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FUNCTIONS RELATED TO JACOBI THETA FUNCTIONS AND APPLICATIONS II

ABD RAOUF CHOUIKHA

ABSTRACT. This paper is devoted to a representation of the Jacobi theta functions $\theta_j(v, \tau)$ as a product $g_j(v, \tau)g_j(-v, \tau)$, $j = 1, 2, 3, 4$ where the complex functions of two variables are defined by infinite products for $Imv < \frac{1}{2}Im\tau$. These functions $g_j(v, \tau)$ which are related between them have interesting properties. In particular, they permit to express the q -analogues trigonometric functions $\sin_q(v), \cos_q(v)$ as infinite products. Moreover, all these functions satisfied duplication formula and multiplication by an integer n (not necessary odd).

1. INTRODUCTION

1.1. Jacobi theta functions and the $g_j(v, \tau)$ functions.

To carry out our investigations, we need to introduce the Jacobi theta functions. The Jacobi theta functions for $j = 1, 2, 3, 4$ are defined as ([7],[8],[13])

$$\theta_1(v, \tau) = 2 \sum_{n \geq 0} (-1)^n q^{(n+\frac{1}{2})^2} \sin((2n+1)\pi v)$$

$$\theta_2(v, \tau) = 2 \sum_{n \geq 0} q^{(n+\frac{1}{2})^2} \cos((2n+1)\pi v)$$

$$\theta_3(v, \tau) = 1 + 2 \sum_{n \geq 1} q^{n^2} \cos(2n\pi v)$$

$$\theta_4(v, \tau) = 1 + 2 \sum_{n \geq 1} (-1)^n q^{n^2} \cos(2n\pi v)$$

These four theta functions can be extended to complex values for v and $q = e^{i\pi\tau}$ such that $|q| < 1$ or $\Im\tau : Im(\tau) > 0$.

These series converge for all complex values of v whenever τ has positive imaginary part. Moreover, these series converge absolutely and uniformly on compact subsets. Theta functions are entire and periodic functions of v , [1], [2], [8].

Using the triple product identity, we can derive the infinite product representations of the Jacobi theta functions [8, (19), p.357] :

$$\theta_1(v, \tau) = 2q_0 q^{1/4} \sin \pi v \prod_{n \geq 1} [1 - 2q^{2n} \cos 2\pi v + q^{4n}],$$

$$\theta_2(v, \tau) = 2q_0 q^{1/4} \cos \pi v \prod_{n \geq 1} [1 + 2q^{2n} \cos 2\pi v + q^{4n}],$$

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$$\theta_3(v, \tau) = q_0 \prod_{n \geq 1} [1 + 2q^{2n-1} \cos 2\pi v + q^{4n-2}],$$

$$\theta_4(v, \tau) = q_0 \prod_{n \geq 1} [1 - 2q^{2n-1} \cos 2\pi v + q^{4n-2}],$$

where $q_0 = \prod_{n \geq 1} [1 - q^n]$.

Corollary 3-3 below will give another infinite product representations of the Jacobi theta functions.

We will consider for that the following functions

$$g_4(v, \tau) = \prod_{k \geq 0} \frac{\sin[(k + \frac{1}{2})\pi\tau + \pi v]}{\sin[(k + \frac{1}{2})\pi\tau]}, \quad g_3(v, \tau) = \prod_{k \geq 0} \frac{\cos[(k + \frac{1}{2})\pi\tau + \pi v]}{\cos[(k + \frac{1}{2})\pi\tau]},$$

$$g_1(v, \tau) = \prod_{k \geq 1} \frac{\sin[k\pi\tau + \pi v]}{\sin[k\pi\tau]}, \quad g_2(v, \tau) = \prod_{k \geq 1} \frac{\cos[k\pi\tau + \pi v]}{\cos[k\pi\tau]}$$

where $v \in \mathbb{C}$ and τ belongs to the half plane ($\Im\tau : \text{Im}(\tau) > 0$).

These functions are connected to theta functions (Propositions 3-1, 3-2):

$$\theta_4(v, \tau) = \theta_4(0, \tau) g_4(v, \tau) g_4(-v, \tau),$$

$$\theta_3(v, \tau) = \theta_3(0, \tau) g_3(v, \tau) g_3(-v, \tau),$$

$$\theta_1(v, \tau) = (\pi \sin(\pi v) \theta_1'(0, \tau) g_1(v, \tau) g_1(-v, \tau),$$

$$\theta_2(v, \tau) = \cos(\pi v) \theta_2(0, \tau) g_2(v, \tau) g_2(-v, \tau).$$

Furthermore, we had considered in [4] another representation of the Jacobi theta functions by mean of $f_i(v, \tau), i = 1, 2, 3, 4$. which appear to have remarkable properties. These functions are naturally connected to $g_i(v, \tau), i = 1, 2, 3, 4$:

$$\frac{f_1(v, \tau)}{g_1(v, \tau)} = \prod_{k \geq 1} \frac{\sin \frac{[k\pi\tau - \pi v]}{2}}{\sin \frac{[k\pi\tau + \pi v]}{2}}, \quad \frac{f_4(v, \tau)}{g_4(v, \tau)} = \prod_{k \geq 0} \frac{\sin \frac{[(k + \frac{1}{2})\pi\tau - \pi v]}{2}}{\sin \frac{[(k + \frac{1}{2})\pi\tau + \pi v]}{2}},$$

$$\frac{f_2(v, \tau)}{g_2(v, \tau)} = \prod_{k \geq 1} \frac{\sin \frac{[k\pi(\tau+1) - \pi v]}{2}}{\sin \frac{[k\pi(\tau+1) + \pi v]}{2}}, \quad \frac{f_3(v, \tau)}{g_3(v, \tau)} = \prod_{k \geq 0} \frac{\sin \frac{[(k + \frac{1}{2})\pi(\tau+1) - \pi v]}{2}}{\sin \frac{[(k + \frac{1}{2})\pi(\tau+1) + \pi v]}{2}}.$$

1.2. The aim of the paper.

The purpose of this paper is to bring out some intrinsic properties of these functions $g_j(v, \tau)$. We will see in the sequel these functions which are related between them seem play a particular role.

On the other hand, we derive the Fourier series expansions for $g_j(v, \tau)$, as well as for $\log(g_j(v, \tau))$. Some other significant properties and equalities will be derived.

We also provide a n -order transformation formulas for these functions and find again those for the Jacobi theta functions.

The last part is devoted to applications in q -trigonometry introduced by Gosper [8]. We will firstly express $\sin_q(v), \cos_q(v)$ as infinite products :

$$\sin_q(v) = (\sin \pi v) \prod_{k \geq 1} \frac{\sin(\pi v + \frac{k\pi}{\tau}) \sin(\frac{k\pi}{\tau} - \pi v)}{(\cos \frac{k\pi}{\tau})^2},$$

$$\cos_q(v) = (\cos \pi v) \prod_{k \geq 1} \frac{\cos(\pi v + \frac{k\pi}{\tau}) \cos(\pi v - \frac{k\pi}{\tau})}{(\cos \frac{k\pi}{\tau})^2},$$

for $|Imv| < \Im\tau$.

All these representations permits to provide some properties and identities, notably for duplication and n -transformations.

1.3. Connection with functions introduced by Eisenstein.

These $g_j(v, \tau)$ functions allowing a representation of theta do not seem to have been explicitly mentioned in the literature. However, a few allusive remarks lead one to believe that this approach has been considered.

Indeed, Eisenstein ([6, p.258] or [12, p.94-116]) mentioned the double product function defined as

$$\phi(v) = v \prod'_{m,k} \left(1 - \frac{v}{m + k\tau}\right),$$

the prime symbol means that the term corresponding to $m = k = 0$ is to be excluded.

This author established a connection between his product function $\phi(v)$ and the theta function

$$\phi(v) = \frac{2}{\pi \theta'_1(0, \tau)} \theta_1(v, \tau)$$

where $\theta'_1(v, \tau) = \frac{\delta \theta}{\delta v}(v, \tau)$.

Eisenstein observed that the infinite product representation of sinus function

$$\sin(\pi v) = \pi v \lim_{M \rightarrow \infty} \prod_{-M \leq m \leq M} \left(1 + \frac{v}{m}\right)$$

would imply

$$\prod'_{m=-\infty}^{\infty} \left(1 - \frac{v}{m + k\tau}\right) = \frac{\sin(k\pi\tau - \pi v)}{\sin(k\pi\tau)}.$$

Therefore one gets the expression

$$\phi(v) = v \prod_{k=-\infty}^{\infty} \frac{\sin(k\pi\tau - \pi v)}{\sin(k\pi\tau)}.$$

The function $g_1(v, \tau)$ we will consider is such that

$$\phi(v) = \sin(\pi v) g_1(v, \tau) g_1(-v, \tau).$$

On the other hand, following Eisenstein [5, p.275-280] who consider the theta product

$$\xi(e^{i\pi v}) = (e^{i\pi v} - e^{-i\pi v}) \prod_{k \geq 1} (1 - q^{2k} e^{2i\pi v})(1 - q^{2k} e^{-2i\pi v}),$$

so that we may write

$$f(u, v) = \prod_{k \geq 1} \prod_m \left(1 - \frac{v}{m + x\tau + x}\right) = \frac{\xi(e^{i\pi(v-u)})}{\xi(e^{i\pi v})}.$$

We then obtain (Proposition 2-7 below)

$$g_1(v, \tau) = \prod_{k \geq 1} \prod_m \left(1 + \frac{v}{m + k\tau} \right) = \lim_{u \rightarrow 0} [-uf(u, v)].$$

Analogous expressions may be found for the other $g_j(v, \tau)$, $j = 2, 3, 4$ since they are related.

2. FUNCTIONS DEFINED BY AN INFINITE PRODUCT

Consider the complex functions of two variables g_j , $j = 1, 2, 3, 4$ defined by the infinite products

$$g_4(v, \tau) = \prod_{k \geq 0} \frac{\sin[(k + \frac{1}{2})\pi\tau + \pi v]}{\sin[(k + \frac{1}{2})\pi\tau]}, \quad g_3(v, \tau) = \prod_{k \geq 0} \frac{\cos[(k + \frac{1}{2})\pi\tau + \pi v]}{\cos[(k + \frac{1}{2})\pi\tau]},$$

$$g_1(v, \tau) = \prod_{k \geq 1} \frac{\sin[k\pi\tau + \pi v]}{\sin[k\pi\tau]}, \quad g_2(v, \tau) = \prod_{k \geq 1} \frac{\cos[k\pi\tau + \pi v]}{\cos[k\pi\tau]}$$

where $v \in C$ and τ belongs to the half plane ($\Im\tau : \text{Im}(\tau) > 0$).

These functions are naturally related. Replacing v by $v + 1, v + \frac{1}{2}$ and τ by $\tau + 1, \tau + \frac{1}{2}$ we then obtain

Proposition 2-1 *The above function $g_j(v, \tau)$ verifies the following properties for $\Im\tau > 0$, $\text{Im}v < \frac{1}{2}\Im\tau$*

$$\text{For } j = 1, 2, 3, 4 \quad g_j(0, \tau) = 1, \quad g_j(v, \tau + 2) = g_j(v, \tau),$$

$$g_4(v, \tau + 1) = g_3(v, \tau), \quad g_3(v, \tau + 1) = g_4(v, \tau),$$

$$g_i(v, \tau + 1) = g_i(v, \tau) \text{ for } i = 1, 2, \quad g_1(v, \tau + \frac{1}{2}) = g_2(v, \tau), \quad g_2(v, \tau + \frac{1}{2}) = g_1(v, \tau).$$

Proposition 2-2 *The following identities hold for $\text{Im}v < \frac{1}{2}\Im\tau$*

$$g_4(v + \frac{1}{2}, \tau) = \prod_{k \geq 0} \cot[(k + \frac{1}{2})\pi\tau] g_3(v, \tau), \quad g_4(v + \frac{\tau}{2}, \tau) = g_1(v, \tau) g_4(\frac{\tau}{2}, \tau),$$

$$g_1(v + \frac{1}{2}, \tau) = \prod_{k \geq 0} \cot(k\pi\tau) g_2(v, \tau), \quad g_3(v + \frac{\tau}{2}, \tau) = g_2(v, \tau) g_3(\frac{\tau}{2}, \tau),$$

$$g_4(v + \tau, \tau) = \frac{g_4(v, \tau)}{\sin(\pi v + \frac{\pi\tau}{2})}, \quad g_1(v + \tau, \tau) = \frac{g_1(v, \tau)}{\sin(\pi v + \pi\tau)}.$$

Proof Indeed, concerning the last identity we may write

$$\prod_{k \geq 1} \frac{\sin[(k + 1)\pi\tau + \pi v]}{\sin[k\pi\tau]} = \frac{1}{\sin(\pi v + \pi\tau)} \prod_{k \geq 1} \frac{\sin[k\pi\tau + \pi v]}{\sin[k\pi\tau]}.$$

Thus

$$g_1(v + \tau, \tau) \sin(\pi v + \pi\tau) = g_1(v, \tau).$$

Proposition 2-3 For any integer n the functions $g_j(v, \tau)$ may also be expressed

$$g_4(v, \tau) = \prod_{0 \leq k \leq n-1} \sin[(k + \frac{1}{2})\pi\tau + \pi v] \quad g_4(v + n\tau, \tau)$$

$$g_1(v, \tau) = \prod_{1 \leq k \leq n-1} \sin[k\pi\tau + \pi v] \quad g_1(v + n\tau, \tau).$$

We may provide also a duplication formula and n -order transformation

Proposition 2-4 The following identities hold

$$g_4(2v, 2\tau) = g_4(v, \tau) \quad g_3(v, \tau) = g_4(v, \tau) \quad g_4(\frac{\pi}{4} - v, \tau) \quad g_3(\frac{\pi}{4} - v, \tau),$$

$$g_1(2v, 2\tau) = g_1(v, \tau) \quad g_2(v, \tau) = g_1(v, \tau) \quad g_1(\frac{\pi}{4} - v, \tau) \quad g_2(\frac{\pi}{4} - v, \tau).$$

For any integer n we get,

$$g_1(v + n\tau, \tau) = \frac{g_1(v, \tau)}{\prod_{m=1}^n \sin[m\pi\tau + \pi v]}, \quad g_2(v + n\tau, \tau) = \frac{g_2(v, \tau)}{\prod_{m=1}^n \cos[m\pi\tau + \pi v]},$$

$$g_4(v + n\tau, \tau) = \frac{g_4(v, \tau)}{\prod_{m=1}^n \sin[(\frac{m\pi}{2})\tau + \pi v]}, \quad g_3(v + n\tau, \tau) = \frac{g_3(v, \tau)}{\prod_{m=1}^n \cos[(\frac{m\pi}{2})\tau + \pi v]}.$$

Proof By Proposition 2-2 we have $g_4(v + \tau, \tau) = \frac{g_4(v, \tau)}{\sin(\pi v + \frac{\pi\tau}{2})}$, then

$$g_4(v + 2\tau, \tau) = \frac{g_4(v + \tau, \tau)}{\sin(\pi v + \frac{3\pi\tau}{2})} = \frac{g_4(v, \tau)}{\sin[\frac{\pi\tau}{2} + \pi v] \sin[(\frac{3\pi}{2})\tau + \pi v]},$$

$$g_1(v + 2\tau, \tau) = \frac{g_1(v + \tau, \tau)}{\sin(\pi v + 3\pi\tau)} = \frac{g_1(v, \tau)}{\sin[\pi\tau + \pi v] \sin[(3\pi)\tau + \pi v]}.$$

Moreover,

$$g_1(v + 3\tau, \tau) = \frac{g_1(v + 2\tau, \tau)}{\sin(\pi v + 4\pi\tau)} = \frac{g_1(v + \tau)}{\sin[2\pi\tau + \pi v] \sin[(4\pi)\tau + \pi v]},$$

thus

$$g_1(v + 3\tau, \tau) = \frac{g_1(v, \tau)}{\prod_{m=1}^4 \sin[m\pi\tau + \pi v]},$$

$$g_4(v + 3\tau, \tau) = \frac{g_4(v, \tau)}{\prod_{m=1}^4 \sin[(\frac{m\pi}{2})\tau + \pi v]}.$$

The last assertion of the Proposition may be proved by induction.

2.1. n -order transformations. These functions have many other interesting properties. For example, we can express in a simple way n -order transformation (for an integer). In the sense that we can highlight a relationship between $g_j(nv, n\tau)$ and $g_j(v + \frac{m}{n}, \tau)$.

Theorem 2-5 *Let n be any integer then the following identities hold*

$$g_1(nv, n\tau) = 2^{n-1} \prod_{k \geq 1} \frac{(\sin k\pi\tau)^n}{\sin nk\pi\tau} \prod_{0 \leq m \leq n-1} g_1(v + \frac{m\pi}{n}, \tau),$$

$$g_4(nv, n\tau) = 2^{n-1} \prod_{k \geq 0} \frac{(\sin(k + \frac{1}{2})\pi\tau)^n}{\sin n(k + \frac{1}{2})\pi\tau} \prod_{0 \leq m \leq n-1} g_4(v + \frac{m\pi}{n}, \tau).$$

When n is an even integer then

$$g_1(nv, n\tau) = (-1)^{\frac{n}{2}} 2^{n-1} \prod_{k \geq 1} \frac{(\cos k\pi\tau)^n}{\sin nk\pi\tau} \prod_{0 \leq m \leq n-1} g_2(v + \frac{m\pi}{n}, \tau),$$

$$g_4(nv, n\tau) = 2^{n-1} \prod_{k \geq 0} \frac{(\cos(k + \frac{1}{2})\pi\tau)^n}{\sin nk\pi\tau} \prod_{0 \leq m \leq n-1} g_3(v + \frac{m\pi}{n}, \tau).$$

When n is odd then the following hold :

$$g_1(nv, n\tau) = 2^{n-1} \prod_{k \geq 1} \frac{(\sin k\pi\tau)^n}{\sin nk\pi\tau} \prod_{0 \leq m \leq n-1} g_1(v + \frac{m\pi}{n}, \tau),$$

$$g_2(nv, n\tau) = (-1)^{\frac{n-1}{2}} 2^{n-1} \prod_{k \geq 1} \frac{(\cos k\pi\tau)^n}{\cos nk\pi\tau} \prod_{0 \leq m \leq n-1} g_2(v + \frac{m\pi}{n}, \tau),$$

$$g_3(nv, n\tau) = (-1)^{\frac{n-1}{2}} 2^{n-1} \prod_{k \geq 0} \frac{(\cos(k + \frac{1}{2})\pi\tau)^n}{\cos n(k + \frac{1}{2})\pi\tau} \prod_{0 \leq m \leq n-1} g_4(v + \frac{m\pi}{n}, \tau),$$

$$g_4(nv, n\tau) = 2^{n-1} \prod_{k \geq 0} \frac{(\sin(k + \frac{1}{2})\pi\tau)^n}{\sin n(k + \frac{1}{2})\pi\tau} \prod_{0 \leq m \leq n-1} g_4(v + \frac{m\pi}{n}, \tau).$$

Proof of Theorem 2-5 Starting from the classical trigonometric product formulas valid for n integer

$$\sin(nz) = 2^{n-1} \sin(z) \sin(z + \frac{\pi}{n}) \sin(z + \frac{2\pi}{n}) \dots \sin(z + \frac{(n-1)\pi}{n}) = 2^{n-1} \prod_{m=0}^{n-1} \sin(z + \frac{m\pi}{n}).$$

When n is even then it yields

$$\begin{aligned} \sin(nz) &= (-1)^{n/2} 2^{n-1} \cos(z) \cos(z + \frac{\pi}{n}) \cos(z + \frac{2\pi}{n}) \dots \cos(z + \frac{(n-1)\pi}{n}) \\ &= (-1)^{\frac{n}{2}} 2^{n-1} \prod_{m=0}^{n-1} \cos(z + \frac{m\pi}{n}). \end{aligned}$$

When n is odd we then obtain

$$\cos(nz) = (-1)^{n-1/2} 2^{n-1} \cos(z) \cos(z + \frac{\pi}{n}) \cos(z + \frac{2\pi}{n}) \dots \cos(z + \frac{(n-1)\pi}{n})$$

$$= (-1)^{\frac{n-1}{2}} 2^{n-1} \prod_{m=0}^{n-1} \cos\left(z + \frac{m\pi}{n}\right).$$

Therefore we may deduce for any integer k

$$\begin{aligned} \frac{\sin[nk\pi\tau + n\pi v]}{\sin(nk\pi\tau)} &= \frac{2^{n-1}}{\sin(nk\pi\tau)} \prod_{m=0}^{n-1} \left(\sin\left[k\pi\tau + \pi v + \frac{m\pi}{n}\right] \right), \\ \frac{\sin\left[n\left(k + \frac{1}{2}\right)\pi\tau + n\pi v\right]}{\sin\left(n\left(k + \frac{1}{2}\right)\pi\tau\right)} &= \frac{2^{n-1}}{\sin\left(n\left(k + \frac{1}{2}\right)\pi\tau\right)} \prod_{m=0}^{n-1} \left(\sin\left[\left(k + \frac{1}{2}\right)\pi\tau + \pi v + \frac{m\pi}{n}\right] \right), \\ \frac{\cos[nk\pi\tau + n\pi v]}{\cos(nk\pi\tau)} &= \frac{2^{n-1}}{\cos(nk\pi\tau)} \prod_{m=0}^{n-1} \left(\cos\left[k\pi\tau + \pi v + \frac{m\pi}{n}\right] \right), \\ \frac{\cos\left[n\left(k + \frac{1}{2}\right)\pi\tau + n\pi v\right]}{\cos\left(n\left(k + \frac{1}{2}\right)\pi\tau\right)} &= \frac{2^{n-1}}{\cos\left(n\left(k + \frac{1}{2}\right)\pi\tau\right)} \prod_{m=0}^{n-1} \left(\cos\left[\left(k + \frac{1}{2}\right)\pi\tau + \pi v + \frac{m\pi}{n}\right] \right). \end{aligned}$$

Then it follows for any integer n

$$\begin{aligned} g_1(nv, n\tau) &= \prod_{k \geq 1} \frac{\sin[nk\pi\tau + n\pi v]}{\sin[nk\pi\tau]} = \prod_{k \geq 1} \frac{2^{n-1}}{\sin(nk\pi\tau)} \prod_{m=0}^{n-1} \left(\sin\left[k\pi\tau + \pi v + \frac{m\pi}{n}\right] \right) \\ &= \prod_{k \geq 1} \frac{2^{n-1} [\sin(k\pi\tau)]^n}{\sin(nk\pi\tau)} \prod_{m=0}^{n-1} \left(\frac{\sin\left[k\pi\tau + \pi v + \frac{m\pi}{n}\right]}{\sin(k\pi\tau)} \right), \end{aligned}$$

which means

$$g_1(nv, n\tau) = \prod_{k \geq 1} \frac{2^{n-1} [\sin(k\pi\tau)]^n}{\sin(nk\pi\tau)} \prod_{0 \leq m \leq n-1} g_1\left(v + \frac{m\pi}{n}, \tau\right),$$

and

$$g_4(nv, n\tau) = \prod_{k \geq 0} \frac{2^{n-1} [\sin\left(\left(k + \frac{1}{2}\right)\pi\tau\right)]^n}{\sin\left(n\left(k + \frac{1}{2}\right)\pi\tau\right)} \prod_{0 \leq m \leq n-1} g_4\left(v + \frac{m\pi}{n}, \tau\right).$$

On the other hand, when n is even we have

$$\frac{\sin[nk\pi\tau + n\pi v]}{\sin[nk\pi\tau]} = \frac{2^{n-1}}{\sin(nk\pi\tau)} \prod_{m=0}^{n-1} \left(\cos\left[k\pi\tau + \pi v + \frac{m\pi}{n}\right] \right),$$

which means

$$g_1(nv, n\tau) = \prod_{k \geq 1} \frac{2^{n-1} [\cos(k\pi\tau)]^n}{\sin(nk\pi\tau)} \prod_{0 \leq m \leq n-1} g_2\left(v + \frac{m\pi}{n}, \tau\right).$$

By the same way we deduce for n even

$$g_4(nv, n\tau) = \prod_{k \geq 0} \prod_{m=0}^{n-1} \frac{2^{n-1} [\cos\left(\left(k + \frac{1}{2}\right)\pi\tau\right)]^n}{\sin\left(n\left(k + \frac{1}{2}\right)\pi\tau\right)} \prod_{1 \leq m \leq n} g_4\left(v + \frac{m\pi}{n}, \tau\right).$$

Turn now to the case n odd, we easily prove by the same method

$$\begin{aligned} g_2(nv, n\tau) &= \prod_{k \geq 1} \frac{2^{n-1} [\cos(k\pi\tau)]^n}{\cos(nk\pi\tau)} \prod_{0 \leq m \leq n-1} g_2\left(v + \frac{m\pi}{n}, \tau\right), \\ g_3(nv, n\tau) &= \prod_{k \geq 0} \frac{2^{n-1} [\cos\left(\left(k + \frac{1}{2}\right)\pi\tau\right)]^n}{\cos\left(n\left(k + \frac{1}{2}\right)\pi\tau\right)} \prod_{0 \leq m \leq n-1} g_3\left(v + \frac{m\pi}{n}, \tau\right). \end{aligned}$$

Taking the logarithmic derivative of functions $g_j(v, \tau)$ it yields

Corollary 2-6 *Let n be any integer then the following hold*

$$\frac{g'_1}{g_1}(nv, n\tau) = \pi \sum_{k \geq 1} \cot(k\pi\tau + \pi v) = \prod_{0 \leq m \leq n-1} \frac{g'_1}{g_1}\left(v + \frac{m\pi}{n}, \tau\right),$$

$$\frac{g'_4}{g_4}(nv, n\tau) = \pi \sum_{k \geq 1} \cot\left(\left(k + \frac{1}{2}\right)\pi\tau + \pi v\right) = \prod_{0 \leq m \leq n-1} \frac{g'_4}{g_4}\left(v + \frac{m\pi}{n}, \tau\right).$$

For any odd integer n :

$$\frac{g'_j}{g_j}(nv, n\tau) = \prod_{0 \leq m \leq n-1} \frac{g'_j}{g_j}\left(v + \frac{m\pi}{n}, \tau\right), \quad j = 1, 2, 3, 4.$$

For any even integer n :

$$\frac{g'_1}{g_1}(nv, n\tau) = \prod_{0 \leq m \leq n-1} \frac{g'_2}{g_2}\left(v + \frac{m\pi}{n}, \tau\right),$$

$$\frac{g'_4}{g_4}(nv, n\tau) = \prod_{0 \leq m \leq n-1} \frac{g'_3}{g_3}\left(v + \frac{m\pi}{n}, \tau\right).$$

Notice that

$$\frac{g'_2}{g_2}(nv, n\tau) = -\pi \sum_{k \geq 1} \tan(k\pi\tau + \pi v), \quad \frac{g'_3}{g_3}(nv, n\tau) = -\pi \sum_{k \geq 1} \tan\left(\left(k + \frac{1}{2}\right)\pi\tau + \pi v\right).$$

Moreover, these functions may also be expressed as double infinite products

Proposition 2-7 *The functions $g_j(v, \tau)$ may be expressed*

$$g_1(v, \tau) = \prod_{k \geq 1} \prod_m \left(1 + \frac{v}{m + k\tau}\right), \quad g_2(v, \tau) = \prod_{k \geq 1} \prod_m \left(1 + \frac{v}{m + k\tau + \frac{k}{2}}\right),$$

$$\frac{g_4(v, \tau)}{g_4\left(\frac{\tau}{2}, \tau\right)} = \prod_{k \geq 1} \prod_m \left(1 + \frac{v - \frac{\tau}{2}}{m + k\tau}\right), \quad \frac{g_3(v, \tau)}{g_3\left(\frac{\tau}{2}, \tau\right)} = \prod_{k \geq 1} \prod_m \left(1 + \frac{v - \frac{\tau}{2}}{m + k\tau + \frac{k}{2}}\right).$$

Proof of Proposition 2-7 Indeed, following Eisenstein [6, p.213] starting from product formula for $\sin \pi x$

$$\sin \pi x = \pi x \prod'_m \left(1 + \frac{x}{m}\right),$$

(the prime means the integer $m \neq 0$) it follows for any integer n

$$\sin[k\pi\tau + n\pi v] = (k\pi\tau + \pi v) \prod'_m \left(1 + \frac{k\tau + v}{m}\right),$$

as well as

$$\sin[k\pi\tau] = (k\pi\tau) \prod'_m \left(1 + \frac{k\tau}{m}\right).$$

Then,

$$\begin{aligned} \frac{\sin[k\pi\tau + n\pi v]}{\sin[k\pi\tau]} &= \frac{(k\tau + v)}{(k\tau)} \prod'_m \left(\frac{1 + \frac{k\tau+v}{m}}{1 + \frac{k\tau}{m}}\right) = \frac{(k\tau + v)}{(k\tau)} \prod'_m \left(\frac{m + k\tau + v}{m + k\tau}\right) \\ &= \prod'_m \left(1 + \frac{v}{m + k\tau}\right). \end{aligned}$$

We then deduce $g_1(v, \tau) = \prod_{k \geq 1} \prod_m \left(1 + \frac{v}{m+k\tau}\right)$. The others expressions for g_2, g_3, g_4 are derived from Proposition 2-2.

Remarks 2-8 To go further and explore other properties of functions g_j we can take inspiration from the method developed by Eisenstein [6, p.275] which consists in expressing g_j as the limit of a function of 2 variables $f(u, v)$. This method has been used successfully to reveal the connection between his product function $\phi(v)$ and the Jacobi function θ_1 . More explicitly, following 1-3 consider again

$$f(u, v) = \prod_{k \geq 1} \prod_m \left(1 - \frac{v}{m + x\tau + x}\right).$$

We observe that

$$\prod_m \left(1 - \frac{v}{m + k\tau + x}\right) = \frac{\sin[k\pi\tau + n\pi v - \pi u]}{\sin[k\pi\tau + \pi v]},$$

thus it could be written

$$f(u, v) = \prod_{k \geq 1} \frac{\sin[k\pi\tau + n\pi v - \pi u]}{\sin[k\pi\tau + \pi v]}.$$

Following Eisenstein [5, p.275-280] who consider the theta product

$$\xi(e^{i\pi v}) = (e^{i\pi v} - e^{-i\pi v}) \prod_{k \geq 1} (1 - q^{2k} e^{2i\pi v})(1 - q^{2k} e^{-2i\pi v}),$$

so that we may write

$$f(u, v) = \prod_{k \geq 1} \prod_m \left(1 - \frac{v}{m + x\tau + x}\right) = \frac{\xi(e^{i\pi(v-u)})}{\xi(e^{i\pi v})}.$$

Notice that since by Proposition 2-7

$$g_1(v, \tau) = \prod_{k \geq 1} \prod_m \left(1 + \frac{v}{m + k\tau}\right),$$

we then obtain the identity

$$f(u, v) = \frac{g_1(v-u)}{g_1(v)},$$

and

$$g_1(v, \tau) = \lim_{u \rightarrow 0} [-u f(u, v)].$$

3. CONNECTION OF g_j WITH THE JACOBI THETA FUNCTIONS

Consider the Ramanujan theta function

$$\phi(a, b) = \sum_{k \in \mathbb{Z}} a^{\frac{k(k+1)}{2}} b^{\frac{k(k-1)}{2}}$$

$$\phi(a, b) = (-a, ab)_{\infty} (-b, ab)_{\infty} (ab, ab)_{\infty}$$

with $|ab| < 1$. We denote here $(\alpha, \beta)_{\infty} = \prod_{i \geq 1} (1 - \alpha\beta^i)$. When $a = -qe^{2i\pi v}$, $b = -qe^{-2i\pi v}$ this function is related for example to the fourth theta function

$$\theta_4(v, q) = \phi(-qe^{2i\pi v}, -qe^{-2i\pi v}).$$

From the knowledge of the zeros of $\theta_4(v, q)$ or by the triple product identity, it is possible to obtain infinite products representing this function. We refer to [4,5] in order to find more ample information as well as other aspects and developments as infinite products concerning these theta functions.

3.1. Connection with $\theta_4(v, \tau)$. A natural relation with $\theta_4(v, \tau)$ is given by

Proposition 3-1 *Let $\theta_4(v, \tau)$ the fourth Jacobi theta function. Then we have*

$$\frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} = g_4(v, \tau) g_4(-v, \tau)$$

where g_4 and its logarithmic derivative satisfy the infinite product

$$g_4(v, \tau) = \prod_{k \geq 0} \frac{\sin[(k + \frac{1}{2})\pi\tau + \pi v]}{\sin[(k + \frac{1}{2})\pi\tau]}$$

$$\frac{1}{\theta_4} \frac{\partial \theta_4}{\partial v}(v, \tau) = \frac{1}{g_4} \frac{\partial g_4}{\partial v}(v, \tau) + \frac{1}{g_4} \frac{\partial g_4}{\partial v}(-v, \tau) = \sum_{k \geq 0} \cot[(k + \frac{1}{2})\pi\tau + \pi v] + \cot[(k + \frac{1}{2})\pi\tau - \pi v].$$

θ_4 is defined in the band $|Imv| < \frac{1}{2}\Im\tau$.

Proof of Proposition 3-1

Let $q = e^{i\pi\tau}$ then we may write

$$\begin{aligned} g_4(v, \tau) &= \prod_{k \geq 0} \frac{\sin[(k + \frac{1}{2})\pi\tau + \pi v]}{\sin[(k + \frac{1}{2})\pi\tau]} = \prod_{k \geq 0} \frac{e^{(k + \frac{1}{2})\pi\tau + i\pi v} - e^{-(k + \frac{1}{2})\pi\tau - i\pi v}}{e^{i\pi(k + \frac{1}{2})\tau} - e^{-i\pi(k + \frac{1}{2})\tau}} \\ &= \prod_{k \geq 0} \frac{q^k e^{i\pi v} - q^{-k} e^{-i\pi v}}{q^k - q^{-k}} = \prod_{k \geq 0} \frac{q^{2k} e^{i\pi v} - e^{-i\pi v}}{q^{2k} - 1}. \end{aligned}$$

Therefore

$$\begin{aligned} g_4(v, \tau) g_4(-v, \tau) &= \prod_{k \geq 0} \frac{(q^{2k+1} e^{i\pi v} - e^{-i\pi v})(q^{2k+1} e^{-i\pi v} - e^{i\pi v})}{(q^{2k+1} - 1)^2} \\ &= \prod_{k \geq 0} \frac{q^{4k+2} - q^{2k+1} e^{2i\pi v} - e^{-2i\pi v} q^{2k+1} - 1}{(q^{2k+1} + 1)^2}. \end{aligned}$$

On the other hand, by [8, (16) p.369] the infinite products representing the theta functions

$$\begin{aligned}\theta_4(v, \tau) &= \prod_{k \geq 1} (q^{2k} - 1) \prod_{k \geq 1} [1 - 2q^{2k-1} \cos 2\pi v + q^{4k-2}], \\ \theta_4(0, \tau) &= \prod_{k \geq 1} (q^{2k} - 1) \prod_{k \geq 1} [1 - 2q^{2k-1} + q^{4k-2}] = (q^2 - 1)^3.\end{aligned}$$

We thus derive expression of theta as infinite product

$$\begin{aligned}\frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} &= \prod_{k \geq 1} \frac{1 - 2q^{2k-1} \cos 2\pi v + q^{4k-2}}{(q^{2k} - 1)^2} \\ &= \prod_{k \geq 0} \frac{\sin[(k + \frac{1}{2})\pi\tau + \pi v] \sin[(k + \frac{1}{2})\pi\tau - \pi v]}{(\sin[(k + \frac{1}{2})\pi\tau])^2} = g_4(v, \tau) g_4(-v, \tau).\end{aligned}$$

3.2. Connection with the other theta functions. Consider again the functions $g_i(v, \tau)$, $i = 1, 2, 3$.

$$\begin{aligned}g_3(v, \tau) &= \prod_{k \geq 0} \frac{\cos[(k + \frac{1}{2})\pi\tau + \pi v]}{\cos[(k + \frac{1}{2})\pi\tau]}, \\ g_1(v, \tau) &= \prod_{k \geq 1} \frac{\sin[k\pi\tau + \pi v]}{\sin[k\pi\tau]}, \quad g_2(v, \tau) = \prod_{k \geq 1} \frac{\cos[k\pi\tau + \pi v]}{\cos[k\pi\tau]}.\end{aligned}$$

Recall that they are related

$$g_3(v, \tau) = g_4(v, \tau + 1), \quad g_4(v, \tau) = g_3(v, \tau + 1), \quad g_2(v, \tau + \frac{1}{2}) = g_1(v, \tau), \quad g_1(v, \tau + \frac{1}{2}) = g_2(v, \tau).$$

By the same manner as we prove Proposition 3-1 and using expansions of theta functions [8, (16) p.369]

$$\begin{aligned}\theta_1(v, \tau) &= 2 \prod_{n \geq 1} [1 - q^n] q^{1/4} \sin \pi v \prod_{n \geq 1} [1 - 2q^{2n} \cos 2\pi v + q^{4n}], \\ \theta_2(v, \tau) &= 2 \prod_{n \geq 1} [1 - q^n] q^{1/4} \cos \pi v \prod_{n \geq 1} [1 + 2q^{2n} \cos 2\pi v + q^{4n}], \\ \theta_3(v, \tau) &= \prod_{n \geq 1} [1 - q^n] \prod_{n \geq 1} [1 + 2q^{2n-1} \cos 2\pi v + q^{4n-2}],\end{aligned}$$

one derives the following

Proposition 3-2 *The other theta functions may be expressed*

$$\begin{aligned}\theta_3(v, \tau) &= \theta_3(0, \tau) g_3(v, \tau) g_3(-v, \tau), \\ \theta_1(v, \tau) &= (\pi \sin(\pi v) \theta_1'(0, \tau) g_1(v, \tau) g_1(-v, \tau), \\ \theta_2(v, \tau) &= \cos(\pi v) \theta_2(0, \tau) g_2(v, \tau) g_2(-v, \tau).\end{aligned}$$

Moreover,

$$\frac{1}{\theta_j} \frac{\partial \theta_j}{\partial v}(v, \tau) = \frac{1}{g_j} \frac{\partial g_j}{\partial v}(v, \tau) + \frac{1}{g_j} \frac{\partial g_j}{\partial v}(-v, \tau), \quad j = 1, 2, 3.$$

Here θ_3 is defined in the band $|Imv| < \frac{1}{2}\Im\tau$, while θ_1 and θ_2 is defined in the band $|Imv| < \Im\tau$.

We then deduce expressions for Jacobi theta functions as infinite products

Corollary 3-3 *Let $q = e^{i\pi\tau}$, $|q| < 1$. The functions θ_j , $j = 1, 2, 3, 4$ may also be expressed as infinite products*

$$\frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} = \prod_{k \geq 0} \left[1 - \left(\frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi\tau} \right)^2 \right] = \prod_{k \geq 0} \frac{\sin[(k + \frac{1}{2})\pi\tau + \pi v] \sin[(k + \frac{1}{2})\pi\tau - \pi v]}{(\sin[(k + \frac{1}{2})\pi\tau])^2} \quad (1)$$

$$\frac{\theta_3(v, \tau)}{\theta_3(0, \tau)} = \prod_{k \geq 0} \left[1 - \left(\frac{\sin \pi v}{\cos(k + \frac{1}{2})\pi\tau} \right)^2 \right] = \prod_{k \geq 0} \frac{\cos[(k + \frac{1}{2})\pi\tau + \pi v] \cos[(k + \frac{1}{2})\pi\tau - \pi v]}{(\cos[(k + \frac{1}{2})\pi\tau])^2} \quad (2)$$

$$\frac{\theta_1(v, \tau)}{(\pi \sin \pi v) \theta_1'(0, \tau)} = \prod_{k \geq 1} \left[1 - \left(\frac{\sin \pi v}{\sin k\pi\tau} \right)^2 \right] = \prod_{k \geq 1} \frac{\sin[k\pi\tau + \pi v] \sin[k\pi\tau - \pi v]}{(\sin[k\pi\tau])^2} \quad (3)$$

$$\frac{\theta_2(v, \tau)}{(\cos \pi v) \theta_2(0, \tau)} = \prod_{k \geq 1} \left[1 - \left(\frac{\sin \pi v}{\cos k\pi\tau} \right)^2 \right] = \prod_{k \geq 1} \frac{\cos[k\pi\tau + \pi v] \cos[k\pi\tau - \pi v]}{(\cos[k\pi\tau])^2} \quad (4)$$

θ_3 and θ_4 is defined in the band $|Imv| < \frac{1}{2}\Im\tau$, while θ_1 and θ_2 are defined in the band $|Imv| < \Im\tau$.

Remark 3-4 This type of expansion was already known to the literature. We refer for that for example to the book of H.A. Schwarz [13 p.36] in which he exposed a complete theory of the elliptic sigma functions as well as various trigonometric expansions but in a different context than we use for theta functions.

As we know that these sigma and theta functions are related, by an elementary calculation we may find again expansions of Corollary 3-3. Note however that our approach is different. It seems also Eisenstein [6, p.252-256] to have been aware of this type of expressions. To be more precise, he used a method of summation (according to his notation) for $s \geq 1$

$$(s, x) = \sum_{m, k = -\infty}^{\infty} \frac{1}{(x + m + k\tau)^s}, \quad (s^*, 0) = \sum'_{m, k = -\infty}^{\infty} \frac{1}{(x + m + k\tau)^s}.$$

These sums are called later by A. Weil as Eisenstein series. Eisenstein observed that by periodicity and after term-by-term differentiation

$$(s, x + 1) = (s, x), \quad \frac{d}{dx}(s, x) = -s(s + 1, x).$$

Since $\sum_{m=-\infty}^{\infty} \frac{1}{(y+m)} = \pi \cot \pi y$ we then deduce

$$\sum_{m=-\infty}^{\infty} \frac{1}{(x + m + k\tau)^2} = -\pi \frac{d}{dx} \cot(\pi(x + k\tau)) = \frac{\pi^2}{[\sin(\pi(x + k\tau))]^2}.$$

Thus, $(2, x + \tau) = (2, x)$ and

$$(2, x) = \sum_{m, k = -\infty}^{\infty} \frac{1}{(x + m + k\tau)^2} = \sum_k \frac{\pi^2}{[\sin(\pi(x + k\tau))]^2}$$

is a double periodic function. Note in addition that Eisenstein also pointed out all the important properties of his function $(2, x) - (2^*, 0)$, called later $\wp(x)$ of Weierstrass.

We may derive from the preceding

Corollary 3-5 For any integer n we have

$$\begin{aligned} (-1)^n e^{-i\pi n(2v+\tau)} &= \prod_{m=1}^n \sin[m\pi\tau + \pi v] \sin[-m\pi\tau - \pi v], \\ e^{-i\pi n(2v+\tau)} &= \prod_{m=1}^n \cos[m\pi\tau + \pi v] \cos[-m\pi\tau - \pi v], \\ e^{-i\pi n(2v+\tau)} &= \prod_{m=1}^n \cos\left[\frac{m\pi}{2}\tau + \pi v\right] \cos\left[-\frac{m\pi}{2}\tau - \pi v\right], \\ (-1)^n e^{-i\pi n(2v+\tau)} &= \prod_{m=1}^n \sin\left[\frac{m\pi}{2}\tau + \pi v\right] \sin\left[-\frac{m\pi}{2}\tau - \pi v\right]. \end{aligned}$$

Proof Recall by Proposition 2-4

$$\begin{aligned} g_1(v + n\tau, \tau) &= \frac{g_1(v, \tau)}{\prod_{m=1}^n \sin[m\pi\tau + \pi v]}, & g_2(v + n\tau, \tau) &= \frac{g_2(v, \tau)}{\prod_{m=1}^n \cos[m\pi\tau + \pi v]}, \\ g_4(v + n\tau, \tau) &= \frac{g_4(v, \tau)}{\prod_{m=1}^n \sin\left[\left(\frac{m\pi}{2}\right)\tau + \pi v\right]}, & g_3(v + n\tau, \tau) &= \frac{g_3(v, \tau)}{\prod_{m=1}^n \cos\left[\left(\frac{m\pi}{2}\right)\tau + \pi v\right]}. \end{aligned}$$

Since

$$\begin{aligned} \frac{\theta_4(v + \tau, \tau)}{\theta_4(v, \tau)} &= (-1)e^{-i\pi(2v+\tau)}, & \frac{\theta_3(v + \tau, \tau)}{\theta_3(v, \tau)} &= e^{-i\pi(2v+\tau)}, \\ \frac{\theta_1(v + \tau, \tau)}{\theta_1(v, \tau)} &= (-1)e^{-i\pi(2v+\tau)}, & \frac{\theta_2(v + \tau, \tau)}{\theta_2(v, \tau)} &= e^{-i\pi(2v+\tau)}. \end{aligned}$$

By induction we obtain for any integer

$$\begin{aligned} \frac{\theta_4(v + n\tau, \tau)}{\theta_4(v, \tau)} &= (-1)^n e^{-i\pi n(2v+\tau)}, & \frac{\theta_3(v + n\tau, \tau)}{\theta_3(v, \tau)} &= e^{-i\pi n(2v+\tau)}, \\ \frac{\theta_1(v + n\tau, \tau)}{\theta_1(v, \tau)} &= (-1)^n e^{-i\pi n(2v+\tau)}, & \frac{\theta_2(v + n\tau, \tau)}{\theta_2(v, \tau)} &= e^{-i\pi n(2v+\tau)}. \end{aligned}$$

Moreover, since

$$\frac{\theta_j(v + n\tau, \tau)}{\theta_j(v, \tau)} = \frac{g_j(v + n\tau, \tau)g_j(-v - n\tau, \tau)}{g_j(v, \tau)g_j(-v, \tau)},$$

We then deduce

$$\begin{aligned} \frac{\theta_1(v + \tau, \tau)}{\theta_1(v, \tau)} &= (-1)^n e^{-i\pi n(2v+\tau)} = \frac{1}{\prod_{m=1}^n \sin[m\pi\tau + \pi v] \sin[-m\pi\tau - \pi v]}, \\ \frac{\theta_2(v + \tau, \tau)}{\theta_2(v, \tau)} &= e^{-i\pi n(2v+\tau)} = \frac{1}{\prod_{m=1}^n \cos[m\pi\tau + \pi v] \cos[-m\pi\tau - \pi v]}, \\ \frac{\theta_3(v + n\tau, \tau)}{\theta_3(v, \tau)} &= e^{-i\pi n(2v+\tau)} = \frac{1}{\prod_{m=1}^n \cos\left[\frac{m\pi}{2}\tau + \pi v\right] \cos\left[-\frac{m\pi}{2}\tau - \pi v\right]}, \end{aligned}$$

$$\frac{\theta_4(v + n\tau, \tau)}{\theta_3(v, \tau)} = (-1)^n e^{-i\pi n(2v+\tau)} = \frac{1}{\prod_{m=1}^n \sin\left[\frac{m\pi}{2}\tau + \pi v\right] \sin\left[-\frac{m\pi}{2}\tau - \pi v\right]}.$$

Recall that the transformation theory of elliptic functions or theta functions deals with the relations between these functions belonging to different pairs of primitive periods. Observe [9, p.252] that any transformation of order $n > 1$ may be represented as a product of transformations of first order and of transformations of higher order with matrix

$$M = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$$

Moreover, any transformation $\tau' = n\tau$ can be separated into a product when n has prime factors. Therefore, we only study the case of transformation when n is a prime and limit our study for this type of matrix. We refer to an n -order modular equation as a relation between $\theta(\tau)$ and $\theta(n\tau)$.

Consequently from Proposition 2-5 and Corollary 3-3 we may derive formulas of multiplication by an integer n for theta functions which are well known for the odd case since Jacobi (see [7, p.374]) who asserted

$$\theta_j(nv, n\tau) = \prod_{s \geq 1} \frac{(1 - q^{2sn})}{(1 - q^{2s})^n} \prod_{0 \leq m \leq (n-1)} \theta_j\left(v + \frac{m}{n}, \tau\right).$$

We will find again this result, but the proof we give here is different and will have various consequences and applications, in particular in q-trigonometry. This will be discussed in part 5.

Theorem 3-6 *Let n be any integer then :*

(i) - For $j = 1, 4$ the following identities hold

$$\frac{\theta_1(nv, n\tau)}{\pi\theta'_1(0, n\tau)} = 2^{2n-2} \frac{\sin n\pi v}{(\sin \pi v)^n [\pi\theta'_1(0, \tau)]^n} \prod_{k \geq 1} \left[\frac{[\sin(k\pi\tau)]^n}{\sin(nk\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \theta_1\left(v + \frac{m}{n}, \tau\right),$$

$$\frac{\theta_4(nv, n\tau)}{\theta_4(0, n\tau)} = 2^{2n-2} \frac{1}{[\theta_4(0, \tau)]^n} \prod_{k \geq 0} \left[\frac{[\sin((k + \frac{1}{2})\pi\tau)]^n}{\sin(n(k + \frac{1}{2})\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \theta_4\left(v + \frac{m}{n}, \tau\right).$$

(ii) - When n is an even integer then

$$\frac{\theta_1(nv, n\tau)}{\pi\theta'_1(0, n\tau)} = 2^{2n-2} \frac{\sin n\pi v}{(\cos \pi v)^n [\theta_2(0, \tau)]^n} \prod_{k \geq 1} \left[\frac{[\sin(k\pi\tau)]^n}{\sin(nk\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \theta_2\left(v + \frac{m}{n}, \tau\right),$$

$$\frac{\theta_4(nv, n\tau)}{\theta_4(0, n\tau)} = 2^{2n-2} \frac{1}{[\theta_3(0, \tau)]^n} \prod_{k \geq 0} \left[\frac{[\sin((k + \frac{1}{2})\pi\tau)]^n}{\sin(n(k + \frac{1}{2})\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \theta_3\left(v + \frac{m}{n}, \tau\right).$$

(iii) - When n is odd then the following hold

$$\frac{\theta_1(nv, n\tau)}{\pi\theta'_1(0, n\tau)} = 2^{2n-2} \frac{\sin n\pi v}{(\sin \pi v)^n [\pi\theta'_1(0, \tau)]^n} \prod_{k \geq 1} \left[\frac{[\sin(k\pi\tau)]^n}{\sin(nk\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \theta_1\left(v + \frac{m}{n}, \tau\right),$$

$$\frac{\theta_4(nv, n\tau)}{\theta_4(0, n\tau)} = 2^{2n-2} \frac{1}{[\theta_4(0, \tau)]^n} \prod_{k \geq 0} \left[\frac{[\sin((k + \frac{1}{2})\pi\tau)]^n}{\sin(n(k + \frac{1}{2})\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \theta_4\left(v + \frac{m}{n}, \tau\right),$$

and

$$\frac{\theta_2(nv, n\tau)}{\theta_2(0, n\tau)} = 2^{2n-2} \frac{\cos n\pi v}{(\cos \pi v)^n [\theta_2(0, \tau)]^n} \prod_{k \geq 1} \left[\frac{[\cos(k\pi\tau)]^n}{\cos(nk\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \theta_2\left(v + \frac{m}{n}, \tau\right),$$

$$\frac{\theta_3(nv, n\tau)}{\theta_3(0, n\tau)} = 2^{2n-2} \frac{1}{[\theta_3(0, \tau)]^n} \prod_{k \geq 0} \left[\frac{[\cos((k + \frac{1}{2})\pi\tau)]^n}{\cos(n(k + \frac{1}{2})\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \theta_3\left(v + \frac{m}{n}, \tau\right).$$

Proof of Theorem 3-6 By Theorem 2-5,

$$g_1(nv, n\tau) = 2^{n-1} \prod_{k \geq 1} \frac{(\sin k\pi\tau)^n}{\sin nk\pi\tau} \prod_{0 \leq m \leq n-1} g_1\left(v + \frac{m\pi}{n}, \tau\right),$$

$$g_4(nv, n\tau) = 2^{n-1} \prod_{k \geq 0} \frac{(\sin(k + \frac{1}{2})\pi\tau)^n}{\sin n(k + \frac{1}{2})\pi\tau} \prod_{0 \leq m \leq n-1} g_4\left(v + \frac{m\pi}{n}, \tau\right).$$

When n is an even integer then

$$g_1(nv, n\tau) = (-1)^{\frac{n}{2}} 2^{n-1} \prod_{k \geq 1} \frac{(\cos k\pi\tau)^n}{\sin nk\pi\tau} \prod_{0 \leq m \leq n-1} g_2\left(v + \frac{m\pi}{n}, \tau\right),$$

$$g_4(nv, n\tau) = 2^{n-1} \prod_{k \geq 0} \frac{(\cos(k + \frac{1}{2})\pi\tau)^n}{\sin nk\pi\tau} \prod_{0 \leq m \leq n-1} g_3\left(v + \frac{m\pi}{n}, \tau\right).$$

When n is odd then the following hold :

$$g_1(nv, n\tau) = 2^{n-1} \prod_{k \geq 1} \frac{(\sin k\pi\tau)^n}{\sin nk\pi\tau} \prod_{0 \leq m \leq n-1} g_1\left(v + \frac{m\pi}{n}, \tau\right),$$

$$g_2(nv, n\tau) = (-1)^{\frac{n-1}{2}} 2^{n-1} \prod_{k \geq 1} \frac{(\cos k\pi\tau)^n}{\cos nk\pi\tau} \prod_{0 \leq m \leq n-1} g_2\left(v + \frac{m\pi}{n}, \tau\right),$$

$$g_3(nv, n\tau) = (-1)^{\frac{n-1}{2}} 2^{n-1} \prod_{k \geq 0} \frac{(\cos(k + \frac{1}{2})\pi\tau)^n}{\cos n(k + \frac{1}{2})\pi\tau} \prod_{0 \leq m \leq n-1} g_4\left(v + \frac{m\pi}{n}, \tau\right),$$

$$g_4(nv, n\tau) = 2^{n-1} \prod_{k \geq 0} \frac{(\sin(k + \frac{1}{2})\pi\tau)^n}{\sin n(k + \frac{1}{2})\pi\tau} \prod_{0 \leq m \leq n-1} g_4\left(v + \frac{m\pi}{n}, \tau\right).$$

Consequently,

$$\frac{\theta_1(nv, n\tau)}{\pi\theta'_1(0, n\tau)} = \frac{\sin(n\pi v)}{[\pi\theta'_1(0, \tau)]^n} \prod_{0 \leq m \leq n-1} \frac{\theta_1\left(v + \frac{m}{n}, \tau\right)}{\sin\left(v + \frac{m}{n}\right)}.$$

n even :

$$\frac{\theta_1(nv, n\tau)}{\pi\theta'_1(0, n\tau)} = \frac{\sin(n\pi v)}{[\theta_2(0, \tau)]^n} \prod_{0 \leq m \leq n-1} \frac{\theta_2\left(v + \frac{m}{n}, \tau\right)}{\cos\left(v + \frac{m}{n}\right)},$$

$$\frac{\theta_4(nv, n\tau)}{\theta_4(0, n\tau)} = \frac{1}{[\theta_3(0, \tau)]^n} \prod_{0 \leq m \leq n-1} \theta_3\left(v + \frac{m}{n}, \tau\right).$$

n odd :

$$\frac{\theta_j(nv, n\tau)}{\theta_j(0, n\tau)} = \frac{1}{[\theta_j(0, \tau)]^n} \prod_{0 \leq m \leq n-1} \theta_j\left(v + \frac{m}{n}, \tau\right),$$

for $j = 3, 4$, and

$$\frac{\theta_2(nv, n\tau)}{\pi\theta_2(0, n\tau)} = \frac{\cos(n\pi v)}{[\theta_2(0, \tau)]^n} \prod_{0 \leq m \leq n-1} \frac{\theta_2(v + \frac{m}{n}, \tau)}{\cos(v + \frac{m}{n})}.$$

The results follow since when n even one has

$$\begin{aligned} \sin(nz) &= (-1)^{n/2} 2^{n-1} \cos(z) \cos(z + \frac{\pi}{n}) \cos(z + \frac{2\pi}{n}) \dots \cos(z + \frac{(n-1)\pi}{n}) \\ &= (-1)^{\frac{n}{2}} 2^{n-1} \prod_{m=0}^{n-1} \cos(z + \frac{m\pi}{n}). \end{aligned}$$

For n odd one has

$$\begin{aligned} \cos(nz) &= (-1)^{n-1/2} 2^{n-1} \cos(z) \cos(z + \frac{\pi}{n}) \cos(z + \frac{2\pi}{n}) \dots \cos(z + \frac{(n-1)\pi}{n}) \\ &= (-1)^{\frac{n-1}{2}} 2^{n-1} \prod_{m=0}^{n-1} \cos(z + \frac{m\pi}{n}). \end{aligned}$$

Remark 3-7 Recall that for the odd case of n the formula of [7, p.249]. For $j = 1, 2, 3, 4$

$$\theta_j(nv, n\tau) \frac{\eta^n(\tau)}{\eta(n\tau)} = \theta_j(v, \tau) \prod_{1 \leq m \leq \frac{n-1}{2}} \theta_j(v + \frac{m}{n}, \tau) \theta_j(v - \frac{m}{n}, \tau).$$

In [6, p.272] there is another analogous formula always for n odd

$$\theta_j(v, \frac{\tau}{n}) \frac{\eta^n(\tau)}{\eta(\frac{\tau}{n})} q^{\frac{1-n^2}{24n}} = \theta_j(v, \tau) \prod_{1 \leq m \leq \frac{n-1}{2}} \theta_j(v + \frac{m\tau}{n}, \tau) \theta_j(v - \frac{m\tau}{n}, \tau).$$

For any integer n Enneper proved [6, p.248]

$$\theta_j(nv, n\tau) = \prod_{s \geq 1} \frac{(1 - q^{2sn})}{(1 - q^{2s})^n} \prod_{0 \leq m \leq n-1} \theta_j(v + \frac{m}{n}, \tau), \quad j = 1, 2, 3, 4.$$

Another alternative expression originally due to Jacobi :

$$\theta_j(nv, n\tau) = \frac{(q^{2n}; q^{2n})_\infty}{(q^2; q^2)_\infty^n} \prod_{0 \leq m \leq n-1} \theta_j(v + \frac{m}{n}, \tau), \quad j = 1, 2, 3, 4.$$

3.3. Connection with $f_j(v, \tau)$.

Recall the functions introduced in [4] allowing us to represent the Jacobi theta functions :

$$f_4(v, \tau) = \prod_{k \geq 0} \left[1 - \left(\frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi\tau} \right) \right], \quad f_3(v, \tau) = \prod_{k \geq 0} \left[1 - \left(\frac{(-1)^k \sin \pi v}{\cos(k + \frac{1}{2})\pi\tau} \right) \right],$$

$$f_1(v, \tau) = \prod_{k \geq 1} \left[1 - \left(\frac{\sin \pi v}{\sin k\pi\tau} \right) \right], \quad f_2(v, \tau) = \prod_{k \geq 1} \left[1 - \left(\frac{(-1)^k \sin \pi v}{\cos k\pi\tau} \right) \right]$$

where $v \in \mathbb{C}$ and τ belongs to the half plane ($\Im\tau : \text{Im}(\tau) > 0$). These functions are connected to $g_j(v, \tau)$

Proposition 3-8 *The following identities hold*

$$\begin{aligned}\frac{f_1(v, \tau)}{g_1(v, \tau)} &= \prod_{k \geq 1} \frac{\sin \frac{[k\pi\tau - \pi v]}{2}}{\sin \frac{[k\pi\tau + \pi v]}{2}}, & \frac{f_4(v, \tau)}{g_4(v, \tau)} &= \prod_{k \geq 0} \frac{\sin \frac{[(k + \frac{1}{2})\pi\tau - \pi v]}{2}}{\sin \frac{[(k + \frac{1}{2})\pi\tau + \pi v]}{2}}, \\ \frac{f_2(v, \tau)}{g_2(v, \tau)} &= \prod_{k \geq 1} \frac{\sin \frac{[k\pi(\tau+1) - \pi v]}{2}}{\sin \frac{[k\pi(\tau+1) + \pi v]}{2}}, & \frac{f_3(v, \tau)}{g_3(v, \tau)} &= \prod_{k \geq 0} \frac{\sin \frac{[(k + \frac{1}{2})\pi(\tau+1) - \pi v]}{2}}{\sin \frac{[(k + \frac{1}{2})\pi(\tau+1) + \pi v]}{2}}. \\ \frac{f_j(v, \tau)}{g_j(v, \tau)} &= \frac{g_j(-v, \tau)}{f_j(-v, \tau)}, & j &= 1, 2, 3, 4.\end{aligned}$$

Proof of Proposition 3-8 Indeed, one has

$$\begin{aligned}\frac{f_4(v, \tau)}{g_4(v, \tau)} &= \frac{\prod_{k \geq 0} \left[1 - \left(\frac{\sin \pi v}{\sin[(k + \frac{1}{2})\pi\tau]} \right) \right]}{\prod_{k \geq 0} \frac{\sin[(k + \frac{1}{2})\pi\tau + \pi v]}{\sin[(k + \frac{1}{2})\pi\tau + \pi v]}} = \frac{\prod_{k \geq 0} [\sin[(k + \frac{1}{2})\pi\tau] - (\sin \pi v)]}{\prod_{k \geq 0} \sin[(k + \frac{1}{2})\pi\tau]} \\ &= \prod_{k \geq 0} \frac{\sin \frac{[(k + \frac{1}{2})\pi\tau - \pi v]}{2}}{\sin \frac{[(k + \frac{1}{2})\pi\tau + \pi v]}{2}}.\end{aligned}$$

We prove in exactly the same way for other quotients $\frac{f_j(v, \tau)}{g_j(v, \tau)}$ $j = 1, 2, 3$.

By Corollary 3-5 we may also other identities as

$$\frac{f_1(v + n\tau, \tau)}{f_1(v, \tau)} = \prod_{k=1}^n \frac{\sin \frac{[k\pi\tau + \pi v]}{2}}{\sin \frac{[k\pi\tau - \pi v]}{2} \sin[k\pi\tau + \pi v]}.$$

Remark 3-9 In [4] we have highlighted the impact of the quotient $\frac{f_j(v, \tau)}{f_j(v+1, \tau)}$ allowing to establish modular relations. These quotients may also be written $\frac{f_j(v, \tau)}{f_j(v+1, \tau)} = \frac{g_j(v, \tau)}{g_j(v+1, \tau)}$.

4. INFINITE PRODUCTS FOR \sin_p AND \cos_p

In this part we are interested in the relations between q -trigonometry and infinite product, as a consequence of the previous study on the developments of the Jacobi theta functions.

Gasper [9] introduced q -analogues to \sin and \cos

$$\begin{aligned}\sin_q(v) &= q^{\binom{v-1}{2}} \prod_{k \geq 1} \frac{(1 - q^{2n-2v})(1 - q^{2n+2v-2})}{(1 - q^{2n-1})^2} = \frac{(q^{2-2v}; q^2)_\infty (q^{2v}; q^2)_\infty}{(q; q^2)_\infty^2} q^{v - \frac{1}{2}}, \\ \cos_q(v) &= q^{\binom{v-1}{2}} \prod_{k \geq 1} \frac{(1 - q^{2n-2v-1})(1 - q^{2n+2v-1})}{(1 - q^{2n-1})^2} = \frac{(q^{1-2v}; q^2)_\infty (q^{1+2v}; q^2)_\infty}{(q; q^2)_\infty^2} q^{v^2},\end{aligned}$$

where $(a; q) = \prod_{k \geq 0} (1 - aq)^k$ and gave relations between them and theta function

$$\sin_q(v) = \frac{\theta_1(v, \tau')}{\theta_1(\frac{1}{2}, \tau')}, \quad \cos_q(v) = \frac{\theta_2(v, \tau')}{\theta_2(0, \tau')},$$

where $\tau' = \frac{-1}{\tau}$. It is proved that $\lim_{q \rightarrow 1} \sin_q(v) = \sin(v)$, $\lim_{q \rightarrow 1} \cos_q(v) = \cos(v)$.

4.1. **Allusive remark on q-trigonometry in Enneper's works.** Notice that these q-analogues to trigonometric functions was already known to the ancients mathematicians. Indeed, Enneper after developing a fine analysis introduced [7, p.114] the quotients

$$\frac{\theta_2(x, q)}{\theta_2(0, q)} = \frac{\theta_2(x, q^4)}{\theta_2(0, q^4)} F(x, q^4), \quad \frac{\theta_1(x, q)}{\theta_2(0, q)} = \frac{\theta_1(x, q^4)}{\theta_2(0, q^4)} F_1(x, q^4)$$

where

$$F(x, q^4) = \frac{\theta_3(x, q^4) (\theta_2(x, q^4))^2 + (\theta_3(x, q^4))^2}{\theta_3(0, q^4) (\theta_2(0, q^4))^2 + (\theta_3(0, q^4))^2},$$

$$F_1(x, q^4) = \frac{\theta_4(x, q^4) (\theta_1(x, q^4))^2 + (\theta_4(x, q^4))^2}{\theta_3(0, q^4) (\theta_2(0, q^4))^2 + (\theta_3(0, q^4))^2}.$$

Moreover, in each of these equations, put $q^4, q^{16}, q^{24} \dots$ instead of q , multiply the results obtained in this way with each other. That gives

$$\frac{\theta_1(x, q)}{\theta_2(0, q)} = \frac{\theta_1(x, q^{16})}{\theta_2(0, q^{16})} F_1(x, q^4) F_1(x, q^{16}) = \dots,$$

$$\frac{\theta_2(x, q)}{\theta_2(0, q)} = \frac{\theta_2(x, q^{16})}{\theta_2(0, q^{16})} F(x, q^4) F(x, q^{16}) = \dots$$

If one goes over to the limit, it follows

$$\frac{\theta_2(x, q)}{\theta_2(0, q)} = \lim_{l \rightarrow \infty} \frac{\theta_2(x, q^l)}{\theta_2(0, q^l)} F(x, q^4) F(x, q^{16}) F(x, q^{24}) \dots,$$

$$\frac{\theta_1(x, q)}{\theta_2(0, q)} = \lim_{l \rightarrow \infty} \frac{\theta_1(x, q^l)}{\theta_2(0, q^l)} F(x, q^4) F(x, q^{16}) F(x, q^{24}) \dots$$

for l unlimited integer. Since $q < 1$, therefore $\lim_{l \rightarrow \infty} q^l = 0$ and thus

$$\lim_{l \rightarrow \infty} \frac{\theta_2(x, q^l)}{\theta_2(0, q^l)} = \cos(x).$$

Changing x by $x + \frac{\pi}{2}$ then

$$\lim_{l \rightarrow \infty} \frac{\theta_1(x, q^l)}{\theta_2(0, q^l)} = \sin(x).$$

These remarks of Enneper undoubtedly deserve to be developed and exploited, since \cos_q and \sin_q appear in fact as limit product of a combination of theta functions.

Replace for example τ by $\tau' = -1/\tau$ then $q = e^{-\frac{i\pi}{\tau}}$. We thus obtain by using modern notations where $q' = e^{i\pi\tau'}$

$$\frac{\theta_2(x, \tau')}{\theta_2(0, \tau')} = \frac{\theta_2(x, 4\tau')}{\theta_2(0, 4\tau')} F(x, q'^4), \quad \frac{\theta_1(x, \tau')}{\theta_2(0, \tau')} = \frac{\theta_1(x, 4\tau')}{\theta_2(0, 4\tau')} F_1(x, q'^4).$$

So,

$$\cos_q(x) = \cos_{q^{1/4}} F(x, q'^4), \quad \sin_q(x) = \sin_{q^{1/4}} F_1(x, q'^4).$$

Or equivalently,

$$\cos_{q^4}(x) = \cos_q(x) F(x, q'), \quad \sin_{q^4}(x) = \sin_q(x) F_1(x, q').$$

Theorem 4-1

$$\sin_q(v) = (\sin \pi v) \prod_{k \geq 1} (\tan k\pi\tau')^2 \left[1 - \left(\frac{\sin \pi v}{\sin \frac{k\pi}{\tau}} \right)^2 \right] = (\sin \pi v) \prod_{k \geq 1} \frac{\sin(\pi v + \frac{k\pi}{\tau}) \sin(\frac{k\pi}{\tau} - \pi v)}{(\cos \frac{k\pi}{\tau})^2},$$

$$\cos_q(v) = (\cos \pi v) \prod_{k \geq 1} \left[1 - \left(\frac{\sin \pi v}{\cos \frac{k\pi}{\tau}} \right)^2 \right] = (\cos \pi v) \prod_{k \geq 1} \frac{\cos(\pi v + \frac{k\pi}{\tau}) \cos(\pi v - \frac{k\pi}{\tau})}{(\cos \frac{k\pi}{\tau})^2}.$$

$\sin_q(v)$ and $\cos_q(v)$ are defined in the band $|Imv| < \Im\tau$.

Proof of Theorem 4-1 Notice that since $\theta_1(v + \frac{1}{2}, \tau) = \theta_2(v, \tau)$ and by Corollary 3-5 one gets

$$\begin{aligned} & \frac{\theta_1(v + \frac{1}{2}, \tau)}{(\pi \cos \pi v) \theta'_1(0, \tau)} = \frac{\theta_2(v, \tau)}{(\pi \cos \pi v) \theta'_1(0, \tau)} \\ &= \frac{\theta_2(0, \tau)}{\pi \theta'_1(0, \tau)} \prod_{k \geq 1} \frac{\cos[k\pi\tau + \pi v] \cos[k\pi\tau - \pi v]}{(\cos[k\pi\tau])^2} = \prod_{k \geq 1} \frac{\cos[k\pi\tau + \pi v] \cos[k\pi\tau - \pi v]}{(\sin[k\pi\tau])^2}. \end{aligned}$$

We thus deduce

$$\theta_2(0, \tau') = \pi \theta'_1(0, \tau') \prod_{k \geq 1} (\cot k\pi\tau')^2.$$

So, we find again by another way the infinite product as a q-series

$$\begin{aligned} \pi \sin'_q(0) &= \frac{\pi \theta'_1(0, \tau')}{\theta_2(0, \tau')} = \prod_{k \geq 1} (\tan k\pi\tau')^2 \\ &= \pi \frac{\sum_{n \geq 0} (-1)^n (2n+1) q^{\binom{n+1}{2}}}{\sum_{n \geq 0} q^{\binom{n+1}{2}}} = \prod_{k \geq 1} \left(\frac{1 - q^{2k}}{1 + q^{2k}} \right)^2. \end{aligned}$$

On the other hand,

$$\frac{\theta_1(v + \frac{1}{2}, \tau)}{\theta_2(0, \tau)} = \frac{\theta_2(v, \tau)}{\theta_2(0, \tau)} = \frac{\theta_1(v + \frac{1}{2}, \tau)}{\pi \theta'_1(0, \tau) \prod_{k \geq 1} (\cot k\pi\tau)^2} = \cos \pi v \prod_{k \geq 1} \left[1 - \left(\frac{\sin \pi v}{\cos k\pi\tau} \right)^2 \right],$$

and

$$\begin{aligned} \theta_2(v, \tau) &= (\cos \pi v) \theta_2(0, \tau) \prod_{k \geq 1} \left[1 - \left(\frac{\sin \pi v}{\cos k\pi\tau} \right)^2 \right] \\ &= (\cos \pi v) \pi \theta'_1(0, \tau) \prod_{k \geq 1} (\cot k\pi\tau)^2 \prod_{k \geq 1} \left[1 - \left(\frac{\sin \pi v}{\cos k\pi\tau} \right)^2 \right]. \end{aligned}$$

Thus,

$$\frac{\theta_2(v, \tau')}{\theta_2(0, \tau')} = (\cos \pi v) \prod_{k \geq 1} \left[1 - \left(\frac{\sin \pi v}{\cos k\pi\tau'} \right)^2 \right] = (\cos \pi v) \prod_{k \geq 1} \left[\frac{(\sin \pi v)^2 - (\cos k\pi\tau')^2}{(\sin k\pi\tau')^2} \right],$$

$$\frac{\theta_1(v, \tau')}{\theta_2(0, \tau')} = (\sin \pi v) \prod_{k \geq 1} (\tan k\pi\tau')^2 \left[1 - \left(\frac{\sin \pi v}{\sin k\pi\tau'} \right)^2 \right]$$

$$= (\sin \pi v) \prod_{k \geq 1} (\tan k\pi\tau')^2 \left[\frac{(\cos \pi v)^2 - (\cos k\pi\tau')^2}{(\sin k\pi\tau')^2} \right].$$

Finally, we have

$$\begin{aligned} \sin_q(v) &= (\sin \pi v) \prod_{k \geq 1} (\tan k\pi\tau')^2 \left[1 - \left(\frac{\sin \pi v}{\sin \frac{k\pi}{\tau}} \right)^2 \right] \\ &= (\sin \pi v) \prod_{k \geq 1} (\tan k\pi\tau')^2 \left[\frac{(\cos \pi v)^2 - (\cos \frac{k\pi}{\tau})^2}{(\sin \frac{k\pi}{\tau})^2} \right], \\ \cos_q(v) &= (\cos \pi v) \prod_{k \geq 1} \left[1 - \left(\frac{\sin \pi v}{\cos \frac{k\pi}{\tau}} \right)^2 \right] = (\cos \pi v) \prod_{k \geq 1} \left[\frac{(\sin \pi v)^2 - (\cos \frac{k\pi}{\tau})^2}{(\sin \frac{k\pi}{\tau})^2} \right]. \end{aligned}$$

Remark $\sin_q(v), \cos_q(v)$ are related to the functions $g_j(v, \tau)$ defined above in Part 2 where $\tau' = \frac{-1}{\tau}$

$$\sin_q(v) = (\sin \pi v) \frac{g_1(v, \tau') g_1(-v, \tau')}{g_1(1/2, \tau') g_1(-1/2, \tau')}, \quad \cos_q(v) = (\cos \pi v) g_2(v, \tau') g_2(-v, \tau').$$

Taking the logarithmic derivative of functions $g_j(v, \tau)$, $j = 1, 2$ (see above Corollary 2-6)

$$g_1(v, \tau') = \prod_{k \geq 1} \frac{\sin[k\pi\tau' + \pi v]}{\sin[k\pi\tau']}, \quad g_2(v, \tau') = \prod_{k \geq 1} \frac{\cos[k\pi\tau' + \pi v]}{\cos[k\pi\tau']},$$

we derive the following

Corollary 4-2 *Let $q = e^{i\pi\tau}$, and $\tau' = \frac{-1}{\tau}$. Then the logarithmic derivative of $\sin_q(v), \cos_q(v)$ are*

$$\begin{aligned} \frac{\cos'_q(v)}{\cos_q(v)} &= \frac{1}{\cos_q} \frac{\partial \cos_q}{\partial v}(v) = -(\tan \pi v) + \sum_{k \geq 1} \tan[k\pi\tau' + \pi v] + \tan[k\pi\tau' - \pi v] \\ &= -(\tan \pi v) + \sum_{k \geq 1} \frac{\sin k\pi\tau'}{\cos k\pi\tau' - \sin \pi v} + \frac{\sin k\pi\tau'}{\cos k\pi\tau' + \sin \pi v}, \end{aligned}$$

$$\begin{aligned} \frac{\sin'_q(v)}{\sin_q(v)} &= \frac{1}{\sin_q} \frac{\partial \sin_q}{\partial v}(v) = (\cot \pi v) - \sum_{k \geq 1} \cot[k\pi\tau' + \pi v] + \cot[k\pi\tau' - \pi v], \\ &= (\cot \pi v) + \sum_{k \geq 1} \frac{\sin k\pi\tau'}{\cos k\pi\tau' - \cos \pi v} + \frac{\sin k\pi\tau'}{\cos k\pi\tau' + \cos \pi v}, \end{aligned}$$

\sin'_q and \cos'_q are defined in the band $|Imv| < \Im\tau$.

Theorem 4-3 *Let $q = e^{i\pi\tau}$, and $\tau' = \frac{-1}{\tau}$. Then the following identities hold*

$$(i) \quad \sin_q(2v) = 8 \prod_{k \geq 1} \left(\frac{\cos(\frac{k\pi}{\tau})}{(\cos(\frac{k\pi}{2\tau}))^2} \right)^2 \sin_{q^2}(v) \cos_{q^2}(v),$$

$$(ii) \quad \tan_q(2v) = \frac{\sin_q(2v)}{\cos_q(2v)} = \tan(2v) \prod_{k \geq 1} \tan(2\pi v + \frac{k\pi}{\tau}) \tan(\frac{k\pi}{\tau} - 2\pi v).$$

$$(iii) \quad \tan_{q^2}(v) = \tan(v) \prod_{k \geq 1} \tan(\pi v + \frac{k\pi}{2\tau}) \tan(\frac{k\pi}{2\tau} - \pi v).$$

$$(iv) \quad \frac{\tan_q(2v)}{\tan_{q^2}(v)} = \prod_{k \geq 1} \left[4 \frac{(\cos(\frac{k\pi}{2\tau}))^4}{(\cos(\frac{k\pi}{\tau}))^2} \right] \frac{\theta_2^2(v, \frac{\tau'}{2})}{\theta_2(2v, \tau)} = 4 \frac{(\cos_{q^2}(v))^2}{\cos_q(2v)} \prod_{k \geq 1} \frac{(\cos \frac{k\pi}{2\tau})^4}{(\cos \frac{k\pi}{\tau})^2}.$$

Proof of Theorem 4-3 Notice that after developing and reducing the following expressions

$$\left[1 - \frac{(\cos \pi v)^2}{(\sin k \frac{\pi}{\tau})^2} \right] \left[1 - \frac{(\sin \pi v)^2}{(\sin k \frac{\pi}{\tau})^2} \right], \quad \left[1 - \frac{(\sin 2\pi v)^2}{(\sin 2k \frac{\pi}{\tau})^2} \right],$$

one easily find

$$\left[1 - \frac{(\cos \pi v)^2}{(\sin k \frac{\pi}{\tau})^2} \right] \left[1 - \frac{(\sin \pi v)^2}{(\sin k \frac{\pi}{\tau})^2} \right] + (\cot \frac{k\pi}{\tau})^2 \left[1 - \frac{(\sin 2\pi v)^2}{(\sin 2k \frac{\pi}{\tau})^2} \right] = 0,$$

as well as

$$\left[1 - \frac{(\sin \pi v)^2}{(\cos k \frac{\pi}{\tau})^2} \right] \left[1 - \frac{(\sin \pi v)^2}{(\sin k \frac{\pi}{\tau})^2} \right] - \left[1 - \frac{(\sin 2\pi v)^2}{(\sin 2k \frac{\pi}{\tau})^2} \right] = 0.$$

Observe now from these trigonometric identities we can deduce

$$\sin_q(2v) = (\sin \pi 2v) \prod_{k \geq 1} \frac{\sin(2\pi v + \frac{k\pi}{\tau}) \sin(\frac{k\pi}{\tau} - 2\pi v)}{(\cos \frac{k\pi}{\tau})^2},$$

$$\sin_{q^2}(v) = (\sin \pi v) \prod_{k \geq 1} \frac{\sin(\pi v + \frac{k\pi}{2\tau}) \sin(\frac{k\pi}{2\tau} - \pi v)}{(\cos \frac{k\pi}{2\tau})^2},$$

$$\cos_{q^2}(v) = (\cos \pi v) \prod_{k \geq 1} \frac{\cos(\pi v + \frac{k\pi}{2\tau}) \cos(\pi v - \frac{k\pi}{2\tau})}{(\cos \frac{k\pi}{2\tau})^2}.$$

Therefore

$$\begin{aligned} \sin_{q^2}(v) \cos_{q^2}(v) &= (\sin \pi v)(\cos \pi v) \prod_{k \geq 1} \frac{\sin(\pi v + \frac{k\pi}{2\tau}) \sin(\frac{k\pi}{2\tau} - \pi v)}{(\cos \frac{k\pi}{2\tau})^2} \frac{\cos(\pi v + \frac{k\pi}{2\tau}) \cos(\pi v - \frac{k\pi}{2\tau})}{(\cos \frac{k\pi}{2\tau})^2} \\ &= \frac{1}{2} (\sin \pi 2v) \prod_{k \geq 1} \frac{\sin(2\pi v + \frac{k\pi}{\tau}) \sin(\frac{k\pi}{\tau} - 2\pi v)}{4(\cos \frac{k\pi}{2\tau})^4} \\ &= \frac{1}{8} \prod_{k \geq 1} \left(\frac{(\cos(\frac{k\pi}{2\tau}))^2}{(\cos(\frac{k\pi}{\tau}))} \right)^2 \sin_q(2v). \end{aligned}$$

We then find again by another way the Gosper's identity

$$\sin_q(2v) = \frac{\Pi_q}{\Pi_{q^2}} \sin_{q^2}(v) \cos_{q^2}(v) = \frac{\theta_1^2(\frac{\pi}{2}, \frac{\tau'}{2})}{\theta_1^2(\frac{\pi}{4}, \frac{\tau'}{2})} \sin_{q^2}(v) \cos_{q^2}(v),$$

where

$$\Pi_q = q^{\frac{1}{4}} \frac{(q^2; q^2)_{\infty}^2}{(q; q^2)_{\infty}^2}, \quad \frac{\Pi_q}{\Pi_{q^2}} = 8 \prod_{k \geq 1} \left(\frac{\cos(\frac{k\pi}{\tau})}{(\cos(\frac{k\pi}{2\tau}))^2} \right)^2.$$

To prove (iv) it suffices to develop and reduce

$$\begin{aligned} & \frac{\tan(\pi(2v + \frac{k}{\tau})) \tan(\pi(-2v + \frac{k}{\tau}))}{\tan(\pi(v + \frac{k}{2\tau})) \tan(\pi(-v + \frac{k}{2\tau}))} \\ &= \frac{\sin(\pi(2v + \frac{k}{\tau})) \sin(\pi(-2v + \frac{k\pi}{\tau})) \cos(\pi(v + \frac{k}{2\tau})) \cos(\pi(-v + \frac{k}{2\tau}))}{\cos(\pi(2v + \frac{k}{\tau})) \cos(\pi(-2v + \frac{k}{\tau})) \sin(\pi(v + \frac{k}{2\tau})) \sin(\pi(-v + \frac{k\pi}{2\tau}))} \\ &= 4 \frac{(\cos(\pi(v + \frac{k}{2\tau})))^2 (\cos(\pi(-v + \frac{k}{2\tau})))^2}{\cos(\pi(2v + \frac{k}{\tau})) \cos(\pi(-2v + \frac{k}{\tau}))}. \end{aligned}$$

Therefore

$$\prod_{k \geq 1} 4 \frac{(\cos(\pi(v + \frac{k}{2\tau})))^2 (\cos(\pi(-v + \frac{k}{2\tau})))^2}{\cos(\pi(2v + \frac{k}{\tau})) \cos(\pi(-2v + \frac{k}{\tau}))} = 4 \frac{(\cos_{q^2}(v))^2 \cos(2\pi v)}{\cos_q(2v) (\cos(\pi v))^2} \prod_{k \geq 1} \frac{(\cos \frac{k\pi}{2\tau})^4}{(\cos \frac{k\pi}{\tau})^2}.$$

On the other hand this could be re-written

$$\begin{aligned} & 4 \prod_{k \geq 1} \frac{(\cos(\pi(v + \frac{k}{2\tau})))^2 (\cos(\pi(-v + \frac{k}{2\tau})))^2}{\cos(\pi(2v + \frac{k}{\tau})) \cos(\pi(-2v + \frac{k}{\tau}))} = 4 \prod_{k \geq 1} \frac{\left((\cos(\frac{k\pi}{2\tau}))^2 - 1 + (\cos(\pi v))^2 \right)^2}{(\cos(\frac{k\pi}{\tau}))^2 - 4(\cos(\pi v))^2 + 4(\cos(\pi v))^4} \\ &= 4 \prod_{k \geq 1} \frac{\left((\cos(\frac{k\pi}{2\tau}))^2 - (\sin(\pi v))^2 \right)^2}{(\cos(\frac{k\pi}{\tau}))^2 - (\sin(2\pi v))^2} = \prod_{k \geq 1} \left[4 \frac{(\cos(\frac{k\pi}{2\tau}))^4}{(\cos(\frac{k\pi}{\tau}))^2} \right] \frac{\cos(2\pi v)}{(\cos(\pi v))^2} \frac{\theta_2^2(v, \frac{\tau'}{2})}{\theta_2(2v, \tau)}. \end{aligned}$$

Theorem 4-4 Let $q = e^{i\pi\tau}$, and $\tau' = \frac{-1}{\tau}$. Then the following hold

$$(\tan_{q^2}(v))^2 = \frac{\Pi_{q^2} \tan_q(2v)}{\Pi_q + \Pi_{q^2} \tan_q(2v)} = 1 - \prod_{k \geq 1} \left[\frac{\cot(\frac{k\pi}{\tau})}{\cot(\frac{k\pi}{2\tau})} \right]^2 \frac{\Pi_q}{4\Pi_{q^2}} \frac{(\cos(\frac{k\pi}{\tau}))^2 - (\sin(2v))^2}{\left((\cos(\frac{k\pi}{2\tau}))^2 - (\sin(v))^2 \right)^2},$$

$$\frac{\cos_q(2v)}{(\cos_{q^2}(v))^2} = \prod_{k \geq 1} \left[\frac{\cot(\frac{k\pi}{\tau})}{\cot(\frac{k\pi}{2\tau})} \right]^2 \frac{\Pi_q}{\Pi_{q^2}} \frac{(\cos(\frac{2k\pi}{\tau}) + \cos(4v))^2}{\cos(\frac{k\pi}{\tau}) + (\cos(2v))^2},$$

where $\Pi_q = \frac{1}{4}\theta_1^2(\frac{\pi}{2}, \frac{\tau'}{2})$ $\Pi_{q^2} = \frac{1}{4}\theta_1^2(\frac{\pi}{4}, \frac{\tau'}{2})$.

Proof of Theorem 4-4 Starting from the Gosper's identities

$$\frac{\sin_q(2v)}{\sin_{q^2}(v)} = \frac{\Pi_q}{\Pi_{q^2}} \cos_{q^2}(v), \quad \cos_q(2v) = \cos_{q^2}(v) - \sin_{q^2}(v),$$

we get

$$\frac{\tan_q(2v)}{\tan_{q^2}(v)} = \frac{\sin_q(2v) \cos_{q^2}(v)}{\cos_q(2v) \sin_{q^2}(v)} = \frac{\Pi_q}{\Pi_{q^2}} \frac{(\cos_{q^2}(v))^2}{\cos_q(2v)} = \frac{\Pi_q}{\Pi_{q^2}} \frac{1}{1 - (\tan_{q^2}(v))^2}.$$

On the other side one has

$$\frac{\tan_q(2v)}{\tan_{q^2}(v)} = 4 \prod_{k \geq 1} \left[\frac{\cot(\frac{k\pi}{\tau})}{\cot(\frac{k\pi}{2\tau})} \right]^2 \frac{\left((\cos(\frac{k\pi}{2\tau}))^2 - (\sin(v))^2 \right)^2}{\left((\cos(\frac{k\pi}{\tau}))^2 - (\sin(2v))^2 \right)^2}.$$

$$\frac{\cos_q(2v)}{(\cos_{q^2}(v))^2} = 1 - (\tan_{q^2}(v))^2 = \frac{\prod_q}{4\prod_{q^2}} \prod_{k \geq 1} \left[\frac{\cot(\frac{k\pi}{\tau})}{\cot(\frac{k\pi}{2\tau})} \right]^2 \frac{(\cos(\frac{k\pi}{\tau}))^2 - (\sin(2v))^2}{\left((\cos(\frac{k\pi}{2\tau}))^2 - (\sin(v))^2 \right)^2}.$$

By Enneper's terminology ([7] p.114)

$$F(x, q) = \frac{\theta_3(x, q)}{\theta_3(0, q)} \frac{(\theta_2(x, q))^2 + (\theta_3(x, q))^2}{(\theta_2(0, q))^2 + (\theta_3(0, q))^2}, \quad F_1(x, q) = \frac{\theta_4(x, q)}{\theta_3(0, q)} \frac{(\theta_1(x, q))^2 + (\theta_4(x, q))^2}{(\theta_2(0, q))^2 + (\theta_3(0, q))^2},$$

then

$$\frac{F_1(x, q)}{F(x, q)} = \frac{\theta_4(x, q)}{\theta_3(x, q)} \frac{(\theta_2(x, q))^2 + (\theta_3(x, q))^2}{(\theta_1(x, q))^2 + (\theta_4(x, q))^2}.$$

We deduce from Corollary 4-3

$$\tan_{q^4}(x) = \tan(v) \frac{F_1(v, q)}{F(v, q)} = \tan(v) \prod_{k \geq 1} \tan(\pi v + \frac{k\pi}{4\tau}) \tan(\frac{k\pi}{4\tau} - \pi v).$$

We thus obtain the following theta identity

Corollary 4-5 *Let $q = e^{i\pi\tau}$, and $\tau' = \frac{-1}{\tau}$. The following equalities hold*

$$\frac{\theta_4(v, \tau')}{\theta_3(v, \tau')} \frac{(\theta_2(v, \tau'))^2 + (\theta_3(v, \tau'))^2}{(\theta_1(v, \tau'))^2 + (\theta_4(v, \tau'))^2} = \prod_{k \geq 1} \tan(\pi v + \frac{k\pi}{4\tau}) \tan(\frac{k\pi}{4\tau} - \pi v) = \frac{\tan_{q^4}(v)}{\tan(v)}.$$

$$\frac{(\theta_2(v, \tau'))^2 + (\theta_3(v, \tau'))^2}{(\theta_1(v, \tau'))^2 + (\theta_4(v, \tau'))^2} = \prod_{k \geq 1} \frac{\tan(\pi v + \frac{k\pi}{4\tau}) \tan(\frac{k\pi}{4\tau} - \pi v)}{\tan(\pi v + \frac{k\pi}{\tau}) \tan(\frac{k\pi}{\tau} - \pi v)}.$$

Corollary 4-6 *Let $q = e^{i\pi\tau}$, and $\tau' = \frac{-1}{\tau}$. The following infinite products hold*

$$\prod_{k \geq 1} \frac{\cos[\pi(v + \frac{k}{2\tau})] \cos[\pi(-v + \frac{k}{2\tau})]}{\cos[\pi(v + \frac{k}{\tau})] \cos[\pi(-v + \frac{k}{\tau})]} = \prod_{k \geq 0} \cos[\pi(v + \frac{(k + \frac{1}{2})}{\tau})] \cos[\pi(-v + \frac{(k + \frac{1}{2})}{\tau})],$$

$$\prod_{k \geq 1} \frac{\sin[\pi(v + \frac{k}{2\tau})] \sin[\pi(-v + \frac{k}{2\tau})]}{\sin[\pi(v + \frac{k}{\tau})] \sin[\pi(-v + \frac{k}{\tau})]} = \frac{\theta_4(0, \tau')}{\theta_3(0, \tau')} \prod_{k \geq 0} \sin[\pi(v + \frac{(k + \frac{1}{2})}{\tau})] \sin[\pi(-v + \frac{(k + \frac{1}{2})}{\tau})],$$

$$\tan(\pi v) \prod_{k \geq 1} \frac{\tan[\pi(v + \frac{k}{2\tau})] \tan[\pi(-v + \frac{k}{2\tau})]}{\tan[\pi(v + \frac{k}{\tau})] \tan[\pi(-v + \frac{k}{\tau})]}$$

$$= \frac{\theta_4(0, \tau')}{\theta_3(0, \tau')} \prod_{k \geq 1} \tan[\pi(v + \frac{(k + \frac{1}{2})}{\tau})] \tan[\pi(-v + \frac{(k + \frac{1}{2})}{\tau})].$$

Proof Indeed, (following Gosper notation) it suffices to notice that

$$ccs_q(v) = \frac{\cos_{q^2}(v)}{\cos_q(v)} = \prod_{k \geq 1} \frac{\cos\left(\pi\left(v + \frac{k}{2\tau}\right)\right) \cos\left(\pi\left(-v + \frac{k}{2\tau}\right)\right)}{\cos\left(\pi\left(v + \frac{k}{\tau}\right)\right) \cos\left(\pi\left(-v + \frac{k}{\tau}\right)\right)},$$

$$ssn_q(v) = \frac{\sin_{q^2}(v)}{\sin_q(v)} = \prod_{k \geq 1} \frac{\sin\left(\pi\left(v + \frac{k}{2\tau}\right)\right) \sin\left(\pi\left(-v + \frac{k}{2\tau}\right)\right)}{\sin\left(\pi\left(v + \frac{k}{\tau}\right)\right) \sin\left(\pi\left(-v + \frac{k}{\tau}\right)\right)},$$

These functions can also be re-written

$$ccs_q(v) = \frac{\theta_3(v, \tau')}{\theta_3(0, \tau')}, \quad ssn_q(v) = \frac{\theta_4(v, \tau')}{\theta_3(0, \tau')}.$$

Therefore, we deduce the identities by Corollary 3-3 since

$$\frac{\theta_3(v, \tau')}{\theta_3(0, \tau')} = \prod_{k \geq 0} \frac{\cos\left[\left(k + \frac{1}{2}\right)\pi\tau' + \pi v\right] \cos\left[\left(k + \frac{1}{2}\right)\pi\tau' - \pi v\right]}{\left(\cos\left[\left(k + \frac{1}{2}\right)\pi\tau'\right]\right)^2}.$$

$$\frac{\theta_4(v, \tau')}{\theta_3(0, \tau')} = \frac{\theta_4(0, \tau')}{\theta_3(0, \tau')} \prod_{k \geq 0} \frac{\sin\left[\left(k + \frac{1}{2}\right)\pi\tau' + \pi v\right] \sin\left[\left(k + \frac{1}{2}\right)\pi\tau' - \pi v\right]}{\left(\sin\left[\left(k + \frac{1}{2}\right)\pi\tau'\right]\right)^2}.$$

4.2. n -order transformations and q -trigonometry.

Notice that Gosper [9, p.92] stated the following identity for any integer n

$$q^{\frac{n^2-1}{12}} \frac{(q, q^2)_{\infty}^2}{(q^n, q^{2n})_{\infty}^{2n}} \sin_q n\pi v = \prod_{k=0}^{n-1} \sin_{q^n}\left(v + \frac{k}{n}\right).$$

In [3] the authors point out that this formula is in fact equivalent to the Jacobi identity :

$$\theta_j(nv, n\tau) = \prod_{s \geq 1} \frac{(1 - q^{2sn})}{(1 - q^{2s})^n} \prod_{0 \leq m \leq (n-1)} \theta_j\left(v + \frac{m}{n}, \tau\right).$$

Another consequence of the infinite product expansions deduced from Theorem 3-6 is the n -order multiplication formulas for $\sin_q(v)$, $\cos_q(v)$ where in addition we distinguish the two cases n odd, n even

Theorem 4-7 *Let n be any integer then :*

(i) - *The following identity holds*

$$\sin_q(nv) = \frac{\pi\theta'_1(0, n\tau')}{\theta_2(0, n\tau')} \frac{2^{2n-2} \sin n\pi v}{(\sin \pi v)^n} \prod_{k \geq 1} \left[\frac{[\sin(k\pi\tau)]^n}{\sin(nk\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \sin_{q^n}\left(v + \frac{m}{n}\right).$$

(ii) - *When n is an even integer then*

$$\sin_q(nv) = \frac{2^{2n-2} \pi \theta'_1(0, n\tau')}{\theta_2(0, n\tau')} \prod_{k \geq 1} \left[\frac{[\sin(k\pi\tau)]^n}{\sin(nk\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \cos_{q^n}\left(v + \frac{m}{n}\right).$$

$$\cos_q(nv) = \frac{2^{2n-2} \pi \theta'_1(0, n\tau')}{\theta_2(0, n\tau')} \prod_{k \geq 1} \left[\frac{[\cos(k\pi\tau)]^n}{\cos(nk\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \sin_{q^n}\left(v + \frac{m}{n}\right).$$

$$\tan_q(nv) = \prod_{k \geq 1} \left[\frac{[\tan(k\pi\tau)]^n}{\tan(nk\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \cot_{q^n}(v + \frac{m}{n}).$$

(iii) - When n is odd then the following hold

$$\cos_q(nv) = \frac{\pi\theta'_1(0, n\tau')}{\theta_2(0, n\tau')} \frac{2^{2n-2} \cos n\pi v}{(\cos \pi v)^n} \prod_{k \geq 1} \left[\frac{[\cos(k\pi\tau)]^n}{\cos(nk\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \cos_{q^n}(v + \frac{m}{n}).$$

$$\tan_q(nv) = \frac{\tan n\pi v}{(\tan \pi v)^n} \prod_{k \geq 1} \left[\frac{[\tan(k\pi\tau)]^n}{\tan(nk\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \tan_{q^n}(v + \frac{m}{n}).$$

For $n = 2$ one find again

$$\begin{aligned} \sin_q(2v) &= \frac{4\pi\theta'_1(0, 2\tau')}{\theta_2(0, 2\tau')} \prod_{k \geq 1} \left[\frac{[\sin(k\pi\tau)]^2}{\sin(2k\pi\tau)} \right]^2 \cos_{q^2} v \cos_{q^2}(v + \frac{1}{2}) \\ &= \frac{\theta_2(0, \tau')\theta_3(0, \tau')}{\theta_2(0, 2\tau')\theta_3(0, 2\tau')} \sin_{q^2} v \cos_{q^2} v = \frac{\Pi_q}{\Pi_q^2} \sin_{q^2} v \cos_{q^2} v. \end{aligned}$$

Proof of Theorem 4-7 Theorem 3-6 yields for any n

$$\frac{\theta_1(nv, n\tau)}{\pi\theta'_1(0, n\tau)} = 2^{2n-2} \frac{\sin n\pi v}{(\sin \pi v)^n [\pi\theta'_1(0, \tau)]^n} \prod_{k \geq 1} \left[\frac{[\sin(k\pi\tau)]^n}{\sin(nk\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \theta_1(v + \frac{m}{n}, \tau),$$

or equivalently

$$\theta_1(nv, n\tau) = \pi\theta'_1(0, n\tau) 2^{2n-2} \frac{\sin n\pi v}{(\sin \pi v)^n [\pi\theta'_1(0, \tau)]^n} \prod_{k \geq 1} \left[\frac{[\sin(k\pi\tau)]^n}{\sin(nk\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \theta_1(v + \frac{m}{n}, \tau),$$

then

$$\begin{aligned} \sin_q(nv) &= \frac{\pi\theta'_1(0, n\tau')}{\theta_2(0, n\tau')} \frac{2^{2n-2} \sin n\pi v}{(\sin \pi v)^n [\pi\theta'_1(0, \tau')]^n} \prod_{k \geq 1} \left[\frac{[\sin(k\pi\tau')]^n}{\sin(nk\pi\tau')} \right]^2 \prod_{0 \leq m \leq n-1} \theta_1(v + \frac{m}{n}, \tau') \\ &= \frac{\pi\theta'_1(0, n\tau')}{\theta_2(0, n\tau')} \frac{2^{2n-2} \sin n\pi v}{(\sin \pi v)^n} \prod_{k \geq 1} \left[\frac{[\sin(k\pi\tau)]^n}{\sin(nk\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \sin_q^n(v + \frac{m}{n}). \end{aligned}$$

For n even we have

$$\frac{\theta_1(nv, n\tau)}{\pi\theta'_1(0, n\tau)} = 2^{2n-2} \frac{\sin n\pi v}{(\cos \pi v)^n [\theta_2(0, \tau)]^n} \prod_{k \geq 1} \left[\frac{[\sin(k\pi\tau)]^n}{\sin(nk\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \theta_2(v + \frac{m}{n}, \tau),$$

then

$$\sin_q(nv) = \frac{2^{2n-2} \pi\theta'_1(0, n\tau')}{\theta_2(0, n\tau')} \prod_{k \geq 1} \left[\frac{[\sin(k\pi\tau')]^n}{\sin(nk\pi\tau')} \right]^2 \prod_{0 \leq m \leq n-1} \cos_q^n(v + \frac{m}{n}).$$

For n odd we have

$$\frac{\theta_2(nv, n\tau)}{\theta_2(0, n\tau)} = 2^{2n-2} \frac{\cos n\pi v}{(\cos \pi v)^n [\theta_2(0, \tau)]^n} \prod_{k \geq 1} \left[\frac{[\cos(k\pi\tau)]^n}{\cos(nk\pi\tau)} \right]^2 \prod_{0 \leq m \leq n-1} \theta_2(v + \frac{m}{n}, \tau),$$

implying

$$\cos_q(nv) = \frac{\pi\theta'_1(0, n\tau')}{\theta_2(0, n\tau')} \frac{2^{2n-2} \cos n\pi v}{(\cos \pi v)^n} \prod_{k \geq 1} \left[\frac{[\cos(k\pi\tau')]^n}{\cos(nk\pi\tau')} \right]^2 \prod_{0 \leq m \leq n-1} \cos_q^n\left(v + \frac{m}{n}\right).$$

By Corollary 2-6 and Theorem 4-2 we deduce the logarithmic derivative of $\sin_q(nv)$, $\cos_q(nv)$. See also [3, Theorem 1]

Corollary 4-8 *Let n be any odd integer then*

$$n \frac{\cos'_q}{\cos_q}(nv) = \sum_{m=0}^{n-1} \frac{\cos'_q}{\cos_q} \left(v + \frac{m\pi}{n}\right),$$

$$n \frac{\sin'_q}{\sin_q}(nv) = -n \frac{\cos'_q}{\cos_q}\left(\frac{\pi}{2} - nv\right) = \sum_{m=0}^{n-1} \frac{\sin'_q}{\sin_q} \left(v + \frac{m\pi}{n}\right).$$

Indeed, this corollary is a consequence of Theorem 4-2 and trigonometric identities

$$n \tan nv = \sum_{m=0}^{n-1} \tan\left(v + \frac{m\pi}{n}\right), \quad n \cot nv = \sum_{m=0}^{n-1} \cot\left(v + \frac{m\pi}{n}\right).$$

Notice that

$$\lim_{q \rightarrow 1} \frac{\cos'_q}{\cos_q}(v) = \tan(v), \quad \lim_{q \rightarrow 1} \frac{\sin'_q}{\sin_q}(v) = \cot(v).$$

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