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► **To cite this version:**

Bernard Candelpergher. An expansion of the Riemann Zeta function on the critical line. 2021. hal-03265364

HAL Id: hal-03265364

<https://hal.science/hal-03265364>

Preprint submitted on 20 Jun 2021

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An expansion of the Riemann zeta function on the critical line

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Abstract

We give an expansion of the Riemann zeta function on the critical line as a converging series $\sum_{m \geq 0} a_m q_m(\frac{1}{2} + it)$ in the space $L^2(\mathbb{R}, \frac{dt}{\cosh(\pi t)})$, where the functions q_m are related to Meixner polynomials of the first kind and the coefficients a_m are linear combinations of the Euler constant γ and the values $\zeta(2), \zeta(3), \dots, \zeta(m+1)$.

1 Laguerre functions

The Laguerre polynomials $x \mapsto L_m(2x)$ are defined by the generating function (cf. [6])

$$\frac{1}{1-a} e^{-\frac{2ax}{1-a}} = \frac{1}{1+a} \sum_{m=0}^{+\infty} L_m(2x) a^m \quad \text{where } |a| < 1.$$

They are given by $L_m(2x) = \sum_{k=0}^m C_m^k (-2)^k \frac{x^k}{k!}$. The Laguerre functions

$$\varphi_m(x) = \sqrt{2} e^{-x} L_m(2x),$$

form an orthonormal basis of $L^2(]0, +\infty[, dx)$, and we have the generating function

$$\frac{\sqrt{2}}{1-u} e^{-x \frac{1+u}{1-u}} = \sum_{m=0}^{+\infty} \varphi_m(x) u^m \quad \text{where } |u| < 1. \quad (1)$$

With $z = \frac{1+u}{1-u}$, we get for $Re(z) > 0$

$$e^{-xz} = \sqrt{2\pi} \sum_{m=0}^{+\infty} \varphi_m(x) \psi_m(z), \quad (2)$$

where ψ_m is the function defined in the half-plane $\{Re(z) > 0\}$ by

$$\psi_m(z) = \frac{1}{\sqrt{\pi}(1+z)} \left(\frac{z-1}{z+1} \right)^m.$$

The function ψ_m is related to φ_m by

$$\psi_m(z) = \int_0^{+\infty} \frac{e^{-xz}}{\sqrt{2\pi}} \varphi_m(x) dx.$$

Thus we have $\psi_m(z) = \frac{1}{\sqrt{2\pi}} \mathcal{L}\varphi_m(z)$, where \mathcal{L} is the Laplace transform

$$\mathcal{L}f(z) = \int_0^{+\infty} e^{-xz} f(x) dx.$$

This transformation maps the space $L^2(]0, +\infty[, dx)$ to the Hardy space $H^2(P)$ of analytic functions g in the half-plane $P = \{Re(z) > 0\}$ such that: there exists $M_g > 0$ with

$$\int_{\mathbb{R}} |g(x+iy)|^2 dy \leq M_g \quad \text{for all } x > 0.$$

2 Mellin transform and Meixner polynomials

For a function f on $]0, +\infty[$, we define the Mellin transform of f by

$$\mathcal{M}(f)(s) = \int_0^{+\infty} x^{s-1} f(x) dx$$

which is supposed to be defined for $s \in \mathbb{C}$ such that $0 < \operatorname{Re}(s) < 1$.

We have $\mathcal{M}\mathcal{L}f(s) = \Gamma(s)\mathcal{M}f(1-s)$ for $0 < \operatorname{Re}(s) < 1$, and if f and g are in $L^2(]0, +\infty[)$, we have the *Parseval-Mellin formula* (cf. [4]) :

$$\frac{1}{2i\pi} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathcal{M}(f)(z) \overline{\mathcal{M}(g)(z)} dz = \int_0^{+\infty} f(x) \overline{g(x)} dx.$$

The Mellin transform of φ_m is given by

$$\int_0^{+\infty} \varphi_m(x) x^{s-1} dx = \sqrt{2} \sum_{k=0}^m C_m^k \frac{(-2)^k}{k!} \int_0^{+\infty} e^{-x} x^{s+k-1} dx = \sqrt{2} \Gamma(s) q_m(s),$$

with

$$q_m(s) = \sum_{k=0}^m C_m^k (-2)^k \frac{(s)_k}{k!} \quad \text{where} \quad (s)_k = s(s+1) \cdots (s+k-1) \quad (\text{with } (s_0) = 1).$$

We have (cf. [2]) $q_m(s) = F(-m, s; 1; 2)$ where F is the Gauss hypergeometric function (also denoted by ${}_2F_1$). We have also $q_k(s) = \frac{1}{k!} m_k(-s, 1, -1)$, where m_k is a Meixner polynomial of the first kind.

Since $\mathcal{M}\varphi_m(s) = \sqrt{2} \Gamma(s) q_m(s)$ and $\psi_m = \frac{1}{\sqrt{2\pi}} \mathcal{L}\varphi_m$, we get for $0 < \operatorname{Re}(s) < 1$

$$\mathcal{M}\psi_m(s) = \frac{1}{\sqrt{2\pi}} \mathcal{M}\mathcal{L}\varphi_m(s) = \frac{1}{\sqrt{2\pi}} \Gamma(s) \Gamma(1-s) \mathcal{M}\varphi_m(1-s) = \frac{\sqrt{\pi}}{\sin(\pi s)} q_m(1-s).$$

By the definition of ψ_m we verify that $\frac{1}{x} \psi_m(\frac{1}{x}) = (-1)^m \psi_m(x)$, thus we have

$$\mathcal{M}\psi_m(1-s) = (-1)^m \mathcal{M}\psi_m(s),$$

this gives

$$q_m(1-s) = (-1)^m q_m(s).$$

By the Parseval-Mellin formula we get

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{M}(\varphi_m)\left(\frac{1}{2} + it\right) \overline{\mathcal{M}(\varphi_n)\left(\frac{1}{2} + it\right)} dt = \int_0^{+\infty} \varphi_m(x) \varphi_n(x) dx = \delta_{m,n}$$

(with $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$ if $m \neq n$). This gives

$$\delta_{m,n} = \int_{-\infty}^{+\infty} \Gamma\left(\frac{1}{2} + it\right) q_m\left(\frac{1}{2} + it\right) \overline{\Gamma\left(\frac{1}{2} + it\right) q_n\left(\frac{1}{2} + it\right)} \frac{dt}{\pi} = \int_{-\infty}^{+\infty} q_m\left(\frac{1}{2} + it\right) \overline{q_n\left(\frac{1}{2} + it\right)} \frac{dt}{\cosh(\pi t)}$$

Thus the polynomials $t \mapsto q_m(\frac{1}{2} + it)$ form an orthonormal basis of $L^2(\mathbb{R}, \frac{dt}{\cosh(\pi t)})$ with respect to the scalar product

$$(f|g) = \int_{-\infty}^{+\infty} f(t)\overline{g(t)} \frac{dt}{\cosh(\pi t)}$$

This implies that all the zeros of the polynomials $t \mapsto q_m(\frac{1}{2} + it)$ are real.

We have $q_0 = 1$ and

$$\begin{aligned} q_1\left(\frac{1}{2} + it\right) &= -2it \\ q_2\left(\frac{1}{2} + it\right) &= 1/2 - 2t^2 \\ q_3\left(\frac{1}{2} + it\right) &= -5/3it + 4/3it^3 \\ q_4\left(\frac{1}{2} + it\right) &= 3/8 - 7/3t^2 + 2/3t^4 \end{aligned}$$

By Mellin transform of (1), we see that the generating function of the polynomials q_m is

$$\sum_{m=0}^{+\infty} q_m(s)u^m = \frac{1}{1-u} \left(\frac{1+u}{1-u}\right)^{-s} \quad \text{for } u \in]-1, 1[. \quad (3)$$

This gives, with $y = \frac{1+u}{1-u}$, the relation

$$y^{-s} = 2\sqrt{\pi} \sum_{m=0}^{+\infty} \psi_m(y)q_m(s) \quad \text{for } y > 0. \quad (4)$$

Let $s = \frac{1}{2} + it$ with $t \in \mathbb{R}$ and $y = e^{-\xi}$, we get

$$e^{it\xi} = 2\sqrt{\pi}e^{-\xi/2} \sum_{m=0}^{+\infty} \psi_m(e^{-\xi})q_m\left(\frac{1}{2} + it\right) \quad \text{for } \xi \in \mathbb{R}.$$

The latter series converges in $L^2(\mathbb{R}, \frac{dt}{\cosh(\pi t)})$ since $\sum_{m=0}^{+\infty} |\psi_m(e^{-\xi})|^2 < +\infty$. And if a function $h \in L^2(\mathbb{R}, \frac{dt}{\cosh(\pi t)})$ has an expansion

$$h(t) = \sum_{n \geq 0} a_n q_n\left(\frac{1}{2} + it\right),$$

then we have

$$(h|e^{it\xi}) = 2\sqrt{\pi}e^{-\xi/2} \sum_{m=0}^{+\infty} a_m \psi_m(e^{-\xi}),$$

that is

$$\mathcal{F}\left(\frac{h(t)}{\cosh(\pi t)}\right)(\xi) = 2\sqrt{\pi}e^{-\xi/2} \sum_{m=0}^{+\infty} a_m \psi_m(e^{-\xi}) \quad (5)$$

where \mathcal{F} is the Fourier transform $\mathcal{F}g(\xi) = \int_{-\infty}^{+\infty} g(t)e^{-it\xi} dt$.

3 An expansion of Zeta

In the critical strip $0 < \operatorname{Re}(s) < 1$, we have (cf. [7])

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} f(x)x^{s-1}dx = \frac{1}{\Gamma(s)}\mathcal{M}f(s) \quad \text{where} \quad f(x) = \frac{1}{e^x - 1} - \frac{1}{x}$$

(also we have $\zeta(s) = s \int_0^{+\infty} ([x] - x)x^{-s-1}dx$, which gives $|\zeta(\frac{1}{2} + it)| = O(|t|)$ for $t \rightarrow \pm\infty$).

Since we have (cf.[2]) for $x > 0$

$$\mathcal{L}(f)(x) = \log(x) - \Psi(1+x) \quad \text{where} \quad \Psi = \frac{\Gamma'}{\Gamma},$$

we get for $0 < \operatorname{Re}(s) < 1$

$$\mathcal{M}(\log(x) - \Psi(1+x))(s) = \mathcal{M}\mathcal{L}(f)(s) = \Gamma(s)\mathcal{M}(f)(1-s) = \Gamma(s)\Gamma(1-s)\zeta(1-s),$$

thus

$$\frac{\pi}{\sin(\pi s)}\zeta(1-s) = \mathcal{M}(\log(x) - \Psi(1+x))(s).$$

By Mellin inversion, we obtain for $x > 0$

$$\log(x) - \Psi(1+x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin(\pi s)}\zeta(1-s)x^{-s} ds \quad \text{for all } 0 < c < 1.$$

With $c = \frac{1}{2}$, we have for $x > 0$

$$\log(x) - \Psi(1+x) = \frac{1}{2} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + it\right) \frac{1}{\sqrt{x}} e^{it \log(x)} \frac{dt}{\cosh(\pi t)}.$$

Let $x = e^{-\xi}$ with $\xi \in \mathbb{R}$, then we get

$$\mathcal{F}\left(\frac{\zeta\left(\frac{1}{2} + it\right)}{\cosh(\pi t)}\right)(\xi) = -2e^{-\xi/2}(\xi + \Psi(1 + e^{-\xi})). \quad (6)$$

By (5), the coefficients a_m of the expansion of $t \mapsto \zeta(\frac{1}{2} + it)$ in the space $L^2(\mathbb{R}, \frac{dt}{\cosh(\pi t)})$

$$\zeta\left(\frac{1}{2} + it\right) = \sum_{m \geq 0} a_m q_m\left(\frac{1}{2} + it\right)$$

are given by

$$-(\xi + \Psi(1 + e^{-\xi})) = \sqrt{\pi} \sum_{m=0}^{+\infty} a_m \psi_m(e^{-\xi}). \quad (7)$$

For an explicit evaluation of a_m , let $u = \frac{e^{-\xi}-1}{e^{-\xi}+1}$ in the relation (7), then we get for $-1 < u < 1$

$$\frac{1}{1-u} \left(\log\left(\frac{1+u}{1-u}\right) - \Psi\left(1 + \frac{1+u}{1-u}\right) \right) = \sum_{m=0}^{+\infty} b_m u^m \quad \text{where} \quad b_m = \frac{a_m}{2}. \quad (8)$$

Now, take the Taylor expansion of the left side of (8). For the logarithmic part, we have simply

$$\frac{1}{1-u} \log\left(\frac{1+u}{1-u}\right) = \frac{1}{1-u} 2 \sum_{n=0}^{+\infty} \frac{1}{2n+1} u^{2n+1} = \sum_{n=1}^{+\infty} \left(2 \sum_{p=0}^{[(n-1)/2]} \frac{1}{2p+1}\right) u^n,$$

For the part involving the function Ψ , we need the help of (cf.[2]) the integral formula

$$\Psi(x+1) = \frac{1}{x} + \Psi(x) = \frac{1}{x} - \gamma + \int_0^{+\infty} \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt,$$

this gives

$$-\frac{1}{1-u} \Psi\left(1 + \frac{1+u}{1-u}\right) = -\frac{1}{1+u} + \frac{1}{1-u} \gamma - \frac{1}{1-u} \int_0^{+\infty} \frac{e^{-t} - e^{-\frac{1+u}{1-u}t}}{1 - e^{-t}} dt,$$

and, with (1), we get

$$-\frac{1}{1-u} \Psi\left(1 + \frac{1+u}{1-u}\right) = -\frac{1}{1+u} + \frac{1}{1-u} \gamma - \sum_{m=1}^{+\infty} \int_0^{+\infty} \frac{e^{-t}(1 - L_m(2t))}{1 - e^{-t}} dt u^m.$$

Since, for k integer ≥ 1 , we have

$$\int_0^{+\infty} \frac{e^{-t} t^k}{1 - e^{-t}} dt = k! \zeta(k+1)$$

we get for $m \geq 1$

$$-\int_0^{+\infty} \frac{e^{-t}(1 - L_m(2t))}{1 - e^{-t}} dt = \int_0^{+\infty} \frac{e^{-t}}{1 - e^{-t}} \left(\sum_{k=1}^m C_m^k (-2)^k \frac{t^k}{k!}\right) dt = \sum_{k=1}^m C_m^k (-2)^k \zeta(k+1).$$

Thus we have proved the following theorem.

Theorem. *The expansion of $t \mapsto \zeta(\frac{1}{2} + it)$ in the space $L^2(\mathbb{R}, \frac{dt}{\cosh(\pi t)})$ is given by*

$$\zeta\left(\frac{1}{2} + it\right) = 2 \sum_{m \geq 0} b_m q_m\left(\frac{1}{2} + it\right) \quad (9)$$

with $b_0 = -1 + \gamma$ et and for $m \geq 1$

$$b_m = 2 \sum_{p=0}^{[(m-1)/2]} \frac{1}{2p+1} + (-1)^{m+1} + \gamma + \sum_{k=1}^m (-2)^k C_m^k \zeta(k+1). \quad (10)$$

For example we have

$$\begin{aligned} b_1 &= 3 + \gamma - 2\zeta(2) \\ b_2 &= 1 + \gamma - 4\zeta(2) + 4\zeta(3) \\ b_3 &= \frac{11}{3} + \gamma - 6\zeta(2) + 12\zeta(3) - 8\zeta(4) \\ b_4 &= \frac{5}{3} + \gamma - 8\zeta(2) + 24\zeta(3) - 32\zeta(4) + 16\zeta(5) \end{aligned}$$

Remark. The Fourier transform given by the relation (6), gives for $g \in L^2(\mathbb{R})$, the relation

$$\int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + it\right) \mathcal{F}g(t) \frac{dt}{\cosh(\pi t)} = -2 \int_{-\infty}^{+\infty} g(\xi) e^{-\xi/2} (\xi + \Psi(1 + e^{-\xi})) d\xi. \quad (11)$$

For example, let $g(t) = t^{s-1} e^{-at} \chi_{[0, +\infty[}(t)$ with $a > 0$ and $Re(s) > \frac{1}{2}$. Then $\mathcal{F}g(t) = \frac{\Gamma(s)}{(a+it)^s}$ and we have

$$-\frac{1}{2} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + it\right) \frac{1}{(a+it)^s} \frac{dt}{\cosh(\pi t)} = \frac{1}{\Gamma(s)} \int_0^{+\infty} \xi^{s-1} e^{-\xi(a+\frac{1}{2})} (\xi + \Psi(1 + e^{-\xi})) d\xi.$$

Expanding the Ψ function as

$$\Psi(1 + e^{-\xi}) = -\gamma + \sum_{n=1}^{+\infty} (-1)^{n+1} \zeta(n+1) e^{-n\xi} \quad \text{since } 0 < e^{-\xi} < 1,$$

we get for $\alpha = a + \frac{1}{2} > \frac{1}{2}$

$$-\frac{1}{2} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + it\right) \frac{1}{(\alpha - \frac{1}{2} + it)^s} \frac{dt}{\cosh(\pi t)} = \frac{s}{\alpha^{s+1}} - \gamma \frac{1}{\alpha^s} + \sum_{n=1}^{+\infty} (-1)^{n+1} \zeta(n+1) \frac{1}{(n+\alpha)^s},$$

Thus, for $x > -\frac{1}{2}$ and $Re(s) > \frac{1}{2}$, we have a generalization of a formula of I.V. Blagouchine (cf.[0])

$$\sum_{n=2}^{+\infty} (-1)^n \zeta(n) \frac{1}{(n+x)^s} = -\frac{s}{(x+1)^{s+1}} + \gamma \frac{1}{(x+1)^s} - \frac{1}{2} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + it\right) \frac{1}{(x + \frac{1}{2} + it)^s} \frac{dt}{\cosh(\pi t)}$$

4 An integral formula

If we use now the notation $\gamma = z_0$ and $2^k \zeta(k+1) = z_k$ for $k \geq 1$, then (10) is simply

$$b_n - c_n = \sum_{k=0}^n (-1)^k C_n^k z_k \quad \text{where} \quad \sum_{n \geq 0} c_n u^n = \frac{1}{1-u} \log\left(\frac{1+u}{1-u}\right) - \frac{1}{1+u} \quad (12)$$

We remind that the binomial transform

$$\widehat{a}_n = \sum_{k=0}^n (-1)^k C_n^k a_k$$

is auto-inverse and is given in terms of generating function by

$$\sum_{n \geq 0} a_n x^n = g(x) \Rightarrow \sum_{n \geq 0} \widehat{a}_n x^n = \frac{1}{1-x} g\left(\frac{-x}{1-x}\right)$$

Then by inversion of the binomial transform, (12) gives

$$z_n = \sum_{k=0}^n (-1)^k C_n^k b_k - d_n \quad \text{where} \quad d_n = \sum_{k=0}^n (-1)^k C_n^k c_k$$

The generating function of (d_n) is

$$\frac{1}{1-x} \left(\frac{1}{1-\frac{-x}{1-x}} \log\left(\frac{1+\frac{-x}{1-x}}{1-\frac{-x}{1-x}}\right) - \frac{1}{1+\frac{-x}{1-x}} \right) = \log(1-2x) - \frac{1}{1-2x}$$

this gives $d_0 = -1$ and

$$d_n = \left(-1 - \frac{1}{n}\right)2^n \text{ for } n \geq 1$$

Thus we have $\gamma = z_0 = b_0 - 1$ and for $m \geq 1$

$$2^m \zeta(m+1) = z_m = \sum_{k=0}^m (-1)^k C_m^k b_k + \left(1 + \frac{1}{m}\right)2^m$$

Since, from (9), we have for $m \geq 0$

$$b_m = \frac{1}{2} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + it\right) q_m\left(\frac{1}{2} - it\right) \frac{dt}{\cosh(\pi t)}, \quad (13)$$

we can evaluate the binomial transform of (b_m) using the binomial transform of $(q_m(s))$. We have

$$q_m(s) = \sum_{k=0}^m C_m^k (-2)^k \frac{(s)_k}{k!} \Rightarrow 2^m \frac{(s)_m}{m!} = \sum_{k=0}^m (-1)^k C_m^k q_k(s)$$

thus

$$\begin{aligned} \sum_{k=0}^m (-1)^k C_m^k b_k &= \frac{1}{2} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + it\right) \sum_{k=0}^m (-1)^k C_m^k q_k\left(\frac{1}{2} - it\right) \frac{dt}{\cosh(\pi t)} \\ &= 2^m \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + it\right) \frac{\left(\frac{1}{2} - it\right)_m}{m!} \frac{dt}{\cosh(\pi t)} \end{aligned}$$

Finally we get the integral expressions

$$\gamma = 1 + \frac{1}{2} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + it\right) \frac{dt}{\cosh(\pi t)},$$

$$\zeta(m+1) = 1 + \frac{1}{m} + \frac{1}{2} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + it\right) \frac{\left(\frac{1}{2} - it\right)_m}{m!} \frac{dt}{\cosh(\pi t)} \text{ for } m \geq 1.$$

which is also

$$\zeta(m+1) = 1 + \frac{1}{m} + \frac{1}{2} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + it\right) \frac{\Gamma\left(\frac{1}{2} + it\right) \Gamma\left(\frac{1}{2} - it + m\right)}{\Gamma(m+1)} dt \text{ for } m \geq 1.$$

We see that the analytic functions defined by

$$f(s) = \zeta(s+1) - \frac{1}{s} \quad \text{for } s \neq 0 \quad \text{and } f(0) = \gamma$$

and

$$g(s) = 1 + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + iu\right) \frac{\Gamma\left(\frac{1}{2} + iu\right)\Gamma\left(\frac{1}{2} - iu + s\right)}{\Gamma(s+1)} du \quad \text{for } \text{Re}(s) > -\frac{1}{2}$$

are such that $f(m) = g(m)$ for all integers $m \geq 0$. Then by the Carlson's theorem we have the integral formula

$$\zeta(s+1) - \frac{1}{s} = 1 + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + iu\right) \frac{\Gamma\left(\frac{1}{2} + iu\right)\Gamma\left(\frac{1}{2} - iu + s\right)}{\Gamma(s+1)} du \quad \text{for } \text{Re}(s) > -\frac{1}{2} \quad (14)$$

For $s = -\frac{1}{2} + it$ we get the integral equation

$$\zeta\left(\frac{1}{2} + it\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + iu\right) \frac{\Gamma(i(t-u))\Gamma\left(\frac{1}{2} + iu\right)}{\Gamma\left(\frac{1}{2} + it\right)} du - \frac{\frac{1}{2} + it}{\frac{1}{2} - it}$$

which is also a convolution equation for the function $t \mapsto (\Gamma\zeta)\left(\frac{1}{2} + it\right)$

$$(\Gamma\zeta)\left(\frac{1}{2} + it\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\Gamma\zeta)\left(\frac{1}{2} + iu\right) \Gamma(i(t-u)) du - \Gamma\left(\frac{1}{2} + it\right) \frac{\frac{1}{2} + it}{\frac{1}{2} - it} \quad (15)$$

Acknowledgments. My warmest thanks go to F.Rouvière for his helpful comments and his encouragements.

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Soit f la fonction définie par

$$f(s) = \zeta(s+1) - \frac{1}{s} \quad \text{pour } s \neq 0 \quad \text{et } f(0) = \gamma$$

La formule d'interpolation de Newton donne

$$f(s) = \sum_{n \geq 0} (-1)^n c_n \frac{s(s-1)\dots(s-n+1)}{n!} \quad \text{où } c_n = \sum_{k=0}^n (-1)^k C_n^k f(k)$$

On a vu précédemment que

$$f(0) = 1 + \frac{1}{2} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + it\right) \frac{dt}{\cosh(\pi t)},$$

et pour $k \geq 1$

$$f(k) = 1 + \frac{1}{2} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + iu\right) \frac{\left(\frac{1}{2} - iu\right)_k}{k!} \frac{du}{\cosh(\pi u)}$$

On en déduit que

$$c_0 = 1 + \frac{1}{2} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + it\right) \frac{dt}{\cosh(\pi t)},$$

et pour $n \geq 1$

$$c_n = \frac{1}{2} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + iu\right) \sum_{k=0}^n (-1)^k C_n^k \frac{\left(\frac{1}{2} - iu\right)_k}{k!} \frac{du}{\cosh(\pi u)}$$

Par définition des polynômes de Legendre, on a

$$\sum_{k=0}^n (-1)^k C_n^k \frac{\left(\frac{1}{2} - iu\right)_k}{k!} = \frac{1}{\Gamma\left(\frac{1}{2} - iu\right)} \mathcal{M}(e^{-x} L_n(x)) \left(\frac{1}{2} - iu\right)$$

on obtient donc

$$c_n = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + iu\right) \Gamma\left(\frac{1}{2} + iu\right) \mathcal{M}(e^{-x} L_n(x)) \left(\frac{1}{2} - iu\right) du$$

La formule d'interpolation de Newton s'écrit alors

$$f(s) = 1 + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + iu\right) \Gamma\left(\frac{1}{2} + iu\right) \mathcal{M}\left(e^{-x} \sum_{n \geq 0} (-1)^n L_n(x) \frac{s \cdots (s-n+1)}{n!}\right) \left(\frac{1}{2} - iu\right) du$$

(il y a eu permutation $\sum \int \mathcal{M} = \int \mathcal{M} \sum$).

Or on a pour $Re(s) > -1$

$$\sum_{n \geq 0} (-1)^n L_n(x) \frac{s(s-1)\dots(s-n+1)}{n!} = \frac{x^s}{\Gamma(s+1)}$$

ce qui donne

$$f(s) = 1 + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + iu\right) \Gamma\left(\frac{1}{2} + iu\right) \mathcal{M}\left(e^{-x} \frac{x^s}{\Gamma(s+1)}\right) \left(\frac{1}{2} - iu\right) du$$

Comme on a immédiatement

$$\mathcal{M}\left(e^{-x} x^s\right) \left(\frac{1}{2} - iu\right) = \Gamma\left(\frac{1}{2} - iu + s\right)$$

Conclusion. Pour $Re(s) > -1$ et $s \neq 0$ on a

$$\zeta(s+1) = 1 + \frac{1}{s} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + iu\right) \frac{\Gamma\left(\frac{1}{2} + iu\right) \Gamma\left(\frac{1}{2} - iu + s\right)}{\Gamma(s+1)} du \quad (16)$$

En particulier avec $s = -\frac{1}{2} + it$ on obtient l'équation intégrale

$$\zeta\left(\frac{1}{2} + it\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \zeta\left(\frac{1}{2} + iu\right) \frac{\Gamma(i(t-u)) \Gamma\left(\frac{1}{2} + iu\right)}{\Gamma\left(\frac{1}{2} + it\right)} du - \frac{\frac{1}{2} + it}{\frac{1}{2} - it}$$

C'est une équation de convolution sur la fonction $t \mapsto (\Gamma\zeta)\left(\frac{1}{2} + it\right)$

$$(\Gamma\zeta)\left(\frac{1}{2} + it\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\Gamma\zeta)\left(\frac{1}{2} + iu\right) \Gamma(i(t-u)) du - \Gamma\left(\frac{1}{2} + it\right) \frac{\frac{1}{2} + it}{\frac{1}{2} - it} \quad (17)$$

6 Generalization

Pour $0 < c < 1$, la **formule de Parseval-Mellin** s'écrit:

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{M}(f)(c + i\tau) \overline{\mathcal{M}(g)(c + i\tau)} d\tau = \int_0^{+\infty} x^{2c-1} f(x) \overline{g(x)} dx$$

On va construire, au moyen des fonctions de Laguerre, des fonctions Φ_m^c qui sont orthogonales pour le produit scalaire qui intervient naturellement dans la relation de Parseval-Mellin

$$(f, g)_c = \int_0^{+\infty} x^{2c-1} f(x) \overline{g(x)} dx$$

Polynômes et fonctions de Laguerre

Soit $0 < c < 1$. On considère les polynômes de Laguerre $x \mapsto L_m^{c-1}(2x)$ qui sont définis par la fonction génératrice

$$\frac{1}{(1-t)^c} e^{-2xt \frac{1}{1-t}} = \sum_{m \geq 0} L_m^{c-1}(2x) t^m$$

Ils sont orthogonaux dans $L^2(\mathbb{R}, x^{c-1} e^{-2x} dx)$. La fonction exponentielle a un développement très simple sur cette famille de polynômes

$$e^{-2ax} = \frac{1}{(1+a)^c} \sum_{m \geq 0} \left(\frac{a}{a+1}\right)^m L_m^{c-1}(2x) \quad (18)$$

Soient les fonctions de Laguerre

$$\varphi_m^c(x) = e^{-x} L_m^{c-1}(2x)$$

Elles sont orthogonales dans $L^2(]0, +\infty[, x^{c-1}dx)$, et sont données par la fonction génératrice

$$\frac{1}{(1-t)^c} e^{-x \frac{1+t}{1-t}} = \sum_{m \geq 0} \varphi_m^c(x) t^m$$

En faisant $t = 1 + 2a$ dans (1) on obtient

$$\boxed{e^{-\pi x^2 t^2} = \sum_{m \geq 0} \Phi_m^c(x) \Psi_m^c(t)} \quad (19)$$

où

$$\boxed{\Psi_m^c(t) = \left(\frac{t^2 - 1}{t^2 + 1}\right)^m \frac{2^c}{(1 + t^2)^c}}$$

La transformée de Mellin de φ_m^c (cf. I.S.Gradshteyn, I.M.Ryzhik. Tables of Integrals, Series and Products. Academic Press, Inc. (1994). p.850) est pour $Re(s) > 0$

$$\int_0^{+\infty} \varphi_m^c(x) x^{s-1} dx = \frac{\Gamma(s)\Gamma(c+m)}{m!\Gamma(c)} q_m^c(s)$$

où q_m^c est le polynôme

$$\boxed{q_m^c(s) = F(-m, s; c; 2)}$$

la fonction F (notée aussi ${}_2F_1$) étant la fonction hypergéométrique de Gauss. ¹

On a

$$q_m^c(c-s) = (-1)^m q_m^c(s)$$

De l'orthogonalité des φ_m^c on déduit que les polynômes $t \mapsto q_m^c(\frac{c}{2} + it)$ sont orthogonaux pour la mesure $|\Gamma(\frac{c}{2} + it)|^2 dt$ sur $]0, +\infty[$. Donc q_m^c a ses racines sur la droite $Re(s) = c/2$.

¹c'est le polynôme de Meixner $M_m(s, -c, 1)$ et aussi une variante du polynôme de Meixner-Pollaczek

$$P_m^{(c/2)}(t, \frac{\pi}{2}) = \frac{i^m \Gamma(c+m)}{m! \Gamma(c)} F(-m, \frac{c}{2} + it; c; 2) = \frac{i^m \Gamma(c+m)}{m! \Gamma(c)} q_m^c(\frac{c}{2} + it)$$