimality: 0.0001, 0.001, 0.01 and 0.1. For each random QP, we set  $ny = 100$ , and the sparsity of matrix  $\mathcal{A}$  is randomly drawn from uniform distribution (0, 1).

Table 2 shows that the predefined suboptimality of all random tests is fulfilled. And the average relative error, in general, is significantly lower than the maximum relative error of all tests.

From Fig. 1, as predefined relative suboptimality increases from  $1 \times 10^{-4}$  to  $1 \times 10^{-1}$ , the boxplot of iteration number ratio of ASBSM to POASIM declines steadily. Since the higher suboptimality, the higher tolerance of incorrectness of  $\mathcal{A}^k$ , thus the higher possibility for (14) or (15) to occur (the gap between  $g_{best}^k$  and  $\mathcal{J}^*$  becomes more tolerated). This interpretation is validated in Table. 3, where the average number of CP (18) calculated and the number of  $\mathbf{y}_{sub}$  grows dramatically as predefined relative suboptimality increases.

In what follows, we point out some noteworthy statistics related to time performance: solving CP (18) generally consumes  $4 - 5 \times 10^4$  more time than one iterate of (30); the solving time ratio of CP (18) to LP (8) is invariant (ranges from 3.1 to 5) to suboptimality setting.

In terms of computation time analysis, the more CP (18) calculated, the longer time ASBSM consumed. Because solving of CP (18) in ASBSM starts only after the LP (8) is failed, resulting 2 solving processes for one  $\mathcal{A}^k$ . Nevertheless, as shown in Fig. 2, ASBSM outperform POASIM statistically on time performance, even though the superiority is blunted somewhat under  $1 \times 10^{-2}$



**Fig. 1:** Iteration number ratio of ASBSM to POASIM of 1000 random tests with  $n_y = 100$  (Sample value exceeded  $\pm 2.7\sigma$  shows as whisker, same setting for other box plots. Sample value less, greater than or equals to 1 (green horizontal line) means that ASBSM consumes less, more or the same iteration as POASIM in the same test. The lower value, the better performance of ASBSM.)

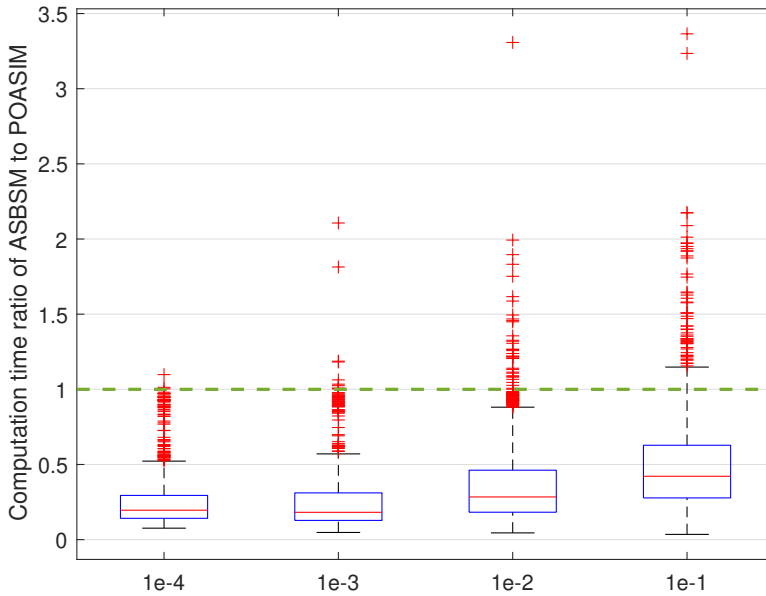
and  $1 \times 10^{-1}$ . Note that the considerable number of  $\mathbf{y}_{\mathcal{A}^k}^*$  and  $\mathbf{y}^*$  (the first 3 columns of Table. 3) are returned as  $\epsilon$  primal suboptimal solution, which leads to time superiority of ASBSM, as it spares the effort of solving CP (18).

**Table 3:** Statistics of 1000 random tests with  $n_y = 100$ : termination by different approaches in ASBSM and calculation number of CP

Predefined Rel. Subopt.	Total # returned				Ave. # of CP Calculated
	$D_{\mathcal{A}^k}$	$\mathbf{y}_{\mathcal{A}^k}^*$	$\mathbf{y}^*$	$\mathbf{y}_{sub}$	
$10^{-4}$	51*	24**	899	26	0.039
$10^{-3}$	210	193	379	218	0.270
$10^{-2}$	109	126	68	697	0.972
$10^{-1}$	27	31	49	893	2.079

\*  $D_{\mathcal{A}^k}$  means that the  $\epsilon$  primal solution returned is  $\mathbf{y}_{\mathcal{A}^k}^*$ , which is found by (29).

\*\*  $\mathbf{y}_{\mathcal{A}^k}^*$  means that the  $\epsilon$  primal solution returned is  $\mathbf{y}_{\mathcal{A}^k}^*$ , which is found by Step 5 in ASBSM.



**Fig. 2:** Ratio of ASBSM to POASIM on computation time of 1000 random tests with  $n_y = 100$  (Sample value less, greater than or equals to 1 (green horizontal line) means that ASBSM spends less, more or the same computation time as POASIM in the same test. The lower value, the better performance of ASBSM.)

## 6 Conclusion

In this paper, combining the primal-dual gradient method and the KKT conditions, we have proposed a proactive method (POASIM) to solve QP by checking whether the dynamically updated active set is optimal during the iterative process. The proactive method can find the optimal solution once the optimal active set is identified, making the iteration number needed fewer than the conventional primal-dual gradient method. Aiming at even fewer iterations required and less computation time consumed, we have turned to search for feasible solutions with guaranteed suboptimality and have then proposed a suboptimal method (ASBSM) based on cone programming. The suboptimal method can be considerably beneficial when the optimal active set is prohibitive to identify during the iterative process. In addition, we have demonstrated the lower bound of suboptimality for ASBSM to be terminated with finite iterations.

Through random numerical experiments, the  $\epsilon$ -suboptimality and feasibility have been verified for the suboptimal method, which has moreover revealed



statistical improvement of computation time and iteration number compared to the proactive method.

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