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# Unrestricted Quantum Moduli Algebras, II: Noetherianity and Simple Fraction Rings at Roots of 1

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**Abstract.** We prove that the quantum graph algebra and the quantum moduli algebra associated to a punctured sphere and complex semisimple Lie algebra  $\mathfrak{g}$  are Noetherian rings and finitely generated rings over  $\mathbb{C}(q)$ . Moreover, we show that these two properties still hold on  $\mathbb{C}[q, q^{-1}]$  for the integral version of the quantum graph algebra. We also study the specializations  $\mathcal{L}_{0,n}^\epsilon$  of the quantum graph algebra at a root of unity  $\epsilon$  of odd order, and show that  $\mathcal{L}_{0,n}^\epsilon$  and its invariant algebra under the quantum group  $U_\epsilon(\mathfrak{g})$  have classical fraction algebras which are central simple algebras of PI degrees that we compute.

*Key words:* quantum groups; invariant theory; character varieties; skein algebras; TQFT

*2020 Mathematics Subject Classification:* 16R30; 17B37; 20G42; 57M27; 57R56; 81R50

## 1 Introduction

This paper is the second part of our work, initiated in [18], on the quantum graph algebra  $\mathcal{L}_{g,n}(\mathfrak{g})$  and the quantum moduli algebra  $\mathcal{M}_{g,n}(\mathfrak{g})$ , which are associated to a surface  $\Sigma_{g,n+1}$  of genus  $g$  with  $n+1$  punctures and a complex semisimple Lie algebra  $\mathfrak{g}$ . As in [18], we focus in this paper on punctured spheres ( $g=0$ ,  $n \geq 1$ ). From now on we fix  $\mathfrak{g}$ , and when no confusion may arise we omit it from the notations of the various algebras.

The algebras  $\mathcal{L}_{g,n}$  and  $\mathcal{M}_{g,n}$  are defined over the field  $\mathbb{C}(q)$ . They were introduced in the mid 90's by Alekseev–Grosse–Schomerus [2, 3] and Buffenoir–Roche [29, 30] by a method called *combinatorial quantization*. By this method, the pair formed by  $\mathcal{L}_{g,n}$  and  $\mathcal{M}_{g,n}$  appear naturally as a  $q$ -deformation of the Fock–Rosly [55] lattice model of the algebra of functions on the “classical” moduli space  $\mathcal{M}_{g,n}^{cl}$  of flat  $\mathfrak{g}$ -connections on the surface  $\Sigma_{g,n+1}$ .

In [18], we showed that both  $\mathcal{L}_{0,n}$  and  $\mathcal{M}_{0,n}$  have integral forms  $\mathcal{L}_{0,n}^A$  and  $\mathcal{M}_{0,n}^A$  defined over the ring  $A = \mathbb{C}[q, q^{-1}]$  (in fact we could have taken  $\mathbb{Q}[q, q^{-1}]$  or  $\mathbb{Z}[q, q^{-1}]$  as ground ring, see Section 1.1). One can thus consider the specializations of these algebras at  $q = \epsilon \in \mathbb{C}^\times$ , which we denote by  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{M}_{0,n}^{A,\epsilon}$  respectively. The algebra  $\mathcal{L}_{0,n}^A$  is endowed with an action of the Lusztig integral form  $U_A^{\text{res}} = U_A^{\text{res}}(\mathfrak{g})$  of the quantum group  $U_q = U_q(\mathfrak{g})$ , and  $\mathcal{M}_{0,n}^A$  is the subalgebra of invariant elements under this action. Therefore,

$$\mathcal{M}_{0,n}^A := (\mathcal{L}_{0,n}^A)^{U_A^{\text{res}}}, \quad \mathcal{M}_{0,n} := \mathcal{L}_{0,n}^{U_q} = \mathcal{M}_{0,n}^A \bigotimes_A \mathbb{C}(q).$$

The definition of  $\mathcal{L}_{0,n}^A$  is based on the original combinatorial quantization method, together with twists of module-algebras and Lusztig’s theory of canonical bases of quantum groups [83]. This allows us to address the structure and representation theory of  $\mathcal{L}_{0,n}^A$  and  $\mathcal{M}_{0,n}^A$  by means of quantum groups, following ideas of classical invariant theory. In particular, we obtained that  $\mathcal{L}_{0,n}$  and  $\mathcal{L}_{0,n}^\epsilon$  have no nontrivial zero divisors (and therefore do as well the subalgebras  $\mathcal{M}_{0,n}$ ,  $\mathcal{L}_{0,n}^A$ ,  $\mathcal{M}_{0,n}^A$ , and  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon^{\text{res}}}$ , where  $U_\epsilon^{\text{res}}$  is the specialization of  $U_A^{\text{res}}$  at  $q = \epsilon$ ).

Also, by extending the quantum coadjoint action of De Concini–Kac–Procesi [39, 40, 42], we described in the  $\mathfrak{sl}_2$  case an action by derivations of the center  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  of  $\mathcal{L}_{0,n}^\epsilon$  on  $\mathcal{L}_{0,n}^\epsilon$ , and we defined a subalgebra  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}} \subset \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ , which is a finite extension of the ring of regular functions on the character variety of the sphere with  $(n+1)$  punctures (see [18, Corollary 7.20 and Theorem 8.8]). Moreover, from these results we derived an action by derivation of  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}}$  on  $\mathcal{M}_{0,n}^{A,\epsilon}(\mathfrak{sl}_2)$ .

Representations of a quotient (the semisimplification) of  $\mathcal{M}_{g,n}^{A,\epsilon}$  were already constructed and classified in [4]; they involve only the irreducible representations of the finite-dimensional “small” quantum group  $u_\epsilon(\mathfrak{g})$ . Moreover, [4] deduced from these representations of  $\mathcal{M}_{g,n}^{A,\epsilon}$  a family of representations of the mapping class groups of surfaces, that is equivalent to the one associated to the Witten–Reshetikhin–Turaev TQFT [95, 106]. Recently, representations of another, larger quotient of  $\mathcal{M}_{g,n}^{A,\epsilon}$ , and the corresponding representations of the mapping class groups of surfaces, were constructed in [52, 53]. These representations are equivalent to those previously obtained by Lyubashenko–Majid [85], and are associated to the TQFT defined in [44, 45]. In the  $\mathfrak{sl}_2$  case, they involve the irreducible and also the principal indecomposable representations of the small quantum group  $u_\epsilon(\mathfrak{sl}_2)$ . The related link and 3-manifold invariants coincide with those of [21, 90].

In general, the representation theory of  $\mathcal{M}_{g,n}^{A,\epsilon}$  is by now far from being understood. Because  $\mathcal{M}_{g,n}^{A,\epsilon}$  deforms the classical moduli space  $\mathcal{M}_{g,n}^{cl}$ , it is natural to expect that its representation theory provides  $(2+1)$ -dimensional TQFTs for 3-manifolds endowed with general flat  $\mathfrak{g}$ -connections, extending the known TQFTs based on quantum groups (where purely topological ones correspond to the trivial connection). A family of such invariants, called *quantum hyperbolic invariants*, has already been defined for  $\mathfrak{g} = \mathfrak{sl}_2$  by means of certain  $6j$ -symbols, *Deus ex machina*; they are closely connected to classical Chern–Simons theory, provide generalized volume conjectures, and contain quantum Teichmüller theory (see [9, 10, 11, 12, 13, 14, 15]). It is part of our present program, initiated in [8], to shed light on these invariants and to generalize them to arbitrary  $\mathfrak{g}$  by developing the representation theory of  $\mathcal{M}_{g,n}^{A,\epsilon}$ .

The quantum moduli algebras have also been recognized as central objects from the viewpoints of factorization homology [22], multiplicative quiver varieties [58] and (stated) skein theory [16, 33, 36, 54]. In another direction, one may expect that the equivalence proved in [89] between combinatorial quantisation for the Drinfeld double  $D(H)$  of a finite-dimensional semisimple Hopf algebra  $H$ , and Kitaev’s lattice model in topological quantum computation, can be extended to the setup of quantum moduli algebras.

In the present paper, we study  $\mathcal{L}_{0,n}$ , its integral form  $\mathcal{L}_{0,n}^A$ , and the specialization  $\mathcal{L}_{0,n}^\epsilon$  of  $\mathcal{L}_{0,n}^A$  at  $q = \epsilon$  a primitive root of unity of odd order. We study also the subalgebras of invariant elements  $\mathcal{M}_{0,n} = \mathcal{L}_{0,n}^{U_q}$  and  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ . Here,  $U_\epsilon$  is the specialization of  $U_A$  at  $q = \epsilon$ , where  $U_A$  is the De Concini–Kac integral form of  $U_q$  (see Section 1.1). Our results hold for every complex semisimple Lie algebra  $\mathfrak{g}$ . The main ones are proofs that  $\mathcal{L}_{0,n}$ ,  $\mathcal{L}_{0,n}^A$  and  $\mathcal{M}_{0,n}$  are Noetherian and finitely generated rings (see Theorem 1.1), and that the classical fraction algebras of  $\mathcal{L}_{0,n}^\epsilon$  and  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  are central simple algebras of PI degrees  $l^{mN}$  and  $l^{N(n-1)-m}$  respectively (see Theorem 1.3). Here,  $m$  and  $N$  are the rank and the number of positive roots of  $\mathfrak{g}$ .

In the sequel [16] to this paper, in collaboration with M. Faitg, we extend Theorem 1.1 to the algebras  $\mathcal{L}_{g,n}$  and  $\mathcal{M}_{g,n}$ , associated to arbitrary finite type surfaces (arbitrary genus and number of punctures). Also, we show that  $\mathcal{M}_{g,n}$  is isomorphic to the  $\mathfrak{g}$ -skein algebra of  $\Sigma_{g,n+1}$ , and  $\mathcal{L}_{g,n}$  to the stated skein algebra of the compact surface  $\bar{\Sigma}_{g,n+1}$  with one boundary component and one marked point on the boundary component. This was proved for  $\mathfrak{g} = \mathfrak{sl}_2$  in [54]. In this specific case  $\mathfrak{g} = \mathfrak{sl}_2$ , the fact that the stated skein algebra of any finite type surface is Noetherian and finitely generated was proved in [80]. Still in the  $\mathfrak{sl}_2$  case, for related results, e.g., on non-zero divisors and computation of PI degrees, see [23, 24, 57, 64, 73, 74, 75, 78]. For recent results on  $\mathfrak{g} = \mathfrak{sl}_n$ , see [79, 105].

By using the analysis developed in the present paper for  $\mathcal{L}_{0,n}^A$ , one can define the integral form  $\mathcal{L}_{g,n}^A$  as well, and show that it is a Noetherian and finitely generated ring. We do not have a proof yet of these properties for the algebra  $\mathcal{M}_{0,n}^A$ , which seems to be much more difficult to handle. We note that there is a strict inclusion  $\mathcal{M}_{0,n}^{A,\epsilon} \subset (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ . This is discussed after Theorem 1.2. In [17], we study further properties of  $(\mathcal{L}_{g,n}^\epsilon)^{U_\epsilon}$ , and we consider also the subalgebra  $\mathcal{M}_{g,n}^{A,\epsilon}$ .

## 1.1 Statement of results

Let us recall a few notations and facts from [18]. Let  $U_q$  be the simply-connected quantum group of  $\mathfrak{g}$ , defined over the field  $\mathbb{C}(q)$ . From  $U_q$  one can define a  $U_q$ -module algebra  $\mathcal{L}_{0,n}$ , called (quantum, daisy) *graph algebra*, where  $U_q$  acts by means of a right coadjoint action. The set of invariant elements of  $\mathcal{L}_{0,n}$  for this action is an algebra; we denote it  $\mathcal{M}_{0,n} := \mathcal{L}_{0,n}^{U_q}$  and call it *quantum moduli algebra*. As a  $\mathbb{C}(q)$ -module  $\mathcal{L}_{0,n}$  is just  $\mathcal{O}_q^{\otimes n}$ , where  $\mathcal{O}_q = \mathcal{O}_q(G)$  is the standard quantum function algebra of the connected and simply-connected Lie group  $G$  with Lie algebra  $\mathfrak{g}$ . The product of  $\mathcal{L}_{0,n}$  is obtained by twisting both the product of each factor  $\mathcal{O}_q$  and the product between them. It is equivariant with respect to a (right) coadjoint action of  $U_q$ , which defines the structure of  $U_q$ -module of  $\mathcal{L}_{0,n}$ .

The module algebra  $\mathcal{L}_{0,n}$  has an integral form  $\mathcal{L}_{0,n}^A$ , which is defined over  $A = \mathbb{C}[q, q^{-1}]$ , and endowed with an (coadjoint) action of the Lusztig [82] integral form  $U_A^{\text{res}}$  of  $U_q$ . It is obtained by replacing  $\mathcal{O}_q$  in the construction of  $\mathcal{L}_{0,n}$  with the restricted dual  $\mathcal{O}_A$  of the integral form  $U_A^{\text{res}}$ , or equivalently with the restricted dual of the integral form  $\Gamma$  of  $U_q$  defined by De Concini–Lyubashenko [41]. Since  $U_A^{\text{res}}$  contains the De Concini–Kac [39] integral form  $U_A$ , and  $U_A$  has the same set of invariant elements in  $\mathcal{L}_{0,n}^A$ , we systematically denote the latter

$$\mathcal{M}_{0,n}^A := (\mathcal{L}_{0,n}^A)^{U_A} \quad (= (\mathcal{L}_{0,n}^A)^{U_A^{\text{res}}}).$$

We call  $\mathcal{M}_{0,n}^A$  the *integral quantum moduli algebra*.

A cornerstone of the theory of  $\mathcal{M}_{0,n}$  is a map  $\Phi_n$  originally due to Alekseev [1], building on works of Drinfeld [48] and Reshetikhin and Semenov-Tian-Shansky [94]. In [18], we showed that  $\Phi_n$  eventually provides isomorphisms of module algebras and algebras respectively,

$$\Phi_n: \mathcal{L}_{0,n} \rightarrow (U_q^{\otimes n})^{\text{lf}}, \quad \Phi_n: \mathcal{M}_{0,n} \rightarrow (U_q^{\otimes n})^{U_q},$$

where  $U_q^{\otimes n}$  is endowed with a right adjoint action of  $U_q$ , and  $(U_q^{\otimes n})^{\text{lf}}$  is the subalgebra of locally finite elements with respect to this action. When  $n = 1$  the algebra  $U_q^{\text{lf}}$  has been studied in great detail by Joseph–Letzter [61, 62, 63]; we will use simplified proofs of their results, obtained in [104].

All the material we need about the results discussed above is described in [18], and recalled in Sections 2.1 and 2.2.

Our first result, proved in Section 3, is the following.

**Theorem 1.1.**  *$\mathcal{L}_{0,n}$ ,  $\mathcal{M}_{0,n}$  and the integral form  $\mathcal{L}_{0,n}^A$  are Noetherian rings, and finitely generated rings.*

It follows immediately from the theorem that the specializations  $\mathcal{L}_{0,n}^\epsilon$ ,  $\epsilon \in \mathbb{C}^\times$ , are Noetherian and finitely generated rings as well. In [18] we proved that all these algebras (and therefore  $\mathcal{M}_{0,n}^A$  and  $\mathcal{M}_{0,n}^{A,\epsilon}$ ) have no nontrivial zero divisors.

Because the construction of the integral form  $\mathcal{L}_{0,n}^A$  is based on the Kashiwara–Lusztig theory of canonical bases, we could have defined  $\mathcal{L}_{0,n}^A$  over the ground ring  $\mathbb{Z}[q, q^{-1}]$ , and Theorem 1.1 for  $\mathcal{L}_{0,n}^A$  holds true as well in this generality. Since we are mainly interested in the representation theory of the specializations  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{M}_{0,n}^{A,\epsilon}$ , which will be addressed in [17], the choice

of  $A = \mathbb{C}[q, q^{-1}]$  is natural. Note however that the proof of Proposition 2.18 uses that  $\mathbb{C}[q, q^{-1}]$  is a PID.

We describe the background material on canonical bases in Section 2.2.2; we have tried to make the exposition pedestrian and self-contained, so as to be more accessible to non experts.

After we finished this work, we discovered that [47] already proved that  $\mathcal{L}_{0,1}(\mathfrak{gl}(n))$  and  $\mathcal{L}_{0,n}(\mathfrak{gl}(2))$  are Noetherian and finitely generated rings. Our approach here is completely different. For  $\mathcal{L}_{0,n}$ , we adapt the proof given by Voigt–Yuncken [104] of a result of Joseph [61], which asserts that  $U_q^{\text{lf}}$  is a Noetherian ring (see Theorem 3.1). For  $\mathcal{M}_{0,n}$ , we deduce the result from the one for  $\mathcal{L}_{0,n}$ , by following a line of proof of the Hilbert–Nagata theorem in classical invariant theory (see Theorem 3.4).

At present, we do not have a proof that  $\mathcal{M}_{0,n}^A$  is a Noetherian and finitely generated ring for arbitrary  $\mathfrak{g}$  and  $n \geq 1$ , though it is natural to expect it is the case. Indeed, when  $\mathfrak{g} = \mathfrak{sl}_2$ ,  $\mathcal{M}_{0,n}^A(\mathfrak{sl}_2)$  is isomorphic to the skein algebra of a sphere with  $n + 1$  punctures (see [18, Theorem 8.6]), which is finitely generated and Noetherian by results of [32] and [93]. In our general situation, key arguments in the proof of Theorem 1.1 for  $\mathcal{M}_{0,n}$  depend on the existence of a Reynolds operator on the  $U_q$ -module  $\mathcal{L}_{0,n}$ , and one can easily show there is no Reynolds operator on  $\mathcal{L}_{0,n}^A$ . This follows from the corresponding fact for the integral quantum coordinate ring  $\mathcal{O}_A$  (see Remark 2.19).

From Section 4, we consider the specializations  $\mathcal{L}_{0,n}^\epsilon$  of  $\mathcal{L}_{0,n}^A$  at  $q = \epsilon$ , a primitive root of unity of odd order  $l$  (and coprime to 3 if  $\mathfrak{g}$  has  $G_2$  components). In [41], De Concini–Lyubashenko introduced a central subalgebra  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  of  $\mathcal{O}_\epsilon$  isomorphic to the coordinate ring  $\mathcal{O}(G)$ , and proved that the  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ -module  $\mathcal{O}_\epsilon$  is projective of rank  $l^{\dim \mathfrak{g}}$ . As observed by Brown–Gordon–Stafford [28], Bass’ cancellation theorem in  $K$ -theory and the fact that  $K_0(\mathcal{O}(G)) \cong \mathbb{Z}$ , proved by Marlin [87], imply that this module is free. Alternatively, this follows also from the fact that  $\mathcal{O}_\epsilon$  is a cleft extension of  $\mathcal{O}(G)$  by the dual of the Frobenius–Lusztig kernel  $u_\epsilon(\mathfrak{g})$ , as proved by Andruskiewitsch–Garcia (see [6, Remark 2.18 (b)], and also [25, Section 3.2]; this argument was explained to us by K.A. Brown).

The Section 4 proves the analogous property for  $\mathcal{L}_{0,n}^\epsilon$ . Namely:

**Theorem 1.2.**  *$\mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$  is a central subalgebra of  $\mathcal{L}_{0,n}^\epsilon$ , and  $\mathcal{L}_{0,n}^\epsilon$  is a free  $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$ -module of rank  $l^{n \cdot \dim \mathfrak{g}}$ , isomorphic to the  $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$ -module  $\mathcal{O}_\epsilon^{\otimes n}$ .*

In the sequel we systematically denote  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) := \mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$ . We prove the first and third claims of Theorem 1.2 in Proposition 4.1. The arguments use results of De Concini–Kac [39], De Concini–Procesi [40, 42], and De Concini–Lyubashenko [41], that we recall in Sections 2.3–2.5. Let us stress that the algebra structures of  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{O}_\epsilon^{\otimes n}$  are completely different.

Since  $\mathcal{Z}_0(\mathcal{O}_\epsilon) \cong \mathcal{O}(G)$ , we may deduce the second claim of Theorem 1.2 from the first and third claims together with the results of [41, 87], or [6], recalled above. Nevertheless, we give a self-contained proof that  $\mathcal{L}_{0,1}^\epsilon$  is finite projective of rank  $l^{\dim \mathfrak{g}}$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , by adapting the original arguments of De Concini–Lyubashenko [41, Theorem 7.2]. In particular, we study the coregular action of the braid group of  $\mathfrak{g}$  on  $\mathcal{L}_{0,1}^\epsilon$ ; by the way, in the appendix, we provide different proofs of some technical facts shown in [41]. Of course, it remains an exciting problem to describe the centralizing extension  $\mathcal{O}(G)^{\otimes n} \subset \mathcal{L}_{0,n}^\epsilon$  (and similarly  $\mathcal{O}(G)^{\otimes n} \subset (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  below), aiming at generalizing the results of [6] and finding a direct, more structural proof of freeness in Theorem 1.2. Also, we note that bases of  $\mathcal{L}_{0,n}^\epsilon$  over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  are complicated. The only case we know is for  $\mathcal{O}_\epsilon(\mathfrak{sl}_2)$ , described in [38], and it is far from being obvious (see (4.4)).

In Section 5, we turn to fraction rings. As mentioned above  $\mathcal{L}_{0,n}^\epsilon$  has no nontrivial zero divisors. Therefore, its center  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is an integral domain. Denote by  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  its fraction field. Denote by  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  the subring of  $\mathcal{L}_{0,n}^\epsilon$  formed by the invariant elements of  $\mathcal{L}_{0,n}^\epsilon$  with respect to the right coadjoint action of  $U_\epsilon$ . The center  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  of  $\mathcal{L}_{0,n}^\epsilon$  is contained in  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  (this follows from [18, Proposition 6.19]). Note also that we trivially have an inclusion

$\mathcal{M}_{0,n}^{A,\epsilon} \subset (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ , and these two algebras are distinct in general. For instance, when  $n = 1$ , we have  $(\mathcal{L}_{0,1}^\epsilon)^{U_\epsilon} = \mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)$ , which is a finite extension of  $\mathcal{Z}_0(\mathcal{O}_\epsilon) \cong \mathcal{O}(G)$  (see Lemma 5.1). On another hand,  $\mathcal{M}_{0,1}^{A,\epsilon}$  is the specialization at  $q = \epsilon$  of  $\mathcal{Z}(\mathcal{L}_{0,1}^A)$ , a polynomial algebra in  $\text{rk}(\mathfrak{g})$  variables, which may be identified via  $\Phi_1$  with the center  $\mathcal{Z}(U_A)$  of the integral form  $U_A$ .

Consider the rings

$$Q(\mathcal{L}_{0,n}^\epsilon) = Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \bigotimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} \mathcal{L}_{0,n}^\epsilon, \quad Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) = Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \bigotimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}.$$

In general, given a ring  $A$  with center  $\mathcal{Z}(A)$  an integral domain we reserve the notation  $Q(A)$  to the central localization of  $A$ , i.e.,  $Q(A) := Q(\mathcal{Z}(A)) \bigotimes_{\mathcal{Z}(A)} A$ . Though the center  $\mathcal{Z}((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  of  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is larger than  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ , the notation  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is valid, for  $\mathcal{Z}((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is an integral domain finite over  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ , and hence the central localization of  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  coincides with  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  as defined above. Throughout the paper, unless we mention it explicitly, we follow the conventions of McConnell–Robson [88] as regards the terminology of ring theory; in particular, for the notions of central simple algebras and PI degrees, see in [88, Sections 5.3 and 13.3.6–13.6.7].

Denote by  $m$  the rank of  $\mathfrak{g}$ , and by  $N$  the number of its positive roots. In Section 5, we prove the following.

**Theorem 1.3.**

- (1)  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a division algebra and a central simple algebra of PI degree  $l^{nN}$ .
- (2)  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ ,  $n \geq 2$ , is a division algebra and a central simple algebra of PI degree  $l^{N(n-1)-m}$ .

The second claim of (1) means that  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a complex subalgebra of a full matrix algebra  $\text{Mat}_d(\mathbb{F})$ , where  $d = l^{nN}$  and  $\mathbb{F}$  is a finite extension of  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  such that

$$\mathbb{F} \bigotimes_{Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} Q(\mathcal{L}_{0,n}^\epsilon) = \text{Mat}_d(\mathbb{F}).$$

That  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a division algebra and a central simple algebra follows from Theorem 1.2 and the fact that  $\mathcal{L}_{0,n}^\epsilon$  has no nontrivial zero divisors (proved in [18]). The computation of  $d = l^{nN}$  uses a lower bound coming from the representation theory of  $U_\epsilon$ , and a lower bound for the degree of  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  as a field extension of  $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ , obtained by using specializations to  $q = \epsilon$  of certain central elements in  $\mathcal{L}_{0,n}$  (for  $q$  generic). In this computation a main role is played by results of De Concini–Kac [39].

We deduce (2) from (1), the double centralizer theorem for central simple algebras, a few results of [18, 41], and Theorem 1.2 again.

## 1.2 Basic notations

Given a ring  $R$ , we denote by  $\mathcal{Z}(R)$  its center. When  $R$  is commutative and has no nontrivial zero divisors,  $Q(R)$  denotes its fraction field.

Given a Hopf algebra  $H$  with product  $m$  and coproduct  $\Delta$ , we denote by  $H^{\text{cop}}$  (resp.  $H_{\text{op}}$ ) the Hopf algebra with the same algebra (resp. coalgebra) structure as  $H$  but the opposite coproduct  $\Delta^{\text{cop}} := \sigma \circ \Delta$  (resp. opposite product  $m \circ \sigma$ ), where  $\sigma(x \otimes y) = y \otimes x$ , and the antipode  $S^{-1}$ . We use Sweedler's coproduct notation,  $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ ,  $x \in H$ , and we set  $\Delta^{(1)} := \text{id}$ ,  $\Delta^{(2)} := \Delta$ , and  $\Delta^{(n)} := (\Delta \otimes \text{id})\Delta^{(n-1)}$  for  $n \geq 3$  (this is not the convention used in [18]).

The results of this paper hold true for any finite-dimensional complex semisimple Lie algebra  $\mathfrak{g}$ , but unless we state it differently, we will assume  $\mathfrak{g}$  is simple. We will denote its rank

by  $m$ , and its Cartan matrix by  $(a_{ij})$ . We fix a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  and a basis of simple roots  $\alpha_i \in \mathfrak{h}_{\mathbb{R}}^*$ , and denote by  $\mathfrak{b}_{\pm}$  the Borel subalgebras associated to it. We denote by  $N$  the number of positive roots of  $\mathfrak{g}$ , and by  $\rho$  half the sum of the positive roots.

We denote by  $d_1, \dots, d_m$  the unique coprime positive integers such that the matrix  $(d_i a_{ij})$  is symmetric, and  $(\ , \ )$  the unique inner product on  $\mathfrak{h}_{\mathbb{R}}^*$  such that  $d_i a_{ij} = (\alpha_i, \alpha_j)$ . For any root  $\alpha$ , the coroot is  $\check{\alpha} = 2\alpha/(\alpha, \alpha)$ ; in particular  $\check{\alpha}_i = d_i^{-1}\alpha_i$ . The root lattice  $Q$  is the  $\mathbb{Z}$ -lattice in  $\mathfrak{h}_{\mathbb{R}}^*$  defined by  $Q = \sum_{i=1}^m \mathbb{Z}\alpha_i$ . The weight lattice  $P$  is the  $\mathbb{Z}$ -lattice formed by all  $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$  such that  $(\lambda, \check{\alpha}_i) \in \mathbb{Z}$  for every  $i = 1, \dots, m$ . So  $P = \sum_{i=1}^m \mathbb{Z}\varpi_i$ , where  $\varpi_i$  is the fundamental weight dual to the simple coroot  $\check{\alpha}_i$ , which satisfies  $(\varpi_i, \check{\alpha}_j) = \delta_{i,j}$ . Note that  $(\lambda, \alpha) \in \mathbb{Z}$  for every  $\lambda \in P$ ,  $\alpha \in Q$ . We denote by  $D$  the cardinality of the quotient lattice  $P/Q$ . Then  $D$  is the smallest positive integer such that  $D(\lambda, \mu) \in \mathbb{Z}$  for every  $\lambda, \mu \in P$ , that is, such that  $DP \subset Q$ .

We denote by

$$P_+ := \sum_{i=1}^m \mathbb{Z}_{\geq 0} \varpi_i$$

the cone of dominant integral weights, and we put

$$Q_+ := \sum_{i=1}^m \mathbb{Z}_{\geq 0} \alpha_i.$$

Though  $Q \subset P$ , it is not true that  $Q_+ \subset P_+$ , but we have  $DP_+ \subset Q_+$ . This last property is not trivial, and follows from the classical fact that the inverse of the Cartan matrix  $(a_{ij})$  has coefficients in  $D^{-1}\mathbb{N}$ .

We will use the standard partial order relation  $\leq$  on  $P$ , defined by:  $\lambda, \mu \in P$  satisfy  $\lambda \leq \mu$  if  $\mu - \lambda \in Q_+$ . In Section 3, we will also use the partial order relation  $\preceq$  on  $P$  defined by:  $\lambda \preceq \mu$  if  $\mu - \lambda \in D^{-1}Q_+$ .

We denote by  $\mathcal{B}(\mathfrak{g})$  the braid group of  $\mathfrak{g}$ ; we recall its standard defining relations in Appendix B.

We denote by  $G$  the connected and simply-connected algebraic group with Lie algebra  $\mathfrak{g}$ , and by  $T_G$  the maximal torus of  $G$  with Lie algebra  $\mathfrak{h}$ ;  $N(T_G)$  is the normalizer of  $T_G$ ,  $W = N(T_G)/T_G$  is the Weyl group,  $B_{\pm}$  are the Borel subgroups of  $G$  with Lie algebra  $\mathfrak{b}_{\pm}$ , and  $U_{\pm} \subset B_{\pm}$  are their unipotent subgroups.

We denote by  $\mathcal{O}(G)$  the coordinate ring of  $G$ . It is a commutative Hopf algebra, which can be identified with the restricted dual of the universal enveloping algebra  $U(\mathfrak{g})$  (see [76, 84]). Similarly we denote by  $\mathcal{O}(B_{\pm})$  the coordinate ring of  $B_{\pm}$ .

Let  $q$  be an indeterminate, let  $q^{1/D}$  be such that  $(q^{1/D})^D = q$ , set  $A = \mathbb{C}[q, q^{-1}]$ ,  $q_i = q^{d_i}$ ,  $q_{\beta} = q^{(\beta, \beta)/2}$  for  $\beta \in Q$ , and given integers  $p, k$  with  $0 \leq k \leq p$ , we put

$$\begin{aligned} [p]_q &= \frac{q^p - q^{-p}}{q - q^{-1}}, & [0]_q! &= 1, & [p]_q! &= [1]_q [2]_q \cdots [p]_q, & \begin{bmatrix} p \\ k \end{bmatrix}_q &= \frac{[p]_q!}{[p-k]_q! [k]_q!}, \\ (p)_q &= \frac{q^p - 1}{q - 1}, & (0)_q! &= 1, & (p)_q! &= (1)_q (2)_q \cdots (p)_q, & \binom{p}{k}_q &= \frac{(p)_q!}{(p-k)_q! (k)_q!}. \end{aligned}$$

We denote by  $\mathcal{A}_0 \subset \mathbb{C}(q)$  the ring of functions regular at  $q = 0$ ; this ring is used only in Section 2.2.2.

We denote by  $\epsilon$  a primitive  $l$ -th root of unity such that  $\epsilon^{2d_i} \neq 1$  is also a primitive  $l$ -th root of unity for all  $i \in \{1, \dots, m\}$ . This means that  $l$  is odd, and coprime to 3 if  $\mathfrak{g}$  is  $G_2$ . We put  $\epsilon_i := \epsilon^{d_i}$ .

In this paper, we use the definition of the unrestricted integral form  $U_A(\mathfrak{g})$  given in [41, 42]; in [18] we used the one of [39, 40]. The two are (trivially) isomorphic, and have the same

specialization at  $q = \epsilon$ . Also, we denote here by  $L_i$  the generators of  $U_q(\mathfrak{g})$  we denoted by  $\ell_i$  in [18].

In order to facilitate the comparison with the results of [41], we note that their generators denoted  $K_i$ ,  $E_i$  and  $F_i$ , that we will denote by  $\tilde{K}_i$ ,  $\tilde{E}_i$  and  $\tilde{F}_i$ , can be written as  $K_i$ ,  $K_i^{-1}E_i$  and  $F_iK_i$  in our notations. They satisfy the same algebra relations.

## 2 Background results

### 2.1 On $U_q$ , $\mathcal{O}_q$ , $\mathcal{L}_{0,n}$ , $\mathcal{M}_{0,n}$ , and $\Phi_n$

Except when stated differently, we refer to [18, Sections 2–4 and 6], and the references therein for details about the material of this section. We stress that the simply-connected quantum group, that we denote  $U_q$  below, was denoted  $\tilde{U}_q$  in [18]. Also, the adjoint quantum group  $U_q^{\text{ad}}$  was denoted  $U_q$ .

The *simply-connected* quantum group  $U_q = U_q(\mathfrak{g})$  is the Hopf algebra over  $\mathbb{C}(q)$  with generators  $E_i$ ,  $F_i$ ,  $L_i$ ,  $L_i^{-1}$ ,  $1 \leq i \leq m$ , and defining relations

$$\begin{aligned} L_i L_j &= L_j L_i, & L_i L_i^{-1} &= L_i^{-1} L_i = 1, & L_i E_j L_i^{-1} &= q_i^{\delta_{i,j}} E_j, & L_i F_j L_i^{-1} &= q_i^{-\delta_{i,j}} F_j, \\ E_i F_j - F_j E_i &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r &= 0 & \text{if } i \neq j, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{ij}-r} F_j F_i^r &= 0 & \text{if } i \neq j, \end{aligned}$$

where for  $\lambda = \sum_{i=1}^m m_i \varpi_i \in P$  we set  $K_\lambda = \prod_{i=1}^m L_i^{m_i}$ , and  $K_i = K_{\alpha_i} = \prod_{j=1}^m L_j^{\alpha_{ji}}$ . The coproduct  $\Delta$ , antipode  $S$ , and counit  $\varepsilon$  of  $U_q$  are given by

$$\begin{aligned} \Delta(L_i) &= L_i \otimes L_i, & \Delta(E_i) &= E_i \otimes K_i + 1 \otimes E_i, & \Delta(F_i) &= F_i \otimes 1 + K_i^{-1} \otimes F_i, \\ S(E_i) &= -E_i K_i^{-1}, & S(F_i) &= -K_i F_i, & S(L_i) &= L_i^{-1}, \\ \varepsilon(E_i) &= \varepsilon(F_i) = 0, & \varepsilon(L_i) &= 1. \end{aligned}$$

We fix a reduced expression  $s_{i_1} \cdots s_{i_N}$  of the longest element  $w_0$  of the Weyl group of  $\mathfrak{g}$ . It induces a total ordering of the positive roots,

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \dots, \quad \beta_N = s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N}).$$

The root vectors of  $U_q$  with respect to such an ordering are defined by

$$E_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(E_{i_k}), \quad F_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k}), \quad (2.1)$$

where  $T_i$  is the Lusztig algebra automorphism of  $U_q$  associated to the simple root  $\alpha_i$  [82, 83] (see also [35, Chapter 8]). The braid group  $\mathcal{B}(\mathfrak{g})$  acts on  $U_q$  by means of the Lusztig automorphisms. In the appendix, we recall the relation between  $T_i$  and the generator  $\hat{w}_i$  of the quantum Weyl group, which we will mostly use. Let us just recall here that the monomials  $F_{\beta_1}^{r_1} \cdots F_{\beta_N}^{r_N} K_\lambda E_{\beta_N}^{t_N} \cdots E_{\beta_1}^{t_1}$  ( $r_i, t_i \in \mathbb{N}$ ,  $\lambda \in P$ ) form a basis of  $U_q$ , the *PBW basis*.

$U_q$  is a *pivotal* Hopf algebra, with pivotal element  $\ell := K_{2\rho} = \prod_{j=1}^m L_j^2$ . So  $\ell$  is group-like, and  $S^2(x) = \ell x \ell^{-1}$  for every  $x \in U_q$ .



The *adjoint* quantum group  $U_q^{\text{ad}} = U_q^{\text{ad}}(\mathfrak{g})$  is the Hopf subalgebra of  $U_q$  generated by the elements  $E_i, F_i$  ( $i = 1, \dots, m$ ) and  $K_\alpha$  with  $\alpha \in Q$ ; so  $\ell \in U_q^{\text{ad}}$ . When  $\mathfrak{g} = \mathfrak{sl}_2$ , we simply write the above generators  $E = E_1, F = F_1, L = L_1, K = K_1$ .

We denote by  $U_q(\mathfrak{n}_+)$ ,  $U_q(\mathfrak{n}_-)$  and  $U_q(\mathfrak{h})$  the subalgebras of  $U_q$  generated respectively by the  $E_i$ , the  $F_i$ , and the  $K_\lambda$  ( $\lambda \in P$ ), and by  $U_q(\mathfrak{b}_+)$  and  $U_q(\mathfrak{b}_-)$  the subalgebras generated by the  $E_i$  and the  $K_\lambda$ , and by the  $F_i$  and the  $K_\lambda$ , respectively. We do similarly with  $U_q^{\text{ad}}$ , where now  $U_q^{\text{ad}}(\mathfrak{h})$  is generated by the  $K_\lambda$  with  $\lambda \in Q$ .

The Hopf algebra  $U_q^{\text{ad}}$  is not braided in a strict sense, but it has braided categorical completions. Let us recall briefly what this means and implies. For details, we refer to [18, Sections 2 and 3] (see also [104, Section 3.10], where  $\mathbb{U}_q$  below is formulated in terms of multiplier Hopf algebras).

A  $U_q^{\text{ad}}$ -module  $V$  is said of *type 1* if it has finite dimension and the generators  $K_i$  are diagonalizable on  $V$  with eigenvalues in  $q_i^{\mathbb{Z}}$ . We denote by  $\mathcal{C}$  the category of  $U_q^{\text{ad}}$ -modules of type 1, by  $\text{Vect}$  the category of finite-dimensional  $\mathbb{C}(q)$ -vector spaces, and by  $F_{\mathcal{C}}: \mathcal{C} \rightarrow \text{Vect}$  the forgetful functor. The category  $\mathcal{C}$  is semisimple. The simple objects are highest weight  $U_q^{\text{ad}}$ -modules; we denote by  $V_\mu$  the simple module with highest weight  $\mu \in P_+$ . In the case  $\mathfrak{g} = \mathfrak{sl}_2$ , we identify  $P_+$  with  $\mathbb{N}$ , and denote by  $V_n$  the simple module of dimension  $n + 1$ . Note that  $V_\mu$  is canonically endowed with a structure of  $U_q$ -module of type 1, the generators  $L_i$  being diagonalizable with eigenvalues in  $q_i^{\mathbb{Z}/D}$ . The *categorical completion*  $\mathbb{U}_q^{\text{ad}}$  of  $U_q^{\text{ad}}$  is the set of natural transformations  $F_{\mathcal{C}} \rightarrow F_{\mathcal{C}}$ . An element of  $\mathbb{U}_q^{\text{ad}}$  is a collection  $(a_V)_{V \in \text{Ob}(\mathcal{C})}$ , where  $a_V \in \text{End}_{\mathbb{C}(q)}(V)$  satisfies  $F_{\mathcal{C}}(f) \circ a_V = a_W \circ F_{\mathcal{C}}(f)$  for any objects  $V, W$  of  $\mathcal{C}$  and any arrow  $f \in \text{Hom}_{U_q^{\text{ad}}}(V, W)$ . It is not hard to see that  $\mathbb{U}_q^{\text{ad}}$  inherits from  $\mathcal{C}$  a natural structure of (completed) Hopf algebra such that the map

$$\iota: U_q^{\text{ad}} \longrightarrow \mathbb{U}_q^{\text{ad}}, \quad x \longmapsto (\pi_V(x))_{V \in \text{Ob}(\mathcal{C})} \quad (2.2)$$

is a morphism of Hopf algebras, where  $\pi_V: U_q^{\text{ad}} \rightarrow \text{End}(V)$  is the representation associated to a module  $V$  in  $\mathcal{C}$ . It is a theorem that this map is injective. From now on, let us extend the coefficient ring of the modules and morphisms in  $\mathcal{C}$  to  $\mathbb{C}(q^{1/D})$ . Put  $\mathbb{U}_q = \mathbb{U}_q^{\text{ad}} \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})$ . The map  $\iota$  above extends to an embedding of  $U_q$  in  $\mathbb{U}_q$ . The category  $\mathcal{C}$ , with coefficients extended to  $\mathbb{C}(q^{1/D})$ , is braided and ribbon; we postpone a discussion of that fact to Section 2.3, where it will be developed. As a consequence, we can regard  $\mathbb{U}_q$  as a quasitriangular and ribbon Hopf algebra in a generalized sense (see [18]). The  $R$ -matrix of  $\mathbb{U}_q$  is the family of morphisms

$$R = (R_{V,W})_{V,W \in \text{Ob}(\mathcal{C})},$$

where  $R_{V,W} \in \text{End}(V \otimes W)$  is the endomorphism defined by the action of Drinfeld's universal  $R$ -matrix on  $V \otimes W$ . The ribbon element of  $\mathbb{U}_q$  is defined similarly by Drinfeld's universal ribbon element. One defines the *categorical tensor product*  $\mathbb{U}_q^{\otimes 2}$  similarly as  $\mathbb{U}_q$ ; in particular it contains all the infinite series of elements of  $\mathbb{U}_q^{\otimes 2}$  having only a finite number of non-zero terms when evaluated on a given module  $V \otimes W$  of  $\mathcal{C}$ . There is an expansion of  $R$  as an infinite series in  $\mathbb{U}_q^{\otimes 2}$ . Adapting Sweedler's coproduct notation  $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ , we find convenient to write this series as

$$R = \sum_{(R)} R_{(1)} \otimes R_{(2)}. \quad (2.3)$$

We put  $R^+ := R$ ,  $R^- := (\sigma \circ R)^{-1}$  where  $\sigma$  is the flip map  $x \otimes y \mapsto y \otimes x$ . We will not use any explicit formula of  $R$ , but the following factorization formula

$$R = \Theta \hat{R}, \quad (2.4)$$

where

$$\Theta = q^{\sum_{i,j=1}^m (B^{-1})_{ij} H_i \otimes H_j} \in \mathbb{U}_q^{\hat{\otimes} 2},$$

with  $B \in M_m(\mathbb{Q})$  the matrix with entries  $B_{ij} := d_j^{-1} a_{ij}$ , and

$$\hat{R} = \sum_{(\hat{R})} \hat{R}_{(1)} \otimes \hat{R}_{(2)} \in \mathbb{U}_q(\mathfrak{n}_+) \hat{\otimes} \mathbb{U}_q(\mathfrak{n}_-)$$

(see [18, Section 3.2], and for details, e.g., [35, Theorem 8.3.9], or [104, Theorem 3.108]). If  $x, y$  are weight vectors of weights  $\mu, \nu$  respectively, then  $\Theta(x \otimes y) = q^{(\mu, \nu)} x \otimes y$ . Moreover,  $\hat{R}$  has weight 0 for the adjoint action of  $U_q(\mathfrak{h})$ ; that is, complementary components  $\hat{R}_{(1)}$  and  $\hat{R}_{(2)}$  have opposite weights.

Recall that we denote by  $G$  the connected and simply-connected algebraic group with Lie algebra  $\mathfrak{g}$ . The quantum function Hopf algebra  $\mathcal{O}_q = \mathcal{O}_q(G)$  is defined as the restricted dual of  $U_q^{\text{ad}}$  with respect to the category  $\mathcal{C}$ , that is, the set of  $\mathbb{C}(q)$ -linear maps  $f: U_q^{\text{ad}} \rightarrow \mathbb{C}(q)$  such that  $\text{Ker}(f)$  contains a cofinite two sided ideal  $I$  (i.e., such that  $I \oplus M = U_q$  for some finite-dimensional vector space  $M$ ), and  $\prod_{s=-r}^r (K_i - q_i^s) \in I$  for some  $r \in \mathbb{N}$  and every  $i$  (see, e.g., [26, Chapter I.7]).

The space  $\mathcal{O}_q$  is a Hopf algebra, with structure maps defined dually to those of  $U_q^{\text{ad}}$ . We denote by  $\star$  its product. The algebras  $\mathcal{O}_q(T_G), \mathcal{O}_q(U_{\pm}), \mathcal{O}_q(B_{\pm})$  are defined similarly, by replacing  $U_q^{\text{ad}}$  with  $U_q^{\text{ad}}(\mathfrak{h}), U_q^{\text{ad}}(\mathfrak{n}_{\pm}), U_q^{\text{ad}}(\mathfrak{b}_{\pm})$ , respectively. As a vector space,  $\mathcal{O}_q$  is generated by the functionals  $x \mapsto w(\pi_V(x)v), x \in U_q^{\text{ad}}$ , for every object  $V \in \text{Ob}(\mathcal{C})$  and vectors  $v \in V, w \in V^*$ . Such functionals are called *matrix coefficients*. Because the morphism  $\iota: U_q^{\text{ad}} \rightarrow \mathbb{U}_q$  is injective (see (2.2)), the Hopf duality pairing  $\langle \cdot, \cdot \rangle: \mathcal{O}_q \times U_q^{\text{ad}} \rightarrow \mathbb{C}(q)$  is non degenerate. By extending the coefficient ring from  $\mathbb{C}(q)$  to  $\mathbb{C}(q^{1/D})$ , we can uniquely extend it to a bilinear pairing

$$\langle \cdot, \cdot \rangle: (\mathcal{O}_q \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})) \times \mathbb{U}_q \rightarrow \mathbb{C}(q^{1/D})$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}_q \otimes U_q^{\text{ad}} & \xrightarrow{\langle \cdot, \cdot \rangle} & \mathbb{C}(q) \\ \text{id} \otimes \iota \downarrow & & \downarrow \\ (\mathcal{O}_q \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})) \otimes \mathbb{U}_q & \xrightarrow{\langle \cdot, \cdot \rangle} & \mathbb{C}(q^{1/D}). \end{array}$$

This pairing is defined by  $\langle Y\phi_v^w, (a_X) \rangle = w(a_Y v)$  for every  $(a_X) \in \mathbb{U}_q$  and  $Y\phi_v^w \in \mathcal{O}_q$ . It is non degenerate.

The maps

$$\Phi^{\pm}: \mathcal{O}_q \longrightarrow U_q^{\text{cop}}, \quad \alpha \longmapsto (\alpha \otimes \text{id})(R^{\pm}) = \sum_{(R^{\pm})} \langle \alpha, R_{(1)}^{\pm} \rangle R_{(2)}^{\pm} \quad (2.5)$$

are well-defined morphisms of Hopf algebras. Here we stress that it is the simply-connected quantum group  $U_q^{\text{cop}}$  that is the range of  $\Phi^{\pm}$ . This will be explained with more details in Section 2.3.

Let us make two simple observations, for future reference. Firstly, because  $\mathcal{O}_q$  is spanned by the matrix coefficients of the objects of  $\mathcal{C}$ , and  $\mathcal{C}$  is semisimple with simple objects the  $U_q^{\text{ad}}$ -modules  $V_{\mu}, \mu \in P_+$ , there is a decomposition of  $U_q$ -bimodule

$$\mathcal{O}_q = \bigoplus_{\mu \in P_+} C(\mu), \quad (2.6)$$

where  $C(\mu) = V_\mu^* \otimes V_\mu$ , the space of matrix coefficients of  $V_\mu$ , is endowed with the left action on the factor  $V_\mu$  and the right action on  $V_\mu^*$ , and  $\mathcal{O}_q$  has the left and right coregular actions  $\triangleleft$  and  $\triangleright$ , defined by

$$x \triangleright \alpha := \sum_{(\alpha)} \alpha_{(1)} \langle \alpha_{(2)}, x \rangle, \quad \alpha \triangleleft x := \sum_{(\alpha)} \langle \alpha_{(1)}, x \rangle \alpha_{(2)}$$

for all  $x \in U_q$  and  $\alpha \in \mathcal{O}_q$ . Here we recall that each  $U_q^{\text{ad}}$ -module  $V_\mu$  can be regarded as a  $U_q$ -module, so the above expressions make sense. The decomposition (2.6) is the *Peter–Weyl* decomposition of  $\mathcal{O}_q$ . It will be refined in Section 2.2.2.

Moreover, the algebra  $\mathcal{O}_q$  is generated by the matrix coefficients of the simple  $U_q^{\text{ad}}$ -modules  $V_{\varpi_k}$  with highest weights the fundamental weights  $\varpi_k$ ,  $k = 1, \dots, m$ ; see, e.g., [26, Proposition I.7.8] for a proof. This relies on the standard fact that, for any  $\mu, \nu \in P_+$  we have a direct sum decomposition of modules (where  $m(\lambda) \in \mathbb{N}$ )

$$V_\mu \otimes V_\nu = V_{\mu+\nu} \oplus \bigoplus_{\lambda < \mu+\nu} V_\lambda^{\oplus m(\lambda)}. \quad (2.7)$$

In particular, this decomposition implies that, up to scalar multiples, there is a unique non-zero morphism  $V_{\mu+\nu} \rightarrow V_\mu \otimes V_\nu$ , which is injective and splits. Dually, this means that, applying the product in  $\mathcal{O}_q$  followed by the projection onto the subspace  $C(\mu + \nu)$  we get a canonical projection map

$$C(\mu) \otimes C(\nu) \rightarrow C(\mu + \nu). \quad (2.8)$$

The *loop algebra*  $\mathcal{L}_{0,1} = \mathcal{L}_{0,1}(\mathfrak{g})$  is defined by twisting the product  $\star$  of  $\mathcal{O}_q$ , keeping the same underlying linear space. The new product is equivariant with respect to the right coadjoint action  $\text{coad}^r$  of  $U_q$ , defined by

$$\text{coad}^r(x)(\alpha) = \sum_{(x)} S(x_{(2)}) \triangleright \alpha \triangleleft x_{(1)}$$

for all  $x \in U_q$  and  $\alpha \in \mathcal{O}_q$ . By equivariant we mean that  $\mathcal{L}_{0,1}$  is a  $U_q$ -module algebra. Let us spell out its product and equivariance property. Using the fact that  $U_q$  can be regarded as a subspace of  $\mathbb{U}_q$ , the actions  $\triangleleft$  and  $\triangleright$  extend naturally to actions of  $\mathbb{U}_q$ , and the product of  $\mathcal{L}_{0,1}$  is expressed in terms of  $\star$  by the formula (see [18, Proposition 4.1]):

$$\alpha\beta = \sum_{(R),(R)} (R_{(2)'} S(R_{(2)}) \triangleright \alpha) \star (R_{(1)'} \triangleright \beta \triangleleft R_{(1)}), \quad (2.9)$$

where  $\sum_{(R)} R_{(1)} \otimes R_{(2)}$  and  $\sum_{(R)} R_{(1)'} \otimes R_{(2)'}$  are expansions of two copies of  $R \in \mathbb{U}_q^{\otimes 2}$ . Note that the sum in (2.9) has only a finite number of non-zero terms. By using that

$$R\Delta = \Delta^{\text{cop}}R,$$

this product can equivalently be expressed as

$$\alpha\beta = \sum_{(R),(R)} (\beta \triangleleft R_{(1)} R_{(1)'}) \star (S(R_{(2)}) \triangleright \alpha \triangleleft R_{(2)'}). \quad (2.10)$$

This product gives  $\mathcal{L}_{0,1}$  (like  $\mathcal{O}_q$ ) a structure of  $U_q$ -module algebra for the actions  $\triangleright$ ,  $\triangleleft$ , but also for  $\text{coad}^r$  (which is not the case of  $\mathcal{O}_q$ ). Spelling this out for  $\text{coad}^r$ , this means

$$\text{coad}^r(x)(\alpha\beta) = \sum_{(x)} \text{coad}^r(x_{(1)})(\alpha) \text{coad}^r(x_{(2)})(\beta).$$

The relations between  $\mathcal{O}_q$ ,  $\mathcal{L}_{0,1}$  and  $U_q$  are encoded by the map

$$\Phi_1: \mathcal{O}_q \longrightarrow \mathbb{U}_q, \quad \alpha \longmapsto (\alpha \otimes \text{id})(RR'), \quad (2.11)$$

where  $R' = \sigma \circ R$ , and as usual  $\sigma: x \otimes y \mapsto y \otimes x$ . Note that

$$\Phi_1 = m \circ (\Phi^+ \otimes (S^{-1} \circ \Phi^-)) \circ \Delta. \quad (2.12)$$

We call  $\Phi_1$  the *RSD* map, for Drinfeld, Reshetikhin and Semenov-Tian-Shansky introduced it first (see [48, 86, 94]). It is a fundamental result of the theory (see [20, 34, 61]) that  $\Phi_1$  affords an isomorphism of  $U_q$ -modules  $\Phi_1: \mathcal{O}_q \rightarrow U_q^{\text{lf}}$ . For full details on that result we refer to [104, Section 3.12]. Here,  $U_q^{\text{lf}}$  is the set of *locally finite* elements of  $U_q$ , endowed with the right adjoint action  $\text{ad}^r$  of  $U_q$ . It is defined by

$$U_q^{\text{lf}} := \{x \in U_q \mid \text{rk}_{\mathbb{C}(q)}(\text{ad}^r(U_q)(x)) < \infty\}$$

and

$$\text{ad}^r(y)(x) = \sum_{(y)} S(y_{(1)})xy_{(2)}$$

for every  $x, y \in U_q$ . The action  $\text{ad}^r$  gives in fact  $U_q^{\text{lf}}$  a structure of right  $U_q$ -module algebra. It is also a right coideal, that is  $\Delta(U_q^{\text{lf}}) \subset U_q^{\text{lf}} \otimes U_q$ . Moreover,  $\Phi_1$  affords an isomorphism of  $U_q$ -module algebras  $\Phi_1: \mathcal{L}_{0,1} \rightarrow U_q^{\text{lf}}$ . It is a fact that  $\Phi_1$  affords an isomorphism between the centers  $\mathcal{Z}(\mathcal{L}_{0,1})$  of  $\mathcal{L}_{0,1}$  and  $\mathcal{Z}(U_q)$  of  $U_q$  [18, Proposition 6.24]. Since  $\Phi_1$  is an isomorphism of  $U_q$ -modules and  $\mathcal{Z}(U_q) = U_q^{U_q}$ , it follows that  $\mathcal{Z}(\mathcal{L}_{0,1}) = \mathcal{L}_{0,1}^{U_q}$ .

Let us recall a few fundamental results about  $U_q^{\text{lf}}$  that we will meet again later. Denote by  $T \subset U_q$  the multiplicative Abelian group formed by the elements  $K_\lambda$ ,  $\lambda \in P$ , and by  $T_2 \subset T$  the subgroup formed by the elements  $K_\lambda$ ,  $\lambda \in 2P$ . Consider the subset  $T_{2-} \subset T_2$  formed by the elements  $K_{-\lambda}$ ,  $\lambda \in 2P_+$ . Clearly,  $T_2 = T_{2-}^{-1}T_{2-}$  and  $\text{Card}(T/T_2) = 2^m$ .

**Theorem 2.1.**

- (1)  $U_q^{\text{lf}} = \bigoplus_{\lambda \in 2P_+} \text{ad}^r(U_q)(K_{-\lambda})$ .
- (2)  $U_q = T_{2-}^{-1}U_q^{\text{lf}}[T/T_2]$ , so  $U_q$  is a free  $T_{2-}^{-1}U_q^{\text{lf}}$ -module of rank  $2^m$ .
- (3) The ring  $U_q^{\text{lf}}$  is (left and right) Noetherian.

These results were proved by Joseph–Letzter in [63, Theorem 4.10], [62, Theorem 6.4], and [61, Theorem 7.4.8], respectively (see also [61, Sections 7.1.6, 7.1.13 and 7.1.25]). For (1) and (3), we refer also to [104, Theorems 3.113 and 3.137], which provides simpler proofs. For instance, in the  $\mathfrak{sl}_2$  case, we have

$$U_q(\mathfrak{sl}_2) = U_q(\mathfrak{sl}_2)^{\text{lf}}[K] \oplus U_q(\mathfrak{sl}_2)^{\text{lf}}[K].L.$$

The actual values of  $\Phi_1$  are complicated in general, however, there is a simple important one, that we describe now. Let  $V_{-\lambda}$  be the type 1 simple  $U_q^{\text{ad}}$ -module of lowest weight  $-\lambda \in -P_+$  (i.e., the highest weight  $U_q^{\text{ad}}$ -module  $V_{-w_0(\lambda)}$  of highest weight  $-w_0(\lambda)$ , where  $w_0$  is the longest element of the Weyl group; note that  $-w_0$  permutes the simple roots). Let  $v \in V_{-\lambda}$  be a lowest weight vector, and  $v^* \in V_{-\lambda}^*$  be such that  $v^*(v) = 1$  and  $v^*$  vanishes on a  $U_q^{\text{ad}}(\mathfrak{h})$ -invariant complement of  $v$ . Define  $\psi_{-\lambda} \in \mathcal{O}_q$  by  $\langle \psi_{-\lambda}, x \rangle = v^*(xv)$ ,  $x \in U_q$ . From the definition (2.11), it is quite easy to see that

$$\Phi_1(\psi_{-\lambda}) = K_{-2\lambda}. \quad (2.13)$$

In particular,  $\Phi_1(\psi_{-\rho}) = \ell^{-1}$ , where as usual  $\ell$  is the pivotal element of  $U_q$ .

**Remark 2.2.** Since  $\mathcal{L}_{0,1} = \mathcal{O}_q$  as a vector space, we still denote by  $C(\mu)$ ,  $\mu \in P^+$ , the linear subspace generated by the matrix coefficients of  $V_\mu$ , the  $U_q^{\text{ad}}$ -module of type 1 and highest weight  $\mu$ . It can be proved (see [61, Section 7.1.22], or [104, p. 156], where different conventions are used) that  $\Phi_1$  yields an isomorphism of  $U_q$ -modules

$$\Phi_1: C(-w_0(\mu)) \rightarrow \text{ad}^r(U_q)(K_{-2\mu}). \quad (2.14)$$

Therefore, the summands in (1) are finite-dimensional  $U_q$ -modules, and the action  $\text{ad}^r$  is completely reducible on  $U_q^{\text{lf}}$ . In fact,  $U_q^{\text{lf}}$  is the socle of  $\text{ad}^r$  on  $U_q$ .

**Remark 2.3.** Because  $\ell = \prod_{j=1}^m L_j^2$  and  $\Phi_1(\psi_{-\rho}) = \ell^{-1}$ , a natural question is the factorization of  $\psi_{-\rho}$  in  $\mathcal{L}_{0,1}$  (see Corollary 2.23). This question is considered in [60], where  $\mathcal{L}_{0,1}(\mathfrak{g})$  for  $\mathfrak{g} = \mathfrak{gl}(r+1)$  is analysed and quantum minors are extensively studied. Let us review here some of their results in relation with  $\psi_{-\rho}$ .

First note that for  $\mathfrak{g} = \mathfrak{sl}(r+1)$  the irreducible representation  $V_{-\rho}$  of lowest weight  $-\rho$  is isomorphic to the representation of highest weight  $V_\rho$  because  $-w_0(\rho) = \rho$ . By the Weyl formula, the dimension of this representation is

$$\prod_{\alpha > 0} \frac{(2\rho, \alpha)}{(\rho, \alpha)} = 2^N.$$

In [71], a presentation of  $U_q(\mathfrak{gl}(r+1))$  is given, which differs from our presentation of  $U_q(\mathfrak{sl}(r+1))$  only by its subalgebra  $U_q(\mathfrak{h})$ , generated by  $r+1$  elements  $\mathbb{K}_1, \dots, \mathbb{K}_{r+1}$ . The inclusion

$$U_q(\mathfrak{sl}(r+1)) \subset U_q(\mathfrak{gl}(r+1))$$

is such that  $K_i = \mathbb{K}_i^2 \mathbb{K}_{i+1}^{-2}$ ,  $i = 1, \dots, r$ . The quantum minors, properly defined in [60], of the matrix of matrix elements of the natural representation of  $U_q(\mathfrak{gl}(r+1))$  are denoted  $\det_q(A_{\geq k})$  for  $k = 1, \dots, r+1$ . We have  $\det_q(A_{\geq 1}) = 1$  in the case of  $\mathfrak{sl}(r+1)$ . Then [60] proves that  $\det_q(A_{\geq k}) = (\mathbb{K}_k \cdots \mathbb{K}_{r+1})^2$ , and there exists an element  $\mathbb{K} \in U_q(\mathfrak{gl}(r+1))$  such that

$$\mathbb{K}^{-2\rho} = \det_q(A_{\geq 1})^{-r} \det_q(A_{\geq 2}) \cdots \det_q(A_{\geq r+1}).$$

This has to be interpreted as  $K_{-2\rho} = \Phi_1(\det_q(A_{\geq 2}) \cdots \det_q(A_{\geq r+1}))$  in the case of  $\mathfrak{sl}(r+1)$ . As a result, this gives the equality

$$\psi_{-\rho} = \det_q(A_{\geq 2}) \cdots \det_q(A_{\geq r+1}).$$

The (*quantum*) *graph algebra*  $\mathcal{L}_{0,n} = \mathcal{L}_{0,n}(\mathfrak{g})$  is the braided tensor product of  $n$  copies of  $\mathcal{L}_{0,1}$  (considered as a  $U_q$ -module algebra). As a linear space and  $U_q$ -bimodule with actions  $\triangleleft$  and  $\triangleangleright$ , it coincides with  $\mathcal{L}_{0,1}^{\otimes n}$ , and thus with  $\mathcal{O}_q^{\otimes n}$ . It is also a right  $U_q$ -module algebra, with the following action of  $U_q$  (extending  $\text{coad}^r$  on  $\mathcal{L}_{0,1}$ ):

$$\text{coad}_n^r(y)(\alpha^{(1)} \otimes \cdots \otimes \alpha^{(n)}) = \sum_{(y)} \text{coad}^r(y_{(1)})(\alpha^{(1)}) \otimes \cdots \otimes \text{coad}^r(y_{(n)})(\alpha^{(n)}) \quad (2.15)$$

for all  $y \in U_q$  and  $\alpha^{(1)} \otimes \cdots \otimes \alpha^{(n)} \in \mathcal{L}_{0,n}$ . The product of  $\mathcal{L}_{0,n}$  can be expressed as follows. For every  $1 \leq a \leq n$ , define  $\mathbf{i}_a: \mathcal{L}_{0,1} \rightarrow \mathcal{L}_{0,n}$  by  $\mathbf{i}_a(x) = 1^{\otimes(a-1)} \otimes x \otimes 1^{\otimes(n-a)}$ ;  $\mathbf{i}_a$  is an embedding of  $U_q$ -module algebras. We will use the notations

$$\mathcal{L}_{0,n}^{(a)} := \text{Im}(\mathbf{i}_a), \quad (\alpha)^{(a)} := \mathbf{i}_a(\alpha). \quad (2.16)$$

Take  $(\alpha)^{(a)}, (\alpha')^{(a)} \in \mathcal{L}_{0,n}^{(a)}$  and  $(\beta)^{(b)}, (\beta')^{(b)} \in \mathcal{L}_{0,n}^{(b)}$  with  $a < b$ . Then the product of  $\mathcal{L}_{0,n}$  is given by the following formula (see [18, Section 6]):

$$\begin{aligned} ((\alpha)^{(a)} \otimes (\beta)^{(b)})((\alpha')^{(a)} \otimes (\beta')^{(b)}) &= \sum_{(R^1), \dots, (R^4)} (\alpha(S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha' \triangleleft R_{(1)}^1 R_{(1)}^2))^{(a)} \\ &\quad \otimes ((S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4) \beta')^{(b)}, \end{aligned} \quad (2.17)$$

where  $R^i = \sum_{(R^i)} R_{(1)}^i \otimes R_{(2)}^i$ ,  $i \in \{1, 2, 3, 4\}$ , are expansions of four copies of  $R \in \mathbb{U}_q^{\otimes 2}$ , and on the right-hand side the product is componentwise that of  $\mathcal{L}_{0,1}$ . Later we will use the fact that the product of  $\mathcal{L}_{0,n}$  is obtained from the standard (componentwise) product of  $\mathcal{L}_{0,1}^{\otimes n}$  by a process that may be inverted. Indeed, (2.17) can be rewritten as

$$((\alpha)^{(a)} \otimes (\beta)^{(b)})((\alpha')^{(a)} \otimes (\beta')^{(b)}) = \sum_{(F)} (\alpha)^{(a)} ((\alpha')^{(a)} \cdot F_{(2)}) \otimes ((\beta)^{(b)} \cdot F_{(1)}) (\beta')^{(b)}, \quad (2.18)$$

where  $F = \sum_{(F)} F_{(1)} \otimes F_{(2)} := (\Delta \otimes \Delta)(R')$ , and the symbol “ $\cdot$ ” stands for the right action of  $\mathbb{U}_q^{\otimes 2}$  on  $\mathcal{L}_{0,1}$  that may be read from (2.17). The tensor  $F$  is known as a twist. Then, by replacing  $F$  with its inverse  $\bar{F} = (\Delta \otimes \Delta)(R'^{-1})$ , one can express the product of  $\mathcal{L}_{0,1}^{\otimes n}$  in terms of the product of  $\mathcal{L}_{0,n}$  by

$$(\alpha)^{(a)} (\alpha')^{(a)} \otimes (\beta)^{(b)} (\beta')^{(b)} = \sum_{(\bar{F})} ((\alpha)^{(a)} \otimes ((\beta)^{(b)} \cdot \bar{F}_{(1)})) (((\alpha')^{(a)} \cdot \bar{F}_{(2)}) \otimes (\beta')^{(b)}). \quad (2.19)$$

We call *quantum moduli algebra* and denote by  $\mathcal{M}_{0,n} = \mathcal{L}_{0,n}^{U_q}$  the subalgebra of  $\mathcal{L}_{0,n}$  formed by the  $U_q$ -invariant elements.

The map  $\Phi_1$  can be extended to  $\mathcal{L}_{0,n}$  as follows. Consider the following action of  $U_q$  on the tensor product algebra  $U_q^{\otimes n}$ , which extends  $\text{ad}^r$  on  $U_q$ :

$$\text{ad}_n^r(y)(x) = \sum_{(y)} \Delta^{(n)}(S(y_{(1)})) x \Delta^{(n)}(y_{(2)})$$

for all  $y \in U_q$ ,  $x \in U_q^{\otimes n}$ . This action gives  $U_q^{\otimes n}$  a structure of right  $U_q$ -module algebra. In [1], Alekseev introduced a morphism of  $U_q$ -module algebras  $\Phi_n: \mathcal{L}_{0,n} \rightarrow U_q^{\otimes n}$  which extends  $\Phi_1$ . In [18, Proposition 6.7], we showed that  $\Phi_n$  affords isomorphisms

$$\Phi_n: \mathcal{L}_{0,n} \rightarrow (U_q^{\otimes n})^{\text{lf}}, \quad \Phi_n: \mathcal{M}_{0,n} \rightarrow (U_q^{\otimes n})^{U_q}, \quad (2.20)$$

where  $(U_q^{\otimes n})^{\text{lf}}$  is the set of  $\text{ad}_n^r$ -locally finite elements of  $U_q^{\otimes n}$ . We call  $\Phi_n$  the *Alekseev map*; we do not recall here the definition of  $\Phi_n$ , for we will not use it. It is a key argument of the proof of (2.20) that the set of locally finite elements of  $U_q^{\otimes n}$  for  $(\text{ad}^r)^{\otimes n} \circ \Delta^{(n)}$  coincides with  $(U_q^{\text{lf}})^{\otimes n}$ ; this follows from the main result of [72]. Using that the map

$$\psi_n = \Phi_n \circ (\Phi_1^{-1})^{\otimes n}: (U_q^{\text{lf}})^{\otimes n} \rightarrow (U_q^{\otimes n})^{\text{lf}} \quad (2.21)$$

intertwines the actions  $(\text{ad}^r)^{\otimes n} \circ \Delta^{(n-1)}$  and  $\text{ad}_n^r$ , we deduced that  $\text{Im}(\Phi_n) = (U_q^{\otimes n})^{\text{lf}}$ .

**Remark 2.4.** We have  $(U_q^{\text{lf}})^{\otimes n} \neq (U_q^{\otimes n})^{\text{lf}}$  and in fact there is not even an inclusion. Indeed, let  $\Omega = (q - q^{-1})^2 FE + qK + q^{-1}K^{-1}$  be the Casimir element of  $U_q(\mathfrak{sl}_2)$ . We trivially have  $\Delta(\Omega) \in (U_q^{\otimes 2})^{\text{lf}}$  but

$$\Delta(\Omega) = (q - q^{-1})^2 (K^{-1}E \otimes FK + F \otimes E) + \Omega \otimes K + K^{-1} \otimes \Omega - (q + q^{-1})K^{-1} \otimes K$$

and therefore  $\Delta(\Omega) \notin (U_q^{\text{lf}})^{\otimes 2}$ , since  $K \notin U_q^{\text{lf}}$  (see, e.g., Theorem 2.1(2)). This reflects the fact that  $U_q^{\text{lf}}$  is only a right coideal of  $U_q$  (and not a subcoalgebra).

As in Remark 2.2, denote by  $C(\mu)$ ,  $\mu \in P^+$ , the linear subspace of  $\mathcal{L}_{0,1}$  generated by the matrix coefficients of  $V_\mu$ . For every tuple  $[\mu] = (\mu_1, \dots, \mu_n) \in P_+^n$  put

$$C([\mu]) = C(\mu_1) \otimes \cdots \otimes C(\mu_n). \quad (2.22)$$

Then  $\mathcal{L}_{0,n} = \bigoplus_{[\mu] \in P_+^n} C([\mu])$ . Each space  $C([\mu])$  is a finite-dimensional  $U_q$ -module under the action  $\text{coad}_n^r$ , whence it is completely reducible. Therefore,  $\mathcal{L}_{0,n} = \mathcal{M}_{0,n} \oplus I$  as  $U_q$ -modules, where  $I$  is the sum of nontrivial isotypical components of  $\mathcal{L}_{0,n}$ . The  $\mathbb{C}(q)$ -linear projection map

$$\mathcal{R}: \mathcal{L}_{0,n} \rightarrow \mathcal{M}_{0,n}, \quad \text{Ker}(\mathcal{R}) = I \quad (2.23)$$

is called the *Reynolds operator*. For all  $\alpha \in \mathcal{M}_{0,n}$ ,  $\beta \in \mathcal{L}_{0,n}$  it satisfies  $\mathcal{R}(\alpha\beta) = \alpha\mathcal{R}(\beta)$ . This property will be crucial in the sequel, so let us recall a (classical) proof of it. We can write  $\beta = \mathcal{R}(\beta) + \gamma$  with  $\gamma \in I$ , and then we have to show  $\alpha\gamma \in I$ . We can reduce to the case where  $\gamma$  is contained in a simple summand  $V$  of  $I$ . Multiplication by the invariant element  $\alpha$  yields a surjective map  $V \rightarrow \alpha V$ , which is a morphism of  $U_q$ -modules. Since  $V$  is simple, it is either the 0 map, or an isomorphism. In either cases it follows  $\alpha V \subset I$  (in fact the first case cannot happen, for  $\mathcal{L}_{0,n}$  has no nontrivial zero divisors, as explained after (2.25)).

We can formulate the Reynolds operator in the following way. Recall that  $\mathcal{O}_q$  has a unique left (or right, or 2-sided) Haar integral, that is a linear map  $h: \mathcal{O}_q \rightarrow \mathbb{C}(q)$  such that

$$h(1) = 1 \quad \text{and} \quad (\text{id} \otimes h)\Delta(\alpha) = h(\alpha)1, \quad \forall \alpha \in \mathcal{O}_q.$$

(See, e.g., [35, Proposition 13.3.6].) It vanishes on all matrix coefficients except the one of the trivial representation, to which it gives the value 1. Denote by  $\Delta_{\mathcal{L}}: \mathcal{L}_{0,n} \rightarrow \mathcal{L}_{0,n} \otimes \mathcal{O}_q$  the right coaction dual to the action  $\text{coad}_n^r$  of  $U_q$  on  $\mathcal{L}_{0,n}$ . Then, in analogy with the formula of the averaging operator  $\mathcal{C}^\infty(G) \rightarrow \mathcal{C}^\infty(G)^G$ ,  $f \rightarrow [f] = \int_G f(g^{-1} \cdot g) d\mu(g)$ , for a locally compact group  $G$  with Haar measure  $d\mu(g)$ , it is straightforward that

$$\mathcal{R} = (\text{id} \otimes h)\Delta_{\mathcal{L}}. \quad (2.24)$$

Note that the complete reducibility of  $\mathcal{L}_{0,n}$  discussed after (2.22) follows also from Theorem 2.1 (1), since by (2.21) we have an isomorphism of  $U_q$ -modules

$$\mathcal{L}_{0,n} \xrightarrow{\Phi_n} (U_q(\mathfrak{g})^{\otimes n})^{\text{lf}} \xrightarrow{\psi_n^{-1}} U_q^{\text{lf}}(\mathfrak{g})^{\otimes n},$$

where lf means respectively locally finite for the action  $\text{ad}_n^r$  of  $U_q(\mathfrak{g})$  on  $U_q(\mathfrak{g})^{\otimes n}$ , and locally finite for the action  $\text{ad}^r$  of  $U_q(\mathfrak{g})$  on  $U_q(\mathfrak{g})$ . An explicit basis of  $\mathcal{M}_{0,n}$  is described in [18, Proposition 6.22].

Finally, let us point out here two important consequences of (2.20). First,  $\Phi_n$  yields isomorphisms between centers,  $\mathcal{Z}(\mathcal{L}_{0,n}) \cong \mathcal{Z}(U_q)^{\otimes n}$  and  $\mathcal{Z}(\mathcal{L}_{0,n}^{U_q}) \cong \mathcal{Z}((U_q^{\otimes n})^{U_q})$ , where one can show that [18, Lemma 6.29]

$$\mathcal{Z}((U_q^{\otimes n})^{U_q}) \cong \Delta^{(n)}(\mathcal{Z}(U_q)) \bigotimes_{\mathbb{C}(q)} \mathcal{Z}(U_q)^{\otimes n}. \quad (2.25)$$

Second,  $\mathcal{L}_{0,n}$  (and therefore  $\mathcal{M}_{0,n}$ ) has no nontrivial zero divisors because of the isomorphisms  $\Phi_n: \mathcal{L}_{0,n} \rightarrow (U_q^{\otimes n})^{\text{lf}} \subset U_q^{\otimes n}$  and  $U_q^{\otimes n} \cong U_q(\mathfrak{g}^{\oplus n})$ , and the fact that  $U_q(\mathfrak{g}^{\oplus n})$  has no nontrivial zero divisors (proved, e.g., in [39]).

## 2.2 Integral forms and specializations

Let  $A = \mathbb{C}[q, q^{-1}]$ . We call integral form of a (Hopf)  $\mathbb{C}(q)$ -algebra  $H$  a (Hopf)  $A$ -subalgebra  ${}_A H$  such that the canonical map  ${}_A H \otimes_A \mathbb{C}(q) \rightarrow H$  is an isomorphism. Note that the standard notion of integral form of  $\mathbb{C}(q)$ -algebra uses  $\mathbb{Z}[q, q^{-1}]$  instead of  $\mathbb{C}[q, q^{-1}]$ ; our choice is made for simplicity ( $\mathbb{C}[q, q^{-1}]$  is a principal ideal domain, whereas  $\mathbb{Z}[q, q^{-1}]$  is not).

### 2.2.1 Definitions

The *unrestricted* integral form of  $U_q$  is the  $A$ -subalgebra  $U_A = U_A(\mathfrak{g})$  introduced by De Concini–Kac–Procesi in [42, Section 12] (and in a differently normalized form in [39, 40]). It is the smallest  $A$ -subalgebra of  $U_q$  which contains the elements  $(i = 1, \dots, m)$

$$\bar{E}_i = (q_i - q_i^{-1})E_i, \quad \bar{F}_i = (q_i - q_i^{-1})F_i, \quad L_i, \quad L_i^{-1} \quad (2.26)$$

and is stable under the action of  $\mathcal{B}(\mathfrak{g})$  given by the Lusztig automorphisms (see (2.1)). Recall the root vectors  $E_{\beta_k}, F_{\beta_k}$  defined in (2.1). Let us put  $q_{\beta} := q^{(\beta, \beta)/2}$ . The algebra  $U_A$  is a free  $A$ -module with basis the monomials  $\bar{E}_{\beta_1}^{p_1} \dots \bar{E}_{\beta_N}^{p_N} K_{\lambda} \bar{F}_{\beta_N}^{n_N} \dots \bar{F}_{\beta_1}^{n_1}$ , where  $\lambda \in P$  and we set

$$\bar{E}_{\beta_k} = (q_{\beta_k} - q_{\beta_k}^{-1})E_{\beta_k}, \quad \bar{F}_{\beta_k} = (q_{\beta_k} - q_{\beta_k}^{-1})F_{\beta_k}.$$

We denote  $U_A^{\text{lf}} := U_A \cap U_q^{\text{lf}}$ . The unrestricted integral form of  $U_q^{\text{ad}}$  is defined similarly, as the smallest  $A$ -subalgebra  $U_A^{\text{ad}} \subset U_A$  which contains the elements  $\bar{E}_i, \bar{F}_i$  and  $K_i^{\pm 1}$ , for  $i = 1, \dots, m$ , and is stable under the Lusztig action of  $\mathcal{B}(\mathfrak{g})$ .

For  $\beta$  a positive root, we define the divided powers

$$E_{\beta}^{(k)} = \frac{E_{\beta}^k}{[k]_{q_{\beta}}!}, \quad F_{\beta}^{(k)} = \frac{F_{\beta}^k}{[k]_{q_{\beta}}!}, \quad k \in \mathbb{N}.$$

The Lusztig *restricted* integral form of  $U_q^{\text{ad}}$  [82, 83] (see also [35, Chapter 9.3]) is the  $A$ -subalgebra  $U_A^{\text{res}}$  generated by the elements  $(i = 1, \dots, m, k \in \mathbb{N}^*)$

$$E_i^{(k)} = \frac{E_i^k}{[k]_{q_i}!}, \quad F_i^{(k)} = \frac{F_i^k}{[k]_{q_i}!}, \quad K_i, \quad K_i^{-1}.$$

The algebra  $U_A^{\text{res}}$  is a free  $A$ -module with Poincaré–Birkhoff–Witt (PBW) basis

$$E_{\beta_1}^{(p_1)} \dots E_{\beta_N}^{(p_N)} \prod_{i=1}^m K_i^{\sigma_i} [K_i; t_i]_{q_i} F_{\beta_N}^{(n_N)} \dots F_{\beta_1}^{(n_1)},$$

where  $\sigma_i \in \{0, 1\}$ ,  $n_i, p_i, t_i \in \mathbb{N}$ , and we set  $[K_i; 0]_{q_i} := 1$  and

$$[K_i; t]_{q_i} = \prod_{s=1}^t \frac{K_i q_i^{-s+1} - K_i^{-1} q_i^{s-1}}{q_i^s - q_i^{-s}}.$$

The integral forms  $U_A(\mathfrak{h}), U_A(\mathfrak{b}_{\pm})$  and  $U_A^{\text{res}}(\mathfrak{h}), U_A^{\text{res}}(\mathfrak{b}_{\pm})$  associated to the subalgebras  $\mathfrak{h}, \mathfrak{b}_{\pm} \subset \mathfrak{g}$  are the subalgebras of  $U_A$  and  $U_A^{\text{res}}$ , respectively, defined in the obvious way. For instance, the ‘‘Cartan’’ subalgebra  $U_A^{\text{res}}(\mathfrak{h}) = U_q(\mathfrak{h}) \cap U_A^{\text{res}}$  is generated as a  $A$ -module by the elements  $\prod_{i=1}^m K_i^{\sigma_i} [K_i; t_i]_{q_i}$ .

Denote by  $\mathcal{C}_A$  the category of  $U_A^{\text{res}}$ -modules of *type 1*, i.e., free  $A$ -modules of finite rank which have a basis where the elements  $K_i$  act diagonally with eigenvalues of the form  $q_i^k$ ,  $k \in \mathbb{Z}$  (in general, finiteness of the rank imposes eigenvalues of the form  $\pm q_i^k$ ,  $k \in \mathbb{Z}$ ). The category  $\mathcal{C}_A$  is a rigid and tensor category. It is not semisimple, and this makes the study of  $\mathcal{C}_A$  a complicated task; for this, see [18], and Section 2.2.2 below. Every type 1 finite-dimensional simple  $U_q$ -module  $V_{\mu}$ ,  $\mu \in P_+$ , has a  $U_A^{\text{res}}$ -invariant full  $A$ -sublattice, that we denote by  ${}_A V_{\mu}$ . These  $U_A^{\text{res}}$ -modules form the simple objects of  $\mathcal{C}_A$ . Moreover,  $\mathcal{C}_A \otimes \mathbb{C}[q^{1/D}, q^{-1/D}]$  is a ribbon category (see Section 2.3).

The *integral* quantum function Hopf algebra  $\mathcal{O}_A = \mathcal{O}_A(G)$  is the (type 1) restricted dual of  $U_A^{\text{res}}$ , that is, the  $A$ -span of the matrix coefficients  $x \mapsto v^i(\pi_V(x)v_i)$ ,  $x \in U_A^{\text{res}}$ , for every module  $V$  in  $\mathcal{C}_A$ , where  $(v_i)$  is an  $A$ -basis of  $V$  and  $(v^i)$  the dual  $A$ -basis of the dual module  $V^*$  (compare with the definition of  $\mathcal{O}_q$ ). We can also regard  $\mathcal{O}_A$  as the set of  $A$ -linear maps  $f: U_A^{\text{res}} \rightarrow A$



such that  $\text{Ker}(f)$  contains a cofinite two sided ideal  $I$ , and  $\prod_{s=-r}^r (K_i - q_i^s) \in I$  for some  $r \in \mathbb{N}$  and every  $i$ . Because of the inclusions of  $U_A^{\text{res}}(\mathfrak{h})$ ,  $U_A^{\text{res}}(\mathfrak{n}_\pm)$ ,  $U_A^{\text{res}}(\mathfrak{b}_\pm)$  in  $U_A^{\text{res}}$ , there are Hopf epimorphisms from  $\mathcal{O}_A$  to the  $A$ -duals of these subalgebras, that we denote by  $\mathcal{O}_A(T_G)$ ,  $\mathcal{O}_A(U_\pm)$  and  $\mathcal{O}_A(B_\pm)$ , respectively.

The algebra  $\mathcal{O}_A$  has been introduced by Lusztig in [82, 83]. It is an integral form of  $\mathcal{O}_q$ , so  $\mathcal{O}_q = \mathcal{O}_A \otimes_A \mathbb{C}(q)$ .

$\mathcal{O}_A$  is also the restricted dual of the integral form  $\Gamma = \Gamma(\mathfrak{g})$  of  $U_q^{\text{ad}}$  introduced by De Concini–Lyubashenko in [41, Sections 2 and 3];  $\Gamma$  is the  $A$ -subalgebra of  $U_q^{\text{ad}}$  generated by the elements  $(i = 1, \dots, m)$

$$E_i^{(k)} = \frac{E_i^k}{[k]_{q_i}!}, \quad F_i^{(k)} = \frac{F_i^k}{[k]_{q_i}!}, \quad (K_i; t)_{q_i} = \prod_{s=1}^t \frac{K_i q_i^{-s+1} - 1}{q_i^s - 1}, \quad K_i^{-1},$$

where  $k \in \mathbb{N}$ ,  $t \in \mathbb{N}$  (setting  $(K_i; 0)_{q_i} = 1$  by convention). Note that the definition of  $\Gamma$  is less symmetric than that of  $U_A^{\text{res}}$ . However,  $\Gamma$  contains the elements  $K_i$ , and the commutation relations between the generators  $E_i^{(k)}$ ,  $F_i^{(k)}$  imply that the symmetrized elements  $[K_i; t]_{q_i}$  belong to  $\Gamma$ . In fact, let us denote  $\Gamma(\mathfrak{h}) = U_q(\mathfrak{h}) \cap \Gamma$  and  $\Gamma(\mathfrak{b}_\pm) = U_q(\mathfrak{b}_\pm) \cap \Gamma$ . It is proved in [41, Theorem 3.1] that  $\Gamma(\mathfrak{h})$  contains  $U_A^{\text{res}}(\mathfrak{h})$  and that the elements  $\prod_{i=1}^m K_i^{-\sigma(t_i)} (K_i; t_i)_{q_i}$ ,  $t_i \in \mathbb{N}$ , where  $\sigma(t)$  is the integer part of  $t/2$ , is an  $A$ -basis of  $\Gamma(\mathfrak{h})$ . A PBW basis of  $\Gamma$  is formed by the monomials

$$E_{\beta_1}^{(p_1)} \dots E_{\beta_N}^{(p_N)} \prod_{i=1}^m K_i^{-\sigma(t_i)} (K_i; t_i)_{q_i} F_{\beta_N}^{(n_N)} \dots F_{\beta_1}^{(n_1)}.$$

The inclusion  $U_A^{\text{res}} \subset \Gamma$  is strict, for the elements  $(K_i; t)_{q_i}$ ,  $t \neq 0$ , do not belong to  $U_A^{\text{res}}$ . However, the restriction functor  $\mathcal{C}_\Gamma \rightarrow \mathcal{C}_A$  is obviously an equivalence, where  $\mathcal{C}_\Gamma$  is the category of  $\Gamma$ -modules of *type 1*, i.e., free  $A$ -modules of finite rank which have a basis where the elements  $K_i$  act diagonally with eigenvalues of the form  $q_i^k$ ,  $k \in \mathbb{Z}$ . Therefore, we can identify the two categories, and  $\mathcal{O}_A$  with the (type 1) restricted dual of  $\Gamma$ . We will thus consider the  $U_A^{\text{res}}$ -modules  ${}_A V_\mu$ ,  $\mu \in P_+$ , equally as  $\Gamma$ -modules. We will sometimes use  $\Gamma$  instead of  $U_A^{\text{res}}$  in order to make direct the connection with results of De Concini–Lyubashenko about the integral pairings  $\pi_A^\pm$  considered in Section 2.3.

The *integral* form  $\mathcal{L}_{0,1}^A$  of  $\mathcal{L}_{0,1}$  is defined as the  $U_A^{\text{res}}$ -module  $\mathcal{O}_A$  endowed with the product of  $\mathcal{L}_{0,1}$ . The *integral* form  $\mathcal{L}_{0,n}^A$  of  $\mathcal{L}_{0,n}$  is the braided tensor product of  $n$  copies of  $\mathcal{L}_{0,1}^A$ ; in particular,  $\mathcal{L}_{0,n}^A = \mathcal{O}_A^{\otimes n}$  as  $U_A^{\text{res}}$ -modules. That the products of  $\mathcal{L}_{0,1}$  and  $\mathcal{L}_{0,n}$  are well defined over  $A$  was shown in [18, Proposition 6.9].

The *integral* quantum moduli algebra is

$$\mathcal{M}_{0,n}^A := (\mathcal{L}_{0,n}^A)^{U_A^{\text{res}}} = (\mathcal{L}_{0,n}^A)^{U_A}.$$

Finally, given  $q = \epsilon \in \mathbb{C}^\times$  we define the *specializations*  $U_\epsilon$ ,  $\Gamma_\epsilon$ ,  $\mathcal{O}_\epsilon$ ,  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{M}_{0,n}^{A,\epsilon}$  as the  $\mathbb{C}$ -algebras obtained by tensoring  $U_A$ ,  $\Gamma$ ,  $\mathcal{O}_A$ ,  $\mathcal{L}_{0,n}^A$  and  $\mathcal{M}_{0,n}^A$  respectively with  $\mathbb{C}_\epsilon$ , the  $A$ -module  $\mathbb{C}$  where  $q$  acts by multiplication by  $\epsilon$ . Each one can also be defined as the quotient by the ideal generated by  $q - \epsilon$ . We find convenient to use the notations

$$(U_A^{\otimes n})_\epsilon^{U_A} := (U_A^{\otimes n})^{U_A} \otimes_A \mathbb{C}_\epsilon, \quad (U^{\otimes n})_\epsilon^{\text{lf}} := (U_A^{\otimes n})^{\text{lf}} \otimes_A \mathbb{C}_\epsilon. \quad (2.27)$$

Let us stress here that when  $\epsilon$  is a root of unity, taking the locally finite part and taking the specialization at  $\epsilon$  are non commuting operations. Indeed, as shown by Theorem 2.27 below,  $U_\epsilon$  is finite over  $\mathcal{Z}_0(U_\epsilon)$  and therefore all its elements are locally finite for  $\text{ad}^r$ ; on another hand  $U_\epsilon^{\text{lf}} = U_A^{\text{lf}} \otimes_A \mathbb{C}_\epsilon$  does not contain the elements  $L_i$ .

Similarly, taking invariants and taking the specialization at  $\epsilon$  are non commuting operations when  $\epsilon$  is a root of unity: indeed, it is easily checked that in this case  $(U_A^{\otimes n})_\epsilon^{U_A}$  and  $(U_\epsilon^{\otimes n})^{U_\epsilon}$ , or  $\mathcal{M}_{0,n}^{A,\epsilon} = \mathcal{M}_{0,n}^A \otimes_A \mathbb{C}_\epsilon$  and  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ , are distinct spaces. When  $\epsilon$  is a root of unity, we will not consider the algebras  $\mathcal{M}_{0,n}^{A,\epsilon}$  in this paper.

Arguments similar to those mentioned at the end of Section 2.1 imply that the algebras  $\mathcal{L}_{0,n}^A$ ,  $\mathcal{M}_{0,n}^A$  and  $\mathcal{L}_{0,n}^{\epsilon'}$ ,  $\mathcal{M}_{0,n}^{A,\epsilon'}$ ,  $\epsilon' \in \mathbb{C}^\times$ , have no nontrivial zero divisors (see [18, Propositions 6.11 and 6.30]).

## 2.2.2 Canonical bases and modified quantum groups

Because the category  $\mathcal{C}_A$  is not semisimple, it is not clear from the above definition of  $\mathcal{O}_A$  whether or not it is a finitely generated algebra, if  $\mathcal{M}_{0,n}^A$  is a direct summand of the  $A$ -module  $\mathcal{L}_{0,n}^A$ , or if the projection map (2.8) may be refined to a morphism between underlying  $A$ -modules.

Such properties, using the appropriate formalism developed by Kashiwara–Lusztig, indeed hold true, and will play a key role later. We state them precisely in Proposition 2.10, Theorem 2.15 and Proposition 2.12. These results are consequences of the existence of an  $A$ -basis of  $\mathcal{O}_A$  with favourable properties, which implies in particular that  $\mathcal{O}_A$  is a free  $A$ -module. In order to introduce this  $A$ -basis it is necessary to consider a variant of  $U_q^{\text{ad}}$  introduced by Lusztig [83], called *modified quantum group*, and use the Kashiwara–Lusztig theory of canonical bases [65, 66, 67, 83]. We are going to recall the background material step by step.

The Lusztig *modified quantum group* is the  $\mathbb{C}(q)$ -algebra  $\dot{\mathbf{U}}$  obtained by replacing  $U_q^{\text{ad}}(\mathfrak{h})$  with the direct sum of infinitely many one-dimensional algebras, generated by orthogonal idempotents  $1_\lambda$  indexed by the elements  $\lambda$  of the weight lattice  $P$  [83, Chapter 23]. Namely, as a vector space  $\dot{\mathbf{U}} = \bigoplus_{\lambda', \lambda'' \in P} \lambda' \dot{\mathbf{U}}_{\lambda''}$ , where

$$\lambda' \dot{\mathbf{U}}_{\lambda''} = U_q^{\text{ad}} \left/ \left( \sum_{\alpha \in Q} (K_\alpha - q^{(\alpha, \lambda')}) U_q^{\text{ad}} + \sum_{\alpha \in Q} U_q^{\text{ad}} (K_\alpha - q^{(\alpha, \lambda'')}) \right) \right.$$

Denote by  $\pi_{\lambda', \lambda''}: U_q^{\text{ad}} \rightarrow \lambda' \dot{\mathbf{U}}_{\lambda''}$  the canonical projection. The product of  $\dot{\mathbf{U}}$  is given by  $\pi_{\lambda'_1, \lambda''_1}(s) \pi_{\lambda'_2, \lambda''_2}(t) = \pi_{\lambda'_1, \lambda''_2}(st)$  if  $\lambda''_1 = \lambda'_2$  and zero otherwise. Set  $1_\lambda := \pi_{\lambda, \lambda}(1)$ . The algebra  $\dot{\mathbf{U}}$  has not unit, but the family  $(1_\lambda)_{\lambda \in P}$  can be regarded as a substitute of it. Denote by  $\Delta$  the collection of maps

$$\Delta_{\lambda'_1, \lambda'_2, \lambda''_1, \lambda''_2}: \lambda'_1 + \lambda'_2 \dot{\mathbf{U}}_{\lambda''_1 + \lambda''_2} \rightarrow \lambda'_1 \dot{\mathbf{U}}_{\lambda''_1} \otimes \lambda'_2 \dot{\mathbf{U}}_{\lambda''_2}$$

such that

$$\Delta_{\lambda'_1, \lambda'_2, \lambda''_1, \lambda''_2} \pi_{\lambda'_1 + \lambda'_2, \lambda''_1 + \lambda''_2} = (\pi_{\lambda'_1, \lambda''_1} \otimes \pi_{\lambda'_2, \lambda''_2}) \Delta_{U_q^{\text{ad}}}, \quad (2.28)$$

where  $\Delta_{U_q^{\text{ad}}}$  is the coproduct of  $U_q^{\text{ad}}$ . We can regard  $\Delta$  as a (categorically completed) coproduct  $\Delta: \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}^{\otimes 2}$ . There is a natural structure of  $U_q^{\text{ad}}$ -bimodule on  $\dot{\mathbf{U}}$ , defined by

$$t' \pi_{\lambda', \lambda''}(s) t'' = \pi_{\lambda' + \nu', \lambda'' - \nu''}(t' s t'') \quad (2.29)$$

for all  $s \in U_q^{\text{ad}}$  and all elements  $t', t'' \in U_q^{\text{ad}}$  of respective weights  $\nu', \nu''$ . This structure affords a triangular decomposition of  $\dot{\mathbf{U}}$ : given bases  $\{b^\pm\}$  of  $U_q^{\text{ad}}(\mathfrak{n}_\pm)$ , the set of elements  $b^+ 1_\lambda b^-$  (or  $b^- 1_\lambda b^+$ , or  $b^+ b^- 1_\lambda$ ), where  $\lambda \in P$ , is a basis of  $\dot{\mathbf{U}}$ .

Given any  $U_q^{\text{ad}}$ -module  $X$  of type 1, and any weight subspace  $X^\lambda \subset X$  of weight  $\lambda \in P$ , one can define the action of an element  $u 1_\lambda \in \dot{\mathbf{U}}$ ,  $u \in U_q^{\text{ad}}$ , on  $X$  as the projection onto  $X^\lambda$  followed by the action of  $u$ . This way, one can identify the category  $\mathcal{C}$  with the one of finite-dimensional

unital  $\dot{\mathbf{U}}$ -modules, where unital means that all elements  $1_\lambda$  act as 0 but a finite number of them, and  $\sum_{\lambda \in P} 1_\lambda$  acts as the identity. It is proved in [83, Section 29.5.1], that

$$\mathcal{O}_q = \left\{ f: \dot{\mathbf{U}} \rightarrow \mathbb{C}(q) \mid \begin{array}{l} f \text{ is } \mathbb{C}(q)\text{-linear and vanishes on some} \\ \text{two-sided ideal of finite codimension of } \dot{\mathbf{U}} \end{array} \right\}.$$

There is an analogous realization of  $\mathcal{O}_A$ , of the form (see [83, Sections 23.2 and 29.5.2], and [84])

$$\mathcal{O}_A = \left\{ f: \dot{\mathbf{U}}_A \rightarrow A \mid \begin{array}{l} f \text{ is } A\text{-linear and vanishes on some} \\ \text{two-sided ideal of finite corank of } \dot{\mathbf{U}}_A \end{array} \right\},$$

where  $\dot{\mathbf{U}}_A$  is the  $A$ -subalgebra of  $\dot{\mathbf{U}}$  generated by the elements  $E_i^{(k)} 1_\lambda$  and  $F_i^{(k)} 1_\lambda$ , for all  $i \in \{1, \dots, m\}$ ,  $k \in \mathbb{N}$  and  $\lambda \in P$ . It is a  $U_A^{\text{res}}$ -subbimodule of  $\dot{\mathbf{U}}$ , and the coproduct restricts to a map  $\Delta: \dot{\mathbf{U}}_A \rightarrow \dot{\mathbf{U}}_A^{\otimes 2}$ . The above identification of the category  $\mathcal{C}$  with the one of finite-dimensional unital  $\dot{\mathbf{U}}$ -modules yields an identification of the category  $\mathcal{C}_A$  of  $U_A^{\text{res}}$ -modules of type 1 with the category of  $\dot{\mathbf{U}}_A$ -modules of finite rank.

The key advantage of this realization of  $\mathcal{O}_A$  is that  $\dot{\mathbf{U}}_A$  can be equipped with a canonical  $A$ -basis  $\dot{\mathbf{B}}$ . The construction of  $\dot{\mathbf{B}}$  is described in [83, Chapter 25]. It relies on the Kashiwara–Lusztig *canonical basis* of  $U_A^{\text{res}}(\mathfrak{n}_-)$ . This last basis, denoted by  $\mathbf{B}^-$ , is defined in [83, Chapter 14], and [65] (a review can be found in [35, Chapter 14]). It enjoys the following nice properties. Denote by  $\bar{\cdot}: \mathbb{C}(q) \rightarrow \mathbb{C}(q)$  the field involution such that  $\bar{q} = q^{-1}$ , and by  $\bar{\cdot}: U_q^{\text{ad}} \rightarrow U_q^{\text{ad}}$  the homomorphism of  $\mathbb{C}$ -algebras such that

$$\bar{E}_i = E_i, \quad \bar{F}_i = F_i, \quad \bar{K}_\lambda = K_{-\lambda}, \quad \overline{f\bar{x}} = \bar{f}x$$

for all  $f \in \mathbb{C}(q)$ ,  $x \in U_q^{\text{ad}}$  ( $\bar{E}_i$  and  $\bar{F}_i$  above, which will not appear elsewhere, should not be confused with the normalized elements in (2.26)). The map  $\bar{\cdot}$  yields a  $\mathbb{C}$ -algebra homomorphism  $\bar{\cdot}: \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$ . Then, we have

- (1) the elements of  $\mathbf{B}^-$  are weight vectors under the adjoint action of  $U_q^{\text{ad}}(\mathfrak{h})$ ;
- (2) for every  $b \in \mathbf{B}^-$ ,  $\bar{b} = b$ ;
- (3) for every  $b, b' \in \mathbf{B}^-$ ,  $bb' = \sum_{b'' \in \mathbf{B}^-} N_{b''}^{bb'} b''$  where  $N_{b''}^{bb'} \in \mathbb{Z}[q, q^{-1}]$ ;
- (4) for every  $b, b' \in \mathbf{B}^-$ ,  $\Delta(b) = \sum_{b', b'' \in \mathbf{B}^-} C_{b', b''}^b b' \otimes b''$  where  $C_{b', b''}^b \in \mathbb{Z}[q, q^{-1}]$ ;
- (5) for every  $\mu \in P^+$ , denoting by  $v_\mu$  the highest weight vector of the  $U_A^{\text{res}}$ -module  ${}_A V_\mu$ , the elements  $bv_\mu$  which are non-zero, where  $b \in \mathbf{B}^-$ , form an  $A$ -basis of  ${}_A V_\mu$ .

When  $\mathfrak{g}$  is simply laced, the coefficients  $N_{b''}^{bb'}$  and  $C_{b', b''}^b$  belong to  $\mathbb{N}[q, q^{-1}]$  [83, Theorem 14.3.13]. In the case of  $\mathfrak{g} = \mathfrak{sl}_2$ , the elements of  $\mathbf{B}^-$  are just the divided powers  $F^{(k)}$ ,  $k \in \mathbb{N}$ . Formulas in terms of PBW basis elements are known also for  $\mathfrak{g} = \mathfrak{sl}_3$  and  $\mathfrak{sl}_4$ , and an algorithm exists in the general case (see [43] and the references therein).

Correspondingly to  $\mathbf{B}^-$ , the set  $\mathbf{B}^+ = \omega(\mathbf{B}^-)$  is a basis of  $U_A^{\text{res}}(\mathfrak{n}_+)$ , where  $\omega: U_q^{\text{ad}} \rightarrow U_q^{\text{ad}}$  is the ( $\mathbb{C}(q)$ -linear) *Cartan automorphism*, defined by

$$\omega(E_i) = F_i, \quad \omega(F_i) = E_i, \quad \omega(K_i) = K_i^{-1}$$

for  $i = 1, \dots, m$ . The triangular decomposition of  $\dot{\mathbf{U}}$  implies that the elements  $b^+ 1_\lambda b'^-$ , where  $b^+ \in \mathbf{B}^+$ ,  $b'^- \in \mathbf{B}^-$  and  $\lambda \in P$ , form a basis of  $\dot{\mathbf{U}}$ . They form in fact an  $A$ -basis of  $\dot{\mathbf{U}}_A$ , and its elements are fixed by the involution  $\bar{\cdot}: \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$ .

Lusztig has constructed another  $A$ -basis of  $\dot{\mathbf{U}}_A$ , denoted  $\dot{\mathbf{B}}$ , and called the *canonical basis* of  $\dot{\mathbf{U}}_A$ . It satisfies numerous properties that we now review. Its elements are denoted by  $b \diamond_\lambda b'$ ,

where  $b, b' \in \mathbf{B}^-$  and  $\lambda \in P$ , and are related to the elements  $b^+b'^-1_\lambda$ , where  $b^+ := \omega(b)$  and  $b'^- := b'$ , by a specific trigonal change of basis with coefficients in  $A$ . Although we always have  $b^+1_\lambda, b'^-1_\lambda \in \mathbf{B}$ , to our knowledge explicit formulas of the elements of  $\mathbf{B}$  as linear combinations of elements  $b^+1_\lambda b'^-$  or  $b'^-1_\lambda b^+$  are known only for  $\mathfrak{g} = \mathfrak{sl}_2$  or  $\mathfrak{sl}_3$  (see [83, Section 25.3] and [37]). In the former case, identifying  $P$  with  $\mathbb{Z}$  and  $Q$  with  $2\mathbb{Z}$  the canonical basis  $\mathbf{B}$  is formed by the elements

$$E^{(k)}1_{-n}F^{(l)} \quad \text{and} \quad F^{(l)}1_nE^{(k)}, \quad k, l, n \in \mathbb{N}, \quad n \geq k + l,$$

where  $E^{(k)}1_{-n}F^{(l)} = F^{(l)}1_nE^{(k)}$  for  $n = k + l$ .

We are going to review Lusztig's construction of  $\mathbf{B}$ , its canonical partition  $\dot{\mathbf{B}} = \bigcup_{\lambda \in P_+} \dot{\mathbf{B}}[\lambda]$ , the dual basis  $\mathbf{B}^*$ , and Kashiwara's approach to  $\mathbf{B}^*$  [66, 67]. The latter is stated in Theorem 2.6 below. At first we need to recall the notions of based module and balanced triple; for details on these notions we refer to [83, Chapter 27] and [66] (see also [68], [104, Sections 3.15 and 3.16], or [35, Chapter 14] for overviews).

Denote by  $\mathcal{A}_0 \subset \mathbb{C}(q)$  the ring of rational functions regular at  $q = 0$ . By applying the involution  $\bar{\phantom{x}}$ , put  $\mathcal{A}_\infty = \overline{\mathcal{A}_0}$ . Since  $\mathcal{A}_0$  is the localization of  $\mathbb{C}[q]$  at  $q = 0$ , we may regard  $\mathcal{A}_\infty$  as the localization of  $\mathbb{C}[q^{-1}]$  at  $q = \infty$ .

Let us recall briefly the definition of crystal basis (see [65]). Denote by  $U_q^{\text{ad}}(\mathfrak{g})_i$  the subalgebra of  $U_q^{\text{ad}}(\mathfrak{g})$  generated by  $E_i, F_i$  and  $K_i^{\pm 1}$ ; thus  $U_q^{\text{ad}}(\mathfrak{g})_i$  is isomorphic to  $U_{q_i}(\mathfrak{sl}_2)$ . Let  $M$  be a  $U_q^{\text{ad}}$ -module of type 1. Denote  $M^\zeta$  the subspace of  $M$  of weight  $\zeta \in P$ . For every  $i = 1, \dots, m$ , we can regard  $M$  as a  $U_q^{\text{ad}}(\mathfrak{g})_i$ -module, so  $M \cong \bigoplus_j V_{\lambda_j}$  for some simple  $U_q^{\text{ad}}(\mathfrak{g})_i$ -modules  $V_{\lambda_j}$ . These being generated by primitive weight vectors, the PBW basis of  $U_q^{\text{ad}}(\mathfrak{g})_i$  yields

$$M = \bigoplus_{\zeta \in P} \bigoplus_{0 \leq n \leq (\check{\alpha}_i, \zeta)} F_i^{(n)}(\text{Ker}(E_i) \cap M^\zeta).$$

The *Kashiwara operators*  $\tilde{e}_i, \tilde{f}_i$  are the endomorphisms of  $M$  defined by, for every  $v \in \text{Ker}(E_i) \cap M^\zeta$  and  $0 \leq n \leq (\check{\alpha}_i, \zeta)$ ,

$$\tilde{f}_i(F_i^{(n)}v) = F_i^{(n+1)}v, \quad \tilde{e}_i(F_i^{(n)}v) = F_i^{(n-1)}v.$$

A *crystal basis* of  $M$  at  $q = 0$  consists of a pair  $(\mathcal{L}, \mathcal{B})$ , where

- $\mathcal{L}$  is a free  $\mathcal{A}_0$ -sublattice of  $M$  such that the canonical map  $\mathcal{L} \otimes_{\mathcal{A}_0} \mathbb{C}(q) \rightarrow M$  is an isomorphism;
- $\mathcal{B}$  is a basis of the  $\mathbb{C}$ -vector space  $\mathcal{L}/q\mathcal{L}$ ;
- $\mathcal{L} = \bigoplus_{\zeta \in P} \mathcal{L}^\zeta$  and  $\mathcal{B} = \coprod_{\zeta \in P} (\mathcal{B} \cap \mathcal{L}^\zeta / q\mathcal{L}^\zeta)$ , where  $\mathcal{L}^\zeta = \mathcal{L} \cap M^\zeta$ ;
- for every  $i = 1, \dots, m$  the Kashiwara operators  $\tilde{e}_i, \tilde{f}_i$  preserve  $\mathcal{L}$ , and the induced maps on  $\mathcal{L}/q\mathcal{L}$  send  $\mathcal{B}$  into  $\mathcal{B} \cup \{0\}$ , and satisfy  $b' = \tilde{f}_i(b)$  if and only if  $b = \tilde{e}_i(b')$  for every  $b, b' \in \mathcal{B}$ .

Crystal bases at  $q = \infty$  are defined similarly, by replacing  $\mathcal{A}_0$  with  $\mathcal{A}_\infty$  and  $q$  with  $q^{-1}$ .

A *based module* consists of a pair  $(M, B)$  where  $M$  is a  $U_q^{\text{ad}}$ -module of type 1 endowed with a  $\mathbb{C}(q)$ -basis  $B$  such that the following conditions hold:

- (i) For every weight  $\zeta \in P$ , the set  $B \cap M^\zeta$  is a basis of the weight subspace  $M^\zeta \subset M$ .
- (ii) The  $A$ -module  ${}_A M$  generated by  $B$  is stable under  $U_A^{\text{res}}$ .

We will denote by  $\mathcal{L}_M$  the  $\mathcal{A}_0$ -submodule of  $M$  generated by  $B$ , and by  $\bar{\mathcal{L}}_M$  the  $\mathcal{A}_\infty$ -submodule of  $M$  generated by  $B$ .

- (iii) The  $\mathbb{C}$ -linear involution  $\bar{\phantom{x}} : M \rightarrow M$  defined by  $\overline{fb} = \bar{f}b$  for all  $f \in \mathbb{C}(q)$  and  $b \in B$  is compatible with the action of  $U_q^{\text{ad}}$  in the sense that  $\overline{x\bar{m}} = \bar{x}m$  for all  $x \in U_q^{\text{ad}}$ ,  $m \in M$ .

- (iv) The  $\mathcal{A}_\infty$ -submodule  $\bar{\mathcal{L}}_M$  of  $M$  together with the image of  $B$  in  $\bar{\mathcal{L}}_M/q^{-1}\bar{\mathcal{L}}_M$  forms a crystal basis of  $M$  at  $q = \infty$ .

If  $(M, B)$  is a based module, we will denote by  $\bar{B}$  the image of  $B$  in  $\bar{\mathcal{L}}_M/q^{-1}\bar{\mathcal{L}}_M$ . From the notion of balanced triple that we recall now, denoting by  $\mathcal{B}$  the image of  $B$  in  $\mathcal{L}_M/q\mathcal{L}_M$ , we see that  $(\mathcal{L}_M, \mathcal{B})$  is a crystal basis at  $q = 0$ .

Indeed, consider more generally a  $\mathbb{C}(q)$ -vector space  $V$ , finite-dimensional or not, a sub- $A$ -module  ${}_A V$ , a sub- $\mathcal{A}_0$ -module  ${}_{\mathcal{A}_0} V$  and a sub- $\mathcal{A}_\infty$ -module  ${}_{\mathcal{A}_\infty} V$  satisfying the conditions (all isomorphisms being the canonical maps)

$$V \cong \mathbb{C}(q) \bigotimes_A {}_A V, \quad V \cong \mathbb{C}(q) \bigotimes_{\mathcal{A}_0} {}_{\mathcal{A}_0} V, \quad V \cong \mathbb{C}(q) \bigotimes_{\mathcal{A}_\infty} {}_{\mathcal{A}_\infty} V.$$

Consider the  $\mathbb{C}$ -vector space  $E := {}_A V \cap {}_{\mathcal{A}_0} V \cap {}_{\mathcal{A}_\infty} V$ . Then  $({}_A V, {}_{\mathcal{A}_0} V, {}_{\mathcal{A}_\infty} V)$  is a *balanced triple* [65, 66] if the canonical maps

$$A \bigotimes_{\mathbb{C}} E \rightarrow {}_A V, \quad \mathcal{A}_0 \bigotimes_{\mathbb{C}} E \rightarrow {}_{\mathcal{A}_0} V, \quad \mathcal{A}_\infty \bigotimes_{\mathbb{C}} E \rightarrow {}_{\mathcal{A}_\infty} V \quad (2.30)$$

are isomorphisms. Equivalently,  $({}_A V, {}_{\mathcal{A}_0} V, {}_{\mathcal{A}_\infty} V)$  is balanced if and only if the canonical map  $E \rightarrow {}_{\mathcal{A}_0} V/q{}_{\mathcal{A}_0} V$  is an isomorphism, if and only if the canonical map  $E \rightarrow {}_{\mathcal{A}_\infty} V/q^{-1}{}_{\mathcal{A}_\infty} V$  is an isomorphism [66, Lemma 2.1.1].

Given a based module  $(M, B)$ , the elements of  $B$  are weight vectors and  $\bar{b} = b$  for every  $b \in B$ . Also, if an element  $m \in {}_A M$  satisfies  $\bar{m} = m$  and  $m \in B + q^{-1}\bar{\mathcal{L}}_M$ , then  $m \in B$  (see [83, Section 27.1.5] for details on this fact). It follows that the canonical quotient map

$${}_A M \cap \mathcal{L}_M \cap \bar{\mathcal{L}}_M \rightarrow \bar{\mathcal{L}}_M/q^{-1}\bar{\mathcal{L}}_M \quad (2.31)$$

is an isomorphism of  $\mathbb{C}$ -vector spaces. This provides another way of viewing based modules: by (2.31),  $({}_A M, \mathcal{L}_M, \bar{\mathcal{L}}_M)$  is a balanced triple, and by (2.30) the  $A$ -lattice  ${}_A M$  is completely determined by the crystal base  $(\bar{\mathcal{L}}_M, \bar{B})$ . We will say that  $(\bar{\mathcal{L}}_M, \bar{B})$  (or the corresponding crystal base  $(\mathcal{L}_M, \mathcal{B})$  at  $q = 0$ ) is *melted into* the based module  $(M, B)$ .

We will indifferently apply the notion of based module to finite-dimensional unital  $\dot{U}$ -modules, since they are equivalent to  $U_q^{\text{ad}}$ -modules of type 1.

Every module  $V_\mu$ ,  $\mu \in P^+$ , supports a structure of based module (see [83, Section 14.4.10] and [65]); the corresponding basis, called *canonical basis* and that we will denote by  $\underline{\mathbf{B}}_\mu$ , is formed by the elements  $bv_\mu \in {}_A V_\mu$  which are non-zero, where  $b \in \mathbf{B}^-$  and  $v_\mu$  is the canonical highest weight vector of  $V_\mu$ , corresponding to the coset of  $1 \in U_q^{\text{ad}}(\mathfrak{n}_-)$  in the Verma module construction of  $V_\mu$ . Note that the involution  $\bar{\cdot} : V_\mu \rightarrow V_\mu$  defined by (iii) above is indeed an automorphism, for the space  $V_\mu$  with action of  $U_q^{\text{ad}}$  defined by  $x \cdot v := \bar{x}v$ , for all  $x \in U_q^{\text{ad}}$ ,  $v \in V_\mu$ , has highest weight  $\mu$ , and is thus isomorphic to  $V_\mu$  via the map  $\bar{\cdot}$ . The crystal base  $(\mathcal{L}_\mu^{\text{low}}, \mathcal{B}_\mu^{\text{low}})$  at  $q = 0$  is formed by the  $\mathcal{A}_0$ -sublattice  $\mathcal{L}_\mu^{\text{low}}$  of  $V_\mu$  generated by  $\underline{\mathbf{B}}_\mu$  (which is eventually the same as the  $\mathcal{A}_0$ -sublattice generated by the vectors of the form  $\tilde{f}_{i_1} \circ \cdots \circ \tilde{f}_{i_k}(v_\mu)$ , where  $i_1, \dots, i_k \in \{1, \dots, m\}$ ), and  $\mathcal{B}_\mu^{\text{low}}$  is the set of non-zero images of these vectors in  $\mathcal{L}_\mu^{\text{low}}/q\mathcal{L}_\mu^{\text{low}}$ .

There is the following uniqueness result [65, Theorem 3].

**Theorem 2.5.** *Let  $M$  be a  $U_q^{\text{ad}}$ -module of type 1, and  $(\mathcal{L}, \mathcal{B})$  a crystal base at  $q = 0$  of  $M$ . Then there exists a  $\mathbb{C}(q)$ -isomorphism  $M \rightarrow \bigoplus_j V_{\lambda_j}$  by which  $(\mathcal{L}, \mathcal{B})$  is  $\mathcal{A}_0$ -isomorphic to  $\bigoplus_j (\mathcal{L}_{\lambda_j}^{\text{low}}, \mathcal{B}_{\lambda_j}^{\text{low}})$ .*

The based modules form a category. Given based modules  $(M, B)$  and  $(M', B')$ , a morphism of  $U_q^{\text{ad}}$ -modules  $f : M \rightarrow M'$  is a morphism of based modules if

- (a)  $f(b) \in B' \cup \{0\}$  for any  $b \in B$ ;
- (b)  $B \cap \text{Ker}(f)$  is a basis of  $\text{Ker}(f)$ .

The direct sum of based modules  $(M, B)$  and  $(M', B')$  is a based module  $(M \oplus M', B \cup B')$ ; and a submodule  $M'$  of a based module  $(M, B)$  spanned over  $\mathbb{C}(q)$  by a subset  $B'$  of  $B$  forms a based module  $(M', B')$ . The quotient module  $M/M'$  together with the image of  $B \setminus B'$  is then a based module.

The tensor product of based modules  $(M, B)$ ,  $(M', B')$  is also defined. Namely, consider the  $\mathbb{C}$ -linear map  $\Psi: M \otimes M' \rightarrow M \otimes M'$  defined by

$$\Psi(m \otimes m') = \hat{R}^{-1}(\bar{m} \otimes \bar{m}'),$$

where  $\hat{R} = \Theta^{-1}R$ , see (2.4) (note that, as we use the coproduct opposite to [83] our quasi- $R$ -matrix is  $\hat{R}^{-1}$ ). It can be checked that  $\Psi$  is an involution compatible with the action of  $\dot{\mathbf{U}}$  in the sense of (iii) above in the definition of based module. Moreover, denote by  $\mathcal{L}_{M, M'}$  the  $\mathbb{C}[q^{-1}]$ -submodule of  $M \otimes M'$  spanned by the basis elements  $b \otimes b'$ , where  $b \in B$ ,  $b' \in B'$ . It is shown in [83, Section 27.3], that for every pair  $(b, b') \in B \times B'$  there is a unique element  $b \diamond b' \in \mathcal{L}_{M, M'}$  such that

- (a)  $\Psi(b \diamond b') = b \otimes b'$ ,
- (b)  $b \diamond b' - b \otimes b' \in q^{-1}\mathcal{L}_{M, M'}$ .

Moreover,  $B_\diamond = \{b \diamond b', b \in B, b' \in B'\}$  is a basis of  $M \otimes M'$ , a  $\mathbb{C}[q^{-1}]$ -basis of  $\mathcal{L}_{M, M'}$ , a  $\mathbb{C}[q, q^{-1}]$ -basis of the  $\mathbb{C}[q, q^{-1}]$ -module  ${}_A\mathcal{L}_{M, M'}$  of  $M \otimes M'$  generated by the elements  $b \otimes b'$ , where  $b \in B$ ,  $b' \in B'$ , and  $(M \otimes M', B_\diamond)$  is a based module.

This construction of  $B_\diamond$  is associative. Since  $(V_\mu, \mathbf{B}_\mu)$  is for every  $\mu \in P_+$  a based module, it follows that any tensor product  $M$  of a finite number of the simple modules  $V_\mu$  is naturally a based module. The corresponding basis is called *the canonical basis* of  $M$ . These canonical basis have been computed explicitly in [56] in the case  $\mathfrak{g} = \mathfrak{sl}_2$ .

Consider now the  $U_q^{\text{ad}}$ -module  ${}^\omega V_\mu$  with underlying space  $V_\mu$ ,  $\mu \in P_+$ , and action defined by  $x \cdot {}^\omega v := \omega(x)v$ , for every  $x \in U_q^{\text{ad}}$  and  $v \in V_\mu$  (as usual  $\omega: U_q^{\text{ad}} \rightarrow U_q^{\text{ad}}$  is the Cartan automorphism). Note that there are isomorphisms  ${}^\omega V_\mu \cong V_{-w_0(\mu)} \cong V_\mu^*$  (endowed with the standard left action of  $U_q^{\text{ad}}$ ). Let us denote by  ${}^\omega v_\mu$  the vector  $v_\mu$  regarded in  ${}^\omega V_\mu$  (i.e., its canonical lowest weight vector), and by  ${}^\omega \mathbf{B}_\mu := \{b \cdot {}^\omega v_\mu, b \in \mathbf{B}^+\} \setminus \{0\}$  its canonical basis; note that  ${}^\omega \mathbf{B}_\mu = \{\omega(b)v_\mu, b \in \omega(\mathbf{B}^-)\} \setminus \{0\} = \{bv_\mu, b \in \mathbf{B}^-\} \setminus \{0\} = \mathbf{B}_\mu$ . Then  ${}^\omega V_{\mu'} \otimes V_{\mu''}$  has the canonical basis  $\mathbf{B}_{\mu', \mu''} := \{\underline{b}' \diamond \underline{b}'', \underline{b}' \in {}^\omega \mathbf{B}_{\mu'}, \underline{b}'' \in \mathbf{B}_{\mu''}\}$ . Since  $\underline{b}' \diamond \underline{b}''$  is canonically determined by the elements  $\underline{b}', \underline{b}'' \in \mathbf{B}^-$  such that  $\underline{b}' = \omega(b') \cdot {}^\omega v_{\mu'}$ ,  $\underline{b}'' = b'' v_{\mu''}$ , following Lusztig we denote it by  $(b' \diamond b'')_{\mu', \mu''}$ .

Denote by  $v_{w_0(\mu)}$  the canonical lowest weight vector of  $V_\mu$ , and by  ${}^\omega v_{w_0(\mu)}$  the vector  $v_{w_0(\mu)}$  regarded in  ${}^\omega V_\mu$ . It is a crucial observation that  ${}^\omega v_{w_0(\mu')} \otimes v_{w_0(\mu'')}$  is a cyclic vector of  ${}^\omega V_{\mu'} \otimes V_{\mu''}$  (see, e.g., [83, Proposition 23.3.6]; note that  ${}^\omega v_{w_0(\mu')} \otimes v_{w_0(\mu'')}$  plays the role of  $\xi_{-\mu'} \otimes \eta_{\mu''} := {}^\omega v_{\mu'} \otimes v_{\mu''}$  in [83], because we use opposite coproducts on  $U_q^{\text{ad}}$  but the factors  ${}^\omega V_{\mu'}$  and  $V_{\mu''}$  are ordered in the same way).

We can now state the definition of the canonical basis  $\dot{\mathbf{B}}$  of  $\dot{\mathbf{U}}$ : each element  $u$  of  $\dot{\mathbf{B}}$  belongs to  $\dot{\mathbf{U}}_A 1_\zeta$  for some (unique)  $\zeta \in P$ , and it is then uniquely determined by the property that, for every  $\mu', \mu'' \in P^+$  such that  $w_0(\mu'' - \mu') = \zeta$ , we have

$$u({}^\omega v_{w_0(\mu')} \otimes v_{w_0(\mu'')}) = (b' \diamond b'')_{\mu', \mu''} \tag{2.32}$$

for some  $(b' \diamond b'')_{\mu', \mu''} \in \mathbf{B}_{\mu', \mu''}$  [83, Section 25.2]. We write  $u = b' \diamond_\zeta b''$ , and as in [84] we denote by  $\dot{\mathbf{B}}_{\mu', \mu''}$  the finite subset of  $\mathbf{B}$  which is in bijection with  $\mathbf{B}_{\mu', \mu''}$  under the map  $u \mapsto u({}^\omega v_{w_0(\mu')} \otimes v_{w_0(\mu'')})$ . So

$$\dot{\mathbf{B}} = \bigcup_{\mu', \mu'' \in P_+} \dot{\mathbf{B}}_{\mu', \mu''}. \tag{2.33}$$

Note in particular that  $\dot{\mathbf{B}}$  is formed by weight vectors for the left and right action of  $U_q^{\text{ad}}(\mathfrak{h})$  (defined as usual by (2.29)).

In a sense, one can view  $\dot{\mathbf{U}}$  as the projective limit of an inverse system formed by the  $(U_q^{\text{ad}} \otimes U_q^{\text{ad}})$ -modules  ${}^\omega V_{\mu'} \otimes V_{\mu''}$ , where  $\mu', \mu'' \in P^+$ ; then  $\dot{\mathbf{B}}$  is the basis resulting from the corresponding inverse system of basis  $\{\dot{\mathbf{B}}_{\mu', \mu''}\}_{\mu', \mu''}$ .

Lusztig has produced a partition of  $\dot{\mathbf{B}}$  as follows. First, consider the situation of a based module  $(M, B)$ . For every  $\lambda \in P_+$  denote by  $M[\lambda]$  the sum of the simple submodules of  $M$  isomorphic to  $V_\lambda$  (i.e., its isotypical component). Set

$$M[\geq \lambda] = \bigoplus_{\lambda' \geq \lambda} M[\lambda']. \quad (2.34)$$

Then, for every base element  $b \in B$  there is a unique  $\lambda \in P_+$  such that  $b \in M[\geq \lambda]$  and  $\lambda$  is maximal with this property [83, Section 27.2]. Denote by  $B[\lambda]$  the set of all  $b \in B$  that give rise to  $\lambda \in P_+$  in this way. Clearly, the sets  $B[\lambda]$ ,  $\lambda \in P_+$ , form a partition of  $B$ .

Now, given  $b \in \dot{\mathbf{B}}$ , let  $\zeta \in P$  be the unique weight such that  $b \in \dot{\mathbf{U}}_A 1_\zeta$ , and let  $\mu', \mu'' \in P^+$  be such that  $w_0(\mu'' - \mu') = \zeta$ , and  $(\check{\alpha}_i, \mu')$  is large enough for all  $i = 1, \dots, m$  so that  $u({}^\omega v_{w_0(\mu')} \otimes v_{w_0(\mu'')})$  is non-zero. This element belongs to the canonical basis  $\underline{\mathbf{B}}_{\mu', \mu''}$  of  ${}^\omega V_{\mu'} \otimes V_{\mu''}$ , and therefore to one of the subsets  $\underline{\mathbf{B}}_{\mu', \mu''}[\lambda]$ , for a unique  $\lambda \in P_+$ . It is a result that  $\lambda$  does not depend on the choice of  $\mu', \mu''$  (see [83, Section 29.1.1]). Hence there is a well-defined map  $\dot{\mathbf{B}} \rightarrow P_+$ ,  $b \mapsto \lambda$ . Denoting by  $\dot{\mathbf{B}}[\lambda]$  the fiber of this map, we thus obtain a partition

$$\dot{\mathbf{B}} = \coprod_{\lambda \in P_+} \dot{\mathbf{B}}[\lambda]. \quad (2.35)$$

The sets  $\dot{\mathbf{B}}[\lambda]$  are called *2-sided cells*. They are finite sets and have the following remarkable properties. For every  $\lambda \in P_+$  denote by  $\dot{\mathbf{U}}[\geq \lambda]$  and  $\dot{\mathbf{U}}[> \lambda]$  the subspaces of  $\dot{\mathbf{U}}$  spanned by  $\coprod_{\lambda' \geq \lambda} \dot{\mathbf{B}}[\lambda']$  and  $\coprod_{\lambda' > \lambda} \dot{\mathbf{B}}[\lambda']$  respectively. Then  $\dot{\mathbf{U}}[\geq \lambda]$  (respectively  $\dot{\mathbf{U}}[> \lambda]$ ) consists of the elements  $u \in \dot{\mathbf{U}}$  such that if  $u$  acts on  $V_\mu$  by a non-zero linear map, then  $\mu \geq \lambda$  (respectively  $\mu > \lambda$ ) [83, Lemmas 29.1.3 and 29.1.4]. Both  $\dot{\mathbf{U}}[\geq \lambda]$  and  $\dot{\mathbf{U}}[> \lambda]$  are two-sided ideals of  $\dot{\mathbf{U}}$ . Moreover, the algebra homomorphism  $\pi_\lambda: \dot{\mathbf{U}}[\geq \lambda] \rightarrow \text{End}(V_\lambda)$  given by the  $\dot{\mathbf{U}}$ -module structure on  $V_\lambda$  descends to an algebra and  $U_q^{\text{ad}}$ -bimodule isomorphism (keeping the same notation) [83, Proposition 29.2.2]

$$\bar{\pi}_\lambda: \dot{\mathbf{U}}[\geq \lambda] / \dot{\mathbf{U}}[> \lambda] \rightarrow \text{End}(V_\lambda). \quad (2.36)$$

For instance, when  $\mathfrak{g} = \mathfrak{sl}_2$  the 2-sided cell  $\dot{\mathbf{B}}[n]$  associated to the simple  $U_q^{\text{ad}}(\mathfrak{sl}_2)$ -module of type 1 and dimension  $n + 1$  is the set of cardinality  $(n + 1)^2$  given by [83, Section 29.4.3]

$$\dot{\mathbf{B}}[n] = \{E^{(k)} 1_{-n} F^{(l)}, n \geq k + l\} \cup \{F^{(l)} 1_n E^{(k)}, n \geq k + l\}, \quad (2.37)$$

with the identification  $E^{(k)} 1_{-n} F^{(l)} = F^{(l)} 1_n E^{(k)}$  when  $n = k + l$ . As we are mainly interested in  $\mathcal{O}_A$ , we are going to describe the dual partition of  $\dot{\mathbf{B}}^*$ , see Theorem 2.6. The duality with (2.35) is discussed after that theorem.

First, we follow the approach of Kashiwara [66, 67]. For every  $\lambda \in P_+$ , denote by  $V_\lambda^r$  the dual space of  $V_\lambda$  endowed with its natural structure of right  $U_q^{\text{ad}}$ -module, defined by  $(fx)(v) = f(xv)$  for every  $f \in V_\lambda^r$ ,  $x \in U_q^{\text{ad}}$ ,  $v \in V_\lambda$ . Clearly,  $V_\lambda^r$  is a simple module of highest weight  $\lambda$ . Let  $\varphi: U_q^{\text{ad}} \rightarrow U_q^{\text{ad}}$  be the anti-automorphism of  $\mathbb{C}(q)$ -algebra given by  $\varphi(E_i) = F_i$ ,  $\varphi(F_i) = E_i$ ,  $\varphi(K_\lambda) = K_\lambda$ . By using  $\varphi$ , any right  $U_q^{\text{ad}}$ -module can be considered as a left  $U_q^{\text{ad}}$ -module. In particular, by the Verma module construction of  $V_\lambda$  it follows

$$V_\lambda^r \cong U_q^{\text{ad}} \left/ \left( \sum_{\mu \in P_+} (K_\mu - q^{(\lambda, \mu)}) U_q^{\text{ad}} + \sum_{i=1}^m E_i^{1+(\check{\alpha}_i, \lambda)} U_q^{\text{ad}} \right) \right.,$$

and  $\varphi$  affords an isomorphism of the right module  $V_\lambda^r$  with the *left* module  $V_\lambda$ . We will denote by  $f_\lambda$  the unique highest weight vector of  $V_\lambda^r$  satisfying  $\langle f_\lambda, v_\lambda \rangle = 1$ .

The space  $V_\lambda^r \otimes V_\lambda$  can be identified with  $\text{End}(V_\lambda)^*$ , and thus acquires by duality a natural structure of  $U_q^{\text{ad}}$ -bimodule (or equivalently left  $U_q^{\text{ad}} \otimes (U_q^{\text{ad}})^{\text{op}}$ -module); the left and right actions are given by

$$x(f \otimes v)y = fy \otimes xv \quad (2.38)$$

for every  $x, y \in U_q^{\text{ad}}$ ,  $f \in V_\lambda^r$ ,  $v \in V_\lambda$ . The space  $V_\lambda^r \otimes V_\lambda$  also acquires by duality a natural ‘‘upper’’ crystal structure over  $U_q^{\text{ad}} \otimes (U_q^{\text{ad}})^{\text{op}}$ , as we explain now. Denote by  $\langle \cdot, \cdot \rangle_\lambda: V_\lambda \times V_\lambda \rightarrow \mathbb{C}(q)$  the unique symmetric bilinear form such that

$$\langle v_\lambda, v_\lambda \rangle_\lambda = 1 \quad \text{and} \quad \langle \varphi(x)u, v \rangle_\lambda = \langle u, xv \rangle_\lambda \quad (2.39)$$

for every  $u, v \in V_\lambda$  and  $x \in U_q^{\text{ad}}$ . Recall the crystal base  $(\mathcal{L}_\mu^{\text{low}}, \mathcal{B}_\mu^{\text{low}})$  at  $q = 0$  introduced before Theorem 2.5. In Kashiwara’s terminology [65, 66], the pair  $(\mathcal{L}_\lambda^{\text{low}}, \mathcal{B}_\lambda^{\text{low}})$  is the *lower crystal base of  $V_\lambda$*  at  $q = 0$ . Applying the involution  $\bar{\cdot}: V_\lambda \rightarrow V_\lambda$ , one obtains the lower crystal base  $(\overline{\mathcal{L}_\lambda^{\text{low}}}, \overline{\mathcal{B}_\lambda^{\text{low}}})$  at  $q = \infty$ . Because the canonical bases are determined by the crystal bases (see the discussion about (2.31)), we call  $(V_\lambda, \underline{\mathbf{B}}_\lambda)$  the *lower based module of  $V_\lambda$* , and  $\underline{\mathbf{B}}_\lambda$  the *lower canonical basis* of  $V_\lambda$ .

Put

$$\begin{aligned} {}_A V_\lambda^{\text{up}} &:= \{v \in V_\lambda, \langle v, {}_A V_\lambda \rangle_\lambda \subset A\}, & \mathcal{L}_\lambda^{\text{up}} &:= \{v \in V_\lambda, \langle v, \mathcal{L}_\lambda^{\text{low}} \rangle_\lambda \subset \mathcal{A}_0\}, \\ \overline{\mathcal{L}_\lambda^{\text{up}}} &:= \{v \in V_\lambda, \langle v, \overline{\mathcal{L}_\lambda^{\text{low}}} \rangle_\lambda \subset \mathcal{A}_\infty\}. \end{aligned} \quad (2.40)$$

Then  $({}_A V_\lambda^{\text{up}}, \mathcal{L}_\lambda^{\text{up}}, \overline{\mathcal{L}_\lambda^{\text{up}}})$  is a balanced triple [66, Lemma 4.2.1]. Denote by  $\mathcal{B}_\lambda^{\text{up}}$  the basis of  $\mathcal{L}_\lambda^{\text{up}}/q\mathcal{L}_\lambda^{\text{up}}$  dual to  $\mathcal{B}_\lambda^{\text{low}}$  by the induced pairing  $\langle \cdot, \cdot \rangle_\lambda: \mathcal{L}_\lambda^{\text{up}}/q\mathcal{L}_\lambda^{\text{up}} \times \mathcal{L}_\lambda^{\text{low}}/q\mathcal{L}_\lambda^{\text{low}} \rightarrow \mathbb{C}$ . The pair  $(\mathcal{L}_\lambda^{\text{up}}, \mathcal{B}_\lambda^{\text{up}})$  is the *upper crystal base* of  $V_\lambda$  at  $q = 0$ . The weight spaces of the  $\mathcal{A}_0$ -modules  $\mathcal{L}_\lambda^{\text{low}}$  and  $\mathcal{L}_\lambda^{\text{up}}$  are related by

$$(\mathcal{L}_\lambda^{\text{up}})^\mu = q^{\frac{\langle \lambda, \lambda \rangle}{2} - \frac{\langle \mu, \mu \rangle}{2}} (\mathcal{L}_\lambda^{\text{low}})^\mu, \quad \mu \in P. \quad (2.41)$$

Correspondingly, denoting  $(\mathcal{B}_\lambda^{\text{up}})^\mu := \mathcal{B}_\lambda^{\text{up}} \cap (\mathcal{L}_\lambda^{\text{up}})^\mu$  and  $(\mathcal{B}_\lambda^{\text{low}})^\mu := \mathcal{B}_\lambda^{\text{low}} \cap (\mathcal{L}_\lambda^{\text{low}})^\mu$ , we have (see [65] and [66, equation (4.2.9)])

$$(\mathcal{B}_\lambda^{\text{up}})^\mu = q^{\frac{\langle \lambda, \lambda \rangle}{2} - \frac{\langle \mu, \mu \rangle}{2}} (\mathcal{B}_\lambda^{\text{low}})^\mu.$$

The  $A$ -module  ${}_A V_\lambda^{\text{up}}$  is characterized by the following two properties [66, equations (4.2.10)–(4.2.12)]:

$$({}_A V_\lambda^{\text{up}})^\lambda = \mathbb{C}[q, q^{-1}]v_\lambda, \quad ({}_A V_\lambda^{\text{up}})^\mu = \{v \in V_\lambda \mid U_A^{\text{res}}(\mathfrak{n}^+)^{\lambda-\mu}v \in \mathbb{C}[q, q^{-1}]v_\lambda\},$$

where  $U_A^{\text{res}}(\mathfrak{n}^+)^\gamma = \{u \in U_A^{\text{res}}(\mathfrak{n}^+) \mid \forall \nu \in P, K_\nu u K_\nu^{-1} = q^{(\nu, \gamma)}u\}$ . Denote by  $\underline{\mathbf{B}}_\lambda^{\text{up}}$  the inverse image of  $\mathcal{B}_\lambda^{\text{up}}$  by the isomorphism  ${}_A V_\lambda^{\text{up}} \cap \mathcal{L}_\lambda^{\text{up}} \cap \overline{\mathcal{L}_\lambda^{\text{up}}} \rightarrow \mathcal{L}_\lambda^{\text{up}}/q\mathcal{L}_\lambda^{\text{up}}$ . By (2.30), the set  $\underline{\mathbf{B}}_\lambda^{\text{up}}$  is a basis of  ${}_A V_\lambda^{\text{up}}$ ; we call it the *upper canonical basis* of  $V_\lambda$ . In the appendix, we describe in details the  $\mathfrak{sl}_2$  case.

Similarly, the right module  $V_\lambda^r$  with its canonical basis  $\underline{\mathbf{B}}_\lambda^r = \{f_\lambda b, b \in \mathbf{B}^+\} \setminus \{0\}$  has the lower crystal base  $(\mathcal{L}_\lambda^{r, \text{low}}, \mathcal{B}_\lambda^{r, \text{low}})$ , and it supports a balanced triple  $({}_A V_\lambda^{\text{up}}, \mathcal{L}_\lambda^{r, \text{up}}, \overline{\mathcal{L}_\lambda^{r, \text{up}}})$  defined again by duality. We denote by  $(\mathcal{L}_\lambda^{r, \text{up}}, \mathcal{B}_\lambda^{r, \text{up}})$  and  $\underline{\mathbf{B}}_\lambda^{r, \text{up}}$  the corresponding crystal base and upper canonical basis of  $V_\lambda^r$ , respectively.

It follows that  $({}_A V_\lambda^{r, \text{up}} \otimes_{{}_A A} V_\lambda^{\text{up}}, \mathcal{L}_\lambda^{r, \text{up}} \otimes_{\mathcal{A}_0} \mathcal{L}_\lambda^{\text{up}}, \overline{\mathcal{L}_\lambda^{r, \text{up}}} \otimes_{\mathcal{A}_\infty} \overline{\mathcal{L}_\lambda^{\text{up}}})$  is a balanced triple; equivalently  $V_\lambda^r \otimes V_\lambda$  with the bimodule structure (2.38) and the basis  $\underline{\mathbf{B}}_\lambda^{r, \text{up}} \otimes \underline{\mathbf{B}}_\lambda^{\text{up}}$  is a based  $(U_q^{\text{ad}} \otimes (U_q^{\text{ad}})^{\text{op}})$ -module.



Denote again by  $\langle \cdot, \cdot \rangle: \mathcal{O}_q \times \dot{\mathbf{U}} \rightarrow \mathbb{C}(q)$  the pairing of  $U_q^{\text{ad}}$ -bimodules induced by the canonical pairing  $\langle \cdot, \cdot \rangle: \mathcal{O}_q \times U_q^{\text{ad}} \rightarrow \mathbb{C}(q)$ , and let  $\Phi_\lambda: V_\lambda^r \otimes V_\lambda \rightarrow \mathcal{O}_q$ ,  $\lambda \in P_+$ , be the ‘‘matrix coefficient’’ map, i.e.,

$$\langle \Phi_\lambda(f \otimes v), x \rangle = \langle f, xv \rangle_\lambda \quad (2.42)$$

for every  $f \in V_\lambda^r$ ,  $x \in U_q^{\text{ad}}$ ,  $v \in V_\lambda$ . The map  $\Phi := \bigoplus_{\lambda \in P_+} \Phi_\lambda$  is an isomorphism of  $U_q^{\text{ad}}$ -bimodules, so let us use it to identify  $\mathcal{O}_q$  with  $\bigoplus_{\lambda \in P_+} V_\lambda^r \otimes V_\lambda$  (which is the content of the Peter–Weyl decomposition (2.6)). Define

$$\begin{aligned} \mathcal{L}(\mathcal{O}_q) &= \bigoplus_{\lambda \in P_+} \left( \mathcal{L}_\lambda^{r \text{ up}} \otimes_{\mathcal{A}_0} \mathcal{L}_\lambda^{\text{up}} \right), & \mathcal{B}(\mathcal{O}_q) &:= \prod_{\lambda \in P_+} \mathcal{B}_\lambda^{r \text{ up}} \otimes \mathcal{B}_\lambda^{\text{up}}, \\ \overline{\mathcal{L}}(\mathcal{O}_q) &= \bigoplus_{\lambda \in P_+} \left( \overline{\mathcal{L}_\lambda^{r \text{ up}}} \otimes_{\mathcal{A}_\infty} \overline{\mathcal{L}_\lambda^{\text{up}}} \right), & \overline{\mathcal{B}}(\mathcal{O}_q) &:= \prod_{\lambda \in P_+} \overline{\mathcal{B}_\lambda^{r \text{ up}}} \otimes \overline{\mathcal{B}_\lambda^{\text{up}}}. \end{aligned}$$

**Theorem 2.6.**

- (i) *The triple  $(\mathcal{O}_A, \mathcal{L}(\mathcal{O}_q), \overline{\mathcal{L}}(\mathcal{O}_q))$  is balanced. Therefore, denoting by  $G$  the inverse of the canonical map  $\mathcal{O}_A \cap \mathcal{L}(\mathcal{O}_q) \cap \overline{\mathcal{L}}(\mathcal{O}_q) \rightarrow \mathcal{L}(\mathcal{O}_q)/q\mathcal{L}(\mathcal{O}_q)$ , we have*

$$\mathcal{O}_A = \bigoplus_{b \in \mathcal{B}(\mathcal{O}_q)} AG(b).$$

- (ii) *The basis  $G(\mathcal{B}(\mathcal{O}_q)) := \{G(b), b \in \mathcal{B}(\mathcal{O}_q)\}$  coincides with the dual canonical basis  $\dot{\mathbf{B}}^*$ , i.e., the elements  $a^* \in \mathcal{O}_A$ , for every  $a \in \dot{\mathbf{B}}$ , defined by  $a^*(a') = \delta_{a,a'}$  for every  $a' \in \dot{\mathbf{B}}$ . Therefore,*

$$\mathcal{O}_A = \bigoplus_{b \in \dot{\mathbf{B}}} Ab^*.$$

The statement (i) is [66, Theorem 1], and (ii) is [67, Theorem 10.1 and Proposition 10.2.2] and [83, Section 29.5]. The basis  $G(\mathcal{B}(\mathcal{O}_q)) = \dot{\mathbf{B}}^*$  is called the *global basis*, or *canonical basis*, of  $\mathcal{O}_q$ . The proof of Theorem 2.6 (ii) in [67] (see also [68]) exhibits an isomorphism of crystals over  $U_q^{\text{ad}} \otimes (U_q^{\text{ad}})^{\text{op}}$ ,

$$\psi: \mathcal{B}(\mathcal{O}_q) \rightarrow \mathcal{B}(\dot{\mathbf{U}}), \quad (2.43)$$

where  $(\mathcal{L}(\dot{\mathbf{U}}), \mathcal{B}(\dot{\mathbf{U}}))$  is the crystal base of  $\dot{\mathbf{U}}$  associated to the canonical basis  $\dot{\mathbf{B}}$ . The isomorphism  $\psi$  satisfies  $\langle G(b), G(b') \rangle = \delta_{\psi(b), b'}$  for every  $b \in \mathcal{B}(\mathcal{O}_q)$ ,  $b' \in \mathcal{B}(\dot{\mathbf{U}})$ . The unit 1 of  $\mathcal{O}_A$  is  $(1_0)^*$ ; the constant structures of  $\mathcal{O}_A$  are studied in [83, 84].

**The canonical basis of  $\mathcal{O}_A$  when  $\mathfrak{g} = \mathfrak{sl}_2$ .** Denote by  $a, b, c, d$  the matrix coefficients in the canonical basis  $(v_+, v_- := Fv_+)$  of  $V_1$ , the simple  $U_q^{\text{ad}}(\mathfrak{sl}_2)$ -module of type 1 and dimension two, read from the top left to the bottom right. In that case of  $V_1$  the upper canonical basis  $\underline{\mathbf{B}}_1^{r \text{ up}}$  and  $\underline{\mathbf{B}}_1^{\text{up}}$  coincide with the lower ones (this is not true in general, see Example 2.17). The basis  $\dot{\mathbf{B}}^*(\mathfrak{sl}_2)$  is formed by the monomials  $c^s a^p b^r$  where  $p, r, s \in \mathbb{N}$ , and  $c^s d^p b^r$  where  $p, r, s \in \mathbb{N}$  and  $p > 0$ ; this is stated in [66, Proposition 9.1.1] (in [41, Proposition 1.3], similar monomials are shown to form an  $A$ -basis of  $\mathcal{O}_A(\text{SL}_2)$ , but without reference to the canonical basis; see the comments before (4.3) below). More precisely, recall the 2-sided cells (2.37). We verified by a tedious though straightforward computation that we have the duality pairing

$$\begin{aligned} \langle c^s d^p b^r, E^{(i)} 1_{-k} F^{(j)} \rangle &= \delta_{p+r+s, k} \delta_{r, i} \delta_{s, j}, & \langle c^s d^p b^r, F^{(j)} 1_k E^{(i)} \rangle &= 0, \\ \langle c^s a^p b^r, E^{(i)} 1_{-k} F^{(j)} \rangle &= 0, & \langle c^s a^p b^r, F^{(j)} 1_k E^{(i)} \rangle &= \delta_{p+r+s, k} \delta_{r, i} \delta_{s, j}. \end{aligned}$$

Therefore,

$$\begin{aligned} \dot{\mathbf{B}}[n]^* &:= \{c^s a^p b^r, p, r, s \in \mathbb{N}, p + r + s = n\} \\ &\cup \{c^s d^p b^r, p, r, s \in \mathbb{N}, p > 0, p + r + s = n\}. \end{aligned}$$

A description of  $\dot{\mathbf{B}}^*$  in the case of  $\mathfrak{g} = \mathfrak{sl}_n$  can be found in [49]. Moreover, denote by  $V_n$  the simple  $U_q^{\text{ad}}(\mathfrak{sl}_2)$ -module of type 1 and dimension  $n + 1$ , by  $(v_k)$  the canonical basis of  $V_n$ , by  $(v^k)$  the dual basis, and by  $\pi_n: \dot{\mathbf{U}}(\mathfrak{sl}_2) \rightarrow \text{End}(V_n)$  the representation morphism. By using the above pairing, it is readily checked that for every  $0 \leq l, m \leq n$ , we have

$$\begin{aligned} &v^l(\pi_n(\cdot) v_m) \\ &= \sum_{\substack{0 \leq i, j, k \\ i+j \leq k \leq n \\ j-i=l-m}} \delta_{-k, n-2(m+j)} \begin{bmatrix} m+j \\ j \end{bmatrix}_q \begin{bmatrix} n-m+i-j \\ i \end{bmatrix}_q (E^{(i)} 1_{-k} F^{(j)})^* \\ &+ \sum_{\substack{0 \leq i, j, k \\ i+j < k \leq n \\ j-i=l-m}} \delta_{k, n-2(m-i)} \begin{bmatrix} m-i+j \\ j \end{bmatrix}_q \begin{bmatrix} n-m+i \\ i \end{bmatrix}_q (F^{(j)} 1_{+k} E^{(i)})^*. \end{aligned} \quad (2.44)$$

In particular, we see in this case of  $\mathfrak{g} = \mathfrak{sl}_2$  that in general the matrix coefficients of simple  $U_A^{\text{res}}$ -modules of type 1 are not elements of the dual canonical basis  $\dot{\mathbf{B}}^*$ . Moreover, these matrix coefficients do not form a basis of  $\mathcal{O}_A$ . For instance, it follows from (2.44) that the matrix of matrix coefficients of  $V_2$  has the following form:

$$\begin{pmatrix} a^2 & [2]_q ab & b^2 \\ ca & [2]_q bc + 1 & db \\ c^2 & [2]_q cd & d^2 \end{pmatrix}. \quad (2.45)$$

The matrix coefficient  $v_0^* \otimes v_0$  being equal to  $[2]_q bc + 1$ , this shows  $bc$  cannot be expressed as a linear combination over  $A$  of matrix coefficients of simple modules.

**The refined Peter–Weyl theorem.** Let us discuss the  $U_A^{\text{res}}$ -bimodule structure of  $\mathcal{O}_A$ , and its relation with the partition (2.35). For every  $\lambda \in P_+$ , put

$${}_A \dot{C}(\lambda) := \bigoplus_{b \in \dot{\mathbf{B}}[\lambda]} Ab^* \quad (2.46)$$

and

$$\mathcal{O}_{A(\leq \lambda)} := \bigoplus_{\lambda' \leq \lambda} {}_A \dot{C}(\lambda'), \quad \mathcal{O}_{A(< \lambda)} := \bigoplus_{\lambda' < \lambda} {}_A \dot{C}(\lambda').$$

In particular, in the  $\mathfrak{sl}_2$  case the  $A$ -module  ${}_A \dot{C}(n\varpi_1)$  has basis  $\dot{\mathbf{B}}[n]^*$  given above, of cardinality  $(n + 1)^2$ .

Recall that  $\dot{\mathbf{U}}[\geq \lambda]$  and  $\dot{\mathbf{U}}[> \lambda]$  are two-sided ideals of  $\dot{\mathbf{U}}$ , and the algebra (whence  $U_q^{\text{ad}}$ -bimodule) isomorphism  $\bar{\pi}_\lambda: \dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda] \rightarrow \text{End}(V_\lambda)$  (see (2.36)). In [83, Section 29.3], Lusztig groups this isomorphism and its properties under the general term of *refined Peter–Weyl theorem*. We are going to reinterpret it in terms of  $\mathcal{O}_A$ . First observe that

**Lemma 2.7.** *The  $A$ -modules  $\mathcal{O}_{A(\leq \lambda)}$  and  $\mathcal{O}_{A(< \lambda)}$  are  $U_A^{\text{res}}$ -bimodules, and the surjective map*

$$d_\lambda: \mathcal{O}_{A(\leq \lambda)} \longrightarrow \text{Hom}(\dot{\mathbf{U}}_A[\geq \lambda]/\dot{\mathbf{U}}_A[> \lambda], A), \quad \alpha \longmapsto \langle \alpha, \cdot \rangle \quad (2.47)$$

*descends to an isomorphism of  $U_A^{\text{res}}$ -bimodules  $\bar{d}_\lambda$  on  $\mathcal{O}_{A(\leq \lambda)}/\mathcal{O}_{A(< \lambda)}$ .*

**Proof.** For every  $\alpha \in \mathcal{O}_A(\leq \lambda)$ ,  $x, y \in U_A^{\text{res}}$ , and  $b \in \dot{\mathbf{B}}[\mu]$  with  $\mu \not\leq \lambda$ , we have  $xb y \in \dot{\mathbf{U}}_A[\geq \mu]$ . Since  $\dot{\mathbf{U}}_A[\geq \mu] = \bigoplus_{\eta \geq \mu} A\dot{\mathbf{B}}[\eta]$  and  $\eta \geq \mu$  implies  $\eta \not\leq \lambda$ , it follows that  $\langle xby, \alpha \rangle = 0$ , i.e.,  $(x \triangleright \alpha \triangleleft y)(b) = 0$ . This shows  $x \triangleright \alpha \triangleleft y \in \mathcal{O}_A(\leq \lambda)$ . The same proof applies as well to  $\mathcal{O}_A(< \lambda)$ , whence the first claim. Since  $\dot{\mathbf{U}}[\geq \lambda]$  and  $\dot{\mathbf{U}}[> \lambda]$  are two-sided ideals of  $\dot{\mathbf{U}}$ ,  $\dot{\mathbf{B}}$  is a basis of  $\dot{\mathbf{U}}_A$ , and the  $A$ -modules  $\dot{\mathbf{U}}_A[\geq \lambda]$  and  $\dot{\mathbf{U}}_A[> \lambda]$  are spanned by  $\prod_{\lambda' \geq \lambda} \dot{\mathbf{B}}[\lambda']$  and  $\prod_{\lambda' > \lambda} \dot{\mathbf{B}}[\lambda']$ , both are two-sided ideals of  $\dot{\mathbf{U}}_A$ , and  $\dot{\mathbf{U}}_A[\geq \lambda]/\dot{\mathbf{U}}_A[> \lambda]$  inherits the quotient  $U_A^{\text{res}}$ -bimodule structure. Clearly, the map  $d_\lambda$  is well defined, it is a morphism of  $U_A^{\text{res}}$ -bimodules, and its kernel contains  $\mathcal{O}_A(< \lambda)$ . Bijectivity of  $\bar{d}_\lambda$  comes by comparing the cardinality of canonical bases:  $\mathcal{O}_A(\leq \lambda)/\mathcal{O}_A(< \lambda)$  has the basis formed by the cosets of the elements of the basis  $(\dot{\mathbf{B}}[\lambda])^*$  of  ${}_A\dot{C}(\lambda)$ , and  $\dot{\mathbf{U}}_A[\geq \lambda]/\dot{\mathbf{U}}_A[> \lambda]$  the basis formed by the cosets of the elements of  $\dot{\mathbf{B}}[\lambda]$ , all cosets being non-zero and pairwise distinct.  $\blacksquare$

Since  $\dot{\mathbf{U}}_A$  preserves the canonical basis  $\mathbf{B}_\lambda$  of  ${}_A V_\lambda$ ,  $\bar{\pi}_\lambda$  descends to an isomorphism of  $U_A^{\text{res}}$ -bimodules  $\bar{\pi}_\lambda: \dot{\mathbf{U}}_A[\geq \lambda]/\dot{\mathbf{U}}_A[> \lambda] \rightarrow \text{End}({}_A V_\lambda)$ . We thus get exact sequences of  $U_A^{\text{res}}$ -bimodules

$$0 \longrightarrow \dot{\mathbf{U}}_A[> \lambda] \longrightarrow \dot{\mathbf{U}}_A[\geq \lambda] \xrightarrow{\bar{\pi}_\lambda} \text{End}({}_A V_\lambda) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_A(< \lambda) \longrightarrow \mathcal{O}_A(\leq \lambda) \xrightarrow{(\bar{\pi}_\lambda^{-1})^* \circ d_\lambda} (\text{End}({}_A V_\lambda))^* \longrightarrow 0. \quad (2.48)$$

They split as sequences of  $A$ -modules but not as sequences of bimodules. In fact,

$$\begin{aligned} (\text{End}({}_A V_\lambda))^* &:= \text{Hom}(\text{End}({}_A V_\lambda), A) \\ &\cong \text{Hom}({}_A V_\lambda \otimes_A {}_A V_\lambda, A) = {}_A V_\lambda^{\text{up}} \otimes_A ({}^\omega {}_A V_\lambda)^{\text{up}}, \end{aligned} \quad (2.49)$$

with the “up” structure defined in (2.40), and corresponding basis  $\mathbf{B}_\lambda^{\text{up}} \otimes ({}^\omega \mathbf{B}_\lambda)^{\text{up}}$ . Moreover, the exact sequence (2.48) shows that this  $A$ -module of matrix coefficients, regarded as an  $A$ -submodule of  $\mathcal{O}_A$  by means of the coefficient map  $\Phi := \bigoplus_{\lambda \in P_+} \Phi_\lambda$  (see (2.42)), is contained in  $\mathcal{O}_A(\leq \lambda)$ . This for all  $\lambda' \leq \lambda$  yields  $\bigoplus_{\lambda' \leq \lambda} (\text{End}({}_A V_{\lambda'}))^* \subset \mathcal{O}_A(\leq \lambda)$ . Now, using the isomorphism  $\bar{\pi}_\lambda$ , we get

$$\text{rank}_A(\mathcal{O}_A(\leq \lambda)) = \sum_{\lambda' \leq \lambda} \text{Card}(\dot{\mathbf{B}}[\lambda']) = \sum_{\lambda' \leq \lambda} \text{rank}({}_A V_{\lambda'})^2$$

and therefore

$$\dim_{\mathbb{C}(q)}(\mathcal{O}_A(\leq \lambda) \otimes_A \mathbb{C}(q)) = \sum_{\lambda' \leq \lambda} \dim(V_{\lambda'})^2 = \sum_{\lambda' \leq \lambda} \dim(C(\lambda')), \quad (2.50)$$

where as usual  $C(\lambda')$  denotes the space of matrix coefficients of  $V_{\lambda'}$  (see (2.22)). It follows

$$\mathcal{O}_A(\leq \lambda) \otimes_A \mathbb{C}(q) = \bigoplus_{\lambda' \leq \lambda} C(\lambda'), \quad \mathcal{O}_A(< \lambda) \otimes_A \mathbb{C}(q) = \bigoplus_{\lambda' < \lambda} C(\lambda'). \quad (2.51)$$

However, in general  ${}_A\dot{C}(\lambda) \otimes_A \mathbb{C}(q)$  is not equal to  $C(\lambda)$ ,  ${}_A\dot{C}(\lambda)$  is not an  $A$ -sublattice of  $C(\lambda)$ , and  ${}_A\dot{C}(\lambda)$  is not a  $U_A^{\text{res}}$ -bimodule (it is because of this discrepancy that we have introduced the dot notation “ $\dot{\cdot}$ ”). For instance, we can see the first two facts in the case of  $\mathfrak{g} = \mathfrak{sl}_2$ , by inverting the system of identities (2.44) for all  $0 \leq l, m \leq n$  (or more simply by considering the identity  $v_0^* \otimes v_0 = [2]_q bc + 1$  from (2.45)). For the third fact, we have  $1_2 E \in \dot{\mathbf{B}}[2]$  (see (2.37)), so  $\langle (1_2 E)^* \triangleleft E \rangle(1_0) = \langle \Delta((1_2 E)^*), E \otimes 1_0 \rangle = \langle (1_2 E)^*, E 1_0 \rangle = \langle (1_2 E)^*, 1_2 E \rangle = 1$  since  $E 1_0 = 1_2 E$ . Therefore,  $(1_2 E)^* \triangleleft E \notin {}_A\dot{C}(2)$ .

From the formulas (2.44) and Appendix A, we can observe the isomorphism (2.49) in the case of  $\mathfrak{g} = \mathfrak{sl}_2$ . More simply, by projecting the matrix (2.45) onto  $(\text{End}({}_A V_2))^*$  the entries are unchanged except the (1, 1) entry, which becomes  $[2]_q bc$ . All factors  $[2]_q$  in the middle column disappear if one uses matrix coefficients in the upper canonical basis of  $V_2$ , which is  $v_0^{\text{up}} := v_0$ ,  $v_1^{\text{up}} := [2]_q^{-1} v_1$ ,  $v_2^{\text{up}} := v_2$  in the notations of (2.44), since we have  $v^l(\pi_2(\cdot) v_m) = [\delta_{m,1} + 1]_q \langle v_l^{\text{up}}, \cdot v_m^{\text{up}} \rangle$  for  $l, m \in \{0, 1, 2\}$ , where  $\langle \cdot, \cdot \rangle$  is the pairing (2.39). Thus, in this particular example of  $(\text{End}({}_A V_2))^*$  we see explicitly the identification of the basis  $(\bar{\pi}_2^*)^{-1} \circ d_2(\dot{\mathbf{B}}[2]^*)$  and  $\underline{\mathbf{B}}_2^{\text{up}} \otimes ({}^\omega \underline{\mathbf{B}}_2)^{\text{up}}$ .

Summing up this discussion, the Lusztig refined Peter–Weyl theorem of [83, Section 29.3], implies the following.

**Theorem 2.8.** *As an  $A$ -module we have a direct sum decomposition*

$$\mathcal{O}_A = \bigoplus_{\lambda \in P_+} {}_A \dot{C}(\lambda), \quad (2.52)$$

as  $U_A^{\text{res}}$ -bimodules we have a (directed by inclusion, and non direct) sum

$$\mathcal{O}_A = \sum_{\lambda \in P_+} \mathcal{O}_A(\leq \lambda), \quad (2.53)$$

and the composition factors of  $\mathcal{O}_A$  are the bimodules

$$(\text{End}({}_A V_\lambda))^* \cong ({}^\omega V_\lambda \otimes {}_A V_\lambda)^* \quad (2.54)$$

for every  $\lambda \in P_+$ , each of multiplicity 1.

**Remark 2.9.** The above filtration and its composition factors appear in disguised manner as *good filtration* in [5] and [91] (see also [103]).

Because  $\dot{\mathbf{B}}$  is formed by weight vectors for the left and right action of  $U_q^{\text{ad}}(\mathfrak{h})$  (see (2.33)), the same is true of  $\dot{\mathbf{B}}^*$  and (2.52) can thus be refined into a weight space decomposition

$$\mathcal{O}_A = \bigoplus_{\mu, \nu \in P} \bigoplus_{\lambda \in P_+} ({}_A \dot{C}(\lambda))_{\mu, \nu}. \quad (2.55)$$

Now recall the property (2.33). Consider in particular the finite subsets  $\dot{\mathbf{B}}_{0, \varpi_i}$  and  $\dot{\mathbf{B}}_{\varpi_i, 0}$  associated to the fundamental weights  $\varpi_i$ ,  $i = 1, \dots, m$ . The map  $u \mapsto u({}^\omega v_0 \otimes v_{w_0(\varpi_i)})$ ,  $u \in \dot{\mathbf{U}}$ , allows one to identify  $\dot{\mathbf{B}}_{0, \varpi_i}$  with the canonical basis  $\underline{\mathbf{B}}_{\varpi_i}$  of  ${}^\omega V_0 \otimes V_{\varpi_i} \cong V_{\varpi_i}$ , and therefore with a uniquely determined finite subset  $\mathbf{B}_{\varpi_i}$  of the canonical basis  $\mathbf{B}^-$  of  $U_q^{\text{ad}}(\mathfrak{n}_-)$ ; similarly, one can identify  $\dot{\mathbf{B}}_{\varpi_i, 0}$  with a uniquely determined finite subset  ${}^\omega \mathbf{B}_{\varpi_i}$  of the canonical basis  $\mathbf{B}^+$  of  $U_q^{\text{ad}}(\mathfrak{n}_+)$ . The elements of  $\dot{\mathbf{B}}_{0, \varpi_i}$  and  $\dot{\mathbf{B}}_{\varpi_i, 0}$  are respectively of the form  $b^- 1_{\varpi_i}$  and  $b^+ 1_{-\varpi_i}$ , where  $b^- \in \mathbf{B}_{\varpi_i}$  and  $b^+ \in {}^\omega \mathbf{B}_{\varpi_i}$ , and we have (see [84, Proposition 3.3 and Section 3.4]):

**Proposition 2.10.** *The algebra  $\mathcal{O}_A$  is finitely generated. A system of generators is provided by the elements  $a^* \in \dot{\mathbf{B}}^*$ , where  $a \in \bigcup_{i=1}^m (\dot{\mathbf{B}}_{0, \varpi_i} \cup \dot{\mathbf{B}}_{\varpi_i, 0})$ .*

Note that the above system of generators of  $\mathcal{O}_A$  has  $2 \sum_{i=1}^m \dim(V_{\varpi_i})$  elements. In fact, recall that  $\varphi: U_q^{\text{ad}} \rightarrow U_q^{\text{ad}}$  is the anti-automorphism given by  $\varphi(E_i) = F_i$ ,  $\varphi(F_i) = E_i$ ,  $\varphi(K_\lambda) = K_\lambda$ . Denote by  $v_{-\varpi_i}$  and  $f_{-\varpi_i}$  the canonical lowest-weight vectors of the highest weight modules  $V_{-w_0(\varpi_i)}$  and  $V_{-w_0(\varpi_i)}^r$ , respectively, and put the superscript “up” for the upper canonical basis vectors.

**Lemma 2.11.** *For every  $b^- \in \mathbf{B}_{\varpi_i}$  and  $b^+ \in {}^\omega \mathbf{B}_{\varpi_i}$ , we have*

$$(b^- 1_{\varpi_i})^* = \Phi_{\varpi_i}((f_{\varpi_i} \varphi(b^-))^{\text{up}} \otimes v_{\varpi_i}), \quad (2.56)$$

$$(b^+ 1_{-\varpi_i})^* = \Phi_{-w_0(\varpi_i)}((f_{-\varpi_i} \varphi(b^+))^{\text{up}} \otimes v_{-\varpi_i}). \quad (2.57)$$

In other words,  $(b^- 1_{\varpi_i})^*$  and  $(b^+ 1_{-\varpi_i})^*$  are the matrix coefficients lying on the first and last columns of the matrix representations in the upper canonical bases of the spaces  $V_{\varpi_i}$ ,  $i = 1, \dots, m$ .

**Proof.** This can be checked by using the isomorphism (2.43). The key observation is that

$$\langle \Phi_\lambda(f_\lambda \otimes v_\lambda), 1_\mu \rangle = \langle f_\lambda, 1_\mu v_\lambda \rangle_\lambda = \delta_{\lambda, \mu}$$

for every  $\lambda \in P_+$ ,  $\mu \in P$ , and therefore  $\Phi_\lambda(f_\lambda \otimes v_\lambda) = 1_\lambda^*$ . Then the computation proceeds by using the equivariance of  $\Phi$  under the action of  $U_q^{\text{ad}} \otimes (U_q^{\text{ad}})^{\text{op}}$ , the fact that  $\langle \cdot, \cdot \rangle$  dualizes the bimodules structures on  $\mathcal{O}_q$  and  $\dot{\mathbf{U}}$ , and the description of the associated Kashiwara operators on  $\mathcal{B}(\mathcal{O}_q)$  and  $\mathcal{B}(\dot{\mathbf{U}})$ . Here is an alternative argument. By the very definition of the sets  $\dot{\mathbf{B}}[\lambda]$  we have  $b^- 1_{\varpi_i} \in \dot{\mathbf{B}}[\varpi_i]$ ,  $b^+ 1_{-\varpi_i} \in \dot{\mathbf{B}}[-w_0(\varpi_i)]$ . We wish to check if their duals  $(b^- 1_{\varpi_i})^*$ ,  $(b^+ 1_{-\varpi_i})^*$  coincide with the elements of  $\mathcal{O}_A$  on the right sides of (2.56) and (2.57). As already noticed after (2.48), by the isomorphism  $\mathcal{O}_A(\leq \lambda)/\mathcal{O}_A(< \lambda) \cong \text{End}({}_A V_\lambda)^*$  every matrix coefficient of  ${}_A V_\lambda$  belongs to  $\mathcal{O}_A(\leq \lambda)$ . Now, the  $A$ -modules  $\mathcal{O}_A(\leq \varpi_i)$  and  $\mathcal{O}_A(\leq -w_0(\varpi_i))$  are generated by  $\dot{\mathbf{B}}[\varpi_i]^*$  and  $\dot{\mathbf{B}}[-w_0(\varpi_i)]^*$ , respectively. Because  $((\bar{\pi}_\lambda^*)^{-1} \circ d_\lambda)(\dot{\mathbf{B}}[\lambda]^*)$  coincides with  $\underline{\mathbf{B}}_\lambda^{\text{up}} \otimes ({}^\omega \underline{\mathbf{B}}_\lambda)^{\text{up}}$ , the conclusion follows.  $\blacksquare$

Note that the same argument implies that, for every  $\lambda \in P_+$ , any matrix coefficient of  $V_\lambda$  in the upper canonical basis and vanishing on the elements of  $\dot{\mathbf{B}}[\lambda']$  for  $\lambda' < \lambda$  must belong to  $\dot{\mathbf{B}}[\lambda]^*$ . For instance, in the  $\mathfrak{sl}_2$  case,  $\mathcal{O}_A(\leq 2)$  has canonical basis  $\dot{\mathbf{B}}[0]^* \llbracket \dot{\mathbf{B}}[2]^*$ , so the matrix coefficients of  $V_2$  vanishing on  $1_0$  belong to  $\dot{\mathbf{B}}[2]^*$ . This can be observed in (2.45), using the comments in the paragraph before (2.52).

Though the  $A$ -module  ${}_A V_\mu \otimes_A {}_A V_\nu$  has no decomposition like (2.7), we can refine the map  $C(\mu) \otimes C(\nu) \rightarrow C(\mu + \nu)$  in (2.8) to an  $A$ -linear map defined on  ${}_A C(\mu) \otimes_A {}_A C(\nu)$ . Indeed, there is a unique injective morphism of  $U_A^{\text{res}}$ -modules  $\mathfrak{T}_{\mu, \nu}: {}_A V_{\mu+\nu} \rightarrow {}_A V_\mu \otimes_A {}_A V_\nu$ , which is given by  $\mathfrak{T}_{\mu, \nu}(v_{\mu+\nu}) = v_\mu \otimes v_\nu$  [83, Proposition 25.1.2 (a)–(b)]. It defines a morphism of based modules

$$(V_{\mu+\nu}, \underline{\mathbf{B}}_{\mu+\nu}) \rightarrow (V_\mu \otimes V_\nu, \underline{\mathbf{B}}_\mu \diamond \underline{\mathbf{B}}_\nu),$$

where  $\underline{\mathbf{B}}_\mu \diamond \underline{\mathbf{B}}_\nu := \{b \diamond b', b \in \underline{\mathbf{B}}_\mu, b' \in \underline{\mathbf{B}}_\nu\}$  [83, Proposition 27.1.7]. Hence,  $\mathfrak{T}_{\mu, \nu}$  is a split  $A$ -linear map, i.e., there exists a  $A$ -linear map  $\mathfrak{S}_{\mu, \nu}: {}_A V_\mu \otimes_A {}_A V_\nu \rightarrow {}_A V_{\mu+\nu}$  such that  $\mathfrak{S}_{\mu, \nu} \circ \mathfrak{T}_{\mu, \nu} = \text{id}$ . Note that  $\mathfrak{S}_{\mu, \nu}$  is not a  $U_A^{\text{res}}$ -morphism. Similarly, the unique morphism of  $U_A^{\text{res}}$ -modules  ${}^\omega \mathfrak{T}_{\mu, \nu}: {}^\omega V_{\mu+\nu} \rightarrow {}^\omega V_\mu \otimes_A {}^\omega V_\nu$  is a split injection. Define  $\rho_{\mu', \mu''}: \dot{\mathbf{U}}_A \rightarrow {}^\omega V_{\mu'} \otimes_A {}^\omega V_{\mu''}$  by

$$\rho_{\mu', \mu''}(u) = u \left( {}^\omega v_{w_0(\mu')} \otimes_A v_{w_0(\mu'')} \right),$$

and  $\rho_{\mu', \mu'', \nu', \nu''}: \dot{\mathbf{U}}_A^{\otimes 2} \rightarrow {}^\omega V_{\mu'} \otimes_A {}^\omega V_{\mu''} \otimes_A {}^\omega V_{\nu'} \otimes_A {}^\omega V_{\nu''}$  by

$$\rho_{\mu', \mu'', \nu', \nu''}(u) = u \left( {}^\omega v_{w_0(\mu')} \otimes_A v_{w_0(\mu'')} \otimes_A {}^\omega v_{w_0(\nu')} \otimes_A v_{w_0(\nu'')} \right).$$

Define  $\tau_{\mu', \mu'', \nu', \nu''}: {}^\omega V_{\mu'+\nu'} \otimes_A {}^\omega V_{\mu''+\nu''} \rightarrow {}^\omega V_{\mu'} \otimes_A {}^\omega V_{\mu''} \otimes_A {}^\omega V_{\nu'} \otimes_A {}^\omega V_{\nu''}$  by

$$\tau_{\mu', \mu'', \nu', \nu''} = (1 \otimes \hat{R}^{-1} \otimes 1) ({}^\omega \mathfrak{T}_{\mu', \nu'} \otimes {}^\omega \mathfrak{T}_{\mu'', \nu''}).$$

It is an injective morphism of  $U_A^{\text{res}}$ -modules. In [84, Section 1.13], Lusztig proved that  $\tau_{\mu', \mu'', \nu', \nu''}$  is a split  $A$ -linear map ([84] uses  $\hat{R}$  instead of  $\hat{R}^{-1}$ , since our coproducts on  $U_q^{\text{ad}}$  are opposite), and that it satisfies

$$\tau_{\mu', \mu'', \nu', \nu''} \rho_{\mu'+\mu'', \nu'+\nu''} = \rho_{\mu', \mu'', \nu', \nu''} \Delta, \quad (2.58)$$

where  $\Delta$  is the coproduct of  $\dot{U}_A$ , see (2.28).

Now take  $\mu := \mu' = \mu''$ ,  $\nu := \nu' = \nu'' \in P_+$ , and put  $\tau_{\mu, \nu} := \tau_{\mu, \mu, \nu, \nu}$ . It follows from the classical decomposition (2.7) over  $\mathbb{C}(q)$ , and (2.8) and (2.51), that the product of  $\mathcal{O}_A$  yields a map  $m: \mathcal{O}_A(\leq \mu) \otimes_A \mathcal{O}_A(\leq \nu) \rightarrow \mathcal{O}_A(\leq \mu + \nu)$ .

Denote the projection map  $p_{\mu+\nu}: \mathcal{O}_A(\leq \mu + \nu) \rightarrow {}_A\dot{C}(\mu + \nu)$ , define  ${}_A\dot{\tau}_{\mu, \nu} := p_{\mu+\nu} \circ m$ , and put

$$\pi'_\lambda: \mathcal{O}_A(\leq \lambda) \longrightarrow \mathcal{O}_A(\leq \lambda) / \mathcal{O}_A(< \lambda) \xrightarrow{(\bar{\pi}'_\lambda)^{-1} \circ \bar{d}_\lambda} (\text{End}({}_A V_\lambda))^*,$$

where the first map is the quotient map. Consider the diagram

$$\begin{array}{ccc} {}_A\dot{C}(\mu) \otimes {}_A\dot{C}(\nu) & \xrightarrow{{}_A\dot{\tau}_{\mu, \nu}} & {}_A\dot{C}(\mu + \nu) \\ \downarrow \pi'_\mu \otimes \pi'_\nu & & \downarrow \pi'_{\mu+\nu} \\ (\text{End}({}_A V_\mu))^* \otimes (\text{End}({}_A V_\nu))^* & \xrightarrow{\tau_{\mu, \nu}^t} & (\text{End}({}_A V_{\mu+\nu}))^*, \end{array}$$

where  $\tau_{\mu, \nu}^t$  is the transpose of Lusztig's map  $\tau_{\mu, \nu}$ .

**Proposition 2.12.** *The map  ${}_A\dot{\tau}_{\mu, \nu}: {}_A\dot{C}(\mu) \otimes_A {}_A\dot{C}(\nu) \rightarrow {}_A\dot{C}(\mu + \nu)$  is split as an  $A$ -linear map and the above diagram is commutative.*

**Proof.** The commutativity of the diagram comes from equation (2.58). The epimorphism  $\pi'_\lambda$  is injective on  ${}_A\dot{C}(\lambda)$ , and maps the canonical basis elements to the elements of the upper canonical basis  $\mathbf{B}_\lambda^{\text{up}} \otimes (\omega \mathbf{B}_\lambda)^{\text{up}}$ . By Lusztig's results recalled above, the epimorphism  $\tau_{\mu, \nu}^t$  splits as an  $A$ -linear map. Therefore, the same is true of  ${}_A\dot{\tau}_{\mu, \nu}$ .  $\blacksquare$

We stress that  ${}_A\dot{\tau}_{\mu, \nu}$  plays for  $\mathcal{O}_A$  the same role as the map (2.8) for  $\mathcal{O}_q$ .

Finally, we consider for any  $n \geq 1$  the invariant elements of  $\mathcal{O}_A^{\otimes n}$  endowed with the action  $\text{coad}_n^r$  of  $U_A^{\text{res}}$ , see (2.15) (recall that  $\mathcal{L}_{0, n} = \mathcal{O}_q^{\otimes n}$  as  $U_q^{\text{ad}}$ -module).

First note that, by definition,  $\mathcal{O}_A(G^n)$  is the restricted dual of the Hopf algebra  $U_A^{\text{res}}(\mathfrak{g}^{\oplus n})$ , associated to its category of type 1 modules. By ordering the summands of  $\mathfrak{g}^{\oplus n}$  we get an isomorphism  $U_A^{\text{res}}(\mathfrak{g}^{\oplus n}) \cong U_A^{\text{res}}(\mathfrak{g})^{\otimes n}$ , and any type 1 simple  $U_A^{\text{res}}(\mathfrak{g})^{\otimes n}$ -module is isomorphic to  $V_{[\lambda]} := \bigotimes_{i=1}^n V_{\lambda_i}$  endowed with the componentwise action, for some  $[\lambda] := (\lambda_1, \dots, \lambda_n) \in P_+^n$  (this is a classical fact; see, e.g., [51, Theorem 3.10.2]). Therefore, we have an isomorphism  $\mathcal{O}_A(G^n) \cong \mathcal{O}_A^{\otimes n}$ . With the same notation  $[\lambda] := (\lambda_1, \dots, \lambda_n) \in P_+^n$ , let us put

$$\begin{aligned} {}_A\dot{C}([\lambda]) &:= \bigotimes_{i=1}^n {}_A\dot{C}(\lambda_i) = \bigoplus_{b \in \bigotimes_{i=1}^n \dot{\mathbf{B}}[\lambda_i]^*} Ab, \\ \mathcal{O}_A(\leq [\lambda]) &:= \bigotimes_{i=1}^n \mathcal{O}_A(\leq \lambda_i) = \bigoplus_{[\lambda'] \in P_+^n, \lambda'_i \leq \lambda_i} {}_A\dot{C}([\lambda']). \end{aligned}$$

We thus obtain a decomposition into based  $(U_A^{\text{res}} \otimes (U_A^{\text{res}})^{\text{op}})^{\otimes n}$ -modules

$$\mathcal{O}_A^{\otimes n} = \sum_{[\lambda] \in P_+^n} \mathcal{O}_A(\leq [\lambda]).$$

Now  $\text{coad}_n^r = (\text{coad}^r)^{\otimes n} \circ \Delta^{(n-1)}$  gives structures of  $U_A^{\text{res}}$ -modules to  $\mathcal{O}_A^{\otimes n}$  and  $\mathcal{O}_A(\leq [\lambda])$ . In order to make it a based module, we give it the “ $\diamond$ ” product of the canonical bases of the factors  $\mathcal{O}_A(\leq \lambda_i)$ , i.e.,

$$\dot{\mathbf{B}}[[\lambda]]^* := \diamond_{i=1}^n \left( \prod_{\lambda'_i \leq \lambda_i} \dot{\mathbf{B}}[\lambda'_i]^* \right).$$

We thus obtain a decomposition into based  $U_A^{\text{res}}$ -modules

$$\mathcal{O}_A^{\otimes n} = \sum_{[\lambda] \in P_+^n} (\mathcal{O}_A(\leq [\lambda]), \dot{\mathbf{B}}[[\lambda]]^*), \quad (2.59)$$

with composition factors  $\otimes_{i=1}^n (\text{End}({}_A V_{\lambda_i}))^*$ . By the properties of “ $\diamond$ ” products of bases of based modules, the underlying  $A$ -module is

$$\mathcal{O}_A^{\otimes n} = \bigoplus_{[\lambda] \in P_+^n} {}_A \dot{C}([\lambda]). \quad (2.60)$$

Finally, we state the last property of based modules we need. Let  $(M, B)$  be a based module. Recall the notations introduced around (2.34). It is proved in [83, Proposition 27.1.8] that for every  $\lambda \in P_+$  the submodule  $M[\geq \lambda]$  is a sub-based module of  $M$ , and that it has the basis

$$B \cap M[\geq \lambda] = \bigcup_{\lambda' \geq \lambda} B[\lambda']. \quad (2.61)$$

Consider  $M[\neq 0] := \bigoplus_{\lambda \neq 0} M[\lambda]$ , the largest proper submodule of  $M$  that contains no non-zero invariant element. Recall that the space of *coinvariants* of  $M$  is

$$M_{U_q^{\text{ad}}} = M/M[\neq 0] = M/\mathbb{C}(q)\{um - \varepsilon(u)m, m \in M, u \in U_q^{\text{ad}}\}$$

that is, the largest quotient of  $M$  with trivial action, where  $\varepsilon: U_q^{\text{ad}} \rightarrow \mathbb{C}(q)$  is the counit. It follows from (2.61) that  $M[\neq 0]$  is a sub-based module of  $M$ , with the basis  $\bigcup_{\lambda \neq 0} B[\lambda]$ , and we have (this is, [83, Proposition 27.2.6]):

**Proposition 2.13.** *The quotient map  $\pi: M \rightarrow M_{U_q^{\text{ad}}}$  is a morphism of based modules, where  $M_{U_q^{\text{ad}}}$  is endowed with the basis  $B_{U_q^{\text{ad}}} := \pi(B[0])$ .*

Keeping the same notations, let  ${}_A M \subset M$  be the  $A$ -module generated by  $B$ , and let  ${}_A M^* \subset M^*$  be the  $A$ -module generated by  $B^*$ . They are  $U_A^{\text{res}}$ -modules. Denote by  $({}_A M^*)^{U_A^{\text{res}}}$  the submodule of  $U_A^{\text{res}}$ -invariant elements of  ${}_A M^*$ , regarded as a right module in the natural way.

**Lemma 2.14.** *We have a direct sum decomposition of  $A$ -modules*

$${}_A M^* = ({}_A M^*)^{U_A^{\text{res}}} \bigoplus_A {}_A N, \quad (2.62)$$

where  ${}_A N \subset {}_A M^*$  is the  $A$ -submodule generated by  $\bigcup_{\lambda \neq 0} B[\lambda]^*$ .

**Proof.** By Proposition 2.13, the transpose map  $\pi^t: (M_{U_q^{\text{ad}}})^* \rightarrow M^*$  is a monomorphism mapping the dual basis  $B_{U_q^{\text{ad}}}^*$  to the subset  $B[0]^*$  of  $B^*$ . The image of  $\pi^t$  is  $(M^*)^{U_q^{\text{ad}}}$ . If we set  ${}_A M_{U_A^{\text{res}}} = \pi({}_A M)$ , then  $\pi^t({}_A M_{U_A^{\text{res}}})^* = ({}_A M^*)^{U_A^{\text{res}}}$  is generated by  $B[0]^*$ , which concludes the proof.  $\blacksquare$

Note that, since  $B[0]$  is in general not invariant under the action of  $U_A^{\text{res}}$ ,  ${}_A N$  need not be stable under this action.

We are now ready to draw consequences of this discussion and the previous results. As usual denote by  $(\mathcal{O}_A^{\otimes n})^{U_A^{\text{res}}}$  the subspace of invariant elements of  $\mathcal{O}_A^{\otimes n}$  for the action  $\text{coad}_n^r$ . In the case  $n = 1$ , it is just the center  $\mathcal{Z}(\mathcal{O}_A)$ .

**Theorem 2.15.**  $(\mathcal{O}_A^{\otimes n})^{U_A^{\text{res}}}$  is a direct summand of the  $A$ -module  $\mathcal{O}_A^{\otimes n}$  for any  $n \geq 1$ .

**Proof.** By equation (2.59), it is enough to show that for every  $[\lambda] \in P_+^n$  the invariant elements of  $\mathcal{O}_A(\leq [\lambda])$  form a direct summand, and these summands are compatible with non-empty intersections  $\mathcal{O}_A(\leq [\lambda]) \cap \mathcal{O}_A(\leq [\lambda'])$ . Using that  $\mathcal{O}_A(G^n) \cong \mathcal{O}_A^{\otimes n}$  and viewing  $P_+^n$  as the weight lattice of  $G^n$ , it is enough to prove these claims for  $n = 1$ . Given  $\lambda \in P_+$  put

$$P_\lambda = \{\lambda' \in P_+, \lambda' \not\leq \lambda\},$$

and denote by  $\dot{\mathbf{U}}_A[P_\lambda]$  the  $A$ -submodule of  $\dot{\mathbf{U}}_A$  generated by  $\coprod_{\lambda' \in P_\lambda} \dot{\mathbf{B}}[\lambda']$ . Also, let us put  $\dot{\mathbf{U}}[P_\lambda] = \dot{\mathbf{U}}_A[P_\lambda] \otimes_A \mathbb{C}(q)$ . The complement  $P_+ \setminus P_\lambda$  is finite, and if  $\lambda' \in P_\lambda$  and  $\lambda'' \geq \lambda'$ , then  $\lambda'' \in P_\lambda$ . By the results of [83, Section 29.2],  $\dot{\mathbf{U}}[P_\lambda]$  is a two-sided ideal, and the quotient algebra  $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]$  is finite-dimensional with unit the coset of  $\sum_{\lambda' \leq \lambda} 1_{\lambda'}$ , and it is semisimple, isomorphic to  $\bigoplus_{\lambda' \leq \lambda} \text{End}(V_{\lambda'})$  (whereas  $\dot{\mathbf{U}}_A/\dot{\mathbf{U}}_A[P_\lambda]$  has indecomposable modules, see Example 2.17). It inherits from  $\dot{\mathbf{U}}$  a canonical basis, formed by the non-zero cosets of elements of  $\dot{\mathbf{B}}$ , and with this basis  $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]$  is a based module for the right adjoint action  $\text{ad}^r$ . Similarly as for (2.47), we have a morphism of  $U_A^{\text{res}}$ -modules

$$\tilde{d}_\lambda: \mathcal{O}_A(\leq \lambda) \longrightarrow \text{Hom}(\dot{\mathbf{U}}_A/\dot{\mathbf{U}}_A[P_\lambda], A), \quad \alpha \longmapsto \langle \alpha, \cdot \rangle,$$

which is an isomorphism by (2.50) and the computation  $\dim(\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]) = \sum_{\lambda' \leq \lambda} \dim(V_{\lambda'})^2$  in [83, Section 29.2]. Applying Proposition 2.13 and (2.62) to the based module  $M = \dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]$ , we obtain that the invariant elements of  $\mathcal{O}_A(\leq \lambda)$  form a direct summand. Finally, for any  $\lambda, \lambda' \in P_+$  we have  $\mathcal{O}_A(\leq \lambda) \cap \mathcal{O}_A(\leq \lambda') \cong \text{Hom}(\dot{\mathbf{U}}_A/(\dot{\mathbf{U}}_A[P_\lambda] + \dot{\mathbf{U}}_A[P_{\lambda'}]), A)$ . Applying Proposition 2.13 and (2.62) to the based module  $M := \dot{\mathbf{U}}/(\dot{\mathbf{U}}[P_\lambda] + \dot{\mathbf{U}}[P_{\lambda'}])$ , we obtain that the invariant elements  $({}_A M^*)^{U_A^{\text{res}}}$  of  $\mathcal{O}_A(\leq \lambda) \cap \mathcal{O}_A(\leq \lambda')$  form a direct  $A$ -summand. Since the latter is a based  $U_A^{\text{res}}$ -submodule of  $\mathcal{O}_A(\leq \lambda)$  and  $\mathcal{O}_A(\leq \lambda')$ , this summand is also a direct  $A$ -summand of  $\mathcal{O}_A(\leq \lambda)^{U_A^{\text{res}}}$  and  $\mathcal{O}_A(\leq \lambda')^{U_A^{\text{res}}}$ . This shows the  $A$ -modules  $\mathcal{O}_A(\leq \lambda)^{U_A^{\text{res}}}$  for all  $\lambda \in P_+$  match to form the  $A$ -summand  $(\mathcal{O}_A)^{U_A^{\text{res}}}$  of  $\mathcal{O}_A$ , and thus concludes the proof. ■

**Remark 2.16.** Let  $(M, B), (M', B')$  be based modules, with tensor product  $(M \otimes M', B_\diamond)$ , and  $B_\diamond[0] \subset B_\diamond$  the subset in bijection with the canonical basis of the space of coinvariants  $(M \otimes M')_{U^{\text{ad}}}$  (see Proposition 2.13). This subset is described in [83, Proposition 27.3.8] in terms of  $B$  and  $B'$ . Since  $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]$  is semisimple with known summands, and the construction of the “ $\diamond$ ” product of canonical bases is associative, one can recursively compute the subset of the canonical basis of  $\bigotimes_{i=1}^n \dot{\mathbf{U}}/\dot{\mathbf{U}}[P_{\lambda_i}]$  (endowed with the action dual to  $\text{coad}_n^r$ ) which is in bijection with the canonical basis of the space of coinvariants. Therefore, a complete (though highly nontrivial) characterization of the basis of  $(\mathcal{O}_A^{\otimes n})^{U_A^{\text{res}}}$  can be obtained. Examples can be found in [83, Section 27.3.10]. In the case  $\mathfrak{g} = \mathfrak{sl}_2$ , the canonical basis of the dual space  $\text{End}(V_1^{\otimes n})^*$  has been identified in [56] with the canonical basis of the Temperley–Lieb algebra  $TL_n(q)$ .

**Example 2.17.** The simplest case is already instructive. Namely, consider  $V_1$  and  $V_2$ , the simple  $U_q^{\text{ad}}(\mathfrak{sl}_2)$ -modules of type 1 and dimension two and three.

On  $V_1$ , we have the lower canonical basis vectors  $v_+$  and  $v_-$ , such that  $Kv_+ = qv_+, Ev_+ = 0, v_- = Fv_+$ . The canonical lower and upper bases of  $V_1$  are both  $\{v_+, v_-\}$ . Using the relation (2.32), we see that the elements of  $\dot{\mathbf{B}}_{0,1}$  and  $\dot{\mathbf{B}}_{1,0}$  are  $1_1, F1_1$  and  $1_{-1}, E1_{-1}$ , respectively;



the dual linear forms generate  $\mathcal{O}_A(\mathrm{SL}_2)$ , they are the matrix coefficients  $a, c, d$  and  $b$  respectively. By (2.37), we have  $\hat{\mathbf{B}}[1] = \hat{\mathbf{B}}_{0,1} \amalg \hat{\mathbf{B}}_{1,0}$ .

Next consider  $V_2$ . On  $V_2$ , we have the canonical highest weight vector  $v_0$  of weight 2, and lower canonical basis  $\mathbf{B}_2 = \{v_0, v_1, v_2\}$ , where  $v_1 = Fv_0$  and  $v_2 = F^{(2)}v_0$ . We have  $\mathbf{B}_2^{\mathrm{up}} = \{v_0, [2]_q^{-1}v_1, v_2\}$  (see Appendix A). We can identify the ambient space of the right module  $V_2^r$  with that of  $V_2$ ; its highest weight vector is then  $v_0$ , and its canonical lower and upper bases are  $\mathbf{B}_2^r = \{v_0, v_1, v_2\}$  and  $\mathbf{B}_2^{r,\mathrm{up}} = \{v_0, [2]_q^{-1}v_1, v_2\}$ .

Consider now the module  ${}^\omega V_1 \otimes V_1$ . We have

$$\hat{R} = \sum_{n=0}^{\infty} \frac{(q - q^{-1})^n}{[n]_q!} q^{n(n-1)/2} E^n \otimes F^n,$$

so the matrix of the involution  $\Psi = \hat{R}^{-1} \circ^-$  in the basis  $v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-$  is

$$(\hat{R}^{-1} \circ^-)_{\omega V_1, V_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q^{-1} - q & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, the canonical basis  $\mathbf{B}_{1,1}$  is formed by the vectors  $v_+ \diamond v_+ = v_+ \otimes v_+ + q^{-1}v_- \otimes v_-$  and  $v_+ \diamond v_- = v_+ \otimes v_-, v_- \diamond v_+ = v_- \otimes v_+, v_- \diamond v_- = v_- \otimes v_-$ . Consider the partition  $\mathbf{B}_{1,1} = \mathbf{B}_{1,1}[2] \cup \mathbf{B}_{1,1}[0]$ . We have  $\mathbf{B}_{1,1}[2] = \{v_- \diamond v_+, v_+ \diamond v_+, v_+ \diamond v_-\}$ , which is a basis of the three-dimensional submodule  $W_2$  of  $V_1 \otimes V_1$ . Since  $\mathbf{B}_{1,1}$  is an  $A$ -basis of  ${}^\omega V_1 \otimes_A V_1$ , it follows that the epimorphism  $\tau_{1,1}^t: {}_A \dot{C}(1) \otimes_A {}_A \dot{C}(1) \rightarrow {}_A \dot{C}(2)$  splits (see Proposition 2.12). The vector  $v_- \diamond v_-$  is cyclic, so  $\mathbf{B}_{1,1}[0] = \{v_- \diamond v_-\}$ . By the definitions, we have  $v_+ \diamond v_+ = (1 \diamond_0 1)_{1,1}$ ,  $v_+ \diamond v_- = (1 \diamond_0 F)_{1,1}$ ,  $v_- \diamond v_+ = (F \diamond_0 1)_{1,1}$ ,  $v_- \diamond v_- = (F \diamond_0 F)_{1,1}$ , so the corresponding elements of  $\hat{\mathbf{B}}_{1,1} \subset \hat{\mathbf{B}}$  are respectively  $1_0, 1_{-2}F, 1_2E$ , and  $F1_2E = E1_{-2}F$ .

The invariant submodule  $W_0$  of  ${}^\omega V_1 \otimes V_1$  is generated by  $v' = v_- \otimes v_- - q^{-1}v_+ \otimes v_+$ . The  $U_A^{\mathrm{res}}$ -modules  ${}^\omega V_1 \otimes_A V_1$  and  $W_2 \oplus W_0$  are not equal, though they are by extending scalars to  $\mathbb{C}(q)$ . Indeed, we have

$$v_+ \otimes v_+ = [2]_q^{-1}(qv_+ \diamond v_+ - v') \notin W_2 \oplus W_0.$$

The module of coinvariants is  $({}^\omega V_1 \otimes V_1)_{U_q^{\mathrm{ad}}} = \mathbb{C}(q)\{\pi(v_- \otimes v_-)\}$ , where as usual  $\pi: {}^\omega V_1 \otimes V_1 \rightarrow ({}^\omega V_1 \otimes V_1)_{U_q^{\mathrm{ad}}}$  is the quotient map. The transpose map  $\pi^t: ({}^\omega V_1 \otimes V_1)_{U_q^{\mathrm{ad}}}^* \rightarrow ({}^\omega V_1 \otimes V_1)^*$  sends  $(v_- \diamond v_-)^*$  to the unique  $U_q^{\mathrm{ad}}$ -invariant linear map

$$\mathrm{ev}_1: {}^\omega V_1 \otimes V_1 \rightarrow \mathbb{C}(q)$$

such that  $\mathrm{ev}_1(v_- \otimes v_-) = 1$ .

Note that, since elements of  $\dot{\mathbf{U}}_A[\lambda > 2]$  act trivially on modules with all isotypical components of highest weight  $\leq 2$ ,  ${}^\omega V_1 \otimes_A V_1$  is an indecomposable module over  $\dot{\mathbf{U}}_A/\dot{\mathbf{U}}_A[\lambda > 2]$  (that is,  $\dot{\mathbf{U}}_A/\dot{\mathbf{U}}_A[P_2]$  in the notations of Theorem 2.15).

### 2.2.3 Some consequences on $\mathcal{L}_{0,n}^A$ and $\mathcal{M}_{0,n}^A$

Recall from Section 2.2.1 the definition of the integral forms  $\mathcal{L}_{0,n}^A$  and  $\mathcal{M}_{0,n}^A$ .

**Proposition 2.18.**  $\mathcal{L}_{0,n}^A$  and  $\mathcal{M}_{0,n}^A$  are free  $A$ -modules, and  $\mathcal{M}_{0,n}^A$  is a direct summand of the  $A$ -module  $\mathcal{L}_{0,n}^A$ . Moreover,  $\mathcal{L}_{0,n}^A$  is a finitely generated ring.

**Proof.** Since  $\mathcal{L}_{0,n}^A = \mathcal{O}_A^{\otimes n}$  as  $U_A^{\text{res}}$ -modules, by (2.60) it has the basis  $\bigcup_{[\lambda] \in P_+^n} \dot{\mathbf{B}}[[\lambda]]^*$ . Therefore,  $\mathcal{L}_{0,n}^A$  is a free  $A$ -module. Since  $A$  is a principal ideal domain, it follows that  $\mathcal{M}_{0,n}^A$  is a free  $A$ -submodule [77, Appendix 2.2]. By Theorem 2.15, there is a direct sum decomposition as  $A$ -module

$$\mathcal{L}_{0,n}^A = \mathcal{M}_{0,n}^A \oplus {}_A N, \quad (2.63)$$

and the proof identifies a basis of  $\mathcal{M}_{0,n}^A$  as a subset of  $\bigcup_{[\lambda] \in P_+^n} \dot{\mathbf{B}}[[\lambda]]^*$ .

Next, consider the question of finite generation. By the formula (2.17), it is enough to verify this for  $\mathcal{L}_{0,1}^A$ , but  $\mathcal{L}_{0,1}^A = \mathcal{O}_A$  as an  $A$ -module, and  $\mathcal{O}_A$  is finitely generated by the matrix coefficients of the fundamental  $U_A^{\text{res}}$ -modules  ${}_A V_{\varpi_k}$ ,  $k \in \{1, \dots, m\}$  (see (2.56) and (2.57)). Any monomials in these generators can be written as a  $A$ -linear combination of monomials in the same generators but with the product of  $\mathcal{L}_{0,1}^A$ , instead of the product  $\star$ . This follows from the integrality properties of the  $R$ -matrix, and the formula inverse to (2.9) (see in [18, Section 3.3 and the formulas (4.6)–(4.8)]). ■

**Remark 2.19.**

- (a) As noted in (2.62), the  $A$ -module  ${}_A N$  in the decomposition (2.63) is in general not a  $U_A^{\text{res}}$ -module. Therefore, the  $A$ -linear projection map  $\mathcal{R}_A: \mathcal{L}_{0,n}^A \rightarrow \mathcal{M}_{0,n}^A$  such that  $\text{Ker}(\mathcal{R}_A) = {}_A N$  is not a Reynolds operator, for it does not satisfy the identity  $\mathcal{R}_A(\alpha\beta) = \alpha\mathcal{R}_A(\beta)$  for all  $\alpha \in \mathcal{M}_{0,n}^A$ ,  $\beta \in \mathcal{L}_{0,n}^A$ .
- (b) Recall (2.24). In coherence with (a) above, there is no normalized Haar measure on  $\mathcal{O}_A$  taking values in  $A$ . Indeed, by extending scalars over  $\mathbb{C}(q)$  it should otherwise coincide with the Haar measure  $h: \mathcal{O}_q \rightarrow \mathbb{C}(q)$ , but in the notations of Example 2.17 (see also the comments after (2.44)), since  $h(v_0^* \otimes v_0) = 0$  we have  $h(bc) = -1/(q + q^{-1})$ , whence  $h$  cannot be defined on  $\mathcal{O}_A$ .
- (c) The Haar measure yields a well-defined  $\mathcal{A}_0$ -linear map  $h: \mathcal{L}(\mathcal{O}_q) \rightarrow \mathcal{A}_0$  (and analogously  $\mathcal{A}_0$ -linear and  $\mathcal{A}_\infty$ -linear maps  $h: \mathcal{L}_\diamond(\mathcal{O}_q^{\otimes n}) \rightarrow \mathcal{A}_0$  and  $\bar{h}: \bar{\mathcal{L}}_\diamond(\mathcal{O}_q^{\otimes n}) \rightarrow \mathcal{A}_\infty$  for any  $n \geq 1$ , where  $(\mathcal{L}_\diamond(\mathcal{O}_q^{\otimes n}), \mathcal{B}[[\lambda]]^*)$  is the crystal basis at  $q = 0$  underlying the based  $U_q^{\text{ad}}$ -module (2.59)). Indeed, by (2.41) the lattice  $\mathcal{L}_\lambda^{\text{up}} \otimes_{\mathcal{A}_0} \mathcal{L}_\lambda^{\text{up}}$  is generated by the matrix coefficients in the canonical bases of  $V_\lambda^r$  and  $V_\lambda$ . Since the normalisation by powers of  $q$  is vacuous on the trivial module  $V_0^* \otimes V_0$ , and  $h$  vanishes on  $V_\lambda^* \otimes V_\lambda$  for  $\lambda \in P_+ \setminus \{0\}$ , the claim follows.

### 2.3 Perfect pairings

We will need to restrict the morphisms  $\Phi^+$ ,  $\Phi^-$  in (2.5) on the integral forms  $\mathcal{O}_A(B_+)$ ,  $\mathcal{O}_A(B_-)$ . We collect their properties in Theorem 2.20 and the discussion thereafter. In order to state it, we recall first a few facts about  $R$ -matrices and related pairings.

Recall that  $\mathcal{C}_A$  is the category of  $U_A^{\text{res}}$ -modules of type 1. In [82, 83], Lusztig proved that  $\mathcal{C}_A \otimes_A \mathbb{C}[q^{\pm 1/D}]$  is braided and ribbon, with braiding given by the collection of endomorphisms

$$R = (R_{V,W})_{V,W \in \text{Ob}(\mathcal{C}_A)}.$$

Actually,  $R_{V,W}$  is represented by a matrix with coefficients in  $q^{\mathbb{Z}/D} \mathbb{C}[q^{\pm 1}]$  on the tensor product of the lower canonical bases of  $V$  and  $W$  (see [83, Corollary 24.1.5]).

This can be rephrased as follows in Hopf algebra terms. Denote by  $\mathbb{U}_\Gamma$  the categorical completion of  $\Gamma$ , i.e., the Hopf algebra of natural transformations  $F_{\mathcal{C}_A} \rightarrow F_{\mathcal{C}_A}$ , where  $F_{\mathcal{C}_A}: \mathcal{C}_A \rightarrow A\text{-Mod}_f$  is the forgetful functor towards the category  $A\text{-Mod}_f$  of finite rank  $A$ -modules. Then

$\mathbb{U}_\Gamma \otimes_A \mathbb{C}[q^{\pm 1/D}]$  is quasi-triangular and ribbon with  $R$ -matrix

$$R \in \mathbb{U}_\Gamma^{\hat{\otimes}^2} \bigotimes_A \mathbb{C}[q^{\pm 1/D}].$$

As in (2.3), we can write

$$R^\pm = \sum_{(R)} R_{(1)}^\pm \otimes R_{(2)}^\pm.$$

There are pairings of Hopf algebras naturally related to the  $R$ -matrix  $R$ , considered as an element of  $\mathbb{U}_q^{\hat{\otimes}^2}$ . What follows is standard (see, e.g., [69, 70, 81]), for details we refer to [104, Proposition 3.73, Lemma 3.75, Theorem 3.92, Propositions 3.106 and 3.107]:

- There is a unique pairing of Hopf algebras  $\rho: U_q(\mathfrak{b}_-)^{\text{cop}} \otimes U_q(\mathfrak{b}_+) \rightarrow \mathbb{C}(q^{1/D})$  such that, for every  $\alpha, \lambda \in P$  and  $l, k \in U_q(\mathfrak{h})$ ,

$$\begin{aligned} \rho(K_\lambda, K_\alpha) &= q^{(\lambda, \alpha)}, & \rho(F_i, E_j) &= \delta_{i,j} (q_i - q_i^{-1})^{-1}, \\ \rho(l, E_j) &= \rho(F_i, k) = 0. \end{aligned} \tag{2.64}$$

- The *Drinfeld pairing*  $\tau: U_q(\mathfrak{b}_+)^{\text{cop}} \otimes U_q(\mathfrak{b}_-) \rightarrow \mathbb{C}(q^{1/D})$  is the bilinear map defined by  $\tau(X, Y) = \rho(S(Y), X)$ ; it satisfies

$$\begin{aligned} \tau(K_\lambda, K_\alpha) &= q^{-(\lambda, \alpha)}, & \tau(E_j, F_i) &= -\delta_{i,j} (q_i - q_i^{-1})^{-1}, \\ \tau(l, F_i) &= \tau(E_j, k) = 0. \end{aligned} \tag{2.65}$$

- $\rho$  and  $\tau$  are perfect pairings; this means that they yield *isomorphisms* of Hopf algebras  $i_\pm: U_q(\mathfrak{b}_\pm) \rightarrow \mathcal{O}_q(B_\mp)_{\text{op}}$  (with coefficients *a priori* extended to  $\mathbb{C}(q^{1/D})$ , but see below) defined by, for every  $X \in U_q(\mathfrak{b}_+)$ ,  $Y \in U_q(\mathfrak{b}_-)$ ,

$$\langle i_+(X), Y \rangle = \tau(S(X), Y), \quad \langle i_-(Y), X \rangle = \tau(X, Y).$$

Since  $\mathcal{O}_q(B_\mp)_{\text{op}}$  is equipped with the *inverse* of the antipode of  $\mathcal{O}_q(B_\mp)$ , which is induced by the antipode  $S_{\mathcal{O}_q}$  of  $\mathcal{O}_q$ , it follows that  $i_\pm \circ S = S_{\mathcal{O}_q}^{-1} \circ i_\pm$ .

- Denote by  $p_\pm: \mathcal{O}_q(G) \rightarrow \mathcal{O}_q(B_\pm)$  the canonical projection map, i.e., the Hopf algebra homomorphism dual to the inclusion map  $U_q(\mathfrak{b}_\pm) \hookrightarrow U_q(\mathfrak{g})$ . For every  $\alpha, \beta \in \mathcal{O}_q(G)$ , we have

$$\langle \alpha \otimes \beta, R \rangle = \tau(i_+^{-1}(p_-(\beta)), i_-^{-1}(p_+(\alpha))). \tag{2.66}$$

Note that it is the use of weights  $\alpha, \lambda \in P$  that forces the pairings  $\rho, \tau$  to be defined over  $\mathbb{C}(q^{1/D})$ , instead of  $\mathbb{C}(q)$ . Then, let us consider the restrictions  $\pi_q^+$  of  $\rho$ , and  $\pi_q^-$  of  $\tau$  defined by the formulas (2.64) and (2.65), where now  $\alpha \in Q$  and  $k \in U_q^{\text{ad}}(\mathfrak{h})$ . They take values in  $\mathbb{C}(q)$ , and define pairings

$$\pi_q^+: U_q(\mathfrak{b}_-)^{\text{cop}} \otimes U_q^{\text{ad}}(\mathfrak{b}_+) \rightarrow \mathbb{C}(q), \quad \pi_q^-: U_q(\mathfrak{b}_+)^{\text{cop}} \otimes U_q^{\text{ad}}(\mathfrak{b}_-) \rightarrow \mathbb{C}(q).$$

By the same arguments as for  $\rho$  and  $\tau$  (e.g., in [104, Proposition 3.92]), it follows that  $\pi_q^\pm$  are perfect pairings. Note also that  $\pi_q^- = \kappa \circ \pi_q^+ \circ (\kappa \otimes \kappa)$ , where  $\kappa: U_q \rightarrow U_q$  is the  $\mathbb{C}$ -linear automorphism extending  $-: U_q^{\text{ad}} \rightarrow U_q^{\text{ad}}$  in Section 2.2.2, so defined by

$$\kappa(E_i) = F_i, \quad \kappa(F_i) = E_i, \quad \kappa(K_\lambda) = K_{-\lambda}, \quad \kappa(q) = q^{-1}. \tag{2.67}$$

In [41], De Concini–Lyubashenko described integral forms of  $\pi_q^\pm$  as follows. Denote by  $m^*: \mathcal{O}_A \rightarrow \mathcal{O}_A(B_+) \otimes \mathcal{O}_A(B_-)$  the map dual to the multiplication map  $\Gamma(\mathfrak{b}_+) \otimes \Gamma(\mathfrak{b}_-) \rightarrow \Gamma$ , so  $m^* = (p_+ \otimes p_-) \circ \Delta_{\mathcal{O}_A}$ . Let  $U_A(G^*)$  be the smallest  $A$ -subalgebra of  $U_A(\mathfrak{b}_-)^{\text{cop}} \otimes U_A(\mathfrak{b}_+)^{\text{cop}}$  which contains the elements

$$1 \otimes K_i^{-1} \bar{E}_i, \quad \bar{F}_i K_i \otimes 1, \quad L_i^{\pm 1} \otimes L_i^{\mp 1}, \quad i = 1, \dots, m,$$

and is stable under the diagonal action of  $\mathcal{B}(\mathfrak{g})$ . The reason for the notation  $U_A(G^*)$  will be explained at the beginning of Section 2.5. Note that  $U_A(G^*)$  is free over  $A$ , a Hopf subalgebra, and that a basis is given by the elements

$$\bar{F}_{\beta_1}^{n_1} \cdots \bar{F}_{\beta_N}^{n_N} K_{n_1 \beta_1 + \cdots + n_N \beta_N} K_\lambda \otimes K_{-\lambda} K_{-p_1 \beta_1 - \cdots - p_N \beta_N} \bar{E}_{\beta_1}^{p_1} \cdots \bar{E}_{\beta_N}^{p_N}, \quad (2.68)$$

where  $\lambda \in P$  and  $n_1, \dots, n_N, p_1, \dots, p_N \in \mathbb{N}$ .

Now, let  $v$  be a lowest weight vector of the lowest weight  $\Gamma$ -module  ${}_A V_{-\lambda}$ ,  $\lambda \in P_+$ . As after Theorem 2.1, denote by  $v^* \in {}_A V_{-\lambda}^*$  the dual vector, and by  $\psi_{-\lambda} \in \mathcal{O}_A$  the matrix coefficient defined by  $\langle \psi_{-\lambda}, x \rangle = v^*(xv)$  for every  $x \in \Gamma$ . Consider the maps  $j_q^\pm: \mathcal{O}_q(B_\pm) \rightarrow U_q(\mathfrak{b}_\mp)^{\text{cop}}$  defined by

$$\langle \alpha_+, X \rangle = \pi_q^+(j_q^+(\alpha_+), X), \quad \langle \alpha_-, Y \rangle = \pi_q^-(j_q^-(\alpha_-), Y),$$

where  $\alpha_\pm \in \mathcal{O}_q(B_\pm)$ ,  $X \in U_q^{\text{ad}}(\mathfrak{b}_+)$ , and  $Y \in U_q^{\text{ad}}(\mathfrak{b}_-)$ .

The following theorem summarizes results proved in [41, Sections 3 and 4]. Denote by  $\mathcal{O}_A[\psi_{-\rho}^{-1}]$  the localization of  $\mathcal{O}_A$  by the element  $\psi_{-\rho}$ ; this localization is well defined, for the set  $\{\psi_{-\rho}^n\}_{n \in \mathbb{N}}$  is a left and right multiplicative Ore subset of  $\mathcal{O}_A$  (see Corollary 2.23 below for an analogous statement for  $\mathcal{L}_{0,1}^A$ ). For the sake of clarity, let us spell out the correspondence of notations between statements:  $\pi_q^+$ ,  $\pi_q^-$ ,  $U_q(\mathfrak{b}_\mp)^{\text{cop}}$ ,  $U_A(\mathfrak{b}_\mp)^{\text{cop}}$ ,  $\mathcal{O}_A(B_\pm)$ ,  $U_A(G^*)$  and  $\Phi$  are denoted in [41] respectively by  $\pi''$ ,  $\bar{\pi}''$ ,  $U_q(\mathfrak{b}_\mp)_{\text{op}}$ ,  $R_q[B_\pm]''$ ,  $R_q[B_\pm]$ ,  $A''$  and  $\mu''$  (the definition of  $j_A^\pm$  is implicit in [41, Section 4.2]).

**Theorem 2.20.**

- (1)  $\pi_q^\pm$  restricts to a perfect Hopf pairing between the unrestricted and restricted integral forms,  $\pi_A^\pm: U_A(\mathfrak{b}_\mp)^{\text{cop}} \otimes \Gamma(\mathfrak{b}_\pm) \rightarrow A$ .
- (2)  $j_q^\pm$  yields an isomorphism of Hopf algebras  $j_A^\pm: \mathcal{O}_A(B_\pm) \rightarrow U_A(\mathfrak{b}_\mp)^{\text{cop}}$ , satisfying  $\langle \alpha_\pm, x_\pm \rangle = \pi_A^\pm(j_A^\pm(\alpha_\pm), x_\pm)$  for every  $\alpha_\pm \in \mathcal{O}_A(B_\pm)$ ,  $x_\pm \in \Gamma(\mathfrak{b}_\pm)$ .
- (3) The map  $\Phi := (j_A^+ \otimes j_A^-) \circ m^*: \mathcal{O}_A \rightarrow U_A(G^*) \subset U_A(\mathfrak{b}_-)^{\text{cop}} \otimes U_A(\mathfrak{b}_+)^{\text{cop}}$  is an embedding of Hopf algebras, and it extends to an isomorphism  $\Phi: \mathcal{O}_A[\psi_{-\rho}^{-1}] \rightarrow U_A(G^*)$ .

For our purposes, it is necessary to reformulate this result. Consider the morphisms of Hopf algebras  $\Phi^\pm: \mathcal{O}_A(B_\pm) \rightarrow U_A(\mathfrak{b}_\mp)^{\text{cop}}$ ,  $\alpha \mapsto (\alpha \otimes \text{id})(R^\pm)$ .

**Lemma 2.21.** *We have  $\Phi^\pm = j_A^\pm$ .*

**Proof.** By definitions, for every  $X \in U_q(\mathfrak{b}_+)^{\text{cop}}$ ,  $Y \in U_q^{\text{ad}}(\mathfrak{b}_-)$ , we have  $\langle i_+(S^{-1}(X)), Y \rangle = \pi_q^-(X, Y)$ , and similarly for every  $X \in U_q^{\text{ad}}(\mathfrak{b}_+)$ ,  $Y \in U_q(\mathfrak{b}_-)^{\text{cop}}$ , we have  $\langle i_-(S^{-1}(Y)), X \rangle = \pi_q^+(Y, X)$ . By keeping these notations for  $X$  and  $Y$ , we deduce  $j_q^-(i_+(S^{-1}(X))) = X$  and  $j_q^+(i_-(S^{-1}(Y))) = Y$ , i.e.,  $j_q^\pm = S \circ i_\mp^{-1}$ . Because  $S_{\mathcal{O}_q}^{-1} \circ i_\pm = i_\pm \circ S$ , it follows that

$$j_q^\pm \circ S_{\mathcal{O}_q} = S^{-1} \circ j_q^\pm. \quad (2.69)$$

Also, for every  $\alpha_- \in \mathcal{O}_q(B_-)$ , we have

$$\begin{aligned} \langle \alpha_-, \Phi^+(i_-(Y)) \rangle &= \langle i_-(Y) \otimes \alpha_-, R \rangle = \tau(i_+^{-1}(\alpha_-), Y) \\ &= \pi_q^-(j_q^-(S_{\mathcal{O}_q}(\alpha_-)), Y) = \langle \alpha_-, S(Y) \rangle, \end{aligned}$$

where the first equality is by definition of  $\Phi^+$  (see (2.5)), the second is (2.66), the third follows from (2.69), and the last from the definition of  $j_q^-$ . Similarly, for every  $\alpha_+ \in \mathcal{O}_q(B_+)$ , we have

$$\begin{aligned} \langle \alpha_+, \Phi^-(i_+(X)) \rangle &= \langle i_+(X) \otimes \alpha_+, R^- \rangle = \langle \alpha_+ \otimes S_{\mathcal{O}_q}^{-1} \circ i_+(X), R \rangle = \langle \alpha_+ \otimes i_+(S(X)), R \rangle \\ &= \tau(S(X), i_-^{-1}(\alpha_+)) = \pi_q^+(S(i_-^{-1}(\alpha_+)), S(X)) \\ &= \pi_q^+(j_q^+(\alpha_+), S(X)) = \langle \alpha_+, S(X) \rangle. \end{aligned}$$

These computations imply  $\Phi^\pm = S \circ i_\mp^{-1} = j_q^\pm$ , and the result follows by taking integral forms.  $\blacksquare$

## 2.4 Integral form and specialization of $\Phi_n$

Recall the isomorphism of  $U_q$ -module algebras  $\Phi_1: \mathcal{L}_{0,1} \rightarrow U_q^{\text{lf}}$ , and that  $U_A^{\text{lf}} = U_A \cap U_q^{\text{lf}}$ . We have:

**Corollary 2.22.** *The map  $\Phi_1$  affords an embedding of  $U_A^{\text{res}}$ -module algebras  $\Phi_1: \mathcal{L}_{0,1}^A \rightarrow U_A^{\text{lf}}$ .*

**Proof.** The only thing to be proved is that  $\Phi_1(\mathcal{O}_A) \subset U_A^{\text{lf}}$ , since  $\mathcal{L}_{0,1}^A = \mathcal{O}_A$  as  $A$ -module. But Lemma 2.21 and (2.12) imply  $\Phi_1 = m \circ (\text{id} \otimes S^{-1}) \circ \Phi$ , and  $\Phi$  maps  $\mathcal{O}_A$  into  $U_A(\mathfrak{b}_-)^{\text{cop}} \otimes U_A(\mathfrak{b}_+)^{\text{cop}}$  by Theorem 2.20. The conclusion follows.  $\blacksquare$

Let us denote

$$d = \psi_{-\rho} \in \mathcal{L}_{0,1}^A.$$

(The linear forms  $\psi_{-\lambda}$  have been introduced before Theorem 2.20.) When  $\mathfrak{g} = \mathfrak{sl}_2$  the element  $d$  is one of the ‘‘standard’’ generators of  $\mathcal{L}_{0,1}(\mathfrak{sl}_2)$  (see (4.5) below). In this case we have shown in [18, Lemma 5.7] that  $\mathcal{L}_{0,1}^A$  has a well-defined localization  $\mathcal{L}_{0,1}^A[d^{-1}]$ , and that  $\Phi_1: \mathcal{L}_{0,1}^A[d^{-1}] \rightarrow U_A^{\text{ad}} = T_{2^-}^{-1}U_A^{\text{lf}}$  is an isomorphism of algebras. A generalization of these facts to any  $\mathfrak{g}$  is provided by the following statement. As usual  $\ell = K_{2\rho}$ , the pivotal element.

**Corollary 2.23.**

- (1) *The set  $\{d^n\}_{n \in \mathbb{N}}$  is a left and right multiplicative Ore set in  $\mathcal{L}_{0,1}^A$ . We can therefore define the localization  $\mathcal{L}_{0,1}^A[d^{-1}]$ .*
- (2)  *$\Phi_1$  extends to an embedding of  $U_A^{\text{res}}$ -module algebras  $\Phi_1: \mathcal{L}_{0,1}^A[d^{-1}] \rightarrow U_A^{\text{lf}}[\ell]$ , and  $U_A^{\text{lf}}[\ell] = T_{2^-}^{-1}U_A^{\text{lf}}$ .*

**Proof.** (1) Because  $\mathcal{L}_{0,1}^A$  has no nontrivial zero divisors,  $d$  is a regular element. We have to show that for all  $x \in \mathcal{L}_{0,1}^A$  there exists elements  $y, y' \in \mathcal{L}_{0,1}^A$  and  $d', d'' \in \{d^n\}_{n \in \mathbb{N}}$  such that  $xd' = dy$  and  $d''x = y'd$ . In fact,  $d' = d'' = d$  in the present situation. Indeed by (2.13), we have  $\Phi_1(x)\Phi_1(d) = \Phi_1(x)K_{-2\rho} = K_{-2\rho}\text{ad}^r(K_{2\rho})(\Phi_1(x))$ , and  $\text{ad}^r(K_{2\rho})(\Phi_1(x)) = \Phi_1(\text{coad}^r(K_{2\rho})(x))$ . Therefore, the left Ore condition is satisfied with  $y = \text{coad}^r(K_{2\rho})(x)$ . Similarly, one finds  $y'$ .

(2) The first claim follows immediately from Corollary 2.22 and  $\Phi_1(d) = \ell^{-1}$ , which is a regular element of  $U_A$ . For the second claim, since  $K_{-2\rho} = \prod_{j=1}^m L_j^{-2}$ , localizing in  $d$  we obtain

$$L_j^2 = \prod_{k \neq j} L_k^{-2} \Phi_1(d^{-1}) = \Phi_1 \left( \prod_{k \neq j} \psi_{-\varpi_k} d^{-1} \right) \in \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}]).$$

Therefore,  $T_{2^-}^{-1} \subset \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$ , which implies the assertion (2).  $\blacksquare$

We expect that the inclusion  $\Phi_1(\mathcal{O}_A) \subset U_A^{\text{lf}}$  is an equality, but have no proof yet. However, recall Joseph–Letzter’s Theorem 2.1 (1) and (2).

**Proposition 2.24.** *We have*

$$U_A = T_2^{-1}U_A^{\text{lf}}[T/T_2] = \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2],$$

and therefore  $\Phi_1: \mathcal{L}_{0,1}^A[d^{-1}] \rightarrow T_2^{-1}U_A^{\text{lf}}$  is an isomorphism. Moreover,

$$\Phi_1(\mathcal{O}_A) = \bigoplus_{\lambda \in 2P_+} \text{ad}^r(U_A^{\text{res}})(K_{-\lambda}).$$

**Proof.** The inclusions  $T \subset U_A$ ,  $U_A^{\text{lf}} \subset U_A$  and  $\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}]) \subset T_2^{-1}U_A^{\text{lf}}$  imply

$$\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2] \subset T_2^{-1}U_A^{\text{lf}}[T/T_2] \subset U_A.$$

For the inverse inclusion, it is enough to show that any PBW basis vector of  $U_A$  lies in  $\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2]$ . This will follow at once if this is true of all root vectors  $\bar{E}_{\beta_k}, \bar{F}_{\beta_k}$ . Let us show this explicitly for the simple root vectors  $\bar{E}_i$  and  $\bar{F}_i$ . For every positive root  $\alpha$ , define elements  $\psi_{-\lambda}^{\alpha}, \psi_{-\lambda}^{-\alpha} \in \mathcal{O}_A$  by the formulas

$$\langle \psi_{-\lambda}^{\alpha}, x \rangle = v^*(xE_{\alpha}v), \quad \langle \psi_{-\lambda}^{-\alpha}, x \rangle = v^*(F_{\alpha}xv),$$

where  $x \in \Gamma$ . It is shown in [41, Lemma 4.5] that

$$\begin{aligned} \Phi(\psi_{-\lambda}^{\alpha}) &= K_{-\lambda} \otimes K_{\lambda}, & \Phi(\psi_{-\varpi_j}^{\alpha_i}) &= -\delta_{i,j} q_i L_i^{-1} \otimes L_i K_i^{-1} \bar{E}_i, \\ \Phi(\psi_{-\varpi_j}^{-\alpha_i}) &= \delta_{i,j} q_i^{-1} \bar{F}_i K_i L_i^{-1} \otimes L_i. \end{aligned}$$

(Note that the generators denoted by  $E_i$  and  $F_i$  in [41] are respectively  $K_i^{-1}E_i$  and  $F_i K_i$  in our notations, which explains the factors  $q_i, q_i^{-1}$  in the formulas below; also  $\kappa$  in (2.67) maps  $\bar{E}_i, \bar{F}_i$  to  $-\bar{F}_i, -\bar{E}_i$ , whence the sign for the expression of  $\Phi(\psi_{-\varpi_j}^{\alpha_i})$ .) Since  $\Phi_1 = m \circ (\text{id} \otimes S^{-1}) \circ \Phi$ , we have

$$\Phi_1(\psi_{-\lambda}^{\alpha}) = K_{-2\lambda}, \quad \Phi_1(\psi_{-\varpi_j}^{\alpha_i}) = \delta_{i,j} L_i^{-2} \bar{E}_i, \quad \Phi_1(\psi_{-\varpi_j}^{-\alpha_i}) = \delta_{i,j} q_i^{-1} \bar{F}_i K_i L_i^{-2}. \quad (2.70)$$

Therefore,

$$\bar{E}_i, \bar{F}_i, L_i^{\pm 1} \in T_2^{-1}\Phi_1(\mathcal{L}_{0,1}^A)[T/T_2] = \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2].$$

These elements do not generate  $U_A$ ; it is necessary to consider general root vectors. By the stability of  $U_A(G^*)$  under  $\mathcal{B}(\mathfrak{g})$  and the isomorphism  $\mathcal{O}_A[\psi_{-\rho}^{-1}] \rightarrow U_A(G^*)$  of Theorem 2.20 (3), for every positive root  $\beta_k$ , we have  $1 \otimes K_{\beta_k}^{-1} \bar{E}_{\beta_k}, \bar{F}_{\beta_k} K_{\beta_k} \otimes 1 \in \Phi(\mathcal{O}_A[\psi_{-\rho}^{-1}]) = \Phi(\mathcal{L}_{0,1}^A[d^{-1}])$ . Therefore,  $\bar{F}_{\beta_k} K_{\beta_k}, S^{-1}(\bar{E}_{\beta_k}) K_{\beta_k} \in \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$ , and  $\bar{F}_{\beta_k}, S^{-1}(\bar{E}_{\beta_k}) \in \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2]$ . The sets  $S^{-1}(\bar{E}_{\beta_k})U_A(\mathfrak{h})$  generate the subalgebra  $U_A(\mathfrak{b}_+)$  of  $U_A$  (in fact, let us quote that a formula of  $S^{-1}(\bar{E}_{\beta_k})$  is given in [107]). From the triangular decomposition  $U_A = U_A(\mathfrak{n}_-)U_A(\mathfrak{h})U_A(\mathfrak{n}_+)$ , the inclusion  $U_A \subset \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2]$  follows, whence the equality too. In particular,  $U_A$  is a free  $\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$ -module with a basis formed by representatives of the cosets in  $T/T_2$ . By the uniqueness of this free decomposition, we find  $\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}]) = T_2^{-1}U_A^{\text{lf}}$ . Therefore,  $\Phi_1$  in Corollary 2.23 (2) is surjective.

For the third claim, recall the isomorphism  $\Phi_1: C(-w_0(\mu)) \rightarrow \text{ad}^r(U_q)(K_{-2\mu})$  (see (2.14)), and that  $\psi_{-\mu}$  is the matrix coefficient dual to the vector  ${}^{\omega}v_{-\mu} \otimes v_{-\mu} \in \text{End}_A(V_{-w_0(\mu)})$ . This vector is cyclic by (2.32), so by equivariance  $\Phi_1: {}_A C(-w_0(\mu)) \rightarrow \text{ad}^r(U_A^{\text{res}})(K_{-2\mu})$  is an isomorphism of  $U_A^{\text{res}}$ -modules. The second claim follows from this and (2.60) for  $n = 1$ .  $\blacksquare$

Recall from (2.20) the isomorphisms of  $U_q$ -module algebras  $\Phi_n: \mathcal{L}_{0,n} \rightarrow (U_q^{\otimes n})^{\text{lf}}$  and of algebras  $\Phi_n: \mathcal{M}_{0,n} \rightarrow (U_q^{\otimes n})^{U_q}$ , and from (2.27) the notations for specializations. Corollary 2.22 can be extended to  $\Phi_n$  as follows:

**Corollary 2.25.** *The map  $\Phi_n$  affords embeddings of module algebras  $\Phi_n: \mathcal{L}_{0,n}^A \rightarrow (U_A^{\otimes n})^{\text{lf}}$  and  $\Phi_n: \mathcal{L}_{0,n}^{\epsilon'} \rightarrow (U^{\otimes n})_{\epsilon'}^{\text{lf}}$ ,  $q = \epsilon' \in \mathbb{C}^\times$ .*

**Proof.** For the first claim, the only thing to prove is the inclusion  $\Phi_n(\mathcal{L}_{0,n}^A) \subset U_A^{\otimes n}$ . It follows from Corollary 2.22 and the expression of  $\Phi_n$  in terms of  $\Phi_1$  and  $R$ -matrices (in particular, the fact that they preserve integrality, see [18, Lemma 6.10]). For the specialization at  $q = \epsilon' \in \mathbb{C}^\times$ , we have to justify that  $\Phi_n$  is injective. One uses the fact, to be developed in Theorem 2.29 below, that  $\Phi: \mathcal{O}_\epsilon \rightarrow U_\epsilon(G^*)$  is an embedding. The algebra  $U_\epsilon(G^*)$  has the basis elements (2.68), and the map  $m \circ (\text{id} \otimes S^{-1})$  sends this basis to a free family of  $U_\epsilon$ . Therefore,  $\Phi_1: \mathcal{L}_{0,1}^\epsilon \rightarrow U_\epsilon$  is injective. Since  $\Phi_n$  differs from  $\Phi_1^{\otimes n}$  by a linear isomorphism (induced by the conjugation action of  $R$ -matrices on the components  ${}_A C([\lambda])$  of  $\mathcal{L}_{0,n}^A$  in (2.60), see [18, equation (6.10)]),  $\Phi_n: \mathcal{L}_{0,n}^\epsilon \rightarrow U_\epsilon^{\otimes n}$  is an embedding as well. ■

**Remark 2.26.**

- (1) It is a natural problem to determine the image of  $\Phi_n$ . One may expect that it would be  $(T_{2-}^{-1}U_A^{\text{lf}})^{\otimes n}$ , because this is true for  $n = 1$ , as well as for any  $n$  in the  $\mathfrak{sl}_2$  case, as shown in [18]. Unfortunately, this is not so. This comes from the fact, e.g., for  $n = 2$ , that the matrix elements of  $R_{02}R_{01}R'_{01}R_{02}^{-1}$  do not belong to  $(T_{2-}^{-1}U_A^{\text{lf}})^{\otimes 2}$  as can be shown by an explicit computation in the  $\mathfrak{sl}(3)$  case.
- (2) In the case of  $\mathfrak{g} = \mathfrak{sl}_2$ , we defined in [18] an algebra  ${}_{\text{loc}}\mathcal{L}_{0,n}^A$  generalizing  $\mathcal{L}_{0,1}^A[d^{-1}]$  above, containing  $\mathcal{L}_{0,n}^A$  as a subalgebra, and such that  $\Phi_n$  extends to  ${}_{\text{loc}}\mathcal{L}_{0,n}^A$  and yields an isomorphism  $\Phi_n: {}_{\text{loc}}\mathcal{L}_{0,n}^A \rightarrow U_A^{\text{ad}}(\mathfrak{sl}_2)^{\otimes n}$ . The definition of  ${}_{\text{loc}}\mathcal{L}_{0,n}^A$  involves elements  $\xi^{(i)} \in \mathcal{L}_{0,n}^A$  ( $i = 1, \dots, n$ ) such that  $\Phi_n(\xi^{(i)}) = (K^{-1})^{(i)} \dots (K^{-1})^{(n)}$ . It may be of interest to study a similar extension of  $\Phi_n$  for general  $\mathfrak{g}$ .

## 2.5 Structure theorems for $U_\epsilon$ and $\mathcal{O}_\epsilon$

As usual, we denote by  $\epsilon$  a primitive  $l$ -th root of unity, where  $l$  is odd, and coprime to 3 if  $\mathfrak{g}$  has  $G_2$ -components.

Recall the subgroups  $T_G$ ,  $U_\pm$  and  $B_\pm$  of  $G$ . Let  $G^0 = B_+B_-$  (the *big cell* of  $G$ ), and define the subgroup

$$G^* = \{(u_+t, u_-t^{-1}), t \in T_G, u_\pm \in U_\pm\} \subset B_+^{\text{op}} \times B_-^{\text{op}},$$

where  $B_\pm^{\text{op}}$  is the group  $B_\pm$  with opposite multiplication. The group  $G^*$  can be naturally identified with the Poisson–Lie dual of  $G$  with its standard structure.

Recall also that there is an injective homomorphism  $\gamma_q^{-1} \circ h_q: \mathcal{Z}(U_q) \rightarrow U_q(\mathfrak{h})$ , defined by means of the quantum Harish-Chandra homomorphism (see, e.g., [35, Section 9.1.C], or [104, Section 3.13]). The image of  $\gamma_q^{-1} \circ h_q$  is the set  $U_q(\mathfrak{h})^{\tilde{W}}$  of invariant elements under  $\tilde{W}$ , the subgroup of  $W \times P_2^*$  generated by the conjugates  $\sigma W \sigma$  of  $W$  by elements  $\sigma \in P_2^*$ . Here,  $P_2^*$  is the group of homomorphisms  $P \rightarrow \mathbb{Z}/2\mathbb{Z}$ , and the semidirect product  $W \times P_2^*$  acts on  $U_q(\mathfrak{h})$  by the standard action of the Weyl group  $W$ , and by the action of  $P_2^*$  given by  $\sigma \cdot K_\lambda := \sigma(\lambda)K_\lambda$ .

Consider the inverse map  $h_q^{-1} \circ \gamma_q: U_q(\mathfrak{h})^{\tilde{W}} \rightarrow \mathcal{Z}(U_q)$ . The elements of the domain and target, when expanded in the PBW basis, have coefficients in  $\mathbb{C}(q)$ . It was shown in [42, Section 21.1] that if an element of  $U_q(\mathfrak{h})^{\tilde{W}}$  has no coefficient with a pole at  $q = \epsilon$ , then its image by  $h_q^{-1} \circ \gamma_q$  has no coefficient with a pole at  $q = \epsilon$ . We therefore have a well-defined injection

$$U_\epsilon(\mathfrak{h})^{\tilde{W}} \rightarrow \mathcal{Z}(U_\epsilon).$$

We denote its image by  $\mathcal{Z}_1(U_\epsilon)$ . For instance, when  $U_\epsilon = U_\epsilon(\mathfrak{sl}_2)$ ,  $\mathcal{Z}_1(U_\epsilon)$  is the polynomial algebra generated by the Casimir element  $\Omega = (\epsilon - \epsilon^{-1})^2 FE + \epsilon K + \epsilon^{-1} K^{-1}$ .

Denote by  $\mathcal{Z}_0(U_\epsilon) \subset U_\epsilon$  the smallest subalgebra containing the elements  $E_i^l, F_i^l, K_\alpha^l$ , for  $i \in \{1, \dots, m\}$ ,  $\alpha \in P$ , and stable under  $\mathcal{B}(\mathfrak{g})$ ; it is also the subalgebra generated by  $E_{\beta_k}^l, F_{\beta_k}^l, L_i^{\pm l}$ , for  $k \in \{1, \dots, N\}$  and  $i \in \{1, \dots, m\}$  [42, Section 18]. We will denote by  $\mathcal{Z}_0(U_\epsilon(\mathfrak{n}_-))$ ,  $\mathcal{Z}_0(U_\epsilon(\mathfrak{h}))$  and  $\mathcal{Z}_0(U_\epsilon(\mathfrak{n}_+))$  the subalgebras of  $\mathcal{Z}_0(U_\epsilon)$  generated by the elements  $F_{\beta_k}^l, K_\lambda^l$  ( $\lambda \in P$ ), and  $E_{\beta_k}^l$ , respectively. In [39, Sections 1.8, 3.3 and 3.8] and [42, Theorem 14.1 and Sections 20–21], the following results are proved:

**Theorem 2.27.**

- (1)  $U_\epsilon$  has no nontrivial zero divisors,  $\mathcal{Z}_0(U_\epsilon)$  is a central Hopf subalgebra of  $U_\epsilon$ , and  $U_\epsilon$  is a free  $\mathcal{Z}_0(U_\epsilon)$ -module of rank  $l^{\dim \mathfrak{g}}$ . Moreover, the classical fraction algebra  $Q(U_\epsilon) = Q(\mathcal{Z}(U_\epsilon)) \otimes_{\mathcal{Z}(U_\epsilon)} U_\epsilon$  is a central simple algebra of PI degree  $l^N$ , and  $U_\epsilon$  is a maximal order of  $Q(U_\epsilon)$ .
- (2)  $\text{Maxspec}(\mathcal{Z}_0(U_\epsilon))$  is a group isomorphic to  $G^*$  above, and the multiplication map yields an isomorphism  $\mathcal{Z}_0(U_\epsilon) \otimes_{\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon)} \mathcal{Z}_1(U_\epsilon) \rightarrow \mathcal{Z}(U_\epsilon)$ .

By this theorem, the dimension of  $Q(U_\epsilon)$  over its center  $Q(\mathcal{Z}(U_\epsilon))$  is  $l^{2N}$ , and its dimension over  $Q(\mathcal{Z}_0(U_\epsilon))$  is  $l^{\dim \mathfrak{g}} = l^{m+2N}$ . Therefore, the field  $Q(\mathcal{Z}(U_\epsilon))$  is an extension of  $Q(\mathcal{Z}_0(U_\epsilon))$  of degree  $l^m$ .

Note that, because  $\mathcal{Z}_0(U_\epsilon)$  is an affine and commutative algebra, the maximal spectrum  $\text{Maxspec}(\mathcal{Z}_0(U_\epsilon))$ , viewed as the set of characters of  $\mathcal{Z}_0(U_\epsilon)$ , acquires by duality a structure of affine algebraic group. Thus, the first claim of (2) in the theorem means precisely that this group can be identified with  $G^*$ . See, for instance, [18, Section 7.2.1] for an explicit description in the  $\mathfrak{sl}_2$  case.

In addition,  $\text{Maxspec}(\mathcal{Z}_0(U_\epsilon))$  and  $G^*$  have natural Poisson structures which correspond one to the other under the isomorphism of (2), and we have the following identifications (see [42, Section 21.2]). The dual isomorphism  $\mathcal{O}(G^*) \rightarrow \mathcal{Z}_0(U_\epsilon)$  identifies  $\mathcal{O}(T_G)$  with  $\mathcal{Z}_0(U_\epsilon) \cap U_\epsilon(\mathfrak{h}) = \mathbb{C}[lP]$ , where as usual  $U_\epsilon(\mathfrak{h}) = U_A(\mathfrak{h}) \otimes_A \mathbb{C}_\epsilon$ . Therefore, we can identify  $\mathbb{C}[P]$  with  $\mathcal{O}(\tilde{T}_G)$ , the coordinate ring of the  $l^m$ -fold covering space  $\tilde{T}_G \rightarrow T_G$ . The quantum Harish-Chandra isomorphism identifies  $\mathcal{Z}_1(U_\epsilon)$  with  $\mathbb{C}[2P]^W \cong \mathcal{O}(\tilde{T}_G/(2))^W$ , where we denote by (2) the subgroup of 2-torsion elements in  $T_G$ . Consider the map

$$\sigma: B_+ \times B_- \longrightarrow G^0, \quad (b_+, b_-) \longmapsto b_+ b_-^{-1}.$$

The restriction of  $\sigma$  to  $G^*$  is an unramified covering map of degree  $2^m$ . Composing  $\sigma: G^* \rightarrow G^0$  with the quotient map under conjugation,  $G^0 \hookrightarrow G \rightarrow G//G$ , we get dually an embedding of  $\mathcal{O}(G//G) = \mathcal{O}(G)^G$  in  $\mathcal{O}(G^*)$ . Collecting these observations, we see that the isomorphism of Theorem 2.27 (2) affords identifications

$$\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon) \cong \mathcal{O}(G)^G$$

as a subalgebra of  $\mathcal{Z}_0(U_\epsilon) \cong \mathcal{O}(G^*)$ , and

$$\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon) = \mathbb{C}[2lP]^W \cong \mathcal{O}(\tilde{T}_G/(2l))^W \cong \mathcal{O}(T_G/(2))^W$$

as a subalgebra of  $\mathcal{Z}_1(U_\epsilon) \cong \mathcal{O}(\tilde{T}_G/(2))^W$ .

We will use the following obvious though crucial fact. Note that  $U_A^{\text{ad}}$  is naturally a subalgebra of  $U_A^{\text{res}}$ , and therefore acts on  $U_\epsilon^{\text{res}}$ -modules. Denote by  $\mathcal{Z}_0(U_A^{\text{ad}}) \subset U_A^{\text{ad}}$  the subalgebra generated by the elements  $\bar{E}_{\beta_k}^l, \bar{F}_{\beta_k}^l, K_i^{\pm l}$ , for  $k \in \{1, \dots, N\}$  and  $i \in \{1, \dots, m\}$ .

**Lemma 2.28.** *For every  $U_A^{\text{res}}$ -module  $V$  of type 1, the action of  $\mathcal{Z}_0(U_A^{\text{ad}})$  on the specialization  $V_\epsilon := V \otimes_A \mathbb{C}_\epsilon$  is trivial.*



**Proof.** This comes from  $E_i^l = [l]_{q_i}! E_i^{(l)}$ ,  $F_i^l = [l]_{q_i}! F_i^{(l)}$  and the fact that  $K_i$  acts on  $V$  by powers of  $q_i$ . Specializing to  $q = \epsilon$  ends the proof.  $\blacksquare$

A result similar to Theorem 2.27 holds true for  $\mathcal{O}_\epsilon$ . Namely, take the specializations at  $q = \epsilon$  in Theorem 2.20. Denote by  $\mathcal{Z}_0(U_\epsilon(G^*))$  the subalgebra of  $U_\epsilon(G^*)$  generated by the elements  $(k \in \{1, \dots, N\}, i \in \{1, \dots, m\})$

$$1 \otimes K_{-l\beta_k} E_{\beta_k}^l, \quad F_{\beta_k}^l K_{l\beta_k} \otimes 1, \quad L_i^{\pm l} \otimes L_i^{\mp l}.$$

It is a central Hopf subalgebra. Recall that the coordinate ring  $\mathcal{O}(G)$  can be identified as a Hopf algebra with  $U(\mathfrak{g})^\circ$ , where as usual  $U(\mathfrak{g})^\circ$  denotes the restricted dual of the enveloping algebra  $U(\mathfrak{g})$  over  $\mathbb{C}$ . In [41, Section 6], De Concini–Lyubashenko introduced an epimorphism of Hopf algebras  $\eta: \Gamma_\epsilon \rightarrow U(\mathfrak{g})$  (essentially a version of Lusztig’s ‘‘Frobenius’’ epimorphism in [82]), defined by

$$\begin{aligned} \eta(E_i^{(p)}) &= \begin{cases} \frac{e_i^{p/l}}{(p/l)!} & \text{if } l \text{ divides } p, \\ 0 & \text{otherwise,} \end{cases} & \eta(F_i^{(p)}) &= \begin{cases} \frac{f_i^{p/l}}{(p/l)!} & \text{if } l \text{ divides } p, \\ 0 & \text{otherwise,} \end{cases} \\ \eta(K_i) &= 1, & \eta((K_i; p)_{q_i}) &= \begin{cases} \frac{h_i(h_i - 1) \cdots (h_i - (p/l) + 1)}{(p/l)!} & \text{if } l \text{ divides } p, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (2.71)$$

where  $p \in \mathbb{N}$ , and  $e_i$ ,  $f_i$  and  $h_i$ ,  $i \in \{1, \dots, m\}$ , denote the standard generators of  $U(\mathfrak{g})$ . The kernel of  $\eta$  is generated by the elements  $E_i$ ,  $F_i$ ,  $K_i - 1$ , and  $(K_i; p)_{q_i}$  where  $l$  does not divide  $p$ . Put

$$\mathcal{Z}_0(\mathcal{O}_\epsilon) := \eta^*(\mathcal{O}(G)), \quad (2.72)$$

where  $\eta^*: U(\mathfrak{g})^\circ \rightarrow \Gamma_\epsilon^\circ$  is the monomorphism dual to  $\eta$ . Let us define special matrix coefficients, analogous to those introduced in Theorem 2.20. Denote by  $v_{\varpi_i}$  and  $v_{w_0(\varpi_i)}$  a highest weight vector and a lowest weight vector of the  $\Gamma$ -module  ${}_A V_{\varpi_i}$ . Denote also by  $v_{w_0(\varpi_i)}^*$  and  $v_{\varpi_i}^*$  a highest and lowest weight vector of the dual module  $\Gamma$ -module  ${}_A V_{\varpi_i}^* \cong {}_A V_{-w_0(\varpi_i)}$ . Define the matrix coefficients  $b_{\varpi_i}, c_{\varpi_i} \in \mathcal{O}_A$  by

$$b_{\varpi_i}(x) = v_{\varpi_i}^*(xv_{w_0(\varpi_i)}), \quad c_{\varpi_i}(x) = v_{w_0(\varpi_i)}^*(xv_{\varpi_i})$$

for all  $x \in \Gamma$ . We consider them as elements of  $\mathcal{O}_\epsilon$ . Denote by  $\mathcal{Z}_1(\mathcal{O}_\epsilon)$  the subalgebra of  $\mathcal{O}_\epsilon$  generated by the elements  $b_{\varpi_i}^k c_{\varpi_i}^{l-k}$  for  $1 \leq i \leq m$  and  $0 \leq k \leq l$ .

**Theorem 2.29.**

- (1)  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  is a central Hopf subalgebra of  $\mathcal{O}_\epsilon \subset \Gamma_\epsilon^\circ$ , and  $Q(\mathcal{Z}(\mathcal{O}_\epsilon))$  is an extension of  $Q(\mathcal{Z}_0(\mathcal{O}_\epsilon))$  of degree  $l^m$ .
- (2)  $\psi_{-l\rho} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ , and  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  is generated by matrix coefficients of irreducible  $\Gamma$ -modules of highest weight  $l\lambda$ ,  $\lambda \in P_+$ . Moreover, the multiplication map yields an isomorphism

$$\mathcal{Z}_0(\mathcal{O}_\epsilon) \otimes_{\mathcal{Z}_0(\mathcal{O}_\epsilon) \cap \mathcal{Z}_1(\mathcal{O}_\epsilon)} \mathcal{Z}_1(\mathcal{O}_\epsilon) \rightarrow \mathcal{Z}(\mathcal{O}_\epsilon),$$

and the map  $\Phi$  in Theorem 2.20 affords an algebra embedding  $\mathcal{Z}_0(\mathcal{O}_\epsilon) \rightarrow \mathcal{Z}_0(U_\epsilon(G^*))$  and algebra isomorphisms  $\mathcal{Z}_0(\mathcal{O}_\epsilon)[\psi_{-l\rho}^{-1}] \rightarrow \mathcal{Z}_0(U_\epsilon(G^*))$  and  $\mathcal{O}_\epsilon[\psi_{-l\rho}^{-1}] \rightarrow U_\epsilon(G^*)$ .

- (3)  $\mathcal{O}_\epsilon$  has no nontrivial zero divisors, and it is a free  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ -module of rank  $l^{\dim \mathfrak{g}}$ . Moreover, the classical fraction algebra  $Q(\mathcal{O}_\epsilon) = Q(\mathcal{Z}(\mathcal{O}_\epsilon)) \otimes_{\mathcal{Z}(\mathcal{O}_\epsilon)} \mathcal{O}_\epsilon$  is a central simple algebra of PI degree  $l^N$ , and  $\mathcal{O}_\epsilon$  is a maximal order of  $Q(\mathcal{O}_\epsilon)$ .

For the proof, see [41]: Proposition 6.4 for the first claim of (1) (where  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  and  $\mathcal{Z}_0(U_\epsilon(G^*))$  are denoted  $F_0$  and  $A_0$  respectively), the appendix of Enriquez and [50] for the second claim of (1) and (2), Propositions 6.4 and 6.5 for the other claims of (2), Theorem 7.2 (where  $\mathcal{O}_\epsilon$  is shown to be projective over  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ ) and [28] (which provides the additional K-theoretic arguments to deduce that  $\mathcal{O}_\epsilon$  is free), or [6, Remark 2.18 (b)], for the second claim of (3), and Corollary 7.3 and Theorem 7.4 for the third claim. The fact that  $\mathcal{O}_\epsilon$  has no nontrivial zero divisors follows from the embedding  $\mathcal{O}_\epsilon \rightarrow U_\epsilon(G^*)$  via  $\Phi$ .

As above for  $U_\epsilon$ , it follows directly from (3) that  $Q(\mathcal{Z}(\mathcal{O}_\epsilon))$  has degree  $l^m$  over  $Q(\mathcal{Z}_0(\mathcal{O}_\epsilon))$ . For a more complete description of  $\mathcal{Z}(\mathcal{O}_\epsilon)$  we refer to [50] and Enriquez' appendix in [41], as well as [27].

We do not know a basis of  $\mathcal{O}_\epsilon$  over  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  for general  $G$ , but see [38] for the case of  $\mathrm{SL}_2$ . We will recall the known results in this case of  $\mathrm{SL}_2$  before Lemma 4.5.

Finally, there is a natural action of the braid group  $\mathcal{B}(\mathfrak{g})$  on  $\mathcal{O}_\epsilon$ , that we will use. Namely, let  $n_i \in N(T_G)$  be a representative of the reflection  $s_i \in W = N(T_G)/T_G$  associated to the simple root  $\alpha_i$ . In [98, 102], Soibelman–Vaksman introduced functionals  $t_i: \mathcal{O}_q \rightarrow \mathbb{C}(q)$  which quantize the elements  $n_i$ . They correspond dually to generators of the quantum Weyl group of  $\mathfrak{g}$ ; in the appendix, we recall their main properties, in particular, they map  $\mathcal{O}_A$  to  $A$  (see also [35, Section 8.2], and [41, 69, 70, 81, 102]). Denote by  $\triangleleft$  the natural right action of functionals on  $\mathcal{O}_A$ , namely (using Sweedler's notation)

$$\alpha \triangleleft h = \sum_{(\alpha)} h(\alpha_{(1)})\alpha_{(2)}$$

for every  $\alpha \in \mathcal{O}_A$  and  $h \in \mathcal{O}_A \rightarrow A$ . Let us identify  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  with  $\mathcal{O}(G)$  by means of (2.72). We have [41, Proposition 7.1]:

**Proposition 2.30.** *The maps  $\triangleleft t_i$  on  $\mathcal{O}_\epsilon$  preserve  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ , and satisfy  $(f \triangleleft t_i)(a) = f(n_i a)$  and  $(f \star \alpha) \triangleleft t_i = (f \triangleleft t_i)(\alpha \triangleleft t_i)$  for every  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ ,  $a \in G$ ,  $\alpha \in \mathcal{O}_\epsilon$ .*

We provide an alternative, non computational, proof of this result in Appendix C.

### 3 Noetherianity and finiteness

In this section, we prove Theorem 1.1. Recall that by Noetherian we mean right and left Noetherian. We begin with

**Theorem 3.1.** *The algebras  $\mathcal{L}_{0,n}$ ,  $\mathcal{L}_{0,n}^A$  and  $\mathcal{L}_{0,n}^{\epsilon'}$ ,  $\epsilon' \in \mathbb{C}^\times$ , are Noetherian.*

By Proposition 2.18, each of the algebras in this theorem is finitely generated.

Theorem 3.1 for  $\mathcal{L}_{0,1}$  and any  $\mathfrak{g}$  follows immediately from Joseph–Letzter's Theorem 2.1, claim (3), by identifying  $\mathcal{L}_{0,1}$  with  $U_q^{\mathrm{lf}}$  via  $\Phi_1$ . The method of proof uses filtration arguments. An alternative proof in the case of  $\mathfrak{sl}(n)$ , which works also for  $\mathcal{L}_{0,1}^A$ , was obtained by Domokos–Lenagan in [47], by exhibiting special sequences of generators of  $\mathcal{L}_{0,1}^A$  satisfying *polynomial* relations, as we define now.

**Definition 3.2** (see [104, Proposition 3.133]). Let  $R$  be a Noetherian Abelian ring, and  $B$  a finitely generated  $R$ -algebra with product  $\circ$ . We call polynormal a set of relations between generators  $u_1, \dots, u_M$  of  $B$ , of the form

$$u_i \circ u_j - q_{ij} u_j \circ u_i = \sum_{s=1}^{j-1} \sum_{t=1}^M (\alpha_{ij}^{st} u_s \circ u_t + \beta_{ij}^{st} u_t \circ u_s) \quad (3.1)$$

for all  $1 \leq j < i \leq M$ , where  $\alpha_{ij}^{st}, \beta_{ij}^{st} \in R$ , and the elements  $q_{ij} \in R$  are invertible.

Note that this definition is more restrictive than the more standard one, e.g., in [26, Definition II.4.1]. If such a set of relations exists in  $B$ , then  $B$  can be endowed with an algebra filtration such that the associated graded algebra is a quotient of a skew-polynomial algebra [26, Proposition I.8.17]. By classical results, we have (see, e.g., [88, Theorems 1.2.9, 1.6.9 and Examples 1.6.11], or [104, Lemmas 3.130–3.131]):

**Theorem 3.3.** *If the algebra filtration is well founded, then  $B$  is a Noetherian ring.*

In [47], Theorem 3.1 is also proved for any  $n \geq 1$  in the case of  $\mathfrak{g} = \mathfrak{sl}_2$  by considering  $\mathcal{L}_{0,n}^A(\mathfrak{sl}_2)$  as an iterated overring of  $\mathcal{L}_{0,1}(\mathfrak{sl}_2)$ .

The proof of Theorem 3.1 that we develop for any  $\mathfrak{g}$  and  $n \geq 1$  is also based on polynormal relations. In our proof, the generating set of  $\mathcal{L}_{0,n}$  that we will consider is evident, as they are matrix coefficients in the modules  $V_{\varpi_k}$ ,  $k \in \{1, \dots, m\}$ ; the task is then to exhibit a set of polynormal relations between them, that hold in a certain graded algebra associated to  $\mathcal{L}_{0,n}$ . Indeed, as explained above this will imply that the graded algebra is Noetherian, and that  $\mathcal{L}_{0,n}$  is Noetherian as well. In the case of  $\mathcal{L}_{0,n}^A$ , the proof is formally similar, but it needs the use of canonical bases discussed in Section 2.2.2.

**Proof of Theorem 3.1.** First, we develop the proof for  $\mathcal{L}_{0,n}$ , and then for  $\mathcal{L}_{0,n}^A$ ; the result for

$$\mathcal{L}_{0,n}^{\epsilon'} = \mathcal{L}_{0,n}^A / (q - \epsilon') \mathcal{L}_{0,n}^A$$

follows immediately by lifting ideals by the quotient map  $\mathcal{L}_{0,n}^A \rightarrow \mathcal{L}_{0,n}^{\epsilon'}$ .

We adapt the proof of Theorem 2.1 (3) given in [104, Theorem 3.137]. Let us begin by recalling these arguments. In doing this, let us stress that [104] takes on  $\mathcal{O}_q$  and  $\mathcal{L}_{0,1}$  the product opposite to ours, and below in (3.7) and (3.8) we respect their convention.

As usual, let  $C(\mu)$  be the vector space generated by the matrix coefficients of  $V_\mu$ , the simple  $U_q^{\text{ad}}$ -module of highest weight  $\mu \in P_+$ . Denote by  $C(\mu)_\lambda \subset C(\mu)$  the subspace of weight  $\lambda$  for the left coregular action of  $U_q(\mathfrak{h})$ ; so  $\alpha \in C(\mu)_\lambda$  if  $K_\nu \triangleright \alpha = q^{(\nu, \lambda)} \alpha$ ,  $\nu \in P$ . Consider the semigroup

$$\Lambda = \{(\mu, \lambda) \in P_+ \times P, \lambda \text{ is a weight of } V_\mu\}.$$

Recall that the partial order  $\preceq$  on  $P$  is defined by  $\mu \preceq \mu'$  if and only if  $\mu' - \mu \in D^{-1}Q_+$ . Define  $\preceq$  on  $\Lambda$  by:  $(\mu, \lambda) \preceq (\mu', \lambda')$  if and only if  $\mu' - \mu \in D^{-1}Q_+$  and  $\lambda' - \lambda \in D^{-1}Q_+$ . If  $(\mu, \lambda) \preceq (\mu', \lambda')$  and  $(\mu, \lambda) \neq (\mu', \lambda')$ , we write  $(\mu, \lambda) \prec (\mu', \lambda')$ . Since  $\mathcal{L}_{0,1}$  and  $\mathcal{O}_q$  are isomorphic vector spaces, we have  $\mathcal{L}_{0,1} = \bigoplus_{\mu \in P_+} C(\mu) = \bigoplus_{(\mu, \lambda) \in \Lambda} C(\mu)_\lambda$ . Consider the family of subspaces

$$\mathcal{F}_2^{\mu, \lambda} := \bigoplus_{(\mu', \lambda') \preceq (\mu, \lambda)} C(\mu')_{\lambda'}, \quad \mathcal{F}_2^{\prec \mu, \lambda} := \bigoplus_{(\mu', \lambda') \prec (\mu, \lambda)} C(\mu')_{\lambda'}, \quad (\mu, \lambda) \in \Lambda.$$

We have

$$\mathcal{L}_{0,1} = \bigcup_{(\mu, \lambda) \in \Lambda} \mathcal{F}_2^{\mu, \lambda}. \quad (3.2)$$

Indeed, clearly

$$\mathcal{L}_{0,1} = \sum_{(\mu,\lambda) \in \Lambda} \mathcal{F}_2^{\mu,\lambda},$$

so (3.2) follows from the following fact: for every  $(\mu, \lambda), (\mu', \lambda') \in \Lambda$ , the element  $(\mu'', \lambda'') := (\mu + \mu', \lambda + \lambda')$  is such that

$$\mathcal{F}_2^{\mu,\lambda} + \mathcal{F}_2^{\mu',\lambda'} \subset \mathcal{F}_2^{\mu'',\lambda''}.$$

Note that in general, since  $Q_+ \not\subset P_+$  (but  $P_+ \subset D^{-1}Q_+$ ), it is not true that there exists an element  $(\mu'', \lambda'')$  satisfying such an inclusion if one replaces  $\preceq$  with the standard ‘‘product’’ partial order  $\leq$  on  $\Lambda$ , defined by  $(\mu, \lambda) \leq (\mu', \lambda')$  if and only if  $\mu' - \mu \in Q_+$  and  $\lambda' - \lambda \in Q_+$ . Note also that  $\preceq$  is finer than  $\leq$ , in the sense that if  $\mu \leq \mu'$ , then  $\mu \preceq \mu'$ . Again, this would not be true if we had replaced  $D^{-1}Q_+$  by  $P_+$  in the definition of  $\preceq$ .

The family  $\mathcal{F}_2 := \{\mathcal{F}_2^{\mu,\lambda}\}_{(\mu,\lambda) \in \Lambda}$  is a filtration of the vector space  $\mathcal{L}_{0,1}$ , which is clearly well founded (i.e., every subset of  $\Lambda$  contains a minimal element, or equivalently any decreasing infinite sequence of elements in  $\Lambda$  is eventually constant).

Consider the associated graded vector space  $\text{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1}) := \bigoplus_{(\mu,\lambda)} \mathcal{F}_2^{\mu,\lambda} / \mathcal{F}_2^{\prec \mu,\lambda}$ . By identifying an element  $x \in C(\mu)_\lambda$  with its coset  $\bar{x} \in \mathcal{F}_2^{\mu,\lambda} / \mathcal{F}_2^{\prec \mu,\lambda}$ , we get an equality of vector spaces  $\text{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1}) = \bigoplus_{(\mu,\lambda) \in \Lambda} C(\mu)_\lambda$ . Now, one has the following facts:

(i) Taking the product in  $\mathcal{L}_{0,1}$ , we have

$$\alpha\beta \in \mathcal{F}_2^{\mu_1 + \mu_2, \lambda_1 + \lambda_2} \quad \text{for } \alpha \in C(\mu_1)_{\lambda_1}, \quad \beta \in C(\mu_2)_{\lambda_2}. \quad (3.3)$$

This follows from (2.7) and the fact that, for every  $\nu \in P_+$  and every summand of the formula (2.9), denoting by  $-r \in -Q_+$  the weight of the  $R$ -matrix component  $R_{(2)}$  we have

$$\begin{aligned} K_\nu \triangleright ((R_{(2)'} S(R_{(2)})) \triangleright \alpha) \star (R_{(1)'} \triangleright \beta \triangleleft R_{(1)}) \\ = q^{(\nu, \lambda_1 + \lambda_2 - r)} (R_{(2)'} S(R_{(2)}) \triangleright \alpha) \star (R_{(1)'} \triangleright \beta \triangleleft R_{(1)}). \end{aligned}$$

(Details of a similar computation are given below (3.12).) It follows from (3.3) that  $\mathcal{F}_2$  is an algebra filtration of  $\mathcal{L}_{0,1}$ , and then  $\text{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  is a graded algebra.

(ii) Denote by  $\alpha \circ \beta$  the product in  $\text{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  of  $\alpha, \beta \in \mathcal{L}_{0,1}$ . The space  $C(\mu_1 + \mu_2)$  has multiplicity one in  $C(\mu_1) \otimes C(\mu_2)$  (again by (2.7)), therefore if  $\alpha \in C(\mu_1)_{\lambda_1}$  and  $\beta \in C(\mu_2)_{\lambda_2}$ , then  $\alpha \circ \beta$  is the projection of  $\alpha\beta$  onto  $C(\mu_1 + \mu_2)_{\lambda_1 + \lambda_2}$ . Denote by  $\bar{\star}$  the product  $\star$  of  $\mathcal{O}_q$  followed by the projection onto the component  $C(\mu + \nu)$ . Then, we have

$$C(\mu) \circ C(\nu) = C(\mu) \bar{\star} C(\nu) = C(\mu + \nu). \quad (3.4)$$

This follows from the formula (2.9), and the fact that it is given by an invertible twist of the product  $\star$ .

(iii) For every  $\mu \in P_+$ , fix a basis of weight vectors  $e_1^\mu, \dots, e_{d(\mu)}^\mu$  of  $V_\mu$ . Denote by  $e_\mu^1, \dots, e_\mu^{d(\mu)} \in V_\mu^*$  the dual basis, and by  $w(e_i^\mu)$  the weight of  $e_i^\mu$ . Consider the matrix coefficients  ${}_\mu \phi_j^i(x) := e_\mu^i(\pi_V(x)(e_j^\mu))$ ,  $x \in U_q$ . By using the formula (2.9) and the explicit form of the  $R$ -matrix, one can check that

$$\begin{aligned} {}_\mu \phi_j^i \circ {}_\nu \phi_l^k &= \sum_{j', l'} c_{j', l'}^{ikjl} {}_\mu \phi_{j'}^i \bar{\star} {}_\nu \phi_{l'}^k \\ &= q^{(w(e_j^\mu), w(e_{l'}^\nu) - w(e_{k'}^\nu))} {}_\mu \phi_j^i \bar{\star} {}_\nu \phi_{l'}^k + \sum_{\substack{j', l' \\ j' \neq j, l' \neq l}} d_{j', l'}^{ikjl} {}_\mu \phi_{j'}^i \circ {}_\nu \phi_{l'}^k, \end{aligned} \quad (3.5)$$

where  $\sum'_{j',l'}$  is the sum over indices with weights satisfying

$$w(e_j^\mu) + w(e_l^\nu) = w(e_{j'}^\mu) + w(e_{l'}^\nu), \quad w(e_{j'}^\mu) \leq w(e_j^\mu) \quad \text{and} \quad w(e_{l'}^\nu) \geq w(e_l^\nu),$$

and the coefficient  $c_{j,l}^{ikjl}$ , equal to  $q^{(w(e_j^\mu), w(e_l^\nu) - w(e_k^\nu))}$ , is computed from the term  $\Theta$  in the  $R$ -matrix factorization (2.4). In general, all the coefficients  $c_{j',l'}^{ikjl}$  and  $d_{j',l'}^{ikjl}$  belong to  $\mathbb{C}(q)$  (see [18, Proposition 4.1]); in particular  $q^{(w(e_j^\mu), w(e_l^\nu) - w(e_k^\nu))} \in q^{\mathbb{Z}}$  since  $w(e_l^\nu) - w(e_k^\nu) \in Q$ . The second equality follows by repeated use of the first and (3.4). Similarly, by using (2.10) one gets

$$\begin{aligned} \nu\phi_l^k \circ \mu\phi_j^i &= \sum'_{i',k'} e_{i',k'}^{kilj} \mu\phi_j^{i'} \bar{\kappa} \nu\phi_l^{k'} \\ &= q^{(w(e_i^\mu), w(e_k^\nu) - w(e_{i'}^\nu))} \mu\phi_j^i \bar{\kappa} \nu\phi_l^k + \sum'_{\substack{i',k' \\ i' \neq i, k' \neq k}} e_{i',k'}^{kilj} \mu\phi_j^{i'} \bar{\kappa} \nu\phi_l^{k'} \\ &= q^{(w(e_i^\mu), w(e_k^\nu) - w(e_{i'}^\nu))} \mu\phi_j^i \bar{\kappa} \nu\phi_l^k + \sum'_{\substack{i',k',j',l' \\ i' \neq i, k' \neq k}} f_{i',k'}^{kilj} \mu\phi_{j'}^{i'} \circ \nu\phi_{l'}^{k'}, \end{aligned}$$

where  $e_{i',k'}^{kilj}, f_{i',k'}^{kilj} \in \mathbb{C}(q)$ , and  $\sum'_{i',k'}$  is the sum over indices with weights satisfying

$$\begin{aligned} w(e_i^\mu) + w(e_k^\nu) &= w(e_{i'}^\mu) + w(e_{k'}^\nu), & w(e_{i'}^\mu) &\leq w(e_i^\mu), \\ w(e_{k'}^\nu) &\geq w(e_k^\nu), & e_{i,k}^{kilj} &= q^{(w(e_i^\mu), w(e_k^\nu) - w(e_{i'}^\nu))}. \end{aligned}$$

The third equality comes from the second and (3.5); the sum is over indices with weights satisfying

$$\begin{aligned} w(e_i^\mu) + w(e_k^\nu) &= w(e_{i'}^\mu) + w(e_{k'}^\nu), \\ w(e_{i'}^\mu) &< w(e_i^\mu), & w(e_{k'}^\nu) &> w(e_k^\nu), & w(e_{j'}^\mu) &\leq w(e_j^\mu), & w(e_{l'}^\nu) &\geq w(e_l^\nu). \end{aligned}$$

By eliminating the leading term  $\mu\phi_j^i \bar{\kappa} \nu\phi_l^k$ , one deduces

$$\nu\phi_l^k \circ \mu\phi_j^i - q_{ijkl} \mu\phi_j^i \circ \nu\phi_l^k = \sum'_{\substack{i',k',j',l' \\ i' \neq i, k' \neq k}} f_{i',k'}^{kilj} \mu\phi_{j'}^{i'} \circ \nu\phi_{l'}^{k'} - \sum'_{\substack{j',l' \\ j' \neq j, l' \neq l}} q_{ijkl} d_{j',l'}^{ikjl} \mu\phi_{j'}^i \circ \nu\phi_{l'}^k, \quad (3.6)$$

where  $q_{ijkl} = q^{(w(e_j^\mu) + w(e_i^\mu), w(e_k^\nu) - w(e_{i'}^\nu))}$ .

(iv) We can always reorder the weight vectors  $e_1^\mu, \dots, e_{d(\mu)}^\mu$  so that  $w(e_i^\mu) > w(e_j^\mu)$  implies  $i < j$ ; then (3.6) reads

$$\begin{aligned} \nu\phi_l^k \circ \mu\phi_j^i - q_{ijkl} \mu\phi_j^i \circ \nu\phi_l^k &= \sum_{r=i}^{d(\mu)} \sum_{s=1}^k \sum_{u=1}^{l-1} \sum_{v=j+1}^{d(\mu)} \delta_{rsuv}^{ijkl} \mu\phi_v^r \circ \nu\phi_u^s \\ &\quad - \sum_{r=i+1}^{d(\mu)} \sum_{s=1}^{k-1} q_{ijkl} \gamma_{rs}^{ijkl} \mu\phi_j^r \circ \nu\phi_l^s, \end{aligned} \quad (3.7)$$

where  $\gamma_{rs}^{ijkl}, \delta_{rsuv}^{ijkl} \in \mathbb{C}(q)$  are such that  $\gamma_{rs}^{ijkl} = 0$  unless  $w(e_r^\mu) < w(e_i^\mu)$  and  $w(e_s^\nu) > w(e_k^\nu)$ , and  $\delta_{rsuv}^{ijkl} = 0$  unless  $w(e_u^\nu) > w(e_l^\nu)$ ,  $w(e_v^\mu) < w(e_j^\mu)$ ,  $w(e_r^\mu) \leq w(e_i^\mu)$  and  $w(e_s^\nu) \geq w(e_k^\nu)$ . Now, from (3.7) one can extract a defining set of polynomial relations for  $\text{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1})$ , as in (3.1). Indeed, like  $\mathcal{L}_{0,1}$  the algebra  $\text{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  is generated by the matrix coefficients  $\varpi_k \phi_i^j$  of the fundamental representations  $V_{\varpi_k}$ . One can list these matrix coefficients, say  $M$  in number, in an ordered sequence  $u_1, \dots, u_M$  such that the following condition holds: if  $w(e_k^{\varpi_s}) < w(e_i^{\varpi_r})$ ,

or  $w(e_k^{\overline{rs}}) = w(e_i^{\overline{rs}})$  and  $w(e_l^{\overline{rs}}) < w(e_j^{\overline{rs}})$ , then  $u_a := \varpi_r \phi_j^i$  and  $u_b := \varpi_s \phi_l^k$  satisfy  $b < a$ . Then denoting  $\mu \phi_j^i$ ,  $\nu \phi_l^k$  in (3.7) by  $u_j$ ,  $u_i$ , respectively, and assuming  $u_j < u_i$ , one finds that all terms  $u_s := \mu \phi_v^r$ ,  $\mu \phi_j^r$  in the sums are  $< u_j$ . Therefore, for all  $1 \leq j < i \leq M$  it takes the form

$$u_i \circ u_j - q_{ij} u_j \circ u_i = \sum_{s=1}^{j-1} \sum_{t=1}^M \alpha_{ij}^{st} u_s \circ u_t \quad (3.8)$$

for some  $q_{ij} \in q^{\mathbb{Z}}$  and  $\alpha_{ij}^{st} \in \mathbb{C}(q)$ . As explained after (3.1), it follows that  $\text{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  is a Noetherian ring, and since the filtration  $\mathcal{F}_2$  is well founded, it implies that  $\mathcal{L}_{0,1}$  is Noetherian too.

We are going to extend all these facts to  $\mathcal{L}_{0,n}$ ,  $n > 1$ . First, we need to refine the filtration  $\mathcal{F}_2$  on  $\mathcal{L}_{0,1}$ . Consider the action of  $U_q(\mathfrak{h})$  on  $C(\mu)_\lambda$  given by

$$K_\nu \cdot \alpha := \text{coad}(K_\nu^{-1})(\alpha), \quad \nu \in P, \quad \alpha \in C(\mu)_\lambda. \quad (3.9)$$

Denote by  $C(\mu)_{\lambda,\gamma} \subset C(\mu)_\lambda$  the subspace of weight  $\gamma$  for this action; so  $\alpha \in C(\mu)_{\lambda,\gamma}$  if  $K_\nu \cdot \alpha = q^{(\nu,\gamma)} \alpha$ . Consider the semigroup

$$\Lambda_P = \{(\mu, \lambda, \gamma) \in P_+ \times P^2, \lambda \text{ is a weight of } V_\mu \text{ for } \triangleright, \gamma \text{ is a weight of } V_\mu \text{ for } \cdot\}$$

with the partial order  $(\mu, \lambda, \gamma) \preceq (\mu', \lambda', \gamma')$  if and only if  $\mu' - \mu, \lambda' - \lambda, \gamma' - \gamma \in D^{-1}Q_+$ . Define

$$\begin{aligned} [\Lambda_P] &= \{([\mu], [\lambda], [\gamma]) \in P_+^n \times P^n \times P^n \\ &\quad | (\mu_i, \lambda_i, \gamma_i) \in \Lambda_P, [\mu] = (\mu_i)_{i=1}^n, [\lambda] = (\lambda_i)_{i=1}^n, [\gamma] = (\gamma_i)_{i=1}^n\}. \end{aligned}$$

Let us put the following lexicographic order on  $[\Lambda_P]$ , starting from the tail:  $([\mu'], [\lambda'], [\gamma']) \preceq ([\mu], [\lambda], [\gamma])$  if  $(\mu'_n, \lambda'_n, \gamma'_n) \prec (\mu_n, \lambda_n, \gamma_n)$ , or  $(\mu_n, \lambda_n, \gamma_n) = (\mu'_n, \lambda'_n, \gamma'_n)$  and  $(\mu'_{n-1}, \lambda'_{n-1}, \gamma'_{n-1}) \prec (\mu_{n-1}, \lambda_{n-1}, \gamma_{n-1}), \dots$ , or  $(\mu_k, \lambda_k, \gamma_k) = (\mu'_k, \lambda'_k, \gamma'_k)$  for all  $1 < k \leq n$  and  $(\mu'_1, \lambda'_1, \gamma'_1) \preceq (\mu_1, \lambda_1, \gamma_1)$ . (As usual, we write  $([\mu'], [\lambda'], [\gamma']) \prec ([\mu], [\lambda], [\gamma])$  for  $([\mu'], [\lambda'], [\gamma']) \preceq ([\mu], [\lambda], [\gamma])$  and  $([\mu'], [\lambda'], [\gamma']) \neq ([\mu], [\lambda], [\gamma])$ .)

Now recall that  $\mathcal{L}_{0,n} = \mathcal{L}_{0,1}^{\otimes n} = \mathcal{O}_q^{\otimes n}$  as vector spaces. For every  $([\mu], [\lambda], [\gamma]) \in [\Lambda_P]$ , consider the subspace  $C([\mu])_{[\lambda],[\gamma]} \subset \mathcal{L}_{0,n}$  defined by

$$C([\mu]) = C(\mu_1) \otimes \cdots \otimes C(\mu_n), \quad C([\mu])_{[\lambda],[\gamma]} = C(\mu_1)_{\lambda_1,\gamma_1} \otimes \cdots \otimes C(\mu_n)_{\lambda_n,\gamma_n}.$$

Then  $\mathcal{L}_{0,n} = \bigoplus_{[\mu] \in P_+^n} C([\mu])$  and  $C([\mu]) = \bigoplus_{([\lambda],[\gamma])} C([\mu])_{[\lambda],[\gamma]}$ . For every  $([\mu], [\lambda], [\gamma]) \in [\Lambda_P]$  define

$$\begin{aligned} \mathcal{F}_3^{[\mu],[\lambda],[\gamma]} &= \bigoplus_{([\mu'],[\lambda'],[\gamma']) \preceq ([\mu],[\lambda],[\gamma])} C([\mu'])_{[\lambda'],[\gamma']}, \\ \mathcal{F}_3^{\prec [\mu],[\lambda],[\gamma]} &= \bigoplus_{([\mu'],[\lambda'],[\gamma']) \prec ([\mu],[\lambda],[\gamma])} C([\mu'])_{[\lambda'],[\gamma']}. \end{aligned} \quad (3.10)$$

Clearly,  $\mathcal{L}_{0,n}$  is the union of the subspaces  $\mathcal{F}_3^{[\mu],[\lambda],[\gamma]}$  over all  $([\mu], [\lambda], [\gamma]) \in [\Lambda_P]$ , so these form a vector space filtration of  $\mathcal{L}_{0,n}$ . Let us denote it  $\mathcal{F}_3$ , and define

$$\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu],[\lambda],[\gamma]} = \mathcal{F}_3^{[\mu],[\lambda],[\gamma]} / \mathcal{F}_3^{\prec [\mu],[\lambda],[\gamma]}.$$

This space is canonically identified with  $C([\mu])_{[\lambda],[\gamma]}$ , so the graded vector space associated to  $\mathcal{F}_3$  is

$$\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n}) = \bigoplus_{([\mu],[\lambda],[\gamma]) \in [\Lambda_P]} \text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu],[\lambda],[\gamma]} = \bigoplus_{([\mu],[\lambda],[\gamma]) \in [\Lambda_P]} C([\mu])_{[\lambda],[\gamma]}. \quad (3.11)$$

We claim that  $\mathcal{F}_3$  is an algebra filtration with respect to the product of  $\mathcal{L}_{0,n}$ , and therefore  $\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})$  is a graded algebra.

For notational simplicity, let us prove it for  $n = 2$ , the general case being strictly similar. Recall the  $R$ -matrix factorization (2.4). Take tuples  $([\mu], [\lambda], [\gamma]) = ((\mu_1, \mu_2), (\lambda_1, \lambda_2), (\gamma_1, \gamma_2))$  and  $([\mu'], [\lambda'], [\gamma']) = ((\mu'_1, \mu'_2), (\lambda'_1, \lambda'_2), (\gamma'_1, \gamma'_2))$  in  $[\Lambda_P]$ , and elements  $\alpha \otimes \beta \in C([\mu])_{[\lambda], [\gamma]}$  and  $\alpha' \otimes \beta' \in C([\mu'])_{[\lambda'], [\gamma']}$ . Recall from (2.17) that the product of  $\mathcal{L}_{0,2}$  is given by the formula

$$\begin{aligned} & (\alpha \otimes \beta)(\alpha' \otimes \beta') \\ &= \sum_{(R^1), \dots, (R^4)} \alpha(S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha' \triangleleft R_{(1)}^1 R_{(1)}^2) \otimes (S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4) \beta'. \end{aligned} \quad (3.12)$$

For every  $\nu \in P$  and any of the components  $R_{(2)}^1, \dots, R_{(2)}^4$ , denoting by  $-r_j \in -Q_+$  the weight of  $R_{(2)}^j$ , we have

$$\begin{aligned} & K_\nu \triangleright (S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4) \\ &= \sum_{(\beta)} \beta_{(1)}(R_{(2)}^2 R_{(2)}^4) (K_\nu S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta_{(2)}) \\ &= q^{-(\nu, r_1 + r_3)} \sum_{(\beta)} \beta_{(1)}(R_{(2)}^2 R_{(2)}^4) (S(R_{(2)}^1 R_{(2)}^3) K_\nu \triangleright \beta_{(2)}) \\ &= q^{(\nu, \lambda_2 - r_1 - r_3)} \sum_{(\beta)} \beta_{(1)}(R_{(2)}^2 R_{(2)}^4) (S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta_{(2)}) \\ &= q^{(\nu, \lambda_2 - r_1 - r_3)} (S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4). \end{aligned}$$

By similar computations for the action  $\text{coad}(K_\nu^{-1})$ , and for all terms in the right-hand side of (3.12), and using (3.3) componentwisely, we find that

$$\alpha(S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha' \triangleleft R_{(1)}^1 R_{(1)}^2) \otimes (S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4) \beta' \in \mathcal{F}_3^{[\mu] + [\mu'], [\lambda] + [\lambda'], [\gamma] + [\gamma']},$$

where

$$\begin{aligned} \lambda'' &= (\lambda_1 + \lambda'_1 + r_3 + r_4, \lambda_2 + \lambda'_2 - r_1 - r_3), \\ \gamma'' &= (\gamma_1 + \gamma'_1 + r_1 + r_2 + r_3 + r_4, \gamma_2 + \gamma'_2 - r_1 - r_2 - r_3 - r_4). \end{aligned}$$

Since  $r_1 + r_2 + r_3 + r_4 = 0$  implies  $r_1 = r_2 = r_3 = r_4 = 0$ , by the order we have put on  $[\Lambda_P]$ , we deduce

$$(\alpha \otimes \beta)(\alpha' \otimes \beta') \in \mathcal{F}_3^{[\mu] + [\mu'], [\lambda] + [\lambda'], [\gamma] + [\gamma']}.$$

Note that the filtration  $\mathcal{F}_3$ , taking the action (3.9) into account, is crucial for this argument to work. Similar arguments work for any  $n \geq 2$ . This proves that  $\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})$  is a graded algebra. We denote its product by  $\circ_n$ .

Next, we show that (3.4) implies the analogous property for the product  $\circ_n$ . For simplicity of notations let us again assume that  $n = 2$ . Recall that the product  $\circ_2$  is defined on homogeneous elements  $\overline{\alpha \otimes \beta} \in \text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu], [\lambda]}$  and  $\overline{\alpha' \otimes \beta'} \in \text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu'], [\lambda']}$  by

$$\overline{\alpha \otimes \beta} \circ_n \overline{\alpha' \otimes \beta'} = (\alpha \otimes \beta)(\alpha' \otimes \beta') + \mathcal{F}_3^{<[\mu + \mu'], [\lambda + \lambda']}.$$

Clearly, (3.4) gives  $(C(\mu_1) \circ C(\mu'_1)) \otimes (C(\mu_2) \circ C(\mu'_2)) = C([\mu + \mu'])$ , and (3.12) gives

$$C([\mu]) \circ_n C([\mu']) \subset (C(\mu_1) \circ C(\mu'_1)) \otimes (C(\mu_2) \circ C(\mu'_2)).$$

The converse inclusion holds true as well, as one can see by expressing, reciprocally, the (componentwise) product of  $\mathcal{L}_{0,1}^{\otimes n}$  in terms of the product of  $\mathcal{L}_{0,n}$  via the formula (2.19). In conclusion,

$$C([\mu]) \circ_n C([\mu']) = C([\mu + \mu']).$$

We are left to show that (3.7) generalizes to  $\mathcal{L}_{0,n}$ . First, note that for every  $1 \leq a \leq n$  the embedding  $i_a: \mathcal{L}_{0,1} \rightarrow \mathcal{L}_{0,n}$  in (2.16) is a morphism of the filtered algebras  $(\mathcal{L}_{0,1}, \mathcal{F}_2)$  and  $(\mathcal{L}_{0,n}, \mathcal{F}_3)$ , in the sense that

$$i_a(\mathcal{F}_2^{\mu, \lambda}) \subset \sum_{\gamma \in P} \mathcal{F}_3^{[\mu_a], [\lambda_a], [\gamma_a]},$$

where by definition  $[\mu_a] = (0, \dots, 0, \mu, 0, \dots, 0)$  with  $\mu$  on the  $a$ -th entry, and similarly  $[\lambda_a] = (0, \dots, 0, \lambda, 0, \dots, 0)$  and  $[\gamma_a] = (0, \dots, 0, \gamma, 0, \dots, 0)$ . Therefore, the relation (3.7) yields in  $\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})$  similar relations between elements of the form (matrix coefficient)  $\otimes 1$ , or  $1 \otimes$  (matrix coefficient).

We now consider the case of “mixed” products. We give the details when  $n = 2$ , the general case being similar. Let us write the twist  $F$  in (2.18) as

$$F = \sum_{(F)} F_{(1)} \otimes F_{(2)} = \sum_{(F)} F_{(1)1} \otimes F_{(1)2} \otimes F_{(2)1} \otimes F_{(2)2},$$

that is, we set  $F_{(1)1} := R_{(2)}^2 R_{(2)}^4$ ,  $F_{(1)2} := R_{(2)}^1 R_{(2)}^3$ ,  $F_{(2)1} := R_{(1)}^1 R_{(1)}^2$ ,  $F_{(2)2} := R_{(1)}^3 R_{(1)}^4$ . Put  $d(\mu) := \dim(V_\mu)$ ,  $\mu \in P_+$ , and

$$\Delta^{(2)}(\mu_2 \phi_{l_2}^{k_2}) = \sum_{p,s=1}^{d(\mu_2)} \mu_2 \phi_p^{k_2} \otimes \mu_2 \phi_s^p \otimes \mu_2 \phi_{l_2}^s, \quad \Delta^{(2)}(\mu'_1 \phi_{l'_1}^{k'_1}) = \sum_{p',s'=1}^{d(\mu'_1)} \mu'_1 \phi_{p'}^{k'_1} \otimes \mu'_1 \phi_{s'}^{p'} \otimes \mu'_1 \phi_{l'_1}^{s'}.$$

From (3.12), one obtains

$$\begin{aligned} (1 \otimes \mu_2 \phi_{l_2}^{k_2})(\mu'_1 \phi_{l'_1}^{k'_1} \otimes 1) &= \sum_{(F)} \sum_{p,s=1}^{d(\mu_2)} \sum_{p',s'=1}^{d(\mu'_1)} (\mu'_1 \phi_{s'}^{p'}(\mu'_1 \phi_{p'}^{k'_1}(F_{(2)1})\mu'_1 \phi_{l'_1}^{s'}(S(F_{(2)2})))) \\ &\quad \otimes (\mu_2 \phi_s^p(\mu_2 \phi_p^{k_2}(F_{(1)1})\mu_2 \phi_{l_2}^s(S(F_{(1)2}))))). \end{aligned} \quad (3.13)$$

It is immediate that

$$\mu'_1 \phi_{s'}^{p'} \otimes \mu_2 \phi_s^p \in C(\mu'_1)_{w(e_{s'}^{\mu'_1}), w(e_{s'}^{\mu'_1}) - w(e_{p'}^{\mu'_1})} \otimes C(\mu_2)_{w(e_s^{\mu_2}), w(e_s^{\mu_2}) - w(e_p^{\mu_2})}.$$

As in (iv) above, for every  $\mu \in P_+$  we order the weight vectors  $e_1^\mu, \dots, e_m^\mu$  so that  $w(e_i^\mu) > w(e_j^\mu)$  implies  $i < j$ . With such an ordering the factorization  $R = \Theta \hat{R}$  (see (2.4)) implies

$$\mu_2 \phi_p^{k_2}(F_{(1)1})\mu_2 \phi_{l_2}^s(S(F_{(1)2})) = 0 \quad \text{unless } k_2 \geq p \text{ and } s \geq l_2,$$

and

$$\mu'_1 \phi_{p'}^{k'_1}(F_{(2)1})\mu'_1 \phi_{l'_1}^{s'}(S(F_{(2)2})) = 0 \quad \text{unless } k'_1 \leq p' \text{ and } s' \leq l'_1.$$

Since  $s > l_2$ , we have  $w(e_s^{\mu_2}) \leq w(e_{l_2}^{\mu_2})$ , and if  $w(e_s^{\mu_2}) < w(e_{l_2}^{\mu_2})$ , then  $\mu_2 \phi_s^p \in \mathcal{F}_2^{< \mu_2, w(e_{l_2}^{\mu_2})}$ . In this last situation, the summands  $\mu'_1 \phi_{s'}^{p'} \otimes \mu_2 \phi_s^p$  in the sum above vanish in  $\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,2})$ . In order to find all the non-zero summands, we have to consider also the weights with respect to the action (3.9).



Since  $k_2 \geq p$  implies  $w(e_{k_2}^{\mu_2}) \leq w(e_p^{\mu_2})$ , we have  $w(e_s^{\mu_2}) - w(e_p^{\mu_2}) \leq w(e_{l_2}^{\mu_2}) - w(e_{k_2}^{\mu_2})$ . Therefore, the summands which are non-zero in  $\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,2})$  have both weights  $w(e_s^{\mu_2}) = w(e_{l_2}^{\mu_2})$  and  $w(e_p^{\mu_2}) = w(e_{k_2}^{\mu_2})$ . Doing similarly with the weights of  $\mu'_1 \phi_{s'}^{p'}$ , we find that also  $w(e_{s'}^{\mu'_1}) = w(e_{l'_1}^{\mu'_1})$  and  $w(e_{p'}^{\mu'_1}) = w(e_{k'_1}^{\mu'_1})$ . When all these conditions on weights are satisfied, the corresponding components  $F_{(1)1}, F_{(1)2}, F_{(2)1}, F_{(2)2}$  have zero weight. Therefore, the sum reduces to

$$\begin{aligned} & \sum_{(F)} \mu_2 \phi_{k_2}^{k_2}(F_{(1)1}) \mu_2 \phi_{l_2}^{l_2}(S(F_{(1)2})) \mu'_1 \phi_{k'_1}^{k'_1}(F_{(2)1}) \mu'_1 \phi_{l'_1}^{l'_1}(S(F_{(2)2})) \\ &= \langle \mu_2 \phi_{k_2}^{k_2} \otimes \mu_2 \phi_{l_2}^{l_2} \otimes \mu'_1 \phi_{k'_1}^{k'_1} \otimes \mu'_1 \phi_{l'_1}^{l'_1}, \Theta_{13} \Theta_{14}^{-1} \Theta_{24} \Theta_{23}^{-1} \rangle = q^{(w(e_{k_2}^{\mu_2}) - w(e_{l_2}^{\mu_2}), w(e_{k'_1}^{\mu'_1}) - w(e_{l'_1}^{\mu'_1}))}. \end{aligned}$$

Denoting by  $q'_{k_2 l_2 k'_1 l'_1}$  this scalar, it follows

$$(1 \otimes \mu_2 \phi_{l_2}^{k_2}) \circ_2 (\mu'_1 \phi_{l'_1}^{k'_1} \otimes 1) = q'_{k_2 l_2 k'_1 l'_1} \mu'_1 \phi_{l'_1}^{k'_1} \otimes \mu_2 \phi_{l_2}^{k_2} = q'_{k_2 l_2 k'_1 l'_1} (\mu'_1 \phi_{l'_1}^{k'_1} \otimes 1) \circ_2 (1 \otimes \mu_2 \phi_{l_2}^{k_2}).$$

This is the relation analogous to (3.7) for mixed products in  $\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,2})$ .

Recall that in (3.8) we denoted by  $u_1, \dots, u_M$  the ordered list of matrix coefficients  $\varpi_k \phi_i^j$ . Let us order in a lexicographic way the elements  $u_i \otimes u_j$ , i.e., as a sequence  $u_1^{(2)}, \dots, u_{M^2}^{(2)}$  such that the following condition holds: if  $\varpi_{l'} \phi_{s'}^{t'} < \varpi_{k'} \phi_{i'}^{j'}$ , or  $\varpi_{l'} \phi_{s'}^{t'} = \varpi_{k'} \phi_{i'}^{j'}$  and  $\varpi_l \phi_s^t < \varpi_k \phi_i^j$ , then  $u_a^{(2)} := \varpi_k \phi_i^j \otimes \varpi_{k'} \phi_{i'}^{j'}$  and  $u_b^{(2)} := \varpi_l \phi_s^t \otimes \varpi_{l'} \phi_{s'}^{t'}$  satisfy  $u_b^{(2)} < u_a^{(2)}$ . Then, for this ordering the polynomial relations (3.8) hold true for all  $1 \leq u_j^{(2)} < u_i^{(2)} \leq M^2$ . As described after (3.1), it follows that  $\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})$  is Noetherian. The filtration  $\mathcal{F}_3$  being well founded, it implies that  $\mathcal{L}_{0,n}$  is Noetherian too.

Finally, we consider the  $A$ -algebra  $\mathcal{L}_{0,n}^A$ , and prove it is Noetherian. We proceed in exactly the same way as for  $\mathcal{L}_{0,n}$ , changing the generators and replacing key arguments of the steps (i)–(iv) by the corresponding results over  $A$ . Let us describe these modifications step by step.

First, consider the case  $n = 1$ . Recall the  $A$ -lattices  ${}_A \dot{C}(\lambda)$  (see (2.46)), and the decomposition (2.55) of  $\mathcal{O}_A$  into weight subspaces. In particular, have a decomposition into weight subspaces for the left coregular action,

$${}_A \dot{C}(\lambda) = \bigoplus_{\lambda' \in P} {}_A \dot{C}(\lambda)_{\lambda'}.$$

Define

$${}_A \mathcal{F}_2^{\mu, \lambda} := \bigoplus_{(\mu', \lambda') \preceq (\mu, \lambda)} {}_A \dot{C}(\mu')_{\lambda'}.$$

Recall that every  $A$ -module of matrix coefficients  $(\text{End}({}_A V_\mu))^*$ ,  $\mu \in P_+$ , is contained in  $\mathcal{O}_A(\leq \mu)$ , and by inverting over  $\mathbb{C}(q)$  the corresponding linear triangular system between basis elements, and using that the order relation  $\preceq$  is finer than  $\leq$ , we obtain an inclusion

$$\bigoplus_{\mu' \preceq \mu} {}_A \dot{C}(\mu') \subset \bigoplus_{\mu' \leq \mu} C(\mu')$$

(see (2.48)–(2.51)). It follows that  ${}_A \mathcal{F}_2^{\mu, \lambda} = \mathcal{F}_2^{\mu, \lambda} \cap \mathcal{O}_A$ , and therefore, like  $\mathcal{F}_2$  the family  ${}_A \mathcal{F}_2 := \{ {}_A \mathcal{F}_2^{\mu, \lambda} \}_{(\mu, \lambda) \in \Lambda}$  is a well-founded filtration of  $\mathcal{O}_A$ . Put  ${}_A \mathcal{F}_2^{\prec \mu, \lambda} = \mathcal{F}_2^{\prec \mu, \lambda} \cap \mathcal{O}_A$ , and consider the graded  $A$ -module  $\text{Gr}_{{}_A \mathcal{F}_2}(\mathcal{L}_{0,1}^A)$  associated to  ${}_A \mathcal{F}_2$ . By (2.52)–(2.54) and the fact that  $\mathcal{O}_A = \mathcal{L}_{0,1}^A$  as an  $A$ -module, we have the  $A$ -module decomposition

$$\text{Gr}_{{}_A \mathcal{F}_2}(\mathcal{L}_{0,1}^A) = \bigoplus_{(\mu, \lambda) \in \Lambda} {}_A C(\mu)_{\lambda},$$

where  ${}_A C(\mu)_\lambda$  is the submodule of weight  $\lambda$  (for the left coregular action) of

$${}_A C(\mu) := (\text{End}({}_A V_\mu))^*.$$

Then, we can proceed as before. By step (i), we deduce that  ${}_A \mathcal{F}_2$  is an algebra filtration of  $\mathcal{L}_{0,1}^A$ . By Proposition 2.12, the  $A$ -module  ${}_A \dot{C}(\mu_1 + \mu_2)$  has multiplicity one in  ${}_A \dot{C}(\mu_1) \otimes {}_A \dot{C}(\mu_2)$ . In fact, by step (ii),  ${}_A C(\mu_1 + \mu_2)$  has multiplicity one in  ${}_A C(\mu_1) \otimes_A {}_A C(\mu_2)$ , so exactly in the same way as (3.4), we obtain in  $\text{Gr}_{{}_A \mathcal{F}_2}(\mathcal{L}_{0,1}^A)$  the equality

$${}_A C(\mu) \circ {}_A C(\nu) = {}_A C(\mu + \nu).$$

In step (iii), we fixed a basis of each space  $C(\mu)$ , consisting of a set of matrix coefficients  $\{\mu \phi_j^i\}$  with respect to dual basis of weight vectors of the modules  $V_\mu$  and  $V_\mu^*$ . In step (iv), the basis elements of  $V_\mu$  and  $V_\mu^*$  were ordered by means of the weights, and we used the fact that the matrix coefficients in the spaces  $C(\varpi_1), \dots, C(\varpi_m)$  form a generating set of the algebra  $\text{Gr}_{{}_A \mathcal{F}_2}(\mathcal{L}_{0,1})$ . The only property of the matrix coefficients used in the computations was that they are weight vectors for the left coregular action (and later, in the case  $n > 1$ , for the action (3.9)).

We can proceed exactly in the same manner by working with the  $A$ -modules of matrix coefficients  ${}_A C(\mu)$ . If one wishes to work at the level of  $\mathcal{O}_A$ , recall that any set of generators of  $\mathcal{O}_A$  generates  $\mathcal{L}_{0,1}^A$  as well (see the proof of Proposition 2.18). Then, one can replace the basis  $\{\mu \phi_j^i\}$  of each space  $C(\mu)$  with the canonical basis  $\dot{\mathbf{B}}[\mu]^*$  of  ${}_A \dot{C}(\mu)$ , and take the generating set of  $\mathcal{O}_A$  formed by the elements in  $\dot{\mathbf{B}}[\varpi_i]^*$ ,  $i = 1, \dots, m$  (see Proposition 2.10 and the comments thereafter). By the integrality properties satisfied by the  $R$ -matrix and the twists, all the computations in the proof of steps (iii) and (iv) can be done using such basis elements, and eventually take place over  $A$  (see [18, Propositions 4.10 and 6.9]). Therefore, we obtain a relation like (3.8) with coefficients  $\alpha_{ij}^{st} \in A$ . Since  $A$  is a Noetherian ring, again this proves  $\text{Gr}_{{}_A \mathcal{F}_2}(\mathcal{L}_{0,1}^A)$ , whence  $\mathcal{L}_{0,1}^A$ , are Noetherian rings.

This being done, the adaptation of the proof when  $n > 1$  is immediate. The filtration  $\mathcal{F}_3$  has to be replaced with  ${}_A \mathcal{F}_3 := \left\{ {}_A \mathcal{F}_3^{[\mu],[\lambda],[\gamma]} \right\}_{([\mu],[\lambda],[\gamma])}$ , where  ${}_A \mathcal{F}_3^{[\mu],[\lambda],[\gamma]}$  is the  $A$ -module defined by

$${}_A \mathcal{F}_3^{[\mu],[\lambda],[\gamma]} = \bigoplus_{([\mu'],[\lambda'],[\gamma']) \preceq ([\mu],[\lambda],[\gamma])} {}_A \dot{C}([\mu'])_{[\lambda'],[\gamma]},$$

where

$${}_A \dot{C}([\mu])_{[\lambda],[\gamma]} = {}_A \dot{C}(\mu_1)_{\lambda_1, \gamma_1} \otimes_A \cdots \otimes_A {}_A \dot{C}(\mu_n)_{\lambda_n, \gamma_n},$$

and  ${}_A \dot{C}(\mu)_{\lambda, \gamma}$  is the subspace of  ${}_A \dot{C}(\mu)_\lambda$  of weight  $\gamma$  for the action (3.9). Then the proof proceeds in exactly the same way, replacing in (3.13) and all subsequent computations the matrix coefficients by the generators of  $\mathcal{O}_A$  provided by Proposition 2.10. This concludes the proof.  $\blacksquare$

**Theorem 3.4.** *The algebra  $\mathcal{M}_{0,n} = \mathcal{L}_{0,n}^{U_q}$  is Noetherian and generated over  $\mathbb{C}(q)$  by a finite number of elements.*

Our method of proof follows closely that of the Hilbert–Nagata theorem (see [46]). Let us recall one version of this theorem. Let  $K$  be an arbitrary field,  $\mathfrak{A}$  a commutative algebra over  $K$  finitely generated by elements  $a_1, \dots, a_n$ , and  $G$  a group of algebra automorphisms of  $\mathfrak{A}$ .

**Theorem 3.5.** *If the action of  $G$  on  $\mathfrak{A}$  is completely reducible on finite-dimensional representations, then the ring  $\mathfrak{A}^G$  of invariants of  $\mathfrak{A}$  with respect to  $G$  is Noetherian and a finitely generated algebra over  $K$ .*

We recall here the main steps of the proof that we will adapt in order to prove Theorem 3.4:

- (a) From the complete reducibility of the action of  $G$  on  $\mathfrak{A}$ , one can define a linear map

$$R: \mathfrak{A} \rightarrow \mathfrak{A}^G$$

namely the projection onto the space of invariant elements along the sum of nontrivial isotypical components of  $\mathfrak{A}$ . This linear map is the Reynolds operator; we already discussed it in (2.23) in the case of  $U_q$  acting on  $\mathcal{L}_{0,n}$ . By the same arguments we developed there, it satisfies  $R(hf) = hR(f)$  for every  $f \in \mathfrak{A}$ ,  $h \in \mathfrak{A}^G$ .

- (b) Let  $I$  be an ideal of  $\mathfrak{A}^G$ . Then  $I = R(\mathfrak{A}I) = \mathfrak{A}I \cap \mathfrak{A}^G$ . Because  $\mathfrak{A}I$  is an ideal of  $\mathfrak{A}$ , and  $\mathfrak{A}$  is Noetherian, there exist elements  $b_1, \dots, b_s$ , that can be chosen in  $I \subset \mathfrak{A}^G$ , such that  $\mathfrak{A}I = \mathfrak{A}b_1 + \dots + \mathfrak{A}b_s$ . Since  $I = R(\mathfrak{A}I) = R(\mathfrak{A}b_1 + \dots + \mathfrak{A}b_s) = \mathfrak{A}^G b_1 + \dots + \mathfrak{A}^G b_s$ ,  $I$  is finitely generated over  $\mathfrak{A}^G$ . Therefore,  $\mathfrak{A}^G$  is Noetherian.
- (c) Let  $\mathfrak{B}$  be an algebra graded over  $\mathbb{N}$  (for simplicity of notations):  $\mathfrak{B} = \bigoplus_{n=0}^{+\infty} \mathfrak{B}_n$ , with  $\mathfrak{B}_m \cdot \mathfrak{B}_n \subset \mathfrak{B}_{m+n}$ . The augmentation ideal of  $\mathfrak{B}$  is  $\mathfrak{B}^+ := \bigoplus_{n=1}^{+\infty} \mathfrak{B}_n$ . If  $\mathfrak{B}^+$  is a Noetherian ideal of  $\mathfrak{B}$ , then  $\mathfrak{B}$  is a finitely generated algebra over  $\mathfrak{B}_0$ . This is [99, Lemma 2.4.5] (in that statement  $\mathfrak{B}$  is commutative, but this hypothesis is not necessary for the proof).
- (d) Assume that  $\mathfrak{A}^G$  is graded over  $\mathbb{N}$  (for simplicity of notations):  $\mathfrak{A}^G = \bigoplus_{n=0}^{+\infty} \mathfrak{A}_n^G$  with  $\mathfrak{A}_0^G = K$ . Then  $\mathfrak{A}^{G+} = \bigoplus_{n=1}^{+\infty} \mathfrak{A}_n^G$  is an ideal of  $\mathfrak{A}^G$ , which is Noetherian by (b) above. Applying (c), we deduce that  $\mathfrak{A}^G$  is a finitely generated algebra over  $K$ .

**Proof of Theorem 3.4.** Consider the filtration  $\mathcal{F}$  of  $\mathcal{L}_{0,n}$  by the subspaces

$$\mathcal{F}^{[\mu]} = \bigoplus_{[\mu'] \preceq [\mu]} C([\mu']), \quad \mu \in P_+^n,$$

where  $P_+^n$  is given the lexicographic partial order induced from  $[\Lambda]$ . It is easily seen that  $\mathcal{F}$  is an algebra filtration: the coregular actions  $\triangleright, \triangleleft$  fix globally each component  $C(\mu)$  of  $\mathcal{L}_{0,1}$ , so the claim follows from (2.9), (2.17) and the fact that  $C(\mu) \star C(\nu) \subset C(\mu + \nu)$  for all  $\mu, \nu \in P_+$ . Denote by  $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})$  the corresponding graded algebra. As a vector space, we have

$$\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n}) = \bigoplus_{[\mu] \in P_+^n} C([\mu]). \quad (3.14)$$

Because each space  $C([\mu])$  is stabilized by the coadjoint action of  $U_q$ , (3.14) has a key advantage on the refined decomposition (3.11). Indeed, since  $\mathcal{L}_{0,n}$  is a  $U_q$ -module algebra, the action of  $U_q$  is well defined on  $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})$  and gives it a structure of  $U_q$ -module algebra. As vector spaces, we have

$$\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q} = \bigoplus_{[\mu] \in P_+^n} C([\mu])^{U_q}.$$

Now we can adapt the different steps (a)–(d) recalled above:

- (a') The action of  $U_q$  on  $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})$  is completely reducible. This follows from (3.14) and the fact that the spaces  $C(\mu)$  are finite-dimensional and thus completely reducible  $U_q$ -modules. We can therefore define the Reynolds operator as in (a),

$$R: \text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n}) \rightarrow \text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}.$$

- (b')  $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})$  is Noetherian, because (3.14) shows it is filtered by  $\mathcal{F}_3$ , and the associated graded algebra  $\text{Gr}_{\mathcal{F}_3}(\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})) = \text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})$  is Noetherian by Theorem 3.1. As in (b), we deduce that  $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$  is Noetherian. But  $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q} = \text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n}^{U_q})$ , which implies that  $\mathcal{L}_{0,n}^{U_q}$  is Noetherian.

- (c') (and (d')) Then we can apply the steps (c)–(d). As a result  $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$  is finitely generated, say by  $k$  non-zero elements  $\bar{x}_1, \dots, \bar{x}_k$ , which we may assume homogeneous.
- (e') We can now deduce that  $\mathcal{L}_{0,n}^{U_q}$  is generated by elements  $x_i$  with leading terms the  $\bar{x}_i$ 's. Indeed, let  $x \in \mathcal{L}_{0,n}^{U_q}$ , and  $[\mu] \in P_+^n$  such that  $x \in \mathcal{F}^{[\mu]} \setminus \mathcal{F}^{<[\mu]}$ , where  $\mathcal{F}^{<[\mu]} := \bigoplus_{[\mu'] < [\mu]} C([\mu'])$ . In  $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})_{[\mu]}^{U_q} = \mathcal{F}^{[\mu]} / \mathcal{F}^{<[\mu]}$ , we have

$$\bar{x} = \sum_{(i_1, \dots, i_k) \in I} \lambda_{(i_1, \dots, i_k)} \bar{x}_1^{i_1} \cdots \bar{x}_k^{i_k}$$

for some finite set  $I \subset \mathbb{N}^k$ , scalars  $\lambda_{(i_1, \dots, i_k)} \in \mathbb{C}(q)$ , and monomials  $\bar{x}_1^{i_1} \cdots \bar{x}_k^{i_k}$  of degree  $[\mu]$ . By definition of the product in  $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$ ,

$$\bar{x}_1^{i_1} \cdots \bar{x}_k^{i_k} = x_1^{i_1} \cdots x_k^{i_k} + \mathcal{F}^{<[\mu]},$$

so  $x_1^{i_1} \cdots x_k^{i_k} \in \mathcal{F}^{[\mu]} \setminus \mathcal{F}^{<[\mu]}$ , whence  $\bar{x}_1^{i_1} \cdots \bar{x}_k^{i_k} = \overline{x_1^{i_1} \cdots x_k^{i_k}}$  and

$$x - \sum_{(i_1, \dots, i_k) \in I} \lambda_{(i_1, \dots, i_k)} x_1^{i_1} \cdots x_k^{i_k} \in \mathcal{F}^{<[\mu]}.$$

The conclusion follows by decreasing induction on  $[\mu]$ , since at last we terminate at  $\mathcal{F}^{[0]} \cong \mathbb{C}(q)$ .

By combining the steps (a') to (e'), we get that  $\mathcal{M}_{0,n}$  is a Noetherian and finitely generated ring. ■

**Remark 3.6.**

- (1) Because  $\mathcal{L}_{0,1}^{U_q}$  is the center of  $\mathcal{L}_{0,1}$ , (e') proves it is finitely generated. Of course this follows also from the isomorphism  $\mathcal{L}_{0,1} \cong U_q^{\text{lf}}$  and the fact that the center of  $U_q^{\text{lf}}$  is the center of  $U_q$  (by Theorem 2.1), plus the well-known description of the latter.
- (2) In the  $\mathfrak{sl}_2$  case the filtration  $\mathcal{F}$  on  $\mathcal{L}_{0,n}^{U_q}$  should be related via the Wilson loop isomorphism (defined in [18, Section 8.2]) to the filtration of skein algebras of spheres with  $n + 1$  punctures used in [93].

## 4 Proof of Theorem 1.2

As usual we let  $\epsilon$  be a primitive  $l$ -th root of unity with  $l$  odd and  $l > d_i$  for all  $i \in \{1, \dots, m\}$ . We now consider the specialization  $\mathcal{L}_{0,n}^\epsilon$  of  $\mathcal{L}_{0,n}$  at  $q = \epsilon$ , defined in Section 2.2.1. Recall the isomorphism of algebras  $\eta^*: \mathcal{O}(G) \rightarrow \mathcal{Z}_0(\mathcal{O}_\epsilon)$  (see (2.71)), and that  $\mathcal{L}_{0,n}^\epsilon = \mathcal{O}_\epsilon^{\otimes n}$  as a vector space. Consider the linear subspace of  $\mathcal{L}_{0,n}^\epsilon$  defined by  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) := \mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$ . This space is naturally a subalgebra of  $\mathcal{O}_\epsilon^{\otimes n}$  (endowed with the componentwise product  $\star$ ). In fact, we also have the following.

**Proposition 4.1.**

- (1)  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a central subalgebra of the algebra  $\mathcal{L}_{0,n}^\epsilon$ , and the  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -modules  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{O}_\epsilon^{\otimes n}$ , with actions defined by the respective products of these algebras, do coincide.
- (2)  $\mathcal{L}_{0,n}^\epsilon$  is a free  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module of rank  $l^{n \cdot \dim \mathfrak{g}}$ .
- (3)  $(\eta^{*-1})^{\otimes n}: \mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) \rightarrow \mathcal{O}(G)^{\otimes n}$  is an isomorphism of algebras, and  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a Noetherian ring.
- (4) The  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module  $\mathcal{L}_{0,n}^\epsilon$  is finite and Noetherian. Therefore,  $\mathcal{L}_{0,n}^\epsilon$  is a Noetherian ring.

Note that the proof we give in (4) of the fact that  $\mathcal{L}_{0,n}^\epsilon$  is Noetherian is independent from the proof of Theorem 3.1.

**Proof.** (1) Let us show that  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a central subalgebra of  $\mathcal{L}_{0,n}^\epsilon$ . In the case  $n = 1$ , the formula (2.9) implies that  $\alpha\beta = \alpha \star \beta$  for all  $\alpha \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$  and  $\beta \in \mathcal{L}_{0,1}^\epsilon$ . Indeed, by (2.9) we have

$$\begin{aligned} \alpha\beta &= \sum_{(R),(R)} (R_{(2')}S(R_{(2)}) \triangleright \alpha) \star (R_{(1')} \triangleright \beta \triangleleft R_{(1)}) \\ &= \sum_{(R),(R),(\alpha),(\beta)} \alpha_{(1)} \star (\beta_{(1)}(R_{(1)}\alpha_{(3)}(S(R_{(2)})))\beta_{(3)}(R_{(1')}\alpha_{(2)}(R_{(2')}))\beta_{(2)}), \end{aligned}$$

where all components  $\alpha_{(1)}, \alpha_{(2)}, \alpha_{(3)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ , since  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  is a Hopf subalgebra of  $\mathcal{O}_\epsilon$ . But

$$\sum_{(R)} R_{(1)}\alpha_{(3)}(S(R_{(2)})) = S^{-1}(\Phi^-(S_{\mathcal{O}_\epsilon}(\alpha_{(3)}))) \in \mathcal{Z}_0(U_\epsilon),$$

since  $\Phi^-(S_{\mathcal{O}_\epsilon}(\alpha_{(3)})) \in \mathcal{Z}_0(U_\epsilon)$  by Theorem 2.29 (2). Similarly,  $\sum_{(R)} R_{(1')}\alpha_{(2)}(R_{(2')}) \in \mathcal{Z}_0(U_\epsilon)$ . In general, these elements belong to  $\mathcal{Z}_0(U_\epsilon)$  and not  $\mathcal{Z}_0(U_\epsilon^{\text{ad}})$  because of the ‘‘diagonal’’ factor  $\Theta$  of the  $R$ -matrix in (2.4). By Lemma 2.28,  $\mathcal{Z}_0(U_A^{\text{ad}})$  acts by the trivial character  $\varepsilon$  (the counit) on specializations of  $\Gamma$ -modules. The action of  $\mathcal{Z}_0(U_A)$  is the counit  $\varepsilon$  multiplied with some powers of  $\epsilon^{1/D}$ . However, [18, Propositions 4.1 and 4.10] show that such powers of  $\epsilon^{1/D}$  eventually disappear in the sum above; this is because the sum can be rewritten in terms of copies of the quasi  $R$ -matrix  $\hat{R}$  in (2.4) and the pivotal element  $\ell$ , instead of copies of  $R$ . Therefore,

$$\alpha\beta = \sum_{(\alpha),(\beta)} \alpha_{(1)} \star (\varepsilon(\beta_{(1)})\varepsilon(\alpha_{(3)})\varepsilon(\beta_{(3)})\varepsilon(\alpha_{(2)})\beta_{(2)}) = \alpha \star \beta. \quad (4.1)$$

This shows  $\mathcal{L}_{0,1}^\epsilon$  and  $\mathcal{O}_\epsilon$  coincide as modules over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$ . Next, we show that the subalgebras  $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$  are central in  $\mathcal{L}_{0,n}^\epsilon$  for all  $a = 1, \dots, n$ . This fact will conclude the proof that  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{O}_\epsilon^{\otimes n}$  coincide as  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -modules, because the subalgebras  $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$  generate the space  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  in  $(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$ , and hence in  $\mathcal{L}_{0,n}^\epsilon$  (this follows from the comment before (2.18)).

In order to show that  $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$  is central in  $\mathcal{L}_{0,n}^\epsilon$  for all  $a = 1, \dots, n$ , it is enough to show  $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$  commutes with the elements of  $\mathcal{L}_{0,n}^\epsilon$  supported by the tensor factors  $(\mathcal{L}_{0,1}^\epsilon)^{(b)}$  with  $b \neq a$ . Since  $(\alpha)^{(a)} \otimes (\beta)^{(b)} = ((\alpha)^{(a)} \otimes 1)(1 \otimes (\beta)^{(b)})$  by the definition, we have to show that  $(1 \otimes (\beta)^{(b)})((\alpha)^{(a)} \otimes 1) = (\alpha)^{(a)} \otimes (\beta)^{(b)}$  whenever  $\alpha \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ . We have (denoting  $\sum_{(\alpha),(\alpha),(\alpha),(\alpha)}$  by  $\sum_{(\alpha)^4}$ ,  $\Delta(\alpha_{(1)}) = \sum_{(\alpha)} \alpha_{(1)(1)} \otimes \alpha_{(1)(2)}$  etc.):

$$\begin{aligned} (1 \otimes (\beta)^{(b)})((\alpha)^{(a)} \otimes 1) &= \sum_{(R^i)} (S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha \triangleleft R_{(1)}^1 R_{(1)}^2)^{(a)} \\ &\quad \otimes (S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4)^{(b)} \\ &= \sum_{(R^i),(\alpha)^4,(\beta)^2} (\alpha_{(2)})^{(a)} \otimes (\beta_{(2)})^{(b)} \\ &\quad \times \beta_{(1)}(\alpha_{(1)(2)}(R_{(1)}^2)R_{(2)}^2\alpha_{(3)(1)}(S(R_{(1)}^4))R_{(2)}^4) \\ &\quad \times \beta_{(3)}(\alpha_{(3)(2)}(R_{(1)}^3)R_{(2)}^3\alpha_{(1)(1)}(R_{(1)}^1)S(R_{(2)}^1)). \end{aligned}$$

By Theorem 2.29 (2), it follows that

$$\alpha_{(1)(2)}(R_{(1)}^2)R_{(2)}^2 = \Phi^+(\alpha_{(1)(2)}) \in \mathcal{Z}_0(U_\epsilon),$$

and similarly

$$\alpha_{(3)(1)}(S(R_{(1)}^4))R_{(2)}^4, \alpha_{(3)(2)}(R_{(1)}^3)R_{(2)}^3, \alpha_{(1)(1)}(R_{(1)}^1)S(R_{(2)}^1) \in \mathcal{Z}_0(U_\epsilon).$$

Denote by  $z$  any such element;  $\mathcal{Z}_0(U_\epsilon^{\text{ad}})$  acts by the trivial character (the counit  $\varepsilon$ ) on specializations of  $\Gamma$ -modules. By using [18, Proposition 6.2], arguing as above (4.1), we obtain that the expression of  $z$  in terms of the corresponding  $\alpha_{(i)(j)}$  involves  $\varepsilon(z) = \varepsilon(\alpha_{(i)(j)})$  only (no root  $\epsilon^{1/D}$ ). It follows

$$\begin{aligned} & \beta_{(1)}(\alpha_{(1)(2)}(R_{(1)}^2)R_{(2)}^2\alpha_{(3)(1)}(S(R_{(1)}^4))R_{(2)}^4) \\ & \quad = \varepsilon(\alpha_{(1)(2)}\alpha_{(3)(1)})\beta_{(1)}(1) = \varepsilon(\alpha_{(1)(2)})\varepsilon(\alpha_{(3)(1)})\varepsilon(\beta_{(1)}), \\ & \beta_{(3)}(\alpha_{(3)(2)}(R_{(1)}^3)R_{(2)}^3\alpha_{(1)(1)}(R_{(1)}^1)S(R_{(2)}^1)) = \varepsilon(\alpha_{(3)(2)})\varepsilon(\alpha_{(1)(1)})\varepsilon(\beta_{(3)}). \end{aligned}$$

Therefore,  $(1 \otimes (\beta)^{(b)})((\alpha)^{(a)} \otimes 1) = (\alpha)^{(a)} \otimes (\beta)^{(b)}$ . It follows that  $\mathcal{L}_{0,n}^\epsilon = \mathcal{O}_\epsilon^{\otimes n}$  as modules over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ ; for instance when  $n = 2$ , given  $\alpha', \beta' \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  we have  $(\alpha' \otimes \beta')(\alpha \otimes \beta) = (\alpha' \otimes 1)(1 \otimes \beta')(\alpha \otimes 1)(1 \otimes \beta)$  immediately by (2.17), and  $(1 \otimes \beta')(\alpha \otimes 1) = \alpha \otimes \beta' = (\alpha \otimes 1)(1 \otimes \beta')$  as above. Then  $(\alpha' \otimes \beta')(\alpha \otimes \beta) = \alpha' \alpha \otimes \beta' \beta$ . In particular,  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a central subalgebra of  $\mathcal{L}_{0,n}^\epsilon$ .

(2) Since  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{O}_\epsilon^{\otimes n}$  coincide as modules over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon^{\otimes n})$ , the claim follows from Theorem 2.29, that is, from [41, Theorem 7.2], which shows that  $\mathcal{O}_\epsilon$  is a finitely generated projective module of rank  $l^{\dim \mathfrak{g}}$  over  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ , and from the arguments of [28] (using that  $K_0(\mathcal{O}(G)) = \mathbb{Z}$  by [87]), which imply that this module is free. Alternatively, it follows from the fact that  $\mathcal{O}_\epsilon$  is a cleft extension of  $\mathcal{O}(G)$  (see [6, Remark 2.18 (b)], and [25, Section 3.2]).

(3) The linear isomorphism  $(\eta^{*-1})^{\otimes n}: \mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) \rightarrow \mathcal{O}(G)^{\otimes n}$  is an isomorphism of algebras because  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is central in  $\mathcal{L}_{0,n}^\epsilon$ . It implies that  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a Noetherian ring, since  $\mathcal{O}(G)^{\otimes n} = \mathcal{O}(G^n)$  and  $G^n$  is an affine algebraic variety.

(4) The fact that  $\mathcal{L}_{0,n}^\epsilon$  is a finitely generated  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module follows from (2); an alternative proof of this fact will be provided at the end of the proof of Theorem 4.9. Since  $\mathcal{L}_{0,n}^\epsilon$  is finite over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  and  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is Noetherian,  $\mathcal{L}_{0,n}^\epsilon$  is a Noetherian  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module (e.g., by [7, Proposition 6.5]). It follows that  $\mathcal{L}_{0,n}^\epsilon$  is a Noetherian ring (e.g., by [88, Chapter 1, Section 1.3]).  $\blacksquare$

Recall that we denote  $U_\epsilon^{\text{lf}} = U_A^{\text{lf}} \otimes_A \mathbb{C}_\epsilon$  (see (2.27)), and  $\mathcal{Z}_0(U_\epsilon) \subset U_\epsilon$  is the central polynomial subalgebra generated by  $E_{\beta_k}^l, F_{\beta_k}^l, L_i^{\pm l}$ , for  $k \in \{1, \dots, N\}$  and  $i \in \{1, \dots, m\}$ . Since  $\Phi_1: \mathcal{L}_{0,1}^\epsilon \rightarrow U_\epsilon^{\text{lf}}$  is an embedding of algebras (see Corollary 2.25), it identifies  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  with a central subalgebra of  $U_\epsilon^{\text{lf}}$ . Put  $\mathcal{Z}_0(U_\epsilon^{\text{lf}}) := \Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))$ . Recall Theorem 2.1, Proposition 2.24, and let  $T^{(l)}, T_{2-}^{(l)}$  and  $T_2^{(l)}$  be the subsets of  $T, T_{2-}$  and  $T_2$  formed by the elements  $K_{\lambda l}$  with  $\lambda \in P, \lambda \in -2P_+$  and  $\lambda \in 2P$ , respectively.

**Proposition 4.2.** *We have  $U_\epsilon = T_{2-}^{-1}U_\epsilon^{\text{lf}}[T/T_2] = \Phi_1(\mathcal{L}_{0,1}^\epsilon[d^{-1}])[T/T_2]$ , and therefore the map  $\Phi_1: \mathcal{L}_{0,1}^\epsilon[d^{-1}] \rightarrow T_{2-}^{-1}U_\epsilon^{\text{lf}}$  is an isomorphism.*

*Moreover,  $\mathcal{Z}(U_\epsilon^{\text{lf}}) = U_\epsilon^{\text{lf}} \cap \mathcal{Z}(U_\epsilon)$ , and*

$$\mathcal{Z}_0(U_\epsilon) = T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}})[T^{(l)}/T_2^{(l)}], \quad \mathcal{Z}(U_\epsilon) = T_{2-}^{(l)-1} \mathcal{Z}(U_\epsilon^{\text{lf}})[T^{(l)}/T_2^{(l)}].$$

**Proof.** The first claim follows immediately from Proposition 2.24 by specialization at  $q = \epsilon$ . For the second claim, the inclusion  $U_\epsilon^{\text{lf}} \cap \mathcal{Z}(U_\epsilon) \subset \mathcal{Z}(U_\epsilon^{\text{lf}})$  is clear, and for the converse inclusion we only have to show that the elements of  $\mathcal{Z}(U_\epsilon^{\text{lf}})$  commute with  $T$ . They commute with  $T_2 \subset U_\epsilon^{\text{lf}}$ , so the conjugation action by elements of  $T$  on  $\mathcal{Z}(U_\epsilon^{\text{lf}})$  has order at most 2. Let  $x \in \mathcal{Z}(U_\epsilon^{\text{lf}})$  with decomposition  $x = \sum_i c_i x_i$  with all  $c_i \in \mathbb{C}$  and  $x_i$  PBW basis vectors, and let  $\lambda \in P$ . We have  $K_\lambda x K_{-\lambda} = \sum_i c_i q(x_i) x_i$ , where  $q(x_i) \in \epsilon^{\mathbb{Z}}$  satisfies  $q(x_i)^2 = 1$  for all  $i$ . Because  $\epsilon$  has odd order the only possibility is  $q(x_i) = 1$ , whence  $K_\lambda x K_{-\lambda} = x$ . The conclusion follows.

The inclusion  $\mathcal{Z}_0(U_\epsilon^{\text{lf}}) \subset \mathcal{Z}_0(U_\epsilon)$  follows from the definition  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$ , the formula  $\Phi_1 = m \circ (\text{id} \otimes S^{-1}) \circ \Phi$ , and the fact that  $\Phi$  affords an embedding  $\mathcal{Z}_0(\mathcal{O}_\epsilon) \rightarrow \mathcal{Z}_0(U_\epsilon(G^*))$

(see Theorem 2.29 (2)). Since  $T^{(l)} \subset \mathcal{Z}_0(U_\epsilon)$ , we obtain

$$T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}})[T^{(l)}/T_2^{(l)}] \subset \mathcal{Z}_0(U_\epsilon).$$

The proof of the converse inclusion is similar to that in Proposition 2.24. The isomorphism  $\mathcal{Z}_0(\mathcal{O}_\epsilon)[\psi_{-l\rho}^{-1}] \rightarrow \mathcal{Z}_0(U_\epsilon(G^*))$  of Theorem 2.29 (2) implies

$$F_{\beta_k}^l K_{\beta_k}^l \otimes 1, 1 \otimes K_{\beta_k}^{-l} E_{\beta_k}^l \in \Phi(\mathcal{Z}_0(\mathcal{O}_\epsilon)[\psi_{-l\rho}^{-1}])$$

for every positive root  $\beta_k$ . Since  $\psi_{-l\rho} = \Phi_1^{-1}(K_{-2l\rho}) = \psi_{-\rho}^l$  (the  $l$ -th power of  $\psi_{-\rho}$  in  $\mathcal{L}_{0,1}^\epsilon$ ), and

$$\Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[\psi_{-\rho}^{-l}]) = T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}}),$$

it follows that

$$F_{\beta_k}^l K_{\beta_k}^l, S^{-1}(E_{\beta_k}^l) K_{\beta_k}^l \in T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}}).$$

Hence  $F_{\beta_k}^l, S^{-1}(E_{\beta_k}^l) \in T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}})[T^{(l)}/T_2^{(l)}]$ . The sets  $S^{-1}(E_{\beta_k}^l) \mathcal{Z}_0(U_\epsilon(\mathfrak{h}))$ ,  $k = 1, \dots, N$ , generate the subalgebra  $\mathcal{Z}_0(U_\epsilon(\mathfrak{b}_+))$  of  $\mathcal{Z}_0(U_\epsilon)$ , so from the triangular decomposition  $\mathcal{Z}_0(U_\epsilon) = \mathcal{Z}_0(U_\epsilon(\mathfrak{n}_-)) \mathcal{Z}_0(U_\epsilon(\mathfrak{h})) \mathcal{Z}_0(U_\epsilon(\mathfrak{n}_+))$  this proves the inclusion  $\mathcal{Z}_0(U_\epsilon) \subset T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}})[T^{(l)}/T_2^{(l)}]$ . From the isomorphism

$$\mathcal{Z}_0(U_\epsilon) \quad \bigotimes_{\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon)} \quad \mathcal{Z}_1(U_\epsilon) \rightarrow \mathcal{Z}(U_\epsilon)$$

(see Theorem 2.27), and the fact that  $\mathcal{Z}(U_q) \subset U_q^{\text{lf}}$  (whence  $\mathcal{Z}_1(U_\epsilon) \subset \mathcal{Z}(U_\epsilon^{\text{lf}})$ ), the equality  $\mathcal{Z}(U_\epsilon) = T_{2-}^{(l)-1} \mathcal{Z}(U_\epsilon^{\text{lf}})[T^{(l)}/T_2^{(l)}]$  follows at once.  $\blacksquare$

**Remark 4.3.** Let us explain how this can be used to give an interpretation of the isomorphism  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \cong \mathcal{O}(G)$ . Recall the notations introduced around Theorem 2.27. Since  $G^* = U_+ T_G U_-$ , we have  $\mathcal{O}(G^*) = \mathcal{O}(U_+) \mathcal{O}(T_G) \mathcal{O}(U_-)$ , and the map  $\sigma$  yields an identification

$$\mathcal{O}(G^0) = \mathcal{O}(U_+) \mathcal{O}(T_G/(2)) \mathcal{O}(U_-). \quad (4.2)$$

We can identify  $\mathcal{O}(G^0)$  with the subalgebra  $(\sigma|_{G^*})^*(\mathcal{O}(G^0)) \subset \mathcal{O}(G^*)$ . Consider the exterior power  $V = \wedge^N \mathfrak{g}$  endowed with the action  $\wedge^N \text{Ad}$  of  $G$ . Put on  $\mathfrak{g}$  a basis consisting of one element  $e_\alpha$  per root space  $\mathfrak{g}_\alpha$ , along with a basis of  $\mathfrak{h}$ . Let  $v \in V$  be the exterior power of the  $e_\alpha$ 's for  $\alpha$  negative, and  $v^*$  a dual vector such that  $v^*(v) = 1$  and  $v^*$  vanishes on a  $T_G$ -invariant complement of  $v$ . It is classical that  $G \setminus G^0$  has defining equation  $\delta(g) = 0$ , where  $\delta$  is the matrix coefficient  $\delta(g) = v^*(\pi_V(g)v)$  (see, e.g., [59, p. 174]). Hence  $\mathcal{O}(G^0) = \mathcal{O}(G)[\delta^{-1}]$ . On  $G^0$  we have  $\delta(u_+ t u_-) = \chi_{-2\rho}(t)$ , where  $\chi_{-2\rho}$  is the character of  $T_G$  associated to the root  $-2\rho$ . Now we can make the connection with  $U_\epsilon$ . The isomorphism  $\mathcal{Z}_0(U_\epsilon) \cong \mathcal{O}(G^*)$  of Theorem 2.27 (2) identifies  $\mathcal{Z}_0(U_\epsilon(\mathfrak{h})) = \mathbb{C}[T^{(l)}]$  with  $\mathcal{O}(T_G)$  by mapping  $K_\lambda$  to the character of  $T_G$  associated to  $\lambda$ . Therefore, it maps  $\mathbb{C}[T^{(l)}]$  to  $\mathcal{O}(T_G/(2))$ , and  $T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}})$  to  $\mathcal{O}(G^0)$  by (4.2) and Proposition 4.2. Since  $\mathcal{O}(G^0) = \mathcal{O}(G)[\delta^{-1}]$  and  $T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}}) = \mathcal{Z}_0(U_\epsilon^{\text{lf}})[\ell^l]$ , it follows that  $\mathcal{Z}_0(U_\epsilon^{\text{lf}})$  and  $\mathcal{O}(G)$  coincide after localization by  $\ell^l$  and  $\delta$  respectively. By using the Bruhat decomposition of  $G$  as in (4.6) in the proof of Theorem 4.9 below, one can deduce  $\mathcal{Z}_0(U_\epsilon^{\text{lf}}) \cong \mathcal{O}(G)$ , whence  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \cong \mathcal{O}(G)$  by injectivity of  $\Phi_1$ .

Let us make the following observation. Since  $\mathcal{L}_{0,n}^\epsilon = \mathcal{L}_{0,n}^A \otimes_A \mathbb{C}_\epsilon$ , with  $\mathcal{L}_{0,n}^A = \mathcal{O}_A^{\otimes n}$  as an  $A$ -module, and a generating system of  $\mathcal{O}_A^{\otimes n}$  is also a generating system of  $\mathcal{L}_{0,n}^A$ , it follows from Proposition 2.10 and the identities (2.56)–(2.57) that  $\mathcal{L}_{0,n}^\epsilon$  is generated by elements of the form  $\alpha_1 \otimes \dots \otimes \alpha_n$ , where  $\alpha_1, \dots, \alpha_n$  belong to the set  $C_{\text{gen}}$  of matrix coefficients lying on the first and last columns of the matrix representations of  $U_A^{\text{res}}$  in the canonical bases of the modules  ${}_A V_{\varpi_i}$ ,  $i = 1, \dots, m$ . Denote by  $\alpha^{*k}$ ,  $k \in \mathbb{N}$ , the  $k$ -th power of an element  $\alpha \in \mathcal{O}_A$ .

**Lemma 4.4.** *For all  $\alpha \in C_{\text{gen}}$ ,  $\alpha^{*l} \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ .*

**Proof.** Recall that the Frobenius epimorphism  $\eta: U_A^{\text{res}} \otimes_A \mathbb{C}_\epsilon \rightarrow U(\mathfrak{g})$  in (2.71) has kernel the ideal  $I$  generated by the elements  $E_i, F_i, K_i - 1$ , and  $(K_i; p)_{q_i}$  where  $l$  does not divide  $p$ ,  $i = 1, \dots, m$ . It follows that an element of  $\mathcal{O}_\epsilon$  belongs to  $\mathcal{Z}_0(\mathcal{O}_\epsilon) = \eta^*(\mathcal{O}(G))$  if and only if it vanishes on  $I$ . But this is immediate to check for the elements of the form  $\alpha^{*l}$  with  $\alpha \in C_{\text{gen}}$ , using that  $K_i$  is grouplike and the pure summands of  $\Delta(E_i)$  and  $\Delta(F_i)$  have one component equal to 1 or  $K_i^{\pm 1}$  and the other component equal to  $E_i$  or  $F_i$ . For instance,

$$\psi_{\varpi_i}^{*l}(K_i - 1) = \psi_{\varpi_i}(K_i)^l - 1 = \epsilon^{l(\alpha_i, \varpi_i)} - 1 = 0.$$

Similarly, for every  $\alpha \in C_{\text{gen}}$ , we find

$$\alpha^{*l}(E_i) = \alpha^{\otimes l}(\Delta^{(l)}(E_i)) = 0 \quad \text{and} \quad \alpha^{*l}(F_i) = \alpha^{*l}(K_i - 1) = 0. \quad \blacksquare$$

We need below explicit descriptions of the centers of  $\mathcal{O}_\epsilon(\text{SL}_2)$  and  $\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2)$  and their  $\mathcal{Z}_0$ -subalgebras. Denote by  $a, b, c, d$  the standard generators of  $\mathcal{O}_q(\text{SL}_2)$ , i.e., the matrix coefficients in the basis of weight vectors  $v_0, v_1 = F.v_0$  of the 2-dimensional irreducible representation  $V_1$  of  $U_q(\mathfrak{sl}_2)$ . As above, denote by  $x^{*k}$ ,  $k \in \mathbb{N}$ , the  $k$ -th power of an element  $x \in \mathcal{O}_A(\text{SL}_2)$ . The algebra  $\mathcal{O}_A(\text{SL}_2)$  is generated by  $a, b, c, d$ ; the monomials  $a^{*i} \star b^{*j} \star d^{*r}$  and  $a^{*i} \star c^{*k} \star d^{*r}$ ,  $i, j, k, r \in \mathbb{N}, k > 0$ , form an  $A$ -basis of  $\mathcal{O}_A(\text{SL}_2)$ . The algebra  $\mathcal{Z}_0(\mathcal{O}_\epsilon(\text{SL}_2))$  is generated by  $a^{*l}, b^{*l}, c^{*l}, d^{*l}$ ; the monomials  $a^{*il} \star b^{*jl} \star d^{*rl}$  and  $a^{*il} \star c^{*kl} \star d^{*rl}$  form a basis of  $\mathcal{Z}_0(\mathcal{O}_\epsilon(\text{SL}_2))$ , and  $\mathcal{Z}(\mathcal{O}_\epsilon(\text{SL}_2))$  is generated by  $\mathcal{Z}_0(\mathcal{O}_\epsilon(\text{SL}_2))$  and the elements  $b^{*(l-k)} \star c^{*k}$ ,  $k = 0, \dots, l$  (see [41, Proposition 1.4 and the appendix]). We have the relation

$$a^{*l} \star d^{*l} - b^{*l} \star c^{*l} = 1 \tag{4.3}$$

and the Frobenius isomorphism of Parshall–Wang (see [92, Chapter 7]) coincides with the map

$$\text{FrpW}: \mathcal{O}(\text{SL}_2) \rightarrow \mathcal{Z}_0(\mathcal{O}_\epsilon(\text{SL}_2))$$

induced by  $\eta^*$ ; it sends the standard generators  $\underline{a}, \underline{b}, \underline{c}, \underline{d}$  of  $\mathcal{O}(\text{SL}_2) = \mathcal{O}_1(\text{SL}_2)$  respectively to  $a^{*l}, b^{*l}, c^{*l}, d^{*l}$ . Finally, we have seen that  $\mathcal{O}_\epsilon(\text{SL}_2)$  is a free  $\mathcal{Z}_0(\mathcal{O}(\text{SL}_2))$ -module of rank  $l^3$  (see Theorem 2.29 (3)). In [38], it is shown that a basis of this module is formed by the monomials  $a^m b^n c^{s'}$  and  $b^n c^{s''} d^r$ , with the integers  $m, n, r, s', s''$  in the range

$$1 \leq m \leq l-1, \quad 0 \leq n, r \leq l-1, \quad m \leq s' \leq l-1, \quad 0 \leq s'' \leq l-r-1. \tag{4.4}$$

Now consider  $\mathcal{L}_{0,1}^A(\mathfrak{sl}_2)$ . Recall that  $\mathcal{L}_{0,1}^A = \mathcal{O}_A$  as  $U_A$ -modules. The algebra  $\mathcal{L}_{0,1}^A(\mathfrak{sl}_2)$  is also generated by  $a, b, c, d$ ; a set of defining relations is (see [18, Section 5]):

$$\begin{aligned} ad &= da, & db &= q^2 bd, & cd &= q^2 dc, & ab - ba &= -(1 - q^{-2})bd, \\ cb - bc &= (1 - q^{-2})(da - d^2), & ac - ca &= (1 - q^{-2})dc, & ad - q^2 bc &= 1. \end{aligned} \tag{4.5}$$

The element  $\omega := qa + q^{-1}d$  is central. Let  $T_k$ ,  $k \in \mathbb{N}$ , be such that  $T_k(x)/2$  is the  $k$ -th Chebyshev polynomial of the first type in the variable  $x/2$ . We have (see [18, Proposition 7.2], for the generalization to  $\mathcal{L}_{0,n}^\epsilon(\mathfrak{sl}_2)$ ):

**Lemma 4.5.** *Let  $\mathcal{I}$  be the ideal of  $\mathbb{C}[\omega, b^l, c^l, d^l]$  generated by  $(T_l(\omega) - d^l)d^l - b^l c^l - 1$ , we have*

$$\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2)) = \mathbb{C}[\omega, b^l, c^l, d^l]/\mathcal{I} \quad \text{and} \quad \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2)) = \mathbb{C}[T_l(\omega), b^l, c^l, d^l]/\mathcal{I}.$$

Here  $b^l, c^l, d^l$  are the  $l$ -th powers of  $b, c, d$  computed using the product of  $\mathcal{L}_{0,1}^A(\mathfrak{sl}_2)$ , not the product  $\star$  of  $\mathcal{Z}_0(\mathcal{O}_\epsilon(\text{SL}_2))$ . The above generator of  $\mathcal{I}$  can be interpreted as a determinant, and  $\omega$  as a quantum trace on  $V_1$ . The following has also been observed in [75].



**Lemma 4.6.** *Viewed as elements of  $\mathcal{O}_A(\mathrm{SL}_2)$ ,  $T_l(\omega) - d^l = a^{*l}$  and  $x^l = x^{*l}$ ,  $x \in \{b, c, d\}$ .*

**Proof.** Let  $\alpha$  and  $\varpi$  be the simple root and fundamental weight of  $\mathfrak{sl}_2$ . In the notations of (2.70), we have  $b = \psi_{-\varpi}^{-\alpha}$ ,  $c = \psi_{-\varpi}^{\alpha}$ ,  $d = \psi_{-\varpi}$ ; the formulas of  $\Phi$  give

$$\Phi_1(b^{*l}) = (q - q^{-1})^l F^l, \quad \Phi_1(c^{*l}) = (q - q^{-1})^l E^l K^{-l}, \quad \Phi_1(d^{*l}) = K^{-l}.$$

These coincide respectively with  $\Phi_1(b^l)$ ,  $\Phi_1(c^l)$ ,  $\Phi_1(d^l)$  (see [18, equation (5.3)]). By passing to the localization  $\mathcal{O}_A(\mathrm{SL}_2)[d^{-1}]$ , and using Parshall–Wang’s relation (4.3), one deduces easily

$$\Phi_1(a^{*l}) = K^l + (q - q^{-1})^{2l} F^l E^l = T_l(\Omega) - K^{-l},$$

where  $\Omega = (\epsilon - \epsilon^{-1})^2 FE + \epsilon K + \epsilon^{-1} K^{-1}$  is the Casimir element, and  $T_l(x)/2$  is the  $l$ -th Chebyshev polynomial of the first type in the variable  $x/2$ . We have  $\Phi_1(\omega) = \Omega$ , so  $\Phi_1(a^{*l}) = T_l(\omega) - d^l$ . The conclusion follows from the injectivity of  $\Phi_1$ .  $\blacksquare$

This lemma proves that we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}(\mathrm{SL}_2) & \xrightarrow{\mathrm{Fr}_{\mathrm{PW}}} & \mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2)) & \hookrightarrow & \mathcal{O}_\epsilon(\mathrm{SL}_2) \\ & \searrow \mathrm{Fr} & \downarrow & & \downarrow \\ & & \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2)) & \hookrightarrow & \mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2), \end{array}$$

where  $\mathrm{Fr}_{\mathrm{PW}}$  is Parshall–Wang’s Frobenius isomorphism recalled above,  $\mathrm{Fr}$  is the Frobenius isomorphism introduced in [18, Definition 7.1], and the vertical arrows are the identifications as vector spaces (the middle one proved by Proposition 4.1).

**Remark 4.7.** By Lemma 4.5, the monomials  $T_l(\omega)^i b^{jl} d^{rl}$  and  $T_l(\omega)^i c^{kl} d^{rl}$ , for  $i, j, k, r \in \mathbb{N}$  and  $k > 0$ , form an  $A$ -basis of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2))$ . It is straightforward (though cumbersome) to express these basis elements in terms of the basis elements  $a^{*il} \star b^{*jl} \star d^{*rl}$  and  $a^{*il} \star c^{*kl} \star d^{*rl}$  of  $\mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2))$ , and conversely; this can be done by using Lemma 4.6, the formula (2.9) or the inverse one (expressing  $\star$  in terms of the product of  $\mathcal{L}_{0,1}$ , see [18, equation (4.6)]), and the formula of the coproduct  $\Delta: \mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2) \rightarrow \mathcal{L}_{0,2}^\epsilon(\mathfrak{sl}_2)$  restricted to  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2))$  (given in [18, Lemma 7.5]).

Since  $\mathcal{L}_{0,1}^A = \mathcal{O}_A$  as an  $A$ -module, the functionals  $t_i$  in Proposition 2.30 can be seen as maps  $t_i: \mathcal{L}_{0,1}^A \rightarrow A$ . Though the algebra structures of  $\mathcal{O}_\epsilon$  and  $\mathcal{L}_{0,1}^\epsilon$  are very different,  $\mathcal{L}_{0,1}^\epsilon$  satisfies a result analogous to Proposition 2.30:

**Proposition 4.8.** *The maps  $\triangleleft t_i$  preserve  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , and they satisfy  $(f \triangleleft t_i)(a) = f(n_i a)$  and  $(f\alpha) \triangleleft t_i = (f \triangleleft t_i)(\alpha \triangleleft t_i)$  for every  $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ ,  $a \in G$ ,  $\alpha \in \mathcal{L}_{0,1}^\epsilon$ .*

**Proof.** The first two claims follow from Proposition 2.30 and the definition  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$ .

The last claim follows from the case  $\mathfrak{g} = \mathfrak{sl}_2$ , as in the proof of [41, Proposition 7.1]. In fact, it is enough to show that  $t(fg) = t(f)t(g)$  for every  $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2))$ ,  $g \in \mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2)$ ; for completeness we explain this in Appendix C, see (C.3). A word of caution is needed: the proof of (C.3) uses that  $\Delta: \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon \otimes \mathcal{O}_\epsilon$  is a morphism of algebras. The analogous property for  $\mathcal{L}_{0,1}^\epsilon$  is that  $\Delta$  yields a morphism of algebras  $\Delta: \mathcal{L}_{0,1}^\epsilon \rightarrow \mathcal{L}_{0,2}^\epsilon$ . Since the algebra structure of  $\mathcal{L}_{0,2}^\epsilon$  is not the product one on  $\mathcal{L}_{0,1}^\epsilon \otimes \mathcal{L}_{0,1}^\epsilon$ , it is not true in general that

$$\sum_{(f),(g)} (f_{(1)} \otimes f_{(2)})(g_{(1)} \otimes g_{(2)}) = \sum_{(f),(g)} f_{(1)} g_{(1)} \otimes f_{(2)} g_{(2)}$$

for every  $f, g \in \mathcal{L}_{0,1}^\epsilon$ . However, it holds whenever  $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , since  $\Delta(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)) \subset \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \otimes \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  and therefore  $f_{(2)} \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$  commutes in  $\mathcal{L}_{0,2}^\epsilon$  with any  $g_{(1)} \in \mathcal{L}_{0,1}^\epsilon = \mathcal{O}_\epsilon$ .

It is enough to prove the identity  $t(fg) = t(f)t(g)$  when  $f$  ranges in a set of generators of the algebra  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2))$ . So one can take  $f$  among, say,  $T_l(\omega) - d^l = a^{*l}$  and  $x^l = x^{*l}$ ,  $x \in \{b, c, d\}$  (using Lemma 4.5). By (2.9) and Proposition C.1, we have

$$t(fg) = \sum_{(R),(R')} t(R_{(2')}S(R_{(2)}) \triangleright f) t(R_{(1')} \triangleright g \triangleleft R_{(1)}).$$

Expanding coproducts and using that  $R^{-1} = (S \otimes \text{id})(R)$ , we deduce

$$\begin{aligned} t(fg) &= \sum_{(f),(R),(R')} t(f_{(1)}) \langle f_{(2)}, R_{(2')}S(R_{(2)}) \rangle t(R_{(1')} \triangleright g \triangleleft R_{(1)}) \\ &= \sum_{(f),(R),(R')} t(f_{(1)}) t(\langle f_{(2)}, R_{(2')} \rangle R_{(1')} \triangleright g \triangleleft \langle f_{(3)}, S(R_{(2)}) \rangle R_{(1)}) \\ &= \sum_{(f)} t(f_{(1)}) t(S^{-1}(\Phi^-(f_{(2)})) \triangleright g \triangleleft S^{-2}(\Phi^-(f_{(3)}))) \\ &= \sum_{(f)} t(f_{(1)}) \langle g, S^{-2}(\Phi^-(f_{(3)})) \underline{w} S^{-1}(\Phi^-(f_{(2)})) \rangle \\ &= \sum_{(f)} t(f_{(1)}) \varepsilon(S^{-2}(\Phi^-(f_{(3)}))) \varepsilon(S^{-1}(\Phi^-(f_{(2)}))) t(g), \end{aligned}$$

where  $\underline{w} \in \mathbb{U}_\Gamma$  is the quantum Weyl group element dual to  $t$  (see Appendix B), and in the last equality we used that  $\Phi^-$  maps  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  into  $\mathcal{Z}_0(U_\epsilon)$  (see Theorem 2.29 (2)), which acts on specializations of  $\Gamma$ -modules by the trivial character (the counit)  $\varepsilon: U_\epsilon \rightarrow \mathbb{C}$ . By (B.6)–(B.7), we have  $t(a^{*l}) = t(d^{*l}) = 0$  and  $t(b^{*l}) = 1$ ,  $t(c^{*l}) = -1$ . Now the computation of  $t(fg)$  follows easily. For instance, taking  $f = b^l = b^{*l}$ , by using  $\Delta(b^{*l}) = a^{*l} \otimes b^{*l} + b^{*l} \otimes d^{*l}$  and  $\Delta(d^{*l}) = c^{*l} \otimes b^{*l} + d^{*l} \otimes d^{*l}$ , we get

$$t(b^l g) = \varepsilon(S^{-2}(\Phi^-(b^{*l}))) \varepsilon(S^{-1}(\Phi^-(c^{*l}))) t(g) + \varepsilon(S^{-2}(\Phi^-(d^{*l}))) \varepsilon(S^{-1}(\Phi^-(d^{*l}))) t(g).$$

Since  $b^{*l} \in \mathcal{O}_\epsilon(U_+)$ ,  $\Phi^-(b^{*l}) = 0$ . Also, it is immediate from the definition of  $\Phi^-$  that  $\Phi^-(d^{*l}) = \Phi^-(d)^l = L^l$ ; alternatively, one can bypass this computation by observing that  $\Phi^-$  sets an isomorphism from  $\mathcal{O}_\epsilon(T_G) = \mathcal{O}_\epsilon(B_+) \cap \mathcal{O}_\epsilon(B_-)$  to  $\mathbb{C}[L^{\pm 1}] = U_\epsilon(\mathfrak{b}_+) \cap U_\epsilon(\mathfrak{b}_-)$ , mapping a generator  $d$  to  $L$  or  $L^{-1}$ . We have  $\varepsilon(L^l) = 1$ , and therefore

$$t(b^l g) = t(g) = t(b^l) t(g).$$

The other cases  $f = T_l(\omega) - d^l, c^l, d^l$  are similar. ■

**Theorem 4.9.**  $\mathcal{L}_{0,n}^\epsilon$  is a free  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module of rank  $l^{n \cdot \dim \mathfrak{g}}$ , and  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is a Noetherian ring and a finite, whence Noetherian,  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module.

**Proof.** We already proved the first claim in Proposition 4.1, and that  $\mathcal{L}_{0,n}^\epsilon$  is a Noetherian  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module. For the second claim, it follows that the  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -submodule  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is necessarily finitely generated. But  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  being Noetherian,  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is then a Noetherian  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module and a Noetherian ring.

For the sake of clarity, let us provide a self-contained proof of the first claim, not invoking directly [28, 41] or [6, 25], but applying the same arguments directly to  $\mathcal{L}_{0,n}^\epsilon$ . Since  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{L}_{0,1}^{\otimes n}$  coincide as modules over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) = \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$  by Proposition 4.1, the result will follow from

the case  $n = 1$ . Then we argue in four steps. First, using Theorem 2.1 we show that a certain localization of  $\mathcal{L}_{0,1}^\epsilon$  is a free module of rank  $l^{\dim \mathfrak{g}}$ . Then, assuming that  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective, we explain why it has constant rank  $l^{\dim \mathfrak{g}}$  (this is very classical). Thirdly, we prove that  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective as in [41, Theorem 7.2]. Finally, we obtain that it is a free module as in [28].

Recall Proposition 4.2:  $U_\epsilon$  is a free  $\Phi_1(\mathcal{L}_{0,1}^\epsilon[d^{-l}])$ -module of rank  $2^m$  (note that  $\mathcal{L}_{0,1}^\epsilon[d^{-l}] = \mathcal{L}_{0,1}^\epsilon[d^{-1}]$ ),  $\mathcal{Z}_0(U_\epsilon)$  is free over

$$T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}}) = \Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon[d^{-l}]))$$

of rank  $2^m$ . Since  $U_\epsilon$  is also free of rank  $l^{\dim \mathfrak{g}}$  over  $\mathcal{Z}_0(U_\epsilon)$  (see Theorem 2.27 (1)), it is free over  $\Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon[d^{-l}]))$  of rank  $2^m l^{\dim \mathfrak{g}}$ . The decomposition being unique,  $\Phi_1(\mathcal{L}_{0,1}^\epsilon[d^{-l}])$  is free of rank  $l^{\dim \mathfrak{g}}$  over  $\Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon[d^{-l}]))$ , and injectivity of  $\Phi_1$  implies that  $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$  is free of rank  $l^{\dim \mathfrak{g}}$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon[d^{-l}])$ .

Assume now that  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective. Let us show that its rank is  $l^{\dim \mathfrak{g}}$ . The localization  $(\mathcal{L}_{0,1}^\epsilon)_P$  of  $\mathcal{L}_{0,1}^\epsilon$  at any prime ideal  $P$  of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  is a free module over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)_P$  [96, Proposition 2.12.15]; the ranks of such modules are finite in number [96, Proposition 2.12.20]. If these ranks are all equal, then, by definition, it is the rank of  $\mathcal{L}_{0,1}^\epsilon$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ . This happens if  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  has no nontrivial (i.e.,  $\neq 1$ ) idempotent [96, Corollary 2.12.23], which is the case since it has no nontrivial zero divisors. To compute the rank, suppose  $P$  does not contain  $d^l = \psi_{-\rho}^l$ . Such ideals  $P$  are in 1-1 correspondence with the prime ideals of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon[d^{-l}])$  by the natural ring monomorphism  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \rightarrow \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon[d^{-l}])$ . The set  $S = \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \setminus P$  is multiplicatively closed, and we have also a ring morphism  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon[d^{-l}]) \rightarrow S^{-1}\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , which is also an injection (there are no zero divisors in  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , whence in  $S$ ). Then

$$(\mathcal{L}_{0,1}^\epsilon)_P = S^{-1}\mathcal{L}_{0,1}^\epsilon = \mathcal{L}_{0,1}^\epsilon[d^{-l}] \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon[d^{-l}])} S^{-1}\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$$

shows that  $(\mathcal{L}_{0,1}^\epsilon)_P$  has over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)_P = S^{-1}\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  the same rank  $l^{\dim \mathfrak{g}}$  as  $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon[d^{-l}])$ . This proves our claim.

In order to show that  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , it is enough to show it is finite locally free, i.e., there are elements  $d_i \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  such that the localizations  $\mathcal{L}_{0,1}^\epsilon[d_i^{-1}]$  are finite free  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon[d_i^{-1}])$ -modules, and  $\text{Maxspec}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))$  is covered by the open sets  $U(d_i)$  made of the ideals not containing  $d_i$  (see [100, Lemma 77.2]).

We have seen above that  $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$  is free of rank  $l^{\dim \mathfrak{g}}$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon[d^{-l}])$ . By Remark 4.3,  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon[d^{-l}]) \cong \mathcal{Z}_0(U_\epsilon^{\text{lf}})[\ell^l]$  is isomorphic to  $\mathcal{O}(G^0)$ , and  $\mathcal{O}(G^0) = \mathcal{O}(G)[\delta^{-1}]$ . Now, given  $w \in W$  with a reduced expression  $s_{i_1} \cdots s_{i_k}$ , put  $t_w = t_{i_1} \cdots t_{i_k}$ . Let  $w$  be represented by  $n_w = n_{i_1} \cdots n_{i_k}$  in  $N(T_G)$ . By Proposition 4.8, we have  $(f \triangleleft t_w)(x) = f(n_w x)$  for every  $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ ,  $x \in G$ . Then

$$\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon[d^{-l}] \triangleleft t_w) \cong \mathcal{O}(n_w^{-1}G^0) \cong \mathcal{O}(G)[(\delta \triangleleft t_w)^{-1}]. \quad (4.6)$$

If  $b_1, \dots, b_r$  ( $r := l^{\dim \mathfrak{g}}$ ) is a basis of  $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon[d^{-l}])$ , then  $\mathcal{L}_{0,1}^\epsilon[d^{-l}] \triangleleft t_w$  is free over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[(\delta \triangleleft t_w)^{-1}] \cong \mathcal{O}(n_w^{-1}G^0)$  with basis  $b_1 \triangleleft t_w, \dots, b_r \triangleleft t_w$ . Consider the Bruhat decomposition of  $G$ : any  $g \in G$  can be written in the form  $g = b_1 n b_2$ , where  $b_1, b_2 \in B_-, n \in W$ . Hence  $g = n n^{-1} b_1 n b_2 \in n B_+ B_- = n G^0$ , and therefore

$$G = \bigcup_{w \in W} (B_- n_w B_-) = \bigcup_{w \in W} (n_w G^0).$$

For every  $w \in W$ , put  $d_w^l := d^l \triangleleft t_w$ . Under the isomorphism of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  with  $\mathcal{O}(G)$ , we thus get that  $\text{Maxspec}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))$  is covered by the open sets  $U(d_w^l) \cong n_w G^0$ , and  $\mathcal{L}_{0,1}^\epsilon[d_w^{-l}]$  is finite free over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d_w^{-l}]$ . Therefore,  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ .

Finally, let us explain why  $\mathcal{L}_{0,1}^\epsilon$  is free over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , following the arguments of [28]. Let  $R$  be a commutative Noetherian ring, put  $X = \text{Maxspec}(R)$ , and let  $P$  be an  $R$ -module. Denote by  $R_I, P_I$  the localizations of  $R, P$  at a maximal ideal  $I \in X$ . Define the *f-rank* of  $P$  as  $\text{f-rank}(P) = \inf_{I \in X} \{ \text{f-rank}_{R_I}(P_I) \}$ , where  $\text{f-rank}_{R_I}(P_I) = \sup \{ r \in \mathbb{N}, R_I^{\otimes r} \subset P_I \} \in \mathbb{N} \cup \{+\infty\}$  (i.e., the maximal dimension of a free summand of  $P_I$ ). Bass' Cancellation theorem asserts that if  $P$  is projective and  $\text{f-rank}(P) > \dim(X)$ , and  $P \oplus Q \cong M \oplus Q$  for some  $R$ -modules  $Q$  and  $M$  such that  $Q$  is finitely generated and projective, then  $P \cong M$  (see [19, Section IV.3.5, p. 167 and p. 170], taking  $A = R$ , or [88, Section 11.7.13]). Let us apply this to  $R = \mathcal{O}(G)$  and  $P = \mathcal{L}_{0,1}^\epsilon$ . We proved above that  $\text{f-rank}_{R_I}(P_I) = l^{\dim \mathfrak{g}}$ , a constant, and we have  $l^{\dim \mathfrak{g}} > \dim \mathfrak{g} = \dim(G)$ . By a result of Marlin [87],  $G$  being semisimple and simply connected the Grothendieck ring  $K_0(\mathcal{O}(G))$  is isomorphic to  $\mathbb{Z}$ . Therefore,  $\mathcal{L}_{0,1}^\epsilon \oplus Q \cong \mathcal{O}(G)^r$  for some free  $\mathcal{O}(G)$ -module  $Q$  and  $r \in \mathbb{N}$ . Then Bass' cancellation implies  $\mathcal{L}_{0,1}^\epsilon$  is free over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \cong \mathcal{O}(G)$ . ■

## 5 Proof of Theorem 1.3

We begin with the following lemma, interesting by itself.

**Lemma 5.1.**  *$\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is a finite  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module and a Noetherian ring. Therefore, the ring  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is integral over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ .*

**Proof.** We know by Proposition 4.1 that  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a Noetherian ring, and  $\mathcal{L}_{0,n}^\epsilon$  is a finite Noetherian  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module. Therefore, the submodule  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is finitely generated. Being finite over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ , it is necessarily a Noetherian ring (e.g., by [7, Proposition 7.2]).

Let  $x \in \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ . The  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -submodule  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)[x]$  of  $\mathcal{L}_{0,n}^\epsilon$  is finitely generated by the same argument. Using the fact that an element  $x$  is integral over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  if and only if  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)[x]$  is a finitely generated  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module (e.g., by [7, Proposition 5.1]), this proves the last claim. ■

We will use the following notations. Let  $A$  be a ring with no nontrivial zero divisors. The center  $Z = Z(A)$  is a commutative integral domain. We denote by  $Q(Z)$  its field of fractions, and put

$$Q(A) := Q(Z) \otimes_Z A.$$

It is an algebra over its center  $Q(Z)$ . Since  $\mathcal{L}_{0,n}^\epsilon$  has no nontrivial zero divisors [18, Proposition 6.30], we can take  $A = \mathcal{L}_{0,n}^\epsilon$ , or  $A = (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ .

By the lemma,  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is finite over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ , so the ring  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon) \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)} Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$  is a field. Necessarily it coincides with  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , and therefore

$$Q(\mathcal{L}_{0,n}^\epsilon) = Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} \mathcal{L}_{0,n}^\epsilon = Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)} \mathcal{L}_{0,n}^\epsilon. \quad (5.1)$$

Recall that we denote by  $N$  the number of positive roots of  $\mathfrak{g}$ .

**Theorem 5.2.**  *$Q(\mathcal{L}_{0,n}^\epsilon)$  is a division algebra and a central simple algebra of PI degree  $l^{Nn}$ .*

**Proof.** It follows from (5.1) and Theorem 4.9 that  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a vector space of dimension  $l^n \cdot \dim \mathfrak{g}$  over  $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ , and therefore has finite dimension over its center  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ . Because  $\mathcal{L}_{0,n}^\epsilon$  has no nontrivial divisors [18, Proposition 6.30] and  $Q(\mathcal{L}_{0,n}^\epsilon)$  is finite-dimensional over  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ ,  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a division algebra, whence a central simple algebra. By classical theory (see, e.g., [88, Section 13.3.5], or [96, Corollary 2.3.25]), there is a finite extension  $\mathbb{F}$  of  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , a *splitting field*, such that

$$\mathbb{F} \otimes_{Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} Q(\mathcal{L}_{0,n}^\epsilon) = M_d(\mathbb{F}),$$

where  $d \in \mathbb{N}$ , the PI degree of  $Q(\mathcal{L}_{0,n}^\epsilon)$ , satisfies

$$d^2 = [Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))] = \frac{[Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))]}{[Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))]} \quad (5.2)$$

We have to show  $d^2 = l^{2nN}$ . We will obtain this equality by proving firstly that  $d^2 \geq l^{2nN}$ , and then  $d^2 \leq l^{2nN}$ .

In order to show that  $d^2 \geq l^{2nN}$ , it is enough to exhibit an irreducible representation  $V$  of  $\mathcal{L}_{0,n}^\epsilon$  of dimension  $k := l^{nN}$ . Indeed, the representation map  $\rho_V : \mathcal{L}_{0,n}^\epsilon \rightarrow \text{End}_{\mathbb{C}}(V)$  being surjective, given basis elements  $v_1, \dots, v_{k^2} \in \text{End}(V)$ , and elements  $\alpha_1, \dots, \alpha_{k^2} \in \mathcal{L}_{0,n}^\epsilon$  such that  $\rho(\alpha_i) = v_i$  for every  $i \in \{1, \dots, k^2\}$ , necessarily  $\alpha_1, \dots, \alpha_{k^2}$  form a free family of  $Q(\mathcal{L}_{0,n}^\epsilon)$ . For, if there was a nontrivial relation  $\sum_i z_i \alpha_i = 0$ , with  $z_i \in Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ , by clearing denominators and then applying the representation map  $\rho_V$ , we would get a nontrivial relation in  $\text{End}_{\mathbb{C}}(V)$  between  $v_1, \dots, v_{k^2}$ .

Now, by Theorem 2.27 (1) (see [42, Section 20]), the dimension of a generic irreducible representation space of  $U_\epsilon$  is  $l^N$ . Because  $U_\epsilon = T_{2-}^{-1} U_\epsilon^{\text{lf}} [T/T_2]$  by Proposition 4.2, an irreducible representation of  $U_\epsilon$  yields an irreducible representation of  $U_\epsilon^{\text{lf}}$ . Moreover, the tensor product of  $n$  irreducible representation spaces of  $U_\epsilon^{\text{lf}}$  of dimension  $l^N$  is an irreducible representation space of  $(U_\epsilon^{\text{lf}})^{\otimes n}$  of dimension  $l^{nN}$  (see, e.g., [51, Theorem 3.10.2]). Applying the linear isomorphism  $\psi_n = \Phi_n \circ (\Phi_1^{-1})^{\otimes n}$  in (2.21) thus provides an irreducible representation of  $\mathcal{L}_{0,n}^\epsilon$  of dimension  $l^{nN}$ .

It remains to show  $d^2 \leq l^{2nN}$ , which by  $[Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = l^{n(2N+m)}$  is equivalent to  $[Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] \geq l^{mn}$ . For this, it is enough to exhibit an extension of  $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$  contained in  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  and of degree  $l^{mn}$ . There is a very natural one, which we denote by  $Q(\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon))$  and is constructed as follows. Consider for every  $\lambda \in P_+$  the matrices

$$M_\lambda := ({}_{AV_\lambda} \phi_{e_k}^{e_l})_{k,l} \in \text{End}({}_A V_\lambda) \otimes \mathcal{L}_{0,n}^A, \quad M_\lambda^{(i)} := (({}_{AV_\lambda} \phi_{e_k}^{e_l})^{(i)})_{k,l} \in \text{End}({}_A V_\lambda) \otimes \mathcal{L}_{0,n}^A,$$

where  $i = 1, \dots, n$ , and as usual  ${}_{AV_\lambda} \phi_{e_k}^{e_l}$  is a matrix coefficient of  ${}_A V_\lambda$ ,  $\{e_k\}$  the canonical basis of  ${}_A V_\lambda$ , and  $({}_{V_\lambda} \phi_{e_k}^{e_l})^{(i)} := 1^{\otimes(i-1)} \otimes {}_{V_\lambda} \phi_{e_k}^{e_l} \otimes 1^{\otimes(n-i)}$ . Set

$$\lambda\omega := \text{Tr}(\pi_{V_\lambda}(\ell)M_\lambda), \quad \lambda\omega^{(i)} := \text{Tr}(\pi_{V_\lambda}(\ell)M_\lambda^{(i)}),$$

where  $\text{Tr}$  is the standard trace on  $\text{End}(V_\lambda)$ . Clearly,  $\lambda\omega \in \mathcal{L}_{0,1}^A$ ,  $\lambda\omega^{(i)} \in \mathcal{L}_{0,n}^A$ . By [18, Propositions 4.8 and 6.24], the family of elements  $\prod_{i=1}^n \lambda_i \omega^{(i)}$ , where  $\lambda_1, \dots, \lambda_n \in P_+$ , is a basis of  $\mathcal{Z}(\mathcal{L}_{0,n})$ ; moreover the Alekseev map  $\Phi_n$  affords an isomorphism from  $\mathcal{Z}(\mathcal{L}_{0,n})$  to  $\mathcal{Z}(U_q)^{\otimes n}$ , and  $\Phi_n(\lambda\omega^{(i)}) = (\Phi_1(\lambda\omega))^{(i)}$ . For  $n = 1$ , specializing  $q$  to  $\epsilon$  it follows

$$\mathcal{Z}_1(U_\epsilon) = \text{Vect}\{\Phi_1(\lambda\omega), \lambda \in P_+\}, \quad (5.3)$$

where  $\mathcal{Z}_1(U_\epsilon)$  is defined before Theorem 2.27. Then, for every  $i = 1, \dots, n$  define

$$\mathcal{Z}_{0,(i)}(\mathcal{L}_{0,n}^\epsilon) := \mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) \left[ \{ \lambda\omega^{(i)}, \lambda \in P_+ \} \right]$$

and let  $\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon) \subset \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  be the algebra generated by  $\mathcal{Z}_{0,(1)}(\mathcal{L}_{0,n}^\epsilon), \dots, \mathcal{Z}_{0,(n)}(\mathcal{L}_{0,n}^\epsilon)$ . The fields  $Q(\mathcal{Z}_{0,(i)}(\mathcal{L}_{0,n}^\epsilon))$  are  $n$  linearly disjoint extensions of  $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ , so

$$[Q(\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = \prod_{i=1}^n [Q(\mathcal{Z}_{0,(i)}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))].$$

Now, by Proposition 4.2, we know that  $\Phi_1$  affords isomorphisms  $Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)) \cong Q(\mathcal{Z}_0(U_\epsilon^{\text{lf}}))$  and  $Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)) \cong Q(\mathcal{Z}(U_\epsilon^{\text{lf}}))$ , and moreover

$$Q(\mathcal{Z}_0(U_\epsilon)) = Q(\mathcal{Z}_0(U_\epsilon^{\text{lf}}))(T^{(l)}/T_2^{(l)}), \quad Q(\mathcal{Z}(U_\epsilon)) = Q(\mathcal{Z}(U_\epsilon^{\text{lf}}))(T^{(l)}/T_2^{(l)}). \quad (5.4)$$

Computing via the field embedding  $\Phi_1^{\otimes n} : Q(\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon)) \rightarrow Q(\mathcal{Z}(U_\epsilon^{\otimes n}))$ , we deduce

$$\begin{aligned} & [Q(\mathcal{Z}_{0,(i)}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] \\ &= [\Phi_1^{\otimes n}(Q(\mathcal{Z}_{0,(i)}(\mathcal{L}_{0,n}^\epsilon))) : \Phi_1^{\otimes n}(Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)))] \\ &= [Q(\mathcal{Z}_0(U_\epsilon^{\text{lf}})^{\otimes n})[\{(\Phi_1(\lambda\omega))^{(i)}, \lambda \in P_+, i = 1, \dots, n\}] : Q(\mathcal{Z}_0(U_\epsilon^{\text{lf}})^{\otimes n})] \\ &= [Q(\mathcal{Z}_0(U_\epsilon)^{\otimes n})[\{(\Phi_1(\lambda\omega))^{(i)}, \lambda \in P_+, i = 1, \dots, n\}] : Q(\mathcal{Z}_0(U_\epsilon)^{\otimes n})] = l^m. \end{aligned}$$

The second and third equalities follow from (5.4) and the properties of  $\Phi_1$  recalled before it, and the last equality follows from Theorem 2.29 (2) and (5.3). As a result, we have

$$[Q(\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = l^{mn},$$

whence

$$[Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] \geq l^{mn}.$$

Since  $[Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = l^{n(m+2N)}$ , by (5.2) we obtain  $d^2 \leq l^{2nN}$ , which concludes the proof.  $\blacksquare$

**Remark 5.3.** It follows  $[Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = l^{mn}$  by the degree computation above, whence  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) = Q(\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon))$ . In [17], we prove that  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon) = \hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon)$ .

**Theorem 5.4.**  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ ,  $n \geq 2$ , is a division algebra and a central simple algebra of PI degree  $l^{N(n-1)-m}$ .

**Proof.** The center of  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  contains  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ , so the finite-dimensionality of  $Q(\mathcal{L}_{0,n}^\epsilon)$  over  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  implies the finite-dimensionality of  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  over its center. Since it has no non-zero divisors, this proves  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is a division algebra.

Now denote by  $\Delta^{(n)} : \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon^{\otimes n}$ ,  $n \geq 2$ , the  $n$ -fold coproduct, i.e.,  $\Delta^{(2)} := \Delta$ , the standard coproduct of  $\mathcal{O}_\epsilon$ , and  $\Delta^{(n)} := (\text{id} \otimes \Delta) \circ \Delta^{(n-1)}$  for  $n \geq 3$ . Identifying  $\mathcal{L}_{0,n}^\epsilon$  with  $\mathcal{O}_\epsilon^{\otimes n}$  as a vector space, we consider  $\Delta^{(n)}$  as a map  $\Delta^{(n)} : \mathcal{L}_{0,1}^\epsilon \rightarrow \mathcal{L}_{0,n}^\epsilon$ . It is an algebra morphism [18, Proposition 6.18], injective because  $(\varepsilon^{\otimes(n-1)} \otimes \text{id})\Delta^{(n)} = \text{id}$ . Then it extends uniquely to the fraction algebra  $Q(\mathcal{L}_{0,1}^\epsilon)$ . As noted above,  $Q(\mathcal{L}_{0,1}^\epsilon) = Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)) \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)} \mathcal{L}_{0,1}^\epsilon$ . Since  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$  is a Hopf subalgebra of  $\mathcal{O}_\epsilon$  [41, Proposition 6.4],  $\Delta^{(n)}$  maps  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  to  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$ . Then, extending the scalars of  $\Delta^{(n)}(Q(\mathcal{L}_{0,1}^\epsilon))$  by the field  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , consider the algebra

$$\begin{aligned} Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) &:= Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))} \Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon) \\ &= Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)))} \Delta^{(n)}(Q(\mathcal{L}_{0,1}^\epsilon)) \end{aligned}$$

$$\begin{aligned}
&= Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)))} \Delta^{(n)}(Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))) \\
&\quad \otimes_{\Delta^{(n)}(Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)))} \Delta^{(n)}(Q(\mathcal{L}_{0,1}^\epsilon)).
\end{aligned}$$

By Proposition 5.2,  $\Delta^{(n)}(Q(\mathcal{L}_{0,1}^\epsilon))$  is a  $\Delta^{(n)}(Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)))$ -central simple algebra. The left factor is a field, so  $Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$  is a central simple algebra over it (see, e.g., [96, Theorem 1.7.27], or [101, Lemma 4.9]). Note that the left factor can also be written as

$$\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) := Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))} \Delta^{(n)}(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))$$

for it contains  $\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , it is contained in its fraction field, and  $\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  is a field because  $\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)$  is finite over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  and has no nontrivial zero divisors. Note that

$$[\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))] = l^m.$$

We proved in [18, Proposition 6.19] that the ring  $(\mathcal{L}_{0,n}^A)^{U_A}$  is the centralizer of  $\Delta^{(n)}(\mathcal{L}_{0,1}^A)$  in  $\mathcal{L}_{0,n}^A$ ; the same arguments show that  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is the centralizer of  $\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)$  in  $\mathcal{L}_{0,n}^\epsilon$ . So the algebra

$$Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) := Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$$

is the centralizer of  $Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$  in  $Q(\mathcal{L}_{0,n}^\epsilon)$ . Since the latter is simple, we can apply the double centralizer theorem (see, e.g., [96, Theorem 7.1.9], or [101, Theorem 7.1]):  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is a simple algebra, we have

$$[Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))] = \frac{[Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]}{[Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]} = l^{2nN-(2N+m)},$$

and the centralizer of  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is  $Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$ . In particular,  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  has center  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) \cap Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$ , which is easily shown to be  $\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ . It then follows

$$\begin{aligned}
[Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) : \tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))] &= \frac{[Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]}{[\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]} \\
&= l^{2nN-(2N+m)} \cdot l^{-m} = l^{2(N(n-1)-m)}.
\end{aligned}$$

Therefore,  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is a central simple algebra of PI degree  $l^{N(n-1)-m}$ . ■

## A Low and up crystal structures in the $\mathfrak{sl}_2$ case

Let  $k \in \mathbb{N}$ , and denote by  $V_k$  the simple  $U_q^{\text{ad}}(\mathfrak{sl}_2)$  module of dimension  $k+1$ . It has a basis  $v_0, \dots, v_k$  such that

$$\begin{aligned}
K.v_j &= q^{k-2j}v_j, & F.v_j &= [j+1]_q v_{j+1} \quad \text{if } j < k, & F.v_k &= 0, \\
E.v_j &= [k-j+1]_q v_{j-1} \quad \text{if } j > 0, & E.v_0 &= 0.
\end{aligned}$$

This basis defines the full  $A$ -sublattice  ${}_A V_k$ , which is left invariant by  $U_A^{\text{res}}$ , and we have

$$F^{(a)}.v_j = \begin{bmatrix} j+a \\ a \end{bmatrix}_q v_{j+a}, \quad E^{(a)}.v_j = \begin{bmatrix} k-j+a \\ a \end{bmatrix}_q v_{j-a}, \quad a \geq 0.$$

The action of the Kashiwara operator  $\tilde{e}, \tilde{f}$  on  $V_k$  are given by  $\tilde{f}(v_j) = v_{j+1}$ ,  $\tilde{e}(v_j) = v_{j-1}$ .

The crystal basis  $(\mathcal{L}^{\text{low}}, \mathcal{B}^{\text{low}})$  at  $q = 0$  is formed by the  $\mathcal{A}_0$ -sublattice  $\mathcal{L}^{\text{low}}$  generated by  $v_0, \dots, v_k$ , and  $\mathcal{B}^{\text{low}}$  by the images  $\bar{v}_0, \dots, \bar{v}_k$  of these vectors in  $\mathcal{L}^{\text{low}}/q\mathcal{L}^{\text{low}}$ .

The bilinear form  $\langle \cdot \rangle_k$  defined by (2.39) is easily computed

$$\langle v_i, v_j \rangle_k = \langle F^{(i)}.v_0, F^{(j)}.v_0 \rangle_k = \langle v_0, E^{(i)}F^{(j)}.v_0 \rangle_k = \begin{bmatrix} k \\ i \end{bmatrix}_q \delta_{i,j}.$$

By definition,

$${}_A V_k^{\text{up}} = \{v \in V_k, \langle v, {}_A V_k \rangle_k \subset A\} = \bigoplus_{j=0}^k A v_j^{\text{up}},$$

where

$$v_j^{\text{up}} = \begin{bmatrix} k \\ j \end{bmatrix}_q^{-1} v_j.$$

The upper crystal basis  $(\mathcal{L}^{\text{up}}, \mathcal{B}^{\text{up}})$  at  $q = 0$  is formed by the  $\mathcal{A}_0$ -sublattice  $\mathcal{L}^{\text{up}}$  generated by  $v_0^{\text{up}}, \dots, v_k^{\text{up}}$ , and  $\mathcal{B}^{\text{up}}$  by the images  $\bar{v}_0^{\text{up}}, \dots, \bar{v}_k^{\text{up}}$  of these vectors in  $\mathcal{L}^{\text{up}}/q\mathcal{L}^{\text{up}}$ .

Using that  $[n]_q \in q^{1-n}(1 + q\mathcal{A}_0)$ , we obtain

$$\begin{bmatrix} k \\ j \end{bmatrix}_q \in q^{j^2 - kj}(1 + q\mathcal{A}_0).$$

As a result, we get  $\bar{v}_j^{\text{up}} = q^{kj - j^2} \bar{v}_j$ , which is exactly the relation (2.41) relating the low and up crystal bases, with  $\lambda = k\varpi_1$ ,  $\mu = (k - 2j)\varpi_1$ .

## B Quantum Weyl group

We recall some of the formulas of [31]. Let  $e_q(z)$  be the formal power series in  $z$  with coefficients in  $\mathbb{C}(q)$  defined by

$$e_q(z) = \sum_{n=0}^{+\infty} \frac{z^n}{(n)_{q^2}!}.$$

We first consider the case of  $\mathfrak{g} = \mathfrak{sl}_2$ . As explained in [18, Section 3], the Cartan element  $H \in \mathfrak{g}$  defines an element of  $\mathbb{U}_q(\mathfrak{sl}_2)$ . Viewed as elements of  $\mathbb{U}_q(\mathfrak{sl}_2)$  we have  $L = q^{H/2}$ . The series  $\Theta = q^{H \otimes H/2}$  defines an element of  $\mathbb{U}_q(\mathfrak{sl}_2)^{\hat{\otimes} 2}$ , its image under multiplication being  $q^{H^2/2}$ . The  $R$ -matrix can be expressed as  $R = \Theta \hat{R}$  where  $\hat{R} = e_{q^{-1}}((q - q^{-1})E \otimes F)$  is a well defined element of  $\mathbb{U}_q(\mathfrak{sl}_2)^{\hat{\otimes} 2}$ . Consider the Lusztig [82] braid group automorphism of  $U_q(\mathfrak{sl}_2)$ , defined by

$$T(L) = L^{-1}, \quad T(E) = -FK^{-1}, \quad T(F) = -KE. \quad (\text{B.1})$$

For every  $x \in U_q(\mathfrak{sl}_2)$  it satisfies:  $\Delta(T(x)) = \hat{R}^{-1}(T \otimes T)(\Delta(x))\hat{R}$ . Define the quantum Weyl group element  $\hat{w} \in \mathbb{U}_q(\mathfrak{sl}_2)$  by Saito's formula [97]:

$$\hat{w} = e_{q^{-1}}(F)q^{-H^2/4}e_{q^{-1}}(-E)q^{-H^2/4}e_{q^{-1}}(F)q^{-H/2}. \quad (\text{B.2})$$

For every  $x \in U_q(\mathfrak{sl}_2)$ , it satisfies

$$T(x) = \hat{w}x\hat{w}^{-1}, \quad \Delta(\hat{w}) = \hat{R}^{-1}(\hat{w} \otimes \hat{w}), \quad (\text{B.3})$$



$$\hat{w}^2 = q^{H^2/2} \xi \theta, \quad (\text{B.4})$$

where  $\theta \in \mathbb{U}_q(\mathfrak{sl}_2)$  is the ribbon element, and  $\xi \in \mathbb{U}_q(\mathfrak{sl}_2)$  is the central group element whose value on the type 1 simple module  $V_k$  of  $U_q^{\text{ad}}(\mathfrak{sl}_2)$  of dimension  $k+1$  is the scalar endomorphism  $(-1)^k \text{id}_{V_k}$ .

In order to compare our setting to the one of [41], we need an explicit formula of  $\hat{w}$ . Using the basis  $v_j$  of  $V_k$  of Appendix A, (B.1), (B.3) and (B.4), we obtain

$$\hat{w}v_j = (-1)^j q^{-j(k-j-1)-k} v_{k-j}. \quad (\text{B.5})$$

In [41], another quantum Weyl group element  $\underline{w}$  is defined. It is dual to the Vaksman–Soibelman functional  $t: \mathcal{O}_q(\text{SL}_2) \rightarrow \mathbb{C}(q)$  of [98, 102], that is,  $t(\alpha) = \langle \alpha, \underline{w} \rangle$  for all  $\alpha \in \mathcal{O}_q(\text{SL}_2)$ . By comparing (B.5) with the formulas defining the action of  $t$  in [41, Section 1.7], we find  $\underline{w} = \xi \hat{w} K$  and the basis vectors  $w_r^p$  of [41], where  $p \in (1/2)\mathbb{N}$  and  $r \in \{-p, -p+1, \dots, p-1, p\}$ , are related to the vectors  $v_j$  above as follows:  $v_j = \lambda_j w_r^p$ , where  $k = 2p$ ,  $j = p - r$ ,  $\lambda_0 = 1$ ,  $\lambda_1 = [k]q^{-k}$ , and

$$\lambda_j = \frac{[k]!}{[j]![k-(j-2)]!} q^{j(j+1)-j(k+2)}, \quad j \geq 2.$$

Explicit formulas of the evaluation of  $t$  on basis vectors of  $\mathcal{O}_q(\text{SL}_2)$  can be computed. We get

$$t(\tilde{a}^{\star m} \star \tilde{b}^{\star n} \star \tilde{d}^{\star p}) = \delta_{m,p} q^{-np} \prod_{i=1}^p (1 - q^{-2i}), \quad (\text{B.6})$$

$$t(\tilde{a}^{\star m} \star \tilde{c}^{\star n} \star \tilde{d}^{\star p}) = (-1)^n \delta_{m,p} q^{-n(p+1)} \prod_{i=1}^p (1 - q^{-2i}), \quad (\text{B.7})$$

where  $\tilde{a} = a$ ,  $\tilde{b} = qb$ ,  $\tilde{c} = q^{-1}c$ ,  $\tilde{d} = d$  and as usual  $a, b, c, d$  are the standard generators of  $\mathcal{O}_q(\text{SL}_2)$ , i.e., the matrix coefficients in the basis of weight vectors  $v_0, v_1$  of the 2-dimensional irreducible representation  $V_1$  of  $U_q(\mathfrak{sl}_2)$  such that  $K.v_0 = qv_0$  and  $v_1 = F.v_0$ . Here we have introduced the generators  $\tilde{a}, \dots, \tilde{d}$  to facilitate the comparison with the formulas in [41]; these generators come naturally in their setup because they use different generators  $E_i$  and  $F_i$  of  $U_q(\mathfrak{g})$ , which in our notations can be written respectively as  $K_i^{-1}E_i$  and  $F_iK_i$ .

The formulas (B.6)–(B.7) can be shown by two independent methods. The first uses a definition of  $t$  as a *GNS* state associated to an infinite-dimensional representation of  $\mathcal{O}_q(\text{SL}_2)$ , as recalled in [41, Section 1.6]. The second is to write, e.g.,

$$t(\tilde{a}^{\star m} \star \tilde{b}^{\star n} \star \tilde{d}^{\star p}) = \langle \tilde{a}^{\otimes m} \otimes \tilde{b}^{\otimes n} \otimes \tilde{d}^{\otimes p}, \Delta^{(m+n+p)}(\underline{w}) \rangle \quad (\text{B.8})$$

and to use explicit expressions of  $\Delta^{(m+n+p)}(\underline{w})$  when represented on  $V_1^{\otimes(m+n+p)}$ . In general, one can check that

$$\begin{aligned} \Delta^{(n)}(\hat{\omega}) &= (\Delta^{(n-1)} \otimes \text{id})(\hat{R}^{-1})((\Delta^{(n-2)} \otimes \text{id})(\hat{R}^{-1}) \otimes \text{id}) \cdots ((\Delta \otimes \text{id})(\hat{R}^{-1}) \otimes \text{id}^{\otimes(n-3)}) \\ &\quad \times (\hat{R}^{-1} \otimes \text{id}^{\otimes(n-2)}) \hat{\omega}^{\otimes n}. \end{aligned}$$

By (B.5) or (B.6)–(B.7), we see that  $\hat{w}$  (or  $\underline{w}$ ) and  $t$  are well defined on the integral forms,

$$\hat{w} \in \mathbb{U}_\Gamma, \quad t: \mathcal{O}_A(\text{SL}_2) \rightarrow A.$$

We now consider the case where  $\mathfrak{g}$  is of rank  $m \geq 2$ . To each simple root  $\alpha_i$ ,  $1 \leq i \leq m$ , is associated the subalgebra of  $U_q$  generated by  $E_i, F_i, L_i, L_i^{-1}$ . It is a copy of  $U_{q_i}(\mathfrak{sl}_2)$ , where  $q_i = q^{d_i}$ . Let  $\hat{w}_i$  be the corresponding quantum Weyl group element in  $\mathbb{U}_q = \mathbb{U}_q(\mathfrak{g})$ , defined by Saito's

formula (B.2), replacing  $H, E, F$  by  $H_i, E_i$  and  $F_i$ . Also, denote by  $\nu_i: \mathcal{O}_q \rightarrow \mathcal{O}_{q_i}(\mathrm{SL}_2)$  the projection map dual to the inclusion  $U_{q_i}(\mathfrak{sl}_2) \otimes_{\mathbb{C}(q_i)} \mathbb{C}(q) \hookrightarrow U_q$ , and put  $t_i = t \circ \nu_i$ . Let  $\underline{w}_i$  be the corresponding quantum Weyl group element in  $\mathbb{U}_q$ , i.e.,  $t_i(\alpha) = \langle \alpha, \underline{w}_i \rangle$  for all  $\alpha \in \mathcal{O}_q$ . On integral forms they yield well-defined elements  $\hat{w}_i, \underline{w}_i \in \mathbb{U}_\Gamma$  and  $t_i: \mathcal{O}_A \rightarrow A$  (see [41, Proposition 5.1], and [84] for a different construction). They satisfy the defining relations of the braid group  $\mathcal{B}(\mathfrak{g})$  of  $\mathfrak{g}$  [70]:

$$\begin{aligned} \hat{w}_i \hat{w}_j \hat{w}_i &= \hat{w}_j \hat{w}_i \hat{w}_j & \text{if } a_{ij} a_{ji} &= 1, \\ (\hat{w}_i \hat{w}_j)^k &= (\hat{w}_j \hat{w}_i)^k & \text{for } k = 1, 2, 3 & \text{ if } a_{ij} a_{ji} = 0, 2, 3, \end{aligned}$$

and similarly by replacing  $\hat{w}_i$  with  $\underline{w}_i$ , or with  $t_i$  (see [98] for the latter). The Weyl group  $W = W(\mathfrak{g}) = N(T_G)/T_G$  is generated by the reflections  $s_i$  associated to the simple roots  $\alpha_i$ . Denote by  $n_i \in N(T_G)$  a representative of  $s_i$ . Let  $w \in W$  and denote by  $w = s_{i_1} \dots s_{i_k}$  a reduced expression. Because of the braid group relations the elements  $\hat{w} = \hat{w}_{i_1} \dots \hat{w}_{i_k}$ ,  $\underline{w} = \underline{w}_{i_1} \dots \underline{w}_{i_k}$  and the functional  $t_w = t_{i_1} \dots t_{i_k}$  do not depend on the choice of reduced expression. The Lusztig [82] braid group automorphism  $T_w: \Gamma \rightarrow \Gamma$  associated to  $w$  satisfies (see [41])

$$T_w(x) = \hat{w}x\hat{w}^{-1}, \quad x \in \Gamma.$$

Let  $w_0$  be the longest element in  $W$ . We have

$$\Delta(\hat{w}_0) = \hat{R}^{-1}(\hat{w}_0 \otimes \hat{w}_0), \tag{B.9}$$

where as usual  $R = \Theta \hat{R}$ .

## C Regular action on $\mathcal{O}_\epsilon$

The following result is proved in [41, Section 1.10]. For completeness, let us give a (different) proof. Recall from (2.72) that we may identify  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  with  $\mathcal{O}(G)$ .

**Proposition C.1.** *For every  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ ,  $g \in \mathcal{O}_\epsilon$ , we have*

$$t_i(f) = f(n_i), \tag{C.1}$$

$$t_i(f \star g) = t_i(f)t_i(g). \tag{C.2}$$

**Proof.** It is sufficient to prove the results for  $\mathrm{SL}_2$  because  $\nu_i: \mathcal{O}_\epsilon \rightarrow \mathcal{O}_{\epsilon_i}(\mathrm{SL}_2)$  is a morphism of Hopf algebras and  $\nu_i(\mathcal{Z}_0(\mathcal{O}_\epsilon)) \subset \mathcal{Z}_0(\mathcal{O}_{\epsilon_i}(\mathrm{SL}_2))$ . In this case, (C.1) can be proved by using (B.6)–(B.7), evaluating  $t$  on basis elements of  $\mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2))$  as is done in [41, Lemma 1.5 (a)]. Such a basis is formed by monomials like in (B.6)–(B.7), with all exponents divisible by  $l$ ; then for instance

$$t(\tilde{a}^{\star ml} \star \tilde{b}^{\star nl} \star \tilde{d}^{\star pl}) = \delta_{p,0} \delta_{m,0} = \underline{a}^m \underline{b}^n \underline{d}^p(n),$$

where  $\underline{a}, \dots, \underline{d}$  are the generators of  $\mathcal{O}(G) = \mathcal{O}_1(G)$  corresponding to  $a, \dots, d$ , and we take

$$n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

as representative of the reflection  $s$  generating the Weyl group  $W(\mathfrak{sl}_2)$ . Here is an alternative proof of (C.1): (C.2) shows that  $t$  is a homomorphism on  $\mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2))$ , so by proving (C.2) at first one is reduced to check (C.1) on the generators  $a^{\star l}, \dots, d^{\star l}$ , which is easy by means of (B.8) and (B.9).

We provide a proof of (C.2) that we find more conceptual than the one in [41, Lemma 1.5 (b)] (which uses again (B.6)–(B.7)). As above, let us denote  $\underline{w} = \xi \hat{w} K$ . For any  $f, g \in \mathcal{O}_\epsilon$ , we have

$$\begin{aligned} t(f \star g) &= (f \otimes g)(\Delta(\underline{w})) = (f \otimes g)(\widehat{R}^{-1}(\underline{w} \otimes \underline{w})) = \sum_{(\widehat{R}^{-1})} f((\widehat{R}^{-1})_{(1)} \underline{w}) g((\widehat{R}^{-1})_{(2)} \underline{w}) \\ &= \sum_{(\widehat{R}^{-1}), (f)} f_{(1)}((\widehat{R}^{-1})_{(1)}) f_{(2)}(\underline{w}) g((\widehat{R}^{-1})_{(2)} \underline{w}) = \sum_{(f)} f_{(2)}(\underline{w}) g((f_{(1)} \otimes \text{id})(\widehat{R}^{-1}) \underline{w}). \end{aligned}$$

Assume now  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon(\text{SL}_2))$ . Since  $\mathcal{Z}_0(\mathcal{O}_\epsilon(\text{SL}_2))$  is a Hopf subalgebra of  $\mathcal{O}_\epsilon(\text{SL}_2)$ , we have  $f_{(1)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon(\text{SL}_2))$ . From Theorem 2.29 (2), we deduce

$$(f_{(1)} \otimes \text{id})(\widehat{R}^{-1}) \in U_\epsilon(\mathfrak{n}_-) \cap \mathcal{Z}_0(U_\epsilon^{\text{ad}}).$$

Denote by  $z$  this element. Note that from its expression we have  $\epsilon(z) = \epsilon(f_{(1)})$ . Now  $g(z\underline{w}) = \sum_{(g)} g_{(1)}(z) g_{(2)}(\underline{w})$ , but  $g_{(1)}$  is a linear combination of matrix elements of  $\Gamma$ -modules, on which  $\mathcal{Z}_0(U_\epsilon^{\text{ad}})$  acts by the trivial character. Therefore,

$$g(z\underline{w}) = \sum_{(g)} \epsilon(z) g_{(1)}(1) g_{(2)}(\underline{w}) = \epsilon(z) g(\underline{w}) = \epsilon(f_{(1)}) g(\underline{w}),$$

and eventually

$$t(f \star g) = \sum_{(f)} f_{(2)}(\underline{w}) \epsilon(f_{(1)}) g(\underline{w}) = t(f) t(g).$$

This concludes the proof. ■

For the sake of completeness, let us show how this result implies:

**Proof of Proposition 2.30 (i.e., [41, Proposition 7.1]).** We have  $f \triangleleft t_i = \sum_{(f)} t_i(f_{(1)}) f_{(2)}$ ,  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ . Since  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  is a Hopf subalgebra of  $\mathcal{O}_\epsilon$ ,  $f_{(2)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$  and therefore the maps  $\triangleleft t_i: \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon$  preserve  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ . Moreover,  $(f \triangleleft t_i)(a) = \sum_{(f)} f_{(1)}(n_i) f_{(2)}(a) = f(n_i a)$ ,  $a \in G$ , by (C.1).

It remains to show that  $(f \star \alpha) \triangleleft t_i = (f \triangleleft t_i)(\alpha \triangleleft t_i)$  for every  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ ,  $\alpha \in \mathcal{O}_\epsilon$ . We have

$$\begin{aligned} (f \star g) \triangleleft t_i &= \sum_{(f \star g)} t_i((f \star g)_{(1)}) (f \star g)_{(2)} = \sum_{(f), (g)} t_i(f_{(1)} \star g_{(1)}) f_{(2)} \star g_{(2)} \\ &= \sum_{(f), (g)} t(\nu_i(f_{(1)}) \nu_i(g_{(1)})) f_{(2)} \star g_{(2)} \\ &= \sum_{(f), (g)} t(\nu_i(f_{(1)})) t(\nu_i(g_{(1)})) f_{(2)} \star g_{(2)}, \end{aligned} \tag{C.3}$$

using that  $\nu_i$  is a homomorphism in the third equality, and (C.2) in the last one. The result is just  $(f \triangleleft t_i)(g \triangleleft t_i)$ . ■

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