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Optimal investment and consumption for financial markets with jumps under transaction costs

Sergei Egorov · Serguei Pergamenchchikov

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Abstract We consider a portfolio optimization problem for financial markets described by semi-martingales with independent increments and jumps defined through Lévy processes. For this problem we show the corresponding verification theorem and construct the optimal consumption/investment strategies. For the power utility functions we find the optimal strategies in the explicit form and then we apply these strategies to markets with transaction costs. Based on the Leland - Lepinette approach we develop asymptotic optimal investment and consumption method when the number of portfolio revision tends to infinity. Finally, we carried out Monte Carlo simulations to illustrate numerically the obtained theoretical results in practice.

MSC: primary 60P05, secondary 62G05

Keywords: Financial markets; Optimal investment/consumption problem; Stochastic control; Dynamical programming; Hamilton–Jacobi–Bellman equation.

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1 Introduction

1.1 Motivations

In this paper, we consider a portfolio optimization problem for Lévy financial markets with non-random time-dependent coefficients. Such problems are very popular in the stochastic financial markets theory. Beginning with the classic work of Merton, where the optimal investment problem for Black-Scholes models was first studied, interest in these problems is constantly growing to the present day (see, for example, [11, 4, 5, 17] and the references therein). It should be emphasized that the financial markets defined by the continuous stochastic processes similar to the geometric Brownian motion are very limited for the practical applications and they do not allow us to describe situations of abrupt, impulsive changes in price processes observed during of the crises and instability in financial markets. It seems that for the first time an optimization portfolio problem for financial markets with jumps was studied in the paper [8], in which the authors using the stochastic Pontryagin maximum principle constructed optimal investment/consumption strategies. Later, these problems were studied for more complex market models and in different settings: in [14, 9, 24] the authors considered maximization utility problems in general semi-martingale settings, in [25, 4, 2, 19] such problems were considered for stochastic volatility markets, in [6, 10] for the markets defined by the affine processes, in [12, 13, 3, 7] the authors considered the portfolio optimization problems with constraints. In [5, 17] the authors considered pure consumption problem on Lévy markets with infinite time horizon under proportional transaction costs where they used geometric approach and viscosity solutions in similar spirit as it was done in [1].

The main goal of our work is to study the classical investment and consumption problem on the finite time interval $[0, T]$ for the financial market model with jumps under transaction costs. Moreover, we are interested to find the optimal solutions in the explicit form and illustrate their behavior by the Monte - Carlo method.

1.2 Main investments

Based on stochastic dynamic programming and Leland - Lepinette approach, we develop a portfolio optimization method for Levy-type financial markets with transaction costs. To this end, first, we deduce and study the Hamilton–Jacobi–Bellman (HJB) equation. The challenge here is that we could not use directly the classical HJB analysis method from [11], which was due to the additional integral term corresponding to jumps in the market model. Therefore, we need to develop a special analytical tool to analyse this equation and to construct optimal strategies. Similar to [12, 13, 4, 2] we study this problem through the verification theorem method. So, in this paper, probably for the first time, we show a special verification theorem in a non-Gaussian financial markets framework. Then, using this theorem we construct optimal strategies, and, finally, for the power utility functions we provide the solutions for such optimization problems in an explicit form. Moreover, to take into account transaction costs in the optimization problems we use the Leland method (see, for example, [15]) in the difference from the geometric approach developed in [5, 17] for Lévy markets. More precisely, using the explicit form for the obtained optimal strategies we construct their discretized versions and, then study the asymptotic behavior as the number of revisions tends to infinity. We provide the conditions on transaction costs for which the discretized strategy is asymptotically optimal, i.e. when the objective function tends to its

optimal value. It turns out, that in the case of large transaction costs, to obtain the optimality property one needs to use the Lepinette approach developed for the asymptotic hedging problem in [16], i.e. one needs to do portfolio revisions not uniformly as in the Leland approach, but in time moments of special power form. Finally, we illustrate the obtained theoretical results by the numeric simulations.

1.3 Organisation of paper

In Section 2 we describe the problem and all necessary definitions. In Section 3 we study optimal control problem for stochastic systems with jumps through the verification theorem method. In Section 4 we state the main results. In Section 5 we the optimal strategies. In Section 6 we give the results of Monte-Carlo simulations. All main results are shown in Section 7. All auxiliary tool is given in Appendix A.

2 Problem

In this paper, we consider optimal investment and consumption problems for financial markets of Lévy type with time-dependent coefficients. More precisely, we consider the financial market on the time interval $[0, T]$ which consists of a risk-free asset (bond) $(B_t)_{0 \leq t \leq T}$ and m risky assets (stocks) $(S_i(t))_{0 \leq t \leq T}$, $1 \leq i \leq m$, defined as

$$\begin{cases} dB_t = r_t B_t dt, & B_0 = 1, \\ dS_i(t) = \mu_i(t) S_i(t) dt + S_i(t-) d\tilde{L}_i(t), \end{cases} \quad (2.1)$$

where the interest rate r_t and the drifts $(\mu_j(t))_{1 \leq j \leq m}$ are non random $[0, T] \rightarrow \mathbb{R}$ integrated functions, i.e.

$$\int_0^T \left(|r_t| + \sum_{i=1}^m |\mu_i(t)| \right) dt < \infty. \quad (2.2)$$

We assume, that the random market sources $(\tilde{L}_i(t))_{0 \leq t \leq T}$ are represented as

$$\tilde{L}_i(t) = \sum_{j=1}^m \int_0^t (\sigma_{ij}(u) dW_j(u) + \varsigma_{ij}(u) dL_j(u)), \quad (2.3)$$

in which the volatilities $\sigma_t = (\sigma_{ij}(t))_{1 \leq i, j \leq m}$ and $\varsigma_t = (\varsigma_{ij}(t))_{1 \leq i, j \leq m}$ are $m \times m$ non random matrices such that for any $1 \leq i, j \leq m$, $0 \leq t \leq T$

$$\int_0^T \sigma_{ij}^2(u) du < \infty \quad \text{and} \quad 0 \leq \varsigma_{ij}(t) \leq 1. \quad (2.4)$$

Moreover, we assume, that $(W_1(t))_{0 \leq t \leq T}, \dots, (W_m(t))_{0 \leq t \leq T}$ are independent standard Brownian motions and $(L_1(t))_{0 \leq t \leq T}, \dots, (L_m(t))_{0 \leq t \leq T}$ are independent pure jumps Lévy processes, i.e.

$$L_j(t) = \int_0^t \int_{\mathbb{R}_*} y (\nu_j(\omega; dy, du) - \tilde{\nu}_j(dy, du)), \quad (2.5)$$

where $\nu_j(\omega; dy, du)$ is random jump measure with compensator $\tilde{\nu}_j(dy, du) = \Pi_j(dy) du$ and $\Pi_j(\cdot)$ is the corresponding Lévy measure on $\mathbb{R}_* = \mathbb{R} \setminus \{0\}$. It should be noted that in case of independent Lévy processes, $\Delta L_j(t) \Delta L_i(t) = 0$, for any $i \neq j$. First, to provide the positivity for the risky price process (2.1) we assume, that $\Pi_j([-\infty, -1]) = 0$. Moreover, we assume also that

$$\Pi_j(|\ln(1+y)|\mathbf{1}_{\{-1 < y < -1/2\}}) < \infty \quad \text{and} \quad \Pi_j(y^2) < \infty, \quad (2.6)$$

where $\mathbf{1}_A$ is an indicator function of the set A . These conditions are technical and used in the market model to guarantee existence of optimal trading strategy. In the sequel we set $\mu_t = (\mu_1(t), \dots, \mu_m(t))'$, $W_t = (W_1(t), \dots, W_m(t))'$ and $L_t = (L_1(t), \dots, L_m(t))'$, where the prime $'$ denotes the transposition. Everywhere below we use natural filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, i.e. $\mathcal{F}_t = \sigma\{W_u, L_u : 0 \leq u \leq t\}$. Similar to [12] in this paper we use the fractional strategies defined as

$$\theta_i(t) = \frac{\alpha_i(t) S_i(t)}{X_t} \quad \text{and} \quad c_t = \frac{\zeta_t}{X_t}, \quad (2.7)$$

where $\alpha_i(t)$ is the amount of investment into i -th risky asset purchased by an investor at the time moment t and X_t is the corresponding wealth process with β_t bond units defined as

$$X_t = \sum_{i=1}^m \alpha_i(t) S_i(t) + \beta_t B_t, \quad (2.8)$$

using (2.7) and (2.8) can be proved the following equality $1 - \sum_{i=1}^m \theta_i(t) = \beta_t B_t / X_t$. Moreover, the process ζ_t in (2.7) is the consumption intensity, i.e. it is non negative integrable process for which the integral $\int_0^t \zeta_s ds$ is the total amount of capital consumed by the investor on time interval $[0, t]$. Now using the self-financing-consumption principle for the wealth process (2.8) (see, for example, [11]) and the definition (2.1), we obtain that

$$\begin{cases} dX_t = X_t (r_t + \theta_t' \check{\mu}_t - c_t) dt + X_{t-} \theta_{t-}' d\tilde{L}_t, \\ X_0 = x > 0, \end{cases} \quad (2.9)$$

where $\check{\mu}_t = (\mu_t - r_t \mathbf{e}_m)$, $\mathbf{e}_m = (1, \dots, 1)' \in \mathbb{R}^m$ and $\theta_t = (\theta_1(t), \dots, \theta_m(t))'$ and $\tilde{L}_t = (\tilde{L}_1(t), \dots, \tilde{L}_m(t))'$. It should be noted that to provide the positivity of portfolio value X_t the jump sizes of the process $\tilde{L}_t = \int_0^t \theta_{u-}' \varsigma_{u-} dL_u$ have to be more than -1 , i.e. $\Delta \tilde{L}_t > -1$. To this end we assume that the financial strategy $\theta_t = (\theta_1(t), \dots, \theta_m(t))'$ is a càdlàg process with values in the set $[0, 1]^m$ such that, for any fixed $0 \leq t \leq T$, almost sure $\sum_{j=1}^m \theta_j(t) \leq 1$. In the sequel we denote by

$$\Theta = \left\{ \theta = (\theta_1, \dots, \theta_m)' \in [0, 1]^m : \sum_{j=1}^m \theta_j \leq 1 \right\}. \quad (2.10)$$

Using this set we introduce admissible strategies.

Definition 2.1. A stochastic process $v = (\theta_t, c_t)_{0 \leq t \leq T}$ is called admissible if the first component $(\theta_t)_{0 \leq t \leq T}$ is a predictable process with values in the set (2.10) and the process $(c_t)_{0 \leq t \leq T}$ is adapted, non negative integrated on $[0, T]$, for which the equation (2.9) has unique strong solution such that $X_{t-} > 0$ and $X_t > 0$ a.s. for $0 \leq t \leq T$.

We denote by \mathcal{V} the set of admissible strategies. It should be noted that for any $v \in \mathcal{V}$ the wealth process (2.9) can be represented as

$$X_t = x e^{\int_0^t (r_s + \theta'_s \tilde{\mu}_s) ds - \int_0^t c_s ds} \mathcal{E}_t(V)$$

and the Doléan exponential

$$\mathcal{E}_t(V) = e^{V_t - \frac{1}{2} \langle V \rangle_t} \prod_{0 \leq s \leq t} (1 + \Delta V_s) e^{-\Delta V_s}, \quad (2.11)$$

where $V_t = \int_0^t \theta'_s d\tilde{L}_s$ and $\langle V \rangle_t = \int_0^t \theta'_s \sigma_s \sigma'_s \theta_s ds$.

Now to formulate an optimal consumption and investment problem we introduce for some $0 < \gamma < 1$, the objective function as

$$J(x, v) := \mathbf{E}_x \left(\int_0^T \zeta_t^\gamma dt + (X_T)_+^\gamma \right), \quad (2.12)$$

where \mathbf{E}_x is the conditional expectation given $X_0 = x$ and $(x)_+$ is the positive part of x , i.e. $(x)_+ = \max(x, 0)$ and $\zeta_t = c_t X_t$ is the intensity of consumption.

The goal is to maximize the objective function on the set \mathcal{V} , i.e. to find a strategy $v^* \in \mathcal{V}$, such that

$$J(x, v^*) = \sup_{v \in \mathcal{V}} J(x, v) =: J^*(x). \quad (2.13)$$

According to the dynamic programming principle to study this problem we have to study the value functions defined on the interval $[t, T]$ as

$$J^*(t, x) := \sup_{v \in \mathcal{V}} J(t, x, v), \quad (2.14)$$

where for $v \in \mathcal{V}$

$$J(t, x, v) := \mathbf{E}_{x,t} \left(\int_t^T (c_u X_u)^\gamma du + X_T^\gamma \right).$$

Here $\mathbf{E}_{x,t}$ is the conditional expectation with respect to $X_t = x$. Note that in our case $J(t, x, v)$ can be equal to ∞ for some strategy v .

Remark 2.2. *It should be noted that the definition (2.12) for the objective function can be applied to any financial strategies not only for admissible ones, i.e. not only to strategies with positive wealth processes. Indeed, in practice, the capital can be negative for a short period of time, such as for markets with transaction costs. In such cases, it is also necessary to calculate the objective function and compare the strategies between them.*

Remark 2.3. *Note that the optimal consumption and investment problem without terminal functional for the model (2.1) is studied in [8] through the maximum Pontryagin principle. The similar problems were considered in [23, 24] on the basis of the dual problems approach.*

3 Verification theorem

In this section we will generalize the controlled process (2.9) and the definition of admissible strategies. For such framework, we show some verification theorem. Let's define diffusion jumps controlled process X_t with values in an open convex set $\mathcal{X} \subseteq \mathbb{R}$ and control process v_t with values in closed set $\mathcal{K} \subseteq \mathbb{R}^m$. Let $X = (X_t)_{0 \leq t \leq T}$, $X_t, X_{t-} \in \mathcal{X}$ be càdlàg process of form

$$\begin{cases} dX_t = a(t, X_t, v_t) dt + b'(t, X_{t-}, v_{t-}) d\tilde{L}_t, \\ X_0 = x \in \mathcal{X}, \end{cases} \quad (3.1)$$

where the process $\tilde{L}_t \in \mathbb{R}^m$ is defined in (2.3) with the Lévy measures Π_j satisfying the second condition in (2.6), i.e. $\Pi_j(y^2) < \infty$. The functions $a : [0, T] \times \mathcal{X} \times \mathcal{K} \rightarrow \mathbb{R}$ and $b : \mathcal{X} \times \mathcal{K} \rightarrow \mathbb{R}^m$ are non random, continuous and such that, for any non random $\mathbf{v} \in \mathcal{K}$, the equation (3.1) with $v_t \equiv \mathbf{v}$ has an unique strong solution for which $X_t \in \mathcal{X}$ and $X_{t-} \in \mathcal{X}$ on the time interval $[0, T]$ and

$$\int_0^T \left(|a(t, X_t, \mathbf{v})| + |b(t, X_t, \mathbf{v})|^2 \right) dt < \infty \quad \text{a.s.}$$

Definition 3.1. A stochastic process $v = (v_t)_{0 \leq t \leq T}$ is called admissible if it is $(\mathcal{F}_t)_{0 \leq t \leq T}$ -adapted, has càdlàg trajectories, takes values in the set \mathcal{K} , and such that the equation (3.1) on time interval $[0, T]$ has an unique strong solution for which $X_t, X_{t-} \in \mathcal{X}$ on the time interval $[0, T]$ and

$$\int_0^T \left(|a(t, X_t, v_t)| + |b(t, X_t, v_t)|^2 \right) dt < \infty \quad \text{a.s.} \quad (3.2)$$

We denote by \mathcal{V} the set of all admissible strategies. Note that the conditions on functions a and b imply $\mathcal{V} \neq \emptyset$, at least the strategy $v_t \equiv \mathbf{v} \in \mathcal{V}$. Now we fix utility functions $U_1 : [0, T] \times \mathcal{X} \times \mathcal{K} \rightarrow \mathbb{R}_+$ and $U_2 : \mathcal{X} \rightarrow \mathbb{R}_+$ and, then, for any $0 < t \leq T$, we set the objective functions as

$$J(t, x, v) := \mathbf{E}_{t,x} \left(\int_t^T U_1(u, X_u, v_u) du + U_2(X_T) \right).$$

Our goal is to find an admissible strategy $v^* \in \mathcal{V}$, such that, for any $0 \leq t < T$,

$$J^*(t, x) := \sup_{v \in \mathcal{V}} J(t, x, v) = J(t, x, v^*). \quad (3.3)$$

To apply the dynamic programming method, we will need to introduce the Hamilton function. To do this, for any $[0, T] \times \mathcal{X} \rightarrow \mathbb{R}$ function $g(t, x)$, twice continuously differentiable in x and continuously differentiated in t , such that

$$\sup_{x \in \mathcal{X}} \frac{|g(t, x)|}{1 + |x|} < \infty, \quad (3.4)$$

we define

$$\begin{aligned} H_0(t, x, g, \mathbf{v}) &= a(t, x, \mathbf{v}) g_x(t, x) + \frac{1}{2} g_{xx}(t, x) \operatorname{tr} \sigma_t' b(t, x, \mathbf{v}) b'(t, x, \mathbf{v}) \sigma_t \\ &\quad + U_1(t, x, \mathbf{v}) + \bar{g}(t, x, \mathbf{v}), \end{aligned} \quad (3.5)$$

where $\bar{g}(t, x, \mathbf{v}) = \sum_{i=1}^m \int_{\mathbb{R}_*} \Upsilon_g(t, x, b'(t, x, \mathbf{v}) \varsigma_{i,t} y) \Pi_i(dy)$, $\varsigma_{i,t} = (\varsigma_{1i}(t), \dots, \varsigma_{mi}(t))'$ and

$$\Upsilon_g(t, x, v) = (g(t, x+v) - g(t, x) - g_x(t, x)v) \mathbf{1}_{\{x+v \in \mathcal{X}\}}$$

for $v \in \mathbb{R}_*$. Notations g_t, g_x, g_{xx} mean corresponding derivatives of function $g(t, x)$.

Remark 3.2. It should be emphasized that one needs to introduce the indicator $\mathbf{1}_{\{x+v \in \mathcal{X}\}}$ in the term $\Upsilon(t, x, v)$ since there are no assumptions on measures $\Pi_i(\cdot)$ or function g to keep the sum $x+v$ in the set \mathcal{X} , for v from \mathbb{R}_* . As it is shown in appendix A.1 the function \bar{g} is bounded and its Lebesgue integral will be used as jumps compensator for the process $g(t, X_t)$.

Now, for any $x \in \mathcal{X}$ and $0 \leq t \leq T$, we set the Hamilton function as

$$H(t, x, g) := \sup_{\mathbf{v} \in \mathcal{K}} H_0(t, x, g, \mathbf{v}). \quad (3.6)$$

Now we introduce the following Hamilton–Jacobi–Bellman equation as

$$\begin{cases} z_t(t, x) + H(t, x, z) = 0, & t \in [0, T], \\ z(T, x) = U_2(x), & x \in \mathcal{X}. \end{cases} \quad (3.7)$$

Next, we need the following conditions.

H₁) There exists solution $z \in C^{1,2}([0, T] \times \mathcal{X}, \mathbb{R})$ of the equation (3.7) such that,

$$\inf_{0 \leq t \leq T} \inf_{x \in \mathcal{X}} z(t, x) > -\infty \quad \text{and} \quad \sup_{0 \leq t \leq T} \sup_{x \in \mathcal{X}} \frac{|z(t, x)|}{1 + |x|} < \infty. \quad (3.8)$$

H₂) There exists $[0, T] \times \mathcal{X} \rightarrow \mathcal{K}$ measurable function \mathbf{v}_0 , such that for the solution $z = z(t, x)$ of the equation (3.7) the Hamilton function $H(t, x, z) = H_0(t, x, z, \mathbf{v}_0(t, x))$.

H₃) For any $x \in \mathcal{X}$, there exists an unique almost surely solution $X^* = (X_t^*)_{0 \leq t \leq T}$ with values in the set \mathcal{X} and $X_{t-}^* \in \mathcal{X}$ of the equation

$$dX_t^* = a^*(t, X_t^*) dt + (b^*(t-, X_{t-}^*))' d\tilde{L}_t, \quad X_0^* = x, \quad (3.9)$$

where $a^*(t, x) = a(t, x, \mathbf{v}_0(t, x))$ and $b^*(t, x) = b(x, \mathbf{v}_0(t, x))$. Moreover, the process $\mathbf{v}_0 = (\mathbf{v}_0(t, X_t^*))_{0 \leq t \leq T}$ is admissible, i.e. belongs to \mathcal{V} .

H₄) For any $0 \leq t \leq T$ and $x \in \mathcal{X}$,

$$\mathbf{E}_{t,x} \sup_{t \leq u \leq T} |z(u, X_u^*)| < \infty. \quad (3.10)$$

Using the approach proposed in [12], we show the following verification theorem.

Theorem 3.3. *Suppose conditions $\mathbf{H}_1) - \mathbf{H}_4)$ are hold. Then, for any $0 \leq t \leq T$ and $x \in \mathcal{X}$,*

$$z(t, x) = J^*(t, x) = J(t, x, v^*),$$

where the optimal strategy $v^* = (v_s^*)_{t \leq s \leq T}$, $v_s^* = \mathbf{v}_0(s, X_s^*)$ is determined in terms of $\mathbf{H}_2) - \mathbf{H}_4)$ and the function $J^*(t, x)$ is defined in (3.3).

Remark 3.4. *Here (3.8) and $\mathbf{H}_4)$ are technical conditions. In particular, the first condition in (3.8) will be used to apply Fatou's lemma for the limit transition in conditional expectations, the last condition provides the finiteness almost sure for the jumps compensator of $z(t, X_t)$ and $\mathbf{H}_4)$ will be used to apply dominated convergence theorem to prove optimality of the process v^* .*

Remark 3.5. *Note, that we don't assume the uniqueness of a solution for the equation (3.7). But if conditions of Theorem 3.3 hold then $z(t, x)$ will be the unique solution of the equation (3.7), since the supremum $J(t, x, v^*)$ is always unique.*

Now we apply Theorem 3.3 to the problem (2.13). In this case the controlled process driven by (2.9) with state space $\mathcal{X} =]0, \infty[$, admissible strategy $v = (\theta_t, c_t)_{0 \leq t \leq T}$ with values in $\mathcal{K} = \Theta \times \mathbb{R}_+$, where the set Θ defined by (2.10). So, to study the optimal consumption and investment problem (2.14) we will apply Theorem 3.3 for the utility functions $U_1(x, \mathbf{v}) = (x \mathbf{c})^\gamma$ and $U_2(x) = x^\gamma$, where $x \in \mathcal{X}$, $\mathbf{v} = (\theta, \mathbf{c}) \in \mathcal{K}$, the vector $\theta = (\theta_1, \dots, \theta_m)'$ and $0 < \gamma < 1$. First note that, the process (2.9) can be obtained as a special case of the model (3.1)

$$a(t, x, \mathbf{v}) = x(r_t + \theta' \tilde{\mu}_t - \mathbf{c}) \quad \text{and} \quad b(t, x, \mathbf{v}) = x \theta. \quad (3.11)$$

In this case one can check directly, that the HJB equation (3.7) has the following form

$$\begin{cases} z_t(t, x) + r_t x z_x(t, x) + \max_{\theta \in \Theta} \Gamma(t, x, z, \theta) + (1 - \gamma) \left(\frac{\gamma}{z_x(t, x)} \right)^{\frac{\gamma}{1-\gamma}} = 0, \\ z(T, x) = x^\gamma, \end{cases} \quad (3.12)$$

where $\Gamma(t, x, z, \theta) = x z_x(t, x) \theta' \tilde{\mu}_t + x^2 z_{xx}(t, x) \theta' \sigma_t \sigma_t' \theta / 2 + \bar{z}(t, x, \theta)$ and $\bar{z}(t, x, \theta)$ is defined by \bar{g} in (3.5) for $g = z$. Moreover, according to Condition $\mathbf{H}_2)$ to find an optimal control function $\mathbf{v}_0 = (\theta_0, \mathbf{c}_0)$ one needs to choose $\theta_0 = \theta_0(t, x, z)$ and $\mathbf{c}_0 = \mathbf{c}_0(t, x, z)$ as

$$\theta_0 = \arg \max_{\theta \in \Theta} \Gamma(t, x, z, \theta) \quad \text{and} \quad \mathbf{c}_0 = \frac{1}{x} \left(\frac{z_x(t, x)}{\gamma} \right)^{1/(\gamma-1)}. \quad (3.13)$$

Using here the Fourier separation variables method we can conclude, that the solution of the HJB equation has the following form

$$z(t, x) = A(t) x^\gamma. \quad (3.14)$$

It should be noted, that if we substitute z with the form (3.14) in (3.12), we obtain that the function $\Gamma(t, x, z, \theta) = A(t) x^\gamma F(t, \theta)$, where

$$F(t, \theta) = \gamma \theta' \tilde{\mu}_t + \frac{\gamma(\gamma-1)}{2} \text{tr} \theta' \sigma_t \sigma_t' \theta + \sum_{i=1}^m \int_{\mathbb{R}^*} \rho(\theta' \varsigma_{i,ty}) \Pi_i(dy), \quad (3.15)$$

where $\rho(x) = (1+x)^\gamma - 1 - \gamma x$ and $\varsigma_{i,t} = (\varsigma_{1i}(t), \dots, \varsigma_{mi}(t))'$ is i -th column of the matrix ς_t . Therefore, from (3.13) we obtain that in this case for the HJB solution (3.14)

$$\theta_0 = \theta_0(t) = \arg \max_{\theta \in \Theta} F(t, \theta) \quad \text{and} \quad \mathbf{c}_0 = \mathbf{c}_0(t) = \left(\int_t^T \Psi_{t,u} du + \Psi_{t,T} \right)^{-1}, \quad (3.16)$$

where $\Psi_{t,u} = e^{\int_t^u h_s^* ds}$ and $h_s^* = (F(s, \theta_s^*) + \gamma r_s) / (1 - \gamma)$. From (2.9) we obtain, that the corresponding wealth process $(X_t^*)_{0 \leq t \leq T}$ is defined as

$$X_t^* = x e^{\int_0^t (r_s + (\theta_s^*)' \tilde{\mu}_s) ds - \int_0^t c_s^* ds} \mathcal{E}_t(V^*) \quad (3.17)$$

and $\mathcal{E}_t(V^*)$ is the Doléan exponential (2.11) for the process $V_t^* = \int_0^t (\theta_s^*)' d\tilde{L}_s$ with its quadratic characteristic $\langle V^* \rangle_t = \int_0^t (\theta_s^*)' \sigma_s \sigma_s' \theta_s^* ds$.

Remark 3.6. *If there exists many solutions in for the maximization problem of the function $F(t, \cdot)$ we chose any point.*

4 Main results

First we study the strategy (3.16).

Theorem 4.1. *The strategy $v^* = (\theta_t^*, c_t^*)_{0 \leq t \leq T}$ defined in (3.16), i.e. $\theta_t^* = \theta_0$ and $c_t^* = c_0(t)$, is a solution for the problem (2.13). Moreover, for any $0 \leq t < T$, optimal value function (2.14) is given*

$$J^*(t, x) = J(t, x, v^*) = x^\gamma \left(\int_t^T \Psi_{t,u} du + \Psi_{t,T} \right)^{1-\gamma}. \quad (4.1)$$

The proof is given in section 7.

Remark 4.2. *Note that for the homogenous market model (2.1), i.e. for the constant coefficients $r_t, \mu_t, \sigma_t, \varsigma_t$ the optimal strategy (3.16) is obtained in [23].*

Now we consider the optimization problem (2.1) for the markets with transaction costs on the basis of the Leland's approach proposed in [15] for hedging problems. In this case we use the optimal strategy

$$\alpha_t^* = (\alpha_1^*(t), \dots, \alpha_m^*(t))' = \left(\frac{\theta_1^*(t) X_t^*}{S_1(t)}, \dots, \frac{\theta_m^*(t) X_t^*}{S_m(t)} \right)',$$

$$\beta_t^* = \frac{(1 - \sum_{j=1}^m \theta_j^*(t)) X_t^*}{B_t} \quad \text{and} \quad \zeta_t^* = c_t^* X_t^*, \quad (4.2)$$

where the fractional strategy $\theta_t^* = (\theta_1^*(t), \dots, \theta_m^*(t))'$ is given in (3.13) and the corresponding wealth process X_t^* is calculated as

$$X_t^* = x + \int_0^t (\alpha_{u-}^*)' dS_u + \int_0^t \beta_u^* dB_u - \int_0^t \zeta_u^* du. \quad (4.3)$$

According to the Leland approach we can do only n portfolio revisions of the strategy (4.2) on the interval $[0, T]$ at the moments $0 < t_1 < \dots < t_n = T$, i.e. we set

$$\alpha_t^{(n)} = \sum_{k=1}^n \alpha_{t_{k-1}}^* \mathbf{1}_{]t_{k-1}, t_k]}, \quad \beta_t^{(n)} = \sum_{k=1}^n \beta_{t_{k-1}}^* \mathbf{1}_{]t_{k-1}, t_k]}$$

and $\zeta_t^{(n)} = \sum_{k=1}^n \zeta_{t_{k-1}}^* \mathbf{1}_{]t_{k-1}, t_k]}$. (4.4)

Moreover, for each revision the investors must pay the transaction costs proportionally to the volume of trade at the moment t_k defined as

$$\kappa \sum_{j=1}^m S_{t_k} |\alpha_j^{(n)}(t_k) - \alpha_j^{(n)}(t_{k-1})| = \kappa \sum_{j=1}^m S_{t_k} |\alpha_j^*(t_{k-1}) - \alpha_j^*(t_{k-2})|,$$

where by the convention $\alpha_j^*(t) = 0$ for $t < 0$. Therefore, the total transaction costs on the interval $[0, T]$ is presented as

$$\mathbf{D}_n = \kappa \sum_{j=1}^m \sum_{k=1}^n S_j(t_k) |\alpha_j^{(n)}(t_k) - \alpha_j^{(n)}(t_{k-1})|, \quad (4.5)$$

where $\kappa > 0$ is a proportional transaction coefficient which is assumed to be a function of the revisions numbers n , i.e. $\kappa = \kappa_n$. Note that, in this case $\alpha_{u-}^{(n)} = \alpha_u^{(n)}$. Therefore, to take into account transaction costs we defined the wealth as

$$X_T^{(n)} = x + \int_0^T (\alpha_u^{(n)})' dS_u + \int_0^T \beta_u^{(n)} dB_u - \int_0^T \zeta_u^{(n)} du - \mathbf{D}_n. \quad (4.6)$$

To study properties of the strategy $\psi^{(n)} = (\alpha_t^{(n)}, \beta_t^{(n)}, \zeta_t^{(n)})_{0 \leq t \leq T}$ we set

$$J(x, \psi^{(n)}) = \mathbf{E}_x \left(\int_0^T (\zeta_u^{(n)})^\gamma du + (X_T^{(n)})_+^\gamma \right). \quad (4.7)$$

Now we study asymptotic properties of the strategy $\psi^{(n)}$ as $n \rightarrow \infty$. To do this we need to assume the following conditions.

A₁ In the model (2.1) the functions r_t , μ_t and σ_t are bounded on the interval $[0, T]$.

A₂ The functions r_t , μ_t , σ_t , ζ_t in the model (2.1) are such that the optimal strategy (3.13) satisfies the following inequality

$$\sup_{n \geq 1} \sup_{0 < t_1 < \dots < t_n = T} \frac{\sum_{j=1}^n |\theta_{t_j}^* - \theta_{t_{j-1}}^*|}{1 + \sum_{j=1}^n \sqrt{t_j - t_{j-1}}} < \infty. \quad (4.8)$$

Note that **A₂**) holds true if the optimal strategy (3.13) is 1/2-Holder function, i.e.

$$\sup_{0 \leq s, t \leq T} \frac{|\theta_t^* - \theta_s^*|}{\sqrt{|t - s|}} < \infty. \quad (4.9)$$

Theorem 4.3. *Assume that Conditions $\mathbf{A}_1\mathbf{A}_2$) hold true and the transaction coefficient is a function of n , i.e. $\kappa = \kappa_n$ such that $\kappa_n = o(n^{-1/2})$, as $n \rightarrow \infty$. Then the strategy (4.4) - (4.6) with the revision moments $(t_j = jT/n)_{1 \leq j \leq n}$ is asymptotically optimal, i.e.*

$$\lim_{n \rightarrow \infty} J(x, \psi^{(n)}) = J^*(x), \quad (4.10)$$

where the optimal functional $J^*(x)$ is defined in (2.13).

Now we study the optimization problem (2.1) for the markets with large transaction costs, i.e. in the case when the normalized transaction coefficient $\sqrt{n}\kappa_n$ does not go to zero as $n \rightarrow \infty$. We assume only that $\kappa = \kappa_n = o(1)$, i.e. $\kappa_n \rightarrow 0$ as $n \rightarrow \infty$. In this case we need to change the strategy (4.4) making use of Lepinette approach proposed in [16] and used then for the markets with jumps in [21]. According to this approach we do the portfolio revisions at the points

$$t_j = \mathbf{t}^*(u_j)T, \quad u_j = \frac{j}{n} \quad \text{and} \quad \mathbf{t}^*(u) = u^{\mathbf{q}}, \quad (4.11)$$

where the power $\mathbf{q} \geq 1$ is a function of n , i.e. $\mathbf{q} = \mathbf{q}_n$ such that

$$\lim_{n \rightarrow \infty} \mathbf{q}_n = +\infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \left(\frac{\mathbf{q}_n}{n} + \kappa_n \sqrt{\frac{n}{\mathbf{q}_n}} \right) = 0. \quad (4.12)$$

Note that, if $\kappa_n \rightarrow 0$ as $n \rightarrow \infty$, then we can take, for example, $\mathbf{q}_n = \sqrt{n} + \kappa_n n$.

Theorem 4.4. *Assume that the conditions $\mathbf{A}_1\mathbf{A}_2$) hold true and the transaction coefficient is a function of n , i.e. $\kappa = \kappa_n$ such that $\kappa_n = o(1)$ as $n \rightarrow \infty$. Then the strategy (4.4) - (4.6) with the revision time moments defined in (4.11) - (4.12) is asymptotically optimal, i.e. satisfies the property (4.10).*

Moreover, we need to find sufficient conditions providing Condition \mathbf{A}_2).

Proposition 4.5. *Assume that, in the model (2.1) $m = 1$, the functions r_t, μ_t, σ_t and ς_t are continuously differentiable and $\varsigma_t > 0$ for all $0 \leq t \leq T$. Then Conditions $\mathbf{A}_1\mathbf{A}_2$) hold true.*

Remark 4.6. *It should be noted that Condition \mathbf{A}_2) hold for any $m \geq 1$ for the homogeneous market (2.1), i.e. for the constant parameters $r_t, \mu_t, \sigma_t, \varsigma_t$.*

5 Properties of the optimal strategies (3.16) - (3.17)

In this section we study the Doéan exponentials $\mathcal{E}_t(V)$ defined in (2.11).

Proposition 5.1. *For any non random measurable $[0, T] \rightarrow \Theta$ function $\theta = (\theta_t)_{0 \leq t \leq T}$ the Doléan exponential (2.11) is square integrated martingale.*

Proof. First note, that the process $V_t = \int_0^t \theta'_s d\tilde{L}_s$ is square integrated martingale. Therefore, taking into account that $d\mathcal{E}_t(V) = \mathcal{E}_{t-}(V) dV_t$, to show this lemma it suffices to check that $\sup_{0 \leq t \leq T} \mathbf{E} \mathcal{E}_t^2(V) < \infty$. To this end we represent the Doléan exponential in the multiplicative form, i.e.

$$\mathcal{E}_t(V) = e^{\int_0^t \theta'_s \sigma_s dW_s - \frac{1}{2} \int_0^t \theta'_s \sigma_s \sigma'_s \theta_s ds} \mathcal{E}_t^{(d)}, \quad (5.1)$$

where $\mathcal{E}_t^{(d)} = e^{\int_0^t \tilde{\zeta}'_s dL_s + \sum_{0 \leq s \leq t} (\ln(1 + \tilde{\zeta}'_s \Delta L_s) - \tilde{\zeta}'_s \Delta L_s)}$ is the jump Doléan exponential and the non random functions $\tilde{\zeta}_s = \zeta'_s \theta_s = (\tilde{\zeta}_{1,s}, \dots, \tilde{\zeta}_{m,s})' \in [0, 1]^m$. Note here, that this exponential can be represented as production $\mathcal{E}_t^{(d)} = \prod_{j=1}^m \mathcal{E}_{j,t}^{(d)}$ of the independent exponentials

$$\mathcal{E}_{j,t}^{(d)} = e^{\int_0^t \tilde{\zeta}'_{j,s} dL_j(s) + \sum_{0 \leq s \leq t} (\ln(1 + \tilde{\zeta}'_{j,s} \Delta L_j(s)) - \tilde{\zeta}'_{j,s} \Delta L_j(s))}.$$

Therefore, to prove this lemma it suffices to show that

$$\max_{1 \leq j \leq m} \sup_{0 \leq t \leq T} \mathbf{E} (\mathcal{E}_{j,t}^{(d)})^2 < \infty. \quad (5.2)$$

To this end, note that we can write that

$$\ln \mathcal{E}_{j,t}^{(d)} = Z_{j,t} + \int_0^t \Pi_j (\mathbf{g}(\tilde{\zeta}_{j,s} y) - \tilde{\zeta}_{j,s} y) ds, \quad (5.3)$$

where the process $Z_{j,t} = \int_0^t \int_{\mathbb{R}_*} \mathbf{g}(\tilde{\zeta}_{j,s} y) d\bar{\nu}_j(\omega; dy, ds) = (\nu_j - \bar{\nu}_j)(\omega; dy, ds)$ and $\mathbf{g}(x) = \ln(1 + x)$. Note here, that for any $0 \leq \zeta \leq 1$

$$\begin{aligned} \Pi_j (|\mathbf{g}(\zeta y) - \zeta y|) &\leq \Pi_j (\mathbf{1}_{\{|y| < -1/2\}} (1 + |\mathbf{g}(y)|) \\ &+ 2\Pi_j (\mathbf{1}_{\{|y| \leq 1/2\}} y^2) + \Pi_j (\mathbf{1}_{\{|y| > 1/2\}} (y + \mathbf{g}(y))) . \end{aligned}$$

Using here the conditions (2.6) we get that

$$\max_{1 \leq j \leq m} \sup_{0 \leq \zeta \leq 1} \Pi_j (|\mathbf{g}(\zeta y) - \zeta y|) < \infty. \quad (5.4)$$

Therefore, for (5.2) it suffices to show, that

$$\max_{1 \leq j \leq m} \sup_{0 \leq t \leq T} \mathbf{E} e^{2Z_{j,t}} < \infty. \quad (5.5)$$

It is clear that a.s.

$$Z_{j,t} = \lim_{\delta \rightarrow 0} Z_{j,t}^{(\delta)} \quad \text{and} \quad Z_{j,t}^{(\delta)} = \int_0^t \int_{\mathbb{R}_*} \mathbf{g}(\tilde{\zeta}_{j,s} y) d\bar{\nu}_j^{(\delta)},$$

where $\bar{\nu}_j^{(\delta)}(\omega; dy, ds) = \mathbf{1}_{\{|y| > \delta\}} \bar{\nu}_j(\omega; dy, ds)$. Note, that we can represent the function $\tilde{\zeta}_{j,s}$ as a limit in the Lebesgue measure on the interval $[0, t]$ of the piece functions

$$\tilde{\zeta}_{j,s}^{(n)} = \sum_{l=1}^n c_{j,l}^{(n)} \mathbf{1}_{]t_{l-1}, t_l]}, \quad 0 \leq c_{j,l}^{(n)} \leq 1 \quad \text{and} \quad t_l = \frac{l}{n} t. \quad (5.6)$$

It should be emphasized, that in view of the boundedness of the functions $\tilde{\zeta}_{j,s}$ and $\tilde{\zeta}_{j,s}^{(n)}$ we can conclude through the dominated convergence theorem that

$$\lim_{n \rightarrow \infty} \int_0^t |\tilde{\zeta}_{j,s} - \tilde{\zeta}_{j,s}^{(n)}| ds = 0. \quad (5.7)$$

Now, setting

$$Z_{j,t}^{(\delta,n)} = \int_0^t \int_{|y|>\delta} \mathbf{g} \left(\tilde{\zeta}_{j,s}^{(n)} y \right) d\bar{v}_j,$$

we can obtain that

$$\mathbf{E} \left| Z_{j,t}^{(\delta)} - Z_{j,t}^{(\delta,n)} \right| \leq 2 \int_0^t \int_{|y|>\delta} \Delta_j^{(n)}(y, s) \Pi(dy) ds \quad (5.8)$$

and $\Delta_j^{(n)}(y, s) = \left| \mathbf{g} \left(\tilde{\zeta}_{j,s} y \right) - \mathbf{g} \left(\tilde{\zeta}_{j,s}^{(n)} y \right) \right|$. Note here, that for $0 < \varepsilon < 1 - \delta$ and for any $y > -1 + \varepsilon$ we get that

$$\Delta_j^{(n)}(y, s) \leq \frac{1}{\varepsilon} |\tilde{\zeta}_{j,s} - \tilde{\zeta}_{j,s}^{(n)}| |y|.$$

Moreover, note, that for $y > -1$ we have $\Delta_j^{(n)}(y, s) \leq 2|\ln(1+y)|$. Therefore, we can estimate from above the integral in the right side of the inequality (5.8) as

$$2t \int_{\{-1 < y < -1 + \varepsilon\}} |\ln(1+y)| \Pi(dy) + U_*(\varepsilon) \int_0^t |\tilde{\zeta}_{j,s} - \tilde{\zeta}_{j,s}^{(n)}| ds,$$

where

$$U_*(\varepsilon) = \frac{1}{\varepsilon} \left(\int_{\{-1 + \varepsilon < y < -\delta\}} |y| \Pi(dy) + \int_{\{y > \delta\}} y \Pi(dy) \right).$$

Now, taking into account the limit (5.7), we obtain that for any $0 < \varepsilon < 1 - \delta$

$$\limsup_{n \rightarrow \infty} \mathbf{E} \left| Z_{j,t}^{(\delta)} - Z_{j,t}^{(\delta,n)} \right| \leq 4t \int_{\{-1 < y < -1 + \varepsilon\}} |\mathbf{g}(y)| \Pi(dy).$$

Letting here $\varepsilon \rightarrow 0$, we obtain through the condition (2.6)

$$\mathbf{P} - \lim_{n \rightarrow \infty} Z_{j,t}^{(\delta,n)} = Z_{j,t}^{(\delta)}.$$

This implies by the Fatou lemma

$$\mathbf{E} e^{2Z_{j,t}} \leq \liminf_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbf{E} e^{2Z_{j,t}^{(\delta,n)}}. \quad (5.9)$$

Note here, that

$$\mathbf{E} e^{2Z_{j,t}^{(\delta,n)}} = \prod_{l=1}^n \mathbf{E} e^{2\eta_l} \quad \text{and} \quad \eta_l = \int_{t_{l-1}}^{t_l} \int_{\mathbb{R}_*} \mathbf{g} \left(c_{j,l}^{(n)} y \right) \mathbf{1}_{\{|y|>\delta\}} d\bar{v}_j.$$

We calculate directly, that

$$\mathbf{E} e^{2\eta_l} = e^{(t_l - t_{l-1}) \varpi_{j,l}^{(\delta,n)}},$$

where

$$\varpi_{j,l}^{(\delta,n)} = \int_{|y|>\delta} \left(e^{2\mathbf{g}(c_{j,l}^{(n)} y)} - 1 - 2\mathbf{g}(c_{j,l}^{(n)} y) \right) \Pi_j(dy).$$

Using the conditions (2.6) we can estimate this term as

$$\begin{aligned} \varpi_{j,l}^{(\delta,n)} &\leq 2 \int_{-1 < y < -1/2} (1 + |\mathbf{g}(y)|) \Pi_j(dy) \\ &\quad + 2e \int_{|y| \leq 1/2} \mathbf{g}^2(y) \Pi_j(dy) + \int_{y > 1/2} (1 + y^2) \Pi_j(dy) := \varpi^* < \infty. \end{aligned}$$

Therefore, using this estimate in (5.9), we get that

$$\mathbf{E} e^{2Z_{j,t}^{(\delta,n)}} \leq e^{t\varpi^*} < \infty.$$

This implies the upper bound (5.5). Hence Proposition 5.1. \square

Proposition 5.2. *For any non random measurable $[0, T] \rightarrow \Theta$ function $\theta = (\theta_t)_{0 \leq t \leq T}$, the Doléan exponential (2.11) is strictly positive, i.e. $\inf_{0 \leq t \leq T} \mathcal{E}_t(V) > 0$ a.s.*

Proof. First note, that the functions σ_{ji} are square integrated and, therefore, through Doob's inequality (A.3) we get

$$\mathbf{E} \max_{0 \leq t \leq T} \left(\int_0^t \theta'_u \sigma_u dW_u \right)^2 \leq 4 \int_0^T \theta'_u \sigma_u \sigma'_u \theta_u du \leq 4 \int_0^T \text{tr} \sigma_u \sigma'_u du < \infty,$$

i.e. $\max_{0 \leq t \leq T} \left| \int_0^t \theta'_u \sigma_u dW_u \right| < \infty$ a.s. This means, that in view of the representations (5.1) and (5.3) and the upper bound (5.4) to prove this proposition it suffices to show that

$$\max_{1 \leq j \leq m} \sup_{0 \leq t \leq T} |Z_{j,t}| < \infty \quad \text{a.s.} \quad (5.10)$$

Indeed, we can represent this process as

$$Z_{j,t} = Z_{j,t}^{(1)} + Z_{j,t}^{(2)},$$

where

$$Z_{j,t}^{(1)} = \int_0^t \int_{\mathbb{R}_*} \mathbf{1}_{\{y < -1/2\}} \mathbf{g}(\tilde{\zeta}_{j,s} y) d\bar{\nu}_j \quad \text{and} \quad Z_{j,t}^{(2)} = \int_0^t \int_{\mathbb{R}_*} \mathbf{1}_{\{y \geq -1/2\}} \mathbf{g}(\tilde{\zeta}_{j,s} y) d\bar{\nu}_j.$$

Note here, that in view of the condition (2.6) we get, that for any $0 \leq t \leq T$

$$\begin{aligned} |Z_{j,t}^{(1)}| &\leq \sum_{0 \leq s \leq T} \mathbf{1}_{\{\Delta L_j(s) < -1/2\}} |\mathbf{g}(\Delta L_j(s))| \\ &\quad + \Pi_j(|\mathbf{g}(y)| \mathbf{1}_{\{-1 < y < -1/2\}}) < \infty \quad \text{a.s.} \end{aligned}$$

Moreover, we estimate the term $Z_{j,t}^{(2)}$ from above through the inequality (A.2) with $p = 2$ and using again the conditions (2.6), i.e.

$$\begin{aligned} \mathbf{E} \sup_{0 \leq t \leq T} (Z_{j,t}^{(2)})^2 &\leq \check{C}_2 \int_0^T \int_{\mathbb{R}_*} \mathbf{1}_{\{y \geq -1/2\}} \mathbf{g}^2(\tilde{\zeta}_{j,s} y) \, ds \Pi_j(dy) \\ &\leq \check{C}_2 T \Pi_j(|\mathbf{g}(y)| \mathbf{1}_{\{y \geq -1/2\}}) < \infty. \end{aligned}$$

Hence Proposition 5.2. \square

Proposition 5.3. For any $0 \leq t \leq T$ and $x > 0$

$$\mathbf{E}_{t,x} \sup_{t \leq s \leq T} (X_s^*)^2 < \infty. \quad (5.11)$$

Moreover,

$$\inf_{0 \leq t \leq T} X_t^* > 0 \quad a.s. \quad (5.12)$$

Proof. Indeed, note that, from (3.17) it is easy to deduce that $X_t^* \leq C \mathcal{E}_t(V^*)$ for some $C > 0$. Therefore, Doob's martingale inequality (A.3) and Proposition 5.1 imply

$$\mathbf{E}_{t,x} \sup_{t \leq s \leq T} (X_s^*)^2 \leq C \mathbf{E} \sup_{t \leq s \leq T} \mathcal{E}_t^2(V^*) \leq 4C \mathbf{E} \mathcal{E}_T^2(V^*) < \infty$$

and we get the bound (5.11). Moreover, using again the representation (3.17) we obtain through Proposition 5.2 the lower bound (5.12). \square

Now we need to study properties of the discrete investment strategy (4.4).

Proposition 5.4. If the conditions (2.2), (2.4) and (2.6) hold true, then

$$\max_{1 \leq j \leq m} \sup_{n \geq 1} \sup_{0 < t_1 < \dots < t_n = T} \mathbf{E} \left| \int_0^T \alpha_j^{(n)}(t) \, dS_j(t) \right| < \infty$$

and

$$\sup_{n \geq 1} \sup_{0 < t_1 < \dots < t_n = T} \mathbf{E} \left| \int_0^T \beta_t^{(n)} \, dB_t \right| < \infty.$$

Proof. First note, that the stock price can be represented as

$$S_j(t) = S_0 \exp \left(\int_0^t \mu_j(u) \, du \right) \mathcal{E}_t(\tilde{L}_j) \quad (5.13)$$

where $\mathcal{E}_t(\tilde{L}_j)$ is defined in (2.11) with $V_t = \tilde{L}_j$ and $\langle V \rangle_t = \sum_{l=1}^m \int_0^t \sigma_{j,l}^2(s) \, ds$. Note now, that for any fixed $0 \leq u \leq T$ the process $\hat{\mathcal{E}}_{u,t}(\tilde{L}_j) = \mathcal{E}_t(\tilde{L}_j) / \mathcal{E}_u(\tilde{L}_j)$ is the Doléan exponential for $u \leq t \leq T$ which is independent from \mathcal{F}_u and, therefore, Proposition 5.1 yields

$$\max_{1 \leq j \leq m} \sup_{0 \leq u < t \leq T} \mathbf{E} \hat{\mathcal{E}}_{u,t}^2(\tilde{L}_j) = \max_{1 \leq j \leq m} \sup_{0 \leq u < t \leq T} \mathbf{E} \left(\hat{\mathcal{E}}_{u,t}^2(\tilde{L}_j) | \mathcal{F}_u \right) < \infty.$$

Taking into account that the functions μ_j are bounded we can get the same property for the process $\widehat{S}_j(t, u) = S_j(t)/S_j(u)$, i.e.

$$\mathbf{U}^* = \max_{1 \leq j \leq m} \sup_{0 \leq u < t \leq T} \mathbf{E} \widehat{S}_j^2(t, u) = \max_{1 \leq j \leq m} \sup_{0 \leq u < t \leq T} \mathbf{E} \left(\widehat{S}_j^2(t, u) | \mathcal{F}_u \right) < \infty.$$

Moreover, note that from the definition (4.2) we obtain that for any $1 \leq l \leq m$

$$\begin{aligned} \mathbf{E} \int_0^T (\alpha_j^{(n)}(t))^2 S_j^2(t) \sigma_{jl}^2(t) dt &\leq \mathbf{E} \sum_{k=1}^n (X_{t_{k-1}}^*)^2 \int_{t_{k-1}}^{t_k} \mathbf{E} \left(\widehat{S}_j^2(t, t_{k-1}) | \mathcal{F}_{t_{k-1}} \right) \sigma_{jl}^2(t) dt \\ &\leq \mathbf{U}^* \mathbf{E} \sup_{0 \leq t \leq T} (X_t^*)^2 \int_0^T \sigma_{jl}^2(t) dt \end{aligned}$$

and through the upper bound (5.11) we get, that

$$\max_{1 \leq j, l \leq m} \sup_{n \geq 1} \sup_{0 < t_1 < \dots < t_n = T} \mathbf{E} \int_0^T (\alpha_j^{(n)}(t))^2 S_j^2(t) \sigma_{jl}^2(t) dt < \infty.$$

In the same way we can show, that

$$\max_{1 \leq j \leq m} \sup_{n \geq 1} \sup_{0 < t_1 < \dots < t_n = T} \mathbf{E} \int_0^T \alpha_j^{(n)}(t) S_j(t) |\mu_j(t)| dt < \infty.$$

Therefore, using the definition of the stock price in (2.1) we obtain that

$$\begin{aligned} \mathbf{E} \left| \int_0^T \alpha_j^{(n)}(t) dS_j(t) \right| &\leq \mathbf{E} \int_0^T \alpha_j^{(n)}(t) S_j(t) |\mu_j(t)| dt \\ &\quad + \left(\mathbf{E} \left(\int_0^T \alpha_j^{(n)}(t) S_j(t-) d\widetilde{L}_i(t) \right)^2 \right)^{1/2}. \end{aligned}$$

Taking into account here, that

$$\begin{aligned} \mathbf{E} \left(\int_0^T \alpha_j^{(n)}(t) S_j(t-) d\widetilde{L}_i(t) \right)^2 &= \sum_{l=1}^m \mathbf{E} \int_0^T (\alpha_j^{(n)}(t))^2 S_j^2(t) \sigma_{jl}^2(t) dt \\ &\quad + \sum_{l=1}^m \Pi_l(y^2) \mathbf{E} \int_0^T (\alpha_j^{(n)}(t))^2 S_j^2(t) \varsigma_{jl}^2(t) dt, \end{aligned}$$

we obtain through the conditions (2.4) the first inequality of this lemma. As to the last inequality, note that

$$\begin{aligned} \left| \int_0^T \beta_t^{(n)} dB_t \right| &\leq \sum_{k=1}^n X_{t_{k-1}}^* \left| \int_{t_{k-1}}^{t_k} r_t \frac{B_t}{B_{t_{k-1}}} dt \right| = \sum_{k=1}^n X_{t_{k-1}}^* \left| e^{\int_{t_{k-1}}^{t_k} r_u du} - 1 \right| \\ &\leq \mathbf{U}_1^* \sup_{0 \leq t \leq T} X_t^* \sum_{k=1}^n \int_{t_{k-1}}^{t_k} |r_u| du = \mathbf{U}_1^* \sup_{0 \leq t \leq T} X_t^* \int_0^T |r_t| dt, \end{aligned}$$

where

$$\mathbf{U}_1^* = \sup_{|x| \leq x^*} \left| \frac{e^x - 1}{x} \right| \quad \text{and} \quad x^* = \int_0^T |r_t| dt.$$

Using again the upper bound (5.11) we get the last inequality of this proposition. \square

6 Numerical example

For Monte Carlo simulations we consider the market (2.1) for $m = 1$ on the interval $[0, T]$ for $T = 3$ in which $B_t = e^{r \cdot t}$ with $r = 0,055$, $\sigma = 0,05$

$$S_t = S_0 \exp \left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t + L_t \right) \prod_{s \leq t} (1 + \Delta L_s) e^{-\Delta L_s}.$$

Here $\mu = 0.06$, $S_0 = 50$, W_t is standard Brownian motion, L_t is a Compound Poisson process defined by $L_t = \sum_{j=1}^{N_t} Y_j$, where N_t is Poisson process with the constant intensity $\lambda = 17$ and the sizes of jumps $(Y_j)_{j \geq 1}$ are i.i.d. random variables with the density function $\mathbf{p}(y) = e^{-\frac{y^2}{2\eta^2}} \left(\int_{-1/2}^{\infty} e^{-\frac{z^2}{2\eta^2}} dz \right)^{-1} \mathbf{1}_{\{y > -1/2\}}$ and $\eta = 0,01$. Note, that in this case the Lévy measure $\Pi(dy) = \lambda p_Y(y) dy$.

In this example we consider the utility functions $U_1(x, \mathbf{v}) = (x \mathbf{c})^\gamma$ and $U_2(x) = x^\gamma$ with $\gamma = 0.3$, where the control variables $\kappa = (\mathbf{v}, \mathbf{c}) \in [0, 1] \times \mathbb{R}_+$. We calculate the strategies (4.2) – (4.4) with the cumulative consumption defined as $C_t^{(n)} = \int_0^t \zeta_u^{(n)} du$.

The graphs in Figure 1 show the market behavior: b) is the log returns of the risky asset, c) – e) are trading strategies with and without transaction costs, f) is investor's wealth and him consumption. Here the portfolios are revised $n = 100$ times with the transaction costs coefficient $\kappa = 0,1 \ln(100)/\sqrt{100} \approx 0,046$. Moreover, the graphs in Figure 2 a) illustrate the convergence (4.10) for the strategy (4.4) and (4.11). To this end, we calculate the objective function $J(x, \psi^{(n)})$ through Monte Carlo simulations as

$$J(x, \psi^{(n)}) \approx \frac{1}{K} \sum_{l=1}^K \left(\int_0^T \zeta_u^{(n,l)} du + \left(X_T^{(n,l)} \right)_+^\gamma \right), \quad K = 50,$$

where $(\zeta_u^{(n,l)})_{0 \leq u \leq T}$ and $X_T^{(n,l)}$ are the consumption and wealth processes calculated at the l -th simulations, and transaction costs coefficient $\kappa_n = 0,5 \ln(n)/\sqrt{n}$ and the initial endowment $x = 100$.

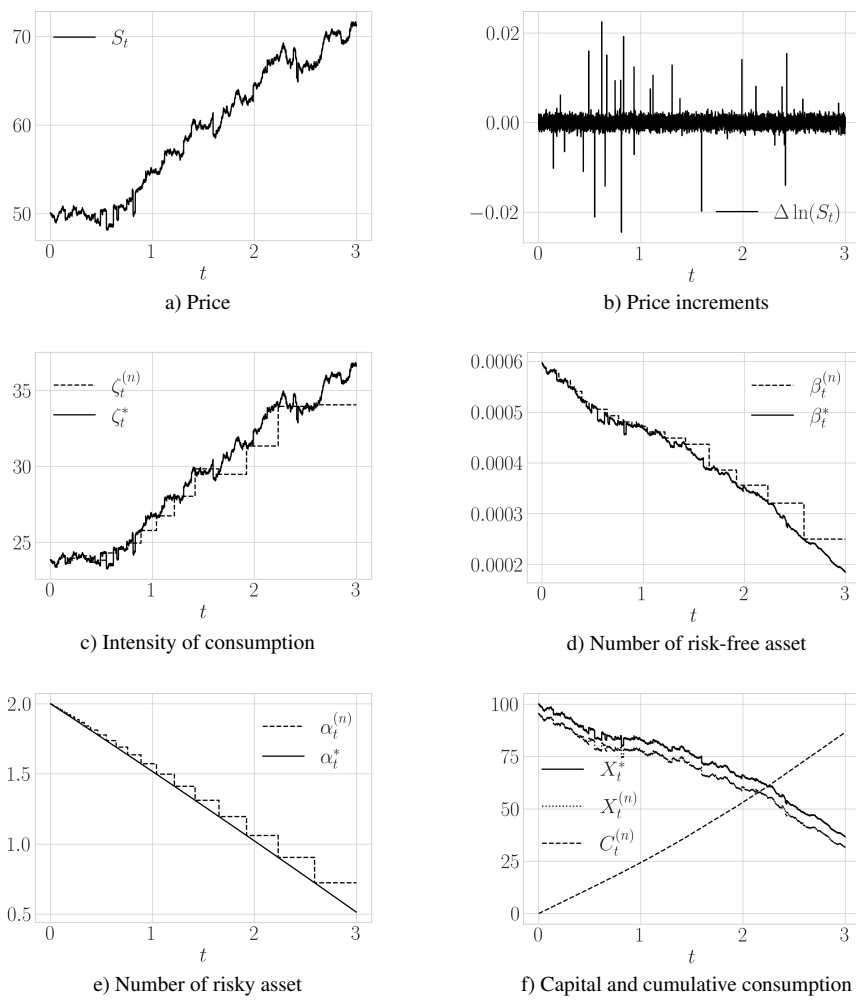


Fig. 1 Results of simulation of the strategies (4.2) – (4.4)

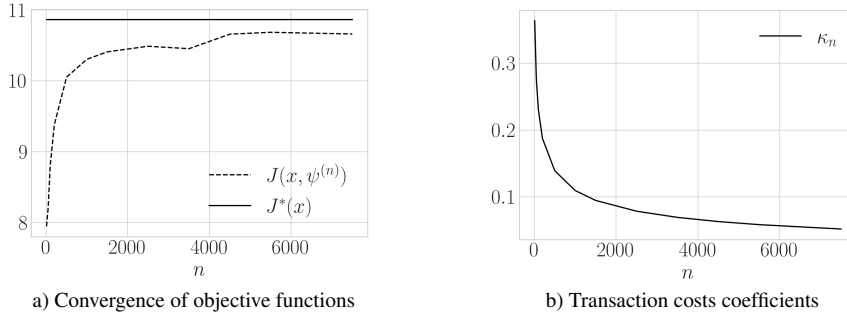


Fig. 2 Monte Carlo simulations

As we can see from the simulation results in Figure 1, the behavior of the optimal strategy means that when the risky asset increases one needs to increase the consumption intensity and decrease the investments in the both assets risky and risky free. It should be noted that such an investor's policy allows better control over the behavior of the financial portfolio, so as not to make its terminal value negative. Usually, after the rise in prices, there is always a decline, and this optimal strategy permits avoiding abrupt changes in the portfolio dynamics. As we can see all the changes in portfolio dynamics generally occur because of consumption, not because of the stochastic behavior of the risky assets. The numerical simulations confirm the natural behavior of the optimal strategies, i.e. the more consumption, the less capital and, moreover, as it is seen in Figure 1, f), this strategy provides an equilibrium-point between consumption and wealth. Finally, as we see In Figure 2, the convergence (4.10) is confirmed in these simulations.

7 Proofs

7.1 Proof of Theorem 3.3

Proof. Let $v \in \mathcal{V}$. Then by the definition of an admissible control process, the equation

$$dX_u = a(u, X_u, v_u) du + b'(X_{u-}, v_{u-}) d\tilde{L}_u, \quad X_t = x \in \mathcal{X}, \quad u \in [t, T] \quad (7.1)$$

has an unique \mathbf{P} -a.s. strong solution. In view of the condition \mathbf{H}_1) the HJB equation (3.7) has a solution z from $\mathbf{C}^{1,2}([0, T] \times \mathbb{R})$. Therefore, the Ito formula implies

$$\begin{aligned} z(u, X_u) &= z(t, x) + \int_t^u z_t(s, X_s) ds + \int_t^u z_x(s, X_{s-}) dX_s \\ &\quad + \frac{1}{2} \int_t^u z_{xx}(s, X_s) \varpi_s ds + \sum_{t \leq s \leq u} (\Delta z(s, X_s) - z_x(s, X_{s-}) \Delta X_s), \end{aligned}$$

where $\varpi_s = \text{tr } \sigma_s' b(s, X_s, \kappa) b'(s, X_s, \kappa) \sigma_s$ and $\Delta z(s, X_s) = z(s, X_s) - z(s, X_{s-})$ and $\Delta X_s = X_s - X_{s-}$. Therefore, using (7.1) we can represent this equation as

$$\begin{aligned} z(u, X_u) &= z(t, x) + \int_t^u \left(z_t(s, X_s) + z_x(s, X_s) a(s, X_s, v_s) + \frac{1}{2} z_{xx}(s, X_s) \varpi_s \right) ds \\ &\quad + \mathbf{M}_u + \sum_{t \leq s \leq u} (\Delta z(s, X_s) - z_x(s, X_{s-}) \Delta X_s), \end{aligned} \quad (7.2)$$

where $\mathbf{M}_u = \int_t^u z_x(s, X_{s-}) b'(s, X_{s-}, v_{s-}) d\tilde{L}_s$.

The key idea of the proof is the following, we need to compensate the jump term, add to and subtract from Ito's formula the integral $\int_t^u U_1(s, X_s, v_s) ds$ and take a conditional expectation given $X_t = x$. But we cannot apply the last operation directly since there are no conditions on finiteness of expectations (in particular, for stochastic integrals). Therefore we introduce a sequence of stopping times $(\tau_n)_{n>0}$, which helps to control the jumps compensator and provides square integrability of stopped stochastic integral,

$$\tau_n := \inf \left\{ u \geq t : \int_t^u (\bar{b}_s z_x^2(s, X_s) + |\bar{z}(s, X_s, v_s)|) ds \geq n \right\} \wedge T, \quad (7.3)$$

where $\bar{b}_s = b'(s, X_s, v_s) (\sigma_s \sigma_s' + \zeta_s \zeta_s') b(s, X_s, v_s)$ and \bar{z} is given as in (3.5) for $g \equiv z$. Note, that $\mathbf{E}_{t,x} \mathbf{M}_{\tau_n} = 0$ and by the properties of the random measures (see, for example, Theorem 1 on the page 153 in [18]) we get

$$\mathbf{E}_{t,x} \sum_{t \leq s \leq \tau_n} |(\Delta z(s, X_s) - z_x(s, X_{s-}) \Delta X_s)| = \mathbf{E}_{t,x} \int_t^{\tau_n} |\bar{z}(s, X_s, v_s)| ds \leq n.$$

Therefore, setting

$$\mathbf{M}_n^* = \mathbf{M}_{\tau_n} + \sum_{t \leq s \leq \tau_n} (\Delta z(s, X_s) - z_x(s, X_{s-}) \Delta X_s) - \int_t^{\tau_n} \bar{z}(s, X_s, v_s) ds,$$

we get $\mathbf{E}_{t,x} \mathbf{M}_n^* = 0$. Furthermore, using the definition (3.5), we can see from (7.2) that

$$\int_t^{\tau_n} U_1(s, X_s, v_s) ds + z_n = z(t, x) + \int_t^{\tau_n} (z_t(s, X_s) + H_0(s, X_s, z, v_s)) ds + \mathbf{M}_n^*,$$

where $z_n = z(\tau_n, X_{\tau_n})$. Note here, that Condition \mathbf{H}_1) and the definition (3.6) yield

$$0 = z_t(s, X_s) + H(s, X_s, z(s, X_s)) \geq z_t(s, X_s) + H_0(s, X_s, z(s, X_s), v_s),$$

i.e. $\int_t^{\tau_n} U_1(s, X_s, v_s) ds + z_n \leq z(t, x) + \mathbf{M}_n^*$. Therefore, for any $n \geq 1$

$$\mathbf{E}_{t,x} \left(\int_t^{\tau_n} U_1(s, X_s, v_s) ds + z_n \right) \leq z(t, x). \quad (7.4)$$

Note here, that according to Definition 3.1 for any $v \in \mathcal{V}$ the wealth process X_t and its left limit X_{t-} belong to \mathcal{X} . Therefore, $\inf_{0 \leq t \leq T} X_t \in \mathcal{X}$ and $\sup_{0 \leq t \leq T} X_t \in \mathcal{X}$ and, in view of Lemma A.1 and the condition (3.2) the integral $\int_0^T |\bar{z}(s, X_s, v_s)| ds < \infty$ a.s. This implies immediately that a.s. $\tau_n \rightarrow T$,

$$\int_t^{\tau_n} U_1(s, X_s, v_s) ds \rightarrow \int_t^T U_1(s, X_s, v_s) ds \quad \text{and} \quad z_n \rightarrow z(T, X_T) = U_2(X_T)$$

as $n \rightarrow \infty$. Moreover, note also, that $U_1 \geq 0$ and by condition \mathbf{H}_1) the function z is uniformly bounded from below. Therefore, the use of the Fatou lemma for the limit transition in the expectation in (7.4) yields, that for any $v \in \mathcal{V}$

$$\mathbf{E}_{t,x} \left(\int_t^T U_1(s, X_s, v_s) ds + U_2(X_T) \right) = J(t, x, v) \leq z(t, x), \quad (7.5)$$

i.e. $\sup_{v \in \mathcal{V}} J(t, x, v) =: J^*(t, x) \leq z(t, x)$. Moreover, it should be noted, that in view of

Conditions \mathbf{H}_2)– \mathbf{H}_3), there exists the measurable function $\mathbf{v}_0 : [0, T] \times \mathcal{X} \rightarrow \mathcal{K}$, such that $H(s, x, z) = H_0(s, x, z, \mathbf{v}_0(s, x))$ and the control process $\mathbf{v}_0 = (\mathbf{v}_0(s, X_s^*))_{0 \leq s \leq T}$ is admissible, i.e. $v^* \in \mathcal{V}$ and the wealth process is defined by the equation (3.9). Now, similarly to (7.4) we can obtain that

$$\mathbf{E}_{t,x} \left(\int_t^{\tau_n^*} U_1(s, X_s, v_s^*) ds + z_n^* \right) = z(t, x),$$

where τ_n^* is the stopping time defined in (7.3) for the strategy v^* and $z_n^* = z(\tau_n^*, X_{\tau_n^*}^*)$. Now, Condition \mathbf{H}_4) implies that the random variable $(z_n^*)_{n \geq 1}$ are uniformly integrable, and therefore, taking into account that a.s. $z_n^* \rightarrow z(T, X_T^*) = U_2(X_T^*)$ as $n \rightarrow \infty$ we can obtain through the monotone convergence theorem that

$$\begin{aligned} z(t, x) &= \lim_{n \rightarrow \infty} \mathbf{E}_{t,x} \left(\int_t^{\tau_n^*} U_1(s, X_s, v_s^*) ds + z_n^* \right) \\ &= \mathbf{E}_{t,x} \left(\int_t^T U_1(s, X_s, v_s^*) ds + U_2(X_T) \right) = J(t, x, v^*). \end{aligned}$$

Therefore, from (7.5) we can conclude that $J(t, x, v^*) = J^*(t, x)$. \square

7.2 Proof of Theorem 4.1

Proof. First note that if use the form (3.14) in (3.12) we obtain that the coefficient A satisfies the following ordinary differential equation

$$\dot{A}(t) + \gamma r_t A(t) + A(t) F^*(t) + (1 - \gamma) A^{\frac{\gamma}{\gamma-1}}(t) = 0, \quad A(T) = 1,$$

where the function $F^*(t) = \max_{\theta \in \Theta} F(t, \theta)$ and $F(t, \theta)$ is given in (3.15). One can find directly, that

$$A(t) = \left(\int_t^T \Psi_{t,u} du + \Psi_{t,T} \right)^{1-\gamma}, \quad (7.6)$$

where $\Psi_{t,u}$ is given in (3.16). Therefore, the Condition \mathbf{H}_1 holds true. Let's check Conditions $\mathbf{H}_2) - \mathbf{H}_4)$. As to Condition $\mathbf{H}_2)$ note that the optimal control function $\mathbf{v}_0 = (\theta_0, c_0)$ is given in (3.13). To check $\mathbf{H}_3)$ note that the inequality (5.12) implies that $X_{t-}^* > 0$ a.s. for all $0 \leq t \leq T$ i.e. the strategy $v^* = (\theta_t^*, c_t^*)_{0 \leq t \leq T}$ belongs to \mathcal{V} . Finally, the form (3.14) and the upper bound (5.11) provide Condition $\mathbf{H}_4)$ and, therefore, Theorem 3.3 implies this theorem. \square

7.3 Proof of Theorem 4.3

Proof. First note, that the process $(\zeta_t^*)_{0 \leq t \leq T}$ defined in (4.2) is càdlàg on the interval $[0, T]$ and through the definition (4.4) one can deduce that for any $\delta > 0$

$$\lim_{n \rightarrow \infty} \int_0^T (\zeta_u^{(n)})^\delta du = \int_0^T (\zeta_u^*)^\delta du = \int_0^T (c_u^* X_u^*)^\delta du \quad \text{a.s.} \quad (7.7)$$

Therefore, using the property (5.11) and the dominated convergence theorem we get

$$\lim_{n \rightarrow \infty} \mathbf{E}_x \int_0^T (\zeta_u^{(n)})^\gamma du = \mathbf{E}_x \int_0^T (c_u^* X_u^*)^\gamma du. \quad (7.8)$$

Now we have to show that

$$\mathbf{P} - \lim_{n \rightarrow \infty} X_T^{(n)} = X_T^*. \quad (7.9)$$

In view of the definition in (4.4) and taking into account that the processes $(\alpha_t^*)_{0 \leq t \leq T}$ and $(\beta_t^*)_{0 \leq t \leq T}$ have càdlàg trajectories on the interval $[0, T]$ we get that in probability

$$\lim_{n \rightarrow \infty} \int_0^T \alpha_u^{(n)} dS_u = \int_0^T \alpha_u^* dS_u \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_0^T \beta_u^{(n)} dB_u = \int_0^T \beta_u^* dB_u. \quad (7.10)$$

Then, in view of the convergence (7.7) for $\delta = 1$, to show (7.9) it is sufficient to prove that

$$\mathbf{P} - \lim_{n \rightarrow \infty} \mathbf{D}_n = 0, \quad (7.11)$$

where the total amount of transactions costs \mathbf{D}_n is defined in (4.5). To do this, using the definitions (4.2) and (4.4) and taking into account that $0 \leq \theta_j^*(t) \leq 1$, we obtain that for $k \geq 2$

$$\begin{aligned} \left| \alpha_j^{(n)}(t_k) - \alpha_j^{(n)}(t_{k-1}) \right| &\leq \left| (\theta_j^*(t_{k-1}) - \theta_j^*(t_{k-2})) \frac{X_{t_{k-1}}^*}{S_j(t_{k-1})} \right| \\ &+ \left| \frac{X_{t_{k-1}}^*}{S_j(t_{k-1})} - \frac{X_{t_{k-2}}^*}{S_j(t_{k-2})} \right|. \end{aligned} \quad (7.12)$$

Now, for $N > x + \max_{1 \leq j \leq m} S_j(0) := N_0$ we introduce the set

$$\mathbf{G}_N = \left\{ \sup_{0 \leq t \leq T} X_t^* \leq N \right\} \cap \left\{ S_* \geq N^{-1} \right\} \cap \left\{ S^* \leq N \right\}, \quad (7.13)$$

where

$$S_* = \min_{1 \leq j \leq m} \inf_{0 \leq t \leq T} S_j(t) \quad \text{and} \quad S^* = \max_{1 \leq j \leq m} \sup_{0 \leq t \leq T} S_j(t).$$

Using the representation (5.13) and taking into account, that the functions μ_j are bounded on the interval $[0, T]$, Propositions 5.1 and 5.2 imply that $0 < S_* \leq S^* < \infty$ a.s. Note also, that the property (5.11) implies $\sup_{0 \leq t \leq T} X_t^* < \infty$ a.s. Therefore, $\lim_{N \rightarrow \infty} \mathbf{P}(\mathbf{G}_N^c) = 0$.

Moreover, note also, that on this set through the Hölder condition (4.9) with some constant $C > 0$ we can rewrite the inequality (7.12) as

$$\begin{aligned} \left| \alpha_j^{(n)}(t_k) - \alpha_j^{(n)}(t_{k-1}) \right| &\leq X_{t_{k-1}}^* \frac{|\theta_j^*(t_{k-1}) - \theta_j^*(t_{k-2})|}{S_j(t_{k-1})} \\ &+ \frac{|X_{t_{k-1}}^* - X_{t_{k-2}}^*|}{S_j(t_{k-1})} + \frac{X_{t_{k-2}}^* |S_j(t_{k-2}) - S_j(t_{k-1})|}{S_j(t_{k-1}) S_j(t_{k-2})} \\ &\leq N^3 \left(|\theta_j^*(t_{k-1}) - \theta_j^*(t_{k-2})| + |X_{t_{k-1}}^* - X_{t_{k-2}}^*| + |S_j(t_{k-2}) - S_j(t_{k-1})| \right). \end{aligned}$$

Therefore, on the set \mathbf{G}_N , using the condition (4.8), we get, that for some $C > 0$

$$\begin{aligned} \mathbf{D}_n &\leq C N^4 m \kappa_n (1 + \mathcal{T}_n) + N^4 \kappa_n \sum_{k=2}^n |X_{t_{k-1}}^* - X_{t_{k-2}}^*| \\ &+ N^4 \kappa_n \sum_{j=1}^m \sum_{k=2}^n |S_j(t_{k-1}) - S_j(t_{k-2})|, \end{aligned} \quad (7.14)$$

where $\mathcal{T}_n = \sum_{k=1}^n \sqrt{t_k - t_{k-1}}$. Now note here, that using the boundedness of the functions r_t and μ_t we can obtain that on the set \mathbf{G}_N for some $C > 0$ and for any $0 < u < t \leq T$

$$(S_j(t) - S_j(u))^2 \leq C N (t - u)^2 + 2 \left(\int_u^t \check{S}_j(v-) d\tilde{L}_j(v) \right)^2$$

and

$$(X_t^* - X_u^*)^2 \leq C N (t - u)^2 + 2 \left(\int_u^t \check{X}_{v-}^*(\theta_v^*)' d\tilde{L}_v \right)^2,$$

where $\check{S}_t = \min(S_t, N)$ and $\check{X}_t^* = \min(X_t^*, N)$. Moreover, the Cauchy - Schwarz - Bunyakovsky inequality yields

$$\begin{aligned} \left(\int_u^t \check{S}_j(v-) d\tilde{L}_j(v) \right)^2 &\leq 2m \sum_{l=1}^m \left(\int_u^t \check{S}_j(v) \sigma_{jl}(v) dW_l(v) \right)^2 \\ &+ 2m \sum_{l=1}^m \left(\int_u^t \check{S}_j(v-) \varsigma_{jl}(v) dL_l(v) \right)^2 \end{aligned}$$

and

$$\begin{aligned} \left(\int_u^t \check{X}_{v-}^*(\theta_v^*)' d\tilde{L}_v \right)^2 &\leq 2m \sum_{j,l=1}^m \left(\int_u^t \check{X}_v^* \theta_j^*(v) \sigma_{jl}(v) dW_l(v) \right)^2 \\ &\quad + 2m \sum_{j,l=1}^m \left(\int_u^t \check{X}_{v-}^* \theta_j^*(v) \varsigma_{jl}(v) dL_l(v) \right)^2. \end{aligned}$$

Using here the boundedness of the functions σ_{ij} and ς_{ij} and the inequality (A.2) for $p = 2$, we obtain that

$$\mathbf{E} \left(\int_u^t \check{S}_j(v-) d\tilde{L}_j(v) \right)^2 \leq CN^2(t-u)$$

and

$$\mathbf{E} \left(\int_u^t \check{X}_{v-}^*(\theta_v^*)' d\tilde{L}_v \right)^2 \leq CN^2(t-u).$$

Therefore, setting

$$\eta_{jn} = \frac{\sum_{k=1}^n \left(\left| \int_{t_{k-1}}^{t_k} \check{S}_j(v-) d\tilde{L}_j(v) \right| + \left| \int_u^t \check{X}_{v-}^* \theta_j^*(v) \varsigma_{jl}(v) dL_l(v) \right| \right)}{\mathcal{T}_n},$$

we obtain that for some constant $C > 0$ the maximum $\eta_n^* = \max_{1 \leq j \leq m} \eta_{jn}$ can be estimated as

$$\sup_{k \geq 1} \mathbf{E} \eta_n^* \leq \sum_{j=1}^m \mathbf{E} \eta_{jn} \leq CN.$$

Therefore, from (7.14) it follows that on the set \mathbf{G}_N for some $C_N > 0$

$$\mathbf{D}_n \leq C_N \kappa_n (1 + \eta_n^*) (1 + \mathcal{T}_n). \quad (7.15)$$

From here we obtain that for any $\varepsilon > 0$,

$$\begin{aligned} \mathbf{P}(\mathbf{D}_n \geq \varepsilon) &\leq \mathbf{P}(C_N \kappa_n (1 + \eta_n^*) (1 + \mathcal{T}_n) \geq \varepsilon) + \mathbf{P}(\mathbf{G}_N^c) \\ &\leq \frac{C_N \kappa_n (1 + \mathcal{T}_n)}{\varepsilon} + \mathbf{P}(\mathbf{G}_N^c). \end{aligned} \quad (7.16)$$

Taking into account that $t_j = jT/n$ and, that $\kappa_n = o(n^{-1/2})$, we obtain, that

$$\kappa_n \mathcal{T}_n = \sqrt{T} \kappa_n \sqrt{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

i.e. we get that for any $N > 1$

$$\limsup_{n \rightarrow \infty} \mathbf{P}(\mathbf{D}_n \geq \varepsilon) \leq \mathbf{P}(\mathbf{G}_N^c).$$

Since $\mathbf{P}(\mathbf{G}_N^c) \rightarrow 0$ as $N \rightarrow \infty$, we get the limit property (7.11). Now, we show the uniform integrability of the sequence $\left((X_T^{(n)})_+^\gamma \right)_{n>0}$. To do this in view of the Vallee-Poussin criteria it suffices to check that

$$\sup_{n \geq 1} \mathbf{E}_x (X_T^{(n)})_+ < \infty. \quad (7.17)$$

Since the term $\int_0^T \zeta_u^{(n)} du + \mathbf{D}_n \geq 0$, we obtain that

$$\begin{aligned} \mathbf{E}_x (X_T^{(n)})_+ &\leq x + \mathbf{E} \left| \sum_{k=1}^n \beta_{t_{k-1}}^* (B_{t_k} - B_{t_{k-1}}) \right| \\ &\quad + \sum_{j=1}^m \mathbf{E} \left| \sum_{k=1}^n \alpha_j^*(t_{k-1}) (S_j(t_k) - S_j(t_{k-1})) \right|. \end{aligned}$$

Proposition 5.4 implies upper bound (7.17). Therefore,

$$\lim_{n \rightarrow \infty} \mathbf{E}_x (X_T^{(n)})_+^\gamma = \mathbf{E}_x (X_T^*)_+^\gamma = \mathbf{E}_x (X_T^*)^\gamma$$

and we obtain Theorem 4.3. \square

7.4 Proof of Theorem 4.4

First of all, note that the time moments (4.11) are such that $0 < t_1 < \dots < t_n = T$ and in view of the condition (4.12)

$$t_k - t_{k-1} = \mathbf{q}T \int_{u_{k-1}}^{u_k} x^{\mathbf{q}-1} dx \leq \frac{\mathbf{q}T}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, the convergences (7.7), (7.8) and (7.10) hold true and, moreover, Proposition 5.4 implies upper bound (7.17). Therefore, to end this proof it suffices to show the convergence (7.9) for which one has to establish the limit property (7.11). To do this, note that in this case

$$t_k - t_{k-1} = \mathbf{q}T \int_{u_{k-1}}^{u_k} x^{\mathbf{q}-1} dx \leq \mathbf{q}T u_k^{\mathbf{q}-1} (u_k - u_{k-1}) = \mathbf{q}T \frac{u_k^{\mathbf{q}-1}}{n}.$$

Therefore, the term \mathcal{T}_n in (7.14) can be estimated as

$$\kappa_n \mathcal{T}_n = \kappa_n \sum_{k=1}^n \sqrt{t_k - t_{k-1}} \leq \sqrt{\mathbf{q}T} \kappa_n \sum_{k=1}^{n-1} \frac{u_k^{(\mathbf{q}-1)/2}}{\sqrt{n}} + \sqrt{T} \kappa_n.$$

Moreover, the first term

$$\begin{aligned} \sqrt{\mathbf{q}T} \kappa_n \sum_{k=1}^{n-1} \frac{u_k^{(\mathbf{q}-1)/2}}{\sqrt{n}} &= \sqrt{\mathbf{q}T} \kappa_n \sum_{k=1}^{n-1} \int_{u_k}^{u_{k+1}} u_k^{(\mathbf{q}-1)/2} dx \\ &\leq \sqrt{\mathbf{q}T} \kappa_n \sum_{k=1}^{n-1} \int_{u_k}^{u_{k+1}} x^{(\mathbf{q}-1)/2} dx \\ &\leq \sqrt{\mathbf{q}T} \kappa_n \int_{u_1}^1 x^{(\mathbf{q}-1)/2} dx \leq 2\sqrt{T} \kappa_n \frac{\sqrt{n} \sqrt{\mathbf{q}}}{\mathbf{q} + 1}. \end{aligned}$$

Therefore, in view of the last condition in (4.12) $\kappa_n \mathcal{T}_n \rightarrow 0$ as $n \rightarrow \infty$. Using this in the bound (7.16) we obtain (7.11). Hence Theorem 4.4.

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A Appendix

A.1 Properties of the mapping Υ .

Lemma A.1. *Let $\mathcal{X} \subset \mathbb{R}$ be an open convex set and g is some fixed arbitrary function from $C^{1,2}([0, T] \times \mathcal{X}, \mathbb{R})$, for which $\sup_{0 \leq t \leq T} \sup_{x \in \mathcal{X}} |g(t, x)| / (1 + |x|) < \infty$. Then for any $x_* \leq x^*$ from \mathcal{X} there exists a constant $C > 0$ such that for all $\varpi \in \mathbb{R}$*

$$\max_{1 \leq i \leq m} \sup_{0 \leq t \leq T} \sup_{x_* \leq x \leq x^*} \int_{\mathbb{R}_*} |\Upsilon_g(t, x, \varpi y)| \Pi_i(dy) \leq C(1 + \varpi^2). \quad (\text{A.1})$$

Proof. First note that, for $\varpi = 0$ the function $\Upsilon_g(t, x, 0) \equiv 0$ and, therefore, (A.1) holds. Let $\varpi \neq 0$. In this case first note, that there exists some $\epsilon > 0$ such that the interval $]x_* - 2\epsilon, x^* + 2\epsilon[\subset \mathcal{X}$. Therefore, $x + v \in \mathcal{X}$ for any $|v| < \epsilon$ and $x \in [x_*, x^*]$. We set $\epsilon_1 = \epsilon/|\varpi|$. It is clear, that $x + \varpi y \in \mathcal{X}$ for $|y| < \epsilon_1$ and

$$|\Upsilon(t, x, \varpi y)| = \left| \int_x^{x+\varpi y} \left(\int_x^z g_{xx}(t, u) du \right) dz \right| \leq \varpi^2 \frac{y^2}{2} \max_{x-2\epsilon \leq u \leq x+2\epsilon_0} |g_{xx}(t, u)|.$$

Therefore, since $\Pi_i(y^2) < \infty$ we get for some $C > 0$

$$\int_{-\epsilon_1}^{\epsilon_1} |\Upsilon(t, x, \varpi y)| \Pi_i(dy) \leq C\varpi^2 \Pi_i(y^2).$$

Moreover, using the condition of this Lemma we can deduce that for some $C > 0$

$$\begin{aligned} \int_{\{|y| \geq \epsilon_1\}} |\Upsilon(t, x, \varpi y)| \Pi_i(dy) &\leq C \int_{\{|y| \geq \epsilon_1\}} (1 + |\varpi y|) \Pi_i(dy) + |g(t, x)| \Pi_i(\{|y| \geq \epsilon_1\}) \\ &+ |g_x(t, x)| |\varpi| \int_{\{|y| \geq \epsilon_1\}} |y| \Pi_i(dy) \\ &\leq C(1 + |\varpi|)(1 + |g(t, x)| + |g_x(t, x)|) \left(\int_{\{|y| \geq \epsilon_1\}} (1 + |y|) \Pi_i(dy) \right). \end{aligned}$$

Therefore, taking into account that

$$\max_{0 \leq t \leq T} \max_{x-2\epsilon \leq u \leq x+2\epsilon_0} (|g(t, u)| + (|g_x(t, u)| + |g_{xx}(t, u)|)) < \infty,$$

we obtain the upper bound (A.1). Hence Lemma A.1. \square

A.2 Technical tools

First, we recall the well-known Novikov inequality for purely discontinuous martingales. To end this we denote by \mathcal{P} the predictable field and by $\mathcal{B}(\mathbb{R}_*)$ the Borel field in \mathbb{R}_* .

Lemma A.2. ([22]) *Let J be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_*)$ a measurable function for which $|J|^2 * \tilde{\nu}_t < \infty$ a.s. Then for any $p \geq 2$ there exists some positive constant $\check{C}_p > 0$ such that for any $t > 0$*

$$\mathbf{E} \sup_{u \leq t} |J * (\nu - \tilde{\nu})_u|^p \leq \check{C}_p \left(\mathbf{E} (|J|^2 * \tilde{\nu}_t)^{p/2} + \mathbf{E} |J|^p * \tilde{\nu}_t \right), \quad (\text{A.2})$$

where ν is a random jump measure with its compensator $\tilde{\nu}$.

Lemma A.3 (Martingales inequalities). (See subsections 3.2 in [18]) *If the process $(X_t)_{t \geq 0}$ is a non negative sub martingale with $\mathbf{E} X_t^p < \infty$ for some $p > 1$, then*

$$\mathbf{P}(X_t^* > \lambda) \leq \frac{\mathbf{E} X_t^p}{\lambda^p} \quad \text{and} \quad \mathbf{E} \left(\sup_{0 \leq u \leq t} X_u \right)^p \leq \left(\frac{p}{p-1} \right)^p \mathbf{E} X_t^p. \quad (\text{A.3})$$

A.3 Proof of Proposition 4.5

Since the functions r_t, μ_t, σ_t and ς_t in the model (2.1) are continuously differentiable, they are bounded. Hence \mathbf{A}_1 . For $m = 1$, the function (3.15) has the following form

$$F(t, \theta) = f(t, \theta) + \mathcal{I}(\varsigma_t \theta), \quad f(t, \theta) = \mathbf{a}_1(t) \theta + \mathbf{a}_2(t) \theta^2, \quad (\text{A.4})$$

where $\mathbf{a}_1(t) = \gamma(\mu_t - r_t)$, $\mathbf{a}_2(t) = \gamma(\gamma - 1)\sigma_t^2/2 \leq 0$ and

$$\mathcal{I}(x) = \int_{-1}^{\infty} ((1 + xy)^\gamma - 1 - \gamma xy) \Pi(dy).$$

Note, that generally this function is defined only for $0 \leq x \leq 1$. First we assume that

$$\int_{-1}^{-1/2} \frac{1}{(1+y)^{2-\gamma}} \Pi(dy) < \infty. \quad (\text{A.5})$$

Then, taking into account here, that $\Pi(y^2) < \infty$, we can show through the dominated convergence theorem, that the function \mathcal{I} is two times continuously differentiable and for $0 \leq x \leq 1$

$$\dot{\mathcal{I}}(x) = \gamma \int_{-1}^{\infty} \left(\frac{y}{(1+xy)^{1-\gamma}} - y \right) \Pi(dy) \quad \text{and} \quad \ddot{\mathcal{I}}(x) = \int_{-1}^{\infty} \frac{\gamma(\gamma-1)y^2}{(1+xy)^{2-\gamma}} \Pi(dy),$$

where $\dot{\mathcal{I}}$ and $\ddot{\mathcal{I}}$ denote the derivatives of the first and second order respectively. Now, we extend the function \mathcal{I} on \mathbb{R} , as follows

$$\mathcal{I}_1(x) = \begin{cases} \ddot{\mathcal{I}}(0)x^2/2 & \text{for } x \leq 0; \\ \mathcal{I}(x) & \text{for } 0 \leq x \leq 1; \\ \mathcal{I}(1) + \dot{\mathcal{I}}(1)(x-1) + \ddot{\mathcal{I}}(1)(x-1)^2/2 & \text{for } x \geq 1. \end{cases} \quad (\text{A.6})$$

It is clear, that this function is two times continuously differentiable and $\ddot{I}_1(x) < 0$ for any $x \in \mathbb{R}$. Note now, that

$$\max_{0 \leq \theta \leq 1} F(t, \theta) = \max_{0 \leq \theta \leq 1} F_1(t, \theta) \quad \text{and} \quad F_1(t, \theta) = f(t, \theta) + \mathcal{I}_1(\varsigma_t \theta). \quad (\text{A.7})$$

We can calculate directly

$$U(t, \theta) = \frac{\partial F_1(t, \theta)}{\partial \theta} = \mathbf{a}_1(t) + 2\mathbf{a}_2(t)\theta + \varsigma_t \dot{I}_1(\theta \varsigma_t)$$

and, moreover, since $\varsigma_t > 0$, then

$$\frac{\partial U(t, \theta)}{\partial \theta} = \frac{\partial^2 F_1(t, \theta)}{\partial \theta^2} = 2\mathbf{a}_2(t) + \varsigma_t^2 \ddot{I}_1(\theta \varsigma_t) < 0.$$

Additionally, it is easy to check that $U(t, \theta) \rightarrow +\infty$ as $\theta \rightarrow -\infty$ and $U(t, \theta) \rightarrow -\infty$ as $\theta \rightarrow +\infty$, therefore, for all $0 \leq t \leq T$ there exist unique root $\theta_0(t)$ of the equation $U(t, \theta_0(t)) = 0$ and by the implicit function theorem we calculate its derivative

$$\dot{\theta}_0(t) = \frac{\dot{\mathbf{a}}_1(t) + 2\dot{\mathbf{a}}_2(t)\theta_0(t) + \dot{\varsigma}_t \dot{I}_1(\theta_0(t)\varsigma_t) + \theta_0(t)\varsigma_t \dot{\varsigma}_t \ddot{I}_1(\theta_0(t)\varsigma_t)}{2|\mathbf{a}_2(t)| + \varsigma_t^2 |\ddot{I}_1(\theta_0(t)\varsigma_t)|}. \quad (\text{A.8})$$

This derivative is continuous on the interval $[0, T]$, i.e. bounded. The solution of the problem (A.7) can be represented as $\theta_t^* = V(\theta_0(t))$ and $V(x) = \min(0, \max(x, 1))$ is Lipschitz function, i.e.

$$\sup_{x, y \in \mathbb{R}} \frac{|V(x) - V(y)|}{|x - y|} \leq 1.$$

Therefore, this implies the condition \mathbf{A}_2). Consider now the case, when the condition (A.5) does not hold, i.e.

$$\int_{-1}^{-1/2} \frac{1}{(1+y)^{2-\gamma}} \Pi(dy) = +\infty.$$

Let now $0 < \delta < 1/2$. Then we set

$$F_\delta(t, \theta) = f(t, \theta) + \mathcal{I}_\delta(\varsigma_t \theta) \quad \text{and} \quad \mathcal{I}_\delta(x) = \int_{-1+\delta}^{\infty} ((1+xy)^\gamma - 1 - \gamma xy) \Pi(dy).$$

Similarly, we get, that \mathcal{I}_δ is two times continuously differentiable and for $0 \leq x \leq 1$

$$\dot{\mathcal{I}}_\delta(x) = \gamma \int_{-1+\delta}^{\infty} \left(\frac{y}{(1+xy)^{1-\gamma}} - y \right) \Pi(dy) \quad \text{and} \quad \ddot{\mathcal{I}}_\delta(x) = \int_{-1+\delta}^{\infty} \frac{\gamma(\gamma-1)y^2}{(1+xy)^{2-\gamma}} \Pi(dy).$$

Let now $\theta_\delta^*(t)$ be the argmax of $F_\delta(t, \theta)$, i.e. $\max_{0 \leq \theta \leq 1} F_\delta(t, \theta) = F_\delta(t, \theta_\delta^*(t))$. Taking into account here, that the function F_δ goes to F as $\delta \rightarrow 0$ uniformly over $[0, T] \times [0, 1]$, we get that $\theta_\delta^*(t) \rightarrow \theta_t^*$ for any $t \in [0, T]$. Moreover, we introduce the function $\mathcal{I}_{1,\delta}$ and $F_{1,\delta}$ by replacing in (A.6) and (A.7) the function \mathcal{I} with \mathcal{I}_δ . Therefore, similarly to the precedent case, the function $\theta_{0,\delta}^*(t) = V(\theta_{0,\delta}(t))$, where the function $\theta_{0,\delta}(t)$ is the root of the partial derivative of $F_{1,\delta}$, i.e.

$$U_\delta(t, \theta_{0,\delta}(t)) = 0 \quad \text{and} \quad U_\delta(t, \theta) = \mathbf{a}_1(t) + \mathbf{a}_2(t)\theta + \varsigma_t \dot{\mathcal{I}}_{1,\delta}(\theta \varsigma_t). \quad (\text{A.9})$$

Note here, that for any $0 \leq x \leq 1$ and for some positive constant C

$$|\dot{I}_\delta(x)| \leq \int_{-1+\delta}^{-1/2} \frac{1}{(1+xy)^{1-\gamma}} \Pi(dy) + C \leq \iota_\delta(x) + C,$$

where

$$\iota_\delta(x) = \int_{-1+\delta}^{-1/2} (1+xy)^{-2+\gamma} \Pi(dy) \geq \Pi(-1+\delta, -1/2) \rightarrow \Pi(-1, -1/2) > 0$$

as $\delta \rightarrow 0$. Note also, that $|\ddot{I}_\delta(x)| > \gamma(1-\gamma)\iota_\delta(x)/4$. Therefore, we get, that for any $\mathbf{x}_* \geq 1$

$$\limsup_{\delta \rightarrow 0} \sup_{|x| \leq \mathbf{x}_*} \frac{|\dot{I}_{1,\delta}(x)|}{|\ddot{I}_{1,\delta}(x)|} < \infty. \quad (\text{A.10})$$

Note now, that from (A.9) for $\theta_{0,\delta}(t)_{\varsigma_t} > 1$ we get, that

$$U_\delta(t, \theta_{0,\delta}(t)) = \mathbf{a}_1(t) + \mathbf{a}_2(t) \theta_{0,\delta}(t) + \varsigma_t \dot{I}_\delta(1) + \ddot{I}_\delta(1)(\varsigma_t^2 \theta_{0,\delta}(t) - \varsigma_t) = 0,$$

i.e.

$$\theta_{0,\delta}(t) = \frac{\mathbf{a}_1(t) + \varsigma_t \dot{I}_\delta(1) - \ddot{I}_\delta(1) \varsigma_t}{|\mathbf{a}_2(t)| + \varsigma_t^2 |\ddot{I}_\delta(1)|}$$

Therefore, taking into account that $\min_{0 \leq t \leq T} \varsigma_t^2 > 0$ and $|\ddot{I}_\delta(1)| \rightarrow +\infty$ as $\delta \rightarrow 0$, we get from (A.10), that $\limsup_{\delta \rightarrow 0} \max_{0 \leq t \leq T} |\theta_{0,\delta}(t)| \mathbf{1}_{\{\theta_{0,\delta}(t)_{\varsigma_t} > 1\}} < \infty$. Moreover, again from (A.9) for $\theta_{0,\delta}(t)_{\varsigma_t} < 0$ we get, that

$$U_\delta(t, \theta_{0,\delta}(t)) = \mathbf{a}_1(t) + \mathbf{a}_2(t) \theta_{0,\delta}(t) + \ddot{I}_\delta(0) \varsigma_t^2 \theta_{0,\delta}(t) = 0,$$

i.e.

$$\theta_{0,\delta}(t) = \frac{\mathbf{a}_1(t)}{|\mathbf{a}_2(t)| + \varsigma_t^2 |\ddot{I}_\delta(0)|}.$$

Taking into account, that $|\ddot{I}_\delta(0)| \rightarrow \gamma(1-\gamma)\Pi(y^2) > 0$ as $\delta \rightarrow 0$, we get, that

$$\limsup_{\delta \rightarrow 0} \max_{0 \leq t \leq T} |\theta_{0,\delta}(t)| \mathbf{1}_{\{\theta_{0,\delta}(t)_{\varsigma_t} < 0\}} < \infty.$$

Finally, we get, that $\limsup_{\delta \rightarrow 0} \max_{0 \leq t \leq T} |\theta_{0,\delta}(t)| < \infty$. Using this bound together with (A.10) in (A.8) we obtain the same property for its derivative, i.e.

$$\limsup_{\delta \rightarrow 0} \max_{0 \leq t \leq T} |\dot{\theta}_{0,\delta}(t)| < \infty.$$

This implies, that $\theta_\delta^*(t)$ satisfies the Lipschitz condition uniformly over δ , i.e.

$$\mathbf{M}^* = \limsup_{\delta \rightarrow 0} \sup_{0 \leq s, t \leq T} \frac{|\theta_\delta^*(t) - \theta_\delta^*(s)|}{|t - s|} < \infty.$$

Therefore, for any $0 \leq s, t \leq T$ we deduce that

$$\frac{|\theta_t^* - \theta_s^*|}{|t - s|} = \lim_{\delta \rightarrow 0} \frac{|\theta_\delta^*(t) - \theta_\delta^*(s)|}{|t - s|} \leq \mathbf{M}^*.$$

This implies the condition **A₂**). Hence Proposition 4.5.

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