# Unrestricted quantum moduli algebras, II: noetherianity and simple fraction rings at roots of 1

Stéphane Baseilhac, Philippe Roche

# UNRESTRICTED QUANTUM MODULI ALGEBRAS, II: NOETHERIANITY AND SIMPLE FRACTION RINGS AT ROOTS OF 1

STÉPHANE BASEILHAC, PHILIPPE ROCHE

IMAG, Univ Montpellier, CNRS, Montpellier, France
stephane.baseilhac@umontpellier.fr, philippe.roche@umontpellier.fr

Abstract. We prove that the quantum graph algebra and the quantum moduli algebra associated to a punctured sphere and complex semisimple Lie algebra $\mathfrak{g}$ are Noetherian rings and finitely generated rings over $\mathbb{C}(q)$. Moreover, we show that these two properties still hold on $\mathbb{C}[q,q^{-1}]$ for the integral version of the quantum graph algebra. We also study the specializations $\mathcal{L}^{\epsilon}_{0,n}$ of the quantum graph algebra at a root of unity $\epsilon$ of odd order, and show that $\mathcal{L}^{\epsilon}_{0,n}$ and its invariant algebra under the quantum group $U_\epsilon(\mathfrak{g})$ have classical fraction algebras which are central simple algebras of PI degrees that we compute.

*Keywords: quantum groups, invariant theory, TQFT*
*AMS subject classification 2020: 16R30, 17B37, 20G42, 57R56*

Contents



## 1. Introduction

This paper is the second part of our work, initiated in [28], on the quantum graph algebra $\mathcal{L}_{g,n}(\mathfrak{g})$ and the quantum moduli algebra $\mathcal{M}_{g,n}(\mathfrak{g})$, which are associated to a surface $\Sigma_{g,n+1}$ of genus $g$ with $n+1$ punctures and a complex semisimple Lie algebra $\mathfrak{g}$. As in [28] we focus in this paper on punctured spheres ($g=0$, $n\geq 1$). From now on we fix $\mathfrak{g}$, and when no confusion may arise we omit it from the notations of the various algebras.

The algebras $\mathcal{L}_{g,n}$ and $\mathcal{M}_{g,n}$ are defined over the field $\mathbb{C}(q)$. They were introduced in the mid '90s by Alekseev-Grosse-Schomerus [2, 3] and Buffenoir-Roche [29, 30] by a method called *combinatorial quantization*. By this method, the pair formed by $\mathcal{L}_{g,n}$ and $\mathcal{M}_{g,n}$ appear naturally as a $q$-deformation of the Fock-Rosly [54] lattice model of the algebra of functions on the "classical" moduli space $\mathcal{M}^{cl}_{g,n}$ of flat $\mathfrak{g}$-connections on the surface $\Sigma_{g,n+1}$.

In [28] we showed that both $\mathcal{L}_{0,n}$ and $\mathcal{M}_{0,n}$ have integral forms $\mathcal{L}^A_{0,n}$ and $\mathcal{M}^A_{0,n}$ defined over the ring $A = \mathbb{C}[q, q^{-1}]$; one can thus consider the specializations of these algebras at $q = \epsilon \in \mathbb{C}^*$, which we denote by $\mathcal{L}^\epsilon_{0,n}$ and $\mathcal{M}^{A,\epsilon}_{0,n}$ respectively. The algebra $\mathcal{L}^A_{0,n}$ is endowed with an action of the Lusztig integral form $U^{res}_A = U^{res}_A(\mathfrak{g})$ of the quantum group $U_q = U_q(\mathfrak{g})$, and $\mathcal{M}^A_{0,n}$ is the subalgebra of invariant elements under this action. Therefore

$$\mathcal{M}^A_{0,n} := (\mathcal{L}^A_{0,n})^{U^{res}_A} \ , \ \mathcal{M}_{0,n} := \mathcal{L}^{U_q}_{0,n} = \mathcal{M}^A_{0,n} \otimes_A \mathbb{C}(q).$$

The definition of $\mathcal{L}^A_{0,n}$ is based on the original combinatorial quantization method, together with twists of module-algebras and Lusztig's theory of canonical basis of quantum groups [73]. This allows us to address the structure and representation theory of $\mathcal{L}^A_{0,n}$ and $\mathcal{M}^A_{0,n}$ by means of quantum groups, following ideas of classical invariant theory. In particular, we obtained that $\mathcal{L}_{0,n}$ and $\mathcal{L}^\epsilon_{0,n}$ have no non trivial zero divisors (and therefore do as well the subalgebras $\mathcal{M}_{0,n}$, $\mathcal{L}^A_{0,n}$, $\mathcal{M}^A_{0,n}$, and $(\mathcal{L}^\epsilon_{0,n})^{U^{res}_\epsilon}$, where $U^{res}_\epsilon$ is the specialization of $U^{res}_A$ at $q = \epsilon$). Also, by extending the quantum coadjoint action of De Concini-Kac-Procesi [39, 40, 41], we described in the $sl(2)$ case an action by derivations of the center $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$ of $\mathcal{L}^\epsilon_{0,n}$ on $\mathcal{L}^\epsilon_{0,n}$, and we defined a subalgebra $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})^{\mathcal{G}} \subset \mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$ which is a finite extension of the ring of regular functions on the character variety of the sphere with $(n+1)$ punctures (see Corollary 7.20 and Theorem 8.8 of [28]). Moreover, from these results we derived an action by derivation of $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})^{\mathcal{G}}$ on $\mathcal{M}^{A,\epsilon}_{0,n}(sl(2))$.

Representations of a quotient (the semisimplification) of $\mathcal{M}^{A,\epsilon}_{g,n}$ were already constructed and classified in [4]; they involve only the irreducible representations of the finite dimensional "small" quantum group $\mathfrak{u}_\epsilon(\mathfrak{g})$. Moreover, [4] deduced from these representations of $\mathcal{M}^{A,\epsilon}_{g,n}$ a family of representations of the mapping class group of surfaces, that is equivalent to the one associated to the Witten-Reshetikin-Turaev TQFT [93, 85]. Recently, representations of another, larger quotient of $\mathcal{M}^{A,\epsilon}_{g,n}$, and the corresponding representations of the mapping class groups of surfaces, were constructed in [50, 51]. These representations are equivalent to those previously obtained by Lyubashenko-Majid [75], and are associated to the TQFT defined in [46, 47]. In the $sl(2)$ case they involve the irreducible and also the principal indecomposable representations of the small quantum group $\mathfrak{u}_\epsilon(sl(2))$. The related link and 3-manifold invariants coincide with those of [77] and [19].

In general, the representation theory of $\mathcal{M}^{A,\epsilon}_{g,n}$ is by now far from being understood. Because $\mathcal{M}^{A,\epsilon}_{g,n}$ deforms the classical moduli space $\mathcal{M}^{cl}_{g,n}$, it is natural to expect that its representation theory provides $(2+1)$-dimensional TQFTs for 3-manifolds endowed with general flat $\mathfrak{g}$-connections, extending the known TQFTs based on quantum groups (where purely topological ones correspond to the trivial connection). A family of such invariants, called *quantum hyperbolic invariants*, has already been defined for $\mathfrak{g} = sl(2)$ by means of certain $6j$-symbols, *Deus ex machina*; they are closely connected to classical Chern-Simons theory, provide generalized Volume Conjectures, and contain quantum Teichmüller theory (see [12]–[18]). It is part of our present program, initiated in [8], to shed light on these invariants and to generalize them to arbitrary $\mathfrak{g}$ by developing the representation theory of $\mathcal{M}^{A,\epsilon}_{g,n}$.

The quantum moduli algebras have also been recognized as central objects from the viewpoints of factorization homology [20], multiplicative quiver varieties [56] and (stated) skein

theory [22, 52, 34, 26]. In another direction, one may expect that the equivalence proved in [76] between combinatorial quantisation for the Drinfeld double $D(H)$ of a finite-dimensional semisimple Hopf algebra $H$, and Kitaev's lattice model in topological quantum computation, can be extended to the setup of quantum moduli algebras.

In the present paper we study $\mathcal{L}_{0,n}$, its integral form $\mathcal{L}^A_{0,n}$, and the specialization $\mathcal{L}^\epsilon_{0,n}$ of $\mathcal{L}^A_{0,n}$ at $q=\epsilon$ a primitive root of unity of odd order. We study also the subalgebras of invariant elements $\mathcal{M}_{0,n}=\mathcal{L}^{U_q}_{0,n}$ and $(\mathcal{L}^\epsilon_{0,n})^{U_\epsilon}$. Here, $U_\epsilon$ is the specialization of $U_A$ at $q=\epsilon$, where $U_A$ is the De Concini-Kac integral form of $U_q$ (see Section 1.1). Our results hold for every complex semisimple Lie algebra $\mathfrak{g}$. The main ones are proofs that $\mathcal{L}_{0,n}$, $\mathcal{L}^A_{0,n}$ and $\mathcal{M}_{0,n}$ are Noetherian and finitely generated rings (Theorem 1.1), and that the classical fraction algebras of $\mathcal{L}^\epsilon_{0,n}$ and $(\mathcal{L}^\epsilon_{0,n})^{U_\epsilon}$ are central simple algebras of PI degrees $l^{nN}$ and $l^{N(n-1)-m}$ respectively (Theorem 1.3). Here, $m$ and $N$ are the rank and the number of positive roots of $\mathfrak{g}$.

In the sequel [26] to this paper, in collaboration with M. Faitg, we extend Theorem 1.1 to the algebras $\mathcal{L}_{g,n}$ and $\mathcal{M}_{g,n}$, associated to arbitrary finite type surfaces (arbitrary genus and number of punctures). Also, we show that $\mathcal{M}_{g,n}$ is isomorphic to the $\mathfrak{g}$-skein algebra of $\Sigma_{g,n+1}$, and $\mathcal{L}_{g,n}$ to the stated skein algebra of the compact surface $\overline{\Sigma}_{g,n+1}$ with one boundary component and one marked point on the boundary component. This was proved for $\mathfrak{g}=sl(2)$ in [52].

By using the analysis developed in the present paper for $\mathcal{L}^A_{0,n}$, one can define the integral form $\mathcal{L}^A_{g,n}$ as well, and show that it is a Noetherian and finitely generated ring. We do not have a proof yet of these properties for the algebra $\mathcal{M}^A_{0,n}$, which seems to be much more difficult to handle. We note that there is a strict inclusion

$$\mathcal{M}^{A,\epsilon}_{0,n}\subset(\mathcal{L}^\epsilon_{0,n})^{U_\epsilon}.$$

This is discussed after Theorem 1.2. In [27] we study further properties of $(\mathcal{L}^\epsilon_{g,n})^{U_\epsilon}$, and we consider also the subalgebra $\mathcal{M}^{A,\epsilon}_{g,n}$.

1.1. **Statement of results.** Let us recall a few notations and facts from [28]. Let $U_q$ be the simply-connected quantum group of $\mathfrak{g}$, defined over the field $\mathbb{C}(q)$. From $U_q$ one can define a $U_q$-module algebra $\mathcal{L}_{0,n}$, called (quantum, daisy) *graph algebra*, where $U_q$ acts by means of a right coadjoint action. The set of invariant elements of $\mathcal{L}_{0,n}$ for this action is an algebra; we denote it

$$\mathcal{M}_{0,n}:=\mathcal{L}^{U_q}_{0,n}$$

and call it *quantum moduli algebra*. As a $\mathbb{C}(q)$-module $\mathcal{L}_{0,n}$ is just $\mathcal{O}_q^{\otimes n}$, where $\mathcal{O}_q=\mathcal{O}_q(G)$ is the standard quantum function algebra of the connected and simply-connected Lie group $G$ with Lie algebra $\mathfrak{g}$. The product of $\mathcal{L}_{0,n}$ is obtained by twisting both the product of each factor $\mathcal{O}_q$ and the product between them. It is equivariant with respect to a (right) coadjoint action of $U_q$, which defines the structure of $U_q$-module of $\mathcal{L}_{0,n}$.

The module algebra $\mathcal{L}_{0,n}$ has an integral form $\mathcal{L}^A_{0,n}$, which is defined over $A=\mathbb{C}[q,q^{-1}]$, and endowed with an (coadjoint) action of the Lusztig [72] integral form $U_A^{res}$ of $U_q$. It is obtained by replacing $\mathcal{O}_q$ in the construction of $\mathcal{L}_{0,n}$ with the restricted dual $\mathcal{O}_A$ of the integral form $U_A^{res}$, or equivalently with the restricted dual of the integral form $\Gamma$ of $U_q$ defined by De Concini-Lyubashenko [43]. Since $U_A^{res}$ contains the De Concini-Kac [39] integral form $U_A$, and $U_A$ has the same set of invariant elements in $\mathcal{L}^A_{0,n}$, we systematically denote the latter

$$\mathcal{M}^A_{0,n}:=(\mathcal{L}^A_{0,n})^{U_A}(=(\mathcal{L}^A_{0,n})^{U_A^{res}}).$$

We call $\mathcal{M}_{0,n}^A$ the *integral* quantum moduli algebra.

A cornerstone of the theory of $\mathcal{M}_{0,n}$ is a map $\Phi_n$ originally due to Alekseev [1], building on works of Drinfeld [37] and Reshetikhin and Semenov-Tian-Shansky [83]. In [28] we showed that $\Phi_n$ eventually provides isomorphisms of module algebras and algebras respectively,

$$\Phi_n \colon \mathcal{L}_{0,n} \to (U_q^{\otimes n})^{lf}, \Phi_n \colon \mathcal{M}_{0,n} \to (U_q^{\otimes n})^{U_q}$$

where $U_q^{\otimes n}$ is endowed with a right adjoint action of $U_q$, and $(U_q^{\otimes n})^{lf}$ is the subalgebra of locally finite elements with respect to this action. When $n = 1$ the algebra $U_q^{lf}$ has been studied in great detail by Joseph-Letzter [59, 60, 58]; we will use simplified proofs of their results, obtained in [92].

All the material we need about the results discussed above is described in [28], and recalled in Section 2.1-2.2.

Our first result, proved in Section 3, is:

**Theorem 1.1.** *$\mathcal{L}_{0,n}$, $\mathcal{M}_{0,n}$ and the integral form $\mathcal{L}_{0,n}^A$ are Noetherian rings, and finitely generated rings.*

It follows immediately from the theorem that the specializations $\mathcal{L}_{0,n}^\epsilon$, $\epsilon \in \mathbb{C} \setminus \{0, 1\}$, are Noetherian and finitely generated rings as well. In [28] we proved that all these algebras (and therefore $\mathcal{M}_{0,n}^A$ and $\mathcal{M}_{0,n}^{A,\epsilon}$) have no non-trivial zero divisors.

After we finished this work we discovered that [44] already proved that $\mathcal{L}_{0,1}(gl(n))$ and $\mathcal{L}_{0,n}(gl(2))$ are Noetherian and finitely generated rings. Our approach here is completely different. For $\mathcal{L}_{0,n}$ we adapt the proof given by Voigt-Yuncken [92] of a result of Joseph [58], which asserts that $U_q^{lf}$ is a Noetherian ring (Theorem 3.1). For $\mathcal{M}_{0,n}$ we deduce the result from the one for $\mathcal{L}_{0,n}$, by following a line of proof of the Hilbert-Nagata theorem in classical invariant theory (Theorem 3.2).

The result for $\mathcal{L}_{0,n}^A$ follows the same line, but uses also in a crucial way the Kashiwara-Lusztig theory of canonical bases of quantum groups. We describe the background material on this theory in Section 2.2.2; we have tried to make the exposition pedestrian and self-contained, so as to be accessible to non experts.

At present we do not have a proof that $\mathcal{M}_{0,n}^A$ is a Noetherian and finitely generated ring for arbitrary $\mathfrak{g}$ and $n \geq 1$, though it is natural to expect it is the case. Indeed, in the case $\mathfrak{g} = sl(2)$, $\mathcal{M}_{0,n}^A(sl(2))$ is isomorphic to the skein algebra of a sphere with $n+1$ punctures (see Theorem 8.6 in [28]), which is finitely generated and Noetherian by results of [11] and [79], or by the particular case $\mathfrak{g} = sl(2)$ of the results in [26]. In our general situation, key arguments in the proof of Theorem 1.1 for $\mathcal{M}_{0,n}$ depend on the existence of a Reynolds operator on the $U_q$-module $\mathcal{L}_{0,n}$, and one can easily show there is no Reynolds operator on $\mathcal{L}_{0,n}^A$. This follows from the corresponding fact for the integral quantum coordinate ring $\mathcal{O}_A$ (see Remark 2.14).

From Section 4 we consider the specializations $\mathcal{L}_{0,n}^\epsilon$ of $\mathcal{L}_{0,n}^A$ at $q = \epsilon$, a primitive root of unity of odd order $l$ (and coprime to 3 if $\mathfrak{g}$ has $G_2$ components). In [43], De Concini-Lyubashenko introduced a central subalgebra $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ of $\mathcal{O}_\epsilon$ isomorphic to the coordinate ring $\mathcal{O}(G)$, and proved that the $\mathcal{Z}_0(\mathcal{O}_\epsilon)$-module $\mathcal{O}_\epsilon$ is projective of rank $l^{dim\mathfrak{g}}$. As observed by Brown-Gordon-Stafford [25], Bass' Cancellation theorem in $K$-theory and the fact that $K_0(\mathcal{O}(G)) \cong \mathbb{Z}$, proved by Marlin [81], imply that this module is free. Alternatively, this follows also from the fact that $\mathcal{O}_\epsilon$ is a cleft extension of $\mathcal{O}(G)$ by the dual of the Hopf algebra $\mathfrak{u}_\epsilon(\mathfrak{g})$, as proved by Andruskiewitsch-Garcia (see [5], Remark 2.18(b), and also Section 3.2 of [21]; this argument was explained to us by K. A. Brown).

The section 4 proves the analogous property for $\mathcal{L}_{0,n}^\epsilon$. Namely:

**Theorem 1.2.** *$\mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$ is a central subalgebra of $\mathcal{L}^\epsilon_{0,n}$, and $\mathcal{L}^\epsilon_{0,n}$ is a free $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$-module of rank $l^{n.dim\mathfrak{g}}$, isomorphic to the $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$-module $\mathcal{O}_\epsilon^{\otimes n}$.*

In the sequel we systematically denote

$$\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}) := \mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}.$$

We prove the first and third claims of Theorem 1.2 in Proposition 4.1. The arguments use results of De Concini-Kac [39], De Concini-Procesi [40, 41], and De Concini-Lyubashenko [43], that we recall in Section 2.3-2.5. Let us stress that the algebra structures of $\mathcal{L}^\epsilon_{0,n}$ and $\mathcal{O}_\epsilon^{\otimes n}$ are completely different.

Since $\mathcal{Z}_0(\mathcal{O}_\epsilon) \cong \mathcal{O}(G)$, we may deduce the second claim of Theorem 1.2 from the first and third claims together with the results of [43] and [81], or [5], recalled above. Nevertheless we give a self-contained proof that $\mathcal{L}^\epsilon_{0,1}$ is finite projective of rank $l^{dim\mathfrak{g}}$ over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$, by adapting the original arguments of Theorem 7.2 of De Concini-Lyubashenko [43]. In particular we study the coregular action of the braid group of $\mathfrak{g}$ on $\mathcal{L}^\epsilon_{0,1}$; by the way, in the Appendix we provide different proofs of some technical facts shown in [43]. Of course, it remains an exciting problem to describe the centralizing extension $\mathcal{O}(G)^{\otimes n} \subset \mathcal{L}^\epsilon_{0,n}$ (and similarly $\mathcal{O}(G)^{\otimes n} \subset (\mathcal{L}^\epsilon_{0,n})^{U_\epsilon}$ below), aiming at generalizing the results of [5] and finding a direct, more structural proof of freeness in Theorem 1.2.

Also, we note that basis of $\mathcal{L}^\epsilon_{0,n}$ over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$ are complicated. The only case we know is for $\mathcal{O}_\epsilon(sl(2))$, described in [45], and it is far from being obvious (see (111)).

In Section 5 we turn to fraction rings. As mentioned above $\mathcal{L}^\epsilon_{0,n}$ has no non-trivial zero divisors. Therefore its center $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$ is an integral domain. Denote by $Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}))$ its fraction field. Denote by $(\mathcal{L}^\epsilon_{0,n})^{U_\epsilon}$ the subring of $\mathcal{L}^\epsilon_{0,n}$ formed by the invariant elements of $\mathcal{L}^\epsilon_{0,n}$ with respect to the right coadjoint action of $U_\epsilon$. The center $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$ of $\mathcal{L}^\epsilon_{0,n}$ is contained in $(\mathcal{L}^\epsilon_{0,n})^{U_\epsilon}$ (this follows from [28], Proposition 6.19). Note also that we trivially have an inclusion $\mathcal{M}^{A,\epsilon}_{0,n} \subset (\mathcal{L}^\epsilon_{0,n})^{U_\epsilon}$, and these two algebras are distinct in general. For instance, when $n = 1$ we have $(\mathcal{L}^\epsilon_{0,1})^{U_\epsilon} = \mathcal{Z}(\mathcal{L}^\epsilon_{0,1})$, which is a finite extension of $\mathcal{Z}_0(\mathcal{O}_\epsilon) \cong \mathcal{O}(G)$ (see Lemma 5.1). On another hand, $\mathcal{M}^{A,\epsilon}_{0,1}$ is the specialization at $q = \epsilon$ of $\mathcal{Z}(\mathcal{L}^A_{0,1})$, a polynomial algebra in $\mathrm{rk}(\mathfrak{g})$ variables, which may be identified via $\Phi_1$ with the center $\mathcal{Z}(U_A)$ of the integral form $U_A$.

Consider the rings

$$Q(\mathcal{L}^\epsilon_{0,n}) = Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})) \otimes_{\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})} \mathcal{L}^\epsilon_{0,n}$$

and

$$Q((\mathcal{L}^\epsilon_{0,n})^{U_\epsilon}) = Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})) \otimes_{\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})} (\mathcal{L}^\epsilon_{0,n})^{U_\epsilon}.$$

In general, given a ring $A$ with center $\mathcal{Z}(A)$ an integral domain we reserve the notation $Q(A)$ to the central localization of $A$, ie. $Q(A) := Q(\mathcal{Z}(A)) \otimes_{\mathcal{Z}(A)} A$. Though the center $\mathcal{Z}((\mathcal{L}^\epsilon_{0,n})^{U_\epsilon})$ of $(\mathcal{L}^\epsilon_{0,n})^{U_\epsilon}$ is larger than $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$, the notation $Q((\mathcal{L}^\epsilon_{0,n})^{U_\epsilon})$ is valid, for $\mathcal{Z}((\mathcal{L}^\epsilon_{0,n})^{U_\epsilon})$ is an integral domain finite over $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$, and hence the central localization of $(\mathcal{L}^\epsilon_{0,n})^{U_\epsilon}$ coincides with $Q((\mathcal{L}^\epsilon_{0,n})^{U_\epsilon})$ as defined above. Throughout the paper, unless we mention it explicitly we follow the conventions of Mc Connell-Robson [82] as regards the terminology of ring theory; in particular, for the notions of central simple algebras and PI degrees, see in [82] the sections 5.3 and 13.3.6-13.6.7.

Denote by $m$ the rank of $\mathfrak{g}$, and by $N$ the number of its positive roots. In section 5 we prove:

**Theorem 1.3.** *(1) $Q(\mathcal{L}^\epsilon_{0,n})$ is a division algebra and a central simple algebra of PI degree $l^{nN}$.*
*(2) $Q((\mathcal{L}^\epsilon_{0,n})^{U_\epsilon})$, $n \geq 2$, is a division algebra and a central simple algebra of PI degree $l^{N(n-1)-m}$.*

The second claim of (1) means that $Q(\mathcal{L}^\epsilon_{0,n})$ is a complex subalgebra of a full matrix algebra $Mat_d(\mathbb{F})$, where $d = l^{nN}$ and $\mathbb{F}$ is a finite extension of $Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}))$ such that

$$\mathbb{F} \otimes_{Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}))} Q(\mathcal{L}^\epsilon_{0,n}) = Mat_d(\mathbb{F}).$$

That $Q(\mathcal{L}^\epsilon_{0,n})$ is a division algebra and a central simple algebra follows from Theorem 1.2 and the fact that $\mathcal{L}^\epsilon_{0,n}$ has no non trivial zero divisors (proved in [28]). The computation of $d = l^{nN}$ uses a lower bound coming from the representation theory of $U_\epsilon$, and a lower bound for the degree of $Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}))$ as a field extension of $Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))$, obtained by using specializations to $q = \epsilon$ of certain central elements in $\mathcal{L}_{0,n}$ (for $q$ generic). In this computation a main role is played by results of De Concini-Kac [39].

We deduce (2) from (1), the double centralizer theorem for central simple algebras, a few results of [28] and [43], and Theorem 1.2 again.

**Acknowledgements.** We are grateful to M. Faitg for many valuable discussions on the subject, especially concerning the filtration arguments in the proof of Theorem 1.1, and the use of the partial order $\preceq$ in the proof of Theorem 1.1. We also thank K. A. Brown for pointing out the references [5] and [21] (see the comments before Theorem 1.2).

1.2. **Basic notations.** Given a ring $R$, we denote by $\mathcal{Z}(R)$ its center. When $R$ is commutative and has no non-trivial zero divisors, $Q(R)$ denotes its fraction field.

Given a Hopf algebra $H$ with product $m$ and and coproduct $\Delta$, we denote by $H^{cop}$ (resp. $H_{op}$) the Hopf algebra with the same algebra (resp. coalgebra) structure as $H$ but the opposite coproduct $\Delta^{cop} := \sigma \circ \Delta$ (resp. opposite product $m \circ \sigma$), where $\sigma(x \otimes y) = y \otimes x$, and the antipode $S^{-1}$. We use Sweedler's coproduct notation, $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$, $x \in H$, and we set $\Delta^{(1)} := id$, $\Delta^{(2)} := \Delta$, and $\Delta^{(n)} := (\Delta \otimes id)\Delta^{(n-1)}$ for $n \geq 3$ (this is not the convention used in [28]).

The results of this paper hold true for any finite dimensional complex semisimple Lie algebra $\mathfrak{g}$, but unless we state it differently, we will assume $\mathfrak{g}$ is simple. We will denote its rank by $m$, and its Cartan matrix by $(a_{ij})$. We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a basis of simple roots $\alpha_i \in \mathfrak{h}^*_{\mathbb{R}}$, and denote by $\mathfrak{b}_\pm$ the Borel subalgebras associated to it. We denote by $N$ the number of positive roots of $\mathfrak{g}$, and by $\rho$ half the sum of the positive roots.

We denote by $d_1, \dots, d_m$ the unique coprime positive integers such that the matrix $(d_i a_{ij})$ is symmetric, and $(\ ,\ )$ the unique inner product on $\mathfrak{h}^*_{\mathbb{R}}$ such that $d_i a_{ij} = (\alpha_i, \alpha_j)$. For any root $\alpha$ the coroot is $\check{\alpha} = 2\alpha/(\alpha, \alpha)$; in particular $\check{\alpha_i} = d_i^{-1}\alpha_i$. The root lattice $Q$ is the $\mathbb{Z}$-lattice in $\mathfrak{h}^*_{\mathbb{R}}$ defined by $Q = \sum_{i=1}^m \mathbb{Z}\alpha_i$. The weight lattice $P$ is the $\mathbb{Z}$-lattice formed by all $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ such that $(\lambda, \check{\alpha_i}) \in \mathbb{Z}$ for every $i = 1, \dots, m$. So $P = \sum_{i=1}^m \mathbb{Z}\varpi_i$, where $\varpi_i$ is the fundamental weight dual to the simple coroot $\check{\alpha_i}$, which satisfies $(\varpi_i, \check{\alpha_j}) = \delta_{i,j}$. Note that $(\lambda, \alpha) \in \mathbb{Z}$ for every $\lambda \in P$, $\alpha \in Q$. We denote by $D$ the cardinal of the quotient lattice $P/Q$. Then $D$ is the smallest positive integer such that $D(\lambda, \mu) \in \mathbb{Z}$ for every $\lambda, \mu \in P$, that is, such that $DP \subset Q$.

We denote by $P_+ := \sum_{i=1}^m \mathbb{Z}_{\geq 0}\varpi_i$ the cone of dominant integral weights, and we put $Q_+ := \sum_{i=1}^m \mathbb{Z}_{\geq 0}\alpha_i$. Though $Q \subset P$, it is not true that $Q_+ \subset P_+$, but we have $DP_+ \subset Q_+$. This last property is not trivial, and follows from the classical fact that the inverse of the Cartan matrix $(a_{ij})$ has coefficients in $D^{-1}\mathbb{N}$.

We will use the standard partial order relation $\leq$ on $P$, defined by: $\lambda, \mu \in P$ satisfy $\lambda \leq \mu$ if $\mu - \lambda \in Q_+$. In section 3, we will also use the partial order relation $\preceq$ on $P$ defined by: $\lambda \preceq \mu$ if $\mu - \lambda \in D^{-1}Q_+$.

We denote by $\mathcal{B}(\mathfrak{g})$ the braid group of $\mathfrak{g}$; we recall its standard defining relations in the Appendix (Section 6.2).

We denote by $G$ the connected and simply-connected algebraic group with Lie algebra $\mathfrak{g}$, and by $T_G$ the maximal torus of $G$ with Lie algebra $\mathfrak{h}$; $N(T_G)$ is the normalizer of $T_G$, $W = N(T_G)/T_G$ is the Weyl group, $B_\pm$ are the Borel subgroups of $G$ with Lie algebra $\mathfrak{b}_\pm$, and $U_\pm \subset B_\pm$ are their unipotent subgroups.

We denote by $\mathcal{O}(G)$ the coordinate ring of $G$. It is a commutative Hopf algebra, which can be identified with the restricted dual of the universal enveloping algebra $U(\mathfrak{g})$ (see [66], and also [74]). Similarly we denote by $\mathcal{O}(B_\pm)$ the coordinate ring of $B_\pm$.

We let $q$ be an indeterminate, set $A = \mathbb{C}[q, q^{-1}]$, $q_i = q^{d_i}$, $q_\beta = q^{(\beta,\beta)/2}$ for $\beta \in Q$, and given integers $p, k$ with $0 \leq k \leq p$ we put

$$[p]_q = \frac{q^p - q^{-p}}{q - q^{-1}} \ , \ [0]_q! = 1 \ , \ [p]_q! = [1]_q[2]_q \dots [p]_q \ , \ \left[ \begin{array}{c} p \\ k \end{array} \right]_q = \frac{[p]_q!}{[p-k]_q![k]_q!}$$

$$(p)_q = \frac{q^p - 1}{q - 1} \ , \ (0)_q! = 1 \ , \ (p)_q! = (1)_q(2)_q \dots (p)_q \ , \ \left( \begin{array}{c} p \\ k \end{array} \right)_q = \frac{(p)_q!}{(p-k)_q!(k)_q!}.$$

We denote by $\mathcal{A}_0 \subset \mathbb{C}(q)$ the ring of functions regular at $q = 0$; this ring is used only in Section 2.2.2.

We denote by $\epsilon$ a primitive $l$-th root of unity such that $\epsilon^{2d_i} \neq 1$ is also a primitive $l$-th root of unity for all $i \in \{1, \dots, m\}$. This means that $l$ is odd, and coprime to 3 if $\mathfrak{g}$ is $G_2$.

In this paper we use the definition of the unrestricted integral form $U_A(\mathfrak{g})$ given in [41], [43]; in [28] we used the one of [39], [40]. The two are (trivially) isomorphic, and have the same specialization at $q = \epsilon$. Also, we denote here by $L_i$ the generators of $U_q(\mathfrak{g})$ we denoted by $\ell_i$ in [28].

In order to facilitate the comparison with the results of [43] we note that their generators denoted $K_i$, $E_i$ and $F_i$, that we will denote by $\tilde{K}_i, \tilde{E}_i$ and $\tilde{F}_i$, can be written as $K_i, K_i^{-1}E_i$ and $F_iK_i$ in our notations. They satisfy the same algebra relations.

## 2. Background results

2.1. **On $U_q$, $\mathcal{O}_q$, $\mathcal{L}_{0,n}$, $\mathcal{M}_{0,n}$, and $\Phi_n$.** Except when stated differently, we refer to [28], Sections 2-4 and 6, and the references therein for details about the material of this section. We stress that the simply-connected quantum group, that we denote $U_q$ below, was denoted $\tilde{U}_q$ in [28]. Also, the adjoint quantum group $U_q^{ad}$ was denoted $U_q$.

The *simply-connected* quantum group $U_q = U_q(\mathfrak{g})$ is the Hopf algebra over $\mathbb{C}(q)$ with generators $E_i$, $F_i$, $L_i$, $L_i^{-1}$, $1 \leq i \leq m$, and defining relations

$$L_iL_j = L_jL_i \ , \ L_iL_i^{-1} = L_i^{-1}L_i = 1 \ , \ L_iE_jL_i^{-1} = q_i^{\delta_{i,j}}E_j \ , \ L_iF_jL_i^{-1} = q_i^{-\delta_{i,j}}F_j$$

$$E_iF_j - F_jE_i = \delta_{i,j}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \begin{array}{c} 1-a_{ij} \\ r \end{array} \right]_{q_i} E_i^{1-a_{ij}-r}E_jE_i^r = 0 \quad \text{if } i \neq j$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \begin{array}{c} 1-a_{ij} \\ r \end{array} \right]_{q_i} F_i^{1-a_{ij}-r}F_jF_i^r = 0 \quad \text{if } i \neq j$$

where for $\lambda = \sum_{i=1}^m m_i \varpi_i \in P$ we set $K_\lambda = \prod_{i=1}^m L_i^{m_i}$, and $K_i = K_{\alpha_i} = \prod_{j=1}^m L_j^{a_{ji}}$. The coproduct $\Delta$, antipode $S$, and counit $\varepsilon$ of $U_q$ are given by

$$\Delta(L_i) = L_i \otimes L_i \ , \ \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i \ , \ \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i$$
$$S(E_i) = -E_i K_i^{-1} \ , \ S(F_i) = -K_i F_i \ , \ S(L_i) = L_i^{-1}$$
$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \ \varepsilon(L_i) = 1.$$

We fix a reduced expression $s_{i_1} \dots s_{i_N}$ of the longest element $w_0$ of the Weyl group of $\mathfrak{g}$. It induces a total ordering of the positive roots,

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \dots, \beta_N = s_{i_1} \dots s_{i_{N-1}}(\alpha_{i_N}).$$

The root vectors of $U_q$ with respect to such an ordering are defined by

$$E_{\beta_k} = T_{i_1} \dots T_{i_{k-1}}(E_{i_k}) \ , \ F_{\beta_k} = T_{i_1} \dots T_{i_{k-1}}(F_{i_k}) \tag{1}$$

where $T_i$ is the Lusztig algebra automorphism of $U_q$ associated to the simple root $\alpha_i$ ([73, 72], see also [35], Ch. 8). The braid group $\mathcal{B}(\mathfrak{g})$ acts on $U_q$ by means of the Lusztig automorphisms. In the Appendix we recall the relation between $T_i$ and the generator $\hat{w}_i$ of the quantum Weyl group, which we will mostly use. Let us just recall here that the monomials $F_{\beta_1}^{r_1} \dots F_{\beta_N}^{r_N} K_\lambda E_{\beta_N}^{t_N} \dots E_{\beta_1}^{t_1}$ ($r_i, t_i \in \mathbb{N}$, $\lambda \in P$) form a basis of $U_q$, the *PBW basis*.

$U_q$ is a *pivotal* Hopf algebra, with pivotal element

$$\ell := K_{2\rho} = \prod_{j=1}^m L_j^2.$$

So $\ell$ is group-like, and $S^2(x) = \ell x \ell^{-1}$ for every $x \in U_q$.

The *adjoint* quantum group $U_q^{ad} = U_q^{ad}(\mathfrak{g})$ is the Hopf subalgebra of $U_q$ generated by the elements $E_i$, $F_i$ ($i = 1, \dots, m$) and $K_\alpha$ with $\alpha \in Q$; so $\ell \in U_q^{ad}$. When $\mathfrak{g} = sl(2)$, we simply write the above generators $E = E_1$, $F = F_1$, $L = L_1$, $K = K_1$.

We denote by $U_q(\mathfrak{n}_+)$, $U_q(\mathfrak{n}_-)$ and $U_q(\mathfrak{h})$ the subalgebras of $U_q$ generated respectively by the $E_i$, the $F_i$, and the $K_\lambda$ ($\lambda \in P$), and by $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$ the subalgebras generated by the $E_i$ and the $K_\lambda$, and by the $F_i$ and the $K_\lambda$, respectively (they are the two-sided ideals generated by $U_q(\mathfrak{n}_\pm)$). We do similarly with $U_q^{ad}$, where now $U_q^{ad}(\mathfrak{h})$ is generated by the $K_\lambda$ with $\lambda \in Q$.

The Hopf algebra $U_q^{ad}$ is not braided in a strict sense, but it has braided categorical completions. Let us recall briefly what this means and implies. For details we refer to the sections 2 and 3 of [28] (see also [92], Section 3.10, where $\mathbb{U}_q$ below is formulated in terms of multiplier Hopf algebras).

A $U_q^{ad}$-module $V$ is said *of type* 1 if it has finite dimension and the generators $K_i$ are diagonalizable on $V$ with eigenvalues in $q_i^{\mathbb{Z}}$. We denote by $\mathcal{C}$ the category of $U_q^{ad}$-modules of type 1, by $Vect$ the category of finite dimensional $\mathbb{C}(q)$-vector spaces, and by $F_{\mathcal{C}} : \mathcal{C} \to Vect$ the forgetful functor. The category $\mathcal{C}$ is semisimple. The simple objects are highest weight $U_q^{ad}$-modules; we denote by $V_\mu$ the simple module with highest weight $\mu \in P_+$. In the case $\mathfrak{g} = sl(2)$ we identify $P_+$ with $\mathbb{N}$, and denote by $V_n$ the simple module of dimension $n+1$. Note that $V_\mu$ is canonically endowed with a structure of $U_q$-module of type 1, the generators $L_i$ being diagonalizable with eigenvalues in $q_i^{\mathbb{Z}/D}$. The *categorical completion* $\mathbb{U}_q^{ad}$ of $U_q^{ad}$ is the set of natural transformations $F_{\mathcal{C}} \to F_{\mathcal{C}}$. An element of $\mathbb{U}_q^{ad}$ is a collection $(a_V)_{V \in Ob(\mathcal{C})}$, where $a_V \in End_{\mathbb{C}(q)}(V)$ satisfies $F_{\mathcal{C}}(f) \circ a_V = a_W \circ F_{\mathcal{C}}(f)$ for any objects $V, W$ of $\mathcal{C}$ and any arrow $f \in Hom_{U_q^{ad}}(V, W)$. It is not hard to see that $\mathbb{U}_q^{ad}$ inherits from $\mathcal{C}$ a natural structure of (completed) Hopf algebra such that the map

$$\begin{array}{rccc} \iota : & U_q^{ad} & \longrightarrow & \mathbb{U}_q^{ad} \\ & x & \longmapsto & (\pi_V(x))_{V \in Ob(\mathcal{C})} \end{array} \tag{2}$$

is a morphism of Hopf algebras, where $\pi_V : U_q^{ad} \to \mathrm{End}(V)$ is the representation associated to a module $V$ in $\mathcal{C}$. It is a theorem that this map is injective. From now on, let us extend the coefficient ring of the modules and morphisms in $\mathcal{C}$ to $\mathbb{C}(q^{1/D})$. Put

$$\mathbb{U}_q = \mathbb{U}_q^{ad} \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D}).$$

The map $\iota$ above extends to an embedding of $U_q$ in $\mathbb{U}_q$. The category $\mathcal{C}$, with coefficients extended to $\mathbb{C}(q^{1/D})$, is braided and ribbon; we postpone a discussion of that fact to Section 2.3, where it will be developed. As a consequence, we can regard $\mathbb{U}_q$ as a quasitriangular and ribbon Hopf algebra in a generalized sense (see [28]). The R-matrix of $\mathbb{U}_q$ is the family of morphisms

$$R = (R_{V,W})_{V,W \in Ob(\mathcal{C})}$$

where $R_{V,W} \in End(V \otimes W)$ is the endomorphism defined by the action of Drinfeld's universal R-matrix on $V \otimes W$. The ribbon element of $\mathbb{U}_q$ is defined similarly by Drinfeld's universal ribbon element. One defines the *categorical tensor product* $\mathbb{U}_q^{\hat{\otimes}2}$ similarly as $\mathbb{U}_q$; in particular it contains all the infinite series of elements of $\mathbb{U}_q^{\otimes 2}$ having only a finite number of non-zero terms when evaluated on a given module $V \otimes W$ of $\mathcal{C}$. There is an expansion of $R$ as an infinite series in $\mathbb{U}_q^{\hat{\otimes}2}$. Adapting Sweedler's coproduct notation $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ we find convenient to write this series as

$$R = \sum_{(R)} R_{(1)} \otimes R_{(2)}. \tag{3}$$

We put $R^+ := R$, $R^- := (\sigma \circ R)^{-1}$ where $\sigma$ is the flip map $x \otimes y \mapsto y \otimes x$. We will not use any explicit formula of $R$, but the following factorization formula

$$R = \Theta \hat{R} \tag{4}$$

where

$$\Theta = q^{\sum_{i,j=1}^m (B^{-1})_{ij} H_i \otimes H_j} \in \mathbb{U}_q^{\hat{\otimes}2},$$

with $B \in M_m(\mathbb{Q})$ the matrix with entries $B_{ij} := d_j^{-1} a_{ij}$, and

$$\hat{R} = \sum_{(\hat{R})} \hat{R}_{(1)} \otimes \hat{R}_{(2)} \in \mathbb{U}_q(\mathfrak{n}_+) \hat{\otimes} \mathbb{U}_q(\mathfrak{n}_-)$$

(see [28], Section 3.2, and for details eg. [35], Theorem 8.3.9, or [92], Theorem 3.108). If $x$, $y$ are weight vectors of weights $\mu$, $\nu$ respectively, then $\Theta(x \otimes y) = q^{(\mu,\nu)} x \otimes y$. Moreover, $\hat{R}$ has weight 0 for the adjoint action of $U_q(\mathfrak{h})$; that is, complementary components $\hat{R}_{(1)}$ and $\hat{R}_{(2)}$ have opposite weights.

Recall that we denote by $G$ the connected and simply-connected algebraic group with Lie algebra $\mathfrak{g}$. The *quantum function Hopf algebra* $\mathcal{O}_q = \mathcal{O}_q(G)$ is defined as the restricted dual of $U_q^{ad}$ with respect to the category $\mathcal{C}$, that is, the set of $\mathbb{C}(q)$-linear maps $f \colon U_q^{ad} \to \mathbb{C}(q)$ such that $\mathrm{Ker}(f)$ contains a cofinite two sided ideal $I$ (ie. such that $I \oplus M = U_q$ for some finite dimensional vector space $M$), and $\prod_{s=-r}^{r} (K_i - q_i^s) \in I$ for some $r \in \mathbb{N}$ and every $i$ (see eg. [23], Chapter I.7).

The space $\mathcal{O}_q$ is a Hopf algebra, with structure maps defined dually to those of $U_q^{ad}$. We denote by $\star$ its product. The algebras $\mathcal{O}_q(T_G)$, $\mathcal{O}_q(U_\pm)$, $\mathcal{O}_q(B_\pm)$ are defined similarly, by replacing $U_q^{ad}$ with $U_q^{ad}(\mathfrak{h})$, $U_q^{ad}(\mathfrak{n}_\pm)$, $U_q^{ad}(\mathfrak{b}_\pm)$ respectively. As a vector space, $\mathcal{O}_q$ is generated by the functionals $x \mapsto w(\pi_V(x)v)$, $x \in U_q^{ad}$, for every object $V \in Ob(\mathcal{C})$ and vectors $v \in V$, $w \in V^*$. Such functionals are called *matrix coefficients*. Because the morphism $\iota : U_q^{ad} \to \mathbb{U}_q$ is injective (see (2)), the Hopf duality pairing $\langle ., . \rangle \colon \mathcal{O}_q \times U_q^{ad} \to \mathbb{C}(q)$ is non degenerate. By

extending the coefficient ring from $\mathbb{C}(q)$ to $\mathbb{C}(q^{1/D})$, we can uniquely extend it to a bilinear pairing

$$\langle .,. \rangle \colon (\mathcal{O}_q \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})) \times \mathbb{U}_q \to \mathbb{C}(q^{1/D})$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}_q \otimes U_q^{ad} & \xrightarrow{\langle .,. \rangle} & \mathbb{C}(q) \\ {\scriptstyle id\otimes\iota}\downarrow & & \downarrow \\ (\mathcal{O}_q \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})) \otimes \mathbb{U}_q & \xrightarrow{\langle .,. \rangle} & \mathbb{C}(q^{1/D}) \end{array}$$

This pairing is defined by

$$\langle {}_Y\phi_v^w, (a_X) \rangle = w(a_Y v)$$

for every $(a_X) \in \mathbb{U}_q$ and ${}_Y\phi_v^w \in \mathcal{O}_q$. It is non degenerate.

The maps

$$\begin{array}{rccl} \Phi^{\pm} : & \mathcal{O}_q & \longrightarrow & U_q^{cop} \\ & \alpha & \longmapsto & (\alpha \otimes id)(R^{\pm}) = \sum\limits_{(R^{\pm})} \langle \alpha, R^{\pm}_{(1)} \rangle R^{\pm}_{(2)} \end{array} \tag{5}$$

are well-defined morphisms of Hopf algebras. Here we stress that it is the simply-connected quantum group $U_q^{cop}$ that is the range of $\Phi^{\pm}$. This will be explained with more details in Section 2.3.

Let us make two simple observations, for future reference. Firstly, because $\mathcal{O}_q$ is spanned by the matrix coefficients of the objects of $\mathcal{C}$, and $\mathcal{C}$ is semisimple with simple objects the $U_q^{ad}$-modules $V_\mu$, $\mu \in P_+$, there is a decomposition of $U_q$-bimodule

$$\mathcal{O}_q = \bigoplus_{\mu \in P_+} C(\mu), \tag{6}$$

where $C(\mu) = V_\mu^* \otimes V_\mu$, the space of matrix coefficients of $V_\mu$, is endowed with the left action on the factor $V_\mu$ and the right action on $V_\mu^*$, and $\mathcal{O}_q$ has the left and right coregular actions $\triangleleft$ and $\triangleright$, defined by

$$x \triangleright \alpha := \sum_{(\alpha)} \alpha_{(1)} \langle \alpha_{(2)}, x \rangle, \ \alpha \triangleleft x := \sum_{(\alpha)} \langle \alpha_{(1)}, x \rangle \alpha_{(2)}$$

for all $x \in U_q$ and $\alpha \in \mathcal{O}_q$. Here we recall that each $U_q^{ad}$-module $V_\mu$ can be regarded as a $U_q$-module, so the above expressions make sense. The decomposition (6) is the *Peter-Weyl* decomposition of $\mathcal{O}_q$. It will be refined in Section 2.2.2.

Moreover, the algebra $\mathcal{O}_q$ is generated by the matrix coefficients of the simple $U_q^{ad}$-modules $V_{\varpi_k}$ with heighest weights the fundamental weights $\varpi_k$, $k = 1, \dots, m$; see eg. Proposition I.7.8 of [23] for a proof. This relies on the standard fact that, for any $\mu, \nu \in P_+$ we have a direct sum decomposition of modules (where $m(\lambda) \in \mathbb{N}$)

$$V_\mu \otimes V_\nu = V_{\mu+\nu} \oplus \bigoplus_{\lambda < \mu+\nu} V_\lambda^{\oplus m(\lambda)}. \tag{7}$$

In particular, this decomposition implies that, up to scalar multiples, there is a unique non zero morphism $V_{\mu+\nu} \to V_\mu \otimes V_\nu$, which is injective and splits. Dually, this means that, applying the product in $\mathcal{O}_q$ followed by the projection onto the subspace $C(\mu+\nu)$ we get a canonical projection map

$$C(\mu) \otimes C(\nu) \to C(\mu+\nu). \tag{8}$$

The *loop algebra* $\mathcal{L}_{0,1} = \mathcal{L}_{0,1}(\mathfrak{g})$ is defined by twisting the product $\star$ of $\mathcal{O}_q$, keeping the same underlying linear space. The new product is equivariant with respect to the right coadjoint action $coad^r$ of $U_q$, defined by

$$coad^r(x)(\alpha) = \sum_{(x)} S(x_{(2)}) \rhd \alpha \lhd x_{(1)}$$

for all $x \in U_q$ and $\alpha \in \mathcal{O}_q$. By equivariant we mean that $\mathcal{L}_{0,1}$ is a $U_q$-module algebra. Let us spell out its product and equivariance property. Using the fact that $U_q$ can be regarded as a subspace of $\mathbb{U}_q$, the actions $\lhd$ and $\rhd$ extend naturally to actions of $\mathbb{U}_q$, and the product of $\mathcal{L}_{0,1}$ is expressed in terms of $\star$ by the formula ([28], Proposition 4.1):

$$\alpha\beta = \sum_{(R),(R)} (R_{(2')} S(R_{(2)}) \rhd \alpha) \star (R_{(1')} \rhd \beta \lhd R_{(1)}), \tag{9}$$

where $\sum_{(R)} R_{(1)} \otimes R_{(2)}$ and $\sum_{(R)} R_{(1')} \otimes R_{(2')}$ are expansions of two copies of $R \in \mathbb{U}_q^{\hat{\otimes} 2}$. Note that the sum in (9) has only a finite number of non zero terms. By using that $R\Delta = \Delta^{cop} R$, this product can equivalently be expressed as

$$\alpha\beta = \sum_{(R),(R)} (\beta \lhd R_{(1)} R_{(1')}) \star (S(R_{(2)}) \rhd \alpha \lhd R_{(2')}). \tag{10}$$

This product gives $\mathcal{L}_{0,1}$ (like $\mathcal{O}_q$) a structure of $U_q$-module algebra for the actions $\rhd$, $\lhd$, but also for $coad^r$ (which is not the case of $\mathcal{O}_q$). Spelling this out for $coad^r$, this means

$$coad^r(x)(\alpha\beta) = \sum_{(x)} coad^r(x_{(1)})(\alpha) coad^r(x_{(2)})(\beta).$$

The relations between $\mathcal{O}_q$, $\mathcal{L}_{0,1}$ and $U_q$ are encoded by the map

$$\begin{array}{rccc} \Phi_1 : & \mathcal{O}_q & \longrightarrow & \mathbb{U}_q \\ & \alpha & \longmapsto & (\alpha \otimes id)(RR') \end{array} \tag{11}$$

where $R' = \sigma \circ R$, and as usual $\sigma : x \otimes y \mapsto y \otimes x$. Note that

$$\Phi_1 = m \circ (\Phi^+ \otimes (S^{-1} \circ \Phi^-)) \circ \Delta. \tag{12}$$

We call $\Phi_1$ the *RSD* map, for Drinfeld, Reshetikhin and Semenov-Tian-Shansky introduced it first (see [37, 83],[80]). It is a fundamental result of the theory (see [32, 58, 10]) that $\Phi_1$ affords an isomorphism of $U_q$-modules

$$\Phi_1 : \mathcal{O}_q \to U_q^{lf}.$$

For full details on that result we refer to Section 3.12 of [92]. Here, $U_q^{lf}$ is the set of *locally finite* elements of $U_q$, endowed with the right adjoint action $ad^r$ of $U_q$. It is defined by

$$U_q^{lf} := \{x \in U_q \mid rk_{\mathbb{C}(q)}(ad^r(U_q)(x)) < \infty\}$$

and

$$ad^r(y)(x) = \sum_{(y)} S(y_{(1)}) x y_{(2)}$$

for every $x, y \in U_q$. The action $ad^r$ gives in fact $U_q^{lf}$ a structure of right $U_q$-module algebra. It is also a right coideal, that is $\Delta(U_q^{lf}) \subset U_q^{lf} \otimes U_q$. Moreover, $\Phi_1$ affords an isomorphism of $U_q$-module algebras

$$\Phi_1 : \mathcal{L}_{0,1} \to U_q^{lf}. \tag{13}$$

It is a fact that $\Phi_1$ affords an isomorphism between the centers $\mathcal{Z}(\mathcal{L}_{0,1})$ of $\mathcal{L}_{0,1}$ and $\mathcal{Z}(U_q)$ of $U_q$ ([28], Proposition 6.24). Since $\Phi_1$ is an isomorphism of $U_q$-modules and $\mathcal{Z}(U_q) = U_q^{U_q}$, it follows that $\mathcal{Z}(\mathcal{L}_{0,1}) = \mathcal{L}_{0,1}^{U_q}$.

Let us recall a few fundamental results about $U_q^{lf}$ that we will meet again later. Denote by $T \subset U_q$ the multiplicative Abelian group formed by the elements $K_\lambda$, $\lambda \in P$, and by $T_2 \subset T$ the subgroup formed by the elements $K_\lambda$, $\lambda \in 2P$. Consider the subset $T_{2-} \subset T_2$ formed by the elements $K_{-\lambda}$, $\lambda \in 2P_+$. Clearly $T_2 = T_{2-}^{-1} T_{2-}$ and $\mathrm{Card}(T/T_2) = 2^m$.

**Theorem 2.1.** *(1)* $U_q^{lf} = \oplus_{\lambda \in 2P_+} ad^r(U_q)(K_{-\lambda})$.
*(2)* $U_q = T_{2-}^{-1} U_q^{lf}[T/T_2]$, *so* $U_q$ *is a free* $T_{2-}^{-1} U_q^{lf}$*-module of rank* $2^m$.
*(3) The ring* $U_q^{lf}$ *is (left and right) Noetherian.*

These results were proved by Joseph-Letzter in Theorem 4.10 of [60], Theorem 6.4 of [59], and Theorem 7.4.8 of [58], respectively (see also the sections 7.1.6, 7.1.13 and 7.1.25 in [58]). For (1) and (3) we refer also to Theorem 3.113 and 3.137 in [92], which provides simpler proofs. For instance, in the $sl(2)$ case we have

$$U_q(sl(2)) = U_q(sl(2))^{lf}[K] \oplus U_q(sl(2))^{lf}[K].L.$$

The actual values of $\Phi_1$ are complicated in general (see Theorem 2.15 and the comments thereafter). However, there is a simple important one, that we describe now. Let $V_{-\lambda}$ be the type 1 simple $U_q^{ad}$-module of lowest weight $-\lambda \in -P_+$ (ie. the highest weight $U_q^{ad}$-module $V_{-w_0(\lambda)}$ of highest weight $-w_0(\lambda)$, where $w_0$ is the longest element of the Weyl group; note that $-w_0$ permutes the simple roots). Let $v \in V_{-\lambda}$ be a lowest weight vector, and $v^* \in V_{-\lambda}^*$ be such that $v^*(v) = 1$ and $v^*$ vanishes on a $U_q^{ad}(\mathfrak{h})$-invariant complement of $v$. Define $\psi_{-\lambda} \in \mathcal{O}_q$ by $\langle \psi_{-\lambda}, x \rangle = v^*(xv)$, $x \in U_q$. From the definition (11) it is quite easy to see that

$$\Phi_1(\psi_{-\lambda}) = K_{-2\lambda}. \tag{14}$$

In particular $\Phi_1(\psi_{-\rho}) = \ell^{-1}$, where as usual $\ell$ is the pivotal element of $U_q$.

**Remark 2.2.** Since $\mathcal{L}_{0,1} = \mathcal{O}_q$ as a vector space, we still denote by $C(\mu)$, $\mu \in P^+$, the linear subspace generated by the matrix coefficients of $V_\mu$, the $U_q^{ad}$-module of type 1 and highest weight $\mu$. It can be proved (see Section 7.1.22 in [58], or page 156 of [92], where different conventions are used) that $\Phi_1$ yields an isomorphism of $U_q$-modules

$$\Phi_1 \colon C(-w_0(\mu)) \to ad^r(U_q)(K_{-2\mu}). \tag{15}$$

Therefore, the summands in (1) are finite-dimensional $U_q$-modules, and the action $ad^r$ is completely reducible on $U_q^{lf}$. In fact, $U_q^{lf}$ is the socle of $ad^r$ on $U_q$.

**Remark 2.3.** Because $\ell = \prod_{j=1}^m L_j^2$ and $\Phi_1(\psi_{-\rho}) = \ell^{-1}$, a natural question is the factorization of $\psi_{-\rho}$ in $\mathcal{L}_{0,1}$ (see Corollary 2.18). This question is considered in [61], where $\mathcal{L}_{0,1}(\mathfrak{g})$ for $\mathfrak{g} = gl(r+1)$ is analysed and quantum minors are extensively studied. Let us review here some of their results in relation with $\psi_{-\rho}$.

First note that for for $\mathfrak{g} = sl(r+1)$ the irreducible representation $V_{-\rho}$ of lowest weight $-\rho$ is isomorphic to the representation of highest weight $V_\rho$ because $-w_0(\rho) = \rho$. By the Weyl formula the dimension of this representation is $\prod_{\alpha>0} \frac{(2\rho,\alpha)}{(\rho,\alpha)} = 2^N$. In [70] a presentation of $U_q(gl(r+1))$ is given, which differs from our presentation of $U_q(sl(r+1))$ only by its subalgebra $U_q(\mathfrak{h})$, generated by $r+1$ elements $\mathbb{K}_1, ..., \mathbb{K}_{r+1}$. The inclusion $U_q(sl(r+1)) \subset U_q(gl(r+1))$ is such that $K_i = \mathbb{K}_i^2 \mathbb{K}_{i+1}^{-2}$, $i = 1, ..., r$. The quantum minors, properly defined in [61], of the matrix of matrix elements of the natural representation of $U_q(gl(r+1))$ are denoted $det_q(A_{\geq k})$

for $k = 1, ..., r+1$. We have $det_q(A_{\geq 1}) = 1$ in the case of $sl(r+1)$. Then [61] proves that $det_q(A_{\geq k}) = (\mathbb{K}_k ... \mathbb{K}_{r+1})^2$, and there exists an element $\mathbb{K} \in U_q(gl(r+1))$ such that

$$\mathbb{K}^{-2\rho} = det_q(A_{\geq 1})^{-r} det_q(A_{\geq 2}) ... det_q(A_{\geq r+1}).$$

This has to be interpreted in the $sl(r+1)$ case as $K_{-2\rho} = \Phi_1(det_q(A_{\geq 2}) ... det_q(A_{\geq r+1}))$. As a result this gives the equality

$$\psi_{-\rho} = det_q(A_{\geq 2}) ... det_q(A_{\geq r+1}).$$

The *(quantum) graph algebra* $\mathcal{L}_{0,n} = \mathcal{L}_{0,n}(\mathfrak{g})$ is the braided tensor product of $n$ copies of $\mathcal{L}_{0,1}$ (considered as a $U_q$-module algebra). As a linear space and $U_q$-bimodule with actions $\triangleleft$ and $\triangleright$, it coincides with $\mathcal{L}_{0,1}^{\otimes n}$, and thus with $\mathcal{O}_q^{\otimes n}$. It is also a right $U_q$-module algebra, with the following action of $U_q$ (extending $coad^r$ on $\mathcal{L}_{0,1}$):

$$coad_n^r(y)(\alpha^{(1)} \otimes \ldots \otimes \alpha^{(n)}) = \sum_{(y)} coad^r(y_{(1)})(\alpha^{(1)}) \otimes \ldots \otimes coad^r(y_{(n)})(\alpha^{(n)}) \tag{16}$$

for all $y \in U_q$ and $\alpha^{(1)} \otimes \ldots \otimes \alpha^{(n)} \in \mathcal{L}_{0,n}$. The product of $\mathcal{L}_{0,n}$ can be explicited as follows. For every $1 \leq a \leq n$ define $\mathfrak{i}_a \colon \mathcal{L}_{0,1} \to \mathcal{L}_{0,n}$ by $\mathfrak{i}_a(x) = 1^{\otimes(a-1)} \otimes x \otimes 1^{\otimes(n-a)}$; $\mathfrak{i}_a$ is an embedding of $U_q$-module algebras. We will use the notations

$$\mathcal{L}_{0,n}^{(a)} := \mathrm{Im}(\mathfrak{i}_a) \ , \ (\alpha)^{(a)} := \mathfrak{i}_a(\alpha). \tag{17}$$

Take $(\alpha)^{(a)}, (\alpha')^{(a)} \in \mathcal{L}_{0,n}^{(a)}$ and $(\beta)^{(b)}, (\beta')^{(b)} \in \mathcal{L}_{0,n}^{(b)}$ with $a < b$. Then the product of $\mathcal{L}_{0,n}$ is given by the following formula (see [28], section 6):

$$\begin{aligned} &\Big((\alpha)^{(a)} \otimes (\beta)^{(b)}\Big) \ \Big((\alpha')^{(a)} \otimes (\beta')^{(b)}\Big) \\ &\qquad = \sum_{(R^1),\ldots,(R^4)} \Big(\alpha \Big(S(R^3_{(1)} R^4_{(1)}) \triangleright \alpha' \triangleleft R^1_{(1)} R^2_{(1)}\Big)\Big)^{(a)} \\ &\qquad\qquad \otimes \Big(\Big(S(R^1_{(2)} R^3_{(2)}) \triangleright \beta \triangleleft R^2_{(2)} R^4_{(2)}\Big) \beta'\Big)^{(b)} \end{aligned} \tag{18}$$

where $R^i = \sum_{(R^i)} R^i_{(1)} \otimes R^i_{(2)}$, $i \in \{1,2,3,4\}$, are expansions of four copies of $R \in \mathbb{U}_q^{\hat{\otimes} 2}$, and on the right-hand side the product is componentwise that of $\mathcal{L}_{0,1}$. Later we will use the fact that the product of $\mathcal{L}_{0,n}$ is obtained from the standard (componentwise) product of $\mathcal{L}_{0,1}^{\otimes n}$ by a process that may be inverted. Indeed, (18) can be rewritten as

$$\Big((\alpha)^{(a)} \otimes (\beta)^{(b)}\Big)\Big((\alpha')^{(a)} \otimes (\beta')^{(b)}\Big) = \sum_{(F)} (\alpha)^{(a)}\Big((\alpha')^{(a)} \cdot F_{(2)}\Big) \otimes \Big((\beta)^{(b)} \cdot F_{(1)}\Big)(\beta')^{(b)} \tag{19}$$

where $F = \sum_{(F)} F_{(1)} \otimes F_{(2)} := (\Delta \otimes \Delta)(R')$, and the symbol "$\cdot$" stands for the right action of $\mathbb{U}_q^{\hat{\otimes} 2}$ on $\mathcal{L}_{0,1}$ that may be read from (18). The tensor $F$ is known as a twist. Then, by replacing $F$ with its inverse $\bar{F} = (\Delta \otimes \Delta)(R'^{-1})$, one can express the product of $\mathcal{L}_{0,1}^{\otimes n}$ in terms of the product of $\mathcal{L}_{0,n}$ by

$$(\alpha)^{(a)}(\alpha')^{(a)} \otimes (\beta)^{(b)}(\beta')^{(b)} = \sum_{(\bar{F})} \Big((\alpha)^{(a)} \otimes \Big((\beta)^{(b)} \cdot \bar{F}_{(1)}\Big)\Big)\Big(\Big((\alpha')^{(a)} \cdot \bar{F}_{(2)}\Big) \otimes (\beta')^{(b)}\Big). \tag{20}$$

We call *quantum moduli algebra* and denote by

$$\mathcal{M}_{0,n} = \mathcal{L}_{0,n}^{U_q}$$

the subalgebra of $\mathcal{L}_{0,n}$ formed by the $U_q$-invariant elements.

The map $\Phi_1$ can be extended to $\mathcal{L}_{0,n}$ as follows. Consider the following action of $U_q$ on the tensor product algebra $U_q^{\otimes n}$, which extends $ad^r$ on $U_q$:

$$ad_n^r(y)(x) = \sum_{(y)} \Delta^{(n)}(S(y_{(1)}))x\Delta^{(n)}(y_{(2)}) \tag{21}$$

for all $y \in U_q$, $x \in U_q^{\otimes n}$. This action gives $U_q^{\otimes n}$ a structure of right $U_q$-module algebra. In [1] Alekseev introduced a morphism of $U_q$-module algebras $\Phi_n \colon \mathcal{L}_{0,n} \to U_q^{\otimes n}$ which extends $\Phi_1$. In Proposition 6.7 of [28] we showed that $\Phi_n$ affords isomorphisms

$$\Phi_n \colon \mathcal{L}_{0,n} \to (U_q^{\otimes n})^{lf} \ , \ \Phi_n : \mathcal{M}_{0,n} \to (U_q^{\otimes n})^{U_q} \tag{22}$$

where $(U_q^{\otimes n})^{lf}$ is the set of $ad_n^r$-locally finite elements of $U_q^{\otimes n}$. We call $\Phi_n$ the *Alekseev map*; we do not recall here the definition of $\Phi_n$, for we will not use it. It is a key argument of the proof of (22) that the set of locally finite elements of $U_q^{\otimes n}$ for $(ad^r)^{\otimes n} \circ \Delta^{(n)}$ coincides with $(U_q^{lf})^{\otimes n}$; this follows from the main result of [69]. Using that the map

$$\psi_n = \Phi_n \circ (\Phi_1^{-1})^{\otimes n} \colon (U_q^{lf})^{\otimes n} \to (U_q^{\otimes n})^{lf} \tag{23}$$

is surjective and intertwines the actions $(ad^r)^{\otimes n} \circ \Delta^{(n-1)}$ and $ad_n^r$, we deduced that $\mathrm{Im}(\Phi_n) = (U_q^{\otimes n})^{lf}$.

**Remark 2.4.** We have $(U_q^{lf})^{\otimes n} \neq (U_q^{\otimes n})^{lf}$, and in fact there is not even an inclusion. Indeed let $\Omega = (q-q^{-1})^2 FE + qK + q^{-1}K^{-1}$ be the Casimir element of $U_q(sl(2))$. We trivially have $\Delta(\Omega) \in (U_q^{\otimes 2})^{lf}$ but

$$\Delta(\Omega) = (q-q^{-1})^2(K^{-1}E \otimes FK + F \otimes E) + \Omega \otimes K + K^{-1} \otimes \Omega - (q+q^{-1})K^{-1} \otimes K$$

and therefore $\Delta(\Omega) \notin (U_q^{lf})^{\otimes 2}$, since $K \notin U_q^{lf}$ (see eg. Theorem 2.1 (2)). This reflects the fact that $U_q^{lf}$ is only a right coideal (and not a subcoalgebra).

As in Remark 2.2, denote by $C(\mu)$, $\mu \in P^+$, the linear subspace of $\mathcal{L}_{0,1}$ generated by the matrix coefficients of $V_\mu$. For every tuple $[\mu] = (\mu_1, \dots, \mu_n) \in P_+^n$ put

$$C([\mu]) = C(\mu_1) \otimes \dots \otimes C(\mu_n) \tag{24}$$

Then $\mathcal{L}_{0,n} = \bigoplus_{[\mu] \in P_+^n} C([\mu])$. Each space $C([\mu])$ is a finite dimensional $U_q$-module under the action $coad_n^r$, whence it is completely reducible. Therefore

$$\mathcal{L}_{0,n} = \mathcal{M}_{0,n} \oplus I \tag{25}$$

as $U_q$-modules, where $I$ the sum of non trivial isotypical components of $\mathcal{L}_{0,n}$. The $\mathbb{C}(q)$-linear projection map

$$\mathcal{R} \colon \mathcal{L}_{0,n} \to \mathcal{M}_{0,n}, \ Ker(\mathcal{R}) = I \tag{26}$$

is called the *Reynolds operator*. For all $\alpha \in \mathcal{M}_{0,n}$, $\beta \in \mathcal{L}_{0,n}$ it satisfies

$$\mathcal{R}(\alpha\beta) = \alpha\mathcal{R}(\beta). \tag{27}$$

This property will be crucial in the sequel, so let us recall a (classical) proof of it. We can write $\beta = \mathcal{R}(\beta) + \gamma$ with $\gamma \in I$, and then we have to show $\alpha\gamma \in I$. We can reduce to the case where $\gamma$ is contained in a simple summand $V$ of $I$. Multiplication by the invariant element $\alpha$ yields a surjective map $V \to \alpha V$, which is a morphism of $U_q$-modules. Since $V$ is simple, it is either the 0 map, or an isomorphism. In either cases it follows $\alpha V \subset I$ (in fact the first case cannot happen, for $\mathcal{L}_{0,n}$ has no non-trivial zero divisors, as explained after (31)).

We can formulate the Reynolds operator in the following way. Recall that $\mathcal{O}_q$ has a unique left (or right, or 2-sided) Haar integral, that is a linear map $h\colon \mathcal{O}_q \to \mathbb{C}(q)$ such that

$$h(1) = 1 \text{ and } (id \otimes h)\Delta(\alpha) = h(\alpha)1,\ \forall \alpha \in \mathcal{O}_q. \tag{28}$$

(See eg. [35], Proposition 13.3.6). It vanishes on all matrix coefficients except the one of the trivial representation, to which it gives the value 1. Denote by $\Delta_{\mathcal{L}}\colon \mathcal{L}_{0,n} \to \mathcal{L}_{0,n} \otimes \mathcal{O}_q$ the right coaction dual to the action $coad_n^r$ of $U_q$ on $\mathcal{L}_{0,n}$. Then, in analogy with the formula of the averaging operator $\mathcal{C}^\infty(G) \to \mathcal{C}^\infty(G)^G$, $f \to [f] = \int_G f(g^{-1}\cdot g)d\mu(g)$, for a locally compact group $G$ with Haar measure $d\mu(g)$, it is straightforward that

$$\mathcal{R} = (id \otimes h)\Delta_{\mathcal{L}}. \tag{29}$$

Note that the complete reducibility of $\mathcal{L}_{0,n}$ discussed after (24) follows also from Theorem 2.1 (1), since by (23) we have an isomorphism of $U_q$-modules

$$\mathcal{L}_{0,n} \xrightarrow{\Phi_n} (U_q(\mathfrak{g})^{\otimes n})^{lf} \xrightarrow{\psi_n^{-1}} U_q^{lf}(\mathfrak{g})^{\otimes n}, \tag{30}$$

where $lf$ means respectively locally finite for the action $ad_n^r$ of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})^{\otimes n}$, and locally finite for the action $ad^r$ of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$. An explicit basis of $\mathcal{M}_{0,n}$ is described in Proposition 6.22 of [28].

Finally, let us point out here two important consequences of (22). First, $\Phi_n$ yields isomorphisms between centers, $\mathcal{Z}(\mathcal{L}_{0,n}) \cong \mathcal{Z}(U_q)^{\otimes n}$ and $\mathcal{Z}(\mathcal{L}_{0,n}^{U_q}) \cong \mathcal{Z}((U_q^{\otimes n})^{U_q})$, where one can show that ([28], Lemma 6.29)

$$\mathcal{Z}((U_q^{\otimes n})^{U_q}) \cong \Delta^{(n)}(\mathcal{Z}(U_q)) \otimes_{\mathbb{C}(q)} \mathcal{Z}(U_q)^{\otimes n}. \tag{31}$$

Second, $\mathcal{L}_{0,n}$ (and therefore $\mathcal{M}_{0,n}$) has no non-trivial zero divisors because of the isomorphisms $\Phi_n\colon \mathcal{L}_{0,n} \to (U_q^{\otimes n})^{lf} \subset U_q^{\otimes n}$ and $U_q^{\otimes n} \cong U_q(\mathfrak{g}^{\oplus n})$, and the fact that $U_q(\mathfrak{g}^{\oplus n})$ has no non-trivial zero divisors (proved eg. in [39]).

2.2. **Integral forms and specializations.** Let $A = \mathbb{C}[q, q^{-1}]$. We call integral form of a (Hopf) $\mathbb{C}(q)$-algebra $H$ a (Hopf) $A$-subalgebra ${}_AH$ such that the canonical map ${}_AH \otimes_A \mathbb{C}(q) \to H$ is an isomorphism. Note that the standard notion of integral form of $\mathbb{C}(q)$-algebra uses $\mathbb{Z}[q, q^{-1}]$ instead of $\mathbb{C}[q, q^{-1}]$; our choice is made for simplicity ($\mathbb{C}[q, q^{-1}]$ is a principal ideal domain, whereas $\mathbb{Z}[q, q^{-1}]$ is not).

2.2.1. *Definitions.* The *unrestricted* integral form of $U_q$ is the $A$-subalgebra $U_A = U_A(\mathfrak{g})$ introduced by De Concini–Kac–Procesi in [41], Section 12 (and in a differently normalized form in [39] and [40]). It is the smallest $A$-subalgebra of $U_q$ which contains the elements $(i = 1, \dots, m)$

$$\bar{E}_i = (q_i - q_i^{-1})E_i\ ,\ \bar{F}_i = (q_i - q_i^{-1})F_i\ , L_i\ ,\ L_i^{-1} \tag{32}$$

and is stable under the action of $\mathcal{B}(\mathfrak{g})$ given by the Lusztig automorphisms (see (1)). Recall the root vectors $E_{\beta_k}, F_{\beta_k}$ defined in (1). Let us put $q_\beta := q^{(\beta,\beta)/2}$. The algebra $U_A$ is a free $A$-module with basis the monomials $\bar{E}_{\beta_1}^{p_1} \dots \bar{E}_{\beta_N}^{p_N} K_\lambda \bar{F}_{\beta_N}^{n_N} \dots \bar{F}_{\beta_1}^{n_1}$, where $\lambda \in P$ and we set

$$\bar{E}_{\beta_k} = (q_{\beta_k} - q_{\beta_k}^{-1})E_{\beta_k}\ , \bar{F}_{\beta_k} = (q_{\beta_k} - q_{\beta_k}^{-1})F_{\beta_k}.$$

We denote

$$U_A^{lf} := U_A \cap U_q^{lf}.$$

The unrestricted integral form of $U_q^{ad}$ is defined similarly, as the smallest $A$-subalgebra $U_A^{ad} \subset U_A$ which contains the elements $\bar{E}_i$, $\bar{F}_i$ and $K_i^{\pm 1}$, for $i = 1, \dots, m$, and is stable under the Lusztig action of $\mathcal{B}(\mathfrak{g})$.

For $\beta$ a positive root, we define the divided powers

$$E_\beta^{(k)} = \frac{E_\beta^k}{[k]_{q_\beta}!}\ ,\ F_\beta^{(k)} = \frac{F_\beta^k}{[k]_{q_\beta}!},\quad k \in \mathbb{N}.$$

The Lusztig *restricted* integral form of $U_q^{ad}$ ([72, 73], see also [35], Chapter 9.3) is the $A$-subalgebra $U_A^{res}$ generated by the elements ($i = 1, \dots, m$, $k \in \mathbb{N}^*$)

$$E_i^{(k)} = \frac{E_i^k}{[k]_{q_i}!}\ ,\ F_i^{(k)} = \frac{F_i^k}{[k]_{q_i}!}\ ,\ K_i\ ,\ K_i^{-1}.$$

The algebra $U_A^{res}$ is a free $A$-module with Poincaré-Birkhoff-Witt (PBW) basis

$$E_{\beta_1}^{(p_1)} \dots E_{\beta_N}^{(p_N)} \prod_{i=1}^m K_i^{\sigma_i} [K_i; t_i]_{q_i} F_{\beta_N}^{(n_N)} \dots F_{\beta_1}^{(n_1)} \tag{33}$$

where $\sigma_i \in \{0, 1\}$, $n_i, p_i, t_i \in \mathbb{N}$, and we set $[K_i; 0]_{q_i} := 1$ and

$$[K_i; t]_{q_i} = \prod_{s=1}^t \frac{K_i q_i^{-s+1} - K_i^{-1} q_i^{s-1}}{q_i^s - q_i^{-s}}.$$

The integral forms $U_A(\mathfrak{h})$, $U_A(\mathfrak{b}_\pm)$ and $U_A^{res}(\mathfrak{h})$, $U_A^{res}(\mathfrak{b}_\pm)$ associated to the subalgebras $\mathfrak{h}$, $\mathfrak{b}_\pm \subset \mathfrak{g}$ are the subalgebras of $U_A$ and $U_A^{res}$, respectively, defined in the obvious way. For instance the "Cartan" subalgebra $U_A^{res}(\mathfrak{h}) = U_q(\mathfrak{h}) \cap U_A^{res}$ is generated as a $A$-module by the elements $\prod_{i=1}^m K_i^{\sigma_i} [K_i; t_i]_{q_i}$.

Denote by $\mathcal{C}_A$ the category of $U_A^{res}$-modules of *type* 1, ie. free $A$-modules of finite rank which have a basis where the elements $K_i$ act diagonally with eigenvalues of the form $q_i^k$, $k \in \mathbb{Z}$ (in general, finiteness of the rank imposes eigenvalues of the form $\pm q_i^k$, $k \in \mathbb{Z}$). The category $\mathcal{C}_A$ is a rigid and tensor category. It is not semisimple, and this makes the study of $\mathcal{C}_A$ a complicated task; for this, see [28], and section 2.2.2 below. Every type 1 finite dimensional simple $U_q$-module $V_\mu$, $\mu \in P_+$, has a $U_A^{res}$-invariant full $A$-sublattice, that we denote by ${}_AV_\mu$. These $U_A^{res}$-modules form the simple objects of $\mathcal{C}_A$. Moreover $\mathcal{C}_A \otimes \mathbb{C}[q^{1/D}, q^{-1/D}]$ is a ribbon category (see Section 2.3).

The *integral* quantum function Hopf algebra $\mathcal{O}_A = \mathcal{O}_A(G)$ is the (type 1) restricted dual of $U_A^{res}$, that is, the $A$-span of the matrix coefficients $x \mapsto v^i(\pi_V(x)v_i)$, $x \in U_A^{res}$, for every module $V$ in $\mathcal{C}_A$, where $(v_i)$ is an $A$-basis of $V$ and $(v^i)$ the dual $A$-basis of the dual module $V^*$ (compare with the definition of $\mathcal{O}_q$). We can also regard $\mathcal{O}_A$ as the set of $A$-linear maps $f\colon U_A^{res} \to A$ such that $\mathrm{Ker}(f)$ contains a cofinite two sided ideal $I$, and $\prod_{s=-r}^r (K_i - q_i^s) \in I$ for some $r \in \mathbb{N}$ and every $i$. Because of the inclusions of $U_A^{res}(\mathfrak{h})$, $U_A^{res}(\mathfrak{n}_\pm)$, $U_A^{res}(\mathfrak{b}_\pm)$ in $U_A^{res}$, there are Hopf epimorphisms from $\mathcal{O}_A$ to the $A$-duals of these subalgebras, that we denote by $\mathcal{O}_A(T_G)$, $\mathcal{O}_A(U_\pm)$ and $\mathcal{O}_A(B_\pm)$ respectively.

The algebra $\mathcal{O}_A$ has been introduced by Lusztig in [72, 73]. It is an integral form of $\mathcal{O}_q$, so

$$\mathcal{O}_q = \mathcal{O}_A \otimes_A \mathbb{C}(q).$$

$\mathcal{O}_A$ is also the restricted dual of the integral form $\Gamma = \Gamma(\mathfrak{g})$ of $U_q^{ad}$ introduced by De Concini-Lyubashenko in [43], Sections 2-3; $\Gamma$ is the $A$-subalgebra of $U_q^{ad}$ generated by the elements ($i = 1, \dots, m$)

$$E_i^{(k)} = \frac{E_i^k}{[k]_{q_i}!}\ ,\ F_i^{(k)} = \frac{F_i^k}{[k]_{q_i}!}\ ,\ (K_i; t)_{q_i} = \prod_{s=1}^t \frac{K_i q_i^{-s+1} - 1}{q_i^s - 1}\ ,\ K_i^{-1}$$

where $k \in \mathbb{N}$, $t \in \mathbb{N}$ (setting $(K_i; 0)_{q_i} = 1$ by convention). Note that the definition of $\Gamma$ is less symmetric than that of $U_A^{res}$. However, $\Gamma$ contains the elements $K_i$, and the commutation

relations between the generators $E_i^{(k)}$, $F_i^{(k)}$ imply that the symmetrized elements $[K_i;t]_{q_i}$ belong to $\Gamma$. In fact, let us denote $\Gamma(\mathfrak{h}) = U_q(\mathfrak{h}) \cap \Gamma$ and $\Gamma(\mathfrak{b}_\pm) = U_q(\mathfrak{b}_\pm) \cap \Gamma$. It is proved in [43], Theorem 3.1, that $\Gamma(\mathfrak{h})$ contains $U_A^{res}(\mathfrak{h})$, and that the elements $\prod_{i=1}^m K_i^{-\sigma(t_i)}(K_i;t_i)_{q_i}$, $t_i \in \mathbb{N}$, where $\sigma(t) \in \{0,1\}$ and $\sigma(t) = t \bmod(2)$, is an $A$-basis of $\Gamma(\mathfrak{h})$. A PBW basis of $\Gamma$ is formed by the monomials

$$E_{\beta_1}^{(p_1)} \dots E_{\beta_N}^{(p_N)} \prod_{i=1}^m K_i^{-\sigma(t_i)}(K_i;t_i)_{q_i} F_{\beta_N}^{(n_N)} \dots F_{\beta_1}^{(n_1)}. \tag{34}$$

The inclusion $U_A^{res} \subset \Gamma$ is strict, for the elements $(K_i;t)_{q_i}$, $t \neq 0$, do not belong to $U_A^{res}$. However the restriction functor $\mathcal{C}_\Gamma \to \mathcal{C}_A$ is obviously an equivalence, where $\mathcal{C}_\Gamma$ is the category of $\Gamma$-modules of *type* 1, ie. free $A$-modules of finite rank which have a basis where the elements $K_i$ act diagonally with eigenvalues of the form $q_i^k$, $k \in \mathbb{Z}$. Therefore we can identify the two categories, and $\mathcal{O}_A$ with the (type 1) restricted dual of $\Gamma$. We will thus consider the $U_A^{res}$-modules ${}_AV_\mu$, $\mu \in P_+$, equally as $\Gamma$-modules. We will sometimes use $\Gamma$ instead of $U_A^{res}$ in order to make direct the connection with results of De Concini-Lyubashenko about the integral pairings $\pi_A^\pm$ considered in Section 2.3.

The *integral* form $\mathcal{L}_{0,1}^A$ of $\mathcal{L}_{0,1}$ is defined as the $U_A^{res}$-module $\mathcal{O}_A$ endowed with the product of $\mathcal{L}_{0,1}$. The *integral* form $\mathcal{L}_{0,n}^A$ of $\mathcal{L}_{0,n}$ is the braided tensor product of $n$ copies of $\mathcal{L}_{0,1}^A$; in particular $\mathcal{L}_{0,n}^A = \mathcal{O}_A^{\otimes n}$ as $U_A^{res}$-modules. That the products of $\mathcal{L}_{0,1}$ and $\mathcal{L}_{0,n}$ are well-defined over $A$ was shown in Proposition 6.9 of [28].

The *integral* quantum moduli algebra is

$$\mathcal{M}_{0,n}^A := (\mathcal{L}_{0,n}^A)^{U_A^{res}} = (\mathcal{L}_{0,n}^A)^{U_A}.$$

Finally, given $q = \epsilon \in \mathbb{C}^\times$ we define the *specializations* $U_\epsilon$, $\Gamma_\epsilon$, $\mathcal{O}_\epsilon$, $\mathcal{L}_{0,n}^\epsilon$ and $\mathcal{M}_{0,n}^{A,\epsilon}$ as the $\mathbb{C}$-algebras obtained by tensoring $U_A$, $\Gamma$, $\mathcal{O}_A$, $\mathcal{L}_{0,n}^A$ and $\mathcal{M}_{0,n}^A$ respectively with $\mathbb{C}_\epsilon$, the $A$-module $\mathbb{C}$ where $q$ acts by multiplication by $\epsilon$. Each one can also be defined as the quotient by the ideal generated by $q - \epsilon$. We find convenient to use the notations

$$(U_A^{\otimes n})_\epsilon^{U_A} := (U_A^{\otimes n})^{U_A} \otimes_A \mathbb{C}_\epsilon \ , \ (U^{\otimes n})_\epsilon^{lf} := (U_A^{\otimes n})^{lf} \otimes_A \mathbb{C}_\epsilon. \tag{35}$$

Let us stress here that when $\epsilon$ is a root of unity, taking the locally finite part and taking the specialization at $\epsilon$ are non commuting operations. Indeed, as shown by Theorem 2.22 below $U_\epsilon$ is finite over $\mathcal{Z}_0(U_\epsilon)$ and therefore all its elements are locally finite for $ad^r$; on another hand $U_\epsilon^{lf} = U_A^{lf} \otimes_A \mathbb{C}_\epsilon$ does not contain the elements $L_i$.

Similarly, taking invariants and taking the specialization at $\epsilon$ are non commuting operations when $\epsilon$ is a root of unity: indeed, it is easily checked that in this case $(U_A^{\otimes n})_\epsilon^{U_A}$ and $(U_\epsilon^{\otimes n})^{U_\epsilon}$, or $\mathcal{M}_{0,n}^{A,\epsilon} = \mathcal{M}_{0,n}^A \otimes_A \mathbb{C}_\epsilon$ and $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$, are distinct spaces. When $\epsilon$ is a root of unity, we will not consider the algebras $\mathcal{M}_{0,n}^{A,\epsilon}$ in this paper.

Arguments similar to those mentioned at the end of Section 2.1 imply that the algebras $\mathcal{L}_{0,n}^A$, $\mathcal{M}_{0,n}^A$ and $\mathcal{L}_{0,n}^{\epsilon'}$, $\mathcal{M}_{0,n}^{A,\epsilon'}$, $\epsilon' \in \mathbb{C}^\times$, have no non-trivial zero divisors (see [28], Proposition 6.11 and 6.30).

2.2.2. *Canonical basis and modified quantum groups.* Because the category $\mathcal{C}_A$ is not semisimple, it is not clear from the above definition of $\mathcal{O}_A$ whether or not it is a finitely generated algebra, or if it satisfies a Peter-Weyl $A$-module direct sum decomposition similar to (6), if $\mathcal{M}_{0,n}^A$ is a direct summand of the $A$-module $\mathcal{L}_{0,n}^A$, or if the projection map (8) may be refined to a morphism between underlying $A$-modules.

Such properties indeed hold true, and will play a key role later. We state them precisely in Proposition 2.7, Theorem 2.6, Theorem 2.10 and Proposition 2.8. These results are consequences of the existence of an $A$-basis of $\mathcal{O}_A$ with favourable properties, which implies in particular that $\mathcal{O}_A$ is a free $A$-module. In order to introduce this $A$-basis it is necessary to consider a variant of $U_q^{ad}$ introduced by Lusztig [73], called *modified quantum group*, and use the Kashiwara-Lusztig theory of canonical bases ([62, 63, 64], [73]). We are going to recall the background material step by step.

The Lusztig *modified quantum group* is the $\mathbb{C}(q)$-algebra $\dot{\mathbf{U}}$ obtained by replacing $U_q^{ad}(\mathfrak{h})$ with the direct sum of infinitely many one-dimensional algebras, generated by orthogonal idempotents $1_\lambda$ indexed by the elements $\lambda$ of the weight lattice $P$ ([73], Chapter 23). Namely, as a vector space $\dot{\mathbf{U}} = \bigoplus_{\lambda',\lambda'' \in P} {}_{\lambda'}\dot{\mathbf{U}}_{\lambda''}$, where

$$ {}_{\lambda'}\dot{\mathbf{U}}_{\lambda''} = U_q^{ad} / \Big( \sum_{\alpha \in Q} (K_\alpha - q^{(\alpha,\lambda')}) U_q^{ad} + \sum_{\alpha \in Q} U_q^{ad} (K_\alpha - q^{(\alpha,\lambda'')}) \Big). $$

Denote by $\pi_{\lambda',\lambda''} \colon U_q^{ad} \to {}_{\lambda'}\dot{\mathbf{U}}_{\lambda''}$ the canonical projection. The product of $\dot{\mathbf{U}}$ is given by $\pi_{\lambda'_1,\lambda''_1}(s)\pi_{\lambda'_2,\lambda''_2}(t) = \pi_{\lambda'_1,\lambda''_2}(st)$ if $\lambda''_1 = \lambda'_2$ and zero otherwise. Set $1_\lambda := \pi_{\lambda,\lambda}(1)$. The algebra $\dot{\mathbf{U}}$ has not unit, but the family $(1_\lambda)_{\lambda \in P}$ can be regarded as a substitute of it. Denote by $\Delta$ the collection of maps $\Delta_{\lambda'_1,\lambda'_2,\lambda''_1,\lambda''_2} \colon {}_{\lambda'_1+\lambda'_2}\dot{\mathbf{U}}_{\lambda''_1+\lambda''_2} \to {}_{\lambda'_1}\dot{\mathbf{U}}_{\lambda''_1} \otimes {}_{\lambda'_2}\dot{\mathbf{U}}_{\lambda''_2}$ such that

$$ \Delta_{\lambda'_1,\lambda'_2,\lambda''_1,\lambda''_2} \pi_{\lambda'_1+\lambda'_2,\lambda''_1+\lambda''_2} = (\pi_{\lambda'_1,\lambda''_1} \otimes \pi_{\lambda'_2,\lambda''_2}) \Delta_{U_q^{ad}}, \tag{36} $$

where $\Delta_{U_q^{ad}}$ is the coproduct of $U_q^{ad}$. We can regard $\Delta$ as a (categorically completed) coproduct $\Delta \colon \dot{\mathbf{U}} \to \dot{\mathbf{U}}^{\hat{\otimes} 2}$. There is a natural structure of $U_q^{ad}$-bimodule on $\dot{\mathbf{U}}$, defined by

$$ t' \pi_{\lambda',\lambda''}(s) t'' = \pi_{\lambda'+\nu',\lambda''-\nu''}(t' s t'') \tag{37} $$

for all $s \in U_q^{ad}$ and all elements $t', t'' \in U_q^{ad}$ of respective weights $\nu', \nu''$. This structure affords a triangular decomposition of $\dot{\mathbf{U}}$: given basis $\{b^\pm\}$ of $U_q^{ad}(\mathfrak{n}_\pm)$, the set of elements $b^+ 1_\lambda b^-$ (or $b^- 1_\lambda b^+$, or $b^+ b^- 1_\lambda$), where $\lambda \in P$, is a basis of $\dot{\mathbf{U}}$.

Given any $U_q^{ad}$-module $X$ of type 1, and any weight subspace $X^\lambda \subset X$ of weight $\lambda \in P$, one can define the action of an element $u 1_\lambda \in \dot{\mathbf{U}}$, $u \in U_q^{ad}$, on $X$ as the projection onto $X^\lambda$ followed by the action of $u$. This way, one can identify the category $\mathcal{C}$ with the one of finite dimensional *unital* $\dot{\mathbf{U}}$-modules, where unital means that all elements $1_\lambda$ act as 0 but a finite number of them, and $\sum_{\lambda \in P} 1_\lambda$ acts as the identity. It is proved in [73], Section 29.5.1, that

$$ \mathcal{O}_q = \left\{ f \colon \dot{\mathbf{U}} \to \mathbb{C}(q) \;\middle|\; \begin{array}{c} f \text{ is } \mathbb{C}(q)-\text{linear and vanishes on some} \\ \text{two} - \text{sided ideal of finite codimension of } \dot{\mathbf{U}} \end{array} \right\}. \tag{38} $$

There is an analogous realization of $\mathcal{O}_A$, of the form (see [73], sections 23.2 and 29.5.2, and [74])

$$ \mathcal{O}_A = \left\{ f \colon \dot{\mathbf{U}}_A \to A \;\middle|\; \begin{array}{c} f \text{ is } A-\text{linear and vanishes on some} \\ \text{two} - \text{sided ideal of finite corank of } \dot{\mathbf{U}}_A \end{array} \right\}, \tag{39} $$

where $\dot{\mathbf{U}}_A$ is the $A$-subalgebra of $\dot{\mathbf{U}}$ generated by the elements $E_i^{(k)} 1_\lambda$ and $F_i^{(k)} 1_\lambda$, for all $i \in \{1, \dots, m\}$, $k \in \mathbb{N}$ and $\lambda \in P$. It is a $U_A^{res}$-subbimodule of $\dot{\mathbf{U}}$, and the coproduct restricts to a map $\Delta \colon \dot{\mathbf{U}}_A \to \dot{\mathbf{U}}_A^{\hat{\otimes} 2}$. The above identification of the category $\mathcal{C}$ with the one of finite dimensional unital $\dot{\mathbf{U}}$-modules yields an identification of the category $\mathcal{C}_A$ of $U_A^{res}$-modules of type 1 with the category of $\dot{\mathbf{U}}_A$-modules of finite rank.

The key advantage of this realization of $\mathcal{O}_A$ is that $\dot{\mathbf{U}}_A$ can be equipped with a canonical $A$-basis $\dot{\mathbf{B}}$. The construction of $\dot{\mathbf{B}}$ is described in [73], Chapter 25. It relies on the Kashiwara-Lusztig *canonical basis* of $U_A^{res}(\mathfrak{n}_-)$. This last basis, denoted by $\mathbf{B}^-$, is defined in [73], Chapter 14, and [62] (a review can be found in [35], Chapter 14). It enjoys the following nice properties. Denote by $^-\colon \mathbb{C}(q) \to \mathbb{C}(q)$ the field involution such that $\overline{q} = q^{-1}$, and by $^-\colon U_q^{ad} \to U_q^{ad}$ the homomorphism of $\mathbb{C}$-algebras such that

$$\bar{E}_i = E_i \ \ \bar{F}_i = F_i \ , \ \bar{K}_\lambda = K_{-\lambda} \ , \ \overline{fx} = \bar{f}\bar{x}$$

for all $f \in \mathbb{C}(q)$, $x \in U_q^{ad}$ ($\bar{E}_i$ and $\bar{F}_i$ above, which will not appear elsewhere, should not be confused with the normalized elements in (32)). The map $^-$ yields a $\mathbb{C}$-algebra homomorphism $^-\colon \dot{\mathbf{U}} \to \dot{\mathbf{U}}$. Then we have:

(1) the elements of $\mathbf{B}^-$ are weight vectors under the adjoint action of $U_q^{ad}(\mathfrak{h})$;
(2) for every $b \in \mathbf{B}^-$, $\bar{b} = b$;
(3) for every $b, b' \in \mathbf{B}^-$, $bb' = \sum_{b'' \in \mathbf{B}^-} N_{b''}^{bb'} b''$ where $N_{b''}^{bb'} \in \mathbb{N}[q, q^{-1}]$;
(4) for every $b, b' \in \mathbf{B}^-$, $\Delta(b) = \sum_{b', b'' \in \mathbf{B}^-} C_{b'b''}^{b} b' \otimes b''$ where $C_{b'b''}^{b} \in \mathbb{N}[q, q^{-1}]$;
(5) For every $\mu \in P^+$, denoting by $v_\mu$ the highest weight vector of the $U_A^{res}$-module ${}_AV_\mu$, the elements $bv_\mu$ which are non zero, where $b \in \mathbf{B}^-$, form an $A$-basis of ${}_AV_\mu$.

In the case of $\mathfrak{g} = sl(2)$ the elements of $\mathbf{B}^-$ are just the divided powers $F^{(k)}$, $k \in \mathbb{N}$. Formulas in terms of PBW basis elements are known also for $\mathfrak{g} = sl(3)$ and $sl(4)$, and an algorithm exists in the general case (see [55] and the references therein).

Correspondingly to $\mathbf{B}^-$, the set $\mathbf{B}^+ = \omega(\mathbf{B}^-)$ is a basis of $U_A^{res}(\mathfrak{n}_+)$, where $\omega\colon U_q^{ad} \to U_q^{ad}$ is the ($\mathbb{C}(q)$-linear) *Cartan automorphism*, defined by

$$\omega(E_i) = F_i \ , \ \omega(F_i) = E_i \ , \ \omega(K_i) = K_i^{-1}$$

for $i = 1, \dots, m$. The triangular decomposition of $\dot{\mathbf{U}}$ implies that the elements $b^+ 1_\lambda b'^-$, where $b^+ \in \mathbf{B}^+$, $b'^- \in \mathbf{B}^-$ and $\lambda \in P$, form a basis of $\dot{\mathbf{U}}$. They form in fact an $A$-basis of $\dot{\mathbf{U}}_A$, and its elements are fixed by the involution $^-\colon \dot{\mathbf{U}} \to \dot{\mathbf{U}}$.

Lusztig has constructed another $A$-basis of $\dot{\mathbf{U}}_A$, denoted $\dot{\mathbf{B}}$, and called the *canonical basis* of $\dot{\mathbf{U}}_A$. It satisfies numerous properties that we now review. Its elements are denoted by $b \diamondsuit_\lambda b'$, where $b, b' \in \mathbf{B}^-$ and $\lambda \in P$, and are related to the elements $b^+ b'^- 1_\lambda$, where $b^+ := \omega(b)$ and $b'^- := b'$, by a specific trigonal change of basis with coefficients in $A$. Although we always have $b^+ 1_\lambda$, $b'^- 1_\lambda \in \dot{\mathbf{B}}$, to our knowledge explicit formulas of the elements of $\dot{\mathbf{B}}$ as linear combinations of elements $b^+ 1_\lambda b'^-$ or $b'^- 1_\lambda b^+$ are known only for $\mathfrak{g} = sl(2)$ or $sl(3)$ (see [73], 25.3, and [33]). In the former case, identifying $P$ with $\mathbb{Z}$ and $Q$ with $2\mathbb{Z}$ the canonical basis $\dot{\mathbf{B}}$ is formed by the elements

$$E^{(k)} 1_{-n} F^{(l)} \quad \text{and} \quad F^{(l)} 1_n E^{(k)}, \quad k, l, n \in \mathbb{N}, n \geq k + l \tag{40}$$

where $E^{(k)} 1_{-n} F^{(l)} = F^{(l)} 1_n E^{(k)}$ for $n = k + l$.

We are going to review Lusztig's construction of $\dot{\mathbf{B}}$, its canonical partition $\dot{\mathbf{B}} = \cup_{\lambda \in P_+} \dot{\mathbf{B}}[\lambda]$, the dual basis $\dot{\mathbf{B}}^*$, and Kashiwara's approach to $\dot{\mathbf{B}}^*$ ([63, 64]). The latter is stated in Theorem 2.6 below. At first we need to recall the notions of based module and balanced triple; for details on these notions we refer to [73], Chapter 27, and [63] (see also [65], the sections 3.15 and 3.16 in [92], or Chapter 14 in [35] for overviews).

Denote by $\mathcal{A}_0 \subset \mathbb{C}(q)$ the ring of rational functions regular at $q = 0$. By applying the involution $^-$, put

$$\mathcal{A}_\infty = \overline{\mathcal{A}}_0.$$

Since $\mathcal{A}_0$ is the localization of $\mathbb{C}[q]$ at $q=0$, we may regard $\mathcal{A}_\infty$ as the localization of $\mathbb{C}[q^{-1}]$ at $q=\infty$.

Let us recall briefly the definition of crystal basis (see [62]). Denote by $U_q^{ad}(\mathfrak{g})_i$ the subalgebra of $U_q^{ad}(\mathfrak{g})$ generated by $E_i$, $F_i$ and $K_i^{\pm 1}$; thus $U_q^{ad}(\mathfrak{g})_i$ is isomorphic to $U_{q_i}(sl(2))$. Let $M$ be a $U_q^{ad}$-module of type 1. Denote $M^\zeta$ the subspace of $M$ of weight $\zeta \in P$. For every $i=1,\dots,m$ we can regard $M$ as a $U_q^{ad}(\mathfrak{g})_i$-module, so $M \cong \oplus_j V_{\lambda_j}$ for some simple $U_q^{ad}(\mathfrak{g})_i$-modules $V_{\lambda_j}$. These being generated by primitive weight vectors, the PBW basis of $U_q^{ad}(\mathfrak{g})_i$ yields

$$M = \bigoplus_{\zeta\in P} \bigoplus_{0\leq n\leq(\check{\alpha}_i,\zeta)} F_i^{(n)}(Ker(E_i)\cap M^\zeta).$$

The *Kashiwara operators* $\tilde{e}_i, \tilde{f}_i$ are the endomorphisms of $M$ defined by, for every $v \in Ker(E_i)\cap M^\zeta$ and $0\leq n\leq(\check{\alpha}_i,\zeta)$,

$$\tilde{f}_i(F_i^{(n)}v) = F_i^{(n+1)}v \ , \ \tilde{e}_i(F_i^{(n)}v) = F_i^{(n-1)}v. \tag{41}$$

A *crystal basis of $M$ at $q=0$* consists of a pair $(\mathcal{L},\mathcal{B})$, where:

- $\mathcal{L}$ is a free $\mathcal{A}_0$-sublattice of $M$ such that the canonical map $\mathcal{L}\otimes_{\mathcal{A}_0}\mathbb{C}(q)\to M$ is an isomorphism;
- $\mathcal{B}$ is a basis of the $\mathbb{C}$-vector space $\mathcal{L}/q\mathcal{L}$;
- $\mathcal{L}=\oplus_{\zeta\in P}\mathcal{L}^\zeta$ and $\mathcal{B}=\coprod_{\zeta\in P}(\mathcal{B}\cap\mathcal{L}^\zeta/q\mathcal{L}^\zeta)$, where $\mathcal{L}^\zeta=\mathcal{L}\cap M^\zeta$;
- for every $i=1,\dots,m$ the Kashiwara operators $\tilde{e}_i,\tilde{f}_i$ preserve $\mathcal{L}$, and the induced maps on $\mathcal{L}/q\mathcal{L}$ send $\mathcal{B}$ into $\mathcal{B}\cup\{0\}$, and satisfy $b'=\tilde{f}_i(b)$ if and only if $b=\tilde{e}_i(b')$ for every $b,b'\in B$.

Crystal basis at $q=\infty$ are defined similarly, by replacing $\mathcal{A}_0$ with $\mathcal{A}_\infty$.

A *based module* consists of a pair $(M,B)$ where $M$ is a $U_q^{ad}$-module of type 1 endowed with a $\mathbb{C}(q)$-basis $B$ such that the following conditions hold:

(i) For every weight $\zeta\in P$, the set $B\cap M^\zeta$ is a basis of the weight subspace $M^\zeta\subset M$.

(ii) The $A$-module ${}_AM$ generated by $B$ is stable under $U_A^{res}$.

We will denote by $\mathcal{L}_M$ the $\mathcal{A}_0$-submodule of $M$ generated by $B$, and by $\bar{\mathcal{L}}_M$ the $\mathcal{A}_\infty$-submodule of $M$ generated by $B$.

(iii) The $\mathbb{C}$-linear involution $^-\colon M\to M$ defined by $\overline{fb}=\overline{f}b$ for all $f\in\mathbb{C}(q)$ and $b\in B$ is compatible with the action of $U_q^{ad}$ in the sense that $\overline{xm}=\bar{x}\bar{m}$ for all $x\in U_q^{ad}$, $m\in M$.

(iv) The $\mathcal{A}_\infty$-submodule $\bar{\mathcal{L}}_M$ of $M$ together with the image of $B$ in $\bar{\mathcal{L}}_M/q^{-1}\bar{\mathcal{L}}_M$ forms a crystal basis of $M$ at $q=\infty$.

If $(M,B)$ is a based module, we will denote by $\overline{\mathcal{B}}$ the image of $B$ in $\bar{\mathcal{L}}_M/q^{-1}\bar{\mathcal{L}}_M$. From the notion of balanced triple that we recall now, denoting by $\mathcal{B}$ the image of $B$ in $\mathcal{L}_M/q\mathcal{L}_M$, we see that $(\mathcal{L}_M,\mathcal{B})$ is a crystal basis at $q=0$.

Indeed, consider more generally a $\mathbb{C}(q)$-vector space $V$, finite dimensional or not, a sub-$A$-module ${}_AV$, a sub-$\mathcal{A}_0$-module ${}_{\mathcal{A}_0}V$ and a sub-$\mathcal{A}_\infty$-module ${}_{\mathcal{A}_\infty}V$ satisfying the conditions (all isomorphisms being the canonical maps)

$$V\cong\mathbb{C}(q)\otimes_A {}_AV \ , \ V\cong\mathbb{C}(q)\otimes_{\mathcal{A}_0}{}_{\mathcal{A}_0}V \ , \ V\cong\mathbb{C}(q)\otimes_{\mathcal{A}_\infty}{}_{\mathcal{A}_\infty}V.$$

Denote the $\mathbb{C}$-vector space $E:={}_AV\cap{}_{\mathcal{A}_0}V\cap{}_{\mathcal{A}_\infty}V$. Then $({}_AV,{}_{\mathcal{A}_0}V,{}_{\mathcal{A}_\infty}V)$ is a *balanced triple* ([62, 63]) if the canonical maps

$$A\otimes_{\mathbb{C}}E\to{}_AV \ , \ \mathcal{A}_0\otimes_{\mathbb{C}}E\to{}_{\mathcal{A}_0}V \ , \ \mathcal{A}_\infty\otimes_{\mathbb{C}}E\to{}_{\mathcal{A}_\infty}V \tag{42}$$

are isomorphisms. Equivalently, $({}_A V, {}_{\mathcal{A}_0}V, {}_{\mathcal{A}_\infty}V)$ is balanced if and only if the canonical map $E \to {}_{\mathcal{A}_0}V/q{}_{\mathcal{A}_0}V$ is an isomorphism, if and only if the canonical map $E \to {}_{\mathcal{A}_\infty}V/q^{-1}{}_{\mathcal{A}_\infty}V$ is an isomorphism ([63], Lemma 2.1.1).

Given a based module $(M, B)$, the elements of $B$ are weight vectors and $\overline{b} = b$ for every $b \in B$. Also, if an element $m \in {}_A M$ satisfies $\overline{m} = m$ and $m \in B + q^{-1}\bar{\mathcal{L}}_M$, then $m \in B$ (see 27.1.5 in [73] for details on this fact). It follows that the canonical quotient map

$$ {}_A M \cap \mathcal{L}_M \cap \bar{\mathcal{L}}_M \to \bar{\mathcal{L}}_M/q^{-1}\bar{\mathcal{L}}_M \tag{43}$$

is an isomorphism of $\mathbb{C}$-vector spaces. This provides another way of viewing based modules: by (43), $({}_A M, \mathcal{L}_M, \bar{\mathcal{L}}_M)$ is a balanced triple, and by (42) the $A$-lattice ${}_A M$ is completely determined by the crystal base $(\bar{\mathcal{L}}_M, \overline{\mathcal{B}})$. We will say that $(\bar{\mathcal{L}}_M, \overline{\mathcal{B}})$ (or the corresponding crystal base $(\mathcal{L}_M, \mathcal{B})$ at $q = 0$) is *melted into* the based module $(M, B)$.

We will indifferently apply the notion of based module to finite-dimensional unital $\dot{\mathbf{U}}$-modules, since they are equivalent to $U_q^{ad}$-modules of type 1.

Every module $V_\mu$, $\mu \in P^+$, supports a structure of based module ([73], Section 14.4.10, [62]); the corresponding basis, called *canonical basis* and that we will denote by $\underline{\mathbf{B}}_\mu$, is formed by the elements $bv_\mu \in {}_A V_\mu$ which are non zero, where $b \in \mathbf{B}^-$ and $v_\mu$ is the canonical highest weight vector of $V_\mu$, corresponding to the coset of $1 \in U_q^{ad}(\mathfrak{n}_-)$ in the Verma module construction of $V_\mu$. Note that the involution $^- \colon V_\mu \to V_\mu$ defined by (iii) above is indeed an automorphism, for the space $V_\mu$ with action of $U_q^{ad}$ defined by $x \cdot v := \bar{x}v$, for all $x \in U_q^{ad}$, $v \in V_\mu$, has highest weight $\mu$, and is thus isomorphic to $V_\mu$ via the map $^-$. The crystal base $(\mathcal{L}_\mu^{low}, \mathcal{B}_\mu^{low})$ at $q = 0$ is formed by the $\mathcal{A}_0$-sublattice $\mathcal{L}_\mu^{low}$ of $V_\mu$ generated by $\underline{\mathbf{B}}_\mu$ (which is eventually the same as the $\mathcal{A}_0$-sublattice generated by the vectors of the form $\tilde{f}_{i_1} \circ \ldots \circ \tilde{f}_{i_k}(v_\mu)$, where $i_1, \ldots, i_k \in \{1, \ldots, m\}$), and $\mathcal{B}_\mu^{low}$ is the set of non-zero images of these vectors in $\mathcal{L}_\mu^{low}/q\mathcal{L}_\mu^{low}$.

There is the following uniqueness result ([62], Theorem 3):

**Theorem 2.5.** *Let $M$ be a $U_q^{ad}$-module of type 1, and $(\mathcal{L}, \mathcal{B})$ a crystal base at $q = 0$ of $M$. Then there exists a $\mathbb{C}(q)$-isomorphism $M \to \oplus_j V_{\lambda_j}$ by which $(\mathcal{L}, \mathcal{B})$ is $\mathcal{A}_0$-isomorphic to $\oplus_j(\mathcal{L}_{\lambda_j}^{low}, \mathcal{B}_{\lambda_j}^{low})$.*

The based modules form a category. Given based modules $(M, B)$ and $(M', B')$, a morphism of $U_q^{ad}$-modules $f \colon M \to M'$ is a morphism of based modules if:

(a) $f(b) \in B' \cup \{0\}$ for any $b \in B$;
(b) $B \cap Ker(f)$ is a basis of $Ker(f)$.

The direct sum of based modules $(M, B)$ and $(M', B')$ is a based module $(M \oplus M', B \cup B')$; and a submodule $M'$ of a based module $(M, B)$ spanned over $\mathbb{C}(q)$ by a subset $B'$ of $B$ forms a based module $(M', B')$. The quotient module $M/M'$ together with the image of $B \setminus B'$ is then a based module.

The tensor product of based modules $(M, B)$, $(M', B')$ is also defined. Namely, consider the $\mathbb{C}$-linear map $\Psi \colon M \otimes M' \to M \otimes M'$ defined by $\Psi(m \otimes m') = \hat{R}^{-1}(\bar{m} \otimes \bar{m}')$, where $\hat{R} = \Theta^{-1}R$, see (4) (note that, as we use the coproduct opposite to [73] our quasi-R-matrix is $\hat{R}^{-1}$). It can be checked that $\Psi$ is an involution compatible with the action of $\dot{\mathbf{U}}$ in the sense of (iii) above in the definition of based module. Moreover, denote by $\mathcal{L}_{M,M'}$ the $\mathbb{C}[q^{-1}]$-submodule of $M \otimes M'$ spanned by the basis elements $b \otimes b'$, where $b \in B$, $b' \in B'$. It is shown in [73], Section 27.3, that for every couple $(b, b') \in B \times B'$ there is a unique element $b \diamondsuit b' \in \mathcal{L}_{M,M'}$ such that

(a) $\Psi(b\Diamond b') = b\Diamond b'$,
(b) $b\Diamond b' - b\otimes b' \in q^{-1}\mathcal{L}_{M,M'}$.

Moreover, $B_\Diamond = \{b\Diamond b', b\in B, b'\in B'\}$ is a basis of $M\otimes M'$, a $\mathbb{C}[q^{-1}]$-basis of $\mathcal{L}_{M,M'}$, a $\mathbb{C}[q,q^{-1}]$-basis of the $\mathbb{C}[q,q^{-1}]$-module ${}_A\mathcal{L}_{M,M'}$ of $M\otimes M'$ generated by the elements $b\otimes b'$, where $b\in B$, $b'\in B'$, and $(M\otimes M', B_\Diamond)$ is a based module.

This construction of $B_\Diamond$ is associative. Since $(V_\mu, \underline{\mathbf{B}}_\mu)$ is for every $\mu\in P_+$ a based module, it follows that any tensor product $M$ of a finite number of the simple modules $V_\mu$ is naturally a based module. The corresponding basis is called *the canonical basis* of $M$. These canonical basis have been computed explicitly in [53] in the case $\mathfrak{g} = sl(2)$.

Consider now the $U_q^{ad}$-module ${}^\omega V_\mu$ with underlying space $V_\mu$, $\mu\in P_+$, and action defined by $x.{}_\omega v := \omega(x)v$, for every $x\in U_q^{ad}$ and $v\in V_\mu$ (as usual $\omega\colon U_q^{ad}\to U_q^{ad}$ is the Cartan automorphism). Note that there are isomorphisms ${}^\omega V_\mu \cong V_{-w_0(\mu)} \cong V_\mu^*$ (endowed with the standard left action of $U_q^{ad}$). Let us denote by ${}^\omega v_\mu$ the vector $v_\mu$ regarded in ${}^\omega V_\mu$ (ie. its canonical lowest weight vector), and by ${}^\omega\underline{\mathbf{B}}_\mu := \{b.{}_\omega{}^\omega v_\mu \neq 0, b\in\mathbf{B}^+\}$ its canonical basis. Then ${}^\omega V_{\mu'}\otimes V_{\mu''}$ has the canonical basis $\underline{\mathbf{B}}_{\mu',\mu''} := \{\underline{b}'\Diamond\underline{b}'', \underline{b}'\in{}^\omega\underline{\mathbf{B}}_{\mu'}, \underline{b}''\in\underline{\mathbf{B}}_{\mu''}\}$. Since $\underline{b}'\Diamond\underline{b}''$ is canonically determined by the elements $b', b''\in\mathbf{B}^-$ such that $\underline{b}' = \omega(b').{}_\omega{}^\omega v_{\mu'}$, $\underline{b}'' = b''v_{\mu''}$, following Lusztig we denote it $(b'\Diamond b'')_{\mu',\mu''}$.

Denote by $v_{w_0(\mu)}$ the canonical lowest weight vector of $V_\mu$, and by ${}^\omega v_{w_0(\mu)}$ the vector $v_{w_0(\mu)}$ regarded in ${}^\omega V_\mu$. It is a crucial observation that ${}^\omega v_{w_0(\mu')}\otimes v_{w_0(\mu'')}$ is a cyclic vector of ${}^\omega V_{\mu'}\otimes V_{\mu''}$ (see eg. [73], 23.3.6; note that ${}^\omega v_{w_0(\mu')}\otimes v_{w_0(\mu'')}$ plays the role of $\xi_{-\mu'}\otimes\eta_{\mu''} := {}^\omega v_{\mu'}\otimes v_{\mu''}$ in [73], because we use opposite coproducts on $U_q^{ad}$ but the factors ${}^\omega V_{\mu'}$ and $V_{\mu''}$ are ordered in the same way).

We can now state the definition of the canonical basis $\dot{\mathbf{B}}$ of $\dot{\mathbf{U}}$: each element $u$ of $\dot{\mathbf{B}}$ belongs to $\dot{\mathbf{U}}_A 1_\zeta$ for some (unique) $\zeta\in P$, and it is then uniquely determined by the property that, for every $\mu'$, $\mu''\in P^+$ such that $w_0(\mu''-\mu') = \zeta$, we have

$$u({}^\omega v_{w_0(\mu')}\otimes v_{w_0(\mu'')}) = (b'\Diamond b'')_{\mu',\mu''} \tag{44}$$

for some $(b'\Diamond b'')_{\mu',\mu''}\in\underline{\mathbf{B}}_{\mu',\mu''}$ ([73], Chapter 25.2). We write $u = b'\Diamond_\zeta b''$, and as in [74] we denote by $\dot{\mathbf{B}}_{\mu',\mu''}$ the finite subset of $\dot{\mathbf{B}}$ which is in bijection with $\underline{\mathbf{B}}_{\mu',\mu''}$ under the map $u\mapsto u({}^\omega v_{w_0(\mu')}\otimes v_{w_0(\mu'')})$. So

$$\dot{\mathbf{B}} = \cup_{\mu',\mu''\in P_+}\ \dot{\mathbf{B}}_{\mu',\mu''}\ . \tag{45}$$

Note in particular that $\dot{\mathbf{B}}$ is formed by weight vectors for the left and right action of $U_q^{ad}(\mathfrak{h})$ (defined as usual by (37)).

In a sense, one can view $\dot{\mathbf{U}}$ as the projective limit of an inverse system formed by the $(U_q^{ad}\otimes U_q^{ad})$-modules ${}^\omega V_{\mu'}\otimes V_{\mu''}$, where $\mu', \mu''\in P^+$; then $\dot{\mathbf{B}}$ is the basis resulting from the corresponding inverse system of basis $\{\dot{\mathbf{B}}_{\mu',\mu''}\}_{\mu',\mu''}$.

Lusztig has produced a partition of $\dot{\mathbf{B}}$ as follows. First, consider the situation of a based module $(M,B)$. For every $\lambda\in P_+$ denote by $M[\lambda]$ the sum of the simple submodules of $M$ isomorphic to $V_\lambda$ (ie. its isotypical component). Set

$$M[\geq\lambda] = \oplus_{\lambda'\geq\lambda} M[\lambda']. \tag{46}$$

Then, for every base element $b\in B$ there is a unique $\lambda\in P_+$ such that $b\in M[\geq\lambda]$ and $\lambda$ is maximal with this property ([73], 27.2). Denote by $B[\lambda]$ the set of all $b\in B$ that give rise to $\lambda\in P_+$ in this way. Clearly the sets $B[\lambda]$, $\lambda\in P_+$, form a partition of $B$.

Now, given $b\in\dot{\mathbf{B}}$, let $\zeta\in P$ be the unique weight such that $b\in\dot{\mathbf{U}}_A 1_\zeta$, and let $\mu'$, $\mu''\in P^+$ be such that $w_0(\mu''-\mu') = \zeta$, and $(\check{\alpha}_i, \mu')$ is large enough for all $i = 1,\dots,m$ so that

$u({}^{\omega}v_{w_0(\mu')} \otimes v_{w_0(\mu'')})$ is non zero. This element belongs to the canonical basis $\underline{\mathbf{B}}_{\mu',\mu''}$ of ${}^{\omega}V_{\mu'} \otimes V_{\mu''}$, and therefore to one of the subsets $\underline{\mathbf{B}}_{\mu',\mu''}[\lambda]$, for a unique $\lambda \in P_+$. It is a result that $\lambda$ does not depend on the choice of $\mu', \mu''$ (see [73], 29.1.1). Hence there is a well-defined map $\dot{\mathbf{B}} \to P_+$, $b \mapsto \lambda$. Denoting by $\dot{\mathbf{B}}[\lambda]$ the fiber of this map, we thus obtain a partition

$$\dot{\mathbf{B}} = \coprod_{\lambda \in P_+} \dot{\mathbf{B}}[\lambda]. \tag{47}$$

The sets $\dot{\mathbf{B}}[\lambda]$ are called 2-*sided cells*. They are finite sets and have the following remarkable properties. For every $\lambda \in P_+$ denote by $\dot{\mathbf{U}}[\geq \lambda]$ and $\dot{\mathbf{U}}[> \lambda]$ the subspaces of $\dot{\mathbf{U}}$ spanned by $\coprod_{\lambda' \geq \lambda} \dot{\mathbf{B}}[\lambda']$ and $\coprod_{\lambda' > \lambda} \dot{\mathbf{B}}[\lambda']$ respectively. Then $\dot{\mathbf{U}}[\geq \lambda]$ (respectively $\dot{\mathbf{U}}[> \lambda]$) consists of the elements $u \in \dot{\mathbf{U}}$ such that if $u$ acts on $V_\mu$ by a non zero linear map, then $\mu \geq \lambda$ (respectively $\mu > \lambda$) (([73], 29.1.3-29.1.4). Both $\dot{\mathbf{U}}[\geq \lambda]$ and $\dot{\mathbf{U}}[> \lambda]$ are two-sided ideals of $\dot{\mathbf{U}}$. Moreover the algebra homomorphism $\pi_\lambda \colon \dot{\mathbf{U}}[\geq \lambda] \to End(V_\lambda)$ given by the $\dot{\mathbf{U}}$-module structure on $V_\lambda$ descends to an algebra and $U_q^{ad}$-bimodule isomorphism (keeping the same notation)

$$\bar{\pi}_\lambda \colon \dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda] \to End(V_\lambda) \tag{48}$$

([73], Proposition 29.2.2).

For instance, when $\mathfrak{g} = sl(2)$ the 2-sided cell $\dot{\mathbf{B}}[n]$ associated to the simple $U_q^{ad}(sl(2))$-module of type 1 and dimension $n+1$ is the set ([73], 29.4.3)

$$\dot{\mathbf{B}}[n] = \{E^{(k)} 1_{-n} F^{(l)}, n \geq k+l\} \cup \{F^{(l)} 1_n E^{(k)}, n \geq k+l\}, \tag{49}$$

with the identification $E^{(k)} 1_{-n} F^{(l)} = F^{(l)} 1_n E^{(k)}$ when $n = k+l$. As we are mainly interested in $\mathcal{O}_A$ we are going to describe the dual partition of $\dot{\mathbf{B}}^*$; see Theorem 2.6. The duality with (47) is discussed after that theorem.

First we follow the approach of Kashiwara [63, 64]. For every $\lambda \in P_+$ denote by $V_\lambda^r$ the dual space of $V_\lambda$ endowed with its natural structure of right $U_q^{ad}$-module, defined by

$$(fx)(v) = f(xv)$$

for every $f \in V_\lambda^r$, $x \in U_q^{ad}$, $v \in V_\lambda$. Clearly $V_\lambda^r$ is a simple module of highest weight $\lambda$. Let $\varphi \colon U_q^{ad} \to U_q^{ad}$ be the anti-automorphism of $\mathbb{C}(q)$-algebra given by $\varphi(E_i) = F_i$, $\varphi(F_i) = E_i$, $\varphi(K_\lambda) = K_\lambda$. By using $\varphi$ any right $U_q^{ad}$-module can be considered as a left $U_q^{ad}$-module. In particular, by the Verma module construction of $V_\lambda$ it follows

$$V_\lambda^r \cong U_q^{ad}/\Big(\sum_{\mu \in P_+} (K_\mu - q^{(\lambda,\mu)}) U_q^{ad} + \sum_{i=1}^{m} E_i^{1+(\check{\alpha}_i,\lambda)} U_q^{ad}\Big),$$

and $\varphi$ affords an isomorphism of the right module $V_\lambda^r$ with the *left* module $V_\lambda$. We will denote by $f_\lambda$ the highest weight vector of $V_\lambda^r$.

The space $V_\lambda^r \otimes V_\lambda$ can be identified with $End(V_\lambda)^*$, and thus acquires by duality a natural structure of $U_q^{ad}$-bimodule (or equivalently left $U_q^{ad} \otimes (U_q^{ad})^{op}$-module); the left and right actions are given by

$$x(f \otimes v)y = fy \otimes xv \tag{50}$$

for every $x, y \in U_q^{ad}$, $f \in V_\lambda^r$, $v \in V_\lambda$. The space $V_\lambda^r \otimes V_\lambda$ also acquires by duality a natural "upper" crystal structure over $U_q^{ad} \otimes (U_q^{ad})^{op}$, as we explain now. Denote by $\langle\ ,\ \rangle_\lambda \colon V_\lambda \times V_\lambda \to \mathbb{C}(q)$ the unique symmetric bilinear form such that

$$\langle v_\lambda, v_\lambda \rangle_\lambda = 1 \quad \text{and} \quad \langle \varphi(x)u, v \rangle_\lambda = \langle u, xv \rangle_\lambda \tag{51}$$

for every $u, v \in V_\lambda$ and $x \in U_q^{ad}$. Recall the crystal base $(\mathcal{L}_\mu^{low}, \mathcal{B}_\mu^{low})$ at $q = 0$ introduced before Theorem 2.5. In Kashiwara's terminology ([62, 63]) the pair $(\mathcal{L}_\lambda^{low}, \mathcal{B}_\lambda^{low})$ is the *lower crystal base* of $V_\lambda$ at $q = 0$. Applying the involution $^-: V_\lambda \to V_\lambda$ one obtains the lower crystal base $(\overline{\mathcal{L}_\lambda^{low}}, \overline{\mathcal{B}_\lambda^{low}})$ at $q = \infty$. Because the canonical bases are determined by the crystal bases (see the discussion about (43)), we call $(V_\lambda, \underline{\mathbf{B}}_\lambda)$ the *lower* based module of $V_\lambda$, and $\underline{\mathbf{B}}_\lambda$ the *lower canonical basis* of $V_\lambda$.

Put

$$
\begin{aligned}
{}_AV_\lambda^{up} :=& \{v \in V_\lambda, \langle v, {}_AV_\lambda\rangle_\lambda \subset A\},\\
\mathcal{L}_\lambda^{up} :=& \{v \in V_\lambda, \langle v, \mathcal{L}_\lambda^{low}\rangle_\lambda \subset \mathcal{A}_0\},\\
\overline{\mathcal{L}_\lambda^{up}} :=& \{v \in V_\lambda, \langle v, \overline{\mathcal{L}_\lambda^{low}}\rangle_\lambda \subset \mathcal{A}_\infty\}.
\end{aligned} \tag{52}
$$

Then $({}_AV_\lambda^{up}, \mathcal{L}_\lambda^{up}, \overline{\mathcal{L}_\lambda^{up}})$ is a balanced triple ([63], Lemma 4.2.1). Denote by $\mathcal{B}_\lambda^{up}$ the basis of $\mathcal{L}_\lambda^{up}/q\mathcal{L}_\lambda^{up}$ dual to $\mathcal{B}_\lambda^{low}$ by the induced pairing $\langle\ ,\ \rangle_\lambda \colon \mathcal{L}_\lambda^{low}/q\mathcal{L}_\lambda^{up} \times \mathcal{L}_\lambda^{low}/q\mathcal{L}_\lambda^{low} \to \mathbb{C}$. The pair $(\mathcal{L}_\lambda^{up}, \mathcal{B}_\lambda^{up})$ is the *upper crystal base* of $V_\lambda$ at $q = 0$. The weight spaces of the $\mathcal{A}_0$-modules $\mathcal{L}_\lambda^{low}$ and $\mathcal{L}_\lambda^{up}$ are related by

$$
(\mathcal{L}_\lambda^{up})^\mu = q^{\frac{(\lambda,\lambda)}{2} - \frac{(\mu,\mu)}{2}} (\mathcal{L}_\lambda^{low})^\mu \quad ,\ \mu \in P. \tag{53}
$$

Correspondingly, denoting $(\mathcal{B}_\lambda^{up})^\mu := \mathcal{B}_\lambda^{up} \cap (\mathcal{L}_\lambda^{up})^\mu$ and $(\mathcal{B}_\lambda^{low})^\mu := \mathcal{B}_\lambda^{low} \cap (\mathcal{L}_\lambda^{low})^\mu$, we have ([62], [63], eq. (4.2.9))

$$
(\mathcal{B}_\lambda^{up})^\mu = q^{\frac{(\lambda,\lambda)}{2} - \frac{(\mu,\mu)}{2}} (\mathcal{B}_\lambda^{low})^\mu.
$$

The $A$-module ${}_AV_\lambda^{up}$ is characterized by the following two properties ([63], eq. (4.2.10)-(4.2.12)):

$$
\begin{aligned}
({}_AV_\lambda^{up})^\lambda &= \mathbb{C}[q, q^{-1}]v_\lambda\\
({}_AV_\lambda^{up})^\mu &= \{v \in V_\lambda \mid U_A^{res}(\mathfrak{n}^+)^{\lambda-\mu} v \in \mathbb{C}[q, q^{-1}]v_\lambda\}
\end{aligned}
$$

where $U_A^{res}(\mathfrak{n}^+)^\gamma = \{u \in U_A^{res}(\mathfrak{n}^+) \mid \forall \nu \in P,\ K_\nu u K_\nu^{-1} = q^{(\nu,\gamma)}u\}$. Denote by $\underline{\mathbf{B}}_\lambda^{up}$ the inverse image of $\mathcal{B}_\lambda^{up}$ by the isomorphism ${}_AV_\lambda^{up} \cap \mathcal{L}_\lambda^{up} \cap \overline{\mathcal{L}_\lambda^{up}} \to \mathcal{L}_\lambda^{up}/q\mathcal{L}_\lambda^{up}$. By (42) the set $\underline{\mathbf{B}}_\lambda^{up}$ is a basis of ${}_AV_\lambda^{up}$; we call it the *upper canonical basis* of $V_\lambda$. In the Appendix we describe in details the $sl(2)$ case.

Similarly, the right module $V_\lambda^r$ with its canonical basis $\underline{\mathbf{B}}_\lambda^r = \{f_\lambda b, b \in \mathbf{B}^+\} \setminus \{0\}$ has the lower crystal base $(\mathcal{L}_\lambda^{r\,low}, \mathcal{B}_\lambda^{r\,low})$, and it supports a balanced triple $({}_AV_\lambda^{r\,up}, \mathcal{L}_\lambda^{r\,up}, \overline{\mathcal{L}_\lambda^{r\,up}})$ defined again by duality. We denote by $(\mathcal{L}_\lambda^{r\,up}, \mathcal{B}_\lambda^{r\,up})$ and $\underline{\mathbf{B}}_\lambda^{r\,up}$ the corresponding crystal base and upper canonical basis of $V_\lambda^r$ respectively.

It follows that $({}_AV_\lambda^{r\,up} \otimes_A {}_AV_\lambda^{up}, \mathcal{L}_\lambda^{r\,up} \otimes_{\mathcal{A}_0} \mathcal{L}_\lambda^{up}, \overline{\mathcal{L}_\lambda^{r\,up}} \otimes_{\mathcal{A}_\infty} \overline{\mathcal{L}_\lambda^{up}})$ is a balanced triple; equivalently $V_\lambda^r \otimes V_\lambda$ with the bimodule structure (50) and the basis $\underline{\mathbf{B}}_\lambda^{r\,up} \otimes \underline{\mathbf{B}}_\lambda^{up}$ is a based $(U_q^{ad} \otimes (U_q^{ad})^{op})$-module.

Denote again by $\langle ., . \rangle \colon \mathcal{O}_q \times \dot{\mathbf{U}} \to \mathbb{C}(q)$ the pairing of $U_q^{ad}$-bimodules induced by the canonical pairing $\langle\ ,\ \rangle \colon \mathcal{O}_q \times U_q^{ad} \to \mathbb{C}(q)$, and let $\Phi_\lambda \colon V_\lambda^r \otimes V_\lambda \to \mathcal{O}_q$, $\lambda \in P_+$, be the "matrix coefficient" map, ie.

$$
\langle \Phi_\lambda(f \otimes v), x\rangle = \langle fx, v\rangle_\lambda \tag{54}
$$

for every $f \in V_\lambda^r$, $x \in U_q^{ad}$, $v \in V_\lambda$. The map $\Phi := \oplus_{\lambda \in P_+} \Phi_\lambda$ is an isomorphism of $U_q^{ad}$-bimodules, so let us use it to identify $\mathcal{O}_q$ with $\oplus_{\lambda \in P_+} V_\lambda^r \otimes V_\lambda$ (which is the content of the

Peter-Weyl decomposition (6)). Define

$$\mathcal{L}(\mathcal{O}_q) = \bigoplus_{\lambda\in P_+} \mathcal{L}_\lambda^{r\,up} \otimes_{\mathcal{A}_0} \mathcal{L}_\lambda^{up}\ ,\ \mathcal{B}(\mathcal{O}_q) := \coprod_{\lambda\in P_+} \mathcal{B}_\lambda^{r\,up} \otimes \mathcal{B}_\lambda^{up}$$

$$\overline{\mathcal{L}}(\mathcal{O}_q) = \bigoplus_{\lambda\in P_+} \overline{\mathcal{L}_\lambda^{r\,up}} \otimes_{\mathcal{A}_\infty} \overline{\mathcal{L}_\lambda^{up}}\ ,\ \overline{\mathcal{B}}(\mathcal{O}_q) := \coprod_{\lambda\in P_+} \overline{\mathcal{B}_\lambda^{r\,up}} \otimes \overline{\mathcal{B}_\lambda^{up}}.$$

**Theorem 2.6.** *(i) The triple $(\mathcal{O}_A, \mathcal{L}(\mathcal{O}_q), \overline{\mathcal{L}}(\mathcal{O}_q))$ is balanced. Therefore, denoting by $G$ the inverse of the canonical map $\mathcal{O}_A \cap \mathcal{L}(\mathcal{O}_q) \cap \overline{\mathcal{L}}(\mathcal{O}_q) \to \mathcal{L}(\mathcal{O}_q)/q\mathcal{L}(\mathcal{O}_q)$, we have*

$$\mathcal{O}_A = \bigoplus_{b\in\mathcal{B}(\mathcal{O}_q)} AG(b).$$

*(ii) The basis $G(\mathcal{B}(\mathcal{O}_q)) := \{G(b), b \in \mathcal{B}(\mathcal{O}_q)\}$ coincides with the dual canonical basis $\dot{\mathbf{B}}^*$, i.e. the elements $a^* \in \mathcal{O}_A$, for every $a \in\dot{\mathbf{B}}$, defined by $a^*(a') = \delta_{a,a'}$ for every $a' \in\dot{\mathbf{B}}$. Therefore*

$$\mathcal{O}_A = \bigoplus_{b\in\dot{\mathbf{B}}} Ab^*$$

The statement (i) is Theorem 1 of [63], and (ii) is Theorem 10.1 and Proposition 10.2.2 of [64] and section 29.5 of [73]. The basis $G(\mathcal{B}(\mathcal{O}_q)) =\dot{\mathbf{B}}^*$ is called the *global basis*, or *canonical basis*, of $\mathcal{O}_q$. The proof of Theorem 2.6 (ii) in [64] (see also [65]) exhibits an isomorphism of crystals over $U_q^{ad} \otimes (U_q^{ad})^{op}$,

$$\psi\colon \mathcal{B}(\mathcal{O}_q) \to \mathcal{B}(\dot{\mathbf{U}}), \tag{55}$$

where $(\mathcal{L}(\dot{\mathbf{U}}), \mathcal{B}(\dot{\mathbf{U}}))$ is the crystal base of $\dot{\mathbf{U}}$ associated to the canonical basis $\dot{\mathbf{B}}$. The isomorphism $\psi$ satisfies $\langle G(b), G(b')\rangle = \delta_{\psi(b),b'}$ for every $b \in \mathcal{B}(\mathcal{O}_q)$, $b' \in \mathcal{B}(\dot{\mathbf{U}})$. The unit 1 of $\mathcal{O}_A$ is $(1_0)^*$; the constant structures of $\mathcal{O}_A$ are studied in [73, 74].

Let us spell out the case $\mathfrak{g} = sl(2)$. Denote by $a$, $b$, $c$, $d$ the matrix coefficients in the canonical basis $(v_+, v_- := Fv_+)$ of $V_1$, the simple $U_q^{ad}(sl(2))$-module of type 1 and dimension two, read from the top left to the bottom right. In that case of $V_1$ the upper canonical basis $\underline{\mathbf{B}}_1^{r\,up}$ and $\underline{\mathbf{B}}_1^{up}$ coincide with the lower ones (this is not true in general, see Example 2.12). The basis $\dot{\mathbf{B}}^*(sl_2)$ is formed by the monomials $c^s a^p b^r$ where $p, r, s \in \mathbb{N}$, and $c^s d^p b^r$ where $p, r, s \in \mathbb{N}$ and $p > 0$; this is stated in [63], Proposition 9.1.1 (in [43], Proposition 1.3, similar monomials are shown to form an $A$-basis of $\mathcal{O}_A(SL_2)$, but without reference to the canonical basis; see the comments before (110) below). More precisely, recall the 2-sided cells (49). A tedious though straightforward computation shows that we have the duality pairing

$$\langle c^s d^p b^r, E^{(i)} 1_{-k} F^{(j)}\rangle = \delta_{p+r+s,k}\delta_{r,i}\delta_{s,j} \quad , \qquad \langle c^s d^p b^r, F^{(j)} 1_k E^{(i)}\rangle = 0$$

$$\langle c^s a^p b^r, E^{(i)} 1_{-k} F^{(j)}\rangle = 0 \qquad , \quad \langle c^s a^p b^r\ ,\ F^{(j)} 1_k E^{(i)}\rangle = \delta_{p+r+s,k}\delta_{r,i}\delta_{s,j}.$$

A generalization of this result to the case of $\mathfrak{g} = sl_n$ can be found in [36]. Moreover, denote by $V_n$ the simple $U_q^{ad}(sl(2))$-module of type 1 and dimension $n+1$, by $(v_k)$ the canonical basis of $V_n$, by $(v^k)$ the dual basis, and by $\pi_n\colon\ \dot{\mathbf{U}}\ (sl_2) \to End(V_n)$ the representation morphism.

By using the above pairing it is readily checked that for every $0 \leq l, m \leq n$ we have:

$$
\begin{aligned}
(56)\quad & v^l(\pi_n(\cdot)\, v_m) = \\
& \sum_{\substack{0 \leq i,j,k \\ i+j \leq k \leq n}} \delta_{m+j-i,l}\delta_{-k,n-2(m+j)} \begin{bmatrix} m+j \\ j \end{bmatrix}_q \begin{bmatrix} n-m+i-j \\ i \end{bmatrix}_q \left(E^{(i)} 1_{-k} F^{(j)}\right)^* \\
& + \sum_{\substack{0 \leq i,j,k \\ i+j < k \leq n}} \delta_{m+j-i,l}\delta_{k,n-2(m-i)} \begin{bmatrix} m-i+j \\ j \end{bmatrix}_q \begin{bmatrix} n-m+i \\ i \end{bmatrix}_q \left(F^{(j)} 1_{+k} E^{(i)}\right)^*.
\end{aligned}
$$

In particular, we see in this case of $\mathfrak{g} = sl(2)$ that in general the matrix coefficients of simple $U_A^{res}$-modules of type 1 are not elements of the dual canonical basis $\dot{\mathbf{B}}^*$. Moreover, these matrix coefficients do not form a basis of $\mathcal{O}_A$. For instance, it follows from (56) that the matrix of matrix coefficients of $V_2$ has the following form:

$$
(57)\qquad \begin{pmatrix} a^2 & [2]_q ab & b^2 \\ ca & [2]_q bc + 1 & db \\ c^2 & [2]_q cd & d^2 \end{pmatrix}.
$$

The matrix coefficient $v_0^* \otimes v_0$ being equal to $[2]_q bc + 1$, this shows $bc$ cannot be expressed as a linear combination over $A$ of matrix coefficients of simple modules.

Let us discuss the $U_A^{res}$-bimodule structure of $\mathcal{O}_A$, and its relation with the partition (47). For every $\lambda \in P_+$ put

$$
(58)\qquad {}_A\dot{C}(\lambda) := \bigoplus_{b \in \dot{\mathbf{B}}[\lambda]} Ab^*
$$

and

$$
\mathcal{O}_A(\leq \lambda) := \bigoplus_{\lambda' \leq \lambda} {}_A\dot{C}(\lambda')\ ,\ \mathcal{O}_A(< \lambda) := \bigoplus_{\lambda' < \lambda} {}_A\dot{C}(\lambda').
$$

Recall that $\dot{\mathbf{U}}[\geq \lambda]$ and $\dot{\mathbf{U}}[> \lambda]$ are two-sided ideals of $\dot{\mathbf{U}}$, and the algebra (whence $U_q^{ad}$-bimodule) isomorphism $\bar{\pi}_\lambda \colon\ \dot{\mathbf{U}}[\geq \lambda]/\,\dot{\mathbf{U}}[> \lambda] \to End(V_\lambda)$ (see (48)). Since $\dot{\mathbf{B}}$ is a basis of $\dot{\mathbf{U}}_A$, the $A$-modules $\dot{\mathbf{U}}_A[\geq \lambda]$ and $\dot{\mathbf{U}}_A[> \lambda]$ spanned by $\coprod_{\lambda' \geq \lambda} \dot{\mathbf{B}}[\lambda']$ and $\coprod_{\lambda' > \lambda} \dot{\mathbf{B}}[\lambda']$ are two-sided ideals of $\dot{\mathbf{U}}_A$. Also, since $\dot{\mathbf{U}}_A$ preserves the canonical basis $\underline{\mathbf{B}}_\lambda$ of ${}_AV_\lambda$, $\bar{\pi}_\lambda$ restricts to an isomorphism of $U_A^{res}$-bimodules $\bar{\pi}_\lambda \colon\ \dot{\mathbf{U}}_A[\geq \lambda]/\,\dot{\mathbf{U}}_A[> \lambda] \to End({}_AV_\lambda)$. In [73], Chapter 29.3, Lusztig groups this isomorphism and its properties under the general term of *refined Peter-Weyl theorem*. We are going to reinterpret it in terms of $\mathcal{O}_A$.

First, observe that $\mathcal{O}_A(\leq \lambda)$ and $\mathcal{O}_A(< \lambda)$ are $U_A^{res}$-bimodules. Indeed, for instance in the case of $\mathcal{O}_A(\leq \lambda)$, if $\alpha \in \mathcal{O}_A(\leq \lambda)$, $x, y \in U_A^{res}$, and $b \in \dot{\mathbf{B}}[\mu]$ for some $\mu \not\leq \lambda$, we have $xby \in \dot{\mathbf{U}}_A[\geq \mu] = \oplus_{\eta \geq \mu} A\ \dot{\mathbf{B}}\ [\eta]$, and since $\eta \geq \mu$ implies $\eta \not\leq \lambda$ it follows that $\langle xby, \alpha \rangle = 0$, i.e $(x \rhd \alpha \lhd y)(b) = 0$. This shows $x \rhd \alpha \lhd y \in \mathcal{O}_A(\leq \lambda)$. The same proof applies as well to $\mathcal{O}_A(< \lambda)$. Moreover, the map

$$
(59)\qquad \begin{array}{rccc} d_\lambda : & \mathcal{O}_A(\leq \lambda) & \longrightarrow & Hom\Big(\dot{\mathbf{U}}_A[\geq \lambda]/\,\dot{\mathbf{U}}_A[> \lambda], A\Big) \\ & \alpha & \longmapsto & \langle \alpha, \cdot\, \rangle \end{array}
$$

descends to an isomorphism of $U_A^{res}$-bimodules $\bar{d}_\lambda$ on $\mathcal{O}_A(\leq \lambda)/\mathcal{O}_A(< \lambda)$ (with the obvious dual structure on $\dot{\mathbf{U}}_A[\geq \lambda]/\,\dot{\mathbf{U}}_A[> \lambda]$). Bijectivity of $\bar{d}_\lambda$ comes by comparing the cardinals of canonical basis: $\mathcal{O}_A(\leq \lambda)/\mathcal{O}_A(< \lambda)$ has the basis formed by the cosets of the elements of the basis $(\dot{\mathbf{B}}\ [\lambda])^*$ of ${}_A\dot{C}(\lambda')$, and $\dot{\mathbf{U}}_A[\geq \lambda]/\,\dot{\mathbf{U}}_A[> \lambda]$ the basis formed by the cosets of

the elements of $\dot{\mathbf{B}}\ [\lambda]$, all cosets being non zero and pairwise distinct. We thus get exact sequences of $U_A^{res}$-bimodules

$$0 \longrightarrow \dot{\mathbf{U}}_A[>\lambda] \longrightarrow \dot{\mathbf{U}}_A[\geq\lambda] \xrightarrow{\bar{\pi}_\lambda} End({}_AV_\lambda) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_A(<\lambda) \longrightarrow \mathcal{O}_A(\leq\lambda) \xrightarrow{(\bar{\pi}_\lambda^{-1})^*\circ d_\lambda} (End({}_AV_\lambda))^* \longrightarrow 0\ . \tag{60}$$

They split as sequences of $A$-modules, but not as sequences of bimodules. We stress that

$$(End({}_AV_\lambda))^* := Hom(End({}_AV_\lambda), A) \cong Hom({}^\omega_AV_\lambda \otimes_A {}_AV_\lambda, A) = {}_AV_\lambda^{up} \otimes_A ({}^\omega_AV_\lambda)^{up},$$

with the " $^{up}$ " structures defined in (52), and corresponding basis $\underline{\mathbf{B}}_\lambda^{up} \otimes ({}^\omega\underline{\mathbf{B}}_\lambda)^{up}$. Moreover, the exact sequence (60) shows that this $A$-module of matrix coefficients, regarded as an $A$-submodule of $\mathcal{O}_A$ by means of the coefficient map $\Phi := \oplus_{\lambda\in P_+}\Phi_\lambda$ (see (54)), is contained in $\mathcal{O}_A(\leq\lambda)$. This for all $\lambda' \leq \lambda$ yields $\oplus_{\lambda'\leq\lambda}(End({}_AV_{\lambda'}))^* \subset \mathcal{O}_A(\leq\lambda)$. Now, note that using the isomorphism $\bar{\pi}_\lambda$ we get

$$\mathrm{rank}_A(\mathcal{O}_A(\leq\lambda)) = \sum_{\lambda'\leq\lambda} \mathrm{Card}\big(\dot{\mathbf{B}}\ [\lambda']\big) = \sum_{\lambda'\leq\lambda} \mathrm{rank}({}_AV_{\lambda'})^2$$

and therefore

$$\dim_{\mathbb{C}(q)}\big(\mathcal{O}_A(\leq\lambda)\otimes_A\mathbb{C}(q)\big) = \sum_{\lambda'\leq\lambda}\dim(V_{\lambda'})^2 = \sum_{\lambda'\leq\lambda}\dim\big((C(\lambda')\big). \tag{61}$$

It follows

$$\mathcal{O}_A(\leq\lambda)\otimes_A\mathbb{C}(q) = \bigoplus_{\lambda'\leq\lambda} C(\lambda'),\ \ \mathcal{O}_A(<\lambda)\otimes_A\mathbb{C}(q) = \bigoplus_{\lambda'<\lambda} C(\lambda'). \tag{62}$$

However, note that in general ${}_A\dot{C}(\lambda)\otimes_A\mathbb{C}(q)$ is not equal to $C(\lambda)$. This is reflected in the fact that in general it is not true that ${}_A\dot{C}(\lambda)$ is a $U_A^{res}$-bimodule.

For instance, we have $1_2E \in\dot{\mathbf{B}}[2]$ (see (49)), so $((1_2E)^* \lhd E)(1_0) = \langle\Delta((1_2E)^*), E\otimes 1_0\rangle = \langle(1_2E)^*, E1_0\rangle = \langle(1_2E)^*, 1_2E\rangle = 1$ since $E1_0 = 1_2E$. Therefore $(1_2E)^* \lhd E \notin {}_A\dot{C}(2)$. Moreover, by just considering the case of $\mathfrak{g} = sl_2$, and inverting the system of identities (56) for all $0 \leq l, m \leq n$ (or simply from the identity $v_0^*\otimes v_0 = [2]_q bc+1$), we see that ${}_A\dot{C}(\lambda)$ is not an $A$-sublattice of the space $C(\lambda)$ of matrix coefficients (defined in (24)). This is emphasized by the dot notation " $\dot{}$ ".

In the case of $\mathfrak{g} = sl_2$ we can observe this in the formulas (56). Also, by projecting the matrix (57) onto $(End({}_AV_2))^*$ the entries are unchanged except the $(1,1)$ entry, which becomes $[2]_q bc$. All factors $[2]_q$ in the middle column disappear if one uses matrix coefficients in the upper canonical basis of $V_2$, which is $v_0^{up} := v_0, v_1^{up} := [2]_q^{-1}v_1, v_2^{up} := v_2$ in the notations of (56), since we have $v^l(\pi_2(\cdot)\ v_m) = [\delta_{m,1}+1]_q\langle v_l^{up},\ \cdot\ v_m^{up}\rangle$ for $l, m \in \{0,1,2\}$, where $\langle\ ,\ \rangle$ is the pairing (51). Thus, in this particular example of $(End({}_AV_2))^*$ we see explicitly the identification of the basis $(\bar{\pi}_2^*)^{-1}\circ d_2(\dot{\mathbf{B}}[2]^*)$ and $\underline{\mathbf{B}}_2^{up}\otimes({}^\omega\underline{\mathbf{B}}_2)^{up}$.

Summing up, as an $A$-module we have a direct sum decomposition

$$\mathcal{O}_A = \bigoplus_{\lambda\in P_+} {}_A\dot{C}(\lambda), \tag{63}$$

as $U_A^{res}$-bimodules we have a (directed by inclusion, and non direct) sum

$$\mathcal{O}_A = \sum_{\lambda\in P_+} \mathcal{O}_A(\leq \lambda), \tag{64}$$

and the composition factors of $\mathcal{O}_A$ are the bimodules

$$(End({}_AV_\lambda))^* \cong ({}^\omega_AV_\lambda \otimes {}_AV_\lambda)^* \tag{65}$$

for every $\lambda \in P_+$, each of multiplicity 1. Because $\dot{\mathbf{B}}$ is formed by weight vectors for the left and right action of $U_q^{ad}(\mathfrak{h})$ (see (45)), the same is true of $\dot{\mathbf{B}}^*$ and (63) can thus be refined into a weight space decomposition

$$\mathcal{O}_A = \bigoplus_{\mu,\nu\in P} \bigoplus_{\lambda\in P_+} \left({}_A\dot{C}(\lambda)\right)_{\mu,\nu}. \tag{66}$$

Now recall the property (45). Consider in particular the finite subsets $\dot{\mathbf{B}}_{0,\varpi_i}$ and $\dot{\mathbf{B}}_{\varpi_i,0}$ associated to the fundamental weights $\varpi_i$, $i = 1,\dots,m$. The map $u \mapsto u({}^\omega v_0 \otimes v_{w_0(\varpi_i)})$, $u \in \dot{\mathbf{U}}$, allows one to identify $\dot{\mathbf{B}}_{0,\varpi_i}$ with the canonical basis $\underline{\mathbf{B}}_{\varpi_i}$ of ${}^\omega V_0 \otimes V_{\varpi_i} \cong V_{\varpi_i}$, and therefore with a uniquely determined finite subset $\mathbf{B}_{\varpi_i}$ of the canonical basis $\mathbf{B}^-$ of $U_q^{ad}(\mathfrak{n}_-)$; similarly, one can identify $\dot{\mathbf{B}}_{\varpi_i,0}$ with a uniquely determined finite subset ${}^\omega\mathbf{B}_{\varpi_i}$ of the canonical basis $\mathbf{B}^+$ of $U_q^{ad}(\mathfrak{n}_+)$. The elements of $\dot{\mathbf{B}}_{0,\varpi_i}$ and $\dot{\mathbf{B}}_{\varpi_i,0}$ are respectively of the form $b^-1_{\varpi_i}$ and $b^+1_{-\varpi_i}$, where $b^- \in \mathbf{B}_{\varpi_i}$ and $b^+ \in {}^\omega\mathbf{B}_{\varpi_i}$, and we have (see [74], Proposition 3.3 and section 3.4):

**Proposition 2.7.** *The algebra $\mathcal{O}_A$ is finitely generated. A system of generators is provided by the elements $a^* \in \dot{\mathbf{B}}^*$, where $a \in \cup_{i=1}^m(\dot{\mathbf{B}}_{0,\varpi_i} \cup \dot{\mathbf{B}}_{\varpi_i,0})$.*

Note that the above system of generators of $\mathcal{O}_A$ has $2\sum_{i=1}^m dim(V_{\varpi_i})$ elements. In fact, recall that $\varphi\colon U_q^{ad} \to U_q^{ad}$ is the anti-automorphism given by $\varphi(E_i) = F_i$, $\varphi(F_i) = E_i$, $\varphi(K_\lambda) = K_\lambda$. Denote by $v_{-\varpi_i}$ and $f_{-\varpi_i}$ the canonical lowest-weight vectors of the highest weight modules $V_{-w_0(\varpi_i)}$ and $V^r_{-w_0(\varpi_i)}$ respectively, and put the upperscript " $^{up}$ " for the upper canonical basis vectors. For every $b^- \in \mathbf{B}_{\varpi_i}$ and $b^+ \in {}^\omega\mathbf{B}_{\varpi_i}$ we have

$$(b^-1_{\varpi_i})^* = \Phi_{\varpi_i}((f_{\varpi_i}\varphi(b^-))^{up} \otimes v_{\varpi_i}) \tag{67}$$

$$(b^+1_{-\varpi_i})^* = \Phi_{-w_0(\varpi_i)}((f_{-\varpi_i}\varphi(b^+))^{up} \otimes v_{-\varpi_i}) \tag{68}$$

In other words, $(b^-1_{\varpi_i})^*$ and $(b^+1_{-\varpi_i})^*$ are the matrix coefficients lying on the first and last columns of the matrix representations in the upper canonical basis of the spaces $V_{\varpi_i}$, $i = 1,\dots,m$. This can be checked by using the isomorphism (55). The key observation is that

$$\langle \Phi_\lambda(f_\lambda \otimes v_\lambda), 1_\mu\rangle = \langle f_\lambda, 1_\mu v_\lambda\rangle_\lambda = \delta_{\lambda,\mu}$$

for every $\lambda \in P_+$, $\mu \in P$, and therefore $\Phi_\lambda(f_\lambda \otimes v_\lambda) = 1^*_\lambda$. Then the computation proceeds by using the equivariance of $\Phi$ under the action of $U_q^{ad} \otimes (U_q^{ad})^{op}$, the fact that $\langle .,. \rangle$ dualizes the bimodules structures on $\mathcal{O}_q$ and $\dot{\mathbf{U}}$, and the description of the associated Kashiwara operators on $\mathcal{B}(\mathcal{O}_q)$ and $\mathcal{B}(\dot{\mathbf{U}})$. Here is an alternative argument. By the very definition of the sets $\dot{\mathbf{B}}[\lambda]$ we have $b^-1_{\varpi_i} \in \dot{\mathbf{B}}[\varpi_i]$, $b^+1_{-\varpi_i} \in \dot{\mathbf{B}}[-\omega_0(\varpi_i)]$. We wish to check if their duals $(b^-1_{\varpi_i})^*$, $(b^+1_{-\varpi_i})^*$ coincide with the elements of $\mathcal{O}_A$ on the right sides of (67) and (68). As already noticed after (60) the isomorphism $O_A(\leq \lambda)/O_A(< \lambda) \cong End({}_AV_\lambda)^*$ implies that every matrix coefficient of ${}_AV_\lambda$ belongs to $O_A(\leq \lambda)$. Now, for $\lambda = \varpi_i$ the $A$-module $O_A(\leq \varpi_i)$ is generated by $\dot{\mathbf{B}}[\varpi_i]^*$, and similarly for $\lambda = -w_0(\varpi_i)$. Because $(\bar{\pi}^*_{\varpi_i})^{-1} \circ d_{\varpi_i}(\dot{\mathbf{B}}[\varpi_i]^*)$ coincides with $\underline{\mathbf{B}}^{up}_{\varpi_i} \otimes ({}^\omega\underline{\mathbf{B}}_{\varpi_i})^{up}$, the conclusion follows.

Note that the same argument implies that, for every $\lambda \in P_+$, any matrix coefficient of $V_\lambda$ in the upper canonical basis and vanishing on the elements of $\dot{\mathbf{B}}[\lambda']$ for $\lambda' < \lambda$ must belong to $\dot{\mathbf{B}}[\lambda]^*$. For instance, in the $sl(2)$ case, $O_A(\leq 2)$ has canonical basis $\dot{\mathbf{B}}[0]^* \coprod \dot{\mathbf{B}}[2]^*$, so the matrix coefficients of $V_2$ vanishing on $1_0$ belong to $\dot{\mathbf{B}}[2]^*$. This can be observed in (57), using the comments in the paragraph before (63).

Though the $A$-module ${}_AV_\mu \otimes_A {}_AV_\nu$ has no decomposition like (7), we can refine the map $C(\mu) \otimes C(\nu) \to C(\mu+\nu)$ in (8) to an $A$-linear map defined on ${}_A\dot{C}(\mu) \otimes_A {}_A\dot{C}(\nu)$. Indeed, there is a unique injective morphism of $U_A^{res}$-modules $\mathfrak{T}_{\mu,\nu}\colon {}_AV_{\mu+\nu} \to {}_AV_\mu \otimes_A {}_AV_\nu$, which is given by $\mathfrak{T}_{\mu,\nu}(v_{\mu+\nu}) = v_\mu \otimes v_\nu$ ([73], Proposition 25.1.2 (a)-(b)). It defines a morphism of based modules

$$(V_{\mu+\nu}, \underline{\mathbf{B}}_{\mu+\nu}) \to (V_\mu \otimes V_\nu, \underline{\mathbf{B}}_\mu \Diamond \underline{\mathbf{B}}_\nu)$$

where $\underline{\mathbf{B}}_\mu \Diamond \underline{\mathbf{B}}_\nu := \{b \Diamond b', b \in \underline{\mathbf{B}}_\mu, b' \in \underline{\mathbf{B}}_\nu\}$ ([73], 27.1.7). Therefore $\mathfrak{T}_{\mu,\nu}$ is a split $A$-linear map, ie. there exists a $A$-linear map $\mathfrak{S}_{\mu,\nu}\colon {}_AV_\mu \otimes_A {}_AV_\nu \to {}_AV_{\mu+\nu}$ such that $\mathfrak{S}_{\mu,\nu} \circ \mathfrak{T}_{\mu,\nu} = id$. Note that $\mathfrak{S}_{\mu,\nu}$ is not a $U_A^{res}$-morphism. Similarly, the unique morphism of $U_A^{res}$-modules ${}^\omega\mathfrak{T}_{\mu,\nu}\colon {}^\omega_AV_{\mu+\nu} \to {}^\omega_AV_\mu \otimes_A {}^\omega_AV_\nu$ is a split injection. Define $\rho_{\mu',\mu''}\colon \dot{\mathbf{U}}_A \to {}^\omega_AV_{\mu'} \otimes_A {}_AV_{\mu''}$ by

$$\rho_{\mu',\mu''}(u) = u({}^\omega v_{w_0(\mu')} \otimes_A v_{w_0(\mu'')}),$$

and $\rho_{\mu',\mu'',\nu',\nu''}\colon \dot{\mathbf{U}}_A{}^{\hat{\otimes}2} \to {}^\omega_AV_{\mu'} \otimes_A {}_AV_{\mu''} \otimes_A {}^\omega_AV_{\nu'} \otimes_A {}_AV_{\nu''}$ by

$$\rho_{\mu',\mu'',\nu',\nu''}(u) = u({}^\omega v_{w_0(\mu')} \otimes_A v_{w_0(\mu'')} \otimes_A {}^\omega v_{w_0(\nu')} \otimes_A v_{w_0(\nu'')}).$$

Define $\tau_{\mu',\mu'',\nu',\nu''}\colon {}^\omega_AV_{\mu'+\nu'} \otimes_A {}_AV_{\mu''+\nu''} \to {}^\omega_AV_{\mu'} \otimes_A {}_AV_{\mu''} \otimes_A {}^\omega_AV_{\nu'} \otimes_A {}_AV_{\nu''}$ by

$$\tau_{\mu',\mu'',\nu',\nu''} = (1 \otimes \hat{R}^{-1} \otimes 1)({}^\omega\mathfrak{T}_{\mu',\nu'} \otimes \mathfrak{T}_{\mu'',\nu''}).$$

It is an injective morphism of $U_A^{res}$-modules. In Section 1.13 of [74], Lusztig proved that $\tau_{\mu',\mu'',\nu',\nu''}$ is a split $A$-linear map ([74] uses $\hat{R}$ instead of $\hat{R}^{-1}$, since our coproducts on $U_q^{ad}$ are opposite), and that it satisfies

$$\tau_{\mu',\mu'',\nu',\nu''} \rho_{\mu'+\mu'',\nu'+\nu''} = \rho_{\mu',\mu'',\nu',\nu''} \Delta \tag{69}$$

where $\Delta$ is the coproduct of $\dot{\mathbf{U}}_A$, see (36). Now take $\mu := \mu' = \mu''$, $\nu := \nu' = \nu'' \in P_+$, and put $\tau_{\mu,\nu} := \tau_{\mu,\mu,\nu,\nu}$.

Using the classical decompositions (7) over $\mathbb{C}(q)$, and (8) and (62), it follows that multiplication in $\mathcal{O}_A$ yields a map $m\colon \mathcal{O}_A(\leq \mu) \otimes_A \mathcal{O}_A(\leq \nu) \to \mathcal{O}_A(\leq \mu+\nu)$. Denote the projection map $p_{\mu+\nu}\colon \mathcal{O}_A(\leq \mu+\nu) \to {}_A\dot{C}(\mu+\nu)$, and put

$$\pi'_\lambda\colon \mathcal{O}_A(\leq \lambda) \longrightarrow \mathcal{O}_A(\leq \lambda)/\mathcal{O}_A(< \lambda) \xrightarrow{(\bar{\pi}^*_\lambda)^{-1} \circ \bar{d}_\lambda} (End({}_AV_\lambda))^*$$

where the first map is the quotient map. Consider the diagram:

$$\begin{array}{ccc} {}_A\dot{C}(\mu) \otimes {}_A\dot{C}(\nu) & \xrightarrow{p_{\mu+\nu} \circ m} & {}_A\dot{C}(\mu+\nu) \\ \downarrow{\scriptstyle \pi'_\mu \otimes \pi'_\nu} & & \downarrow{\scriptstyle \pi'_{\mu+\nu}} \\ (End({}_AV_\mu))^* \otimes (End({}_AV_\nu))^* & \xrightarrow{\tau^t_{\mu,\nu}} & (End({}_AV_{\mu+\nu}))^* \end{array}$$

where $\tau^t_{\mu,\nu}$ is the transpose of Lusztig's map $\tau_{\mu,\nu}$. By Lusztig's results that we have recalled above, $\tau^t_{\mu,\nu}$ is a surjective morphism of $U_A^{res}$-modules, which is split as an $A$-linear map. The epimorphism $\pi'_\lambda$ is injective on ${}_A\dot{C}(\lambda)$, and maps canonical basis elements to elements of the

upper canonical basis $\underline{\mathbf{B}}^{up}_{\lambda} \otimes ({}^{\omega}\underline{\mathbf{B}}_{\lambda})^{up}$. Therefore it is split as an $A$-linear map. Let us denote by ${}_A\dot{\tau}_{\mu,\nu}$ the top map $p_{\mu+\nu} \circ m$. The diagram being commutative, using (69) it follows that:

**Proposition 2.8.** *The map ${}_A\dot{\tau}_{\mu,\nu} \colon {}_A\dot{C}(\mu) \otimes_A {}_A\dot{C}(\nu) \to {}_A\dot{C}(\mu+\nu)$ is split as an $A$-linear map, and it coincides with the product in $\mathcal{O}_A$ followed by the projection onto the component ${}_A\dot{C}(\mu+\nu)$.*

We stress that ${}_A\dot{\tau}_{\mu,\nu}$ will play for $\mathcal{O}_A$ the same role as the map (8) for $\mathcal{O}_q$.

Finally, we consider for any $n \geq 1$ the invariant elements of $\mathcal{O}_A^{\otimes n}$ endowed with the action $coad^r_n$ of $U_A^{res}$, see (16) (recall that $\mathcal{L}_{0,n} = \mathcal{O}_q^{\otimes n}$ as $U_q^{ad}$-module).

First note that, by definition, $\mathcal{O}_A(G^n)$ is the restricted dual of the Hopf algebra $U_A^{res}(\mathfrak{g}^{\oplus n})$, associated to its category of type 1 modules. By ordering the summands of $\mathfrak{g}^{\oplus n}$ we get an isomorphism $U_A^{res}(\mathfrak{g}^{\oplus n}) \cong U_A^{res}(\mathfrak{g})^{\otimes n}$, and any type 1 simple $U_A^{res}(\mathfrak{g})^{\otimes n}$-module is isomorphic to $V_{[\lambda]} := \otimes_{i=1}^n V_{\lambda_i}$ endowed with the componentwise action, for some $[\lambda] := (\lambda_1, \dots, \lambda_n) \in P_+^n$ (this is a classical fact; see eg. Theorem 3.10.2 in [49]). Therefore we have an isomorphism $\mathcal{O}_A(G^n) \cong \mathcal{O}_A^{\otimes n}$. With the same notation $[\lambda] := (\lambda_1, \dots, \lambda_n) \in P_+^n$, let us put

$$ {}_A\dot{C}([\lambda]) := \otimes_{i=1}^n {}_A\dot{C}(\lambda_i) = \bigoplus_{b \in \otimes_{i=1}^n \dot{\mathbf{B}}[\lambda_i]^*} Ab \tag{70} $$

$$ \mathcal{O}_A(\leq [\lambda]) := \otimes_{i=1}^n \mathcal{O}_A(\leq \lambda_i) = \bigoplus_{[\lambda'] \in P_+^n, \lambda'_i \leq \lambda_i} {}_A\dot{C}([\lambda']). \tag{71} $$

We thus obtain a decomposition into based $(U_A^{res} \otimes (U_A^{res})^{op})^{\otimes n}$-modules

$$ \mathcal{O}_A^{\otimes n} = \sum_{[\lambda] \in P_+^n} \mathcal{O}_A(\leq [\lambda]). \tag{72} $$

Now $coad^r_n = (coad^r)^{\otimes n} \circ \Delta^{(n-1)}$ gives structures of $U_A^{res}$-modules to $\mathcal{O}_A^{\otimes n}$ and $\mathcal{O}_A(\leq [\lambda])$. In order to make it a based module, we give it the "$\Diamond$" product of the canonical basis of the factors $\mathcal{O}_A(\leq \lambda_i)$, i.e.

$$ \dot{\mathbf{B}}[[\lambda]]^* := \Diamond_{i=1}^n \left( \coprod_{\lambda'_i \leq \lambda_i} \dot{\mathbf{B}}[\lambda'_i]^* \right). $$

We thus obtain a decomposition into based $U_A^{res}$-modules

$$ \mathcal{O}_A^{\otimes n} = \sum_{[\lambda] \in P_+^n} \left( \mathcal{O}_A(\leq [\lambda]), \dot{\mathbf{B}}[[\lambda]]^* \right), \tag{73} $$

with composition factors $\otimes_{i=1}^n (End({}_AV_{\lambda_i}))^*$. By the properties of "$\Diamond$" products of basis of based modules, the underlying $A$-module is

$$ \mathcal{O}_A^{\otimes n} = \bigoplus_{[\lambda] \in P_+^n} {}_A\dot{C}([\lambda]). \tag{74} $$

Finally we state the last property of based modules we need. Let $(M, B)$ be a based module. Recall the notations introduced around (46). It is proved in Proposition 27.1.8 of [73] that for every $\lambda \in P_+$ the submodule $M[\geq \lambda]$ is a sub-based module of $M$, and that it has the basis

$$ B \cap M[\geq \lambda] = \cup_{\lambda' \geq \lambda} B[\lambda']. \tag{75} $$

Consider $M[\neq 0] := \oplus_{\lambda\neq 0} M[\lambda]$, the largest proper submodule of $M$ that contains no non-zero invariant element. Recall that the space of *coinvariants of* $M$ is

$$\begin{aligned} M_{U_q^{ad}} &= M/M[\neq 0] \\ &= M/\mathbb{C}(q)\{um - \varepsilon(u)m, m \in M, u \in U_q^{ad}\} \end{aligned}$$

that is, the largest quotient of $M$ with trivial action, where $\varepsilon\colon U_q^{ad} \to \mathbb{C}(q)$ is the counit. It follows from (75) that $M[\neq 0]$ is a sub-based module of $M$, with the basis $\cup_{\lambda\neq 0} B[\lambda]$, and we have (this is [73], Proposition 27.2.6):

**Proposition 2.9.** *The quotient map $\pi\colon M \to M_{U_q^{ad}}$ is a morphism of based modules, where $M_{U_q^{ad}}$ is endowed with the basis $B_{U_q^{ad}} := \pi(B[0])$.*

As a consequence, the transpose map $\pi^t\colon M^*_{U_q^{ad}} \to M^*$ is a monomorphism mapping the dual basis $B^*_{U_q^{ad}}$ to the subset $B[0]^*$ of $B^*$. The image of $\pi^t$ is $(M^*)^{U_q^{ad}}$, the subspace of $U_q^{ad}$-invariant elements of $M^*$, regarded as a right module in the natural way. Denote by ${}_AM \subset M$ the $A$-module generated by $B$, and by ${}_AM^* \subset M^*$ the $A$-module generated by $B^*$. They are $U_A^{res}$-modules. Set ${}_AM_{U_A^{res}} = \pi({}_AM)$. Then $\pi^t(({}_AM_{U_A^{res}})^*) = ({}_AM^*)^{U_A^{res}}$ is the $A$-submodule of ${}_AM^*$ generated by $B[0]^*$, and therefore

$$ {}_AM^* = ({}_AM^*)^{U_A^{res}} \oplus_A {}_AN \tag{76}$$

where ${}_AN \subset {}_AM^*$ is the $A$-submodule generated by $\cup_{\lambda\neq 0} B[\lambda]^*$. Note that, since $B[0]$ is in general not invariant under the action of $U_A^{res}$, ${}_AN$ need not be stable under this action.

We are now ready to draw consequences of this discussion and the previous results. As usual denote by $(\mathcal{O}_A^{\otimes n})^{U_A^{res}}$ the subspace of invariant elements of $\mathcal{O}_A^{\otimes n}$ for the action $coad_n^r$. In the case $n = 1$ it is just the center $\mathcal{Z}(\mathcal{O}_A)$.

**Theorem 2.10.** *$(\mathcal{O}_A^{\otimes n})^{U_A^{res}}$ is a direct summand of the $A$-module $\mathcal{O}_A^{\otimes n}$ for any $n \geq 1$.*

*Proof.* By (73) it is enough to show that for every $[\lambda] \in P_+^n$ the invariant elements of $\mathcal{O}_A(\leq [\lambda])$ form a direct summand, and these summands are compatible with non-empty intersections $\mathcal{O}_A(\leq [\lambda]) \cap \mathcal{O}_A(\leq [\lambda'])$. Using that $\mathcal{O}_A(G^n) \cong \mathcal{O}_A^{\otimes n}$ and viewing $P_+^n$ as the weight lattice of $G^n$, it is enough to prove these claims for $n = 1$. Given $\lambda \in P_+$ put

$$P_\lambda = \{\lambda' \in P_+, \lambda' \not\leq \lambda\},$$

and denote by $\dot{\mathbf{U}}_A[P_\lambda]$ the $A$-submodule of $\dot{\mathbf{U}}_A$ generated by $\coprod_{\lambda'\in P_\lambda} \dot{\mathbf{B}}[\lambda']$. Also, let us put $\dot{\mathbf{U}}[P_\lambda] = \dot{\mathbf{U}}_A[P_\lambda] \otimes_A \mathbb{C}(q)$. The complement $P_+ \setminus P_\lambda$ is finite, and if $\lambda' \in P_\lambda$ and $\lambda'' \geq \lambda'$, then $\lambda'' \in P_\lambda$. By the results of [73, Chapter 29.2], $\dot{\mathbf{U}}[P_\lambda]$ is a two-sided ideal, and the quotient algebra $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]$ is finite dimensional with unit the coset of $\sum_{\lambda'\leq\lambda} 1_{\lambda'}$, and it is semisimple, isomorphic to $\oplus_{\lambda'\leq\lambda} End(V_{\lambda'})$ (whereas $\dot{\mathbf{U}}_A/\dot{\mathbf{U}}_A[P_\lambda]$ has indecomposable modules, see Example 2.12). It inherits from $\dot{\mathbf{U}}$ a canonical basis, formed by the non-zero cosets of elements of $\dot{\mathbf{B}}$, and with this basis $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]$ is a based module for the right adjoint action $ad^r$. Similarly as for (59) we have a morphism of $U_A^{res}$-modules

$$\begin{array}{rccc} \tilde{d}_\lambda : & \mathcal{O}_A(\leq \lambda) & \longrightarrow & Hom\Big(\dot{\mathbf{U}}_A/\dot{\mathbf{U}}_A[P_\lambda], A\Big) \\ & \alpha & \longmapsto & \langle \alpha, \cdot\, \rangle \end{array}$$

which is an isomorphism by (61) and the computation $\dim\big(\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]\big) = \sum_{\lambda'\leq\lambda} \dim(V_{\lambda'})^2$ in [73, Chapter 29.2]. Applying Proposition 2.9 and (76) to the based module $M = \dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]$ we obtain that the invariant elements of $\mathcal{O}_A(\leq \lambda)$ form a direct summand. Finally, for

any $\lambda, \lambda' \in P_+$ we have $\mathcal{O}_A(\leq\lambda) \cap \mathcal{O}_A(\leq\lambda') \cong Hom\Big(\dot{\mathbf{U}}_A/(\dot{\mathbf{U}}_A[P_\lambda] + \dot{\mathbf{U}}_A[P_{\lambda'}]), A\Big)$. Applying Proposition 2.9 and (76) to the based module $M := \dot{\mathbf{U}}/(\dot{\mathbf{U}}[P_\lambda] + \dot{\mathbf{U}}[P_{\lambda'}])$ we obtain that the invariant elements $({}_AM^*)^{U_A^{res}}$ of $\mathcal{O}_A(\leq\lambda) \cap \mathcal{O}_A(\leq\lambda')$ form a direct $A$-summand. Since the latter is a based $U_A^{res}$-submodule of $\mathcal{O}_A(\leq\lambda)$ and $\mathcal{O}_A(\leq\lambda')$, this summand is also a direct $A$-summand of $\mathcal{O}_A(\leq\lambda)^{U_A^{res}}$ and $\mathcal{O}_A(\leq\lambda')^{U_A^{res}}$. This shows the $A$-modules $\mathcal{O}_A(\leq\lambda)^{U_A^{res}}$ for all $\lambda \in P_+$ match to form the $A$-summand $(\mathcal{O}_A)^{U_A^{res}}$ of $\mathcal{O}_A$, and thus concludes the proof. □

**Remark 2.11.** Let $(M, B)$, $(M', B')$ be based modules, with tensor product $(M \otimes M', B_\diamondsuit)$, and $B_\diamondsuit[0] \subset B_\diamondsuit$ the subset in bijection with the canonical basis of the space of coinvariants $(M \otimes M')_{U_q^{ad}}$ (see Proposition 2.9). This subset is described in Proposition 27.3.8 of [73] in terms of $B$ and $B'$. Since $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]$ is semisimple with known summands, and the construction of the "$\diamondsuit$" product of canonical basis is associative, one can recursively compute the subset of the canonical basis of $\otimes_{i=1}^n \dot{\mathbf{U}}/\dot{\mathbf{U}}[P_{\lambda_i}]$ (endowed with the action dual to $coad_n^r$) which is in bijection with the canonical basis of the space of coinvariants. Therefore a complete (though highly non trivial) characterization of the basis of $(\mathcal{O}_A^{\otimes n})^{U_A^{res}}$ can be obtained. Examples can be found in [73], 27.3.10. In the case $\mathfrak{g} = sl(2)$, the canonical basis of the dual space $End(V_1^{\otimes n})^*$ has been identified in [53] with the canonical basis of the Temperley-Lieb algebra $TL_n(q)$.

**Example 2.12.** The simplest case is already instructive. Namely, consider $V_1$ and $V_2$, the simple $U_q^{ad}(sl(2))$-modules of type 1 and dimension two and three.

On $V_1$ we have the lower canonical basis vectors $v_+$ and $v_-$, such that $Kv_+ = qv_+$, $Ev_+ = 0$, $v_- = Fv_+$. The canonical lower and upper basis of $V_1$ are both $\{v_+, v_-\}$. Using the relation (44) we see that the elements of $\dot{\mathbf{B}}_{0,1}$ and $\dot{\mathbf{B}}_{1,0}$ are $1_1$, $F1_1$ and $1_{-1}$, $E1_{-1}$ respectively; the dual linear forms generate $\mathcal{O}_A(SL_2)$, they are the matrix coefficients $a$, $c$, $d$ and $b$ respectively. By (49) we have $\dot{\mathbf{B}}[1] = \dot{\mathbf{B}}_{0,1} \coprod \dot{\mathbf{B}}_{1,0}$.

Next consider $V_2$. On $V_2$ we have the canonical highest weight vector $v_0$ of weight 2, and lower canonical basis $\underline{\mathbf{B}}_2 = \{v_0, v_1, v_2\}$, where $v_1 = Fv_0$ and $v_2 = F^{(2)}v_0$. We have $\underline{\mathbf{B}}_2^{up} = \{v_0, [2]_q^{-1}v_1, v_2\}$ (see the Appendix). We can identify the ambient space of the right module $V_2^r$ with that of $V_2$; its highest weight vector is then $v_0$, and its canonical lower and upper basis are $\underline{\mathbf{B}}_2^r = \{v_0, v_1, v_2\}$ and $\underline{\mathbf{B}}_2^{rup} = \{v_0, [2]_q^{-1}v_1, v_2\}$.

Consider now the module ${}^\omega V_1 \otimes V_1$. We have $\hat{R} = \sum_{n=0}^\infty \frac{(q-q^{-1})^n}{[n]_q!} q^{n(n-1)/2} E^n \otimes F^n$, so the matrix of the involution $\Psi = \hat{R}^{-1} \circ^-$ in the basis $v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-$ is

$$(\hat{R}^{-1} \circ^-)_{\omega V_1, V_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q^{-1} - q & 0 & 0 & 1 \end{pmatrix}.$$

Therefore the canonical basis $\underline{\mathbf{B}}_{1,1}$ is formed by the vectors $v_+ \diamondsuit v_+ = v_+ \otimes v_+ + q^{-1} v_- \otimes v_-$ and $v_+ \diamondsuit v_- = v_+ \otimes v_-$, $v_- \diamondsuit v_+ = v_- \otimes v_+$, $v_- \diamondsuit v_- = v_- \otimes v_-$. Consider the partition $\underline{\mathbf{B}}_{1,1} = \underline{\mathbf{B}}_{1,1}[2] \cup \underline{\mathbf{B}}_{1,1}[0]$. We have $\underline{\mathbf{B}}_{1,1}[2] = \{v_- \diamondsuit v_+, v_+ \diamondsuit v_+, , v_+ \diamondsuit v_-\}$, which is a basis of the three-dimensional submodule of $V_1 \otimes V_1$. Since $\underline{\mathbf{B}}_{1,1}$ is an $A$-basis of ${}^\omega_A V_1 \otimes_A {}_A V_1$, it follows that the epimorphism $\tau_{1,1}^t \colon {}_AC(1) \otimes_A {}_AC(1) \to {}_AC(2)$ splits (see Proposition 2.8). The vector $v_- \diamondsuit v_-$ is cyclic, so $\underline{\mathbf{B}}_{1,1}[0] = \{v_- \diamondsuit v_-\}$. By the definitions we have $v_+ \diamondsuit v_+ = (1 \diamondsuit_0 1)_{1,1}$, $v_+ \diamondsuit v_- = (1 \diamondsuit_0 F)_{1,1}$, $v_- \diamondsuit v_+ = (F \diamondsuit_0 1)_{1,1}$, $v_- \diamondsuit v_- = (F \diamondsuit_0 F)_{1,1}$, so the corresponding elements of $\dot{\mathbf{B}}_{1,1} \subset \dot{\mathbf{B}}$ are respectively $1_0$, $1_{-2}F$, $1_2E$, and $F1_2E = E1_{-2}F$.

The invariant submodule of ${}^\omega V_1 \otimes V_1$ is generated by $v' = v_- \otimes v_- - q^{-1} v_+ \otimes v_+$; in particular $v_+ \otimes v_+ = [2]_q^{-1}(qv_+ \diamondsuit v_+ - v') \notin {}_AV_2 \oplus {}_AV_1$, and therefore ${}^\omega_A V_1 \otimes_A {}_A V_1 \neq {}_AV_2 \oplus {}_AV_1$ though

both sides are equal over $\mathbb{C}(q)$. The module of coinvariants is

$$({}^{\omega}V_1 \otimes V_1)_{U_q^{ad}} = \mathbb{C}(q)\{\pi(v_- \otimes v_-)\},$$

where as usual $\pi\colon {}^{\omega}V_1 \otimes V_1 \to ({}^{\omega}V_1 \otimes V_1)_{U_q^{ad}}$ is the quotient map. The transpose map $\pi^t\colon ((^{\omega}V_1 \otimes V_1)_{U_q^{ad}})^* \to ({}^{\omega}V_1 \otimes V_1)^*$ sends $(v_-\Diamond v_-)^*$ to the unique $U_q^{ad}$-invariant linear map $ev_1\colon {}^{\omega}V_1 \otimes V_1 \to \mathbb{C}(q)$ such that $ev_1(v_- \otimes v_-) = 1$.

Note that, since elements of $\dot{\mathbf{U}}_A[\lambda > 2]$ act trivially on modules with all isotypical components of highest weight $\leq 2$, ${}^{\omega}_{A}V_1 \otimes_A {}_AV_1$ is an indecomposable module over $\dot{\mathbf{U}}_A/\dot{\mathbf{U}}_A[\lambda > 2]$ (that is, $\dot{\mathbf{U}}_A/\dot{\mathbf{U}}_A[P_2]$ in the notations of Theorem 2.10).

2.2.3. *Some consequences on $\mathcal{L}^A_{0,n}$ and $\mathcal{M}^A_{0,n}$.* Recall from Section 2.2.1 the definition of the integral forms $\mathcal{L}^A_{0,n}$ and $\mathcal{M}^A_{0,n}$.

**Proposition 2.13.** *$\mathcal{L}^A_{0,n}$ and $\mathcal{M}^A_{0,n}$ are free $A$-modules, and $\mathcal{M}^A_{0,n}$ is a direct summand of the $A$-module $\mathcal{L}^A_{0,n}$. Moreover, $\mathcal{L}^A_{0,n}$ is a finitely generated ring.*

*Proof.* Since $\mathcal{L}^A_{0,n} = \mathcal{O}_A^{\otimes n}$ as $U_A^{res}$-modules, by (74) it has the basis $\cup_{[\lambda]\in P_+^n} \dot{\mathbf{B}}[[\lambda]]^*$. Therefore $\mathcal{L}^A_{0,n}$ is a free $A$-module. Since $A$ is a principal ideal domain, $\mathcal{M}^A_{0,n}$ is a free $A$-submodule. By Theorem 2.10 there is a direct sum decomposition as $A$-module

$$\mathcal{L}^A_{0,n} = \mathcal{M}^A_{0,n} \oplus {}_AN, \tag{77}$$

and the proof identifies a basis of $\mathcal{M}^A_{0,n}$ as a subset of $\cup_{[\lambda]\in P_+^n} \dot{\mathbf{B}}[[\lambda]]^*$.

Next, consider the question of finite generation. By the formula (18) it is enough to verify this for $\mathcal{L}^A_{0,1}$, but $\mathcal{L}^A_{0,1} = \mathcal{O}_A$ as an $A$-module, and $\mathcal{O}_A$ is finitely generated by the matrix coefficients of the fundamental $U_A^{res}$-modules ${}_AV_{\varpi_k}$, $k \in \{1, \dots, m\}$ (see Proposition 2.7 and (67)-(68)). Then our claim follows from the formula inverse to (9), expressing the product $\star$ in terms of the product of $\mathcal{L}_{0,1}$, and the integrality properties of the R-matrix (see in [28] the formula (4.8) and Section 3.3). □

**Remark 2.14.** (a) As noted in (76), the $A$-module ${}_AN$ in the decomposition (77) is in general not a $U_A^{res}$-module. Therefore the $A$-linear projection map $\mathcal{R}_A\colon \mathcal{L}^A_{0,n} \to \mathcal{M}^A_{0,n}$ such that $Ker(\mathcal{R}_A) = {}_AN$ is not a Reynolds operator, for it does not satisfy the identity $\mathcal{R}_A(\alpha\beta) = \alpha\mathcal{R}_A(\beta)$ for all $\alpha \in \mathcal{M}^A_{0,n}$, $\beta \in \mathcal{L}^A_{0,n}$.

(b) Recall (29). In coherence with (a) above, there is no normalized Haar measure on $\mathcal{O}_A$ taking values in $A$. Indeed, by extending scalars over $\mathbb{C}(q)$ it should otherwise coincide with the Haar measure $h\colon \mathcal{O}_q \to \mathbb{C}(q)$, but in the notations of Example 2.12 (see also the comments after (55)), since $h(v_0^* \otimes v_0) = 0$ we have $h(bc) = -1/(q + q^{-1})$, whence $h$ cannot be defined on $\mathcal{O}_A$.

(c) The Haar measure yields a well-defined $\mathcal{A}_0$-linear map $h\colon \mathcal{L}(\mathcal{O}_q) \to \mathcal{A}_0$ (and analogously $\mathcal{A}_0$-linear and $\mathcal{A}_\infty$-linear maps $h\colon \mathcal{L}_\Diamond(\mathcal{O}_q^{\otimes n}) \to \mathcal{A}_0$ and $\bar{h}\colon \bar{\mathcal{L}}_\Diamond(\mathcal{O}_q^{\otimes n}) \to \mathcal{A}_\infty$ for any $n \geq 1$, where $(\mathcal{L}_\Diamond(\mathcal{O}_q^{\otimes n}), \mathcal{B}[[\lambda]]^*)$ is the crystal base at $q = 0$ underlying the based $U_q^{ad}$-module (73)). Indeed, by (53) the lattice $\mathcal{L}^{r\,up}_\lambda \otimes_{\mathcal{A}_0} \mathcal{L}^{up}_\lambda$ is generated by the matrix coefficients in the canonical basis of $V^r_\lambda$ and $V_\lambda$. Since the normalisation by powers of $q$ is vacuous on the trivial module $V_0^* \otimes V_0$, and $h$ vanishes on $V^*_\lambda \otimes V_\lambda$ for $\lambda \in P_+ \setminus \{0\}$, the claim follows.

2.3. **Perfect pairings.** We will need to restrict the morphisms $\Phi^+$, $\Phi^-$ in (5) on the integral forms $\mathcal{O}_A(B_+)$, $\mathcal{O}_A(B_-)$. We collect their properties in Theorem 2.15 and the discussion thereafter. In order to state it, we recall first a few facts about $R$-matrices and related pairings.

Recall that $\mathcal{C}_A$ is the category of $U_A^{res}$-modules of type 1. In [72, 73] Lusztig proved that $\mathcal{C}_A \otimes_A \mathbb{C}[q^{\pm 1/D}]$ is braided and ribbon, with braiding given by the collection of endomorphisms

$$R = (R_{V,W})_{V,W \in Ob(\mathcal{C}_A)}.$$

Actually, $R_{V,W}$ is represented by a matrix with coefficients in $q^{\mathbb{Z}/D}\mathbb{C}[q^{\pm 1}]$ on the tensor product of the lower canonical basis of $V$ and $W$ (see Corollary 24.1.5 of [73]).

This can be rephrased as follows in Hopf algebra terms. Denote by $\mathbb{U}_\Gamma$ the categorical completion of $\Gamma$, ie. the Hopf algebra of natural transformations $F_{\mathcal{C}_A} \to F_{\mathcal{C}_A}$, where $F_{\mathcal{C}_A} \colon \mathcal{C}_A \to A\text{-}Mod_f$ is the forgetful functor towards the category $A\text{-}Mod_f$ of finite rank $A$-modules. Then $\mathbb{U}_\Gamma \otimes_A \mathbb{C}[q^{\pm 1/D}]$ is quasi-triangular and ribbon with R-matrix

$$R \in \mathbb{U}_\Gamma^{\hat{\otimes} 2} \otimes_A \mathbb{C}[q^{\pm 1/D}].$$

As in (3), we can write

$$R^{\pm} = \sum_{(R)} R^{\pm}_{(1)} \otimes R^{\pm}_{(2)}.$$

There are pairings of Hopf algebras naturally related to the R-matrix $R$, considered as an element of $\mathbb{U}_q^{\hat{\otimes} 2}$. What follows is standard (see eg. [67, 68, 71]), for details we refer to the results 3.73, 3.75, 3.92, 3.106 and 3.107 in [92]:

- There is a unique pairing of Hopf algebras $\rho \colon U_q(\mathfrak{b}_-)^{cop} \otimes U_q(\mathfrak{b}_+) \to \mathbb{C}(q^{1/D})$ such that, for every $\alpha, \lambda \in P$ and $l, k \in U_q(\mathfrak{h})$,

$$\rho(K_\lambda, K_\alpha) = q^{(\lambda,\alpha)} \ , \ \rho(F_i, E_j) = \delta_{i,j}(q_i - q_i^{-1})^{-1} \ , \rho(l, E_j) = \rho(F_i, k) = 0. \tag{78}$$

- The *Drinfeld pairing* $\tau \colon U_q(\mathfrak{b}_+)^{cop} \otimes U_q(\mathfrak{b}_-) \to \mathbb{C}(q^{1/D})$ is the bilinear map defined by $\tau(X, Y) = \rho(S(Y), X)$; it satisfies

$$\tau(K_\lambda, K_\alpha) = q^{-(\lambda,\alpha)} \ , \ \tau(E_j, F_i) = -\delta_{i,j}(q_i - q_i^{-1})^{-1} \ , \ \tau(l, F_i) = \tau(E_j, k) = 0. \tag{79}$$

- $\rho$ and $\tau$ are perfect pairings; this means that they yield *isomorphisms* of Hopf algebras $i_\pm \colon U_q(\mathfrak{b}_\pm) \to \mathcal{O}_q(B_\mp)_{op}$ (with coefficients *a priori* extended to $\mathbb{C}(q^{1/D})$, but see below) defined by, for every $X \in U_q(\mathfrak{b}_+)$, $Y \in U_q(\mathfrak{b}_-)$,

$$\langle i_+(X), Y \rangle = \tau(S(X), Y) \ , \ \langle i_-(Y), X \rangle = \tau(X, Y).$$

  Since $\mathcal{O}_q(B_\mp)_{op}$ is equipped with the *inverse* of the antipode of $\mathcal{O}_q(B_\mp)$, which is induced by the antipode $S_{\mathcal{O}_q}$ of $\mathcal{O}_q$, it follows that $i_\pm \circ S = S_{\mathcal{O}_q}^{-1} \circ i_\pm$.
- Denote by $p_\pm \colon \mathcal{O}_q(G) \to \mathcal{O}_q(B_\pm)$ the canonical projection map, ie. the Hopf algebra homomorphism dual to the inclusion map $U_q(\mathfrak{b}_\pm) \hookrightarrow U_q(\mathfrak{g})$. For every $\alpha, \beta \in \mathcal{O}_q(G)$ we have

$$\langle \alpha \otimes \beta, R \rangle = \tau(i_+^{-1}(p_-(\beta)), i_-^{-1}(p_+(\alpha))). \tag{80}$$

Note that it is the use of weights $\alpha, \lambda \in P$ that forces the pairings $\rho$, $\tau$ to be defined over $\mathbb{C}(q^{1/D})$, instead of $\mathbb{C}(q)$. Then, let us consider the restrictions $\pi_q^+$ of $\rho$, and $\pi_q^-$ of $\tau$ defined by the formulas (78) and (79), where now $\alpha \in Q$ and $k \in U_q^{ad}(\mathfrak{h})$. They take values in $\mathbb{C}(q)$, and define pairings

$$\pi_q^+ \colon U_q(\mathfrak{b}_-)^{cop} \otimes U_q^{ad}(\mathfrak{b}_+) \to \mathbb{C}(q) \ , \ \pi_q^- \colon U_q(\mathfrak{b}_+)^{cop} \otimes U_q^{ad}(\mathfrak{b}_-) \to \mathbb{C}(q).$$

By the same arguments as for $\rho$ and $\tau$ (eg. in [92], Proposition 3.92), it follows that $\pi_q^\pm$ are perfect pairings. Note also that $\pi_q^- = \kappa \circ \pi_q^+ \circ (\kappa \otimes \kappa)$, where $\kappa \colon U_q \to U_q$ is the $\mathbb{C}$-linear automorphism extending $^- \colon U_q^{ad} \to U_q^{ad}$ in Section 2.2.2, so defined by

$$\kappa(E_i) = F_i \ , \ \kappa(F_i) = E_i \ , \ \kappa(K_\lambda) = K_{-\lambda} \ , \ \kappa(q) = q^{-1}. \tag{81}$$

In [43], De Concini-Lyubashenko described integral forms of $\pi_q^{\pm}$ as follows. Denote by $m^*\colon \mathcal{O}_A \to \mathcal{O}_A(B_+)\otimes\mathcal{O}_A(B_-)$ the map dual to the multiplication map $\Gamma(\mathfrak{b}_+)\otimes\Gamma(\mathfrak{b}_-)\to\Gamma$, so $m^* = (p_+\otimes p_-)\circ\Delta_{\mathcal{O}_A}$. Let $U_A(G^*)$ be the smallest $A$-subalgebra of $U_A(\mathfrak{b}_-)^{cop}\otimes U_A(\mathfrak{b}_+)^{cop}$ which contains the elements $(i=1,\dots,m)$

$$1\otimes K_i^{-1}\bar{E}_i\ ,\ \bar{F}_iK_i\otimes 1\ ,\ L_i^{\pm 1}\otimes L_i^{\mp 1}.$$

and is stable under the diagonal action of $\mathcal{B}(\mathfrak{g})$. The reason for the notation $U_A(G^*)$ will be explained at the beginning of Section 2.5. Note that $U_A(G^*)$ is free over $A$, a Hopf subalgebra, and that a basis is given by the elements

$$\bar{F}_{\beta_1}^{n_1}\dots\bar{F}_{\beta_N}^{n_N}K_{n_1\beta_1+\dots+n_N\beta_N}K_\lambda\otimes K_{-\lambda}K_{-p_1\beta_1\dots-p_N\beta_N}\bar{E}_{\beta_1}^{p_1}\dots\bar{E}_{\beta_N}^{p_N} \tag{82}$$

where $\lambda\in P$ and $n_1,\dots,n_N,p_1,\dots,p_N\in\mathbb{N}$.

Now, let $v$ be a lowest weight vector of the lowest weight $\Gamma$-module ${}_AV_{-\lambda}$, $\lambda\in P_+$. As after Theorem 2.1, denote by $v^*\in {}_AV_{-\lambda}^*$ the dual vector, and by $\psi_{-\lambda}\in\mathcal{O}_A$ the matrix coefficient defined by $\langle\psi_\lambda,x\rangle = v^*(xv)$ for every $x\in\Gamma$. Consider the maps $j_q^{\pm}\colon \mathcal{O}_q(B_\pm)\to U_q(\mathfrak{b}_\mp)^{cop}$ defined by

$$\langle\alpha_+,X\rangle = \pi_q^+(j_q^+(\alpha_+),X)\ ,\ \langle\alpha_-,Y\rangle = \pi_q^-(j_q^-(\alpha_-),Y)$$

where $\alpha_\pm\in\mathcal{O}_q(B_\pm)$, $X\in U_q^{ad}(\mathfrak{b}_+)$, and $Y\in U_q^{ad}(\mathfrak{b}_-)$.

The following theorem summarizes results proved in the sections 3 and 4 of [43]. Denote by $\mathcal{O}_A[\psi_{-\rho}^{-1}]$ the localization of $\mathcal{O}_A$ by the element $\psi_{-\rho}$; this localization is well-defined, for the set $\{\psi_{-\rho}^n\}_{n\in\mathbb{N}}$ is a left and right multiplicative Ore subset of $\mathcal{O}_A$ (see Corollary 2.18 below for an analogous statement for $\mathcal{L}_{0,1}^A$). For the sake of clarity let us spell out the correspondence of notations between statements: $\pi_q^+$, $\pi_q^-$, $U_q(\mathfrak{b}_\mp)^{cop}$, $U_A(\mathfrak{b}_\mp)^{cop}$, $\mathcal{O}_A(B_\pm)$, $U_A(G^*)$ and $\Phi$ are denoted in [43] respectively by $\pi''$, $\bar{\pi}''$, $U_q(\mathfrak{b}_\mp)_{op}$, $R_q[B_\pm]''$, $R_q[B_\pm]$, $A''$ and $\mu''$ (the definition of $j_A^{\pm}$ is implicit in the section 4.2 of [43]).

**Theorem 2.15.** *(1) $\pi_q^{\pm}$ restricts to a perfect Hopf pairing between the unrestricted and restricted integral forms, $\pi_A^{\pm}\colon U_A(\mathfrak{b}_\mp)^{cop}\otimes\Gamma(\mathfrak{b}_\pm)\to A$.*
*(2) $j_q^{\pm}$ yields an isomorphism of Hopf algebras $j_A^{\pm}\colon \mathcal{O}_A(B_\pm)\to U_A(\mathfrak{b}_\mp)^{cop}$, satisfying $\langle\alpha_\pm,x_\pm\rangle = \pi_A^{\pm}(j_A^{\pm}(\alpha_\pm),x_\pm)$ for every $\alpha_\pm\in\mathcal{O}_A(B_\pm)$, $x_\pm\in\Gamma(\mathfrak{b}_\pm)$.*
*(3) The map $\Phi := (j_A^+\otimes j_A^-)\circ m^*\colon \mathcal{O}_A\to U_A(G^*)\subset U_A(\mathfrak{b}_-)^{cop}\otimes U_A(\mathfrak{b}_+)^{cop}$ is an embedding of Hopf algebras, and it extends to an isomorphism $\Phi\colon \mathcal{O}_A[\psi_{-\rho}^{-1}]\to U_A(G^*)$.*

For our purposes it is necessary to reformulate this result. Consider the morphisms of Hopf algebras $\Phi^{\pm}\colon \mathcal{O}_A(B_\pm)\to U_A(\mathfrak{b}_\mp)^{cop}$, $\alpha\mapsto(\alpha\otimes id)(R^{\pm})$.

**Lemma 2.16.** *We have $\Phi^{\pm} = j_A^{\pm}$.*

*Proof.* By definitions, for every $X\in U_q(\mathfrak{b}_+)^{cop}$, $Y\in U_q^{ad}(\mathfrak{b}_-)$ we have $\langle i_+(S^{-1}(X)),Y\rangle = \pi_q^-(X,Y)$, and similarly for every $X\in U_q^{ad}(\mathfrak{b}_+)$, $Y\in U_q(\mathfrak{b}_-)^{cop}$ we have $\langle i_-(S^{-1}(Y)),X\rangle = \pi_q^+(Y,X)$. By keeping these notations for $X$ and $Y$, we deduce $j_q^-(i_+(S^{-1}(X))) = X$ and $j_q^+(i_-(S^{-1}(Y))) = Y$, ie.

$$j_q^{\pm} = S\circ i_{\mp}^{-1}. \tag{83}$$

Because $S_{\mathcal{O}_q}^{-1}\circ i_\pm = i_\pm\circ S$, it follows that

$$j_q^{\pm}\circ S_{\mathcal{O}_q} = S^{-1}\circ j_q^{\pm}. \tag{84}$$

Also, for every $\alpha_-\in\mathcal{O}_q(B_-)$ we have

$$\langle\alpha_-,\Phi^+(i_-(Y))\rangle = \langle i_-(Y)\otimes\alpha_-,R\rangle = \tau(i_+^{-1}(\alpha_-),Y) = \pi_q^-(j_q^-(S_{\mathcal{O}_q}(\alpha_-)),Y) = \langle\alpha_-,S(Y)\rangle$$

where the first equality is by definition of $\Phi^+$ (see (5)), the second is (80), the third follows from (84), and the last from the definition of $j_q^-$. Similarly, for every $\alpha_+ \in \mathcal{O}_q(B_+)$ we have

$$\begin{aligned}\langle \alpha_+, \Phi^-(i_+(X))\rangle &= \langle i_+(X)\otimes\alpha_+, R^-\rangle \\ &= \langle \alpha_+ \otimes S^{-1}_{\mathcal{O}_q}\circ i_+(X), R\rangle \\ &= \langle \alpha_+ \otimes i_+(S(X)), R\rangle \\ &= \tau(S(X), i_-^{-1}(\alpha_+)) \\ &= \pi_q^+(S(i_-^{-1}(\alpha_+)), S(X)) = \pi_q^+(j_q^+(\alpha_+), S(X)) = \langle \alpha_+, S(X)\rangle.\end{aligned}$$

These computations imply $\Phi^\pm = S\circ i_\mp^{-1} = j_q^\pm$, and the result follows by taking integral forms. □

2.4. **Integral form, specialization and localization of $\Phi_n$.** Recall the isomorphism of $U_q$-module algebras $\Phi_1\colon \mathcal{L}_{0,1} \to U_q^{lf}$, and that $U_A^{lf} = U_A \cap U_q^{lf}$. We have:

**Corollary 2.17.** *The map $\Phi_1$ affords an embedding of $U_A^{res}$-module algebras $\Phi_1\colon \mathcal{L}_{0,1}^A \to U_A^{lf}$.*

*Proof.* The only thing to be proved is that $\Phi_1(\mathcal{O}_A) \subset U_A^{lf}$, since $\mathcal{L}_{0,1}^A = \mathcal{O}_A$ as $A$-module. But Lemma 2.16 and (12) imply $\Phi_1 = m\circ(id\otimes S^{-1})\circ\Phi$, and $\Phi$ maps $\mathcal{O}_A$ into $U_A(\mathfrak{b}_-)^{cop}\otimes U_A(\mathfrak{b}_+)^{cop}$ by Theorem 2.15. The conclusion follows. □

Let us denote

$$d = \psi_{-\rho} \in \mathcal{L}_{0,1}^A.$$

(The linear forms $\psi_{-\lambda}$ have been introduced before Theorem 2.15.) When $\mathfrak{g} = sl(2)$ the element $d$ is one of the "standard" generators of $\mathcal{L}_{0,1}(sl(2))$ (see (112) below). In this case we have shown in Lemma 5.7 of [28] that $\mathcal{L}_{0,1}^A$ has a well-defined localization $\mathcal{L}_{0,1}^A[d^{-1}]$, and that $\Phi_1\colon \mathcal{L}_{0,1}^A[d^{-1}] \to U_A^{ad} = T_{2-}^{-1}U_A^{lf}$ is an isomorphism of algebras. A generalization of these facts to any $\mathfrak{g}$ is provided by the following statement. As usual $\ell = K_{2\rho}$, the pivotal element.

**Corollary 2.18.** *(1) The set $\{d^n\}_{n\in\mathbb{N}}$ is a left and right multiplicative Ore set in $\mathcal{L}_{0,1}^A$. We can therefore define the localization $\mathcal{L}_{0,1}^A[d^{-1}]$.*
*(2) $\Phi_1$ extends to an embedding of $U_A^{res}$-module algebras $\Phi_1\colon \mathcal{L}_{0,1}^A[d^{-1}] \to U_A^{lf}[\ell]$, and $U_A^{lf}[\ell] = T_{2-}^{-1}U_A^{lf}$.*

*Proof.* (1) Because $\mathcal{L}_{0,1}^A$ has no non-trivial zero divisors, $d$ is a regular element. We have to show that for all $x \in \mathcal{L}_{0,1}^A$ there exists elements $y, y' \in \mathcal{L}_{0,1}^A$ and $d', d'' \in \{d^n\}_{n\in\mathbb{N}}$ such that $xd' = dy$ and $d''x = y'd$. In fact $d' = d'' = d$ in the present situation. Indeed by (14) we have $\Phi_1(x)\Phi_1(d) = \Phi_1(x)K_{-2\rho} = K_{-2\rho}ad^r(K_{2\rho})(\Phi_1(x))$, and $ad^r(K_{2\rho})(\Phi_1(x)) = \Phi_1(coad^r(K_{2\rho})(x))$. Therefore the left Ore condition is satisfied with $y = coad^r(K_{2\rho})(x)$. Similarly one finds $y'$.

(2) The first claim follows immediately from Corollary 2.17 and $\Phi_1(d) = \ell^{-1}$, which is a regular element of $U_A$. For the second claim, since $K_{-2\rho} = \prod_{j=1}^m L_j^{-2}$, localizing in $d$ we obtain $L_j^2 = \prod_{k\neq j} L_k^{-2}\Phi_1(d^{-1}) = \Phi_1(\prod_{k\neq j}\psi_{-\varpi_k}d^{-1})) \in \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$. Therefore $T_{2-}^{-1} \subset \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$, which implies the assertion (2). □

We expect that the inclusion $\Phi_1(\mathcal{O}_A) \subset U_A^{lf}$ is an equality, but have no proof yet. However, recall Joseph-Letzter's Theorem 2.1 (1) and (2).

**Proposition 2.19.** *We have $U_A = T_{2-}^{-1}U_A^{lf}[T/T_2] = \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2]$, and therefore $\Phi_1\colon \mathcal{L}_{0,1}^A[d^{-1}] \to T_{2-}^{-1}U_A^{lf}$ is an isomorphism. Moreover $\Phi_1(\mathcal{O}_A) = \oplus_{\lambda\in 2P_+} ad^r(U_A^{res})(K_{-\lambda})$.*

*Proof.* The inclusions $T \subset U_A$, $U_A^{lf} \subset U_A$ and $\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}]) \subset T_{2-}^{-1}U_A^{lf}$ imply

$$\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2] \subset T_{2-}^{-1}U_A^{lf}[T/T_2] \subset U_A.$$

For the inverse inclusion, it is enough to show that any PBW basis vector of $U_A$ lies in $\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2]$. This will follow at once if this is true of all root vectors $\bar{E}_{\beta_k}, \bar{F}_{\beta_k}$. Let us show this explicitly for the simple root vectors $\bar{E}_i$ and $\bar{F}_i$. For every positive root $\alpha$ define elements $\psi_{-\lambda}^{\alpha}, \psi_{-\lambda}^{-\alpha} \in \mathcal{O}_A$ by the formulas, where $x \in \Gamma$:

$$\langle \psi_{-\lambda}^{\alpha}, x\rangle = v^*(xE_\alpha v)\ ,\ \langle \psi_{-\lambda}^{-\alpha}, x\rangle = v^*(F_\alpha x v).$$

It is shown in [43], Lemma 4.5, that

$$\Phi(\psi_{-\lambda}) = K_{-\lambda}\otimes K_\lambda\ ,\ \Phi(\psi_{-\varpi_j}^{\alpha_i}) = -\delta_{i,j}q_iL_i^{-1}\otimes L_iK_i^{-1}\bar{E}_i\ ,\ \Phi(\psi_{-\varpi_j}^{-\alpha_i}) = \delta_{i,j}q_i^{-1}\bar{F}_iK_iL_i^{-1}\otimes L_i.$$

(Note that the generators denoted by $E_i$ and $F_i$ in [43] are respectively $K_i^{-1}E_i$ and $F_iK_i$ in our notations, which explains the factors $q_i, q_i^{-1}$ in the formulas below; also $\kappa$ in (81) maps $\bar{E}_i$, $\bar{F}_i$ to $-\bar{F}_i$, $-\bar{E}_i$, whence the sign for the expression of $\Phi(\psi_{-\varpi_j}^{\alpha_i})$). Since $\Phi_1 = m\circ(id\otimes S^{-1})\circ\Phi$, we have

$$\Phi_1(\psi_{-\lambda}) = K_{-2\lambda}\ ,\ \Phi_1(\psi_{-\varpi_j}^{\alpha_i}) = \delta_{i,j}L_i^{-2}\bar{E}_i\ ,\ \Phi_1(\psi_{-\varpi_j}^{-\alpha_i}) = \delta_{i,j}q_i^{-1}\bar{F}_iK_iL_i^{-2}. \tag{85}$$

Therefore $\bar{E}_i$, $\bar{F}_i, L_i^{\pm 1} \in T_{2-}^{-1}\Phi_1(\mathcal{L}_{0,1}^A)[T/T_2] = \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2]$. These elements do not generate $U_A$; it is necessary to consider general root vectors. By the stability of $U_A(G^*)$ under $\mathcal{B}(\mathfrak{g})$ and the isomorphism $\mathcal{O}_A[\psi_{-\rho}^{-1}] \to U_A(G^*)$ of Theorem 2.15 (3), for every positive root $\beta_k$ we have $1\otimes K_{\beta_k}^{-1}\bar{E}_{\beta_k}\ ,\ \bar{F}_{\beta_k}K_{\beta_k}\otimes 1 \in \Phi(\mathcal{O}_A[\psi_{-\rho}^{-1}]) = \Phi(\mathcal{L}_{0,1}^A[d^{-1}])$. Therefore $\bar{F}_{\beta_k}K_{\beta_k}, S^{-1}(\bar{E}_{\beta_k})K_{\beta_k} \in \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$, and $\bar{F}_{\beta_k}, S^{-1}(\bar{E}_{\beta_k}) \in \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2]$. The sets $S^{-1}(\bar{E}_{\beta_k})U_A(\mathfrak{h})$ generate the subalgebra $U_A(\mathfrak{b}_+)$ of $U_A$ (in fact, let us quote that a formula of $S^{-1}(\bar{E}_{\beta_k})$ is given in [94]). From the triangular decomposition $U_A = U_A(\mathfrak{n}_-)U_A(\mathfrak{h})U_A(\mathfrak{n}_+)$, the inclusion $U_A \subset \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2]$ follows, whence the equality too. In particular, $U_A$ is a free $\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$-module with a basis formed by representatives of the cosets in $T/T_2$. By the uniqueness of this free decomposition, we find $\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}]) = T_{2-}^{-1}U_A^{lf}$. Therefore $\Phi_1$ in Corollary 2.18 (2) is surjective.

For the third claim, recall the isomorphism $\Phi_1\colon C(-w_0(\mu)) \to ad^r(U_q)(K_{-2\mu})$ (see (15)), and that $\psi_{-\mu}$ is the matrix coefficient dual to the vector ${}^{\omega}v_{-\mu}\otimes v_{-\mu} \in End_A(V_{-w_0(\mu)})$. This vector is cyclic by (44), so by equivariance $\Phi_1\colon {}_AC(-w_0(\mu)) \to ad^r(U_A^{res})(K_{-2\mu})$ is an isomorphism of $U_A^{res}$-modules. The second claim follows from this and (74) for $n=1$. □

Recall from (22) the isomorphisms of $U_q$-module algebras $\Phi_n\colon \mathcal{L}_{0,n} \to (U_q^{\otimes n})^{lf}$ and of algebras $\Phi_n : \mathcal{M}_{0,n} \to (U_q^{\otimes n})^{U_q}$, and from (35) the notations for specializations. Corollary 2.17 can be extended to $\Phi_n$ as follows:

**Corollary 2.20.** *The map $\Phi_n$ affords embeddings of module algebras $\Phi_n\colon \mathcal{L}_{0,n}^A \to (U_A^{\otimes n})^{lf}$ and $\Phi_n\colon \mathcal{L}_{0,n}^{\epsilon'} \to (U^{\otimes n})_{\epsilon'}^{lf}$, $q=\epsilon' \in \mathbb{C}^\times$.*

*Proof.* For the first claim the only thing to prove is the inclusion $\Phi_n(\mathcal{L}_{0,n}^A) \subset U_A^{\otimes n}$. It follows from Corollary 2.17 and the expression of $\Phi_n$ in terms of $\Phi_1$ and $R$-matrices (in particular, the fact that they preserve integrality, see [28], Lemma 6.10). For the specialization at $q=\epsilon' \in \mathbb{C}^\times$, we have to justify that $\Phi_n$ is injective. One uses the fact, to be developed in Theorem 2.24 below, that $\Phi\colon \mathcal{O}_\epsilon \to U_\epsilon(G^*)$ is an embedding. The algebra $U_\epsilon(G^*)$ has the basis elements (82), and the map $m\circ(id\otimes S^{-1})$ sends this basis to a free family of $U_\epsilon$. Therefore $\Phi_1\colon \mathcal{L}_{0,1}^\epsilon \to U_\epsilon$ is injective. Since $\Phi_n$ differs from $\Phi_1^{\otimes n}$ by a linear isomorphism

(induced by the conjugation action of $R$-matrices on the components ${}_AC([\lambda])$ of $\mathcal{L}^A_{0,n}$ in (74), see equation (6.10) in [28]), $\Phi_n : \mathcal{L}^\epsilon_{0,n} \to U_\epsilon^{\otimes n}$ is an embedding as well. □

**Remark 2.21.** (1) It is a natural problem to determine the image of $\Phi_n$. One may expect that it would be $(T_{2-}^{-1}U_A^{lf})^{\otimes n}$, because this is true for $n = 1$, as well as for any $n$ in the $sl(2)$ case, as shown in [28]. Unfortunately this is not so. This comes from the fact, eg. for $n = 2$, that the matrix elements of $R_{02}R_{01}R'_{01}R^{-1}_{02}$ do not belong to $(T_{2-}^{-1}U_A^{lf})^{\otimes 2}$ as can be shown by an explicit computation in the $sl(3)$ case.

(2) In the case of $\mathfrak{g} = sl(2)$ we defined in [28] an algebra ${}_{loc}\mathcal{L}^A_{0,n}$ generalizing $\mathcal{L}^A_{0,1}[d^{-1}]$ above, containing $\mathcal{L}^A_{0,n}$ as a subalgebra, and such that $\Phi_n$ extends to ${}_{loc}\mathcal{L}^A_{0,n}$ and yields an isomorphism $\Phi_n \colon {}_{loc}\mathcal{L}^A_{0,n} \to U_A^{ad}(sl(2))^{\otimes n}$. The definition of ${}_{loc}\mathcal{L}^A_{0,n}$ involves elements $\xi^{(i)} \in \mathcal{L}^A_{0,n}$ $(i = 1, ..., n)$ such that $\Phi_n(\xi^{(i)}) = (K^{-1})^{(i)} \cdots (K^{-1})^{(n)}$. It may be of interest to study a similar extension of $\Phi_n$ for general $\mathfrak{g}$.

2.5. **Structure theorems for $U_\epsilon$ and $\mathcal{O}_\epsilon$.** As usual we denote by $\epsilon$ a primitive $l$-th root of unity, where $l$ is odd, and coprime to 3 if $\mathfrak{g}$ has $G_2$-components.

Recall the subgroups $T_G$, $U_\pm$ and $B_\pm$ of $G$. Let $G^0 = B_+B_-$ (the *big cell* of $G$), and define the subgroup

$$G^* = \{(u_+t, u_-t^{-1}), t \in T_G, u_\pm \in U_\pm\} \subset B_+^{op} \times B_-^{op}$$

where $B_\pm^{op}$ is the group $B_\pm$ with opposite multiplication. The group $G^*$ can be naturally identified with the Poisson-Lie dual of $G$ with its standard structure.

Recall also that there is an injective homomorphism $\gamma_q^{-1} \circ h_q \colon \mathcal{Z}(U_q) \to U_q(\mathfrak{h})$, defined by means of the quantum Harish-Chandra homomorphism (see eg. [35], section 9.1.C, or [92], Section 3.13). The image of $\gamma_q^{-1} \circ h_q$ is the set $U_q(\mathfrak{h})^{\tilde{W}}$ of invariant elements under $\tilde{W}$, the subgroup of $W \ltimes P_2^*$ generated by the conjugates $\sigma W \sigma$ of $W$ by elements $\sigma \in P_2^*$. Here, $P_2^*$ is the group of homomorphisms $P \to \mathbb{Z}/2\mathbb{Z}$, and the semidirect product $W \ltimes P_2^*$ acts on $U_q(\mathfrak{h})$ by the standard action of the Weyl group $W$, and by the action of $P_2^*$ given by $\sigma \cdot K_\lambda := \sigma(\lambda)K_\lambda$.

It was shown in [41], section 21.1, that the inverse map $h_q^{-1} \circ \gamma_q \colon U_q(\mathfrak{h})^{\tilde{W}} \to \mathcal{Z}(U_q)$ induces by specializing $q$ to $\epsilon$ a well-defined injection

$$U_\epsilon(\mathfrak{h})^{\tilde{W}} \to \mathcal{Z}(U_\epsilon).$$

We denote its image by $\mathcal{Z}_1(U_\epsilon)$. For instance, when $U_\epsilon = U_\epsilon(sl(2))$, $\mathcal{Z}_1(U_\epsilon)$ is the polynomial algebra generated by the Casimir element $\Omega = (\epsilon - \epsilon^{-1})^2 FE + \epsilon K + \epsilon^{-1}K^{-1}$.

Denote by $\mathcal{Z}_0(U_\epsilon) \subset U_\epsilon$ the smallest subalgebra containing the elements $E_i^l$, $F_i^l$, $K_\alpha^l$, for $i \in \{1, \dots m\}$, $\alpha \in P$, and stable under $\mathcal{B}(\mathfrak{g})$; it is also the subalgebra generated by $E^l_{\beta_k}$, $F^l_{\beta_k}$, $L_i^{\pm l}$, for $k \in \{1, \dots, N\}$ and $i \in \{1, \dots m\}$ ([41], Section 18). We will denote by $\mathcal{Z}_0(U_\epsilon(\mathfrak{n}_-))$, $\mathcal{Z}_0(U_\epsilon(\mathfrak{h}))$ and $\mathcal{Z}_0(U_\epsilon(\mathfrak{n}_+))$ the subalgebras of $\mathcal{Z}_0(U_\epsilon)$ generated by the elements $F^l_{\beta_k}$, $K^l_\lambda$ $(\lambda \in P)$, and $E^l_{\beta_k}$ respectively. In [39], Section 1.8-3.3-3.8, and [41], Theorem 14.1 and Section 20-21, the following results are proved:

**Theorem 2.22.** *(1) $U_\epsilon$ has no non-trivial zero divisors, $\mathcal{Z}_0(U_\epsilon)$ is a central Hopf subalgebra of $U_\epsilon$, and $U_\epsilon$ is a free $\mathcal{Z}_0(U_\epsilon)$-module of rank $l^{dim\mathfrak{g}}$. Moreover the classical fraction algebra $Q(U_\epsilon) = Q(\mathcal{Z}(U_\epsilon)) \otimes_{\mathcal{Z}(U_\epsilon)} U_\epsilon$ is a central simple algebra of PI degree $l^N$, and $U_\epsilon$ is a maximal order of $Q(U_\epsilon)$.*
*(2) Maxspec$(\mathcal{Z}_0(U_\epsilon))$ is a group isomorphic to $G^*$ above, and the multiplication map yields an isomorphism $\mathcal{Z}_0(U_\epsilon) \otimes_{\mathcal{Z}_0(U_\epsilon)\cap\mathcal{Z}_1(U_\epsilon)} \mathcal{Z}_1(U_\epsilon) \to \mathcal{Z}(U_\epsilon)$.*

By this theorem the dimension of $Q(U_\epsilon)$ over its center $Q(\mathcal{Z}(U_\epsilon))$ is $l^{2N}$, and its dimension over $Q(\mathcal{Z}_0(U_\epsilon))$ is $l^{dim\mathfrak{g}} = l^{m+2N}$. Therefore the field $Q(\mathcal{Z}(U_\epsilon))$ is an extension of $Q(\mathcal{Z}_0(U_\epsilon))$ of degree $l^m$.

Note that, because $\mathcal{Z}_0(U_\epsilon)$ is an affine and commutative algebra, the maximal spectrum $\mathrm{Maxspec}(\mathcal{Z}_0(U_\epsilon))$, viewed as the set of characters of $\mathcal{Z}_0(U_\epsilon)$, acquires by duality a structure of affine algebraic group. Thus, the first claim of (2) in the theorem means precisely that this group can be identified with $G^*$. See for instance Section 7.2.1 of [28] for an explicit description in the $sl(2)$ case.

In addition, $\mathrm{Maxspec}(\mathcal{Z}_0(U_\epsilon))$ and $G^*$ have natural Poisson structures which correspond one to the other under the isomorphism of (2), and we have the following identifications (see [41], Section 21.2). The dual isomorphism $\mathcal{O}(G^*) \to \mathcal{Z}_0(U_\epsilon)$ identifies $\mathcal{O}(T_G)$ with $\mathcal{Z}_0(U_\epsilon) \cap U_\epsilon(\mathfrak{h}) = \mathbb{C}[lP]$, where as usual $U_\epsilon(\mathfrak{h}) = U_A(\mathfrak{h}) \otimes_A \mathbb{C}_\epsilon$. Therefore we can identify $\mathbb{C}[P]$ with $\mathcal{O}(\tilde{T}_G)$, the coordinate ring of the $l^m$-fold covering space $\tilde{T}_G \to T_G$. The quantum Harish-Chandra isomorphism identifies $\mathcal{Z}_1(U_\epsilon)$ with $\mathbb{C}[2P]^W \cong \mathcal{O}(\tilde{T}_G/(2))^W$, where we denote by (2) the subgroup of 2-torsion elements in $\tilde{T}_G$. Consider the map

$$\begin{array}{rccc} \sigma: & B_+ \times B_- & \longrightarrow & G^0 \\ & (b_+, b_-) & \longmapsto & b_+ b_-^{-1}. \end{array}$$

The restriction of $\sigma$ to $G^*$ is an unramified covering map of degree $2^m$. Composing $\sigma : G^* \to G^0$ with the quotient map under conjugation, $G^0 \hookrightarrow G \to G//G$, we get dually an embedding of $\mathcal{O}(G//G) = \mathcal{O}(G)^G$ in $\mathcal{O}(G^*)$. Collecting these observations, we see that the isomorphism of Theorem 2.22 (2) affords identifications

$$\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon) \cong \mathcal{O}(G)^G$$

as a subalgebra of $\mathcal{Z}_0(U_\epsilon) \cong \mathcal{O}(G^*)$, and

$$\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon) = \mathbb{C}[2lP]^W \cong \mathcal{O}(\tilde{T}_G/(2l))^W \cong \mathcal{O}(T_G/(2))^W$$

as a subalgebra of $\mathcal{Z}_1(U_\epsilon) \cong \mathcal{O}(\tilde{T}_G/(2))^W$.

We will use the following obvious though crucial fact. Note that $U_A^{ad}$ is naturally a subalgebra of $U_A^{res}$, and therefore acts on $U_\epsilon^{res}$-modules.

**Lemma 2.23.** *For every $U_A^{res}$-module $V$ of type 1, the action of $\mathcal{Z}_0(U_\epsilon^{ad})$ on the specialization $V_\epsilon := V \otimes_A \mathbb{C}_\epsilon$ is trivial.*

*Proof.* This comes from $E_i^l = [l]!E_i^{(l)}$, $F_i^l = [l]!F_i^{(l)}$ and the fact that $K_i$ acts on $V$ by powers of $q_i$. Specializing to $q = \epsilon$ ends the proof. □

A result similar to Theorem 2.22 holds true for $\mathcal{O}_\epsilon$. Namely, take the specializations at $q = \epsilon$ in Theorem 2.15. Denote by $\mathcal{Z}_0(U_\epsilon(G^*))$ the subalgebra of $U_\epsilon(G^*)$ generated by the elements $(k \in \{1, \dots, N\}, i \in \{1, \dots m\})$

$$1 \otimes K_{-l\beta_k} E_{\beta_k}^l \ , \ F_{\beta_k}^l K_{l\beta_k} \otimes 1 \ , \ L_i^{\pm l} \otimes L_i^{\mp l}.$$

It is a central Hopf subalgebra. Recall that the coordinate ring $\mathcal{O}(G)$ can be identified as a Hopf subalgebra with $U(\mathfrak{g})^\circ$, where as usual $U(\mathfrak{g})^\circ$ denotes the restricted dual of the envelopping algebra $U(\mathfrak{g})$ over $\mathbb{C}$. In [43], Section 6, De Concini-Lyubashenko introduced an epimorphism of Hopf algebras $\eta : \Gamma_\epsilon \to U(\mathfrak{g})$ (essentially a version of Lusztig's "Frobenius"

epimorphism in [72]), defined by

$$\eta(E_i^{(p)}) = \begin{cases} \dfrac{e_i^{p/l}}{(p/l)!} & \text{if } l \text{ divides } p \\ 0 & \text{otherwise} \end{cases} \ , \ \eta(F_i^{(p)}) = \begin{cases} \dfrac{f_i^{p/l}}{(p/l)!} & \text{if } l \text{ divides } p \\ 0 & \text{otherwise} \end{cases}$$

(86)

$$\eta(K_i) = 1 \ , \ \eta((K_i;p)_{q_i}) = \begin{cases} \dfrac{h_i(h_i-1)\dots(h_i-(p/l)+1)}{(p/l)!} & \text{if } l \text{ divides } p \\ 0 & \text{otherwise} \end{cases}$$

where $p \in \mathbb{N}$, and $e_i$, $f_i$ and $h_i$, $i \in \{1, \dots, m\}$, denote the standard generators of $U(\mathfrak{g})$. The kernel of $\eta$ is generated by the elements $E_i$, $F_i$, $K_i - 1$, and $(K_i;p)_{q_i}$ where $l$ does not divide $p$. Put

$$\mathcal{Z}_0(\mathcal{O}_\epsilon) := \eta^*(\mathcal{O}(G)) \tag{87}$$

where $\eta^* \colon U(\mathfrak{g})^\circ \to \Gamma_\epsilon^\circ$ is the monomorphism dual to $\eta$. Let us define special matrix coefficients, analogous to those introduced in Theorem 2.15. Denote by $v_{\varpi_i}$ and $v_{w_0(\varpi_i)}$ a highest weight vector and a lowest weight vector of the $\Gamma$-module ${}_AV_{\varpi_i}$. Denote also by $v^*_{w_0(\varpi_i)}$ and $v^*_{\varpi_i}$ a highest and lowest weight vector of the dual module $\Gamma$-module ${}_AV^*_{\varpi_i} \cong {}_AV_{-w_0(\varpi_i)}$. Define the matrix coefficients $b_{\varpi_i}, c_{\varpi_i} \in \mathcal{O}_A$ by

$$b_{\varpi_i}(x) = v^*_{\varpi_i}(xv_{w_0(\varpi_i)}) \ , \ c_{\varpi_i}(x) = v^*_{w_0(\varpi_i)}(xv_{\varpi_i})$$

for all $x \in \Gamma$. We consider them as elements of $\mathcal{O}_\epsilon$. Denote by $\mathcal{Z}_1(\mathcal{O}_\epsilon)$ the subalgebra of $\mathcal{O}_\epsilon$ generated by the elements $b^k_{\varpi_i} c^{l-k}_{\varpi_i}$ for $1 \le i \le m$ and $0 \le k \le l$.

**Theorem 2.24.** *(1) $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ is a central Hopf subalgebra of $\mathcal{O}_\epsilon \subset \Gamma_\epsilon^\circ$, and $Q(\mathcal{Z}(\mathcal{O}_\epsilon))$ is an extension of $Q(\mathcal{Z}_0(\mathcal{O}_\epsilon))$ of degree $l^m$.*
*(2) $\psi_{-l\rho} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$, and $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ is generated by matrix coefficients of irreducible $\Gamma$-modules of highest weight $l\lambda$, $\lambda \in P_+$. Moreover the multiplication map yields an isomorphism $\mathcal{Z}_0(\mathcal{O}_\epsilon) \otimes_{\mathcal{Z}_0(\mathcal{O}_\epsilon)\cap\mathcal{Z}_1(\mathcal{O}_\epsilon)} \mathcal{Z}_1(\mathcal{O}_\epsilon) \to \mathcal{Z}(\mathcal{O}_\epsilon)$, and the map $\Phi$ in Theorem 2.15 affords an algebra embedding $\mathcal{Z}_0(\mathcal{O}_\epsilon) \to \mathcal{Z}_0(U_\epsilon(G^*))$ and algebra isomorphisms $\mathcal{Z}_0(\mathcal{O}_\epsilon)[\psi_{-l\rho}^{-1}] \to \mathcal{Z}_0(U_\epsilon(G^*))$ and $\mathcal{O}_\epsilon[\psi_{-l\rho}^{-1}] \to U_\epsilon(G^*)$.*
*(3) $\mathcal{O}_\epsilon$ has no non-trivial zero divisors, and it is a free $\mathcal{Z}_0(\mathcal{O}_\epsilon)$-module of rank $l^{dim\mathfrak{g}}$. Moreover the classical fraction algebra $Q(\mathcal{O}_\epsilon) = Q(\mathcal{Z}(\mathcal{O}_\epsilon)) \otimes_{\mathcal{Z}(\mathcal{O}_\epsilon)} \mathcal{O}_\epsilon$ is a central simple algebra of PI degree $l^N$, and $\mathcal{O}_\epsilon$ is a maximal order of $Q(\mathcal{O}_\epsilon)$.*

For the proof, see in [43]: the proposition 6.4 for the first claim of (1) (where $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ and $\mathcal{Z}_0(U_\epsilon(G^*))$ are denoted $F_0$ and $A_0$ respectively), the appendix of Enriquez and [48] for the second claim of (1) and (2), the propositions 6.4-6.5 for the other claims of (2), the theorem 7.2 (where $\mathcal{O}_\epsilon$ is shown to be projective over $\mathcal{Z}_0(\mathcal{O}_\epsilon)$) and [25] (which provides the additional K-theoretic arguments to deduce that $\mathcal{O}_\epsilon$ is free), or Remark 2.18(b) of [5], for the second claim of (3), and the corollary 7.3 and theorem 7.4 for the third claim. The fact that $\mathcal{O}_\epsilon$ has no non-trivial zero divisors follows from the embedding $\mathcal{O}_\epsilon \to U_\epsilon(G^*)$ via $\Phi$.

As above for $U_\epsilon$, it follows directly from (3) that $Q(\mathcal{Z}(\mathcal{O}_\epsilon))$ has degree $l^m$ over $Q(\mathcal{Z}_0(\mathcal{O}_\epsilon))$. For a more complete description of $\mathcal{Z}(\mathcal{O}_\epsilon)$ we refer to [48] and Enriquez' Appendix in [43], as well as [24].

We do not know a basis of $\mathcal{O}_\epsilon$ over $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ for general $G$, but see [45] for the case of $SL_2$. We will recall the known results in this case of $SL_2$ before Lemma 4.5.

Finally, there is a natural action of the braid group $\mathcal{B}(\mathfrak{g})$ on $\mathcal{O}_\epsilon$, that we will use. Namely, let $n_i \in N(T_G)$ be a representative of the reflection $s_i \in W = N(T_G)/T_G$ associated to the

simple root $\alpha_i$. In [89, 88] Soibelman-Vaksman introduced functionals $t_i : \mathcal{O}_q \to \mathbb{C}(q)$ which quantize the elements $n_i$. They correspond dually to generators of the quantum Weyl group of $\mathfrak{g}$; in the Appendix we recall their main properties, in particular they map $\mathcal{O}_A$ to $A$ (see also [35], Section 8.2, and [67, 89, 71, 68, 43]). Denote by $\lhd$ the natural right action of functionals on $\mathcal{O}_A$, namely (using Sweedler's notation)

$$\alpha \lhd h = \sum_{(\alpha)} h(\alpha_{(1)})\alpha_{(2)}$$

for every $\alpha \in \mathcal{O}_A$ and $h \in \mathcal{O}_A \to A$. Let us identify $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ with $\mathcal{O}(G)$ by means of (87). We have ([43], Proposition 7.1):

**Proposition 2.25.** *The maps $\lhd t_i$ on $\mathcal{O}_\epsilon$ preserve $\mathcal{Z}_0(\mathcal{O}_\epsilon)$, and satisfy $(f \lhd t_i)(a) = f(n_i a)$ and $(f \star \alpha) \lhd t_i = (f \lhd t_i)(\alpha \lhd t_i)$ for every $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$, $a \in G$, $\alpha \in \mathcal{O}_\epsilon$.*

We provide an alternative, non computational, proof of this result in the Appendix (Section 6.3).

## 3. Noetherianity and finiteness

In this section we prove Theorem 1.1. Recall that by Noetherian we mean right and left Noetherian. We begin with:

**Theorem 3.1.** *The algebras $\mathcal{L}_{0,n}$, $\mathcal{L}^A_{0,n}$ and $\mathcal{L}^{\epsilon'}_{0,n}$, $\epsilon' \in \mathbb{C}^\times$, are Noetherian.*

By Proposition 2.13 each of the algebras in this theorem is finitely generated.

The theorem 3.1 for $\mathcal{L}_{0,1}$ and any $\mathfrak{g}$ follows immediately from Joseph-Letzter's Theorem 2.1, claim (3), by identifying $\mathcal{L}_{0,1}$ with $U_q^{lf}$ via $\Phi_1$. The method of proof uses filtration arguments. An alternative proof in the case of $sl(n)$, which works also for $\mathcal{L}^A_{0,1}$, was obtained by Domokos-Lenagan in [44], by exhibiting special sequences of generators of $\mathcal{L}^A_{0,1}$ satisfying *polynormal* relations. Here, given a Noetherian Abelian ring $R$ and a finitely generated $R$-algebra $B$ with product $\circ$, we call polynormal a set of relations between generators $u_1, \dots, u_M$ of $B$, of the form (see [92], Proposition 3.133)

$$u_i \circ u_j - q_{ij} u_j \circ u_i = \sum_{s=1}^{j-1} \sum_{t=1}^{M} \left( \alpha_{ij}^{st} u_s \circ u_t + \beta_{ij}^{st} u_t \circ u_s \right) \tag{88}$$

for all $1 \leq j < i \leq M$, where $\alpha_{ij}^{st}, \beta_{ij}^{st} \in R$, and the elements $q_{ij} \in R$ are invertible (note that this definition is more restrictive than the more standard one, in e.g. [23], II.4.1). If such a set of relations exists in $B$, then $B$ can be endowed with an algebra filtration such that the associated graded algebra is a quotient of a skew-polynomial algebra ([23], Proposition I.8.17). By classical results, if the algebra filtration is well-founded, then $B$ is a Noetherian ring (see eg. [82], 1.2.9-1.6.9-1.6.11, or [92], Lemma 3.130-3.131). In [44], the theorem 3.1 is also proved for any $n \geq 1$ in the case of $\mathfrak{g} = sl(2)$ by considering $\mathcal{L}^A_{0,n}(sl(2))$ as an iterated overring of $\mathcal{L}_{0,1}(sl(2))$.

The proof of Theorem 3.1 that we develop for any $\mathfrak{g}$ and $n \geq 1$ is also based on polynormal relations. In our proof, the generating set of $\mathcal{L}_{0,n}$ that we will consider is evident, as they are matrix coefficients in the modules $V_{\varpi_k}$, $k \in \{1, \dots, m\}$; the task is then to exhibit a set of polynormal relations between them, that hold in a certain graded algebra associated to $\mathcal{L}_{0,n}$. Indeed, as explained above this will imply that the graded algebra is Noetherian, and that $\mathcal{L}_{0,n}$ is Noetherian as well. In the case of $\mathcal{L}^A_{0,n}$, the proof is formally similar, but it needs the use of canonical basis discussed in Section 2.2.2.

*Proof of Theorem 3.1.* First we develop the proof for $\mathcal{L}_{0,n}$, and then for $\mathcal{L}^A_{0,n}$; the result for $\mathcal{L}^{\epsilon'}_{0,n} = \mathcal{L}^A_{0,n}/(q-\epsilon')\mathcal{L}^A_{0,n}$ follows immediately by lifting ideals by the quotient map $\mathcal{L}^A_{0,n} \to \mathcal{L}^{\epsilon'}_{0,n}$.

We adapt the proof of Theorem 2.1 (3) given in Theorem 3.137 of [92]. Let us begin by recalling these arguments. In doing this, let us stress that [92] takes on $\mathcal{O}_q$ and $\mathcal{L}_{0,1}$ the product opposite to ours, and below in (95) and (96) we respect their convention.

As usual let $C(\mu)$ be the vector space generated by the matrix coefficients of $V_\mu$, the simple $U_q^{ad}$-module of highest weight $\mu \in P_+$. Denote by $C(\mu)_\lambda \subset C(\mu)$ the subspace of weight $\lambda$ for the left coregular action of $U_q(\mathfrak{h})$; so $\alpha \in C(\mu)_\lambda$ if $K_\nu \triangleright \alpha = q^{(\nu,\lambda)}\alpha$ , $\nu \in P$. Consider the semigroup

$$\Lambda = \{(\mu, \lambda) \in P_+ \times P, \lambda \text{ is a weight of } V_\mu\}.$$

Recall that the partial order $\preceq$ on $P$ is defined by: $\mu \preceq \mu'$ if and only if $\mu' - \mu \in D^{-1}Q_+$. Define $\preceq$ on $\Lambda$ by: $(\mu, \lambda) \preceq (\mu', \lambda')$ if and only if $\mu' - \mu \in D^{-1}Q_+$ and $\lambda' - \lambda \in D^{-1}Q_+$. If $(\mu, \lambda) \preceq (\mu', \lambda')$ and $(\mu, \lambda) \neq (\mu', \lambda')$, we write $(\mu, \lambda) \prec (\mu', \lambda')$. Since $\mathcal{L}_{0,1}$ and $\mathcal{O}_q$ are isomorphic vector spaces, we have $\mathcal{L}_{0,1} = \bigoplus_{\mu \in P_+} C(\mu) = \bigoplus_{(\mu,\lambda)\in\Lambda} C(\mu)_\lambda$. Consider the family of subspaces

$$\mathcal{F}_2^{\mu,\lambda} := \bigoplus_{(\mu',\lambda') \preceq (\mu,\lambda)} C(\mu')_{\lambda'} \ , (\mu, \lambda) \in \Lambda,$$

$$\mathcal{F}_2^{\prec\mu,\lambda} := \bigoplus_{(\mu',\lambda') \prec (\mu,\lambda)} C(\mu')_{\lambda'} \ , (\mu, \lambda) \in \Lambda.$$

We have

$$\mathcal{L}_{0,1} = \cup_{(\mu,\lambda)\in\Lambda} \mathcal{F}_2^{\mu,\lambda}. \tag{89}$$

Indeed, clearly $\mathcal{L}_{0,1} = \sum_{(\mu,\lambda)\in\Lambda} \mathcal{F}_2^{\mu,\lambda}$, so (89) follows from the following fact: for every $(\mu, \lambda), (\mu', \lambda') \in \Lambda$, the element $(\mu'', \lambda'') := (\mu + \mu', \lambda + \lambda')$ is such that

$$\mathcal{F}_2^{\mu,\lambda} + \mathcal{F}_2^{\mu',\lambda'} \subset \mathcal{F}_2^{\mu'',\lambda''}.$$

Note that in general, since $Q_+ \nsubseteq P_+$ (but $P_+ \subset D^{-1}Q_+$), it is not true that there exists an element $(\mu'', \lambda'')$ satisfying such an inclusion if one replaces $\preceq$ with the standard "product" partial order $\leq$ on $\Lambda$, defined by: $(\mu, \lambda) \leq (\mu', \lambda')$ if and only if $\mu' - \mu \in Q_+$ and $\lambda' - \lambda \in Q_+$. Note also that $\preceq$ is finer than $\leq$, in the sense that if $\mu \leq \mu'$, then $\mu \preceq \mu'$. Again, this would not be true if we had replaced $D^{-1}Q_+$ by $P_+$ in the definition of $\preceq$.

The family $\mathcal{F}_2 := \{\mathcal{F}_2^{\mu,\lambda}\}_{(\mu,\lambda)\in\Lambda}$ is a filtration of the vector space $\mathcal{L}_{0,1}$, which is clearly well-founded (ie. every subset of $\Lambda$ contains a minimal element, or equivalently any decreasing infinite sequence of elements in $\Lambda$ is eventually constant).

Consider the associated graded vector space $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1}) := \oplus_{(\mu,\lambda)} \mathcal{F}_2^{\mu,\lambda}/\mathcal{F}_2^{\prec\mu,\lambda}$. By identifying an element $x \in C(\mu)_\lambda$ with its coset $\bar{x} \in \mathcal{F}_2^{\mu,\lambda}/\mathcal{F}_2^{\prec\mu,\lambda}$ we get an equality of vector spaces $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1}) = \bigoplus_{(\mu,\lambda)\in\Lambda} C(\mu)_\lambda$. Now, one has the following facts:

**(i)** Taking the product in $\mathcal{L}_{0,1}$ we have

$$\alpha\beta \in \mathcal{F}_2^{\mu_1+\mu_2,\lambda_1+\lambda_2} \quad \text{for } \alpha \in C(\mu_1)_{\lambda_1}, \beta \in C(\mu_2)_{\lambda_2}. \tag{90}$$

This follows from (7) and the fact that, for every $\nu \in P_+$ and every summand of the formula (9), denoting by $-r \in -Q_+$ the weight of the $R$-matrix component $R_{(2)}$ we have

$$K_\nu \triangleright \big((R_{(2')}S(R_{(2)}) \triangleright \alpha) \star (R_{(1')} \triangleright \beta \triangleleft R_{(1)})\big) = q^{(\nu,\lambda_1+\lambda_2-r)}(R_{(2')}S(R_{(2)}) \triangleright \alpha) \star (R_{(1')} \triangleright \beta \triangleleft R_{(1)}). \tag{91}$$

(Details of a similar computation are given below (101)). It follows from (90) that $\mathcal{F}_2$ is an algebra filtration of $\mathcal{L}_{0,1}$, and then $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$ is a graded algebra.

**(ii)** Denote by $\alpha \circ \beta$ the product in $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$ of $\alpha, \beta \in \mathcal{L}_{0,1}$. The space $C(\mu_1+\mu_2)$ has multiplicity one in $C(\mu_1)\otimes C(\mu_2)$ (again by (7)), therefore if $\alpha \in C(\mu_1)_{\lambda_1}$ and $\beta \in C(\mu_2)_{\lambda_2}$, then $\alpha\circ\beta$ is the projection of $\alpha\beta$ onto $C(\mu_1+\mu_2)_{\lambda_1+\lambda_2}$. Denote by $\bar\star$ the product $\star$ of $\mathcal{O}_q$ followed by the projection onto the component $C(\mu+\nu)$. Then we have

$$C(\mu)\circ C(\nu) = C(\mu)\ \bar\star\ C(\nu) = C(\mu+\nu). \tag{92}$$

This follows from the formula (9), and the fact that it is given by an invertible twist of the product $\star$.

**(iii)** For every $\mu \in P_+$ fix a basis of weight vectors $e^\mu_1,\dots,e^\mu_{d(\mu)}$ of $V_\mu$. Denote by $e^1_\mu,\dots,e^{d(\mu)}_\mu \in V^*_\mu$ the dual basis, and by $w(e^\mu_i)$ the weight of $e^\mu_i$. Consider the matrix coefficients ${}_\mu\phi^i_j(x) := e^i_\mu(\pi_V(x)(e^\mu_j))$, $x\in U_q$. By using the formula (9) and the explicit form of the $R$-matrix one can check that

$$\begin{aligned} {}_\mu\phi^i_j \circ {}_\nu\phi^k_l &= \sum_{j',l'}{}' c^{ikjl}_{j',l'}\ {}_\mu\phi^i_{j'}\ \bar\star\ {}_\nu\phi^k_{l'} \\ &= q^{(w(e^\mu_j),w(e^\nu_l)-w(e^\nu_k))}\,{}_\mu\phi^i_j\ \bar\star\ {}_\nu\phi^k_l + \sum_{\substack{j',l'\\ j'\neq j,l'\neq l}}{}' d^{ikjl}_{j',l'}\ {}_\mu\phi^i_{j'}\circ {}_\nu\phi^k_{l'} \end{aligned} \tag{93}$$

where $\sum'_{j',l'}$ is the sum over indices with weights satisfying $w(e^\mu_j)+w(e^\nu_l)=w(e^\mu_{j'})+w(e^\nu_{l'})$, $w(e^\mu_{j'})\le w(e^\mu_j)$ and $w(e^\nu_{l'})\ge w(e^\nu_l)$, and the coefficient $c^{ikjl}_{j,l}$, equal to $q^{(w(e^\mu_j),w(e^\nu_l)-w(e^\nu_k))}$, is computed from the term $\Theta$ in the $R$-matrix factorization (4). In general, all the coefficients $c^{ikjl}_{j',l'}$ and $d^{ikjl}_{j',l'}$ belong to $\mathbb{C}(q)$ (see Proposition 4.1 of [28]); in particular $q^{(w(e^\mu_j),w(e^\nu_l)-w(e^\nu_k))}\in q^{\mathbb{Z}}$ since $w(e^\nu_l)-w(e^\nu_k)\in Q$. The second equality follows by repeated use of the first and (92). Similarly, by using (10) one gets

$$\begin{aligned} {}_\nu\phi^k_l \circ {}_\mu\phi^i_j &= \sum_{i',k'}{}' e^{kilj}_{i',k'}\ {}_\mu\phi^{i'}_j\ \bar\star\ {}_\nu\phi^{k'}_l \\ &= q^{(w(e^\mu_i),w(e^\nu_k)-w(e^\nu_l))}\,{}_\mu\phi^i_j\ \bar\star\ {}_\nu\phi^k_l + \sum_{\substack{i',k'\\ i'\neq i,k'\neq k}}{}' e^{kilj}_{i',k'}\ {}_\mu\phi^{i'}_j\ \bar\star\ {}_\nu\phi^{k'}_l \\ &= q^{(w(e^\mu_i),w(e^\nu_k)-w(e^\nu_l))}\,{}_\mu\phi^i_j\ \bar\star\ {}_\nu\phi^k_l + \sum_{\substack{i',k',j',l'\\ i'\neq i,k'\neq k}}{}' f^{kilj}_{i',k'}\ {}_\mu\phi^{i'}_{j'}\ \circ\ {}_\nu\phi^{k'}_{l'} \end{aligned}$$

where $e^{kilj}_{i',k'}$, $f^{kilj}_{i',k'}\in\mathbb{C}(q)$, and $\sum'_{i',k'}$ is the sum over indices with weights satisfying $w(e^\mu_i)+w(e^\nu_k)=w(e^\mu_{i'})+w(e^\nu_{k'})$, $w(e^\mu_{i'})\le w(e^\mu_i)$ and $w(e^\nu_{k'})\ge w(e^\nu_k)$, and $e^{kilj}_{i,k}=q^{(w(e^\mu_i),w(e^\nu_k)-w(e^\nu_l))}$. The third equality comes from the second and (93); the sum is over indices with weights satisfying $w(e^\mu_i)+w(e^\nu_k)=w(e^\mu_{i'})+w(e^\nu_{k'})$, $w(e^\mu_{i'})<w(e^\mu_i)$, $w(e^\nu_{k'})>w(e^\nu_k)$, $w(e^\mu_{j'})\le w(e^\mu_j)$ and $w(e^\nu_{l'})\ge w(e^\nu_l)$. By eliminating the leading term ${}_\mu\phi^i_j\ \bar\star\ {}_\nu\phi^k_l$ one deduces

$$\begin{aligned} {}_\nu\phi^k_l \circ {}_\mu\phi^i_j - q_{ijkl}\ {}_\mu\phi^i_j\circ {}_\nu\phi^k_l = \\ \sum_{\substack{i',k',j',l'\\ i'\neq i,k'\neq k}}{}' f^{kilj}_{i',k'}\ {}_\mu\phi^{i'}_{j'}\ \circ\ {}_\nu\phi^{k'}_{l'} - \sum_{\substack{j',l'\\ j'\neq j,l'\neq l}}{}' q_{ijkl}d^{ikjl}_{j',l'}\ {}_\mu\phi^i_{j'}\circ {}_\nu\phi^k_{l'} \end{aligned} \tag{94}$$

where $q_{ijkl} = q^{(w(e_j^\mu)+w(e_i^\mu),w(e_k^\nu)-w(e_l^\nu))}$.

**(iv)** We can always always reorder the weight vectors $e_1^\mu,\dots,e_{d(\mu)}^\mu$ so that $w(e_i^\mu) > w(e_j^\mu)$ implies $i<j$; then (94) reads

$$
\begin{aligned}
(95)\qquad {}_\nu\phi_l^k \circ {}_\mu\phi_j^i - q_{ijkl}\ {}_\mu\phi_j^i \circ {}_\nu\phi_l^k = \sum_{r=i}^{d(\mu)}\sum_{s=1}^{k}\sum_{u=1}^{l-1}\sum_{v=j+1}^{d(\mu)} \delta_{rsuv}^{ijkl}\ {}_\mu\phi_v^r \circ {}_\nu\phi_u^s \\
- \sum_{r=i+1}^{d(\mu)}\sum_{s=1}^{k-1} q_{ijkl}\gamma_{rs}^{ijkl}\ {}_\mu\phi_j^r \circ {}_\nu\phi_l^s
\end{aligned}
$$

where $\gamma_{rs}^{ijkl}, \delta_{rsuv}^{ijkl} \in \mathbb{C}(q)$ are such that $\gamma_{rs}^{ijkl} = 0$ unless $w(e_r^\mu) < w(e_i^\mu)$ and $w(e_s^\nu) > w(e_k^\nu)$, and $\delta_{rsuv}^{ijkl} = 0$ unless $w(e_u^\nu) > w(e_l^\nu)$, $w(e_v^\mu) < w(e_j^\mu)$, $w(e_r^\mu) \leq w(e_i^\mu)$ and $w(e_s^\nu) \geq w(e_k^\nu)$. Now, from (95) one can extract a defining set of polynormal relations for $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$, as in (88). Indeed, like $\mathcal{L}_{0,1}$ the algebra $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$ is generated by the matrix coefficients ${}_{\varpi_k}\phi_i^j$ of the fundamental representations $V_{\varpi_k}$. One can list these matrix coefficients, say $M$ in number, in an ordered sequence $u_1,\dots,u_M$ such that the following condition holds: if $w(e_k^{\varpi_s}) < w(e_i^{\varpi_r})$, or $w(e_k^{\varpi_s}) = w(e_i^{\varpi_r})$ and $w(e_l^{\varpi_s}) < w(e_j^{\varpi_r})$, then $u_a := {}_{\varpi_r}\phi_j^i$ and $u_b := {}_{\varpi_s}\phi_l^k$ satisfy $b<a$. Then denoting ${}_\mu\phi_j^i$, ${}_\nu\phi_l^k$ in (95) by $u_j$, $u_i$ respectively, and assuming $u_j < u_i$, one finds that all terms $u_s := {}_\mu\phi_v^r$, ${}_\mu\phi_j^r$ in the sums are $< u_j$. Therefore, for all $1 \leq j < i \leq M$ it takes the form:

$$
(96)\qquad u_i \circ u_j - q_{ij}u_j \circ u_i = \sum_{s=1}^{j-1}\sum_{t=1}^{M} \alpha_{ij}^{st} u_s \circ u_t
$$

for some $q_{ij} \in q^{\mathbb{Z}}$ and $\alpha_{ij}^{st} \in \mathbb{C}(q)$. As explained after (88) it follows that $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$ is a Noetherian ring, and since the filtration $\mathcal{F}_2$ is well-founded, it implies that $\mathcal{L}_{0,1}$ is Noetherian too.

We are going to extend all these facts to $\mathcal{L}_{0,n}$, $n>1$. First we need to refine the filtration $\mathcal{F}_2$ on $\mathcal{L}_{0,1}$. Consider the action of $U_q(\mathfrak{h})$ on $C(\mu)_\lambda$ given by

$$
(97)\qquad K_\nu.\alpha := coad(K_\nu^{-1})(\alpha)\ ,\ \nu \in P,\ \alpha \in C(\mu)_\lambda.
$$

Denote by $C(\mu)_{\lambda,\gamma} \subset C(\mu)_\lambda$ the subspace of weight $\gamma$ for this action; so $\alpha \in C(\mu)_{\lambda,\gamma}$ if $K_\nu.\alpha = q^{(\nu,\gamma)}\alpha$. Consider the semigroup

$$
\Lambda_P = \{(\mu,\lambda,\gamma) \in P_+ \times P^2, \lambda \text{ is a weight of } V_\mu \text{ for } \triangleright,\ \gamma \text{ is a weight of } V_\mu \text{ for } .\}
$$

with the partial order $(\mu,\lambda,\gamma) \preceq (\mu',\lambda',\gamma')$ if and only if $\mu'-\mu, \lambda'-\lambda, \gamma'-\gamma \in D^{-1}Q_+$. Define

$$
\begin{aligned}
(98)\quad [\Lambda_P] = \big\{([\mu],[\lambda],[\gamma]) \in P_+^n \times P^n \times P^n \\
\mid\ (\mu_i,\lambda_i,\gamma_i) \in \Lambda_P,\ [\mu]=(\mu_i)_{i=1}^n, [\lambda]=(\lambda_i)_{i=1}^n, [\gamma]=(\gamma_i)_{i=1}^n\big\}.
\end{aligned}
$$

Let us put the following lexicographic order on $[\Lambda_P]$, starting from the tail: $([\mu'],[\lambda'],[\gamma']) \preceq ([\mu],[\lambda],[\gamma])$ if $(\mu_n',\lambda_n',\gamma_n') \prec (\mu_n,\lambda_n,\gamma_n)$, or $(\mu_n,\lambda_n,\gamma_n) = (\mu_n',\lambda_n',\gamma_n')$ and $(\mu_{n-1}',\lambda_{n-1}',\gamma_{n-1}') \prec (\mu_{n-1},\lambda_{n-1},\gamma_{n-1})$,..., or $(\mu_k,\lambda_k,\gamma_k) = (\mu_k',\lambda_k',\gamma_k')$ for all $1<k\leq n$ and $(\mu_1',\lambda_1',\gamma_1') \preceq (\mu_1,\lambda_1,\gamma_1)$. (As usual we write $([\mu'],[\lambda'],[\gamma']) \prec ([\mu],[\lambda],[\gamma])$ for $([\mu'],[\lambda'],[\gamma']) \preceq ([\mu],[\lambda],[\gamma])$ and $([\mu'],[\lambda'],[\gamma']) \neq ([\mu],[\lambda],[\gamma])$.)

Now recall that $\mathcal{L}_{0,n} = \mathcal{L}_{0,1}^{\otimes n} = \mathcal{O}_q^{\otimes n}$ as vector spaces. For every $([\mu],[\lambda],[\gamma]) \in [\Lambda_P]$ consider the subspace $C([\mu])_{[\lambda],[\gamma]} \subset \mathcal{L}_{0,n}$ defined by

$$C([\mu]) = C(\mu_1) \otimes \ldots \otimes C(\mu_n)$$
$$C([\mu])_{[\lambda],[\gamma]} = C(\mu_1)_{\lambda_1,\gamma_1} \otimes \ldots \otimes C(\mu_n)_{\lambda_n,\gamma_n}.$$

Then $\mathcal{L}_{0,n} = \bigoplus_{[\mu]\in P_+^n} C([\mu])$ and $C([\mu]) = \bigoplus_{([\lambda],[\gamma])} C([\mu])_{[\lambda],[\gamma]}$. For every $([\mu],[\lambda],[\gamma]) \in [\Lambda_P]$ define

$$\mathcal{F}_3^{[\mu],[\lambda],[\gamma]} = \bigoplus_{([\mu'],[\lambda'],[\gamma']) \preceq ([\mu],[\lambda],[\gamma])} C([\mu'])_{[\lambda'],[\gamma']} \tag{99}$$
$$\mathcal{F}_3^{\prec[\mu],[\lambda],[\gamma]} = \bigoplus_{([\mu'],[\lambda'],[\gamma']) \prec ([\mu],[\lambda],[\gamma])} C([\mu'])_{[\lambda'],[\gamma']}.$$

Clearly $\mathcal{L}_{0,n}$ is the union of the subspaces $\mathcal{F}_3^{[\mu],[\lambda],[\gamma]}$ over all $([\mu],[\lambda],[\gamma]) \in [\Lambda_P]$, so these form a vector space filtration of $\mathcal{L}_{0,n}$. Let us denote it $\mathcal{F}_3$, and define

$$Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu],[\lambda],[\gamma]} = \mathcal{F}_3^{[\mu],[\lambda],[\gamma]} / \mathcal{F}_3^{\prec[\mu],[\lambda],[\gamma]}.$$

This space is canonically identified with $C([\mu])_{[\lambda],[\gamma]}$, so the graded vector space associated to $\mathcal{F}_3$ is

$$Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n}) = \bigoplus_{([\mu],[\lambda],[\gamma])\in[\Lambda_P]} Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu],[\lambda],[\gamma]} = \bigoplus_{([\mu],[\lambda],[\gamma])\in[\Lambda_P]} C([\mu])_{[\lambda],[\gamma]}. \tag{100}$$

We claim that $\mathcal{F}_3$ is an algebra filtration with respect to the product of $\mathcal{L}_{0,n}$, and therefore $Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})$ is a graded algebra.

For notational simplicity let us prove it for $n = 2$, the general case being strictly similar. Recall the $R$-matrix factorization (4). Take tuples $([\mu],[\lambda],[\gamma]) = ((\mu_1,\mu_2),(\lambda_1,\lambda_2),(\gamma_1,\gamma_2))$ and $([\mu'],[\lambda'],[\gamma']) = ((\mu_1',\mu_2'),(\lambda_1',\lambda_2'),(\gamma_1',\gamma_2'))$ in $[\Lambda_P]$, and elements $\alpha\otimes\beta \in C([\mu])_{[\lambda],[\gamma]}$ and $\alpha'\otimes\beta' \in C([\mu'])_{[\lambda'],[\gamma']}$. Recall from (18) that the product of $\mathcal{L}_{0,2}$ is given by the formula

$$(\alpha\otimes\beta)(\alpha'\otimes\beta') = \tag{101}$$
$$\sum_{(R^1),\ldots,(R^4)} \alpha\left(S(R^3_{(1)}R^4_{(1)}) \rhd \alpha' \lhd R^1_{(1)}R^2_{(1)}\right) \otimes \left(S(R^1_{(2)}R^3_{(2)}) \rhd \beta \lhd R^2_{(2)}R^4_{(2)}\right)\beta'.$$

For every $\nu \in P$ and any of the components $R^1_{(2)},\ldots,R^4_{(2)}$, denoting by $-r_j \in -Q_+$ the weight of $R^j_{(2)}$ we have

$$\begin{aligned}
K_\nu \rhd \left(S(R^1_{(2)}R^3_{(2)}) \rhd \beta \lhd R^2_{(2)}R^4_{(2)}\right) &= \sum_{(\beta),(\beta)} \beta_{(1)}(R^2_{(2)}R^4_{(2)}) \left(K_\nu S(R^1_{(2)}R^3_{(2)}) \rhd \beta_{(2)}\right) \\
&= q^{-(\nu,r_1+r_3)} \sum_{(\beta),(\beta)} \beta_{(1)}(R^2_{(2)}R^4_{(2)}) \left(S(R^1_{(2)}R^3_{(2)})K_\nu \rhd \beta_{(2)}\right) \\
&= q^{(\nu,\lambda_2-r_1-r_3)} \sum_{(\beta),(\beta)} \beta_{(1)}(R^2_{(2)}R^4_{(2)}) \left(S(R^1_{(2)}R^3_{(2)}) \rhd \beta_{(2)}\right) \\
&= q^{(\nu,\lambda_2-r_1-r_3)} \left(S(R^1_{(2)}R^3_{(2)}) \rhd \beta \lhd R^2_{(2)}R^4_{(2)}\right).
\end{aligned}$$

By similar computations for the action $coad(K_\nu^{-1})$, and for all terms in the right-hand side of (101), and using (90) componentwisely, we find that

$$\alpha\left(S(R^3_{(1)}R^4_{(1)}) \rhd \alpha' \lhd R^1_{(1)}R^2_{(1)}\right) \otimes \left(S(R^1_{(2)}R^3_{(2)}) \rhd \beta \lhd R^2_{(2)}R^4_{(2)}\right)\beta' \in \mathcal{F}_3^{[\mu]+[\mu'],[\lambda''],[\gamma'']},$$

where

$$\lambda'' = (\lambda_1 + \lambda_1' + r_3 + r_4, \lambda_2 + \lambda_2' - r_1 - r_3)$$
$$\gamma'' = (\gamma_1 + \gamma_1' + r_1 + r_2 + r_3 + r_4, \gamma_2 + \gamma_2' - r_1 - r_2 - r_3 - r_4).$$

Since $r_1 + r_2 + r_3 + r_4 = 0$ implies $r_1 = r_2 = r_3 = r_4 = 0$, by the order we have put on $[\Lambda_P]$ we deduce

$$(\alpha \otimes \beta)(\alpha' \otimes \beta') \in \mathcal{F}_3^{[\mu]+[\mu'],[\lambda]+[\lambda'],[\gamma]+[\gamma']}.$$

Note that the filtration $\mathcal{F}_3$, taking the action (97) into account, is crucial for this argument to work. Similar arguments work for any $n \geq 2$. This proves that $Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})$ is a graded algebra. We denote its product by $\circ_n$.

Next we show that (92) implies the analogous property for the product $\circ_n$. For simplicity of notations let us again assume that $n = 2$. Recall that the product $\circ_2$ is defined on homogeneous elements $\overline{\alpha \otimes \beta} \in Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu],[\lambda]}$ and $\overline{\alpha' \otimes \beta'} \in Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu'],[\lambda']}$ by

$$\overline{\alpha \otimes \beta} \circ_n \overline{\alpha' \otimes \beta'} = (\alpha \otimes \beta)(\alpha' \otimes \beta') + \mathcal{F}_3^{\prec[\mu+\mu'],[\lambda+\lambda']}.$$

Clearly (92) gives $\big(C(\mu_1) \circ C(\mu_1')\big) \otimes \big(C(\mu_2) \circ C(\mu_2')\big) = C([\mu + \mu'])$, and (101) gives

$$C([\mu]) \circ_n C([\mu']) \subset \big(C(\mu_1) \circ C(\mu_1')\big) \otimes \big(C(\mu_2) \circ C(\mu_2')\big).$$

The converse inclusion holds true as well, as one can see by expressing, reciprocally, the (componentwise) product of $\mathcal{L}_{0,1}^{\otimes n}$ in terms of the product of $\mathcal{L}_{0,n}$ via the formula (20). In conclusion

$$C([\mu]) \circ_n C([\mu']) = C([\mu + \mu']). \tag{102}$$

We are left to show that (95) generalizes to $\mathcal{L}_{0,n}$. First note that for every $1 \leq a \leq n$ the embedding $\mathfrak{i}_a \colon \mathcal{L}_{0,1} \to \mathcal{L}_{0,n}$ in (17) is a morphism of the filtered algebras $(\mathcal{L}_{0,1}, \mathcal{F}_2)$ and $(\mathcal{L}_{0,n}, \mathcal{F}_3)$, in the sense that $\mathfrak{i}_a(\mathcal{F}_2^{\mu,\lambda}) \subset \sum_{\gamma \in P} \mathcal{F}_3^{[\mu_a],[\lambda_a],[\gamma_a]}$, where by definition $[\mu_a] = (0, \dots, 0, \mu, 0, \dots, 0)$ with $\mu$ on the $a$-th entry, and similarly $[\lambda_a] = (0, \dots, 0, \lambda, 0, \dots, 0)$ and $[\gamma_a] = (0, \dots, 0, \gamma, 0, \dots, 0)$. Therefore the relation (95) yields in $Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})$ similar relations between elements of the form (matrix coefficient)$\otimes 1$, or $1 \otimes$(matrix coefficient).

We now consider the case of "mixed" products. We give the details when $n = 2$, the general case being similar. Let us write the twist $F$ in (19) as

$$F = \sum_{(F)} F_{(1)} \otimes F_{(2)} = \sum_{(F)} F_{(1)1} \otimes F_{(1)2} \otimes F_{(2)1} \otimes F_{(2)2}$$

that is, we set $F_{(1)1} := R^2_{(2)} R^4_{(2)}$, $F_{(1)2} := R^1_{(2)} R^3_{(2)}$, $F_{(2)1} := R^1_{(1)} R^2_{(1)}$, $F_{(2)2} := R^3_{(1)} R^4_{(1)}$. Put $d(\mu) := dim(V_\mu)$, $\mu \in P_+$, and

$$\Delta^{(2)}({}_{\mu_2}\phi^{k_2}_{l_2}) = \sum_{p,s=1}^{d(\mu_2)} {}_{\mu_2}\phi^{k_2}_{p} \otimes {}_{\mu_2}\phi^{p}_{s} \otimes {}_{\mu_2}\phi^{s}_{l_2} \ , \ \Delta^{(2)}({}_{\mu_1'}\phi^{k_1'}_{l_1'}) = \sum_{p',s'=1}^{d(\mu_1')} {}_{\mu_1'}\phi^{k_1'}_{p'} \otimes {}_{\mu_1'}\phi^{p'}_{s'} \otimes {}_{\mu_1'}\phi^{s'}_{l_1'}.$$

From (101) one obtains

$$\begin{aligned} &\Big(1 \otimes {}_{\mu_2}\phi^{k_2}_{l_2}\Big) \Big({}_{\mu_1'}\phi^{k_1'}_{l_1'} \otimes 1\Big) \\ &\qquad = \sum_{(F)} \sum_{p,s=1}^{d(\mu_2)} \sum_{p',s'=1}^{d(\mu_1')} \Big({}_{\mu_1'}\phi^{p'}_{s'} \Big({}_{\mu_1'}\phi^{k_1'}_{p'}(F_{(2)1})_{\mu_1'}\phi^{s'}_{l_1'}(S(F_{(2)2}))\Big)\Big) \\ &\qquad\qquad \otimes \Big({}_{\mu_2}\phi^{p}_{s} \Big({}_{\mu_2}\phi^{k_2}_{p}(F_{(1)1})_{\mu_2}\phi^{s}_{l_2}(S(F_{(1)2}))\Big)\Big). \end{aligned} \tag{103}$$

It is immediate that

$$ {}_{\mu_1'}\phi^{p'}_{s'} \otimes {}_{\mu_2}\phi^{p}_{s} \in C(\mu_1')_{w(e^{\mu_1'}_{s'}),w(e^{\mu_1'}_{1})-w(e^{\mu_1'}_{p'})} \otimes C(\mu_2)_{w(e^{\mu_2}_{s}),w(e^{\mu_2}_{s})-w(e^{\mu_2}_{p})}. $$

As in **(iv)** above, for every $\mu \in P_+$ we order the weight vectors $e^\mu_1, \dots, e^\mu_m$ so that $w(e^\mu_i) > w(e^\mu_j)$ implies $i < j$. With such an ordering the factorization $R = \Theta\hat{R}$ (see (4)) implies ${}_{\mu_2}\phi^{k_2}_p(F_{(1)1})_{\mu_2}\phi^{s}_{l_2}(S(F_{(1)2})) = 0$ unless $k_2 \geq p$ and $s \geq l_2$, and ${}_{\mu_1'}\phi^{k_1'}_{p'}(F_{(2)1})_{\mu_1'}\phi^{s'}_{l_1'}(S(F_{(2)2}) = 0$ unless $k_1' \leq p'$ and $s' \leq l_1'$. Since $s > l_2$ we have $w(e^{\mu_2}_s) \leq w(e^{\mu_2}_{l_2})$, and if $w(e^{\mu_2}_s) < w(e^{\mu_2}_{l_2})$ then ${}_{\mu_2}\phi^p_s \in \mathcal{F}_2^{<\mu_2, w(e^{\mu_2}_{l_2})}$. In this last situation the summands ${}_{\mu_1'}\phi^{p'}_{s'} \otimes {}_{\mu_2}\phi^p_s$ in the sum above vanish in $Gr_{\mathcal{F}_3}(\mathcal{L}_{0,2})$. In order to find all the non zero summands we have to consider also the weights with respect to the action (97). Since $k_2 \geq p$ implies $w(e^{\mu_2}_{k_2}) \leq w(e^{\mu_2}_p)$, we have $w(e^{\mu_2}_s) - w(e^{\mu_2}_p) \leq w(e^{\mu_2}_{l_2}) - w(e^{\mu_2}_{k_2})$. Therefore the summands which are non zero in $Gr_{\mathcal{F}_3}(\mathcal{L}_{0,2})$ have both weights $w(e^{\mu_2}_s) = w(e^{\mu_2}_{l_2})$ and $w(e^{\mu_2}_p) = w(e^{\mu_2}_{k_2})$. Doing similarly with the weights of ${}_{\mu_1'}\phi^{p'}_{s'}$, we find that also $w(e^{\mu_1'}_{s'}) = w(e^{\mu_1'}_{l_1'})$ and $w(e^{\mu_1'}_{p'}) = w(e^{\mu_1'}_{k_1'})$. When all these conditions on weights are satisfied, the corresponding components $F_{(1)1}$, $F_{(1)2}$, $F_{(2)1}$, $F_{(2)2}$ have zero weight. Therefore the sum reduces to

$$ \begin{aligned} \sum_{(F)} {}_{\mu_2}\phi^{k_2}_{k_2}(F_{(1)1})_{\mu_2}\phi^{l_2}_{l_2}(S(F_{(1)2}))_{\mu_1'}\phi^{k_1'}_{k_1'}(F_{(2)1})_{\mu_1'}\phi^{l_1'}_{l_1'}(S(F_{(2)2}) & \\ = \left\langle {}_{\mu_2}\phi^{k_2}_{k_2} \otimes {}_{\mu_2}\phi^{l_2}_{l_2} \otimes {}_{\mu_1'}\phi^{k_1'}_{k_1'} \otimes {}_{\mu_1'}\phi^{l_1'}_{l_1'}, \Theta_{13}\Theta^{-1}_{14}\Theta_{24}\Theta^{-1}_{23} \right\rangle & \\ = q^{\left(w(e^{\mu_2}_{k_2})-w(e^{\mu_2}_{l_2}), w(e^{\mu_1'}_{k_1'})-w(e^{\mu_1'}_{l_1'})\right)}. & \end{aligned} $$

Denoting by $q'_{k_2l_2k_1'l_1'}$ this scalar, it follows

$$ \begin{aligned} \left(1 \otimes {}_{\mu_2}\phi^{k_2}_{l_2}\right) \circ_2 \left({}_{\mu_1'}\phi^{k_1'}_{l_1'} \otimes 1\right) &= q'_{k_2l_2k_1'l_1'} \ {}_{\mu_1'}\phi^{k_1'}_{l_1'} \otimes {}_{\mu_2}\phi^{k_2}_{l_2} \\ &= q'_{k_2l_2k_1'l_1'} \ \left({}_{\mu_1'}\phi^{k_1'}_{l_1'} \otimes 1\right) \circ_2 \left(1 \otimes {}_{\mu_2}\phi^{k_2}_{l_2}\right). \end{aligned} $$

This is the relation analogous to (95) for mixed products in $Gr_{\mathcal{F}_3}(\mathcal{L}_{0,2})$.

Recall that in (96) we denoted by $u_1, \dots, u_M$ the ordered list of matrix coefficients ${}_{\varpi_k}\phi^j_i$. Let us order in a lexicographic way the elements $u_i \otimes u_j$, ie. as a sequence $u^{(2)}_1, \dots, u^{(2)}_{M^2}$ such that the following condition holds: if ${}_{\varpi_{l'}}\phi^{t'}_{s'} < {}_{\varpi_{k'}}\phi^{j'}_{i'}$, or ${}_{\varpi_{l'}}\phi^{t'}_{s'} = {}_{\varpi_{k'}}\phi^{j'}_{i'}$ and ${}_{\varpi_l}\phi^t_s < {}_{\varpi_k}\phi^j_i$, then $u^{(2)}_a := {}_{\varpi_k}\phi^j_i \otimes {}_{\varpi_{k'}}\phi^{j'}_{i'}$ and $u^{(2)}_b := {}_{\varpi_l}\phi^t_s \otimes {}_{\varpi_{l'}}\phi^{t'}_{s'}$ satisfy $u^{(2)}_b < u^{(2)}_a$. Then, for this ordering the polynormal relations (96) hold true for all $1 \leq u^{(2)}_j < u^{(2)}_i \leq M^2$. As described after (88), it follows that $Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})$ is Noetherian. The filtration $\mathcal{F}_3$ being well-founded, it implies that $\mathcal{L}_{0,n}$ is Noetherian too.

Finally we consider the $A$-algebra $\mathcal{L}^A_{0,n}$, and prove it is Noetherian. We proceed in exactly the same way as for $\mathcal{L}_{0,n}$, changing the generators and replacing key arguments of the steps **(i)**–**(iv)** by the corresponding results over $A$. Let us describe these modifications step by step.

First consider the case $n = 1$. Recall the $A$-lattices ${}_A\dot{C}(\lambda)$ (see (58)), and the decomposition (66) of $\mathcal{O}_A$ into weight subspaces. In particular have a decomposition into weight subspaces for the left coregular action,

$$ {}_A\dot{C}(\lambda) = \bigoplus_{\lambda' \in P} {}_A\dot{C}(\lambda)_{\lambda'}. $$

Define

$$ {}_A\mathcal{F}_2^{\mu,\lambda} := \bigoplus_{(\mu',\lambda')\preceq(\mu,\lambda)} {}_A\dot{C}(\mu')_{\lambda'}. $$

Recall that every $A$-module of matrix coefficients $(End({}_AV_\mu))^*$, $\mu \in P_+$, is contained in $\mathcal{O}_A(\leq \mu)$, and by inverting over $\mathbb{C}(q)$ the corresponding linear triangular system between basis elements, and using that the order relation $\preceq$ is finer than $\leq$, we obtain an inclusion $\oplus_{\mu'\preceq\mu}\ {}_A\dot{C}(\mu') \subset \oplus_{\mu'\preceq\mu}\ C(\mu')$ (see (60)–(62)). It follows that ${}_A\mathcal{F}_2^{\mu,\lambda} = \mathcal{F}_2^{\mu,\lambda} \cap \mathcal{O}_A$, and therefore, like $\mathcal{F}_2$ the family ${}_A\mathcal{F}_2 := \{{}_A\mathcal{F}_2^{\mu,\lambda}\}_{(\mu,\lambda)\in\Lambda}$ is a well-founded filtration of $\mathcal{O}_A$. Put ${}_A\mathcal{F}_2^{\prec\mu,\lambda} = \mathcal{F}_2^{\prec\mu,\lambda} \cap \mathcal{O}_A$, and consider the graded $A$-module $Gr_{{}_A\mathcal{F}_2}(\mathcal{L}_{0,1}^A)$ associated to ${}_A\mathcal{F}_2$. By (63)-(64)-(65) and the fact that $\mathcal{O}_A = \mathcal{L}_{0,1}^A$ as an $A$-module we have the $A$-module decomposition

$$ Gr_{{}_A\mathcal{F}_2}(\mathcal{L}_{0,1}^A) = \bigoplus_{(\mu,\lambda)\in\Lambda} {}_AC(\mu)_\lambda $$

where ${}_AC(\mu)_\lambda$ is the submodule of weight $\lambda$ (for the left coregular action) of

$$ {}_AC(\mu) := (End({}_AV_\mu))^*. $$

Then we can proceed as before. By step **(i)** we deduce that ${}_A\mathcal{F}_2$ is an algebra filtration of $\mathcal{L}_{0,1}^A$. By Proposition 2.8 the $A$-module ${}_A\dot{C}(\mu_1+\mu_2)$ has multiplicity one in ${}_A\dot{C}(\mu_1)\otimes {}_A\dot{C}(\mu_2)$. In fact, by step **(ii)**, ${}_AC(\mu_1+\mu_2)$ has multiplicity one in ${}_AC(\mu_1)\otimes_A {}_AC(\mu_2)$, so exactly in the same way as 92 we obtain in $Gr_{{}_A\mathcal{F}_2}(\mathcal{L}_{0,1}^A)$ the equality

$$ {}_AC(\mu)\circ {}_AC(\nu) = {}_AC(\mu+\nu). \tag{104} $$

In step **(iii)** we fixed a basis of each space $C(\mu)$, consisting of a set of matrix coefficients $\{{}_\mu\phi_j^i\}$ with respect to dual basis of weight vectors of the modules $V_\mu$ and $V_\mu^*$. In step **(iv)** the basis elements of $V_\mu$ and $V_\mu^*$ were ordered by means of the weights, and we used the fact that the matrix coefficients in the spaces $C(\varpi_1),\dots,C(\varpi_m)$ form a generating set of the algebra $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$. The only property of the matrix coefficients used in the computations was that they are weight vectors for the left coregular action (and later, in the case $n>1$, for the action (97)).

We can proceed exactly in the same manner by working with the $A$-modules of matrix coefficients ${}_AC(\mu)$. If one wishes to work at the lever of $\mathcal{O}_A$, recall that any set of generators of $\mathcal{O}_A$ generates $\mathcal{L}_{0,1}^A$ as well (see the comment after Theorem 3.1). Then, one can replace the basis $\{{}_\mu\phi_j^i\}$ of each space $C(\mu)$ with the canonical basis $\dot{\mathbf{B}}[\mu]^*$ of ${}_A\dot{C}(\mu)$, and take the generating set of $\mathcal{O}_A$ formed by the elements in $\dot{\mathbf{B}}[\varpi_i]^*$, $i=1,\dots,m$ (see Proposition 2.7 and the comments thereafter). By the integrality properties satisfied by the $R$-matrix and the twists, all the computations in the proof of steps **(iii)** and **(iv)** can be done using such basis elements, and eventually take place over $A$ (see the propositions 4.10 and 6.9 in [28]). Therefore we obtain a relation like (96) with coefficients $\alpha_{ij}^{st}\in A$. Since $A$ is a Noetherian ring, again this proves $Gr_{{}_A\mathcal{F}_2}(\mathcal{L}_{0,1}^A)$, whence $\mathcal{L}_{0,1}^A$, are Noetherian rings.

This being done, the adaptation of the proof when $n>1$ is immediate. The filtration $\mathcal{F}_3$ has to be replaced with ${}_A\mathcal{F}_3 := \{{}_A\mathcal{F}_3^{[\mu],[\lambda],[\gamma]}\}_{([\mu],[\lambda],[\gamma])}$, where ${}_A\mathcal{F}_3^{[\mu],[\lambda],[\gamma]}$ is the $A$-module defined by

$$ {}_A\mathcal{F}_3^{[\mu],[\lambda],[\gamma]} = \bigoplus_{([\mu'],[\lambda'],[\gamma'])\preceq([\mu],[\lambda],[\gamma])} {}_A\dot{C}([\mu'])_{[\lambda'],[\gamma']} $$

where ${}_A\dot{C}([\mu])_{[\lambda],[\gamma]} = {}_A\dot{C}(\mu_1)_{\lambda_1,\gamma_1} \otimes_A \ldots \otimes_A {}_A\dot{C}(\mu_n)_{\lambda_n,\gamma_n}$, and ${}_A\dot{C}(\mu)_{\lambda,\gamma}$ is the subspace of ${}_A\dot{C}(\mu)_\lambda$ of weight $\gamma$ for the action (97). Then the proof proceeds in exactly the same way, replacing in (103) and all subsequent computations the matrix coefficients by the generators of $\mathcal{O}_A$ provided by Proposition 2.7. This concludes the proof. □

**Theorem 3.2.** *The algebra $\mathcal{M}_{0,n} = \mathcal{L}_{0,n}^{U_q}$ is Noetherian and generated over $\mathbb{C}(q)$ by a finite number of elements.*

Our method of proof follows closely that of the Hilbert-Nagata theorem (see [38]). Let us recall one version of this theorem. Let $K$ be an arbitrary field, $\mathfrak{A}$ a commutative algebra over $K$ finitely generated by elements $a_1, \ldots, a_n$, and $G$ a group of algebra automorphisms of $\mathfrak{A}$.

**Theorem 3.3.** *If the action of $G$ on $\mathfrak{A}$ is completely reducible on finite dimensional representations, then the ring $\mathfrak{A}^G$ of invariants of $\mathfrak{A}$ with respect to $G$ is Noetherian and a finitely generated algebra over $K$.*

We recall here the main steps of the proof that we will adapt in order to prove Theorem 3.2:

(a) From the complete reducibility of the action of $G$ on $\mathfrak{A}$ one can define a linear map

$$R : \mathfrak{A} \to \mathfrak{A}^G$$

namely the projection onto the space of invariant elements along the sum of non-trivial isotypical components of $\mathfrak{A}$. This linear map is the Reynolds operator; we already discussed it in (26) in the case of $U_q$ acting on $\mathcal{L}_{0,n}$. By the same arguments we developed there, it satisfies

$$R(hf) = hR(f) \tag{105}$$

for every $f \in \mathfrak{A}$, $h \in \mathfrak{A}^G$.

(b) Let $I$ be an ideal of $\mathfrak{A}^G$. Then $I = R(\mathfrak{A}I) = \mathfrak{A}I \cap \mathfrak{A}^G$. Because $\mathfrak{A}I$ is an ideal of $\mathfrak{A}$, and $\mathfrak{A}$ is Noetherian, there exist elements $b_1, ..., b_s$, that can be chosen in $I \subset \mathfrak{A}^G$, such that $\mathfrak{A}I = \mathfrak{A}b_1 + \ldots + \mathfrak{A}b_s$. Since $I = R(\mathfrak{A}I) = R(\mathfrak{A}b_1 + \ldots + \mathfrak{A}b_s) = \mathfrak{A}^G b_1 + \ldots + \mathfrak{A}^G b_s$, $I$ is finitely generated over $\mathfrak{A}^G$. Therefore $\mathfrak{A}^G$ is Noetherian.

(c) Let $\mathfrak{B}$ be an algebra graded over $\mathbb{N}$ (for simplicity of notations): $\mathfrak{B} = \bigoplus_{n=0}^{+\infty} \mathfrak{B}_n$, with $\mathfrak{B}_m.\mathfrak{B}_n \subset \mathfrak{B}_{m+n}$. The augmentation ideal of $\mathfrak{B}$ is $\mathfrak{B}^+ := \bigoplus_{n=1}^{+\infty} \mathfrak{B}_n$. If $\mathfrak{B}^+$ is a Noetherian ideal of $\mathfrak{B}$, then $\mathfrak{B}$ is a finitely generated algebra over $\mathfrak{B}_0$. This is Lemma 2.4.5 of [87] (in that statement $\mathfrak{B}$ is commutative, but this hypothesis is not necessary for the proof).

(d) Assume that $\mathfrak{A}^G$ is graded over $\mathbb{N}$ (for simplicity of notations): $\mathfrak{A}^G = \bigoplus_{n=0}^{+\infty} \mathfrak{A}_n^G$ with $\mathfrak{A}_0^G = K$. Then $\mathfrak{A}^{G+} = \bigoplus_{n=1}^{+\infty} \mathfrak{A}_n^G$ is an ideal of $\mathfrak{A}^G$, which is Noetherian by (b) above. Applying (*c*) we deduce that $\mathfrak{A}^G$ is a finitely generated algebra over $K$.

*Proof of Theorem 3.2.* Consider the filtration $\mathcal{F}$ of $\mathcal{L}_{0,n}$ by the subspaces

$$\mathcal{F}^{[\mu]} = \bigoplus_{[\mu'] \preceq [\mu]} C([\mu']), \mu \in P_+^n$$

where $P_+^n$ is given the lexicographic partial order induced from $[\Lambda]$. It is easily seen that $\mathcal{F}$ is an algebra filtration: the coregular actions $\triangleright$, $\triangleleft$ fix globally each component $C(\mu)$ of $\mathcal{L}_{0,1}$, so the claim follows from (9), (18) and the fact that $C(\mu) \star C(\nu) \subset C(\mu + \nu)$ for all $\mu, \nu \in P_+$. Denote by $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})$ the corresponding graded algebra. As a vector space we have:

$$Gr_{\mathcal{F}}(\mathcal{L}_{0,n}) = \bigoplus_{[\mu] \in P_+^n} C([\mu]). \tag{106}$$

Because each space $C([\mu])$ is stabilized by the coadjoint action of $U_q$, (106) has a key advantage on the refined decomposition (100). Indeed, since $\mathcal{L}_{0,n}$ is a $U_q$-module algebra, the action of $U_q$ is well-defined on $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})$ and gives it a structure of $U_q$-module algebra. As vector spaces we have

$$Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q} = \bigoplus_{[\mu]\in P_+^n} C([\mu])^{U_q}. \tag{107}$$

Now we can adapt the differents steps (a)–(d) recalled above:

(a') The action of $U_q$ on $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})$ is completely reducible. This follows from (106) and the fact that the spaces $C(\mu)$ are finite-dimensional and thus completely reducible $U_q$-modules. We can therefore define the Reynolds operator as in (a),

$$R\colon Gr_{\mathcal{F}}(\mathcal{L}_{0,n}) \to Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}.$$

(b') $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})$ is Noetherian, because (106) shows it is filtered by $\mathcal{F}_3$, and the associated graded algebra $Gr_{\mathcal{F}_3}(Gr_{\mathcal{F}}(\mathcal{L}_{0,n})) = Gr_{\mathcal{F}_3}(\mathcal{L}_{0,n})$ is Noetherian by Theorem 3.1. As in (b) we deduce that $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$ is Noetherian. But $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q} = Gr_{\mathcal{F}}(\mathcal{L}_{0,n}^{U_q})$, which implies that $\mathcal{L}_{0,n}^{U_q}$ is Noetherian.

(c'-d') Then we can apply the steps (c)-(d). As a result $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$ is finitely generated, say by $k$ non zero elements $\bar{x}_1, \dots, \bar{x}_k$, which we may assume homogeneous.

(e') We can now deduce that $\mathcal{L}_{0,n}^{U_q}$ is generated by elements $x_i$ with leading terms the $\bar{x}_i$'s. Indeed, let $x \in \mathcal{L}_{0,n}^{U_q}$, and $[\mu] \in P_+^n$ such that $x \in \mathcal{F}^{[\mu]} \setminus \mathcal{F}^{\prec[\mu]}$, where $\mathcal{F}^{\prec[\mu]} := \oplus_{[\mu']\prec[\mu]} C([\mu'])$. In $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})_{[\mu]}^{U_q} = \mathcal{F}^{[\mu]}/\mathcal{F}^{\prec[\mu]}$ we have $\bar{x} = \sum_{(i_1,\dots,i_k)\in I} \lambda_{(i_1,\dots,i_k)} \bar{x}_1^{i_1} \dots \bar{x}_k^{i_k}$ for some finite set $I \subset \mathbb{N}^k$, scalars $\lambda_{(i_1,\dots,i_k)} \in \mathbb{C}(q)$, and monomials $\bar{x}_1^{i_1} \dots \bar{x}_k^{i_k}$ of degree $[\mu]$. By definition of the product in $\underline{Gr_{\mathcal{F}}(\mathcal{L}_{0,n})}^{U_q}$, $\bar{x}_1^{i_1} \dots \bar{x}_k^{i_k} = x_1^{i_1} \dots x_k^{i_k} + \mathcal{F}^{\prec[\mu]}$, so $x_1^{i_1} \dots x_k^{i_k} \in \mathcal{F}^{[\mu]} \setminus \mathcal{F}^{\prec[\mu]}$, whence $\bar{x}_1^{i_1} \dots \bar{x}_k^{i_k} = \underline{x_1^{i_1} \dots x_k^{i_k}}$ and $x - \sum_{(i_1,\dots,i_k)\in I} \lambda_{(i_1,\dots,i_k)} x_1^{i_1} \dots x_k^{i_k} \in \mathcal{F}^{\prec[\mu]}$. The conclusion follows by decreasing induction on $[\mu]$, since at last we terminate at $\mathcal{F}^{[0]} \cong \mathbb{C}(q)$.

By combining the steps (a') to (e'), we get that $\mathcal{M}_{0,n}$ is a Noetherian and finitely generated ring. □

**Remark 3.4.** (1) Because $\mathcal{L}_{0,1}^{U_q}$ is the center of $\mathcal{L}_{0,1}$, (e') proves it is finitely generated. Of course this follows also from the isomorphism $\mathcal{L}_{0,1} \cong U_q^{lf}$ and the fact that the center of $U_q^{lf}$ is the center of $U_q$ (by Theorem 2.1), plus the well-known description of the latter.

(2) In the $sl(2)$ case the filtration $\mathcal{F}$ on $\mathcal{L}_{0,n}^{U_q}$ should be related via the Wilson loop isomorphism (defined in [28], Section 8.2) to the filtration of skein algebras of spheres with $n+1$ punctures used in [79].

## 4. Proof of Theorem 1.2

As usual we let $\epsilon$ be a primitive $l$-th root of unity with $l$ odd and $l > d_i$ for all $i \in \{1, \dots, m\}$. We now consider the specialization $\mathcal{L}_{0,n}^{\epsilon}$ of $\mathcal{L}_{0,n}$ at $q = \epsilon$, defined in Section 2.2.1. Recall the isomorphism of algebras $\eta^*\colon \mathcal{O}(G) \to \mathcal{Z}_0(\mathcal{O}_\epsilon)$ (see (86)), and that $\mathcal{L}_{0,n}^{\epsilon} = \mathcal{O}_\epsilon^{\otimes n}$ as a vector space. Consider the linear subspace of $\mathcal{L}_{0,n}^{\epsilon}$ defined by

$$\mathcal{Z}_0(\mathcal{L}_{0,n}^{\epsilon}) := \mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}.$$

This space is naturally a subalgebra of $\mathcal{O}_\epsilon^{\otimes n}$ (endowed with the componentwise product $\star$). In fact we also have:

**Proposition 4.1.** *(1) $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$ is a central subalgebra of the algebra $\mathcal{L}^\epsilon_{0,n}$, and the $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-modules $\mathcal{L}^\epsilon_{0,n}$ and $\mathcal{O}_\epsilon^{\otimes n}$, with actions defined by the respective products of these algebras, do coincide.*
*(2) $\mathcal{L}^\epsilon_{0,n}$ is a free $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-module of rank $l^{n.dim\mathfrak{g}}$.*
*(3) $(\eta^{*-1})^{\otimes n}\colon \mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}) \to \mathcal{O}(G)^{\otimes n}$ is an isomorphism of algebras, and $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$ is a Noetherian ring.*
*(4) The $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-module $\mathcal{L}^\epsilon_{0,n}$ is finite and Noetherian. Therefore $\mathcal{L}^\epsilon_{0,n}$ is a Noetherian ring.*

Note that the proof we give in (4) of the fact that $\mathcal{L}^\epsilon_{0,n}$ is Noetherian is independent from the proof of Theorem 3.1.

*Proof.* (1) Let us show that $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$ is a central subalgebra of $\mathcal{L}^\epsilon_{0,n}$. In the case $n = 1$ the formula (9) implies that $\alpha\beta = \alpha \star \beta$ for all $\alpha \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ and $\beta \in \mathcal{L}^\epsilon_{0,1}$. Indeed, by (9) we have

$$\begin{aligned}\alpha\beta &= \sum_{(R),(R)} (R_{(2')}S(R_{(2)}) \rhd \alpha) \star (R_{(1')} \rhd \beta \lhd R_{(1)}) \\ &= \sum_{(R),(R),(\alpha),(\beta)} \alpha_{(1)} \star \Big(\beta_{(1)}\big(R_{(1)}\alpha_{(3)}(S(R_{(2)}))\beta_{(3)}\big(R_{(1')}\alpha_{(2)}(R_{(2')})\big)\beta_{(2)}\Big)\end{aligned}$$

where all components $\alpha_{(1)}, \alpha_{(2)}, \alpha_{(3)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$, since $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ is a Hopf subalgebra of $\mathcal{O}_\epsilon$. But $\sum_{(R)} R_{(1)}\alpha_{(3)}(S(R_{(2)})) = S^{-1}\big(\Phi^-(S_{\mathcal{O}_\epsilon}(\alpha_{(3)}))\big) \in \mathcal{Z}_0(U_\epsilon)$, since $\Phi^-(S_{\mathcal{O}_\epsilon}(\alpha_{(3)})) \in \mathcal{Z}_0(U_\epsilon)$ by Theorem 2.24 (2). Similarly, $\sum_{(R)} R_{(1')}\alpha_{(2)}(R_{(2')}) \in \mathcal{Z}_0(U_\epsilon)$. In general these elements belong to $\mathcal{Z}_0(U_\epsilon)$ and not $\mathcal{Z}_0(U_\epsilon^{ad})$ because of the "diagonal" factor $\Theta$ of the $R$-matrix in (4). By Lemma 2.23, $\mathcal{Z}_0(U_\epsilon^{ad})$ acts by the trivial character $\varepsilon$ (the counit) on $\Gamma_\epsilon$-modules. The action of $\mathcal{Z}_0(U_\epsilon)$ is $\varepsilon$ multiplied with some powers of $\epsilon^{1/D}$. However, Proposition 4.1 and 4.10 of [28] show that such powers of $\epsilon^{1/D}$ eventually disappear in the sum above; this is because the sum can be rewritten in terms of copies of the quasi $R$-matrix $\hat{R}$ in (4) and the pivotal element $\ell$, instead of copies of $R$. Therefore

$$\alpha\beta = \sum_{(R),(R),(\alpha),(\beta)} \alpha_{(1)} \star \Big(\varepsilon(\beta_{(1)})\varepsilon(\alpha_{(3)})\varepsilon(\beta_{(3)})\varepsilon(\alpha_{(2)})\beta_{(2)}\Big) = \alpha \star \beta. \tag{108}$$

This shows $\mathcal{L}^\epsilon_{0,1}$ and $\mathcal{O}_\epsilon$ coincide as modules over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1}) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$. Next we show that the subalgebras $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$ are central in $\mathcal{L}^\epsilon_{0,n}$ for all $a = 1, \dots, n$. This fact will conclude the proof that $\mathcal{L}^\epsilon_{0,n}$ and $\mathcal{O}_\epsilon^{\otimes n}$ coincide as $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-modules, because the subalgebras $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$ generate the space $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$ in $(\mathcal{L}^\epsilon_{0,1})^{\otimes n}$, and hence in $\mathcal{L}^\epsilon_{0,n}$ (this follows from the comment before (19)).

In order to show that $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$ is central in $\mathcal{L}^\epsilon_{0,n}$ for all $a = 1, \dots, n$, it is enough to show $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$ commutes with the elements of $\mathcal{L}^\epsilon_{0,n}$ supported by the tensor factors $(\mathcal{L}^\epsilon_{0,1})^{(b)}$ with $b \neq a$. Since $(\alpha)^{(a)} \otimes (\beta)^{(b)} = ((\alpha)^{(a)} \otimes 1)(1 \otimes (\beta)^{(b)})$ by the definition, we have to show that $(1 \otimes (\beta)^{(b)})((\alpha)^{(a)} \otimes 1) = (\alpha)^{(a)} \otimes (\beta)^{(b)}$ whenever $\alpha \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$. We have (denoting

$\sum_{(\alpha),(\alpha),(\alpha),(\alpha)}$ by $\sum_{(\alpha)^4}$, $\Delta(\alpha_{(1)}) = \sum_{(\alpha)} \alpha_{(1)(1)} \otimes \alpha_{(1)(2)}$ etc.):

$$\begin{aligned}\Big(1 \otimes (\beta)^{(b)}\Big)\Big((\alpha)^{(a)} \otimes 1\Big) &= \sum_{(R^i)} \Big(S(R^3_{(1)}R^4_{(1)}) \triangleright \alpha \triangleleft R^1_{(1)}R^2_{(1)}\Big)^{(a)} \\ &\qquad \otimes \Big(S(R^1_{(2)}R^3_{(2)}) \triangleright \beta \triangleleft R^2_{(2)}R^4_{(2)}\Big)^{(b)} \\ &= \sum_{(R^i),(\alpha)^4,(\beta)^2} (\alpha_{(2)})^{(a)} \otimes (\beta_{(2)})^{(b)} \\ &\qquad \times \beta_{(1)}\Big(\alpha_{(1)(2)}(R^2_{(1)})R^2_{(2)}\alpha_{(3)(1)}(S(R^4_{(1)}))R^4_{(2)}\Big) \\ &\qquad \times \beta_{(3)}\Big(\alpha_{(3)(2)}(R^3_{(1)})R^3_{(2)}\alpha_{(1)(1)}(R^1_{(1)})S(R^1_{(2)})\Big).\end{aligned}$$

By Theorem 2.24 (2) it follows that $\alpha_{(1)(2)}(R^2_{(1)})R^2_{(2)} = \Phi^+(\alpha_{(1)(2)}) \in \mathcal{Z}_0(U_\epsilon)$, and similarly $\alpha_{(3)(1)}(S(R^4_{(1)}))R^4_{(2)}, \alpha_{(3)(2)}(R^3_{(1)})R^3_{(2)}, \alpha_{(1)(1)}(R^1_{(1)})S(R^1_{(2)}) \in \mathcal{Z}_0(U_\epsilon)$. Denote by $z$ any such element; $\mathcal{Z}_0(U_\epsilon^{ad})$ acts by the trivial character (the counit $\varepsilon$) on $\Gamma_\epsilon$-modules. By using Proposition 6.2 of [28], arguing as above (108), we obtain that the expression of $z$ in terms of the corresponding $\alpha_{(i)(j)}$ involves $\varepsilon(z) = \varepsilon(\alpha_{(i)(j)})$ only (no root $\epsilon^{1/D}$). It follows

$$\begin{aligned}\beta_{(1)}\Big(\alpha_{(1)(2)}(R^2_{(1)})R^2_{(2)}\alpha_{(3)(1)}(S(R^4_{(1)}))R^4_{(2)}\Big) &= \varepsilon(\alpha_{(1)(2)}\alpha_{(3)(1)})\beta_{(1)}(1) \\ &= \varepsilon(\alpha_{(1)(2)})\varepsilon(\alpha_{(3)(1)})\varepsilon(\beta_{(1)}) \\ \beta_{(3)}\Big(\alpha_{(3)(2)}(R^3_{(1)})R^3_{(2)}\alpha_{(1)(1)}(R^1_{(1)})S(R^1_{(2)})\Big) &= \varepsilon(\alpha_{(3)(2)})\varepsilon(\alpha_{(1)(1)})\varepsilon(\beta_{(3)}).\end{aligned}$$

Therefore $(1 \otimes (\beta)^{(b)})((\alpha)^{(a)} \otimes 1) = (\alpha)^{(a)} \otimes (\beta)^{(b)}$. It follows that $\mathcal{L}^\epsilon_{0,n} = \mathcal{O}^{\otimes n}_\epsilon$ as modules over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$; for instance when $n = 2$, given $\alpha', \beta' \in \mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$ we have $(\alpha' \otimes \beta')(\alpha \otimes \beta) = (\alpha' \otimes 1)(1 \otimes \beta')(\alpha \otimes 1)(1 \otimes \beta)$ immediately by (18), and $(1 \otimes \beta')(\alpha \otimes 1) = \alpha \otimes \beta' = (\alpha \otimes 1)(1 \otimes \beta')$ as above. Then $(\alpha' \otimes \beta')(\alpha \otimes \beta) = \alpha'\alpha \otimes \beta'\beta$. In particular $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$ is a central subalgebra of $\mathcal{L}^\epsilon_{0,n}$.

(2) Since $\mathcal{L}^\epsilon_{0,n}$ and $\mathcal{O}^{\otimes n}_\epsilon$ coincide as modules over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}) = \mathcal{Z}_0(\mathcal{O}^{\otimes n}_\epsilon)$, the claim follows from Theorem 2.24, that is: from Theorem 7.2 of [43], which shows that $\mathcal{O}_\epsilon$ is a finitely generated projective module of rank $l^{dim\mathfrak{g}}$ over $\mathcal{Z}_0(\mathcal{O}_\epsilon)$, and from the arguments of [25] (using that $K_0(\mathcal{O}(G)) = \mathbb{Z}$ by [81]), which imply that this module is free. Alternatively, it follows from the fact that $\mathcal{O}_\epsilon$ is a cleft extension of $\mathcal{O}(G)$ (see [5], Remark 2.18 (b), and Section 3.2 of [21]).

(3) The linear isomorphism $(\eta^{*-1})^{\otimes n} \colon \mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}) \to \mathcal{O}(G)^{\otimes n}$ is an isomorphism of algebras because $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$ is central in $\mathcal{L}^\epsilon_{0,n}$. It implies that $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$ is a Noetherian ring, since $\mathcal{O}(G)^{\otimes n} = \mathcal{O}(G^n)$ and $G^n$ is an affine algebraic variety.

(4) The fact that $\mathcal{L}^\epsilon_{0,n}$ is a finitely generated $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-module follows from (2); an alternative proof of this fact will be provided at the end of the proof of Theorem 4.9. Since $\mathcal{L}^\epsilon_{0,n}$ is finite over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$ and $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$ is Noetherian, $\mathcal{L}^\epsilon_{0,n}$ is a Noetherian $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-module (eg. by [6] Proposition 6.5). It follows that $\mathcal{L}^\epsilon_{0,n}$ is a Noetherian ring (by eg. [82], Section 1.3 of Chapter 1). □

Recall that we denote $U^{lf}_\epsilon = U^{lf}_A \otimes_A \mathbb{C}_\epsilon$ (see (35)), and $\mathcal{Z}_0(U_\epsilon) \subset U_\epsilon$ is the central polynomial subalgebra generated by $E^l_{\beta_k}$, $F^l_{\beta_k}$, $L^{\pm l}_i$, for $k \in \{1, \dots, N\}$ and $i \in \{1, \dots m\}$. Since $\Phi_1 \colon \mathcal{L}^\epsilon_{0,1} \to U^{lf}_\epsilon$ is an embedding of algebras (see Corollary 2.20), it identifies $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$ with

a central subalgebra of $U_\epsilon^{lf}$. Put

$$\mathcal{Z}_0(U_\epsilon^{lf}) := \Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)).$$

Recall Theorem 2.1, Proposition 2.19, and let $T^{(l)}$, $T_{2-}^{(l)}$ and $T_2^{(l)}$ be the subsets of $T$, $T_{2-}$ and $T_2$ formed by the elements $K_{\lambda l}$ with $\lambda \in P$, $\lambda \in -2P_+$ and $\lambda \in 2P$ respectively.

**Proposition 4.2.** *We have $U_\epsilon = T_{2-}^{-1}U_\epsilon^{lf}[T/T_2] = \Phi_1(\mathcal{L}_{0,1}^\epsilon[d^{-1}])[T/T_2]$, and therefore the map $\Phi_1 \colon \mathcal{L}_{0,1}^\epsilon[d^{-1}] \to T_{2-}^{-1}U_\epsilon^{lf}$ is an isomorphism. Moreover $\mathcal{Z}(U_\epsilon^{lf}) = U_\epsilon^{lf} \cap \mathcal{Z}(U_\epsilon)$, and*

$$\mathcal{Z}_0(U_\epsilon) = T_{2-}^{(l)-1}\mathcal{Z}_0(U_\epsilon^{lf})[T^{(l)}/T_2^{(l)}]\ ,\ \mathcal{Z}(U_\epsilon) = T_{2-}^{(l)-1}\mathcal{Z}(U_\epsilon^{lf})[T^{(l)}/T_2^{(l)}].$$

*Proof.* The first claim follows immediately from Proposition 2.19 by specialization at $q = \epsilon$. For the second claim, the inclusion $U_\epsilon^{lf} \cap \mathcal{Z}(U_\epsilon) \subset \mathcal{Z}(U_\epsilon^{lf})$ is clear, and for the converse inclusion we only have to show that the elements of $\mathcal{Z}(U_\epsilon^{lf})$ commute with $T$. They commute with $T_2 \subset U_\epsilon^{lf}$, so the conjugation action by elements of $T$ on $\mathcal{Z}(U_\epsilon^{lf})$ has order at most 2. Let $x \in \mathcal{Z}(U_\epsilon^{lf})$ with decomposition $x = \sum_i c_i x_i$ with all $c_i \in \mathbb{C}$ and $x_i$ PBW basis vectors, and let $\lambda \in P$. We have $K_\lambda x K_{-\lambda} = \sum_i c_i q(x_i) x_i$, where $q(x_i) \in \epsilon^{\mathbb{Z}}$ satisfies $q(x_i)^2 = 1$ for all $i$. Because $\epsilon$ has odd order the only possibility is $q(x_i) = 1$, whence $K_\lambda x K_{-\lambda} = x$. The conclusion follows.

The inclusion $\mathcal{Z}_0(U_\epsilon^{lf}) \subset \mathcal{Z}_0(U_\epsilon)$ follows from the definition $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$, the formula $\Phi_1 = m \circ (id \otimes S^{-1}) \circ \Phi$, and the fact that $\Phi$ affords an embedding $\mathcal{Z}_0(\mathcal{O}_\epsilon) \to \mathcal{Z}_0(U_\epsilon(G^*))$ (see Theorem 2.24 (2)). Since $T^{(l)} \subset \mathcal{Z}_0(U_\epsilon)$, we obtain $T_{2-}^{(l)-1}\mathcal{Z}_0(U_\epsilon^{lf})[T^{(l)}/T_2^{(l)}] \subset \mathcal{Z}_0(U_\epsilon)$. The proof of the converse inclusion is similar to that in Proposition 2.19. The isomorphism $\mathcal{Z}_0(\mathcal{O}_\epsilon)[\psi_{-l\rho}^{-1}] \to \mathcal{Z}_0(U_\epsilon(G^*))$ of Theorem 2.24 (2) implies $F_{\beta_k}^l K_{\beta_k}^l \otimes 1, 1 \otimes K_{\beta_k}^{-l} E_{\beta_k}^l \in \Phi(\mathcal{Z}_0(\mathcal{O}_\epsilon)[\psi_{-l\rho}^{-1}])$ for every positive root $\beta_k$. Since $\psi_{-l\rho} = \Phi_1^{-1}(K_{-2l\rho}) = \psi_{-\rho}^l$ (the $l$-th power of $\psi_{-\rho}$ in $\mathcal{L}_{0,1}^\epsilon$), and $\Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[\psi_{-\rho}^{-l}]) = T_{2-}^{(l)-1}\mathcal{Z}_0(U_\epsilon^{lf})$, it follows that $F_{\beta_k}^l K_{\beta_k}^l, S^{-1}(E_{\beta_k}^l)K_{\beta_k}^l \in T_{2-}^{(l)-1}\mathcal{Z}_0(U_\epsilon^{lf})$, hence $F_{\beta_k}^l, S^{-1}(E_{\beta_k}^l) \in T_{2-}^{(l)-1}\mathcal{Z}_0(U_\epsilon^{lf})[T^{(l)}/T_2^{(l)}]$. The sets $S^{-1}(E_{\beta_k}^l)\mathcal{Z}_0(U_\epsilon(\mathfrak{h}))$, $k = 1, \dots, N$, generate the subalgebra $\mathcal{Z}_0(U_\epsilon(\mathfrak{b}_+))$ of $\mathcal{Z}_0(U_\epsilon)$, so from the triangular decomposition $\mathcal{Z}_0(U_\epsilon) = \mathcal{Z}_0(U_\epsilon(\mathfrak{n}_-))\mathcal{Z}_0(U_\epsilon(\mathfrak{h}))\mathcal{Z}_0(U_\epsilon(\mathfrak{n}_+))$ this proves the inclusion $\mathcal{Z}_0(U_\epsilon) \subset T_{2-}^{(l)-1}\mathcal{Z}_0(U_\epsilon^{lf})[T^{(l)}/T_2^{(l)}]$. By the isomorphism $\mathcal{Z}_0(U_\epsilon) \otimes_{\mathcal{Z}_0(U_\epsilon)\cap\mathcal{Z}_1(U_\epsilon)} \mathcal{Z}_1(U_\epsilon) \to \mathcal{Z}(U_\epsilon)$ (see Theorem 2.22), and the fact that $\mathcal{Z}(U_q) \subset U_q^{lf}$ (whence $\mathcal{Z}_1(U_\epsilon) \subset \mathcal{Z}(U_\epsilon^{lf})$), the equality $\mathcal{Z}(U_\epsilon) = T_{2-}^{(l)-1}\mathcal{Z}(U_\epsilon^{lf})[T^{(l)}/T_2^{(l)}]$ follows at once. □

**Remark 4.3.** Let us explain how this can be used to give an interpretation of the isomorphism $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \cong \mathcal{O}(G)$. Recall the notations introduced around Theorem 2.22. Since $G^* = U_+T_GU_-$, we have $\mathcal{O}(G^*) = \mathcal{O}(U_+)\mathcal{O}(T_G)\mathcal{O}(U_-)$, and the map $\sigma$ yields an identification

$$\mathcal{O}(G^0) = \mathcal{O}(U_+)\mathcal{O}(T_G/(2))\mathcal{O}(U_-). \tag{109}$$

We can identify $\mathcal{O}(G^0)$ with the subalgebra $(\sigma_{|G^*})^*(\mathcal{O}(G^0)) \subset \mathcal{O}(G^*)$. Consider the exterior power $V = \wedge^N\mathfrak{g}$ endowed with the action $\wedge^N Ad$ of $G$. Put on $\mathfrak{g}$ a basis consisting of one element $e_\alpha$ per root space $\mathfrak{g}_\alpha$, along with a basis of $\mathfrak{h}$. Let $v \in V$ be the exterior power of the $e_\alpha$'s for $\alpha$ negative, and $v^*$ a dual vector such that $v^*(v) = 1$ and $v^*$ vanishes on a $T_G$-invariant complement of $v$. It is classical that $G \setminus G^0$ has defining equation $\delta(g) = 0$, where $\delta$ is the matrix coefficient $\delta(g) = v^*(\pi_V(g)v)$ (see eg. [57], page 174). Hence $\mathcal{O}(G^0) = \mathcal{O}(G)[\delta^{-1}]$. On $G^0$ we have $\delta(u_+tu_-) = \chi_{-2\rho}(t)$, where $\chi_{-2\rho}$ is the character of $T_G$ associated to the root $-2\rho$. Now we can make the connection with $U_\epsilon$. The isomorphism $\mathcal{Z}_0(U_\epsilon) \cong \mathcal{O}(G^*)$ of Theorem 2.22 (2) identifies $\mathcal{Z}_0(U_\epsilon(\mathfrak{h})) = \mathbb{C}[T^{(l)}]$ with $\mathcal{O}(T_G)$ by mapping $K_{\lambda l}$ to the character of $T_G$ associated to $\lambda$. Therefore it maps $\mathbb{C}[T_2^{(l)}]$ to $\mathcal{O}(T_G/(2))$, and $T_{2-}^{(l)-1}\mathcal{Z}_0(U_\epsilon^{lf})$ to $\mathcal{O}(G^0)$

by (109) and Proposition 4.2. Since $\mathcal{O}(G^0) = \mathcal{O}(G)[\delta^{-1}]$ and $T_{2-}^{(l)-1}\mathcal{Z}_0(U_\epsilon^{lf}) = \mathcal{Z}_0(U_\epsilon^{lf})[\ell^l]$, it follows that $\mathcal{Z}_0(U_\epsilon^{lf})$ and $\mathcal{O}(G)$ coincide after localization by $\ell^l$ and $\delta$ respectively. By using the Bruhat decomposition of $G$ as in (114) in the proof of Theorem 4.9 below, one can deduce $\mathcal{Z}_0(U_\epsilon^{lf}) \cong \mathcal{O}(G)$, whence $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \cong \mathcal{O}(G)$ by injectivity of $\Phi_1$.

Let us make the following observation. Since $\mathcal{L}_{0,n}^\epsilon = \mathcal{L}_{0,n}^A \otimes_A \mathbb{C}_\epsilon$, with $\mathcal{L}_{0,n}^A = \mathcal{O}_A^{\otimes n}$ as an $A$-module, and a generating system of $\mathcal{O}_A^{\otimes n}$ is also a generating system of $\mathcal{L}_{0,n}^A$, it follows from Proposition 2.7 and the identities 67-68 that $\mathcal{L}_{0,n}^\epsilon$ is generated by elements of the form $\alpha_1 \otimes \ldots \otimes \alpha_n$, where $\alpha_1, \ldots, \alpha_n$ belong to the set $C_{gen}$ of matrix coefficients lying on the first and last columns of the matrix representations of $U_A^{res}$ in the canonical basis of the modules ${}_AV_{\varpi_i}$, $i = 1, \ldots, m$. Denote by $\alpha^{\star k}$, $k \in \mathbb{N}$, the $k$-th power of an element $\alpha \in \mathcal{O}_A$.

**Lemma 4.4.** *For all $\alpha \in C_{gen}$, $\alpha^{\star l} \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$.*

*Proof.* Recall that the Frobenius epimorphism $\eta\colon U_A^{res} \otimes_A \mathbb{C}_\epsilon \to U(\mathfrak{g})$ in (86) has kernel the ideal $I$ generated by the elements $E_i$, $F_i$, $K_i - 1$, and $(K_i; p)_{q_i}$ where $l$ does not divide $p$, $i = 1, \ldots, m$. It follows that an element of $\mathcal{O}_\epsilon$ belongs to $\mathcal{Z}_0(\mathcal{O}_\epsilon) = \eta^*(\mathcal{O}(G))$ if and only if it vanishes on $I$. But this is immediate to check for the elements of the form $\alpha^{\star l}$ with $\alpha \in C_{gen}$, using that $K_i$ is grouplike and the pure summands of $\Delta(E_i)$ and $\Delta(F_i)$ have one component equal to 1 or $K_i^{\pm 1}$ and the other component equal to $E_i$ or $F_i$. For instance, $\psi_{\varpi_i}^{\star l}(K_i - 1) = \psi_{\varpi_i}(K_i)^l - 1 = \epsilon^{l(\alpha_i, \varpi_i)} - 1 = 0$. Similarly, for every $\alpha \in C_{gen}$ we find $\alpha^{\star l}(E_i) = \alpha^{\otimes l}(\Delta^{(l)}(E_i)) = 0$, and $\alpha^{\star l}(F_i) = \alpha^{\star l}(K_i - 1) = 0$. □

We need below explicit descriptions of the centers of $\mathcal{O}_\epsilon(SL_2)$ and $\mathcal{L}_{0,1}^\epsilon(sl(2))$ and their $\mathcal{Z}_0$-subalgebras. Denote by $a$, $b$, $c$, $d$ the standard generators of $\mathcal{O}_q(SL_2)$, ie. the matrix coefficients in the basis of weight vectors $v_0$, $v_1 = F.v_0$ of the 2-dimensional irreducible representation $V_1$ of $U_q(sl(2))$. As above, denote by $x^{\star k}$, $k \in \mathbb{N}$, the $k$-th power of an element $x \in \mathcal{O}_A(SL_2)$. The algebra $\mathcal{O}_A(SL_2)$ is generated by $a$, $b$, $c$, $d$; the monomials $a^{\star i} \star b^{\star j} \star d^{\star r}$ and $a^{\star i} \star c^{\star k} \star d^{\star r}$, $i, j, k, r \in \mathbb{N}, k > 0$, form an $A$-basis of $\mathcal{O}_A(SL_2)$. The algebra $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ is generated by $a^{\star l}$, $b^{\star l}$, $c^{\star l}$, $d^{\star l}$; the monomials $a^{\star il} \star b^{\star jl} \star d^{\star rl}$ and $a^{\star il} \star c^{\star kl} \star d^{\star rl}$ form a basis of $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$, and $\mathcal{Z}(\mathcal{O}_\epsilon(SL_2))$ is generated by $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ and the elements $b^{\star(l-k)} \star c^{\star k}$, $k = 0, \ldots, l$ (see [43], Proposition 1.4 and the Appendix). We have the relation

$$a^{\star l} \star d^{\star l} - b^{\star l} \star c^{\star l} = 1 \tag{110}$$

and the Frobenius isomorphism of Parshall-Wang (see [78], Chapter 7) coincides with the map

$$Fr_{PW}\colon \mathcal{O}(SL_2) \to \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$$

induced by $\eta^*$; it sends the standard generators $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ of $\mathcal{O}(SL_2) = \mathcal{O}_1(SL_2)$ respectively to $a^{\star l}$, $b^{\star l}$, $c^{\star l}$, $d^{\star l}$. Finally, we have seen that $\mathcal{O}_\epsilon(SL_2)$ is a free $\mathcal{Z}_0(\mathcal{O}(SL_2))$-module of rank $l^3$ (see Theorem 2.24 (3)). In [45] it is shown that a basis of this module is formed by the monomials $a^m b^n c^{s'}$ and $b^n c^{s''} d^r$, with the integers $m, n, r, s', s''$ in the range

$$1 \leq m \leq l-1\ , 0 \leq n, r \leq l-1\ , m \leq s' \leq l-1\ , 0 \leq s'' \leq l-r-1. \tag{111}$$

Now consider $\mathcal{L}_{0,1}^A(sl(2))$. Recall that $\mathcal{L}_{0,1}^A = \mathcal{O}_A$ as $U_A$-modules. The algebra $\mathcal{L}_{0,1}^A(sl(2))$ is also generated by $a$, $b$, $c$, $d$; a set of defining relations is (see [28], Section 5):

$$\begin{array}{lll} ad = da & , & ab - ba = -(1 - q^{-2})bd \\ db = q^2 bd & , & cb - bc = (1 - q^{-2})(da - d^2) \\ cd = q^2 dc & , & ac - ca = (1 - q^{-2})dc \\ & & ad - q^2 bc = 1. \end{array} \tag{112}$$

The element $\omega := qa + q^{-1}d$ is central. Let $T_k$, $k \in \mathbb{N}$, be such that $T_k(x)/2$ is the $k$-th Chebyshev polynomial of the first type in the variable $x/2$. We have (see [28], Proposition 7.2, for the generalization to $\mathcal{L}^\epsilon_{0,n}(sl(2))$):

**Lemma 4.5.** *$\mathcal{Z}(\mathcal{L}^\epsilon_{0,1}(sl(2))) = \mathbb{C}[\omega, b^l, c^l, d^l]/\mathcal{I}$ and $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1}(sl(2))) = \mathbb{C}[(T_l(\omega), b^l, c^l, d^l]/\mathcal{I}$, where $\mathcal{I}$ is the ideal of $\mathbb{C}[\omega, b^l, c^l, d^l]$ generated by $(T_l(\omega) - d^l)d^l - b^l c^l - 1$.*

Here $b^l, c^l, d^l$ are the $l$-th powers of $b, c, d$ computed using the product of $\mathcal{L}^A_{0,1}(sl(2))$, not the product $\star$ of $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$. The above generator of $\mathcal{I}$ can be interpreted as a determinant, and $\omega$ as a quantum trace on $V_1$.

**Lemma 4.6.** *Viewed as elements of $\mathcal{O}_A(SL_2)$, $T_l(\omega) - d^l = a^{\star l}$ and $x^l = x^{\star l}$, $x \in \{b, c, d\}$.*

*Proof.* Let $\alpha$ and $\varpi$ be the simple root and fundamental weight of $sl(2)$. In the notations of (85) we have $b = \psi^{-\alpha}_{-\varpi}$, $c = \psi^{\alpha}_{-\varpi}$, $d = \psi_{-\varpi}$; the formulas give $\Phi_1(b^{\star l}) = (q - q^{-1})^l F^l$, $\Phi_1(c^{\star l}) = (q - q^{-1})^l E^l K^{-l}$, $\Phi_1(d^{\star l}) = K^{-l}$. These coincide respectively with $\Phi_1(b^l)$, $\Phi_1(c^l)$, $\Phi_1(d^l)$ (see (32) in [28]). By passing to the localization $\mathcal{O}_A(SL_2)[d^{-1}]$, and using Parshall-Wang's relation (110), one deduces easily $\Phi_1(a^{\star l}) = K^l + (q - q^{-1})^{2l} F^l E^l = T_l(\Omega) - K^{-l}$, where $\Omega = (\epsilon - \epsilon^{-1})^2 FE + \epsilon K + \epsilon^{-1} K^{-1}$ is the Casimir element, and $T_l(x)/2$ is the $l$-th Chebyshev polynomial of the first type in the variable $x/2$. We have $\Phi_1(\omega) = \Omega$, so $\Phi_1(a^{\star l}) = T_l(\omega) - d^l$. The conclusion follows from the injectivity of $\Phi_1$. □

This lemma proves that we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}(SL_2) & \xrightarrow{Fr_{PW}} & \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2)) & \hookrightarrow & \mathcal{O}_\epsilon(SL_2) \\ & \searrow^{Fr} & \downarrow & & \downarrow \\ & & \mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1}(sl(2))) & \hookrightarrow & \mathcal{L}^\epsilon_{0,1}(sl(2)) \end{array}$$

where $Fr_{PW}$ is Parshall-Wang's Frobenius isomorphism recalled above, $Fr$ is the Frobenius isomorphism introduced in [28], Definition 7.1, and the vertical arrows are the identifications as vector spaces (the middle one proved by Proposition 4.1).

**Remark 4.7.** By Lemma 4.5 the monomials $T_l(\omega)^i b^{jl} d^{rl}$ and $T_l(\omega)^i c^{kl} d^{rl}$, for $i, j, k, r \in \mathbb{N}$ and $k > 0$, form an $A$-basis of $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1}(sl(2)))$. It is straightforward (though cumbersome) to express these basis elements in terms of the basis elements $a^{\star il} \star b^{\star jl} \star d^{\star rl}$ and $a^{\star il} \star c^{\star kl} \star d^{\star rl}$ of $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$, and conversely; this can be done by using Lemma 4.6, the formula (9) or the inverse one (expressing $\star$ in terms of the product of $\mathcal{L}_{0,1}$, see (18) in [28]), and the formula of the coproduct $\Delta\colon \mathcal{L}^\epsilon_{0,1}(sl(2))) \to \mathcal{L}^\epsilon_{0,2}(sl(2)))$ restricted to $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1}(sl(2)))$ (given in Lemma 7.5 of [28]).

Since $\mathcal{L}^A_{0,1} = \mathcal{O}_A$ as an $A$-module, the functionals $t_i$ in Proposition 2.25 can be seen as maps $t_i : \mathcal{L}^A_{0,1} \to A$. Though the algebra structures of $\mathcal{O}_\epsilon$ and $\mathcal{L}^\epsilon_{0,1}$ are very different, $\mathcal{L}^\epsilon_{0,1}$ satisfies a result analogous to Proposition 2.25:

**Proposition 4.8.** *The maps $\lhd t_i$ preserve $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$, and they satisfy $(f \lhd t_i)(a) = f(n_i a)$ and $(f\alpha) \lhd t_i = (f \lhd t_i)(\alpha \lhd t_i)$ for every $f \in \mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$, $a \in G$, $\alpha \in \mathcal{L}^\epsilon_{0,1}$.*

*Proof.* The first two claims follow from Proposition 2.25 and the definition $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1}) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$.

The last claim follows from the case $\mathfrak{g} = sl(2)$, as in the proof of Proposition 7.1 of [43]. In fact it is enough to show that $t(fg) = t(f)t(g)$ for every $f \in \mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1}(sl(2))$, $g \in \mathcal{L}^\epsilon_{0,1}(sl(2))$; for completeness we explain this in the Appendix, see (138). A word of caution

is needed: the proof of (138) uses that $\Delta\colon \mathcal{O}_\epsilon \to \mathcal{O}_\epsilon \otimes \mathcal{O}_\epsilon$ is a morphism of algebras. The analogous property for $\mathcal{L}^\epsilon_{0,1}$ is that $\Delta$ yields a morphism of algebras $\Delta\colon \mathcal{L}^\epsilon_{0,1} \to \mathcal{L}^\epsilon_{0,2}$. Since the algebra structure of $\mathcal{L}^\epsilon_{0,2}$ is not the product one on $\mathcal{L}^\epsilon_{0,1} \otimes \mathcal{L}^\epsilon_{0,1}$, it is not true in general that $\sum_{(f),(g)}(f_{(1)} \otimes f_{(2)})(g_{(1)} \otimes g_{(2)}) = \sum_{(f),(g)} f_{(1)}g_{(1)} \otimes f_{(2)}g_{(2)}$ for every $f, g \in \mathcal{L}^\epsilon_{0,1}$. However it holds whenever $f \in \mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$, since $\Delta(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})) \subset \mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1}) \otimes \mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$ and therefore $f_{(2)} \in \mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1}) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$ commutes in $\mathcal{L}^\epsilon_{0,2}$ with any $g_{(1)} \in \mathcal{L}^\epsilon_{0,1} = \mathcal{O}_\epsilon$.

It is enough to prove the identity $t(fg) = t(f)t(g)$ when $f$ ranges in a set of generators of the algebra $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1}(sl(2)))$. So one can take $f$ among, say, $T_l(\omega) - d^l = a^{\star l}$ and $x^l = x^{\star l}$, $x \in \{b, c, d\}$ (using Lemma 4.5). By (9) and Proposition 6.1 in the Appendix we have

$$t(fg) = \sum_{(R),(R)} t\left(R_{(2')}S(R_{(2)}) \rhd f\right) t\left(R_{(1')} \rhd g \lhd R_{(1)}\right).$$

Expanding coproducts and using that $R^{-1} = (S \otimes id)(R)$ we deduce

$$\begin{aligned} t(fg) &= \sum_{(f),(R),(R)} t\big(f_{(1)}\big) \left\langle f_{(2)}, R_{(2')}S(R_{(2)})\right\rangle \; t\left(R_{(1')} \rhd g \lhd R_{(1)}\right) \\ &= \sum_{(f),(R),(R)} t\big(f_{(1)}\big)\; t\Big( \left\langle f_{(2)}, R_{(2')}\right\rangle R_{(1')} \rhd g \lhd \left\langle f_{(3)}, S(R_{(2)})\right\rangle R_{(1)}\Big) \\ &= \sum_{(f)} t(f_{(1)})\; t\Big(S^{-1}(\Phi^-(f_{(2)})) \rhd g \lhd S^{-2}(\Phi^-(f_{(3)}))\Big) \\ &= \sum_{(f)} t(f_{(1)}) \left\langle g, S^{-2}(\Phi^-(f_{(3)}))\underline{w}S^{-1}(\Phi^-(f_{(2)}))\right\rangle \\ &= \sum_{(f)} t(f_{(1)})\; \varepsilon\Big(S^{-2}(\Phi^-(f_{(3)}))\Big)\varepsilon\Big(S^{-1}(\Phi^-(f_{(2)}))\Big)t(g) \end{aligned}$$

where $\underline{w} \in \mathbb{U}_\Gamma$ is the quantum Weyl group element dual to $t$ (see Section 6.2), and in the last equality we used that $\Phi^-$ maps $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ into $\mathcal{Z}_0(U_\epsilon)$ (see Theorem 2.24 (2)), which acts on $\Gamma$-modules by the trivial character (the counit) $\varepsilon\colon U_\epsilon \to \mathbb{C}$. By (130)-(131) in the Appendix we have $t(a^{\star l}) = t(d^{\star l}) = 0$ and $t(b^{\star l}) = 1$, $t(c^{\star l}) = -1$. Now the computation of $t(fg)$ follows easily. For instance, taking $f = b^l = b^{\star l}$, by using $\Delta(b^{\star l}) = a^{\star l} \otimes b^{\star l} + b^{\star l} \otimes d^{\star l}$ and $\Delta(d^{\star l}) = c^{\star l} \otimes b^{\star l} + d^{\star l} \otimes d^{\star l}$ we get

$$t(b^l g) = \varepsilon\Big(S^{-2}(\Phi^-(b^{\star l}))\Big)\varepsilon\Big(S^{-1}(\Phi^-(c^{\star l}))\Big)t(g) + \varepsilon\Big(S^{-2}(\Phi^-(d^{\star l}))\Big)\varepsilon\Big(S^{-1}(\Phi^-(d^{\star l}))\Big)t(g)$$

Since $b^{\star l} \in \mathcal{O}_\epsilon(U_+)$, $\Phi^-(b^{\star l}) = 0$. Also, it is immediate from the definition of $\Phi^-$ that $\Phi^-(d^{\star l}) = \Phi^-(d)^l = L^l$; alternatively, one can bypass this computation by observing that $\Phi^-$ sets an isomorphism from $\mathcal{O}_\epsilon(T_G) = \mathcal{O}_\epsilon(B_+) \cap \mathcal{O}_\epsilon(B_-)$ to $\mathbb{C}[L^{\pm 1}] = U_\epsilon(\mathfrak{b}_+) \cap U_\epsilon(\mathfrak{b}_-)$, mapping a generator $d$ to $L$ or $L^{-1}$. We have $\varepsilon(L^l) = 1$, and therefore

$$t(b^l g) = t(g) = t(b^l)t(g).$$

The other cases $f = T_l(\omega) - d^l, c^l, d^l$ are similar. □

**Theorem 4.9.** *$\mathcal{L}^\epsilon_{0,n}$ is a free $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-module of rank $l^{n.dim\mathfrak{g}}$, and $(\mathcal{L}^\epsilon_{0,n})^{U_\epsilon}$ is a Noetherian ring and a finite, whence Noetherian, $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-module.*

*Proof.* We already proved the first claim in Proposition 4.1, and that $\mathcal{L}^\epsilon_{0,n}$ is a Noetherian $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-module. For the second claim, it follows that the $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-submodule $(\mathcal{L}^\epsilon_{0,n})^{U_\epsilon}$ is necessarily finitely generated. But $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$ being Noetherian, $(\mathcal{L}^\epsilon_{0,n})^{U_\epsilon}$ is then a Noetherian $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-module and a Noetherian ring.

For the sake of clarity let us provide a self-contained proof of the first claim, not invoking directly [43, 25] or [5, 21], but applying the same arguments directly on $\mathcal{L}^\epsilon_{0,n}$. Since $\mathcal{L}^\epsilon_{0,n}$ and $\mathcal{L}^{\otimes n}_{0,1}$ coincide as modules over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}) = \mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})^{\otimes n}$ by Proposition 4.1, the result will follow from the case $n = 1$. Then we argue in four steps. First, using Theorem 2.1 we show that a certain localization of $\mathcal{L}^\epsilon_{0,1}$ is a free module of rank $l^{dim\mathfrak{g}}$. Then, assuming that $\mathcal{L}^\epsilon_{0,1}$ is finitely generated and projective, we explain why it has constant rank $l^{dim\mathfrak{g}}$ (this is very classical). Thirdly, we prove that $\mathcal{L}^\epsilon_{0,1}$ is finitely generated and projective as in Theorem 7.2 of [43]. Finally we obtain that it is a free module as in [25].

Recall Proposition 4.2: $U_\epsilon$ is a free $\Phi_1(\mathcal{L}^\epsilon_{0,1}[d^{-l}])$-module of rank $2^m$ (note that $\mathcal{L}^\epsilon_{0,1}[d^{-l}] = \mathcal{L}^\epsilon_{0,1}[d^{-1}]$), and $\mathcal{Z}_0(U_\epsilon)$ is free over $T^{(l)-1}_{2-}\mathcal{Z}_0(U^{lf}_\epsilon) = \Phi_1(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})[d^{-l}])$ of rank $2^m$. Since $U_\epsilon$ is also free of rank $l^{dim\mathfrak{g}}$ over $\mathcal{Z}_0(U_\epsilon)$ (Theorem 2.22 (1)), it is free over $\Phi_1(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})[d^{-l}])$ of rank $2^m l^{dim\mathfrak{g}}$. The decomposition being unique, $\Phi_1(\mathcal{L}^\epsilon_{0,1}[d^{-l}])$ is free of rank $l^{dim\mathfrak{g}}$ over $\Phi_1(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})[d^{-l}])$, and injectivity of $\Phi_1$ implies that $\mathcal{L}^\epsilon_{0,1}[d^{-l}]$ is free of rank $l^{dim\mathfrak{g}}$ over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})[d^{-l}]$.

Assume now that $\mathcal{L}^\epsilon_{0,1}$ is finitely generated and projective. Let us show that its rank is $l^{dim\mathfrak{g}}$. The localization $(\mathcal{L}^\epsilon_{0,1})_P$ of $\mathcal{L}^\epsilon_{0,1}$ at any prime ideal $P$ of $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$ is a free module over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})_P$ ([84], Proposition 2.12.15); the ranks of such modules are finite in number ([84], Proposition 2.12.20). If these ranks are all equal, then, by definition, it is the rank of $\mathcal{L}^\epsilon_{0,1}$ over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$. This happens if $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$ has no non-trivial (ie. $\neq 1$) idempotent ([84], Corollary 2.12.23), which is the case since it has no non-trivial zero divisors. To compute the rank, suppose $P$ does not contain $d^l = \psi^l_{-\rho}$. Such ideals $P$ are in 1-1 correspondence with the prime ideals of $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})[d^{-l}]$ by the natural ring monomorphism $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1}) \to \mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})[d^{-l}]$. The set $S = \mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1}) \setminus P$ is multiplicatively closed, and we have also a ring morphism $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})[d^{-l}] \to S^{-1}\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$, which is also an injection (there are no zero divisors in $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$, whence in $S$). Then

$$(\mathcal{L}^\epsilon_{0,1})_P = S^{-1}\mathcal{L}^\epsilon_{0,1} = \mathcal{L}^\epsilon_{0,1}[d^{-l}] \otimes_{\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})[d^{-l}]} S^{-1}\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1}) \tag{113}$$

shows that $(\mathcal{L}^\epsilon_{0,1})_P$ has over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})_P = S^{-1}\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$ the same rank $l^{dim\mathfrak{g}}$ as $\mathcal{L}^\epsilon_{0,1}[d^{-l}]$ over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})[d^{-l}]$. This proves our claim.

In order to show that $\mathcal{L}^\epsilon_{0,1}$ is finitely generated and projective over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$ it is enough to show it is finite locally free, ie. there are elements $d_i \in \mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$ such that the localizations $\mathcal{L}^\epsilon_{0,1}[d_i^{-1}]$ are finite free $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})[d_i^{-1}]$-modules, and $\mathrm{Maxspec}(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1}))$ is covered by the open sets $U(d_i)$ made of the ideals not containing $d_i$ (see Lemma 77.2 of [90]).

We have seen above that $\mathcal{L}^\epsilon_{0,1}[d^{-l}]$ is free of rank $l^{dim\mathfrak{g}}$ over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})[d^{-l}]$. By Remark 4.3, $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})[d^{-l}] \cong \mathcal{Z}_0(U^{lf}_\epsilon)[\ell^l]$ is isomorphic to $\mathcal{O}(G^0)$, and $\mathcal{O}(G^0) = \mathcal{O}(G)[\delta^{-1}]$. Now, given $w \in W$ with a reduced expression $s_{i_1} \dots s_{i_k}$, put $t_w = t_{i_1} \dots t_{i_k}$. Let $w$ be represented by $n_w = n_{i_1} \dots n_{i_k}$ in $N(T_G)$. By Proposition 4.8 we have $(f \lhd t_w)(x) = f(n_w x)$ for every $f \in \mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$, $x \in G$. Then

$$\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})[d^{-l}] \lhd t_w \cong \mathcal{O}(n_w^{-1}G^0) \cong \mathcal{O}(G)[(\delta \lhd t_w)^{-1}]. \tag{114}$$

If $b_1, \dots, b_r$ ($r := l^{dim\mathfrak{g}}$) is a basis of $\mathcal{L}^\epsilon_{0,1}[d^{-l}]$ over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})[d^{-l}]$, then $\mathcal{L}^\epsilon_{0,1}[d^{-l}] \lhd t_w$ is free over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})[(d \lhd t_w)^{-l}] \cong \mathcal{O}(n_w^{-1}G^0)$ with basis $b_1 \lhd t_w, \dots, b_r \lhd t_w$. Consider the Bruhat decomposition of $G$: any $g \in G$ can be written in the form $g = b_1 n b_2$, where $b_1, b_2 \in B_-$, $n \in W$. Hence $g = nn^{-1}b_1 n b_2 \in nB_+B_- = nG^0$, and therefore

$$G = \cup_{w\in W}(B_- n_w B_-) = \cup_{w\in W}(n_w G^0).$$

For every $w \in W$ put

$$d^l_w := d^l \lhd t_w.$$

Under the isomorphism of $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$ with $\mathcal{O}(G)$, we thus get that $\mathrm{Maxspec}(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1}))$ is covered by the open sets $U(d^l_w) \cong n_w G^0$, and $\mathcal{L}^\epsilon_{0,1}[d_w^{-l}]$ is finite free over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})[d_w^{-l}]$. Therefore $\mathcal{L}^\epsilon_{0,1}$ is finitely generated and projective over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$.

Finally, let us explain why $\mathcal{L}^\epsilon_{0,1}$ is free over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$, following the arguments of [25]. Let $R$ be a commutative Noetherian ring, put $X = \mathrm{Maxspec}(R)$, and let $P$ be an $R$-module. Denote by $R_I$, $P_I$ the localizations of $R$, $P$ at a maximal ideal $I \in X$. Define the *f-rank* of $P$ as f-rank$(P) = \inf_{I\in X}\{$ f-rank$_{R_I}(P_I)\}$, where f-rank$_{R_I}(P_I) = \sup\{r \in \mathbb{N}, R_I^{\otimes r} \subset P_I\} \in \mathbb{N} \cup \{+\infty\}$ (ie. the maximal dimension of a free summand of $P_I$). Bass' Cancellation theorem asserts that if $P$ is projective and f-rank$(P) > \dim(X)$, and $P \oplus Q \cong M \oplus Q$ for some $R$-modules $Q$ and $M$ such that $Q$ is finitely generated and projective, then $P \cong M$ (see [9], IV.3.5 and pages 167 and 170, taking $A = R$, or [82], section 11.7.13). Let us apply this to $R = \mathcal{O}(G)$ and $P = \mathcal{L}^\epsilon_{0,1}$. We proved above that f-rank$_{R_I}(P_I) = l^{dim\mathfrak{g}}$, a constant, and we have $l^{dim\mathfrak{g}} > dim\mathfrak{g} = \dim(G)$. By a result of Marlin [81], $G$ being semisimple and simply connected the Grothendieck ring $K_0(\mathcal{O}(G))$ is isomorphic to $\mathbb{Z}$. Therefore $\mathcal{L}^\epsilon_{0,1} \oplus Q \cong \mathcal{O}(G)^r$ for some free $\mathcal{O}(G)$-module $Q$ and $r \in \mathbb{N}$. Then Bass' Cancellation implies $\mathcal{L}^\epsilon_{0,1}$ is free over $\mathcal{Z}_0(\mathcal{L}_{0,1}) \cong \mathcal{O}(G)$. □

## 5. Proof of Theorem 1.3

We begin with the following lemma, interesting by itself.

**Lemma 5.1.** *$\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$ is a finite $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-module and a Noetherian ring. Therefore the ring $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$ is integral over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$.*

*Proof.* We know by Proposition 4.1 that $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$ is a Noetherian ring, and $\mathcal{L}^\epsilon_{0,n}$ is a finite Noetherian $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-module. Therefore the submodule $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$ is finitely generated. Being finite over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$, it is necessarily a Noetherian ring (by eg. Proposition 7.2 of [6]).

Let $x \in \mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$. The $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-submodule $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})[x]$ of $\mathcal{L}^\epsilon_{0,n}$ is finitely generated by the same argument. Using the fact that an element $x$ is integral over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$ if and only if $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})[x]$ is a finitely generated $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$-module (by eg. Proposition 5.1 of [6]), this proves the last claim. □

We will use the following notations. Let $A$ be a ring with no non-trivial zero divisors. The center $Z = Z(A)$ is a commutative integral domain. We denote by $Q(Z)$ its field of fractions, and put

$$Q(A) := Q(Z) \otimes_Z A.$$

It is an algebra over its center $Q(Z)$. Since $\mathcal{L}^\epsilon_{0,n}$ has no non-trivial zero divisors ([28], Proposition 6.30), we can take $A = \mathcal{L}^\epsilon_{0,n}$, or $A = (\mathcal{L}^\epsilon_{0,n})^{U_\epsilon}$.

By the lemma $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$ is finite over $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$, so the ring $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}) \otimes_{\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})} Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))$ is a field. Necessarily it coincides with $Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}))$, and therefore

$$Q(\mathcal{L}^\epsilon_{0,n}) = Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})) \otimes_{\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})} \mathcal{L}^\epsilon_{0,n} = Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})) \otimes_{\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})} \mathcal{L}^\epsilon_{0,n}. \tag{115}$$

Recall that we denote by $N$ the number of positive roots of $\mathfrak{g}$.

**Theorem 5.2. (Theorem 1.3(1))** *$Q(\mathcal{L}^\epsilon_{0,n})$ is a division algebra and a central simple algebra of PI degree $l^{Nn}$.*

*Proof.* It follows from (115) and Theorem 4.9 that $Q(\mathcal{L}^\epsilon_{0,n})$ is a vector space of dimension $l^{n.dim\mathfrak{g}}$ over $Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))$, and therefore has finite dimension over its center $Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}))$. Because $\mathcal{L}^\epsilon_{0,n}$ has no non-trivial divisors ([28], Proposition 6.30) and $Q(\mathcal{L}^\epsilon_{0,n})$ is finite-dimensional over $Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}))$, $Q(\mathcal{L}^\epsilon_{0,n})$ is a division algebra, whence a central simple algebra. By classical theory (see eg. Section 13.3.5 of [82], or [84], Corollary 2.3.25) there is a finite extension $\mathbb{F}$ of $Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}))$, a *splitting field*, such that

$$\mathbb{F} \otimes_{Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}))} Q(\mathcal{L}^\epsilon_{0,n}) = M_d(\mathbb{F})$$

where $d \in \mathbb{N}$, the PI degree of $Q(\mathcal{L}^\epsilon_{0,n})$, satisfies

$$d^2 = [Q(\mathcal{L}^\epsilon_{0,n}) : Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}))] = \frac{[Q(\mathcal{L}^\epsilon_{0,n}) : Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))]}{[Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})) : Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))]}. \tag{116}$$

We have to show $d^2 = l^{2nN}$. We will obtain this equality by proving firstly that $d^2 \geq l^{2nN}$, and then $d^2 \leq l^{2nN}$.

In order to show that $d^2 \geq l^{2nN}$, it is enough to exhibit an irreducible representation $V$ of $\mathcal{L}^\epsilon_{0,n}$ of dimension $k := l^{nN}$. Indeed, the representation map $\rho_V \colon \mathcal{L}^\epsilon_{0,n} \to End_{\mathbb{C}}(V)$ being surjective, given basis elements $v_1, \ldots, v_{k^2} \in End(V)$, and elements $\alpha_1, \ldots, \alpha_{k^2} \in \mathcal{L}^\epsilon_{0,n}$ such that $\rho(\alpha_i) = v_i$ for every $i \in \{1, \ldots, k^2\}$, necessarily $\alpha_1, \ldots, \alpha_{k^2}$ form a free family of $Q(\mathcal{L}^\epsilon_{0,n})$. For, if there was a non trivial relation $\sum_i z_i \alpha_i = 0$, with $z_i \in Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))$, by clearing denominators and then applying the representation map $\rho_V$ we would get a non trivial relation in $End_{\mathbb{C}}(V)$ between $v_1, \ldots, v_{k^2}$.

Now, by Theorem 2.22 (1) (see [41], section 20), the dimension of a generic irreducible representation space of $U_\epsilon$ is $l^N$. Because $U_\epsilon = T_{2-}^{-1} U_\epsilon^{lf}[T/T_2]$ (Proposition 4.2), an irreducible representation of $U_\epsilon$ yields an irreducible representation of $U_\epsilon^{lf}$. Moreover, the tensor product of $n$ irreducible representation spaces of $U_\epsilon^{lf}$ of dimension $l^N$ is an irreducible representation space of $(U_\epsilon^{lf})^{\otimes n}$ of dimension $l^{nN}$ (see eg. Theorem 3.10.2 in [49]). Applying the linear isomorphism $\psi_n = \Phi_n \circ (\Phi_1^{-1})^{\otimes n}$ in (23) thus provides an irreducible representation of $\mathcal{L}^\epsilon_{0,n}$ of dimension $l^{nN}$.

It remains to show $d^2 \leq l^{2nN}$, which by $[Q(\mathcal{L}^\epsilon_{0,n}) : Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))] = l^{n(2N+m)}$ is equivalent to $[Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})) : Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))] \geq l^{mn}$. For this, it is enough to exhibit an extension of $Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))$ contained in $Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}))$ and of degree $l^{mn}$. There is a very natural one, which we denote by $Q(\hat{\mathcal{Z}}_0(\mathcal{L}^\epsilon_{0,n}))$ and is constructed as follows. Consider for every $\lambda \in P_+$ the matrices

$$M_\lambda := ({}_{A}V_\lambda \phi^{e_l}_{e_k})_{k,l} \in End({}_AV_\lambda) \otimes \mathcal{L}^A_{0,n} \ , \ M^{(i)}_\lambda := (({}_{A}V_\lambda \phi^{e_l}_{e_k})^{(i)})_{k,l} \in End({}_AV_\lambda) \otimes \mathcal{L}^A_{0,n},$$

where $i = 1, \ldots, n$, and as usual ${}_{A}V_\lambda \phi^{e_l}_{e_k}$ is a matrix coefficient of ${}_AV_\lambda$, $\{e_k\}$ the canonical basis of ${}_AV_\lambda$, and $({}_{V_\lambda}\phi^{e_l}_{e_k})^{(i)} := 1^{\otimes(i-1)} \otimes {}_{V_\lambda}\phi^{e_l}_{e_k} \otimes 1^{\otimes(n-i)}$. Set

$${}_\lambda\omega := Tr(\pi_{V_\lambda}(\ell) M_\lambda) \ , \ {}_\lambda\omega^{(i)} := Tr(\pi_{V_\lambda}(\ell) M^{(i)}_\lambda)$$

where $Tr$ is the standard trace on $End(V_\lambda)$. Clearly ${}_\lambda\omega \in \mathcal{L}^A_{0,1}$, ${}_\lambda\omega^{(i)} \in \mathcal{L}^A_{0,n}$. By Proposition 4.8 and 6.24 of [28], the family of elements $\prod_{i=1}^n {}_{\lambda_i}\omega^{(i)}$, where $\lambda_1, \ldots, \lambda_n \in P_+$, is a basis of $\mathcal{Z}(\mathcal{L}_{0,n})$; moreover the Alekseev map $\Phi_n$ affords an isomorphism from $\mathcal{Z}(\mathcal{L}_{0,n})$ to $\mathcal{Z}(U_q)^{\otimes n}$, and $\Phi_n({}_\lambda\omega^{(i)}) = (\Phi_1({}_\lambda\omega))^{(i)}$. For $n = 1$, specializing $q$ to $\epsilon$ it follows

$$\mathcal{Z}_1(U_\epsilon) = Vect\,\{\Phi_1({}_\lambda\omega),\ \lambda \in P_+\}\,, \tag{117}$$

where $\mathcal{Z}_1(U_\epsilon)$ is defined before Theorem 2.22. Then, for every $i = 1, \dots, n$ define

$$\mathcal{Z}_{0,(i)}(\mathcal{L}^\epsilon_{0,n}) := \mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})[\{\lambda\omega^{(i)}, \lambda \in P_+\}]$$

and let $\hat{\mathcal{Z}}_0(\mathcal{L}^\epsilon_{0,n}) \subset \mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$ be the algebra generated by $\mathcal{Z}_{0,(1)}(\mathcal{L}^\epsilon_{0,n}), \dots, \mathcal{Z}_{0,(n)}(\mathcal{L}^\epsilon_{0,n})$. The fields $Q(\mathcal{Z}_{0,(i)}(\mathcal{L}^\epsilon_{0,n}))$ are $n$ linearly disjoint extensions of $Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))$, so

$$[Q(\hat{\mathcal{Z}}_0(\mathcal{L}^\epsilon_{0,n})) : Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))] = \prod_{i=1}^{n} [Q(\mathcal{Z}_{0,(i)}(\mathcal{L}^\epsilon_{0,n})) : Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))].$$

Now, by Proposition 4.2 we know that $\Phi_1$ affords isomorphisms $Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})) \cong Q(\mathcal{Z}_0(U^{lf}_\epsilon))$ and $Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,1})) \cong Q(\mathcal{Z}(U^{lf}_\epsilon))$, and moreover

$$Q(\mathcal{Z}_0(U_\epsilon)) = Q(\mathcal{Z}_0(U^{lf}_\epsilon))(T^{(l)}/T^{(l)}_2) \ , \ Q(\mathcal{Z}(U_\epsilon)) = Q(\mathcal{Z}(U^{lf}_\epsilon))(T^{(l)}/T^{(l)}_2). \tag{118}$$

Computing via the field embedding $\Phi_1^{\otimes n} \colon Q(\hat{\mathcal{Z}}_0(\mathcal{L}^\epsilon_{0,n})) \to Q(\mathcal{Z}(U^{\otimes n}_\epsilon))$, we deduce

$$\begin{aligned}
&[Q(\mathcal{Z}_{0,(i)}(\mathcal{L}^\epsilon_{0,n})) : Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))] \\
&\qquad = [\Phi_1^{\otimes n}(Q(\mathcal{Z}_{0,(i)}(\mathcal{L}^\epsilon_{0,n}))) : \Phi_1^{\otimes n}(Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})))] \\
&\qquad = \Big[Q(\mathcal{Z}_0(U^{lf}_\epsilon)^{\otimes n})[\{(\Phi_1(\lambda\omega))^{(i)}, \lambda \in P_+, i = 1, \dots, n\}] : Q(\mathcal{Z}_0(U^{lf}_\epsilon)^{\otimes n})\Big] \\
&\qquad = \Big[Q(\mathcal{Z}_0(U_\epsilon)^{\otimes n})[\{(\Phi_1(\lambda\omega))^{(i)}, \lambda \in P_+, i = 1, \dots, n\}] : Q(\mathcal{Z}_0(U_\epsilon)^{\otimes n})\Big] \\
&\qquad = l^m.
\end{aligned}$$

The second and third equalities follow from (118) and the properties of $\Phi_1$ recalled before it, and the last equality follows from Theorem 2.24 (2) and (117). As a result we have $[Q(\hat{\mathcal{Z}}_0(\mathcal{L}^\epsilon_{0,n})) : Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))] = l^{mn}$, whence $[Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})) : Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))] \geq l^{mn}$. Since $[Q(\mathcal{L}^\epsilon_{0,n}) : Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))] = l^{n(m+2N)}$, by (116) we obtain $d^2 \leq l^{2nN}$, which concludes the proof. □

**Remark 5.3.** By the degree computation above, it follows $[Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})) : Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))] = l^{mn}$, whence $Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})) = Q(\hat{\mathcal{Z}}_0(\mathcal{L}^\epsilon_{0,n}))$. In [27] we prove that $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}) = \hat{\mathcal{Z}}_0(\mathcal{L}^\epsilon_{0,n})$.

**Theorem 5.4. (Theorem 1.3(2))** *$Q((\mathcal{L}^\epsilon_{0,n})^{U_\epsilon})$, $n \geq 2$, is a division algebra and a central simple algebra of PI degree $l^{N(n-1)-m}$.*

*Proof.* The center of $(\mathcal{L}^\epsilon_{0,n})^{U_\epsilon}$ contains $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$, so the finite dimensionality of $Q(\mathcal{L}^\epsilon_{0,n})$ over $Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}))$ implies the finite dimensionality of $Q((\mathcal{L}^\epsilon_{0,n})^{U_\epsilon})$ over its center. Since it has no non-zero divisors, this proves $Q((\mathcal{L}^\epsilon_{0,n})^{U_\epsilon})$ is a division algebra.

Now denote by $\Delta^{(n)} \colon \mathcal{O}_\epsilon \to \mathcal{O}_\epsilon^{\otimes n}$, $n \geq 2$, the $n$-fold coproduct, ie. $\Delta^{(2)} := \Delta$, the standard coproduct of $\mathcal{O}_\epsilon$, and $\Delta^{(n)} := (id \otimes \Delta) \circ \Delta^{(n-1)}$ for $n \geq 3$. Identifying $\mathcal{L}^\epsilon_{0,n}$ with $\mathcal{O}_\epsilon^{\otimes n}$ as a vector space, we consider $\Delta^{(n)}$ as a map $\Delta^{(n)} \colon \mathcal{L}^\epsilon_{0,1} \to \mathcal{L}^\epsilon_{0,n}$. It is an algebra morphism ([28], Proposition 6.18), injective because $(\varepsilon^{\otimes(n-1)} \otimes id)\Delta^{(n)} = id$. Then it extends uniquely to the fraction algebra $Q(\mathcal{L}^\epsilon_{0,1})$. As noted above $Q(\mathcal{L}^\epsilon_{0,1}) = Q(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})) \otimes_{\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})} \mathcal{L}^\epsilon_{0,1}$. Since $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1}) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$ is a Hopf subalgebra of $\mathcal{O}_\epsilon$ ([43], Proposition 6.4), $\Delta^{(n)}$ maps $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$ to $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})^{\otimes n}$. Then, extending the scalars of $\Delta^{(n)}(Q(\mathcal{L}^\epsilon_{0,1}))$ by the field $Q(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}))$, consider

the algebra

$$\begin{aligned} Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}^{\epsilon}_{0,1})) &:= Q(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n})) \otimes_{\Delta^{(n)}(\mathcal{Z}_0(\mathcal{L}^{\epsilon}_{0,1}))} \Delta^{(n)}(\mathcal{L}^{\epsilon}_{0,1}) \\ &= Q(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n})) \otimes_{\Delta^{(n)}(Q(\mathcal{Z}_0(\mathcal{L}^{\epsilon}_{0,1})))} \Delta^{(n)}(Q(\mathcal{L}^{\epsilon}_{0,1})) \\ &= \Big(Q(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n})) \otimes_{\Delta^{(n)}(Q(\mathcal{Z}_0(\mathcal{L}^{\epsilon}_{0,1})))} \Delta^{(n)}(Q(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,1})))\Big) \\ &\qquad \otimes_{\Delta^{(n)}(Q(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,1})))} \Delta^{(n)}(Q(\mathcal{L}^{\epsilon}_{0,1})). \end{aligned}$$

By Proposition 5.2, $\Delta^{(n)}(Q(\mathcal{L}^{\epsilon}_{0,1}))$ is a $\Delta^{(n)}(Q(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,1})))$-central simple algebra. The left factor is a field, so $Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}^{\epsilon}_{0,1}))$ is a central simple algebra over it (see eg. [84], Theorem 1.7.27, or [91], Lemma 4.9). Note that the left factor can also be written as

$$\tilde{Q}(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n})) := Q(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n})) \otimes_{\Delta^{(n)}(\mathcal{Z}_0(\mathcal{L}^{\epsilon}_{0,1}))} \Delta^{(n)}(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,1}))$$

for it contains $\tilde{Q}(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n}))$, it is contained in its fraction field, and $\tilde{Q}(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n}))$ is a field because $\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,1})$ is finite over $\mathcal{Z}_0(\mathcal{L}^{\epsilon}_{0,1})$ and has no non trivial zero divisors. Note that

$$[\tilde{Q}(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n})) : Q(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n}))] = l^m$$

We proved in Proposition 6.19 of [28] that the ring $(\mathcal{L}^{A}_{0,n})^{U_A}$ is the centralizer of $\Delta^{(n)}(\mathcal{L}^{A}_{0,1})$ in $\mathcal{L}^{A}_{0,n}$; the same arguments show that $(\mathcal{L}^{\epsilon}_{0,n})^{U_\epsilon}$ is the centralizer of $\Delta^{(n)}(\mathcal{L}^{\epsilon}_{0,1})$ in $\mathcal{L}^{\epsilon}_{0,n}$. So the algebra

$$Q((\mathcal{L}^{\epsilon}_{0,n})^{U_\epsilon}) := Q(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n})) \otimes_{\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n})} (\mathcal{L}^{\epsilon}_{0,n})^{U_\epsilon}.$$

is the centralizer of $Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}^{\epsilon}_{0,1}))$ in $Q(\mathcal{L}^{\epsilon}_{0,n})$. Since the latter is simple, we can apply the double centralizer theorem (see eg. [84], Theorem 7.1.9, or [91], Theorem 7.1): $Q((\mathcal{L}^{\epsilon}_{0,n})^{U_\epsilon})$ is a simple algebra, we have

$$[Q((\mathcal{L}^{\epsilon}_{0,n})^{U_\epsilon}) : Q(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n}))] = \frac{[Q(\mathcal{L}^{\epsilon}_{0,n}) : Q(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n}))]}{[Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}^{\epsilon}_{0,1})) : Q(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n}))]} = l^{2nN-(2N+m)},$$

and the centralizer of $Q((\mathcal{L}^{\epsilon}_{0,n})^{U_\epsilon})$ is $Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}^{\epsilon}_{0,1}))$. In particular $Q((\mathcal{L}^{\epsilon}_{0,n})^{U_\epsilon})$ has center $Q((\mathcal{L}^{\epsilon}_{0,n})^{U_\epsilon}) \cap Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}^{\epsilon}_{0,1}))$, which is easily shown to be $\tilde{Q}(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n}))$. It then follows

$$\begin{aligned} [Q((\mathcal{L}^{\epsilon}_{0,n})^{U_\epsilon}) : \tilde{Q}(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n}))] &= \frac{[Q((\mathcal{L}^{\epsilon}_{0,n})^{U_\epsilon}) : Q(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n}))]}{[\tilde{Q}(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n})) : Q(\mathcal{Z}(\mathcal{L}^{\epsilon}_{0,n}))]} \\ &= l^{2nN-(2N+m)}.l^{-m} = l^{2(N(n-1)-m)}. \end{aligned}$$

Therefore $Q((\mathcal{L}^{\epsilon}_{0,n})^{U_\epsilon})$ is a central simple algebra of PI degree $l^{N(n-1)-m}$. □

## 6. Appendix

6.1. **Low and Up crystal structures in the $sl(2)$ case.** Let $k \in \mathbb{N}$, and denote by $V_k$ the simple $U_q^{ad}(sl(2))$ module of dimension $k+1$. It has a basis $v_0, \dots, v_k$ such that:

$$\begin{aligned} &K.v_j = q^{k-2j} v_j, \\ &F.v_j = [j+1]_q v_{j+1} \text{ if } j < k,\ F.v_k = 0, \\ &E.v_j = [k-j+1]_q v_{j-1} \text{ if } j > 0,\ E.v_0 = 0. \end{aligned}$$

This basis defines the full $A$-sublattice ${}_AV_k$, which is left invariant by $U_A^{res}$, and we have:

$$F^{(a)}.v_j = \left[\begin{array}{c} j+a \\ a \end{array}\right]_q v_{j+a}, E^{(a)}.v_j = \left[\begin{array}{c} k-j+a \\ a \end{array}\right]_q v_{j-a}, a \geq 0. \tag{119}$$

The action of the Kashiwara operator $\tilde{e}, \tilde{f}$ on $V_k$ are given by:

$$\tilde{f}(v_j) = v_{j+1}, \tilde{e}(v_j) = v_{j-1}. \tag{120}$$

The crystal basis $(\mathcal{L}^{low}, \mathcal{B}^{low})$ at $q = 0$ is formed by the $\mathcal{A}_0$-sublattice $\mathcal{L}^{low}$ generated by $v_0, ..., v_k$, and $\mathcal{B}^{low}$ by the images $\bar{v}_0, ..., \bar{v}_k$ of these vectors in $\mathcal{L}^{low}/q\mathcal{L}^{low}$.

The bilinear form $\langle , \rangle_k$ defined by (51) is easily computed:

$$\langle v_i, v_j \rangle_k = \langle F^{(i)}.v_0, F^{(j)}.v_0 \rangle_k = \langle v_0, E^{(i)}F^{(j)}.v_0 \rangle_k = \left[ \begin{array}{c} k \\ i \end{array} \right]_q \delta_{i,j}. \tag{121}$$

By definition ${}_AV_k^{up} = \{v \in V_k, \langle v, {}_AV_k \rangle_k \subset A\} = \bigoplus_{j=0}^{k} Av_j^{up}$ where

$$v_j^{up} = \left[ \begin{array}{c} k \\ j \end{array} \right]_q^{-1} v_j.$$

The upper crystal basis $(\mathcal{L}^{up}, \mathcal{B}^{up})$ at $q = 0$ is formed by the $\mathcal{A}_0$-sublattice $\mathcal{L}^{up}$ generated by $v_0^{up}, ..., v_k^{up}$, and $\mathcal{B}^{low}$ by the images $\bar{v}_0^{up}, ..., \bar{v}_k^{up}$ of these vectors in $\mathcal{L}^{up}/q\mathcal{L}^{up}$.

Using that $[n]_q \in q^{1-n} + \mathcal{A}_0$, we obtain

$$\left[ \begin{array}{c} k \\ j \end{array} \right]_q \in q^{k^2-kj} + \mathcal{A}_0.$$

As a result we get $\bar{v}_j^{up} = q^{kj-k^2}\bar{v}_j$, which is exactly the relation (53) relating the low and up crystal basis, with $\lambda = k\varpi_1, \mu = (k-2j)\varpi_1$.

6.2. **Quantum Weyl group.** We recall some of the formulas of [31]. Let $e_q(z)$ be the formal power series in $z$ with coefficients in $\mathbb{C}(q)$ defined by:

$$e_q(z) = \sum_{n=0}^{+\infty} \frac{z^n}{(n)_q!}. \tag{122}$$

We first consider the case of $\mathfrak{g} = sl(2)$. As explained in [28], Section 3, the Cartan element $H \in \mathfrak{g}$ defines an element of $\mathbb{U}_q(sl(2))$. Viewed as elements of $\mathbb{U}_q$ we have $L = q^{H/2}$. The series $\Theta = q^{H \otimes H/2}$ defines an element of $\mathbb{U}_q(sl(2))^{\hat{\otimes}2}$, its image under multiplication being $q^{H^2/2}$. The $R$-matrix can be expressed as $R = \Theta\hat{R}$ where $\hat{R} = e_{q^{-1}}((q-q^{-1})E \otimes F)$ is a well defined element of $\mathbb{U}_q^{\hat{\otimes}2}$. Consider the Lusztig [72] braid group automorphism of $U_q(sl(2))$, defined by

$$T(L) = L^{-1}, T(E) = -FK^{-1}, T(F) = -KE. \tag{123}$$

For every $x \in U_q(sl(2))$ it satisfies:

$$\Delta(T(x)) = \hat{R}^{-1}(T \otimes T)(\Delta(x))\hat{R}. \tag{124}$$

Define the quantum Weyl group element $\hat{w} \in \mathbb{U}_q(sl(2))$ by Saito's formula [86]:

$$\hat{w} = e_{q^{-1}}(F)q^{-H^2/4}e_{q^{-1}}(-E)q^{-H^2/4}e_{q^{-1}}(F)q^{-H/2}. \tag{125}$$

For every $x \in U_q(sl(2))$ it satisfies:

$$T(x) = \hat{w}x\hat{w}^{-1}, \tag{126}$$

$$\Delta(\hat{w}) = \hat{R}^{-1}(\hat{w} \otimes \hat{w}), \tag{127}$$

$$\hat{w}^2 = q^{H^2/2}\xi\theta, \tag{128}$$

where $\theta \in \mathbb{U}_q(sl(2))$ is the ribbon element, and $\xi \in \mathbb{U}_q(sl(2))$ is the central group element whose value on the type 1 simple module $V_k$ of $U_q^{ad}(sl(2))$ of dimension $k+1$ is the scalar endomorphism $(-1)^k id_{V_k}$.

In order to compare our setting to the one of [43] we need an explicit formula of $\hat{w}$. Using (123), (126) and (128), we obtain:

$$\hat{w}v_j = (-1)^j q^{-j(k-j-1)-k} v_{k-j}. \tag{129}$$

In [43] another quantum Weyl group element $\underline{w}$ is defined. It is dual to the Vaksman-Soibelman functional $t\colon \mathcal{O}_q(SL_2) \to \mathbb{C}(q)$ of [89, 88], that is, $t(\alpha) = \langle \alpha, \underline{w} \rangle$ for all $\alpha \in \mathcal{O}_q(SL_2)$. By comparing (129) with the formulas defining the action of $t$ in Section 1.7 of [43], we find

$$\underline{w} = \xi \hat{w} K$$

and the basis vectors $w_r^p$ of [43], where $p \in (1/2)\mathbb{N}$ and $r \in \{-p, -p+1, \dots, p-1, p\}$, are related to the vectors $v_j$ above as follows:

$$v_j = \lambda_j w_r^p$$

where $k = 2p$, $j = p - r$, $\lambda_0 = 1$, $\lambda_1 = [k]q^{-k}$, and

$$\lambda_j = \frac{[k]!}{[j]![k-(j-2)]!} q^{j(j+1)-j(k+2)}, \quad j \geq 2.$$

Explicit formulas of the evaluation of $t$ on basis vectors of $\mathcal{O}_q(SL_2)$ can be computed. We get:

$$t(\tilde{a}^{\star m} \star \tilde{b}^{\star n} \star \tilde{d}^{\star p}) = \delta_{m,p} q^{-np} \prod_{i=1}^{p} (1 - q^{-2i}), \tag{130}$$

$$t(\tilde{a}^{\star m} \star \tilde{c}^{\star n} \star \tilde{d}^{\star p}) = (-1)^n \delta_{m,p} q^{-n(p+1)} \prod_{i=1}^{p} (1 - q^{-2i}) \tag{131}$$

where

$$\tilde{a} = a \ , \ \tilde{b} = qb \ , \ \tilde{c} = q^{-1}c \ , \ \tilde{d} = d \tag{132}$$

and as usual $a$, $b$, $c$, $d$ are the standard generators of $\mathcal{O}_q(SL_2)$, ie. the matrix coefficients in the basis of weight vectors $v_0$, $v_1$ of the 2-dimensional irreducible representation $V_1$ of $U_q(sl(2))$ such that $K.v_0 = qv_0$ and $v_1 = F.v_0$. Here we have introduced the generators $\tilde{a}, \dots, \tilde{d}$ to facilitate the comparison with the formulas in [43]; these generators come naturally in their setup because they use different generators $E_i$ and $F_i$ of $U_q(\mathfrak{g})$, which in our notations can be written respectively as $K_i^{-1}E_i$ and $F_iK_i$.

The formulas (130)-(131) can be shown by two independent methods. The first uses a definition of $t$ as a $GNS$ state associated to an infinite dimensional representation of $\mathcal{O}_q(SL_2)$, as recalled in Section 1.6 of [43]. The second is to write eg.

$$t(\tilde{a}^{\star m} \star \tilde{b}^{\star n} \star \tilde{d}^{\star p}) = \left\langle \tilde{a}^{\otimes m} \otimes \tilde{b}^{\otimes n} \otimes \tilde{d}^{\otimes p}, \Delta^{(m+n+p-1)}(\underline{w}) \right\rangle \tag{133}$$

and to use explicit expressions of $\Delta^{(m+n+p-1)}(\underline{w})$ when represented on $V_1^{\otimes(m+n+p)}$. In general one can check that

$$\begin{aligned} \Delta^{(n)}(\hat{\omega}) = \left(\Delta^{(n-1)} \otimes id\right)\left(\widehat{R}^{-1}\right)\Big(\Big(\Delta^{(n-2)} \otimes id\Big)\left(\widehat{R}^{-1}\right) \otimes id\Big) \\ \dots \Big((\Delta \otimes id)\left(\widehat{R}^{-1}\right) \otimes id^{\otimes(n-3)}\Big)\left(\widehat{R}^{-1} \otimes id^{\otimes(n-2)}\right)\hat{\omega}^{\otimes n}. \end{aligned} \tag{134}$$

By (129) or (130)-(131) we see that $\hat{w}$ (or $\underline{w}$) and $t$ are well-defined on the integral forms,

$$\hat{w} \in \mathbb{U}_\Gamma \ , \ t\colon \mathcal{O}_A(SL_2) \to A.$$

We now consider the case where $\mathfrak{g}$ is of rank $m \geq 2$. To each simple root $\alpha_i$, $1 \leq i \leq m$, is associated the subalgebra of $U_q$ generated by $E_i, F_i, L_i, L_i^{-1}$. It is a copy of $U_{q_i}(sl(2))$, where $q_i = q^{d_i}$. Let $\hat{w}_i$ be the corresponding quantum Weyl group element in $\mathbb{U}_q = \mathbb{U}_q(\mathfrak{g})$, defined by Saito's formula (125), replacing $H$, $E$, $F$ by $H_i$, $E_i$ and $F_i$. Also, denote by $\nu_i\colon \mathcal{O}_q \to \mathcal{O}_{q_i}(SL_2)$ the projection map dual to the inclusion $U_{q_i}(sl(2)) \otimes_{\mathbb{C}(q_i)} \mathbb{C}(q) \hookrightarrow U_q$, and put $t_i = t \circ \nu_i$. Let $\underline{w}_i$ be the corresponding quantum Weyl group element in $\mathbb{U}_q$, ie. $t_i(\alpha) = \langle \alpha, \underline{w}_i \rangle$ for all $\alpha \in \mathcal{O}_q$. On integral forms they yield well-defined elements $\hat{w}_i, \underline{w}_i \in \mathbb{U}_\Gamma$ and $t_i\colon \mathcal{O}_A \to A$ (see [43], Proposition 5.1, and [74] for a different construction). They satisfy the defining relations of the braid group $\mathcal{B}(\mathfrak{g})$ of $\mathfrak{g}$ [67]:

$$\hat{w}_i\hat{w}_j\hat{w}_i = \hat{w}_j\hat{w}_i\hat{w}_j \quad \text{if } a_{ij}a_{ji} = 1$$
$$(\hat{w}_i\hat{w}_j)^k = (\hat{w}_j\hat{w}_i)^k \quad \text{for } k = 1,2,3 \text{ if } a_{ij}a_{ji} = 0,2,3$$

and similarly by replacing $\hat{w}_i$ with $\underline{w}_i$, or with $t_i$ (see [88] for the latter). The Weyl group $W = W(\mathfrak{g}) = N(T_G)/T_G$ is generated by the reflexions $s_i$ associated to the simple roots $\alpha_i$. Denote by $n_i \in N(T_G)$ a representative of $s_i$. Let $w \in W$ and denote by $w = s_{i_1} \dots s_{i_k}$ a reduced expression. Because of the braid group relations the elements $\hat{w} = \hat{w}_{i_1} \dots \hat{w}_{i_k}$, $\underline{w} = \underline{w}_{i_1} \dots \underline{w}_{i_k}$ and the functional $t_w = t_{i_1} \dots t_{i_k}$ do not depend on the choice of reduced expression. The Lusztig [72] braid group automorphism $T_w\colon \Gamma \to \Gamma$ associated to $w$ satisfies (see [43]):

$$T_w(x) = \hat{w}x\hat{w}^{-1}, \quad x \in \Gamma.$$

Let $w_0$ be the longest element in $W$. We have

$$\Delta(\hat{w}_0) = \hat{R}^{-1}(\hat{w}_0 \otimes \hat{w}_0) \tag{135}$$

where as usual $R = \Theta\hat{R}$.

6.3. **Regular action on $\mathcal{O}_\epsilon$.** The following result is proved in Section 1.10 of [43]. For completeness let us give a (different) proof. Recall from (87) that we may identify $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ with $\mathcal{O}(G)$.

**Proposition 6.1.** *For every $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$, $g \in \mathcal{O}_\epsilon$ we have*

$$t_i(f) = f(n_i) \tag{136}$$
$$t_i(f \star g) = t_i(f)t_i(g). \tag{137}$$

*Proof.* It is sufficient to prove the results for $SL_2$ because $\nu_i : \mathcal{O}_\epsilon \to \mathcal{O}_\epsilon(SL_2)$ is a morphism of Hopf algebras and $\nu_i(\mathcal{Z}_0(\mathcal{O}_\epsilon)) \subset \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$. In this case (136) can be proved by using (130)-(131), evaluating $t$ on basis elements of $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ as is done in Lemma 1.5 (a) of [43]. Such a basis is formed by monomials like in (130)-(131), with all exponents divisible by $l$; then for instance

$$t(\tilde{a}^{\star ml} \star \tilde{b}^{\star nl} \star \tilde{d}^{\star pl}) = \delta_{p,0}\delta_{m,0} = \underline{a}^m\underline{b}^n\underline{d}^p(n)$$

where $\underline{a}, \dots, \underline{d}$ are the generators of $\mathcal{O}(G) = \mathcal{O}_1(G)$ corresponding to $a, \dots, d$, and we take

$$n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

as representative of the reflexion $s$ generating the Weyl group $W(sl(2))$. Here is an alternative proof of (136): (137) shows that $t$ is a homomorphism on $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$, so by proving (137) at first one is reduced to check (136) on the generators $a^{\star l}, \dots, d^{\star l}$, which is easy by means of (133) and (135).

We provide a proof of (137) that we find more conceptual than the one in Lemma 1.5 (b) of [43] (which uses again (130)-(131)). As above let us denote $\underline{w} = \xi \hat{w} K$. For any $f, g \in \mathcal{O}_\epsilon$ we have

$$\begin{aligned} t(f \star g) &= (f \otimes g)(\Delta(\underline{w})) \\ &= (f \otimes g)\left(\widehat{R}^{-1}(\underline{w} \otimes \underline{w})\right) \\ &= \sum_{(\widehat{R}^{-1})} f\left((\widehat{R}^{-1})_{(1)}\underline{w}\right) g\left((\widehat{R}^{-1})_{(2)}\underline{w}\right) \\ &= \sum_{(\widehat{R}^{-1}),(f)} f_{(1)}\left((\widehat{R}^{-1})_{(1)}\right) f_{(2)}(\underline{w})\ g\left((\widehat{R}^{-1})_{(2)}\underline{w}\right) \\ &= \sum_{(f)} f_{(2)}(\underline{w})\ g\left((f_{(1)} \otimes id)(\widehat{R}^{-1})\underline{w}\right). \end{aligned}$$

Assume now $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$. Since $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ is a Hopf subalgebra of $\mathcal{O}_\epsilon(SL_2)$ we have $f_{(1)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$. From Theorem 2.24 (2) we deduce

$$(f_{(1)} \otimes id)(\widehat{R}^{-1}) \in U_\epsilon(\mathfrak{n}_-) \cap \mathcal{Z}_0(U_\epsilon^{ad}).$$

Denote by $z$ this element. Note that from its expression we have $\epsilon(z) = \epsilon(f_{(1)})$. Now $g(z\underline{w}) = \sum_{(g)} g_{(1)}(z) g_{(2)}(\underline{w})$, but $g_{(1)}$ is a linear combination of matrix elements of $\Gamma$-modules, on which $\mathcal{Z}_0(U_\epsilon^{ad})$ acts by the trivial character. Therefore

$$g(z\underline{w}) = \textstyle\sum_{(g)} \epsilon(z) g_{(1)}(1) g_{(2)}(\underline{w}) = \epsilon(z) g(\underline{w}) = \epsilon(f_{(1)}) g(\underline{w})$$

and eventually

$$t(f \star g) = \sum_{(f)} f_{(2)}(\underline{w}) \epsilon(f_{(1)}) g(\underline{w}) = t(f) t(g).$$

This concludes the proof. □

For the sake of completeness, let us show how this result implies:

*Proof of Proposition 2.25 (ie. Proposition 7.1 of* [43]). We have $f \lhd t_i = \sum_{(f)} t_i(f_{(1)}) f_{(2)}$, $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$. Since $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ is a Hopf subalgebra of $\mathcal{O}_\epsilon$, $f_{(2)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ and therefore the maps $\lhd t_i \colon \mathcal{O}_\epsilon \to \mathcal{O}_\epsilon$ preserve $\mathcal{Z}_0(\mathcal{O}_\epsilon)$. Moreover, $(f \lhd t_i)(a) = \sum_{(f)} f_{(1)}(n_i) f_{(2)}(a) = f(n_i a)$, $a \in G$, by (136).

It remains to show that $(f \star \alpha) \lhd t_i = (f \lhd t_i)(\alpha \lhd t_i)$ for every $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$, $\alpha \in \mathcal{O}_\epsilon$. We have

$$\begin{aligned} (f \star g) \lhd t_i = \textstyle\sum_{(f\star g)} t_i\left((f \star g)_{(1)}\right)(f \star g)_{(2)} &= \sum_{(f),(g)} t_i\left(f_{(1)} \star g_{(1)}\right) f_{(2)} \star g_{(2)} \\ &= \sum_{(f),(g)} t\left(\nu_i(f_{(1)})\nu_i(g_{(1)})\right) f_{(2)} \star g_{(2)} \\ &= \sum_{(f),(g)} t\left(\nu_i(f_{(1)})\right) t\left(\nu_i(g_{(1)})\right) f_{(2)} \star g_{(2)} \end{aligned} \tag{138}$$

using that $\nu_i$ is a homomorphism in the third equality, and (137) in the last one. The result is just $(f \lhd t_i)(g \lhd t_i)$. □

## References


[1] A. Yu. Alekseev, *Integrability in the Hamiltonian Chern-Simons theory*, Alg. i Anal. 6 (2) (1994) 53–66

[2] A. Y. Alekseev, H. Grosse, V. Schomerus, *Combinatorial quantization of the Hamiltonian Chern-Simons theory I*, Comm. Math. Phys. 172 (1995) 317–358

[3] A. Y. Alekseev, H. Grosse, V. Schomerus, *Combinatorial quantization of the Hamiltonian Chern-Simons theory II*, Comm. Math. Phys. 174 (1996) 561–604

[4] A.Y. Alekseev, V.Schomerus, *Representation theory of Chern-Simons observables*, Duke Math. J. 85 (1996), 447-510

[5] N. Andruskiewitsch, G. A. Garcia, *Quantum subgroups of a simple quantum group at roots of* 1, Comp. Math. 145 (2009), 476–500

[6] M. F. Atiyah, I.G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley (1969)

[7] A. Y. Alekseev, V. Schomerus, *Representation theory of Chern-Simons observables*, Duke Math. J. 28 (2) (1996) 447–510

[8] S. Baseilhac, *Quantum coadjoint action and the* $6j$*-symbols of* $U_q(sl(2))$, in "Interactions between Hyperbolic Geometry, Quantum Topology and Number Theory", AMS Cont. Math. Proc. Ser. 541 (2011) 103–144

[9] H. Bass, *Algebraic K-theory*, W.A. Benjamin Inc. (1968)

[10] P. Baumann, *Another proof of Joseph and Letzter's separation of variables theorem for quantum groups*, Transf. Groups 5 (1) (2000) 3–20

[11] D. Bullock, *A finite set of generators for the Kauffman bracket skein algebra*, Math. Zeitschrift vol 231 (1999), 91-101

[12] S. Baseilhac, R. Benedetti, *Quantum hyperbolic invariants of 3-manifolds with PSL(2,C)-characters*, Topology 43 (6) (2004) 1373–1423

[13] S. Baseilhac, R. Benedetti, *Classical and quantum dilogarithmic invariants of flat PSL(2,C)-bundles over 3-manifolds*, Geom. Topol. 9 (2005) 493–569

[14] S. Baseilhac, R. Benedetti, *Quantum hyperbolic geometry*, Alg. Geom. Topol. 7 (2007) 845–917

[15] S. Baseilhac, R. Benedetti, *The Kashaev and quantum hyperbolic link invariants*, J. Gökova Geom. Topol. 5 (2011) 31–85

[16] S. Baseilhac, R. Benedetti, *Analytic families of quantum hyperbolic invariants*, Alg. Geom. Topol. 15-4 (2015) 1983–2063, DOI 10.2140/agt.2015.15.1983

[17] S. Baseilhac, R. Benedetti, *Non ambiguous structures on 3-manifolds and quantum symmetry defects*, Quantum Topol. 8-4 (2017) 749–846, DOI: 10.4171/QT/101

[18] S. Baseilhac, R. Benedetti, *On the quantum Teichmüller invariants of fibred cusped 3-manifolds*, Geom. Dedicata 197(1) (2018) 1–32, DOI 10.1007/s10711-017-0315-0

[19] A. Beliakova, C. Blanchet, N. Geer: Logarithmic Hennings invariants for restricted quantum sl(2), Algebr. Geom. Topol. 18(7), 4329-4358 (2018)

[20] A. Brochier, D. Ben-Zvi, D. Jordan, *Quantum character varieties and braided module categories*, Selecta Math. 24 (5) (2018) 4711–4748

[21] K. A. Brown, M. Couto, *Affine commutative-by-finite Hopf algebras*, J. Algebra 573 (2021), 56–94

[22] D. Bullock, C. Frohman, J. Kania-Bartoszynska, *Topological interpretations of lattice gauge field theory*, Comm. Math. Phys 198 (1998) 47–81

[23] K. A. Brown, K. R. Goodearl, *Lectures on algebraic quantum groups*, Advanced Courses in Mathematics, CRM Barcelona, Birkhaüser Verlag, Basel (2002)

[24] K. A. Brown, I. Gordon, *The ramifications of the centres: Quantised function algebras at roots of unity*, Proc. London Math. Soc. 84 (2002) 147–178

[25] K. A. Brown, I. Gordon, J. T. Stafford, $\mathcal{O}_\epsilon[G]$ *is a free module over* $\mathcal{O}[G]$, `arXiv:math.QA/0007179` (2000)

[26] S. Baseilhac, M. Faitg, P. Roche, *Unrestricted quantum moduli algebras, III: surfaces of arbitrary genus and skein algebras*, `arXiv:math.QA/2302.00396` (2023)

[27] S. Baseilhac, M. Faitg, P. Roche, *Unrestricted quantum moduli algebras, IV: representations at roots of unity*, in preparation

[28] S. Baseilhac, P. Roche, *Unrestricted quantum moduli algebras, I: the case of punctured spheres*, SIGMA 18 (2022), 025, 78 pages, `arXiv:1912.02440`

[29] E. Buffenoir, P. Roche, *Two dimensional lattice gauge theory based on a quantum group*, Comm. Math. Phys. 170 (1995) 669–698

[30] E. Buffenoir, P. Roche, *Link invariants and combinatorial quantization of Hamiltonian Chern-Simons theory*, Comm. Math. Phys. 181 (1996) 331–365

[31] E. Buffenoir, P. Roche, V.Terras, *Quantum dynamical coboundary equation for finite dimensional simple Lie algebras*, Adv. Math. 214 (2007) 181–229
[32] P. Caldero, *Eléments ad-finis de certains groupes quantiques*, C. R. Acad. Sci. Paris Sér. I. Math 316 (1993), 327–329
[33] W. Cui, *Canonical bases of modified quantum algebras for type $A_2$*, Journal of Algebra and Its Applications Vol. 17, No. 06, 1850113 (2018)
[34] F. Costantino, T.T.Q. Lê, *Stated skein algebras of surfaces*, to appear in Journal Eur. Math. Soc. (2019) `arXiv:1907.11400 [math.GT]`
[35] V. Chari, A. Pressley, *A Guide To Quantum Groups*, Cambridge Univ. Press (1994)
[36] J. Du, *Global IC bases for quantum linear groups*, J. Pure App. Alg. 114 (1996) 25–37
[37] V. G. Drinfeld, *On almost cocommutative Hopf algebras*, Leningrad Math. J. 1 (1999) 321–342
[38] J. A. Dieudonné, J. B. Carrell, *Invariant theory, Old and New*, Adv. Math. 4 (1) (1970) 1–80
[39] C. De Concini, V. Kac, *Representations of quantum groups at roots of 1*, Colloque Dixmier 1989, Birkhäuser, Progr. in Math. 92 (1990) 471–506
[40] C. De Concini, V. Kac, C. Procesi, *Quantum coadjoint action*, J. Amer. Math. Soc. 5 (1992) 151–189
[41] C. De Concini, C. Procesi, *Quantum groups*, in *D-modules, representation theory and quantum groups (Venice, 1992)*, Springer-Verlag Lect. Notes Math. Vol. 1565 (1993) 31–140
[42] C. De Concini, V. Kac, C. Procesi, *Some remarkable degenerations of quantum groups*, Commun. Math. Phys. 157 (1993) 405–427
[43] C. De Concini, V. Lyubashenko, *Quantum function algebra at roots of 1*, Adv. Math. 108 (2) (1994) 205–262
[44] M. Domokos, T. H. Lenagan, *Quantized trace rings*, Quart. J. Math. 56 (4) (2005) 507–523
[45] L. Dabrowski, C. Reina, A. Zampa, *$A[SL_q(2)]$ at roots of unity is a free module over $A[SL(2)]$*, Lett. Math. Phys. 52(4) (2000) 339–342
[46] M. De Renzi, N. Geer, B. Patureau-Mirand, *Renormalized Hennings Invariants and 2+1-TQFTs*, `arXiv:1707.08044 [math.GT]`
[47] M. De Renzi, A. M. Gainutdinov, N. Geer, B. Patureau-Mirand, I. Runkel, *3-Dimensional TQFTs from Non-Semisimple Modular Categories*, `arXiv:1912.02063 [math.GT]`
[48] B. Enriquez, *Le centre des algèbres de coordonnées des groupes quantiques aux racines $p^\alpha$-ièmes de l'unité*, Bull. Soc. Math. France 122(4) (1994) 443–485
[49] P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, D. Vaintrob, E.Yudovina, *Introduction to representation theory*, AMS Student Math. Library 59 (2011)
[50] M. Faitg, *Mapping class groups, skein algebras and combinatorial quantization*, `arXiv:1910.04110 [math.QA]`, PhD Thesis, Université de Montpellier (2019)
[51] M. Faitg, *Projective representations of mapping class groups in combinatorial quantization*, Commun. Math. Phys. 377 (2020) 161–198, https://doi.org/10.1007/s00220-019-03470-z
[52] M. Faitg, *Holonomy and (stated) skein algebras in combinatorial quantization*, to appear in Quantum Topology, `arXiv:2003.08992 [math.QA]`
[53] I. Frenkel, M. Khovanov, *Canonical basis in tensor products and graphical calculus for $U_q(sl(2))$*, Duke Math. J. 87 (3) (1997) 409–480
[54] V. V. Fock, A. A. Rosly, *Poisson structure on moduli of flat connections on Riemann surfaces and r-matrix*, preprint ITEP-72-92 (1992), `arXiv:math/9802054v2`
[55] W. A. de Graaf, *Constructing canonical bases of quantized enveloping algebras*, Experimental Math. 11 (2) (2002) 161–170
[56] I. Ganev, D. Jordan, P. Safronov, *The quantum Frobenius for character varieties and multiplicative quiver varieties*, 2019, `arXiv:1901.11450v4`
[57] J. E. Humphreys, *Linear Algebraic Groups*, Graduate Texts In Maths 21, Springer (1998)
[58] A. Joseph, *Quantum Groups and their Primitive Ideals*, Springer, Ergebn. Math. Grenz. 3 (29) (1995)
[59] A. Joseph, G. Letzter, *Local finiteness of the adjoint action for quantized enveloping algebras*, Journal of Algebra 153 (1992) 289–318
[60] A. Joseph, G. Letzter, *Separation of variables for quantized enveloping algebras*, Amer. J. Math. 116 (1994) 127–177
[61] D. Jordan, N. White, *The center of the reflection equation algebra via quantum minors*, Journal of Algebra 542 (2020) 308–342
[62] M. Kashiwara, *On crystal basis of the q-analog of universal enveloping algebras*, Duke Math. J. 63 (1991) 465–516
[63] M. Kashiwara, *Global crystal bases of quantum groups*, Duke Math. J. 69 (2) (1993) 455–485
[64] M. Kashiwara, *Crystal bases of modified quantized enveloping algebra*, Duke Math. J. 73 (2) (1994) 383–413

[65] M. Kashiwara, *On crystals bases*, Representations of groups (Banff, AB, 1994), 155–197, CMS Conf. Proc., 16, Amer. Math. Soc., Providence, RI, 1995
[66] B. Kostant, *Groups over* $\mathbb{Z}$, in *Algebraic Groups and their Discontinuous Subgroups*, Proc. Symp. Pure Math. Vol. 8, Amer. Math. Soc. (1966), 90–98
[67] A. N. Kirillov, N. Reshetikhin, *q-Weyl Group and a multiplicative formula for universal R-Matrices*, Commun. Math. Phys. 134 (1990), 421–431
[68] S.M. Khoroshkin, V.N. Tolstoy, *Universal R-matrix for quantized (super) algebras*, Commun. Math. Phys. 141 (1991) 599–617
[69] S. Kolb, M. Lorenz, B. Nguyen, R. Yammine, *On the adjoint representation of a Hopf algebra*, Proc. Edinb. Math. Soc. (2) 63 (2020), no. 4, 1092–1099
[70] A. Klimik, K. Schmudgen, *Quantum groups and their representations*, Springer (1997) https://doi.org/10.1007/978-3-642-60896-4
[71] S. Z. Levendorski, Y. S. Soibelman, *Some applications of the quantum Weyl groups*, J. Geom. Phys. 7 (2) (1990) 241–254
[72] G. Lusztig, *Quantum groups at roots of* 1, Geom. Dedicata 35 (1990) 89–113
[73] G. Lusztig, *Introduction To Quantum Groups*, Birkhaüser, Prog. in Math. 110 (1993)
[74] G. Lusztig, *Study of a* $\mathbb{Z}$*-form of the coordinate ring of a reductive group*, Journal Amer. Math. Soc. Vol. 22 (3) (2009) 739–69
[75] V. Lyubashenko, S.Majid, *Braided Groups and Quantum Fourier Transform*, Journal of Alg. 166 (3) 506–528 (1994)
[76] C. Meusburger, *Kitaev Lattice Models as a Hopf Algebra Gauge Theory*, Commun. Math. Phys. 353:413 (2017) 413–468, https://dx.doi.org/10.1007/s00220-017-2860-7
[77] J. Murakami, *Generalized Kashaev invariants for knots in three manifolds*, Quantum Topol. 8(1) 35–73 (2017)
[78] B. Parshall, J. P. Wang, *Quantum linear group*, Memoirs of the AMS 89 (439) (1991)
[79] J. H. Przytycki, A. S. Sikora, *Skein Algebras of Surfaces*, Trans. of AMS 371(2) (2019) 1309–1332
[80] S. Majid, *Braided matrix structure of the Sklyanin algebra and of the quantum Lorentz group*, Comm. Math. Phys. 156 (1993) 607–638
[81] R. Marlin, *Anneaux de Grothendieck des variétés de drapeaux*, Bull. Soc. Math. France 104 (1976) 337–348
[82] J. C. McConnell, J. C. Robson, *Noncommutative Noetherian Rings*, 2nd edition, AMS Grad. Stud. Math. 30 (2001)
[83] N. Yu. Reshetikhin, M. A. Semenov-Tian-Shansky, *Quantum R-matrices and factorization problems*, J. Geom. Phys. 5 (1988) 533-550
[84] L. Rowen, *Ring Theory*, Student Edition, Academic Press (1991)
[85] N. Y. Reshetikhin, V. G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. math. 103(1) 547–597 (1991)
[86] Y. Saito, *PBW Basis of Quantized Universal Envelopping Algebra*, Kyoto Univ Publ. RIMS 30 (1994) 209–232
[87] T. A. Springer, *Invariant Theory*, Springer-Verlag, Lect. Notes Math. Vol. 585 (1977)
[88] Y. S. Soibelman, *The algebra of functions on a compact quantum group, and its representations*, Leningr. Math. J. 2 (1) (1991) 161–178, translated from Algebra Anal. 2 (1) (1990) 190–212
[89] Y. S. Soibelman, L. L. Vaksman, *Algebra of functions on the quantum group* $SU(2)$, Funct. Anal. Appl. 22 (3) (1988) 170–181
[90] *Commutative Algebra*, The Stack project, Chapter 10, https://stacks.math.columbia.edu/browse
[91] *Brauer groups*, The Stack project, Chapter 11, https://stacks.math.columbia.edu/browse
[92] C. Voigt, R. Yuncken, *Complex Semisimple Quantum Groups and Representation Theory*, Lecture Notes in Mathematics 2264, Springer (2020)
[93] E. Witten, *Topological quantum field theory*, Comm. Math. Phys. 117(3): 353-386 (1988)
[94] N. Xi, *Root vectors in quantum groups*, Commentarii Mathematici Helvetici 69 (1994) 612–639, https://doi.org/10.1007/BF02564506