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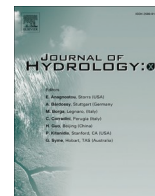
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A unifying model framework for soil erosion, river bedload and chemical transport

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ABSTRACT

A unified framework for simulating various transport processes in the environment is presented. It consists in a single set of partial differential equations. The main feature of this model framework is its exchange layer, which allows to treat several types of transfer between the soil and the surface water.

The model framework equations, termed transfer equations, is shown to reproduce three independently-published models developed for soil erosion, river bedload, and chemical transport respectively. By allowing the different processes to be represented within a single model framework, the transfer equations are therefore unifying the representation of particles and chemical fluxes in the environment. The transfer equations are implemented into the open-source software FullSWOF_1D. The code is verified against the approximation of an exact solution, assuring its proper functioning. A good adequacy is found between our numerical results and those published in the literature, attesting the capability of the transfer equations to unify modeling of soil erosion, river bedload, and chemical transport. Hence, the transfer equations can decrease the number of models to be used for simulating transfer of materials in the environment, and limit the number of computer codes to be developed and maintained. The transfer equations could also help in drawing parallels between different fields of hydrology.

1. Introduction

In hydrology, for a given process, many models may co-exist: for water infiltration at the soil surface, we have the equations of Green and Ampt (1911), Richards (1931), Horton (1939), etc., and for a given model, different numerical implementations are possible. For example, several computer codes exist for solving the shallow water equations (Barré de Saint-Venant, 1871): Hack and Jakob (1992), Breuer and Bader (2012), Delestre et al. (2017), etc. We emphasize the difference we make between a model, i.e. a set of equations, and a computer code: a given model can be solved by different codes (which can involve different numerical schemes or different mesh types).

Because different models were developed for different processes, it seems that each model has a very specific description. For example, numerous equations have been proposed to model the transport of a material in a water flow. These various equations can represent different types of materials, such as particles or dissolved chemicals, and account

for a wide range of processes: interrill erosion, rill erosion, gully erosion, suspended transport, bedload transport, etc. While this gives hydrologists and environment specialists a large choice of models and computer codes, this diversity may be more apparent than real.

Soil erosion, river bedload and chemical transport are typically seen as different processes, hence they are usually accounted for by different models, even if a few exceptions exist, such as Sander et al. (2011), where the Hairsine and Rose model (dedicated to soil erosion) was applied to the specific case of a sand-bed river. However, to simulate two transfer processes, two models usually have to be coupled and their interaction accounted for, leading to an additional complexity. For example, if one wants to simulate transfer processes within a watershed, such as interrill and rill erosions on the hillslopes, and bedload transport in the river network, it currently requires distinct functions to be implemented (one for each process), as well as dedicated coupling functions between these processes. The unifying model framework alleviates these needs: a single function can deal with all the processes,

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these processes being represented through a set of parameters. This obviously solves the coupling issues. Finally, one gets a computer code that is simpler to develop, to debug and to maintain, that simulates multiple processes, and that has a wide range of applications.

In this article, we describe a single model framework that can represent soil erosion, river bedload and chemical transport. Hence, this model framework unifies the description of what is usually regarded as three different processes, and would typically lead to the use of three separate models.

The proposed transfer model framework is based on a general mass balance and has been used previously in nonlinear chromatography (James et al., 2000). In the present paper, we consider a material located in the fluid or supplied by the original soil surface. The material can be stored in an exchange layer or released from this exchange layer. An exchange function is used to model the fluxes between the exchange layer and the water flow. The transfer model equations express the evolution in time and space of the concentrations of the material in the fluid and in the exchange layer. The model framework is not process-specific: various processes and materials can be considered. We show that this model framework is able to represent particle or chemical transport on hillslope or in rivers. Moreover, it can account for multiple classes of particles and their interactions. The model framework is of general purpose, while the equations are limited to the transport of a specific material in a specific type of water flow. This justifies the unifying character of the proposed model framework.

After describing the transfer model framework, its numerical implementation is presented. For a particular configuration, we are able to express a nearly-exact solution of the transfer model equations, for each time and space point. This is used to check that the software computes a good approximation of the solution and that no bug is present in the implementation. This is called a verification of proper functioning. Then, to exemplify its capabilities, we show that the unifying transfer model framework (and its computer code) can represent:

1. Soil erosion by water on a hillslope as done in the model of Hairsine et al. (2002). This model considers erosion and deposition processes separately, taking into account several sizes of particles.
2. Bedload transport in rivers, as done in the model of Lajeunesse et al. (2013). This model considers the spreading of a plume of tracer particles.
3. Chemical transport by overland flow with rain and diffusion in the soil, as done in the model of Gao et al. (2004).

While these three models (and for sure a lot of other models) are apparently very different, we show that they can be rewritten in the same formalism (i.e. the unifying transfer framework presented in this paper), and hence they can be run with a single computer code (e.g. FullSWOF_1D, or any other shallow water software through a coupling with the unifying transfer model equations).

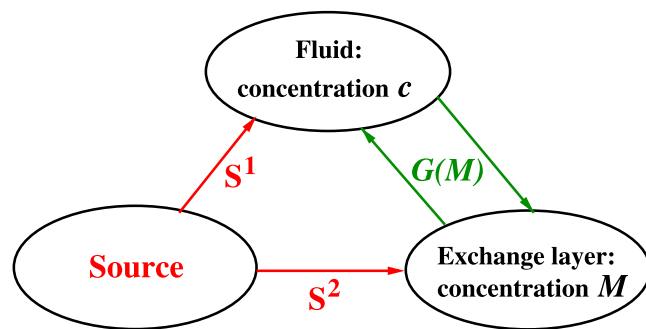
Note that several works are limited to the case of model equilibriums (Beuselinck et al., 2002; Sander et al., 2002). In this article, as an extra challenge, our goal is not only to catch the equilibrium, but also to get the evolution toward this equilibrium: starting from given initial conditions, the transitory solution towards the equilibrium is expected to be simulated.

2. Material and methods

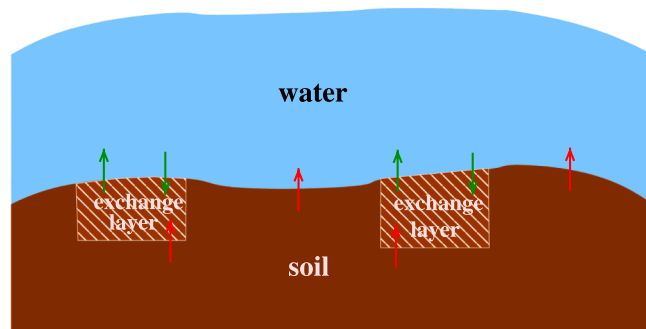
2.1. The unifying transfer model framework

2.1.1. Description

We propose a unifying transfer model framework to compute the concentrations of materials in the environment as functions of time and space, based on equations previously used in nonlinear chromatography (James et al., 2000). For the sake of clarity, we expose the unifying transfer model framework in one space dimension, its extension to two dimensions being straightforward.



(a) Conceptual representation.



(b) An example of physical representation.

Fig. 1. Conceptual representation (a) and an example of physical representation (b) of the transfer equations for each class of materials. The material has a concentration c in the fluid and M in the exchange layer. This material can be moved from the original soil to the water (red arrows and source term S^1), and from the original soil to the exchange layer (red arrows and source term S^2). An exchange, as defined by $G(M)$, can take place between the water and the exchange layer (green arrows). The exchange layer does not have to be continuous over the domain.

The transfer model framework accounts for a fluid containing some materials which can be exchanged between the water column and an exchange layer (Fig. 1a). A source can supply additional materials to both the fluid and the exchange layer. As an example, from a physical point-of-view (Fig. 1b), we consider a medium, such as the soil, from which the materials (particles, chemicals) are released (due to the flow, to the rain, etc.). These materials enter the flow, are transported, and can be accreted to a layer covering part of the initial surface. Eventually, these materials can be released again from this layer, hence this layer is termed “exchange layer”.

To write the transfer model equations, we consider the concentrations of the materials: c stands for the concentration in the fluid and M stands for the concentration in the exchange layer (Fig. 1a). We can consider several classes of materials (such as particles of different sizes): we denote the class by a i subscript ($i = 1 \dots N$). All the quantities are expressed per unit width.

Then, assuming the height of the fluid is h [m] and its discharge is q [$\text{m}^2 \text{s}^{-1}$], by writing the mass balance for the materials, we get the following system:

$$\begin{cases} \frac{\partial(hc_i)}{\partial t} + \frac{\partial(qc_i)}{\partial x} = \frac{1}{ts_i}(G_i(M) - hc_i) + S_i^1 \\ A \frac{\partial M_i}{\partial t} = -\frac{1}{ts_i}(G_i(M) - hc_i) + S_i^2, \end{cases} \quad (1)$$

where c_i [kg m^{-2}] is the concentration of the material class i in the fluid, M represents the vector of the M_i values, with M_i [kg m^{-1}] the

concentration of the material class i in the exchange layer, S_i^1 and S_i^2 [$\text{kg s}^{-1} \text{m}^{-1}$] are source terms representing the material from the original soil that supplies the flow and the exchange layer respectively (the signs of S_i^1 and S_i^2 define the direction of the supply), the function G_i is an equilibrium function, τ_i [s] is a relaxation time, and A a constant coefficient. The parameters G_i , τ_i and A are defined by the physical problem under consideration, and consequently they depend on the intended application.

In case of null source terms ($S_i^1 = S_i^2 = 0$), summing the two equations of (1) cancels the right-hand side, and the overall mass balance becomes

$$\frac{\partial(hc_i)}{\partial t} + A \frac{\partial M_i}{\partial t} + \frac{\partial(qc_i)}{\partial x} = 0. \quad (2)$$

The transfer equations (1) are considered as a system and, in this formulation, we cannot impose one concentration to be at the equilibrium in one equation but not the other (if the τ_i are of the same magnitude). Some studies, such as Sander et al. (2002), consider so-called ‘‘partial equilibria’’ of the Hairsine and Rose model, i.e. they cancel some terms in one equation but not in the other. That can lead to states of the system that could not be attained by the complete model, questioning the relevancy of such an approach. The transfer equations (1) couple the evolution of the concentration of materials in the fluid and the concentration of materials in the exchange layer. This draws two remarks related to the capabilities of the transfer equations (1): First, if an equilibrium state is to be achieved, it will be related to both the concentration in the fluid and the concentration in the exchange layer. Second, an equilibrium will be achieved only if the prescribed conditions permit; otherwise the system will not converge to an equilibrium.

2.1.2. Implementation

FullSWOF (for ‘‘Full Shallow Water equations for Overland Flow’’) is a free open-source software (GPL-compatible licence) that solves the shallow water equations and gives the evolution of a shallow layer of water in space and time (Delestre et al., 2017). Written in C++, it implements the finite-volume numerical method. It is developed by IDP (ex-MAPMO) and INRAE. There exists a one-dimensional version (for one-dimensional flows) where the quantities (such as the water height) depend only on one space dimension x and time, and a two-dimensional version, to simulate more general flows, where functions depend on the two space dimensions x and y and time. Currently, the transfer model framework (1) is integrated in the one-dimensional version of FullSWOF (FullSWOF_1D version 2.00.00) with a splitting method, meaning that, at each time step, the evolution of the water is computed first, giving the water height h and the water discharge q in space, and then the transfer equations (1) are solved. As implemented, the transfer equations (1) could also be used as a library by another flow software. In fact, for any other software giving the evolution of water height and discharge as functions of time, one can extract the resolution of the transfer model framework and make the coupling easily. For details about the implementation, see Appendix A.

In the following, the calculations are done using FullSWOF_1D. The parameters files are not fully detailed here but are available in the source code repository of FullSWOF_1D (<https://sourcesup.renater.fr/projects/fullswof-1d/>), either in the Benchmarks folder (for bedload transport in river) or in the Examples folder (for the other three cases).

2.1.3. Verification of proper functioning

Before doing comparisons with other physical models, we perform an academic test to check the proper functioning of the software. This is possible because we were able to write a nearly-exact solution at each time and at each point, for a given choice of parameters. A software computing the solutions of the transfer equations (1) should return a good approximation of this solution under the corresponding conditions, and the results should be improved when decreasing the space step.

We consider two sizes of particles inside the domain, distributed in the flowing layer and in the exchange layer, without any source, and we let the system evolve in time. We choose $G_i(M) = K_i M_i$, with $K_1 = 243$ and $K_2 = 0.3$, $\tau_1 = 0.087$ s and $\tau_2 = 7.17 \times 10^{-4}$ s, and $A = 1$. We solve the system

$$\begin{cases} \frac{\partial(hc_1)}{\partial t} + \frac{\partial(qc_1)}{\partial x} = \frac{1}{0.087}(243M_1 - hc_1), \\ \frac{\partial M_1}{\partial t} = -\frac{1}{0.087}(243M_1 - hc_1), \\ \frac{\partial(hc_2)}{\partial t} + \frac{\partial(qc_2)}{\partial x} = \frac{1}{0.000717}(0.3M_2 - hc_2), \\ \frac{\partial M_2}{\partial t} = -\frac{1}{0.000717}(0.3M_2 - hc_2), \end{cases} \quad (3)$$

starting from $h(t = 0, x) = 3.75 \times 10^{-3}$ m and $u(t = 0, x) = 0.3391$ m s $^{-1}$ to be at a steady-state for the water over an inclined plane with slope -0.02 (the velocity u is given by $u = q/h$ and stays constant as well as the water height), and

$$c_1(t = 0, x) = \begin{cases} x - 1 & \text{for } 1 \leq x \leq 2 \\ 1 & \text{for } 2 \leq x \leq 3 \\ 4 - x & \text{for } 3 \leq x \leq 4 \\ 0 & \text{elsewhere,} \end{cases} \quad (4)$$

$$c_2(t = 0, x) = \begin{cases} x - 1.5 & \text{for } 1.5 \leq x \leq 2 \\ 0.5 & \text{for } 2 \leq x \leq 3 \\ 3.5 - x & \text{for } 3 \leq x \leq 3.5 \\ 0 & \text{elsewhere,} \end{cases}$$

and $M_1(t = 0, x) = hc_1(t = 0, x)/K_1$, $M_2(t = 0, x) = hc_2(t = 0, x)/K_2$ for the particle concentrations (Fig. 4).

Summing the two equations for the same class of particles i , the right-hand sides cancel two by two (see Eq. (2)), and we get

$$\frac{\partial(hc_i + M_i)}{\partial t} + \frac{\partial(qc_i)}{\partial x} = 0. \quad (5)$$

In our configuration, as the τ_i are small, using an asymptotic development such as in James et al. (2000) and Lajeunesse et al. (2018), one can prove that a good approximation of the solution is given by $M_i(t, x) \approx hc_i(t, x)/K_i$ and, consequently, $c_i(t, x) \approx c_i(t = 0, x - \frac{q/h}{1 + K_i} t)$.

These theoretical approximations are useful to validate the code and to check the propagation velocities of the particles.

2.2. Erosion on an hillslope: the Hairsine and Rose model

2.2.1. Description

Hairsine et al. (2002) described a model of hillslope erosion by rainfall for a set of particle sizes. The model includes erosion, deposition and reentrainment. The eroded particles can be deposited, forming a cohesive deposition layer. This deposition layer can be eroded too,

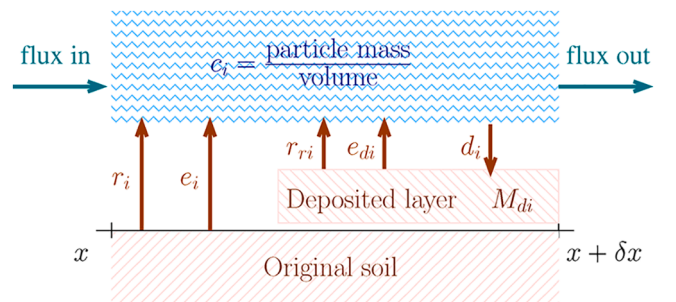


Fig. 2. Conceptual representation of the processes interacting between the original soil, the deposited layer and the particles in overland flow on a cell of the domain, from Hairsine et al. (2002).

hence the possible reentrainment.

Built-up on Hairsine and Rose (1991) and Hairsine and Rose (1992), the model of Hairsine et al. (2002) takes into account spatial variations, such as varying slopes. It is written as (Fig. 2):

$$\begin{cases} \frac{\partial(hc_i)}{\partial t} + \frac{\partial(qc_i)}{\partial x} = r_i + r_{ri} + e_i + e_{di} - d_i, \\ \frac{\partial M_{di}}{\partial t} = d_i - r_{ri} - e_{di}. \end{cases} \quad (6)$$

The variables h and q denote the water height and the discharge of the fluid respectively. The equations are written considering N different classes of particle sizes, denoted by the subscript i from 1 to N . The c_i term stands for the particle concentration in the fluid for the i^{th} size class, and M_{di} for the mass of particles of size class i in the deposited layer per unit area. The r_i term is the rate of entrainment of the original soil, r_{ri} is the rate of reentrainment of the deposited layer. The term e_i is the rate of rainfall detachment of the original soil, and e_{di} is the rate of rainfall detachment of the deposited layer. Finally, the term d_i is the deposition rate.

Explicit formulas are detailed in the next section and can be found in the original article. This model was evaluated and confronted to experimental results obtained with a laboratory rainfall simulator on a bare soil, and published in Proffitt et al. (1991).

2.2.2. Rewriting of the model of Hairsine et al. (2002)

The Hairsine and Rose equations (6) can be rewritten in the form of the unifying transfer equations (1). The deposited layer of the Hairsine and Rose model is easily identified to the exchange layer of the transfer equations. The terms r_i and e_i are source terms from the original soil, while r_{ri} , e_{di} and d_i are exchange terms between the fluid and the exchange layer. We obtain:

$$\begin{cases} \frac{\partial(hc_i)}{\partial t} + \frac{\partial(qc_i)}{\partial x} = r_{ri} + e_{di} - d_i + r_i + e_i = \frac{1}{ts_i}(G_i(M) - hc_i) + S_i^1 \\ A \frac{\partial M_i}{\partial t} = d_i - r_{ri} - e_{di} = -\frac{1}{ts_i}(G_i(M) - hc_i) + S_i^2 \end{cases} \quad (7)$$

with $A = 1$, $M_i = M_{di}$ the mass of particles of the class i per unit area of the deposited layer, $S_i^1 = r_i + e_i$ and $S_i^2 = 0$ the source terms, and the exchange is given thanks to the relation $\frac{1}{ts_i}(G_i(M) - hc_i) = r_{ri} + e_{di} - d_i$. More precisely, the deposition term d_i expresses as $d_i = v_i c_i$, where v_i is the settling velocity of the i^{th} class of particles, such that $ts_i = h/v_i$ (where we recall that h is the water height). The r_{ri} and e_{di} terms are functions of M_i (the explicit formulas are given below) and the function G_i reads $G_i(M) = ts_i(r_{ri} + e_{di})$. Following Hairsine et al. (2002) and Sander et al. (2007), the source and exchange terms can be written as:

$$\begin{cases} r_i = (1 - H)p_i \frac{F}{J}(\Omega - \Omega_0), \\ e_i = (1 - H)p_i a R^p, \\ r_{ri} = H \frac{F}{gh} \frac{\rho_s}{\rho_s - \rho} (\Omega - \Omega_0) \frac{M_i}{M_t}, \\ e_{di} = H a_d R^p \frac{M_i}{M_t}, \end{cases} \quad (8)$$

where:

- p_i is the proportion of particles in the size class i of the original (i.e. uneroded) soil,
- F is the fraction of excess stream power effective in entrainment and reentrainment,
- J is the specific energy of entrainment,
- $M_t = \sum_{i=1}^N M_i$ is the total mass of particles in the deposited layer,

- H is the fractional shielding of the underlying soil provided by the deposited layer. $H = \min(1, M_t/M_t^*)$, with M_t^* the mass of deposited particles required to shield the original soil completely,
- a is the detachability of the original soil,
- a_d is the detachability of the deposited layer,
- ρ_s and ρ are the particle and water densities respectively,
- g is the acceleration due to gravity,
- $\Omega = \rho g S q$ is the stream power (S is the slope of the domain and q the water discharge) and Ω_0 the critical threshold stream power,
- R is the rain intensity and p is an exponent that was evaluated to be close to unity (Proffitt et al., 1991). In the following, we take $p = 1$.

As a consequence, the source term S_i^1 is given by $S_i^1 = (1 - H)p_i \left(\frac{F}{J}(\Omega - \Omega_0) + a R^p \right)$ and the exchange function G_i writes $G_i(M) = \left(\frac{HF}{g v_i} \frac{\rho_s}{\rho_s - \rho} \frac{(\Omega - \Omega_0)}{M_t} + \frac{h}{v_i} \frac{H a_d R^p}{M_t} \right) M_i$. Note that, because there is no direct particle flux between the soil and the exchange layer in the model of Hairsine et al. (2002), the source term S_i^2 equals zero.

2.2.3. Numerical comparison

Hogarth et al. (2004) simulated the experiments of Proffitt et al. (1991), and specifically, temporal changes in sediment concentrations along a flume. According to Proffitt et al. (1991), the evolution of sediment concentrations is thought to be due to the development of a coarser deposited layer that shields an increasing fraction of the original soil surface until an equilibrium is achieved.

To numerically reproduce the experiments of Proffitt et al. (1991), Hogarth et al. (2004) propose a numerical model for rainfall-driven erosion over a slope $S = 0.4\%$ considering $N = 10$ classes of particles uniformly distributed, with a constant rain of $R = 100 \text{ mm h}^{-1}$. They fit the values of (1) the detachabilities a and a_d , and of (2) the mass of deposited particles required to shield the original soil M_t^* . As their numerical solution is in good agreement with the experiments, we take the same parameters as the ones they use. They found the value of M_t^* to be $M_t^* = 0.0767 \text{ kg m}^{-2}$; M_t^* is larger than M_t (so $H = M_t/M_t^*$), such that G_i becomes a linear function. As they also get an approximation of the water height as a linear function, we choose to initialize our flow solver with this expression and let it evolve in time towards the equilibrium.

2.3. Bedload transport in rivers: the model of Lajeunesse et al. (2013)

2.3.1. Description

Bedload transport results from the particle entrainment near the bed by a flow. A model for the spreading of a plume of tracer particles in a river by bedload transport was proposed in Lajeunesse et al. (2013) for the case of a steady-state transport over a flat bed of uniform particles. They denote by n the number of particles in motion per unit bed area, V the mean particle velocity, t_s the characteristic settling time of a particle, and $\alpha < 1$ the ratio of mobile to immobile particles. For steady-state uniform particle transport, erosion and deposition are balanced, the erosion and deposition rates being the same, both equal to n/t_s .

Lajeunesse et al. (2013) are interested in the evolution in time and space of a plume of marked particles (tracers). In their test case, the tracers are initially immobile on the upstream part of the domain. As they are entrained, the tracers move downstream. This phenomenon is modeled by the following system:

$$\begin{cases} \frac{\partial \phi}{\partial t} + V \frac{\partial \phi}{\partial x} = \frac{1}{t_s}(\psi - \phi) \\ \frac{1}{\alpha} \frac{\partial \psi}{\partial t} = -\frac{1}{t_s}(\psi - \phi), \end{cases} \quad (9)$$

where ϕ is the proportion of marked particles in the moving layer and ψ is the proportion of marked particles on the bed surface.

2.3.2. Rewriting of the model of Lajeunesse et al. (2013)

Let us explain how to write the model of Lajeunesse et al. (2013) as a particular case of the unifying transfer equations (1). Multiplying the two equations of system (9) by $m_g n$, where m_g is the mass of one particle (and n the number of particles in motion per unit bed area), we get:

$$\begin{cases} \frac{\partial C}{\partial t} + V \frac{\partial C}{\partial x} = \frac{1}{t_s}(\alpha M - C) \\ \frac{\partial M}{\partial t} = -\frac{1}{t_s}(\alpha M - C), \end{cases} \quad (10)$$

with $C = m_g n \phi$ and $M = m_g n \psi$ the mass concentration of mobile and immobile marked particles per unit bed area, respectively.

Note that these equations can also be written in a dimensionless form with the characteristic time $T = t/t_s$ and the characteristic space $X = x/(t_s V)$:

$$\begin{cases} \frac{\partial C}{\partial T} + \frac{\partial C}{\partial X} = (\alpha M - C) \\ \frac{\partial M}{\partial T} = -(\alpha M - C). \end{cases} \quad (11)$$

If we identify $C = hc$, $G(M) = \alpha M$, $S^1 = S^2 = 0$ and $A = 1$, the bedload transport model of Lajeunesse et al. (2013) can be written in the form of the transfer equations (1). This clearly shows that the proposed transfer equations are really a unifying model framework.

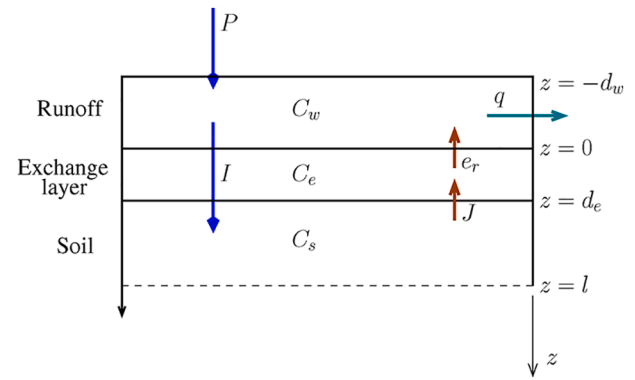


Fig. 3. Conceptual model of chemical transport processes, from Gao et al. (2004).

- the upper layer of height d_w is the runoff or surface water ponding zone,
- the exchange layer of height d_e , from which chemicals leave the soil and enter surface runoff,
- the lower layer of height $l - d_e$ is the soil, where chemicals are transported by diffusion and infiltration.

They proposed the following model:

$$\begin{cases} \frac{\partial(d_w C_w)}{\partial t} + \frac{\partial(q C_w)}{\partial x} = e_r(C_e - \lambda C_w) - I C_w & \text{for } -d_w < z \leq 0 \\ \frac{\partial(\alpha d_e C_e)}{\partial t} = J + e_r(\lambda C_w - C_e) + I(C_w - C_e) & \text{for } 0 < z \leq d_e \\ \frac{\partial(\alpha C_s)}{\partial t} = \frac{\partial}{\partial z} \left(D_s \frac{\partial C_s}{\partial z} - I C_s \right) & \text{for } d_e < z < l, \\ \text{with } \frac{\partial C_s}{\partial z} = 0 \text{ for } z = l \text{ and } C_s = C_e \text{ for } z = d_e, \end{cases} \quad (12)$$

2.3.3. Numerical comparison

Let us consider the tracer dispersion in bedload transport proposed by Lajeunesse et al. (2013). At the beginning of the experiment, the tracers are only immobile particles located between 0 and 20 (dimensionless distance), the proportion of marked particles on the bed surface ψ being equal to 1. We denote by α the ratio of the surface concentration of moving particles to the concentration of static particles. Following Lajeunesse et al. (2013), we choose α equal to 0.3: we obtain an initial concentration of tracer immobile particles of $\frac{1}{1+\alpha}\psi \approx 0.77$ (Fig. 6, black line). Then, we solve Eqs. (11) in terms of the variables ϕ and ψ , the proportions of marked particles in the moving layer and on the bed surface respectively (see the parameters file in the Benchmarks folder of the software repository).

2.4. Chemical transport: the model of Gao et al. (2004)

2.4.1. Description

The considered material can be dissolved chemicals (such as tracers, pollutants, nutrients, etc.). This approach is studied in Gao et al. (2004) with a model that takes into account both raindrop-driven processes and diffusion in the soil.

Gao et al. (2004) consider three different vertically-distributed horizontal layers (Fig. 3):

where the unknowns are the chemical concentrations: C_w is the chemical solute concentration in the runoff, C_e is the chemical solute concentration in the exchange layer, and C_s is the dissolved chemical concentration in the soil water. The total chemical concentration in the soil is αC_s , where α is equal to the soil moisture in case of non-adsorbed chemicals, or takes into account a partition coefficient in case of adsorbed chemicals. The variable J is the diffusion rate of solute from the soil at $z = d_e$ and is given by Fick's law:

$$J = D_s \frac{\partial C_s}{\partial z} \Big|_{z=d_e}. \quad (13)$$

The parameter I is the infiltration rate in the soil, P is the rainfall rate, q is the volumetric runoff flux per unit width, D_s is the dispersivity of chemicals in the soil and e_r , function of P , is the rate of soil water feeded into the runoff. As D_s is very small compared to e_r , the diffusion between the exchange layer and the runoff is neglected. Finally, λ (between 0 and 1) is the proportion of chemical concentration C_w entering the exchange layer: $\lambda = 0$ if there is only rainwater (without chemicals) and $\lambda = 1$ if the water that enters in the exchange layer is only runoff.

Through comparisons between experiments and numerical results, Gao et al. (2004) show that their model reproduces well the experiments and can be fitted using only measured parameters (i.e. without calibration).

2.4.2. Rewriting of the model of Gao et al. (2004)

The model (12) of Gao et al. (2004) can be written as a particular case of the unifying transfer equations (1): the equation on C_s (the third equation of (12)) can be considered as a secondary equation to be solved at each time step to obtain the value of the diffusion J . The water height is d_w , which corresponds to the variable h , and the water discharge is denoted by q in both models. If we denote by $M = d_e C_e$ (to get a concentration per surface unit) and $c = C_w$, we get:

$$\begin{cases} \frac{\partial(hc)}{\partial t} + \frac{\partial(qc)}{\partial x} = \frac{e_r}{d_e} M - \frac{\lambda e_r + I}{d_w} hc \\ \frac{\partial(\alpha M)}{\partial t} = -\left(\frac{e_r}{d_e} M - \frac{\lambda e_r + I}{d_w} hc\right) + J - \frac{I}{d_e} M, \end{cases} \quad (14)$$

which takes the form of the transfer equations (1) with $A = \alpha$, $S^1 = 0$, $S^2 = J - \frac{I}{d_e} M$, $ts = \frac{d_w}{\lambda e_r + I}$ and $G(M) = KM = \frac{d_w}{d_e} \frac{e_r}{\lambda e_r + I} M$.

2.4.3. Numerical comparison

Gao et al. (2004) proposed a numerical scheme to solve their model in a case with no infiltration ($I = 0$) and no water flux ($q = 0$) which fits their experiments. They apply the Crank-Nicolson finite-difference scheme, with a spatial discretization of 0.02 cm and a time step of 1 s, to get C_s as a function of z at each time step. C_e and C_w can be computed easily at each time from the values at the previous time. As their numerical results are in very good agreement with their experiments, we consider them as the reference solution to be compared to the results of the unifying transfer equations as computed by FullSWOF_1D.

We reproduce the first experiment of Gao et al. (2004), taking their values of the parameters, namely a rain intensity of 7.4 cm h^{-1} , an initial concentration of Cl^- in the soil and in the exchange layer of 29.82 g L^{-1} , a water height of 7 mm, a height of the exchange layer of 7.6 mm, a bulk density of $\rho_b = 1.5 \text{ g cm}^{-3}$, a soil moisture of $\theta = 0.37 = \alpha$, a soil erodibility of $a = 0.40 \text{ g cm}^{-3}$, a soil diffusivity coefficient of $4.2 \times 10^{-6} \text{ cm}^2 \text{ s}^{-1}$, a parameter $\lambda = 1$, and a raindrop chemical transfer rate of $e_r = 2.1 \times 10^{-4} \text{ cm s}^{-1}$.

3. Results

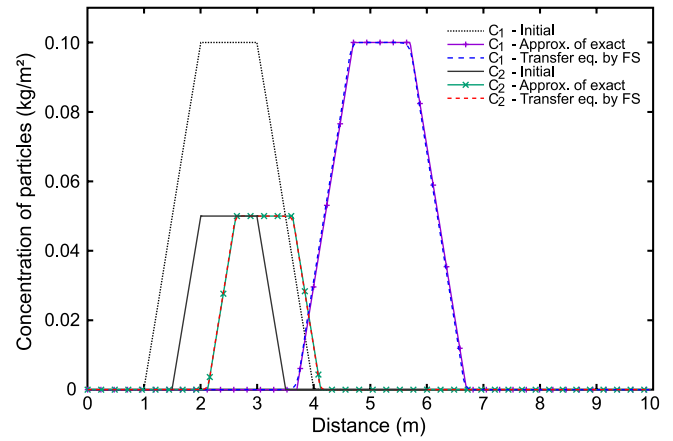
3.1. Verification of proper functioning

As explained in Section 2.1.3, we have an approximation of the exact solution for the unifying transfer equations in the case of two classes of particles without any source. In Fig. 4, we compare the values of c and M for the two classes of particles with the numerical solution obtained by FullSWOF_1D (see the parameters file in the Examples/Sediment_deposition folder of the repository). The results are given at time $T = 8 \text{ s}$. The values of M_1 are multiplied by 100 in order to display them within the same graph as M_2 values. The approximation of the exact solution and the numerical solution are in good agreement. Note that we took 1000 cells in space (i.e. the space step is equal to 1 cm) to have a good precision. If the number of cells is decreased, the results get less accurate, and the numerical errors accumulate with time.

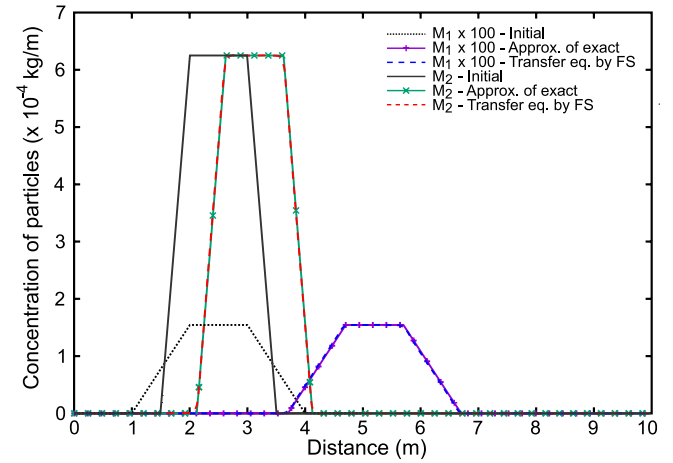
3.2. Rainfall detachment and deposition

In the context of rainfall-driven erosion over a slope, the experiments of Proffitt et al. (1991) were reproduced numerically by Hogarth et al. (2004), and now by the unifying transfer model framework.

In Fig. 5, we plot the total concentration of particles detached by the rain and suspended in the fluid as a function of space for several times (Fig. 5a), and at the output as a function of time (Fig. 5b). The numerical concentrations obtained with FullSWOF_1D and by Hogarth's computations are in good agreement, except for the first times ($T < 5 \text{ min}$). This discrepancy is due to the difference in the initial condition: we start from flow conditions near the equilibrium, and, by solving the Shallow



(a) In the water.

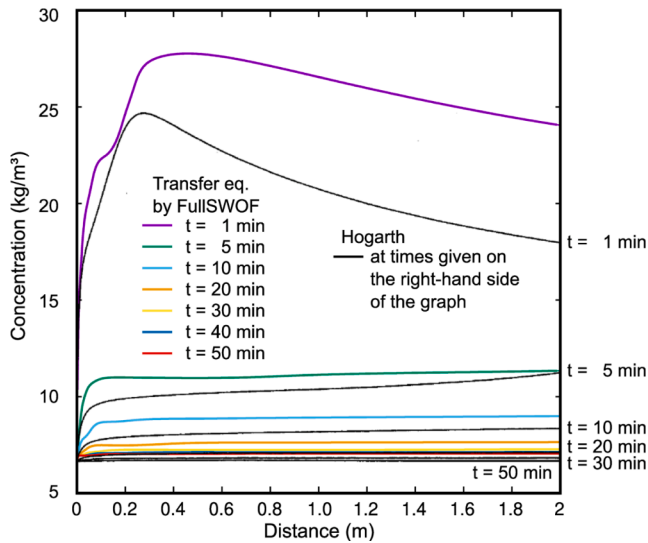


(b) In the exchange layer.

Fig. 4. Verification of proper functioning for two classes of particles evolving by erosion and deposition without sources. Concentrations (a) c_1 and c_2 in the water and (b) M_1 and M_2 in the exchange layer, at $T = 8 \text{ s}$, for the approximation of the exact solution and for the numerical solution of the unifying transfer equations obtained by FullSWOF_1D. The initial concentrations are also given.

Water equations coupled with the Hairsine and Rose equations for the particles, the steady-state for the water flow is attained after some time, while, in their calculations, Hogarth et al. (2004) consider the steady-state for the flow and only solve the equation for the particles.

As explained in the description of the unifying transfer model framework (Section 2.1.1), it is not possible to catch a partial equilibrium of the complete system where the fluid is at a steady-state but not the particles. The global equilibrium can be attained only if the experimental conditions allow it. In this experiment, the solution at the first times (namely, on Fig. 5, for $t = 1 \text{ min}$) is very dependent on the initial condition taken for the water height. One can see on Fig. 5b that the coupled Shallow Water – unifying transfer equations catches the first experimental point ($T = 1 \text{ min}$) in a much better way than Hogarth et al. (2004) while using the same parameters. We can also note, at the end of the experiment, near the steady-state, that the results of Hogarth et al. (2004) are quite similar to the ones of the unifying transfer equations solved by FullSWOF_1D: for large times, the same system is solved in both cases (the flow is at the equilibrium) and the results are the same. Thus, our approach can simulate both equilibrium and non-equilibrium conditions.



(a) Concentration along the domain for different times.

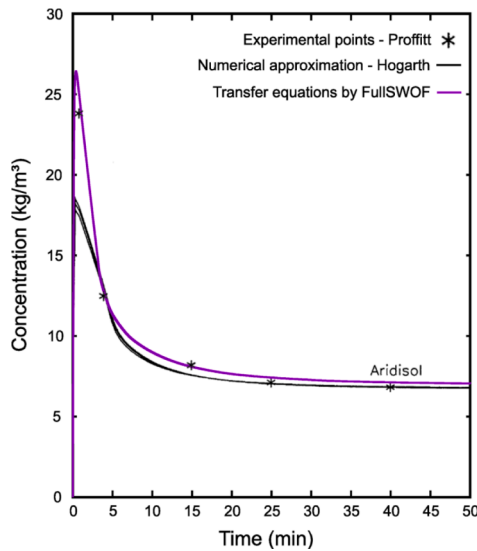


Fig. 6. Bedload transport in rivers. Results comparison for the tracer dispersion proposed by Lajeunesse et al. (2013) with $\alpha = 0.3$, at 4 times (dimensionless). Laj: results of Lajeunesse et al. (2013). FS: resolution of the unifying equations by FullSWOF_1D.

Fig. 5. Rainfall detachment and transport. Comparison of the experimental data of Proffitt et al. (1991) (stars), the numerical resolution of Hogarth et al. (2004) (black lines) and the numerical resolution of the unifying transfer equations by FullSWOF_1D (color lines).

3.3. Bedload transport in rivers

The dispersion of tracer particles by bedload transport, as proposed by Lajeunesse et al. (2013), was simulated using the unifying transfer model framework.

In Fig. 6, we plot the concentrations of tracer particles (Section 2.3.3), namely $\frac{1}{1+\alpha}\psi$ and $\frac{\alpha}{1+\alpha}\phi$, against distance (dimensionless). To better see the particles and their actual behavior in the two layers (mobile and immobile particles), we decided to perform the same plot as in Lajeunesse et al. (2013): we plot $\frac{1}{1+\alpha}\psi$ (immobile particles) and the sum $\frac{1}{1+\alpha}\psi + \frac{\alpha}{1+\alpha}\phi$ to see the concentration of mobile particles above the immobile particles. For all times ($T = 100, 700$ and 1400 – dimensionless), the unifying transfer equations solved by FullSWOF_1D match exactly the results obtained by Lajeunesse et al. (2013).

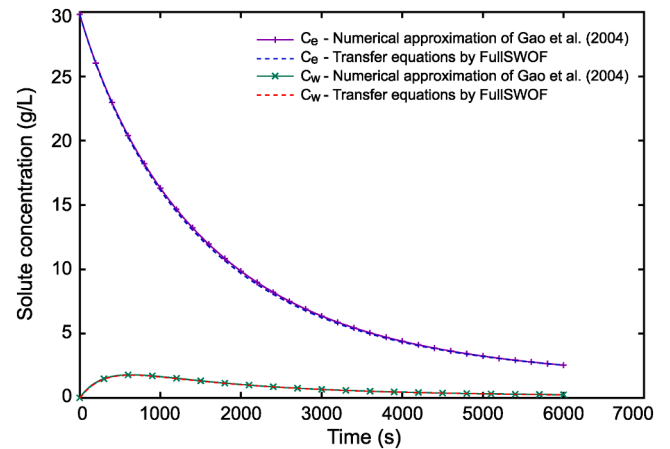


Fig. 7. Chemical transport. Solute concentrations in the exchange layer (C_e) and in the ponding water layer (C_w) at $x = 0.025$ m, as found by the numerical approximations of Gao et al. (2004) and by the resolution of the unifying transfer equations by FullSWOF_1D.

3.4. Chemical transport

An experiment of chemical transport with rain was simulated numerically by Gao et al. (2004), and now with the unifying transfer model framework.

In Fig. 7, we plot the solute concentration (i.e. Cl^-) in the exchange layer and in the runoff layer corresponding to this experiment with the numerical approximation proposed by Gao et al. (2004) represented as cross marks. FullSWOF_1D solves the shallow water equations and the unifying transfer equations with the same parameters, and we choose a point of the domain to plot the evolution in time. In both layers, the concentrations as computed by the unifying transfer equations match very well the results of Gao et al. (2004).

4. Discussion

The first point we want to underline in this discussion is the correctness of the results and the comparison with exact solutions. The writing of a code from a set of equations that models physical phenomena can be affected by two types of errors: either a bug or an issue in the numerical scheme used to solve the equations. In both cases, it negatively affects the results. However, this may easily go unnoticed:

assessing that the results are faithful to the equations is not a trivial task. The use of (quasi-)analytical solutions allows to verify the proper functioning (Delestre et al., 2013). As shown in the present paper, this approach has been applied successfully to the unifying transfer equations as implemented in FullSWOF_1D. Such a verification was previously carried out to check that FullSWOF_1D accurately solves the shallow water equations. FullSWOF_1D is routinely tested via a benchmarking script using ten analytical solutions of the shallow water equations taken from the library SWASHES (Delestre et al., 2013). Following the implementation of the unifying transfer equations, the exact solution of Lajeunesse et al. (2013) was added to the list of benchmarks. These analytical solutions, representing a diversity of flow conditions, enable users and developers to assess the quality of the numerical results (see <https://sourcesup.renater.fr/projects/fullswof-1d/>). On the one hand, they allow developers to check that code changes do not decrease the simulation quality. On the other hand, they allow users to check that FullSWOF_1D (and the unifying transfer equations) runs faithfully with their computer and compiler. We believe such an approach should be carried out as a standard procedure before any scientific results be reported. Since different computer codes of the same model can lead to non-identical results, we strongly recommend that other implementations of the transfer equations are validated using exact solutions.

Let us come now to the unifying transfer model framework proposed in this article. For soil erosion by rainfall, bedload transport in rivers and chemical transport, the numerical results of the unifying transfer equations coupled to the shallow water equations are in good agreement with the results reported in the literature. This clearly shows that the transfer equations are able to simulate soil erosion by rainfall, bedload transport in rivers and chemical transport. This underlines the unifying character of this model framework. Up to now, to simulate within a catchment both soil erosion (on hillslopes) and bedload transport (in rivers), two set of equations had to be implemented together with their boundary conditions, such as the boundary condition between hillslopes and rivers. With the unifying transfer model framework, both processes could be simulated within the same domain by the same computer code, just by changing the parameters of the unifying transfer equations (depending if the grid cells are within the hillslope or the river). Note that the current implementation of the transfer equations of FullSWOF_1D allows for a unique set of parameters within the domain. The implementation of the unifying transfer equations in the 2D version of FullSWOF (namely FullSWOF_2D – see <https://sourcesup.renater.fr/projects/fullswof-2d/>) will overcome this limitation.

In the case of soil erosion by rainfall, our results are not the same as the ones of Hogarth et al. (2004) because the authors simplified the equations, considering that one part of the model (the fluid) is at the equilibrium and not the other part (the particles). As a consequence, the transitory evolutions towards the equilibrium are different, while the same equilibrium is attained. Our numerical results outperform the results of Hogarth et al. (2004) because our implementation in FullSWOF_1D can simulate transitory states. Our approach is more complete, giving the evolution of the physical system in time as well as the final steady-state. In the environment, most of the processes are transitory; steady-states being more of an exception than a common case. Thus, the capability to simulate transitory states should be a requirement for most of the computer codes.

The unifying transfer model framework could also simulate processes not considered in the present paper, as long as they can be described as an exchange between two layers, following the scheme of Fig. 1a. While we show that the unifying transfer model framework is able to simulate soil erosion by rainfall, bedload transport in rivers and chemical transport, this set of equations may have application for other processes, in the environment or in other fields of science. Our approach, consisting in unifying various models from different fields of application, opens up interesting horizons: the same code can be used for different processes, alleviating the need to develop a specific code for

each process. As detailed before, this also makes the combined simulation of different processes easier. Finally, for each specific problem, the reader can identify the layers and the fluxes between the layers, as represented in Fig. 1a, and the corresponding parameters. Thus, the physical meaning of the parameters is not lost.

In all the cases considered in this paper, the relaxation term was linear, leading to the advection of symmetric distributions of concentration (e.g. Fig. 6). Indeed, Lajeunesse et al. (2018) showed that, for this system, the distribution of concentration tends asymptotically (i.e. for long time) to a Gaussian curve. For future works, it would be interesting to investigate the physical relevance of non-linear terms for hydrology, as it has already been done for the field of chromatography (James et al., 2000). That will lead to asymmetric distribution of concentrations: depending on the type of non-linearity, the front edge or the back edge of the concentration distribution could get sharper, skewing the distribution. Thus, the unifying transfer model framework could have an even wider range of applications than presently illustrated.

5. Conclusions

A unifying framework was proposed to model different transfers between a fluid layer and the soil. It is based on a general mass balance between the fluid layer and an exchange layer.

Implemented in FullSWOF_1D, its proper functioning was verified by comparison with a nearly-exact solution. This ascertained the adequacy of the computer programming.

Based on three examples, with various parameters and physical conditions, the obtained results were similar to or better than those in the literature. These examples covered three environmental transfer processes: soil erosion on a hillslope, bedload transport in rivers and chemical transport with rain and diffusion in the soil. While these examples are usually simulated using apparently different models and completely different computer codes, the unifying transfer model framework (and the same computer code) was able to simulate them. This exemplifies the unifying character of the proposed transfer model framework.

The unifying transfer equations relate the evolution of the concentrations of particles or chemicals in a fluid and in an exchange layer. They can account for several classes of particles or chemicals. They are proposed as a unifying transfer model framework for soil erosion, river bedload and chemical transport. The unifying transfer equations could open new prospects to understand one model thanks to another and to use a single model framework for several processes.

CRedit authorship contribution statement

Amina Nouhou Bako: Conceptualization, Methodology, Software, Formal analysis, Investigation, Writing - original draft. **Carine Lucas:** Conceptualization, Methodology, Software, Validation, Formal analysis, Writing - review & editing, Supervision, Funding acquisition. **Frédéric Darboux:** Conceptualization, Methodology, Validation, Writing - review & editing, Supervision, Funding acquisition. **François James:** Conceptualization, Methodology, Formal analysis, Writing - review & editing, Supervision, Funding acquisition. **Noémie Gaveau:** Validation.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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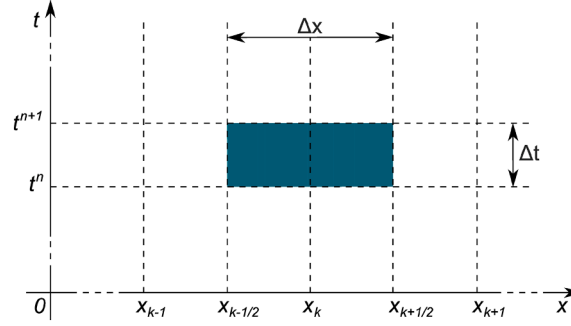


Fig. 8. Space-time discretization grid (1D in space).

Appendix A. Numerical method

We write the numerical scheme for the homogeneous part of the first equation of (1). The right-hand side terms and the second equation of (1) are added at the end this section. The scheme described here is performed for each class i of materials, one class after the other.

Denoting by $V = hc_i$ and $F(V) = uV = \frac{q}{h}V = qc_i$, the homogeneous part of the first equality of the transfer equations (1) reads:

$$\frac{\partial V}{\partial t} + \frac{\partial F(V)}{\partial x} = 0. \quad (\text{A.1})$$

The velocity u is the eigenvalue of system (1).

We use a finite-volume method to discretize Eq. (A.1) over a 1D grid, with a space step denoted by Δx and a time step denoted by Δt (Fig. 8). The discrete time is $t^0 = 0, t^1, \dots, t^n, t^{n+1} = t^n + \Delta t$, with t^0 the initial time. On the colored cell $[t^n, t^{n+1} \times]x_{k-1/2}, x_{k+1/2}]$, the value of V is considered as constant and is denoted by V_k^n .

Integrating Eq. (A.1) over each space-time cell, we get the relation:

$$V_k^{n+1} = V_k^n - \frac{\Delta t}{\Delta x} (F_{k+1/2}^n - F_{k-1/2}^n), \quad (\text{A.2})$$

where $F_{k+1/2}^n$ and $F_{k-1/2}^n$ are the approximations of the fluxes at the interfaces $x_{k+1/2}$ and $x_{k-1/2}$, respectively. They depend on the values of V_k^n on the left and right boundaries of the cell, denoted by V_L^n and V_R^n respectively.

A.1. Spatial scheme at order 1

To compute the values of the $F_{k+1/2}^n$ and $F_{k-1/2}^n$ fluxes, we choose the HLL scheme (Harten, Lax, van Leer, detailed in Harten et al. (1983)) that was proved to be robust and simple (Delestre, 2010). It is one of the fluxes already implemented in FullSWOF_1D for the hydrodynamic part.

For a given interface, we denote by $\varphi(V_L^n, V_R^n)$ the numerical flux that reads:

$$\varphi(V_L^n, V_R^n) = \begin{cases} F(V_L^n) & \text{if } 0 < c_L \\ \frac{c_R F(V_L^n) - c_L F(V_R^n)}{c_R - c_L} + \frac{c_L c_R}{c_R - c_L} (V_R^n - V_L^n) & \text{if } c_L < 0 < c_R \\ F(V_R^n) & \text{if } c_R < 0, \end{cases} \quad (\text{A.3})$$

where c_L and c_R are propagating speeds computed by the following formulas (Delestre et al., 2014):

$$c_L = \min_{V_L^n, V_R^n} (\min(u, 0)), \quad c_R = \max_{V_L^n, V_R^n} (\max(u, 0)). \quad (\text{A.4})$$

The values of V_L^n and V_R^n are taken equal to the values of V in the neighbor cells and consequently the fluxes are approximated by $F_{k+1/2}^n = \varphi(V_{k+1}^n, V_k^n)$ and $F_{k-1/2}^n = \varphi(V_k^n, V_{k-1}^n)$, which means that the scheme is at order one in space.

A.2. Spatial scheme at order 2

To improve the numerical scheme, one can linearly reconstruct the variables V_L^n et V_R^n in order to obtain a second order scheme in space. We choose MUSCL method (introduced in van Leer (1979)): the variables are reconstructed using a piece-wise linear function depending on the slope of each cell of the grid, denoted by σ_k^n . On each side of the interface located at $x_{k+1/2}$, the reconstructed values are:

¹ The time and space steps must be adapted to the parameters of the system; in particular, they must be small if the values of ts_i are small.

$$V_{k+1/2,L}^n = V_k^n + \frac{\Delta x}{2} \sigma_k^n, \quad V_{k+1/2,R}^n = V_{k+1}^n - \frac{\Delta x}{2} \sigma_{k+1}^n, \quad (\text{A.5})$$

which gives $F_{k+1/2}^n = \varphi(V_{k+1/2,L}^n, V_{k+1/2,R}^n)$ and $F_{k-1/2}^n = \varphi(V_{k-1/2,L}^n, V_{k-1/2,R}^n)$. To assure the stability of the scheme, one can compute three slopes from the neighbor cells with:

$$\alpha_{k-1} = \frac{V_k^n - V_{k-1}^n}{\Delta x}, \quad \alpha_k = \frac{V_{k+1}^n - V_{k-1}^n}{2\Delta x}, \quad \alpha_{k+1} = \frac{V_{k+1}^n - V_k^n}{\Delta x}, \quad (\text{A.6})$$

and compute σ_k^n as $\sigma_k^n = \text{minmod}(\alpha_{k-1}, \alpha_k, \alpha_{k+1})$ where the minmod function is defined by:

$$\text{minmod}(a, b, c) = \begin{cases} \min(a, b, c) & \text{if } a, b, c > 0 \\ \max(a, b, c) & \text{if } a, b, c < 0 \\ 0 & \text{else,} \end{cases} \quad (\text{A.7})$$

as described in [Delestre \(2010\)](#) and [Godlewski and Raviart \(1996\)](#).

A.3. Second order in time

In the previous Eq. (A.2), we perform an Euler scheme, of order 1 in time. To improve the accuracy of the scheme and get a second order in time, we choose the Heun method, as it is also the one already implemented in FullSWOF ([Delestre et al., 2014](#)). It is a predictor–corrector method that computes first some predicted values V_k^* and V_k^{**} and then performs a correction step to obtain the value at the following time $n + 1$:

$$\begin{cases} V_k^* = V_k^n - \frac{\Delta t}{\Delta x} (F_{k+1/2}^n - F_{k-1/2}^n) \\ V_k^{**} = V_k^* - \frac{\Delta t}{\Delta x} (F_{k+1/2}^* - F_{k-1/2}^*) \end{cases} \quad \text{and} \quad V_k^{n+1} = \frac{V_k^n + V_k^{**}}{2}. \quad (\text{A.8})$$

A.4. Coupling with the hydrodynamic part of FullSWOF_1D

To implement a numerical scheme for the full system (1), we must deal with the right-hand side of the equations, that is the relaxation term and the sources. This leads to the following numerical scheme:

$$\begin{cases} V_k^{n+1} = V_k^n + \Delta t \left(\frac{-F_{k+1/2}^n + F_{k-1/2}^n}{\Delta x} + \frac{(G_i(M))_k^n - V_k^n}{(ts_i)_k^n} + (S_i^1)_k^n \right) \\ (M_i)_k^{n+1} = (M_i)_k^n + \frac{\Delta t}{A} \left(\frac{-(G_i(M))_k^n + V_k^n}{(ts_i)_k^n} + (S_i^2)_k^n \right). \end{cases} \quad (\text{A.9})$$

We couple this numerical scheme with the hydrodynamic part of FullSWOF_1D with a splitting method. The variables h and $q = hu$ are computed at time n by FullSWOF_1D and then are used by the transfer part to compute the concentrations c_i and M_i at time $n + 1$.

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