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# UNRESTRICTED QUANTUM MODULI ALGEBRAS, II: NOETHERIANITY AND SIMPLE FRACTION RINGS AT ROOTS OF 1

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ABSTRACT. We prove that the unrestricted quantum moduli algebra of a punctured sphere and complex simple Lie algebra  $\mathfrak{g}$  is a finitely generated ring and a Noetherian ring, and that its specialization at a root of unity of odd order  $l$ , coprime to 3 if  $\mathfrak{g}$  has type  $G_2$ , embeds in a natural way in a maximal order of a central simple algebra of PI degree  $l^{(n-1)N-m}$ , where  $N$  is the number of positive roots of  $\mathfrak{g}$ ,  $m$  its rank, and  $n + 1 \geq 3$  the number of punctures.

*Keywords:* quantum groups, invariant theory, TQFT

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## 1. INTRODUCTION

This paper is the second part of our work on the unrestricted quantum moduli algebras, that we initiated in [28]. These algebras, denoted by  $\mathcal{M}_{g,n}^A(\mathfrak{g})$  hereafter, are defined over the ground ring  $A = \mathbb{C}[q, q^{-1}]$  and associated to unrestricted quantum groups of complex simple Lie algebras  $\mathfrak{g}$ , and surfaces of genus  $g$  with  $n + 1$  punctures (thus,  $n = -1$  corresponds to closed surfaces). We are in particular interested in the specializations  $\mathcal{M}_{g,n}^{A,\epsilon}(\mathfrak{g})$  of  $\mathcal{M}_{g,n}^A(\mathfrak{g})$  at roots of unity  $q = \epsilon$ .

As in [28] we focus in this paper on the algebras  $\mathcal{M}_{0,n}^A(\mathfrak{g})$  associated to punctured spheres. From now on we fix a complex simple Lie algebra  $\mathfrak{g}$ , and when no confusion may arise we omit  $\mathfrak{g}$  from the notation of the various algebras.

The rational form  $\mathcal{M}_{g,n}$  of  $\mathcal{M}_{g,n}^A = \mathcal{M}_{g,n}^A(\mathfrak{g})$ , which is an algebra over  $\mathbb{C}(q)$ , has been introduced in the mid '90s by Alekseev-Grosse-Schomerus [2, 3] and Buffenoir-Roche [30, 31]. They defined  $\mathcal{M}_{g,n}$  by a  $q$ -deformation of the algebra of functions on the Fock-Rosly lattice models of the moduli spaces  $\mathcal{M}_{g,n}^{cl}$  of flat  $\mathfrak{g}$ -connections on surfaces of genus  $g$  with  $n + 1$  punctures. Because of this geometric input, it is quite natural to expect that the representation theory of the specializations  $\mathcal{M}_{g,n}^{A,\epsilon}(\mathfrak{g})$  of  $\mathcal{M}_{g,n}^A(\mathfrak{g})$  at roots of unity  $q = \epsilon$  provides a  $(2+1)$ -dimensional TQFT for 3-manifolds endowed with general flat  $\mathfrak{g}$ -connections, extending the known TQFTs based on quantum groups (where purely topological ones correspond to the trivial connection).

For instance, representations of the semisimplification of  $\mathcal{M}_{g,n}^{A,\epsilon}$  have been constructed and classified in [4]; they involve only the irreducible representations of the finite dimensional “small”, also called “restricted”, quantum group  $u_\epsilon(\mathfrak{g})$ , which is a quotient of  $U_\epsilon(\mathfrak{g})$  below, and a version of the Frobenius-Lusztig kernel of  $\mathfrak{g}$  at  $\epsilon$  (see [23], III.6.4). Moreover, [4] deduced from their representations of  $\mathcal{M}_{g,n}^{A,\epsilon}$  a family of representations of the mapping class groups of surfaces, that is equivalent to the one associated to the Witten-Reshetikin-Turaev TQFT [81, 73].

Recently, representations of another quotient of  $\mathcal{M}_{g,n}^{A,\epsilon}$  have been constructed in [47]. The corresponding representations of the mapping class groups of surfaces are equivalent to those previously obtained by Lyubashenko-Majid [62], and are associated to the so called non-semisimple TQFT defined by Geer, Patureau-Mirand and their collaborators (see eg. [44, 45]). In the  $sl(2)$  case they involve the irreducible and also the principal indecomposable representations of  $u_\epsilon(sl(2))$ . The related link and 3-manifold invariants coincide with those of [64] and [19].

In general, the representation theory of  $\mathcal{M}_{g,n}^{A,\epsilon}$  is by now far from being completely understood. As mentioned above, it is expected to provide a good framework to construct and study quantum invariants of 3-manifolds equipped with general flat  $\mathfrak{g}$ -connections. A family of such invariants, called *quantum hyperbolic invariants*, has already been defined for  $\mathfrak{g} = sl(2)$  by means of certain  $6j$ -symbols, *Deus ex machina*; they are closely connected to classical Chern-Simons theory, provide generalized Volume Conjectures, and contain quantum Teichmüller theory (see [12]–[18]). It is part of our present program, initiated in [9], to shed light on these invariants and to generalize them to arbitrary  $\mathfrak{g}$  by developing the representation theory of  $\mathcal{M}_{g,n}^{A,\epsilon}$ .

Besides, the quantum moduli algebras are now recognized as central objects from the viewpoints of factorization homology [20] and (stated) skein theory [22, 49, 34]. As already suggested above, their underlying formalism of *combinatorial quantisation* is very-well suited to the construction of mapping class group representations [48]. In another direction, one may expect that the equivalence proved in [63] between combinatorial quantisation for the Drinfeld double  $D(H)$  of a finite-dimensional semisimple Hopf algebra  $H$ , and Kitaev’s lattice model in topological quantum computation, can be extended to the setup of quantum moduli algebras.

We introduced  $\mathcal{M}_{0,n}^A$  and began its study in [28]. Its definition is based on the original combinatorial quantization method of [2, 3] and [30, 31], and uses also twists of module-algebras. This allows us to exploit fully the representation theory of quantum groups, by following ideas of classical invariant theory. Namely, as we shall describe more precisely below,  $\mathcal{M}_{0,n}^A$  can be regarded as the invariant subalgebra of a certain module-algebra  $\mathcal{L}_{0,n}^A$ , endowed with an action of the unrestricted (De Concini-Kac) integral form  $U_A = U_A(\mathfrak{g})$  of the quantum group  $U_q = U_q(\mathfrak{g})$ . We therefore study  $\mathcal{L}_{0,n}^A$  and its specializations  $\mathcal{L}_{0,n}^\epsilon$  at  $q = \epsilon$  a

root of unity. Under such a specialization  $\mathcal{M}_{0,n}^A$  embeds in  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ , the invariant subalgebra of  $\mathcal{L}_{0,n}^\epsilon$  under the action of the specialization  $U_\epsilon$  of  $U_A$  at  $q = \epsilon$ .

In [28], for  $q$  a root of unity we focused on the case  $\mathfrak{g} = sl(2)$  and described a Poisson action of the center on  $\mathcal{M}_{0,n}^{A,\epsilon}(sl(2))$ , derived from the quantum coadjoint action of De Concini-Kac-Procesi [38, 39, 40]. The results we prove in the present paper hold for every complex simple Lie algebra  $\mathfrak{g}$ . The main ones are a proof that  $\mathcal{L}_{0,n}^A$  and  $\mathcal{M}_{0,n}^A$  are Noetherian, finitely generated rings (Theorem 1.1), and  $\mathcal{L}_{0,n}^\epsilon$  and  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  are maximal orders of their central localizations (Theorem 1.3). We conclude with an application to their representation theories (Corollary 1.4).

Let us now state precisely and comment our results. First we need to fix notations. Let  $U_q$  be the simply-connected quantum group of  $\mathfrak{g}$ , defined over the field  $\mathbb{C}(q)$ . From  $U_q$  one can define a  $U_q$ -module algebra  $\mathcal{L}_{0,n}$ , called *graph algebra*, where  $U_q$  acts by means of a right coadjoint action. The *quantum moduli algebra*  $\mathcal{M}_{0,n}$  is the subalgebra  $\mathcal{L}_{0,n}^{U_q}$  of invariant elements of  $\mathcal{L}_{0,n}$  for this action. The *unrestricted quantum moduli algebra*  $\mathcal{M}_{0,n}^A$  is an integral form of  $\mathcal{M}_{0,n}$  (thus, defined over  $A = \mathbb{C}[q, q^{-1}]$ ). As a  $\mathbb{C}(q)$ -module  $\mathcal{L}_{0,n}$  is just  $\mathcal{O}_q^{\otimes n}$ , where  $\mathcal{O}_q = \mathcal{O}_q(G)$  is the standard quantum function algebra of the connected and simply-connected Lie group  $G$  with Lie algebra  $\mathfrak{g}$ . The product of  $\mathcal{L}_{0,n}$  is obtained by twisting both the product of each factor  $\mathcal{O}_q$  and the product between them. It is equivariant with respect to a (right) coadjoint action of  $U_q$ , which defines the structure of  $U_q$ -module of  $\mathcal{L}_{0,n}$ . The module algebra  $\mathcal{L}_{0,n}$  has an integral form  $\mathcal{L}_{0,n}^A$ , defined over  $A$ , endowed with a coadjoint action of the *unrestricted* integral form  $U_A$  of  $U_q$  introduced by De Concini-Kac [38]. The algebra  $\mathcal{L}_{0,n}^A$  is obtained by replacing  $\mathcal{O}_q$  in the construction of  $\mathcal{L}_{0,n}$  with the restricted dual  $\mathcal{O}_A$  of the integral form  $U_A^{res}$  of  $U_q$  defined by Lusztig [60], or equivalently with the restricted dual of the integral form  $\Gamma$  of  $U_q$  defined by De Concini-Lyubashenko [42]. The unrestricted integral form  $\mathcal{M}_{0,n}^A$  of  $\mathcal{M}_{0,n}$  is defined as the subalgebra of invariant elements,

$$\mathcal{M}_{0,n}^A := (\mathcal{L}_{0,n}^A)^{U_A}.$$

A cornerstone of the theory of  $\mathcal{M}_{0,n}^A$  is a map originally due to Alekseev [1], building on works of Drinfeld [36] and Reshetikhin and Semenov-Tian-Shansky [70]. In [28] we showed that it eventually provides isomorphisms of module algebras and algebras respectively,

$$\Phi_n: \mathcal{L}_{0,n}^A \rightarrow (U_A^{\otimes n})^{lf}, \Phi_n: \mathcal{M}_{0,n}^A \rightarrow (U_A^{\otimes n})^{U_A}$$

where  $U_A^{\otimes n}$  is endowed with a right adjoint action of  $U_A$ , and  $(U_A^{\otimes n})^{lf}$  is the subalgebra of locally finite elements with respect to this action. When  $n = 1$  the algebra  $U_A^{lf}$  has been studied in great detail by Joseph-Letzter [52, 53, 51]; their results we use have been greatly simplified in [80].

All the material we need about the results discussed above is described in [28], and recalled in Section 2.1-2.2.

Our first result, proved in Section 3, is:

**Theorem 1.1.**  *$\mathcal{L}_{0,n}$ ,  $\mathcal{M}_{0,n}$  and their unrestricted integral forms and specializations at  $q \in \mathbb{C} \setminus \{0, 1\}$  are Noetherian rings, and finitely generated rings.*

In [28] we proved that these algebras have no non-trivial zero divisors. Also, we deduced Theorem 1.1 in the  $sl(2)$  case by using an isomorphism between  $\mathcal{M}_{0,n}(sl(2))$  and the skein algebra of a sphere with  $n + 1$  punctures, which by a result of [66] is Noetherian and finitely generated. Our approach here is completely different. For  $\mathcal{L}_{0,n}$  we adapt the proof given by Voigt-Yuncken [80] of a result of Joseph [51], which asserts that  $U_q^{lf}$  is a Noetherian ring

(Theorem 3.1). For  $\mathcal{M}_{0,n}$  we deduce the result from the one for  $\mathcal{L}_{0,n}$ , by following a line of proof of the Hilbert-Nagata theorem in classical invariant theory (Theorem 3.2).

From Section 4 we consider the specializations  $\mathcal{L}_{0,n}^\epsilon$  of  $\mathcal{L}_{0,n}^A$  at  $q = \epsilon$ , a root of unity of odd order  $l$  coprime to 3 if  $\mathfrak{g}$  has  $G_2$  components. In [42], De Concini-Lyubashenko introduced a central subalgebra  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  of  $\mathcal{O}_\epsilon$  isomorphic to the coordinate ring  $\mathcal{O}(G)$ , and proved that the  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ -module  $\mathcal{O}_\epsilon$  is projective of rank  $l^{\dim \mathfrak{g}}$ . As observed by Brown-Gordon-Stafford [25], Bass' Cancellation theorem in  $K$ -theory and the fact that  $K_0(\mathcal{O}(G)) \cong \mathbb{Z}$ , proved by Marlin [68], imply that this module is free. Alternatively, this follows also from the fact that  $\mathcal{O}_\epsilon$  is a cleft extension of  $\mathcal{O}(G)$  by the dual of the Hopf algebra  $u_\epsilon(\mathfrak{g})$ , as proved by Andruskiewitsch-Garcia (see [6], Remark 2.18(b), and also Section 3.2 of [21]; this argument was explained to us by K. A. Brown).

The section 4 proves the analogous property for  $\mathcal{L}_{0,n}^\epsilon$ . Namely:

**Theorem 1.2.**  *$\mathcal{L}_{0,n}^\epsilon$  has a central subalgebra  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  isomorphic to  $\mathcal{O}(G)^{\otimes n}$ , and it is a free  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module of rank  $l^{n \cdot \dim \mathfrak{g}}$ , isomorphic to the  $\mathcal{O}(G)^{\otimes n}$ -module  $\mathcal{O}_\epsilon^{\otimes n}$ .*

A similar statement for  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is in Theorem 1.3 (3) below.

We prove the first and third claims of Theorem 1.2 in Proposition 4.2. Since  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{O}_\epsilon^{\otimes n}$  are the same modules over  $\mathcal{O}(G)^{\otimes n}$ , at this point we can just deduce the second claim from the results of [42] and [68], or [6], recalled above. Nevertheless we give a self-contained proof that  $\mathcal{L}_{0,1}^\epsilon$  is finite projective of rank  $l^{\dim \mathfrak{g}}$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  by adapting the original arguments of Theorem 7.2 of De Concini-Lyubashenko [42]. In particular we study the coregular action of the braid group of  $\mathfrak{g}$  on  $\mathcal{L}_{0,1}^\epsilon$ ; by the way, in the Appendix we provide different proofs of some technical facts shown in [42]. Of course, it remains an exciting problem to describe the centralizing extension  $\mathcal{O}(G)^{\otimes n} \subset \mathcal{L}_{0,n}^\epsilon$  (and similarly  $\mathcal{O}(G)^{\otimes n} \subset (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  below), aiming at generalizing the results of [6] and finding a direct, more structural proof of freeness in Theorem 1.2.

It is worth noticing that the most natural definition of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  is  $\Phi_1^{-1}(U_\epsilon^{lf} \cap \mathcal{Z}_0(U_\epsilon))$ , where  $\mathcal{Z}_0(U_\epsilon)$  is the De Concini-Kac-Procesi central subalgebra of  $U_\epsilon$ , and  $U_\epsilon^{lf}$  the specialization at  $q = \epsilon$  of the algebra  $U_A^{lf}$ . Thus it is not directly connected to  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ , and the algebra structures of  $\mathcal{L}_{0,1}^\epsilon$  and  $\mathcal{O}_\epsilon$  are completely different indeed. For arbitrary  $n$  we set  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) = \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$ . The fact that  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is central in  $\mathcal{L}_{0,n}^\epsilon$ , and  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  and  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  coincide and give  $\mathcal{L}_{0,1}^\epsilon$  and  $\mathcal{O}_\epsilon$  the same module structures over these subalgebras, relies on results of De Concini-Kac [38], De Concini-Procesi [39, 40], and De Concini-Lyubashenko [42], that we recall in Section 2.3-2.4.

Also, we note that basis of  $\mathcal{L}_{0,n}^\epsilon$  over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  are complicated. The only case we know is  $\mathfrak{g} = sl(2)$ , described in [43], and it is far from being obvious (see (43)).

In Section 5 we turn to fraction rings. As mentioned above  $\mathcal{L}_{0,n}^\epsilon$  has no non-trivial zero divisors. Therefore its center  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is an integral domain. Denote by  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  its fraction field. Denote by  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  the subring of  $\mathcal{L}_{0,n}^\epsilon$  formed by the invariant elements of  $\mathcal{L}_{0,n}^\epsilon$  with respect to the right coadjoint action of  $U_\epsilon$ . Note that we trivially have an inclusion  $\mathcal{M}_{0,n}^{A,\epsilon} \subset (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ , and these two algebras are distinct in general; for instance, when  $n = 1$  we have by definition  $(\mathcal{L}_{0,1}^\epsilon)^{U_\epsilon} = \mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)$ , which is a finite extension of  $\mathcal{O}(G)$  by Theorem 1.2 and Corollary 5.7 discussed below, whereas  $\mathcal{M}_{0,n}^{A,\epsilon}$  is the specialization at  $q = \epsilon$  of  $\mathcal{Z}(\mathcal{L}_{0,1}^A)$ , a polynomial algebra which may be identified via  $\Phi_1$  with  $\mathcal{Z}(U_A)$ , generated by the quantum Casimir elements. Also the center  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  of  $\mathcal{L}_{0,n}^\epsilon$  is contained in  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  (this follows from

[28], Proposition 6.17). Consider the rings

$$Q(\mathcal{L}_{0,n}^\epsilon) = Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} \mathcal{L}_{0,n}^\epsilon$$

and

$$Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) = Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}.$$

In general, given a ring  $A$  with center  $Z$  an integral domain we reserve the notation  $Q(A)$  to the central localization of  $A$ , ie.  $Q(A) := Q(Z) \otimes_Z A$ . Though the center  $\mathcal{Z}((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  of  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is larger than  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ , the notation  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is not ambiguous, for  $\mathcal{Z}((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is an integral domain finite over  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ , and hence the central localization of  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  coincides with  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  as defined above. Throughout the paper, unless we mention it explicitly we follow the conventions of Mc Connell-Robson [69] as regards the terminology of ring theory; in particular, for the notions of central simple algebras, (maximal) orders and PI degrees, see in [69] the sections 5.3 and 13.3.6-13.6.7.

Denote by  $m$  the rank of  $\mathfrak{g}$ , and by  $N$  the number of its positive roots. We prove:

**Theorem 1.3.** (1)  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a central simple algebra of PI degree  $l^{nN}$ , and  $\mathcal{L}_{0,n}^\epsilon$  is a maximal order of  $Q(\mathcal{L}_{0,n}^\epsilon)$ .

(2)  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ ,  $n \geq 2$ , is a central simple algebra of PI degree  $l^{N(n-1)-m}$ , and  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is a maximal order of  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ .

(3)  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is a Noetherian ring, its center is  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon) \otimes_{\Delta^{(n)}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))} \Delta^{(n)}(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))$ , and as a  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ ,  $n \geq 2$ , is free of rank  $l^{(n-1) \cdot \dim \mathfrak{g}}$ .

The first claim of the statement (1) means that  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a complex subalgebra of a full matrix algebra  $Mat_d(\mathbb{F})$ , where  $d = l^{nN}$  and  $\mathbb{F}$  is a finite extension of  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  such that

$$\mathbb{F} \otimes_{Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} Q(\mathcal{L}_{0,n}^\epsilon) = Mat_d(\mathbb{F}).$$

We deduce it from Theorem 1.2 and the computation of the degree of  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  as a field extension of  $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ . This computation uses  $\Phi_n$  and the computation of the degree of  $Q(\mathcal{Z}(U_\epsilon))$  over  $Q(\mathcal{Z}_0(U_\epsilon))$  by De Concini-Kac [38] (see Proposition 5.3).

The second claim of (1) is proved in Theorem 5.6. More precisely we prove that  $\mathcal{L}_{0,n}^\epsilon$  is integrally closed in  $Q(\mathcal{L}_{0,n}^\epsilon)$ , in the sense of [38, 40]. So, before the theorem we show in Lemma 5.5 that a ring  $A$  with no non-trivial zero divisors, Noetherian center, and finite dimensional classical fraction algebra  $Q(A)$ , which is the case of  $\mathcal{L}_{0,n}^\epsilon$  and  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ , is integrally closed in  $Q(A)$  if and only if it is maximal as a (classical) order. For the sake of clarity we have included a general discussion of these notions before Theorem 5.6. The proof of that theorem uses the facts that  $\mathcal{O}_\epsilon$  is a maximal order of its classical fraction algebra, which is Theorem 7.4 of [42], and that the twist which defines the algebra structure of  $\mathcal{L}_{0,n}^\epsilon$  from  $\mathcal{O}_\epsilon^{\otimes n}$  keeps the  $\mathcal{Z}_0$ -module structure unchanged. It seems harder to prove directly that  $\mathcal{L}_{0,n}^\epsilon$  is a maximal order, without this twist argument, essentially because we know only one localization of  $\mathcal{L}_{0,n}^\epsilon$  which is a maximal order (and thus cannot apply the Serre argument as in Theorem 7.4 of [42]), and, in another direction, we lack of a complete set of defining relations, allowing for degeneration arguments as in [40, 41]. However, as an example we do it in the  $sl(2)$  case when  $n = 1$ .

As a consequence of the maximality of  $\mathcal{L}_{0,n}^\epsilon$  and the fact that  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is Noetherian, it is an integrally closed domain, equal to the trace ring of  $\mathcal{L}_{0,n}^\epsilon$ . In fact  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon) = \mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$ , and it is a free  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module of rank  $l^{mn}$  (see Corollary 5.7).

We deduce the first claim of (2) and the second of (3) from the assertion (1), the double centralizer theorem for central simple algebras, a few results of [28] and [42], and Theorem 1.2 again.

The first claim of (3) follows directly from the fact that  $\mathcal{O}(G)$  and  $\mathcal{L}_{0,n}^\epsilon$  are Noetherian rings (the latter by Theorem 1.1; see the proof of Theorem 4.7 for details). Finally, the left regular action of  $\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon) \otimes_{\Delta^{(n)}(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  on  $\mathcal{L}_{0,n}^\epsilon$  yields the following decomposition into simple components,

$$\mathcal{L}_{0,n}^\epsilon \cong \left( \Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon) \otimes_{\Delta^{(n)}(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon} \right)^{\oplus l^m}.$$

From this and our previous results for  $\mathcal{L}_{0,n}^\epsilon$  we deduce the last claims of (2) and (3). We note that  $\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon) = (\mathcal{L}_{0,1}^\epsilon)^{U_\epsilon}$ , and a certain localization of  $\mathcal{L}_{0,1}^\epsilon$  is a direct summand of  $U_\epsilon$  (see Theorem 2.2 (2) and Corollary 2.5 (2)). So one can view the freeness of the  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module  $(\mathcal{L}_{0,1}^\epsilon)^{U_\epsilon}$  as a generalization of the fact that  $\mathcal{Z}(U_\epsilon)$  is free of rank  $l^m$  over  $\mathcal{Z}_0(U_\epsilon)$  (proved in [40], Proposition 20.2).

We conclude with an application of Theorem 1.3, providing a characterization of the irreducible representations of maximal dimension. Recall that given a classical order  $A$  of PI degree  $d$  and with center  $Z$  a Noetherian and integrally closed domain, the *discriminant*  $D(A)$  is the ideal of  $Z$  generated by the elements  $\det((t_{red}(x_i x_j))_{1 \leq i, j \leq d})$ , where  $x_1, \dots, x_d \in A$  and  $t_{red} : A \rightarrow Z$  is the reduced trace map of  $Q(A)$  restricted to  $A$  (see [71], Section 10). Given a central character  $\chi \in \text{Maxspec}(Z)$  denote by  $I^\chi$  the ideal of  $A$  generated by the kernel of  $\chi$ , and let  $A^\chi := A/I^\chi$ . Our results show all this applies in particular to  $A := \mathcal{L}_{0,n}^\epsilon$  or  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ . Classical arguments then imply that if  $A$  has no non-trivial zero divisors, then  $A^\chi \neq 0$ , and moreover we have (see eg. Lemma 3.7 of [38]):

**Corollary 1.4.** (a)  $(\mathcal{L}_{0,n}^\epsilon)^\chi$  is isomorphic to  $M_d(\mathbb{C})$ ,  $d := l^{nN}$ , if and only if  $\chi \notin D(\mathcal{L}_{0,n}^\epsilon)$ , and if  $\chi \in D(\mathcal{L}_{0,n}^\epsilon)$  every irreducible representation of  $(\mathcal{L}_{0,n}^\epsilon)^\chi$  has dimension less than  $d$ .  
(b) Same statement for  $((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})^\chi$ , putting  $d := l^{N(n-1)-m}$  and replacing  $D(\mathcal{L}_{0,n}^\epsilon)$  with  $D((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ .

Much more can be said on irreducible representations of dimension  $< d$ , eg. by using lower discriminant ideals (see the Main Theorem of [26]). Also, it follows from Theorem 7.18 of [28] that  $\mathcal{L}_{0,n}^\epsilon(sl(2))$  is a Poisson order relative to its center, which is a Poisson central finite extension of  $\mathcal{O}(SL(2, \mathbb{C})^n)$  endowed with the Fock-Rosly Poisson structure. This should extend without difficulty to all  $\mathfrak{g}$  beyond the  $sl(2)$  case. By the results of [24], the zero locus of  $D(\mathcal{L}_{0,n}^\epsilon(sl(2)))$  is then a union of symplectic leaves in  $\text{Maxspec}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon(sl(2))))$  (a determined, finite covering space of  $SL(2, \mathbb{C})^n$ ). There is a similar result for  $\mathcal{M}_{0,n}^\epsilon(sl(2))$  (Corollary 7.21 of [28]), in terms of the Atiyah-Bott-Goldman Poisson structure on the invariant coordinate ring  $\mathcal{O}(SL(2, \mathbb{C})^n)^{SL(2, \mathbb{C})}$ .

In [29] we use all this to describe the subalgebra  $\mathcal{M}_{0,n}^{A,\epsilon} \subset (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  and its representations, and we give applications to skein algebras (which is the  $sl(2)$  case). In [27] we consider the algebras  $\mathcal{M}_{g,n}^{A,\epsilon}$  for genus  $g \neq 0$ .

**Acknowledgements.** We are grateful to K. A. Brown for pointing out the reference [26] above, and [6] and [21] (see the comments before Theorem 1.2).

**1.1. Basic notations.** Given a ring  $R$ , we denote by  $\mathcal{Z}(R)$  its center, by  $\text{Spec}(R)$  its spectrum, and by  $\text{Maxspec}(R)$  its *maximal* spectrum. When  $R$  is commutative and has no non-trivial zero divisors,  $Q(R)$  denotes its fraction field.

Given a Hopf algebra  $H$  with product  $m$  and coproduct  $\Delta$ , we denote by  $H^{cop}$  (resp.  $H_{op}$ ) the Hopf algebra with the same algebra (resp. coalgebra) structure as  $H$  but the opposite coproduct  $\sigma \circ \Delta$  (resp. opposite product  $m \circ \sigma$ ), where  $\sigma(x \otimes y) = y \otimes x$ , and the antipode  $S^{-1}$ . We use Sweedler's coproduct notation,  $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ ,  $x \in H$ .

We let  $\mathfrak{g}$  be a finite dimensional complex simple Lie algebra of rank  $m$ , with Cartan matrix  $(a_{ij})$ . We fix a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  and a basis of simple roots  $\alpha_i \in \mathfrak{h}_{\mathbb{R}}^*$ ; we denote by  $d_1, \dots, d_m$  the unique coprime positive integers such that the matrix  $(d_i a_{ij})$  is symmetric, and  $(\cdot, \cdot)$  the unique inner product on  $\mathfrak{h}_{\mathbb{R}}^*$  such that  $d_i a_{ij} = (\alpha_i, \alpha_j)$ . For any root  $\alpha$  the coroot is  $\check{\alpha} = 2\alpha/(\alpha, \alpha)$ ; in particular  $\check{\alpha}_i = d_i^{-1} \alpha_i$ . The root lattice  $Q$  is the  $\mathbb{Z}$ -lattice in  $\mathfrak{h}_{\mathbb{R}}^*$  defined by  $Q = \sum_{i=1}^m \mathbb{Z} \alpha_i$ . The weight lattice  $P$  is the  $\mathbb{Z}$ -lattice formed by all  $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$  such that  $(\lambda, \check{\alpha}_i) \in \mathbb{Z}$  for every  $i = 1, \dots, m$ . So  $P = \sum_{i=1}^m \mathbb{Z} \varpi_i$ , where  $\varpi_i$  is the fundamental weight dual to the simple coroot  $\check{\alpha}_i$ , ie. satisfying  $(\varpi_i, \check{\alpha}_j) = \delta_{i,j}$ . We denote by  $P_+ := \sum_{i=1}^m \mathbb{Z}_{\geq 0} \varpi_i$  the cone of dominant integral weights, by  $N$  the number of positive roots of  $\mathfrak{g}$ , by  $\rho$  half the sum of the positive roots, and by  $D$  the smallest positive integer such that  $D(\lambda, \mu) \in \mathbb{Z}$  for every  $\lambda, \mu \in P$ . Note that  $(\lambda, \alpha) \in \mathbb{Z}$  for every  $\lambda \in P$ ,  $\alpha \in Q$ , and  $D$  is the smallest positive integer such that  $DP \subset Q$ . We denote by  $\mathcal{B}(\mathfrak{g})$  the braid group of  $\mathfrak{g}$ ; we recall its standard defining relations in the Appendix (Section 6.1).

We let  $G$  be the connected and simply-connected Lie group with Lie algebra  $\mathfrak{g}$ . We put  $T_G = \exp(\mathfrak{h})$ , the maximal torus of  $G$  generated by  $\mathfrak{h}$ ;  $N(T_G)$  is the normalizer of  $T_G$ ,  $W = N(T_G)/T_G$  is the Weyl group,  $B_{\pm}$  the unique Borel subgroups such that  $B_+ \cap B_- = T_G$ , and  $U_{\pm} \subset B_{\pm}$  their unipotent subgroups.

We let  $q$  be an indeterminate, set  $A = \mathbb{C}[q, q^{-1}]$ ,  $q_i = q^{d_i}$ , and given integers  $p, k$  with  $0 \leq k \leq p$  we put

$$[p]_q = \frac{q^p - q^{-p}}{q - q^{-1}}, \quad [0]_q! = 1, \quad [p]_q! = [1]_q [2]_q \cdots [p]_q, \quad \begin{bmatrix} p \\ k \end{bmatrix}_q = \frac{[p]_q!}{[p-k]_q! [k]_q!}$$

$$(p)_q = \frac{q^p - 1}{q - 1}, \quad (0)_q! = 1, \quad (p)_q! = (1)_q (2)_q \cdots (p)_q, \quad \binom{p}{k}_q = \frac{(p)_q!}{(p-k)_q! (k)_q!}.$$

We denote by  $\epsilon$  a primitive  $l$ -th root of unity such that  $\epsilon^{2d_i} \neq 1$  is also a primitive  $l$ -th root of unity for all  $i \in \{1, \dots, m\}$ . This means that  $l$  is odd, and coprime to 3 if  $\mathfrak{g}$  has  $G_2$ -components.

In this paper we use the definition of the unrestricted integral form  $U_A(\mathfrak{g})$  given in [40], [42]; in [28] we used the one of [38], [39]. The two are (trivially) isomorphic, and have the same specialization at  $q = \epsilon$ . Also, we denote here by  $L_i$  the generators of  $U_q(\mathfrak{g})$  we denoted by  $\ell_i$  in [28].

To facilitate the comparison with [42] we note that their generators, that we will denote by  $\tilde{K}_i, \tilde{E}_i$  and  $\tilde{F}_i$ , can be written respectively as  $K_i, K_i^{-1} E_i$  and  $F_i K_i$  in our notations. They satisfy the same algebra relations.

## 2. BACKGROUND RESULTS

**2.1. On  $U_q, \mathcal{O}_q, \mathcal{L}_{0,n}, \mathcal{M}_{0,n}$ , and  $\Phi_n$ .** Except when stated differently, we refer to [28], Sections 2-4 and 6, and the references therein for details about the material of this section.



The *simply-connected* quantum group  $U_q = U_q(\mathfrak{g})$  is the Hopf algebra over  $\mathbb{C}(q)$  with generators  $E_i, F_i, L_i, L_i^{-1}$ ,  $1 \leq i \leq m$ , and defining relations

$$\begin{aligned} L_i L_j &= L_j L_i, \quad L_i L_i^{-1} = L_i^{-1} L_i = 1, \quad L_i E_j L_i^{-1} = q_i^{\delta_{i,j}} E_j, \quad L_i F_j L_i^{-1} = q_i^{-\delta_{i,j}} F_j \\ E_i F_j - F_j E_i &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r &= 0 \quad \text{if } i \neq j \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{ij}-r} F_j F_i^r &= 0 \quad \text{if } i \neq j. \end{aligned}$$

where for  $\lambda = \sum_{i=1}^m m_i \varpi_i \in P$  we set  $K_\lambda = \prod_{i=1}^m L_i^{m_i}$ , and  $K_i = K_{\alpha_i} = \prod_{j=1}^m L_j^{a_{ji}}$ . The coproduct  $\Delta$ , antipode  $S$ , and counit  $\varepsilon$  of  $U_q$  are given by

$$\begin{aligned} \Delta(L_i) &= L_i \otimes L_i, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = K_i^{-1} \otimes F_i + F_i \otimes 1 \\ S(E_i) &= -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i, \quad S(L_i) = L_i^{-1} \\ \varepsilon(E_i) &= \varepsilon(F_i) = 0, \quad \varepsilon(L_i) = 1. \end{aligned}$$

We fix a reduced expression  $s_{i_1} \dots s_{i_N}$  of the longest element  $w_0$  of the Weyl group of  $\mathfrak{g}$ . It induces a total ordering of the positive roots,

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \dots, \beta_N = s_{i_1} \dots s_{i_{N-1}}(\alpha_{i_N}).$$

The root vectors of  $U_q$  with respect to such an ordering are defined by

$$E_{\beta_k} = T_{i_1} \dots T_{i_{k-1}}(E_{i_k}), \quad F_{\beta_k} = T_{i_1} \dots T_{i_{k-1}}(F_{i_k})$$

where  $T_i$  is Lusztig's algebra automorphism of  $U_q$  associated to the simple root  $\alpha_i$  ([61, 60], see also [35], Ch. 8). In the Appendix we recall the relation between  $T_i$  and the generator  $\hat{w}_i$  of the quantum Weyl group, which we will mostly use. Let us just recall here that the monomials  $F_{\beta_1}^{r_1} \dots F_{\beta_N}^{r_N} K_\lambda E_{\beta_N}^{t_N} \dots E_{\beta_1}^{t_1}$  ( $r_i, t_i \in \mathbb{N}$ ,  $\lambda \in P$ ) form a basis of  $U_q$ .

$U_q$  is a *pivotal* Hopf algebra, with pivotal element

$$\ell := K_{2\rho} = \prod_{j=1}^m L_j^2.$$

So  $\ell$  is group-like, and  $S^2(x) = \ell x \ell^{-1}$  for every  $x \in U_q$ .

The *adjoint* quantum group  $U_q^{ad} = U_q^{ad}(\mathfrak{g})$  is the Hopf subalgebra of  $U_q$  generated by the elements  $E_i, F_i$  ( $i = 1, \dots, m$ ) and  $K_\alpha$  with  $\alpha \in Q$ ; so  $\ell \in U_q^{ad}$ . When  $\mathfrak{g} = \mathfrak{sl}(2)$ , we simply write the above generators  $E = E_1, F = F_1, L = L_1, K = K_1$ .

We denote by  $U_q(\mathfrak{n}_+)$ ,  $U_q(\mathfrak{n}_-)$  and  $U_q(\mathfrak{h})$  the subalgebras of  $U_q$  generated respectively by the  $E_i$ , the  $F_i$ , and the  $K_\lambda$  ( $\lambda \in P$ ), and by  $U_q(\mathfrak{b}_+)$  and  $U_q(\mathfrak{b}_-)$  the subalgebras generated by the  $E_i$  and the  $K_\lambda$ , and by the  $F_i$  and the  $K_\lambda$ , respectively (they are the two-sided ideals generated by  $U_q(\mathfrak{n}_\pm)$ ). We do similarly with  $U_q^{ad}$ .

$U_q^{ad}$  is not a braided Hopf algebra in a strict sense, but it has braided categorical completions. Namely, denote by  $\mathcal{C}$  the category of type 1 finite dimensional  $U_q^{ad}$ -modules, by  $Vect$  the category of finite dimensional  $\mathbb{C}(q)$ -vector spaces, and by  $F_{\mathcal{C}} : \mathcal{C} \rightarrow Vect$  the forgetful functor. The *categorical completion*  $\mathbb{U}_q^{ad}$  of  $U_q^{ad}$  is the set of natural transformations  $F_{\mathcal{C}} \rightarrow F_{\mathcal{C}}$ .

Let us recall briefly what this means and implies. For details we refer to the sections 2 and 3 of [28] (see also [80], Section 2.10, where  $\mathbb{U}_q$  below is formulated in terms of multiplier Hopf algebras). An element of  $\mathbb{U}_q^{ad}$  is a collection  $(a_V)_{V \in Ob(\mathcal{C})}$ , where  $a_V \in End_{\mathbb{C}(q)}(V)$  satisfies  $F_{\mathcal{C}}(f) \circ a_V = a_W \circ F_{\mathcal{C}}(f)$  for any objects  $V, W$  of  $\mathcal{C}$  and any arrow  $f \in Hom_{U_q^{ad}}(V, W)$ . It is

not hard to see that  $\mathbb{U}_q^{ad}$  inherits from  $\mathcal{C}$  a natural structure of Hopf algebra such that the map

$$\begin{aligned} \iota : U_q^{ad} &\longrightarrow \mathbb{U}_q^{ad} \\ x &\longmapsto (\pi_V(x))_{V \in \text{Ob}(\mathcal{C})} \end{aligned}$$

is a morphism of Hopf algebras, where  $\pi_V : U_q^{ad} \rightarrow \text{End}(V)$  is the representation associated to a module  $V$  in  $\mathcal{C}$ . It is a theorem that this map is injective;  $\mathbb{U}_q^{ad}$  can be understood as a weak- $*$  completion of  $U_q^{ad}$  by means of the pairing  $\langle \cdot, \cdot \rangle$  introduced below. From now on, let us extend the coefficient ring of the modules and morphisms in  $\mathcal{C}$  to  $\mathbb{C}(q^{1/D})$ . Put

$$\mathbb{U}_q = \mathbb{U}_q^{ad} \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})$$

One can show that the map  $\iota$  above extends to an embedding of  $U_q \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})$  in  $\mathbb{U}_q$ . The category  $\mathcal{C}$ , with coefficients in  $\mathbb{C}(q^{1/D})$ , is braided and ribbon. We postpone a discussion of that fact to Section 2.3, where it will be developed. As a consequence,  $\mathbb{U}_q$  is a quasitriangular and ribbon Hopf algebra. The R-matrix of  $\mathbb{U}_q$  is the family of morphisms

$$R = ((R_h)_{V,W})_{V,W \in \text{Ob}(\mathcal{C})}$$

where  $q = e^h$ ,  $R_h$  is the universal R-matrix of the quantized universal enveloping algebra  $U_h(\mathfrak{g})$ , and  $(R_h)_{V,W} \in \text{End}(V \otimes W)$ , for every modules  $V, W$  in  $\mathcal{C}$ , is the endomorphism defined by the action of  $R_h$  on  $V \otimes W$  (which is well-defined). The ribbon element  $v_h$  of  $U_h(\mathfrak{g})$  defines similarly the ribbon element  $v = ((v_h)_V)_V$  of  $\mathbb{U}_q$ . One defines the categorical tensor product  $\mathbb{U}_q^{\otimes 2}$  similarly as  $\mathbb{U}_q$ ; it contains all the infinite series of elements of  $\mathbb{U}_q^{\otimes 2}$  having only a finite number of non-zero terms when evaluated on a given module  $V \otimes W$  of  $\mathcal{C}$ . The expansion of  $R_h$  as an infinite series in  $U_h(\mathfrak{g})^{\otimes 2}$  induces an expansion of  $R$  as an infinite series in  $\mathbb{U}_q^{\otimes 2}$ . Adapting Sweedler's coproduct notation  $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$  we find convenient to write this series as

$$(1) \quad R = \sum_{(R)} R_{(1)} \otimes R_{(2)}.$$

We put  $R^+ := R$ ,  $R^- := (\sigma \circ R)^{-1}$  where  $\sigma$  is the flip map  $x \otimes y \mapsto y \otimes x$ .

The *quantum function Hopf algebra*  $\mathcal{O}_q = \mathcal{O}_q(G)$  is the restricted dual of  $U_q^{ad}$ , ie. the set of  $\mathbb{C}(q)$ -linear maps  $f : U_q^{ad} \rightarrow \mathbb{C}(q)$  such that  $\text{Ker}(f)$  contains a cofinite two sided ideal  $I$  (ie. such that  $I \oplus M = U_q^{ad}$  for some finite dimensional vector space  $M$ ), and  $\prod_{s=-r}^r (K_i - q_i^s) \in I$  for some  $r \in \mathbb{N}$  and every  $i$ . The structure maps of  $\mathcal{O}_q$  are defined dually to those of  $U_q^{ad}$ . We denote by  $\star$  its product. The algebras  $\mathcal{O}_q(T_G)$ ,  $\mathcal{O}_q(U_{\pm})$ ,  $\mathcal{O}_q(B_{\pm})$  are defined similarly, by replacing  $U_q^{ad}$  with  $U_q^{ad}(\mathfrak{h})$ ,  $U_q^{ad}(\mathfrak{n}_{\pm})$ ,  $U_q^{ad}(\mathfrak{b}_{\pm})$  respectively.  $\mathcal{O}_q$  is generated as an algebra by the functionals  $x \mapsto w(\pi_V(x)v)$ ,  $x \in U_q^{ad}$ , for every object  $V \in \text{Ob}(\mathcal{C})$  and vectors  $v \in V$ ,  $w \in V^*$ . Such functionals are called *matrix coefficients*. We can uniquely extend the (non-degenerate) evaluation pairing  $\langle \cdot, \cdot \rangle : \mathcal{O}_q \otimes U_q^{ad} \rightarrow \mathbb{C}(q)$  to a bilinear pairing  $\langle \cdot, \cdot \rangle : \mathcal{O}_q \otimes \mathbb{U}_q \rightarrow \mathbb{C}(q^{1/D})$  such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}_q \otimes U_q^{ad} & \xrightarrow{\langle \cdot, \cdot \rangle} & \mathbb{C}(q) \\ \text{id} \otimes \iota \downarrow & \nearrow \langle \cdot, \cdot \rangle & \\ \mathcal{O}_q \otimes \mathbb{U}_q & & \end{array}$$

This pairing is defined by

$$\langle Y \phi_v^w, (a_X)_X \rangle = w(a_Y v)$$

for every  $(a_X)_X \in \mathbb{U}_q$ ,  ${}_Y\phi_v^w \in \mathcal{O}_q$ . It is a perfect pairing, and reflects the properties of the R-matrix  $R \in \mathbb{U}_q^{\otimes 2}$  in a subtle way. In particular, these properties imply that the maps

$$(2) \quad \begin{aligned} \Phi^\pm : \mathcal{O}_q &\longrightarrow U_q^{cop} \\ \alpha &\longmapsto (\alpha \otimes id)(R^\pm) = \sum_{(R^\pm)} \langle \alpha, R_{(1)}^\pm \rangle R_{(2)}^\pm \end{aligned}$$

are well-defined morphisms of Hopf algebras. Here we stress that it is the simply-connected quantum group  $U_q^{cop}$  that is the range of  $\Phi^\pm$ . This will be explained in more details in Section 2.3.

The *quantum loop algebra*  $\mathcal{L}_{0,1} = \mathcal{L}_{0,1}(\mathfrak{g})$  is defined by twisting the product  $\star$  of  $\mathcal{O}_q$ , keeping the same underlying linear space. The new product is equivariant with respect to the right coadjoint action  $coad^r$  of  $U_q^{ad}$ ; noting that  $coad^r$  extends to an action of the simply-connected quantum group  $U_q$ , the new product thus gives  $\mathcal{L}_{0,1}$  a structure of  $U_q$ -module algebra. Recall that

$$coad^r(x)(\alpha) = \sum_{(x)} S(x_{(2)}) \triangleright \alpha \triangleleft x_{(1)}$$

for all  $x \in U_q$  and  $\alpha \in \mathcal{O}_q$ , where  $\triangleright, \triangleleft$  are the left and right coregular actions of  $U_q$  on  $\mathcal{O}_q$ , defined by

$$x \triangleright \alpha := \sum_{(\alpha)} \alpha_{(1)} \langle \alpha_{(2)}, x \rangle, \quad \alpha \triangleleft x := \sum_{(\alpha)} \langle \alpha_{(1)}, x \rangle \alpha_{(2)}.$$

Using the fact that  $U_q \otimes \mathbb{C}(q^{1/D})$  can be regarded as a subspace of  $\mathbb{U}_q$ , these actions extend naturally to actions of  $\mathbb{U}_q$ . The product of  $\mathcal{L}_{0,1}$  is expressed in terms of  $\star$  by the formula ([28], Proposition 4.1):

$$(3) \quad \alpha\beta = \sum_{(R),(R')} (R_{(2')} S(R_{(2)}) \triangleright \alpha) \star (R_{(1')} \triangleright \beta \triangleleft R_{(1)}),$$

where  $\sum_{(R)} R_{(1)} \otimes R_{(2)}$  and  $\sum_{(R)} R_{(1')} \otimes R_{(2')}$  are expansions of two copies of  $R \in \mathbb{U}_q^{\otimes 2}$ . Note that the sum in (3) has only a finite number of non zero terms. This product gives  $\mathcal{L}_{0,1}$  (like  $\mathcal{O}_q$ ) a structure of module algebra for the actions  $\triangleright, \triangleleft$ , and also for  $coad^r(x)$ . Spelling this out for  $coad^r$ , this means

$$coad^r(x)(\alpha\beta) = \sum_{(x)} coad^r(x_{(1)})(\alpha) coad^r(x_{(2)})(\beta).$$

The relations between  $\mathcal{O}_q$ ,  $\mathcal{L}_{0,1}$  and  $U_q$  (the simply-connected quantum group) are encoded by the map

$$(4) \quad \begin{aligned} \Phi_1 : \mathcal{O}_q &\longrightarrow U_q \\ \alpha &\longmapsto (\alpha \otimes id)(RR') \end{aligned}$$

where  $R' = \sigma \circ R$ , and as usual  $\sigma : x \otimes y \mapsto y \otimes x$ . Note that

$$\Phi_1 = m \circ (\Phi^+ \otimes (S^{-1} \circ \Phi^-)) \circ \Delta.$$

We call  $\Phi_1$  the *RSD* map, for Drinfeld, Reshetikhin and Semenov-Tian-Shansky introduced it first (see [36, 70],[67]). Recall that  $U_q$  embeds in  $\mathbb{U}_q$ . It is a fundamental result of the theory ([33, 51, 11]) that  $\Phi_1$  affords an isomorphism of  $U_q$ -modules

$$\Phi_1 : \mathcal{O}_q \rightarrow U_q^{lf}.$$

For full details on that result we refer to Section 2.12 of [80] (where different conventions are used). Here,  $U_q^{lf}$  is the set of *locally finite* elements of  $U_q$ , endowed with the right adjoint action  $ad^r$  of  $U_q$ . It is defined by

$$U_q^{lf} := \{x \in U_q \mid rk_{\mathbb{C}(q)}(ad^r(U_q)(x)) < \infty\}$$

and

$$ad^r(y)(x) = \sum_{(y)} S(y_{(1)})xy_{(2)}$$

for every  $x, y \in U_q$ . The action  $ad^r$  gives in fact  $U_q^{lf}$  a structure of right  $U_q$ -module algebra. Moreover,  $\Phi_1$  affords an isomorphism of  $U_q$ -module algebras

$$(5) \quad \Phi_1: \mathcal{L}_{0,1} \rightarrow U_q^{lf}.$$

The centers  $\mathcal{Z}(\mathcal{L}_{0,1})$  of  $\mathcal{L}_{0,1}$ , and  $\mathcal{Z}(U_q)$  of  $U_q$ , coincide respectively with  $\mathcal{L}_{0,1}^{U_q}$  and  $U_q^{U_q}$ , the subsets of  $U_q$ -invariants elements of  $\mathcal{L}_{0,1}$  and  $U_q$ . As a consequence,  $\Phi_1$  affords an isomorphism between  $\mathcal{Z}(\mathcal{L}_{0,1})$  and  $\mathcal{Z}(U_q)$ .

The *quantum graph algebra*  $\mathcal{L}_{0,n} = \mathcal{L}_{0,n}(\mathfrak{g})$  is the braided tensor product of  $n$  copies of  $\mathcal{L}_{0,1}$  (considered as a  $\mathbb{U}_q$ -module algebra). Thus it coincides with  $\mathcal{L}_{0,1}^{\otimes n}$  as a linear space, and it is a right  $U_q$ -module algebra, the action of  $U_q$  (extending  $coad^r$  on  $\mathcal{L}_{0,1}$ ) being given by

$$coad_n^r(y)(\alpha^{(1)} \otimes \dots \otimes \alpha^{(n)}) = \sum_{(y)} coad^r(y_{(1)})(\alpha^{(1)}) \otimes \dots \otimes coad^r(y_{(n)})(\alpha^{(n)})$$

for all  $y \in U_q$  and  $\alpha^{(1)} \otimes \dots \otimes \alpha^{(n)} \in \mathcal{L}_{0,n}$ . The algebra structure can be explicitated as follows. For every  $1 \leq a \leq n$  define  $\mathbf{i}_a: \mathcal{L}_{0,1} \rightarrow \mathcal{L}_{0,n}$  by  $\mathbf{i}_a(x) = 1^{\otimes(a-1)} \otimes x \otimes 1^{\otimes(n-a)}$ ;  $\mathbf{i}_a$  is an embedding of  $U_q$ -module algebras. We will use the notations

$$\mathcal{L}_{0,n}^{(a)} := \text{Im}(\mathbf{i}_a), \quad (\alpha)^{(a)} := \mathbf{i}_a(\alpha).$$

Take  $(\alpha)^{(a)}, (\alpha')^{(a)} \in \mathcal{L}_{0,n}^{(a)}$  and  $(\beta)^{(b)}, (\beta')^{(b)} \in \mathcal{L}_{0,n}^{(b)}$  with  $a < b$ . Then the product of  $\mathcal{L}_{0,n}$  is given by the following formula (see in [28] the proposition 6.2-6.3 and the formulas (13)-(41)-(42)):

$$(6) \quad \begin{aligned} & \left( (\alpha)^{(a)} \otimes (\beta)^{(b)} \right) \left( (\alpha')^{(a)} \otimes (\beta')^{(b)} \right) \\ &= \sum_{(R^1), \dots, (R^4)} \left( \alpha \left( S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha' \triangleleft R_{(1)}^1 R_{(1)}^2 \right) \right)^{(a)} \\ & \quad \otimes \left( \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \right) \beta' \right)^{(b)} \end{aligned}$$

where  $R^i = \sum_{(R^i)} R_{(1)}^i \otimes R_{(2)}^i$ ,  $i \in \{1, 2, 3, 4\}$ , are expansions of four copies of  $R \in \hat{\mathbb{U}}_q^{\otimes 2}$ , and on the right-hand side the product is componentwise that of  $\mathcal{L}_{0,1}$ . Later we will use the fact that the product of  $\mathcal{L}_{0,n}$  is obtained from the standard (componentwise) product of  $\mathcal{L}_{0,1}^{\otimes n}$  by a process that may be inverted. Indeed, (6) can be rewritten as

$$(7) \quad \left( (\alpha)^{(a)} \otimes (\beta)^{(b)} \right) \left( (\alpha')^{(a)} \otimes (\beta')^{(b)} \right) = \sum_{(F)} (\alpha)^{(a)} \left( (\alpha')^{(a)} \cdot F_{(2)} \right) \otimes \left( (\beta)^{(b)} \cdot F_{(1)} \right) (\beta')^{(b)}$$

where  $F = \sum_{(F)} F_{(1)} \otimes F_{(2)} := (\Delta \otimes \Delta)(R')$ , and the symbol “ $\cdot$ ” stands for the right action of  $\mathbb{U}_q^{\otimes 2}$  on  $\mathcal{L}_{0,1}$  that may be read from (6). The tensor  $F$  is known as a twist. Then, by replacing

$F$  with its inverse  $\bar{F} = (\Delta \otimes \Delta)(R'^{-1})$ , one can express the product of  $\mathcal{L}_{0,1}^{\otimes n}$  in terms of the product of  $\mathcal{L}_{0,n}$  by

$$(8) \quad (\alpha)^{(a)}(\alpha')^{(a)} \otimes (\beta)^{(b)}(\beta')^{(b)} = \sum_{(\bar{F})} \left( (\alpha)^{(a)} \otimes \left( (\beta)^{(b)} \cdot \bar{F}_{(1)} \right) \right) \left( \left( (\alpha')^{(a)} \cdot \bar{F}_{(2)} \right) \otimes (\beta')^{(b)} \right).$$

We call *quantum moduli algebra* and denote by

$$\mathcal{M}_{0,n} = \mathcal{M}_{0,n}(\mathfrak{g})$$

the subalgebra  $\mathcal{L}_{0,n}^{U_q}$  of  $\mathcal{L}_{0,n}$  formed by the  $U_q$ -invariant elements.

Consider the following action of  $U_q$  on the tensor product algebra  $U_q^{\otimes n}$ , which extends  $ad^r$  on  $U_q$ :

$$(9) \quad ad_n^r(y)(x) = \sum_{(y)} \Delta^{(n)}(S(y_{(1)}))x\Delta^{(n)}(y_{(2)})$$

for all  $y \in U_q$ ,  $x \in U_q^{\otimes n}$ . This action gives  $U_q^{\otimes n}$  a structure of right  $U_q$ -module algebra. In [1] Alekseev introduced a morphism of  $U_q$ -module algebras  $\Phi_n: \mathcal{L}_{0,n} \rightarrow U_q^{\otimes n}$  which extends  $\Phi_1$ . In Proposition 6.5 and Lemma 6.8 of [28] we showed that  $\Phi_n$  affords isomorphisms

$$(10) \quad \Phi_n: \mathcal{L}_{0,n} \rightarrow (U_q^{\otimes n})^{lf}, \quad \Phi_n: \mathcal{M}_{0,n} \rightarrow (U_q^{\otimes n})^{U_q}$$

where  $(U_q^{\otimes n})^{lf}$  is the set of  $ad_n^r$ -locally finite elements of  $U_q^{\otimes n}$ . We call  $\Phi_n$  the *Alekseev map*; we will not use the definition of  $\Phi_n$  in this paper.

It is a key argument of the proof of (10), to be used later, that the set of locally finite elements of  $U_q^{\otimes n}$  for  $(ad^r)^{\otimes n} \circ \Delta^{(n-1)}$  coincides with  $(U_q^{lf})^{\otimes n}$ ; this follows from the main result of [57]. Using that the map

$$(11) \quad \psi_n = \Phi_n \circ (\Phi_1^{-1})^{\otimes n}$$

extends to a linear automorphism of  $U_q^{\otimes n}$  which intertwines the actions  $(ad^r)^{\otimes n} \circ \Delta^{(n-1)}$  and  $ad_n^r$  of  $U_q$ , we deduced that  $\psi_n((U_q^{lf})^{\otimes n}) = (U_q^{\otimes n})^{lf}$ , whence  $\text{Im}(\Phi_n) = (U_q^{\otimes n})^{lf}$ .

**Remark 2.1.** We have  $(U_q^{lf})^{\otimes n} \neq (U_q^{\otimes n})^{lf}$ , and in fact there is not even an inclusion. Indeed let  $\Omega = (q - q^{-1})^2 FE + qK + q^{-1}K^{-1}$  be the standard Casimir element of  $U_q(sl(2))$ . We trivially have  $\Delta(\Omega) \in (U_q^{\otimes 2})^{lf}$  but

$$\Delta(\Omega) = (q - q^{-1})^2(K^{-1}E \otimes FK + F \otimes E) + \Omega \otimes K + K^{-1} \otimes \Omega - (q + q^{-1})K^{-1} \otimes K$$

and therefore  $\Delta(\Omega) \notin (U_q^{lf})^{\otimes 2}$ , since  $K \notin U_q^{lf}$  (see eg. Theorem 2.2 (2)).

Let us point out here two important consequences of (10). First,  $\Phi_n$  yields isomorphisms between centers,  $\mathcal{Z}(\mathcal{L}_{0,n}) \cong \mathcal{Z}(U_q)^{\otimes n}$  and  $\mathcal{Z}(\mathcal{L}_{0,n}^{U_q}) \cong \mathcal{Z}((U_q^{\otimes n})^{U_q})$ , where one can show that ([28], Lemma 6.25)

$$\mathcal{Z}((U_q^{\otimes n})^{U_q}) \cong \Delta^{(n-1)}(\mathcal{Z}(U_q)) \otimes_{\mathbb{C}(q)} \mathcal{Z}(U_q)^{\otimes n}.$$

Second, we see that  $\mathcal{L}_{0,n}$  (and therefore  $\mathcal{M}_{0,n}$ ) has no non-trivial zero divisors, by using the isomorphisms  $\Phi_n: \mathcal{L}_{0,n} \rightarrow (U_q^{\otimes n})^{lf} \subset U_q^{\otimes n}$  and  $U_q^{\otimes n} \cong U_q(\mathfrak{g}^{\oplus n})$ , and the fact that  $U_q(\mathfrak{g}^{\oplus n})$  has no non-trivial zero divisors (proved eg. in [38]).

**2.2. Integral forms and specializations.** An integral form of a (Hopf)  $\mathbb{C}(q)$ -algebra is a (Hopf)  $A$ -subalgebra, where  $A = \mathbb{C}[q, q^{-1}]$ , that becomes isomorphic to the algebra after tensoring it with  $\mathbb{C}(q)$ . We consider three integral forms related by the pairing  $\langle \cdot, \cdot \rangle$ , one of  $U_q$ , one of  $U_q^{ad}$ , and one of  $\mathcal{O}_q$ .

The *unrestricted integral form* of  $U_q$  is the  $A$ -subalgebra  $U_A = U_A(\mathfrak{g})$  introduced by De Concini–Kac–Procesi in [40], Section 12 (and in a differently normalized form in [38] and [39]). It is generated by the elements  $(i = 1, \dots, m)$

$$\bar{E}_i = (q_i - q_i^{-1})E_i, \quad \bar{F}_i = (q_i - q_i^{-1})F_i, \quad L_i, \quad L_i^{-1}.$$

Clearly, the subalgebra of locally finite elements of  $U_A$  is  $U_A^{lf} = U_A \cap U_q^{lf}$ . Similarly, we define the unrestricted integral form of  $U_q^{ad}$  as the  $A$ -subalgebra  $U_A^{ad} \subset U_q^{ad}$  generated by the elements  $\bar{E}_i, \bar{F}_i$  and  $K_i^{\pm 1}$ , for  $i = 1, \dots, m$ .

The *restricted integral form* of  $U_q^{ad}$  is the  $A$ -subalgebra  $\Gamma = \Gamma(\mathfrak{g})$  introduced by De Concini–Lyubashenko in [42], Sections 2-3. It is generated by the elements  $(i = 1, \dots, m)$

$$E_i^{(k)} = \frac{E_i^k}{[k]_{q_i}!}, \quad F_i^{(k)} = \frac{F_i^k}{[k]_{q_i}!}, \quad (K_i; t)_{q_i} = \prod_{s=1}^t \frac{K_i q_i^{-s+1} - 1}{q_i^s - 1}, \quad K_i^{-1}$$

where  $k \in \mathbb{N}$ ,  $t \in \mathbb{N}$  (setting  $(K_i; 0)_{q_i} = 1$  by convention).

Note that  $\Gamma$  contains the elements  $K_i$ , and the unrestricted integral form  $U_A^{ad}$ . It plays a fundamental rôle in relation with the integral pairings  $\pi_A^{\pm}$  considered in Section 2.3; it is by this rôle that  $\Gamma$  is more suited to our purposes than the more standard restricted integral form  $U_A^{res}$  defined by Lusztig, and discussed below.

The integral forms  $U_A(\mathfrak{h})$ ,  $U_A(\mathfrak{b}_{\pm})$  and  $\Gamma(\mathfrak{h})$ ,  $\Gamma(\mathfrak{b}_{\pm})$  associated to the subalgebras  $\mathfrak{h}$ ,  $\mathfrak{b}_{\pm} \subset \mathfrak{g}$  are the subalgebras of  $U_A$  and  $\Gamma$  defined in the obvious way. For instance the ‘‘Cartan’’ subalgebra  $\Gamma(\mathfrak{h})$  is generated by the elements  $(K_i; t)_{q_i}$  and  $K_i^{-1}$ .

Denote by  $\mathcal{C}_A$  the category of  $\Gamma$ -modules which are free  $A$ -modules of finite rank, and semisimple as  $\Gamma(\mathfrak{h})$ -modules; so they have a basis where  $K_i$  and  $(K_i; t)_{q_i}$  act diagonally with respective eigenvalues of the form

$$q_i^k, \quad \binom{k}{t}_{q_i} \quad k \in \mathbb{Z}, t \in \mathbb{N}^*.$$

The *integral quantum function Hopf algebra*  $\mathcal{O}_A = \mathcal{O}_A(G)$  is the restricted dual of  $\Gamma$ , ie. the set of  $A$ -linear maps  $f: \Gamma \rightarrow A$  such that  $\text{Ker}(f)$  contains a cofinite two sided ideal  $I$ , and  $\prod_{s=-r}^r (K_i - q_i^s) \in I$  for some  $r \in \mathbb{N}$  and every  $i$ .  $\mathcal{O}_A$  is an integral form of  $\mathcal{O}_q$ . The algebras  $\mathcal{O}_A(T_G)$ ,  $\mathcal{O}_A(U_{\pm})$ ,  $\mathcal{O}_A(B_{\pm})$  are defined similarly, by replacing  $\Gamma$  with  $\Gamma(\mathfrak{h})$ ,  $\Gamma(\mathfrak{n}_{\pm})$ ,  $\Gamma(\mathfrak{b}_{\pm})$  respectively.  $\mathcal{O}_A$  is generated as an algebra by the matrix coefficients  $x \mapsto v^i(\pi_V(x)v_i)$ ,  $x \in \Gamma$ , for every module  $V$  in  $\mathcal{C}_A$  where  $(v_i)$  is an  $A$ -basis of  $V$  and  $(v^i)$  the dual  $A$ -basis of the dual module  $V^*$ .

It is immediate that the  $U_q$ -module structure of  $\mathcal{O}_q$  restricts to an  $U_A$ -module structure on  $\mathcal{O}_A$ .

We note that  $\mathcal{O}_A$  is also the restricted dual of  $U_A^{res}$ , the Lusztig integral form of  $U_q^{ad}$  [60, 61], defined as  $\Gamma$  except that the  $(K_i; t)_{q_i}$  ( $i = 1, \dots, m$ ), are replaced by the elements

$$[K_i; t]_{q_i} = \prod_{s=1}^t \frac{K_i q_i^{-s+1} - K_i^{-1} q_i^{s-1}}{q_i^s - q_i^{-s}}.$$

Indeed,  $\Gamma(\mathfrak{h})$  contains  $U_A^{res}(\mathfrak{h})$  strictly, but the restriction functor  $\mathcal{C}_A \rightarrow \mathcal{C}_A^{res}$  is an equivalence of categories, where  $\mathcal{C}_A^{res}$  is the category of  $U_A^{res}$ -modules defined as  $\mathcal{C}_A$  above, but replacing

the condition on  $(K_i; t)_{q_i}$  by its analog for  $[K_i; t]_{q_i}$ , ie. that it acts diagonally with eigenvalues

$$\begin{bmatrix} k \\ t \end{bmatrix}_{q_i} \quad k \in \mathbb{Z}, t \in \mathbb{N}^*.$$

The *integral* form  $\mathcal{L}_{0,1}^A$  of  $\mathcal{L}_{0,1}$  is defined as the  $U_A$ -module  $\mathcal{O}_A$  endowed with the product of  $\mathcal{L}_{0,1}$ , and the *integral* form  $\mathcal{L}_{0,n}^A$  of  $\mathcal{L}_{0,n}$  is the braided tensor product of  $n$  copies of  $\mathcal{L}_{0,1}^A$ . That these two products are well-defined over  $A$  is elementary (see Definition 4.10 and 6.7 of [28] for the details). The *integral* quantum moduli algebra is

$$\mathcal{M}_{0,n}^A = (\mathcal{L}_{0,n}^A)^{U_A}.$$

Finally, given  $q = \epsilon' \in \mathbb{C}^\times$  we define  $U_{\epsilon'}, \Gamma_{\epsilon'}, \mathcal{O}_{\epsilon'}, \mathcal{L}_{0,n}^{\epsilon'}$  and  $\mathcal{M}_{0,n}^{A,\epsilon'}$  as the  $\mathbb{C}$ -algebras obtained by tensoring  $U_A, \Gamma, \mathcal{O}_A, \mathcal{L}_{0,n}^A$  and  $\mathcal{M}_{0,n}^A$  respectively with  $\mathbb{C}_{\epsilon'}$ , the  $A$ -module  $\mathbb{C}$  where  $q$  acts by multiplication by  $\epsilon'$ . They are the *specializations* of the latter algebras at  $q = \epsilon'$ ; they can also be defined as the quotients by the ideal generated by  $q - \epsilon'$ . We find convenient to use the notations

$$(12) \quad (U_A^{\otimes n})_{\epsilon'}^{U_A} := (U_A^{\otimes n})^{U_A} \otimes_A \mathbb{C}_{\epsilon'}, \quad (U^{\otimes n})_{\epsilon'}^{lf} := (U_A^{\otimes n})^{lf} \otimes_A \mathbb{C}_{\epsilon'}.$$

Let us stress here that when  $\epsilon'$  is a root of unity, taking the locally finite part and taking the specialization at  $\epsilon'$  are non commuting operations. Indeed, when  $\epsilon'$  has odd order, it follows from Theorem 2.14 below that  $U_{\epsilon'}$  is finite over  $\mathcal{Z}_0(U_{\epsilon'})$  and therefore has all its elements locally finite for  $ad^r$ ; on another hand  $U_A^{lf} \otimes_A \mathbb{C}_{\epsilon'}$ , ie.  $U_{\epsilon'}^{lf}$  in the notations above, does not contain the elements  $L_i$ .

In a similar manner, taking invariants and taking the specialization at  $\epsilon'$  are non commuting operations when  $\epsilon'$  is a root of unity: indeed, it is easily checked that in this case  $(U_A^{\otimes n})_{\epsilon'}^{U_A}$  and  $(U_{\epsilon'}^{\otimes n})^{U_{\epsilon'}}$ , or  $\mathcal{M}_{0,n}^{A,\epsilon'} = \mathcal{M}_{0,n}^A \otimes_A \mathbb{C}_{\epsilon'}$  and  $(\mathcal{L}_{0,n}^{\epsilon'})^{U_{\epsilon'}}$ , are distinct spaces. As explained in the introduction, when  $\epsilon'$  is a root of unity, we will not consider the algebras  $\mathcal{M}_{0,n}^{A,\epsilon'}$  in this paper.

The morphism  $\Phi_n$  has also an integral form. In order to define it, we first consider the relations between  $U_A$  and  $U_A^{lf}$ . Denote by  $T \subset U_A$  the multiplicative Abelian group formed by the elements  $K_\lambda, \lambda \in P$ , and by  $T_2 \subset T$  the subgroup formed by the  $K_\lambda, \lambda \in 2P$ . Consider the subset  $T_{2-} \subset T_2$  formed by the elements  $K_{-\lambda}, \lambda \in 2P_+$ . It is easily seen to be an Ore subset of  $U_A$ . Clearly  $T_2 = T_{2-}^{-1}T_{2-}$  and  $\text{Card}(T/T_2) = 2^m$ .

- Theorem 2.2.** (1)  $U_A^{lf} = \bigoplus_{\lambda \in 2P_+} ad^r(U_A)(K_{-\lambda})$ .  
(2)  $U_A = T_{2-}^{-1}U_A^{lf}[T/T_2]$ , so  $U_A$  is free of rank  $2^m$  over  $T_{2-}^{-1}U_A^{lf}$ .  
(3) The ring  $U_A^{lf}$  is (left and right) Noetherian.

*Proof.* These results are immediate adaptations to  $U_A^{lf}$  of those for  $U_q^{lf}$ , proved in Theorem 4.10 of [53], Theorem 6.4 of [52], and Theorem 7.4.8 of [51], respectively (see also the sections 7.1.6, 7.1.13 and 7.1.25 in [51]). For (1) and (3) we refer to Theorem 2.113 and 2.137 in [80], which provides simpler proofs.  $\square$

**Remark 2.3.** The summands in (1) are finite-dimensional  $U_A$ -modules (by eg. (14) below), so the action  $ad^r$  is completely reducible on  $U_A^{lf}$ . In fact,  $U_A^{lf}$  is the socle of  $ad^r$  on  $U_A$ , and by the theorem of separation of variables ([53, 51, 11], see also [80]),  $U_A^{lf}$  has an  $U_A$ -invariant subspace  $\mathbb{H}$  such that the multiplication in  $U_A$  affords an isomorphism of  $U_A$ -modules from  $\mathbb{H} \otimes_{\mathbb{C}(q)} \mathcal{Z}(U_A)$  onto  $U_A^{lf}$ . In particular,  $U_A^{lf}$  is free over  $\mathcal{Z}(U_A)$ . Moreover, any simple finite

dimensional  $U_A$ -module has in  $\mathbb{H}$  a multiplicity equal to the dimension of its zero-weight subspace.

Recall the RSD map  $\Phi_1: \mathcal{O}_q \rightarrow U_q^{lf}$ . By construction  $\langle \cdot, \cdot \rangle$  induces a perfect pairing  $\langle \cdot, \cdot \rangle: \mathcal{O}_A \otimes \mathbb{U}_\Gamma \rightarrow A$ . Let  $V_{-\lambda}$  be the lowest weight  $\Gamma$ -module of lowest weight  $-\lambda \in -P_+$  (ie. the highest weight  $\Gamma$ -module  $V_{-w_0(\lambda)}$  of highest weight  $-w_0(\lambda)$ , where  $w_0$  is the longest element of the Weyl group; note that  $-w_0$  permutes the simple roots). Let  $v \in V_{-\lambda}$  be a lowest weight vector, and  $v^* \in V_{-\lambda}^*$  be such that  $v^*(v) = 1$  and  $v^*$  vanishes on a  $\Gamma(\mathfrak{h})$ -invariant complement of  $v$ . Define  $\psi_{-\lambda} \in \mathcal{O}_A$  by  $\langle \psi_{-\lambda}, x \rangle = v^*(xv)$ ,  $x \in \Gamma$ . From the definition (4) it is quite easy to see that

$$(13) \quad \Phi_1(\psi_{-\lambda}) = K_{-2\lambda}.$$

**Corollary 2.4.**  $\Phi_1$  restricts on  $\mathcal{O}_A$  to an isomorphism of  $U_A$ -modules  $\Phi_1: \mathcal{O}_A \rightarrow U_A^{lf}$  and an isomorphism of  $U_A$ -module algebras  $\Phi_1: \mathcal{L}_{0,1}^A \rightarrow U_A^{lf}$ .

*Proof.* An elementary computational proof of this result in the  $sl(2)$  case is given in Section 5 of [28]. A proof of the general case can be found in Lemma 4.11 of [28]. It uses Theorem 2.2 (1). We point out an alternative proof in Remark 2.13 (1).  $\square$

**Corollary 2.5.** Let us denote  $d = \psi_{-\rho} \in \mathcal{L}_{0,1}^A$ . We have:

(1) The set  $\{d^n\}_{n \in \mathbb{N}}$  is a left and right multiplicative Ore set in  $\mathcal{L}_{0,1}^A$ . We can therefore define the localization  $\mathcal{L}_{0,1}^A[d^{-1}]$ .

(2)  $\Phi_1$  extends to an isomorphism of  $U_A$ -module algebras  $\Phi_1: \mathcal{L}_{0,1}^A[d^{-1}] \rightarrow T_{2-}^{-1}U_A^{lf}$ .

*Proof.* (1) Because  $\mathcal{L}_{0,1}^A$  has no non-trivial zero divisors,  $d$  is a regular element. It is enough to show that for all  $x \in \mathcal{L}_{0,1}^A$  there exists elements  $y, y' \in \mathcal{L}_{0,1}^A$  such that  $xd = dy$  and  $dx = y'd$ . But  $\Phi_1(x)\Phi_1(d) = \Phi_1(x)K_{-2\rho} = K_{-2\rho}ad^r(K_{2\rho})(\Phi_1(x))$ , and  $ad^r(K_{2\rho})(\Phi_1(x)) = \Phi_1(\text{coad}^r(K_{2\rho})(x))$ . Therefore the left Ore condition is satisfied with  $y = \text{coad}^r(K_{2\rho})(x)$ . Similarly one finds  $y'$ .

(2) Because  $\Phi_1(d) = K_{-2\rho} = \prod_{j=1}^m L_j^{-2}$ , localizing in  $d$  we obtain  $L_j^2 = \prod_{k \neq j} L_k^{-2} \Phi_1(d^{-1}) = \Phi_1(\prod_{k \neq j} \psi_{-\varpi_k} d^{-1}) \in \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$ . Therefore  $T_{2-}^{-1} \subset \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$ , which implies the assertion (2).  $\square$

**Remark 2.6.** When  $\mathfrak{g} = sl(2)$  the element  $d$  is the generator of  $\mathcal{L}_{0,1}(sl(2))$  appearing in (44) below. In this case we had already shown in [28] that  $\Phi_1: \mathcal{L}_{0,1}^A[d^{-1}] \rightarrow U_A^{ad} = T_{2-}^{-1}U_A^{lf}$  is an isomorphism of algebras.

Denote by  $C(\mu)$ ,  $\mu \in P^+$ , the linear subspace of  $\mathcal{L}_{0,1}$  generated by the matrix coefficients of  $V_\mu$ , the  $U_q$ -module of type 1 and highest weight  $\mu$ . The formula (13) can be used to prove (see Section 7.1.22 in [51], or page 112 of [80]) that  $\Phi_1$  yields the following linear isomorphism, which illuminates the claim (1) of Theorem 2.2:

$$(14) \quad \Phi_1: C(\mu) \rightarrow ad^r(U_q)(K_{-2w_0(\mu)}).$$

Working over the ground ring  $A$  one has to consider for  $V_\mu$  the highest weight  $\Gamma$ -module of highest weight  $\mu$ . In that situation  $\Phi_1$  affords an isomorphism from  $C(\mu)_A = \text{End}_A(V_\mu)^*$  to  $ad^r(U_A)(K_{-2w_0(\mu)})$ .

By (13) we have  $\Phi_1(\psi_{-\rho}) = \ell^{-1}$ , where as usual  $\ell$  is the pivotal element of  $U_A$ . Because the latter has the elementary factorization  $\ell = \prod_{j=1}^m L_j^2$ , this naturally raises the question of the factorization of  $\psi_{-\rho}$ . This question is considered in [54], where  $\mathcal{L}_{0,1}(\mathfrak{g})$  for  $\mathfrak{g} = gl(r+1)$  is analysed and quantum minors are extensively studied. Let us review here some of their results in relation with  $\psi_{-\rho}$ .



First note that for  $\mathfrak{g} = sl(r+1)$  the irreducible representation  $V_{-\rho}$  of lowest weight  $-\rho$  is isomorphic to the representation of highest weight  $V_{\rho}$  because  $-w_0(\rho) = \rho$ . By the Weyl formula the dimension of this representation is  $\prod_{\alpha>0} \frac{(2\rho, \alpha)}{(\rho, \alpha)} = 2^N$ . In [58] a presentation of  $U_q(gl(r+1))$  is given, which differs from our presentation of  $U_q(sl(r+1))$  only by its subalgebra  $U_q(\mathfrak{h})$ , generated by  $r+1$  elements  $\mathbb{K}_1, \dots, \mathbb{K}_{r+1}$ . The inclusion  $U_q(sl(r+1)) \subset U_q(gl(r+1))$  is such that  $K_i = \mathbb{K}_i^2 \mathbb{K}_{i+1}^{-2}, i = 1, \dots, r$ . The quantum minors, properly defined in [54], of the matrix of matrix elements of the natural representation of  $U_q(gl(r+1))$  are denoted  $det_q(A_{\geq k})$  for  $k = 1, \dots, r+1$ . We have  $det_q(A_{\geq 1}) = 1$  in the case of  $sl(r+1)$ . Then [54] proves that  $det_q(A_{\geq k}) = (\mathbb{K}_k \dots \mathbb{K}_{r+1})^2$ , and there exists an element  $\mathbb{K} \in U_q(gl(r+1))$  such that

$$\mathbb{K}^{-2\rho} = det_q(A_{\geq 1})^{-r} det_q(A_{\geq 2}) \dots det_q(A_{\geq r+1}).$$

This has to be interpreted in the  $sl(r+1)$  case as  $K_{-2\rho} = \Phi_1(det_q(A_{\geq 2}) \dots det_q(A_{\geq r+1}))$ . As a result this gives the equality

$$\psi_{-\rho} = det_q(A_{\geq 2}) \dots det_q(A_{\geq r+1}).$$

Corollary 2.4 can be extended as follows:

**Theorem 2.7.**  $\Phi_n$  restricts to an isomorphism of  $U_A$ -module algebras  $\Phi_n: \mathcal{L}_{0,n}^A \rightarrow (U_A^{\otimes n})^{lf}$ , and it restricts to an isomorphism of algebras  $\Phi_n: \mathcal{M}_{0,n}^A \rightarrow (U_A^{\otimes n})^{U_A}$ .

The proof relies on (10) and the expression of  $\Phi_n$  in terms of  $\Phi_1$  and  $R$ -matrices (see [28], Proposition 6.5 and Lemma 6.8).

In the case of  $\mathfrak{g} = sl(2)$  we proved in [30] the existence of elements  $\xi^{(i)} \in \mathcal{L}_{0,n}^A$  ( $i = 1, \dots, n$ ), and we defined an algebra  $loc \mathcal{L}_{0,n}^A$  generalizing  $\mathcal{L}_{0,1}^A[d^{-1}]$  above, containing  $\mathcal{L}_{0,n}^A$  as a subalgebra and the inverses of the elements  $\xi^{(i)}$ . We showed that  $\Phi_n$  extends to  $loc \mathcal{L}_{0,n}^A$ , and that  $\Phi_n(loc \mathcal{L}_{0,n}^A) = U_A^{ad}(sl(2))^{\otimes n}$ . The key property of  $\xi^{(i)}$  is

$$(15) \quad \Phi_n(\xi^{(i)}) = (K^{-1})^{(i)} \dots (K^{-1})^{(n)}.$$

For general  $\mathfrak{g}$  we now describe a partial generalization of this result. Define elements  $\xi_j^{(i)} \in \mathcal{L}_{0,n}^A$ , for  $i = 1, \dots, n$  and  $j = 1, \dots, m$ , by

$$(16) \quad \xi_j^{(i)} = v^*(M_j^{(i)} \dots M_j^{(n)})(v)$$

where  $M_j^{(i)} \in End(V_{-\varpi_j}) \otimes \mathcal{L}_{0,n}^A$  is the matrix of matrix coefficients  $1^{\otimes(i-1)} \otimes_{V_{-\varpi_j}} \phi_{e_k}^{e_l} \otimes 1^{\otimes(n-i)}$ , where  $\{e_k\}$  is the canonical basis of weight vectors of  $V_{-\varpi_j}$ ,  $v$  is a lowest non-zero weight vector of  $V_{-\varpi_j}$ , and  $v^*$  the associated linear form, vanishing on a  $\Gamma(\mathfrak{h})$ -invariant complement of  $v$ . Similarly to (15) the elements  $\xi_j^{(i)}$  satisfy

$$(17) \quad \Phi_n(\xi_j^{(i)}) = (L_j^{-2})^{(i)} \dots (L_j^{-2})^{(n)}.$$

The elements  $\xi_j^{(i)}$  commute, and the argument in Corollary 2.5 (1) shows that  $\{\xi_j^{(1)k}\}_{k \in \mathbb{N}}$  is an Ore subset of  $\mathcal{L}_{0,n}^A$ . For  $i \geq 2$  this argument implies only that  $\{\xi_j^{(i)k}\}_{k \in \mathbb{N}}$  is an Ore subset of the subalgebra of  $\mathcal{L}_{0,n}^A$  generated by the subalgebras  $\mathcal{L}_{0,n}^{(a)}$ ,  $a \geq i$ . Nevertheless, one can show it satisfies the left and right Ore conditions in all of  $\mathcal{L}_{0,n}^A$  by using the exchange relations (30) in the graded algebra  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n}^A)$  (see Section 3). For simplicity we omit the details, and sketch hereafter the idea behind the resulting construction of the localization of  $\mathcal{L}_{0,n}^A$  with respect to the elements  $\xi_j^{(i)}$ .

Let us explain the case  $n = 2$ . Since the elements  $\xi_j^{(1)}$ ,  $j \in \{1, \dots, m\}$ , are commuting regular Ore elements of  $\mathcal{L}_{0,2}^A$  we can define the localisation of  $\mathcal{L}_{0,2}^A$  with respect to the multiplicative sets  $\{\xi_1^{(1)k}\}_{k \in \mathbb{N}}, \dots, \{\xi_m^{(1)k}\}_{k \in \mathbb{N}}$ . Denote it  $\mathcal{L}_{0,2}^A[\{\xi_{j_1}^{(1)-1}\}]$ . Let us add new elements  $\nu_{j_1}^{(1)}$  such that  $(\nu_{j_1}^{(1)})^2 = \xi_{j_1}^{(1)}$  and  $\Phi_2(\nu_{j_1}^{(1)}) = (L_{j_1}^{-1})^{(1)}(L_{j_1}^{-1})^{(2)}$ . They are Ore elements, and we can define similarly the localisation  $\mathcal{L}_{0,2}^A[\{\nu_{j_1}^{(1)-1}\}]$  (see Remark 2.10 for an explanation of this additional construction). We want to define the inverses of the elements  $\xi_j^{(2)}$ ,  $j \in \{1, \dots, m\}$ , and a new algebra  $\mathcal{L}_{0,2}^A[\{\xi_{j_1}^{(1)-1}\}][\{\xi_{j_2}^{(2)-1}\}]$  such that  $\mathcal{L}_{0,2}^A[\{\xi_{j_1}^{(1)-1}\}] \subset \mathcal{L}_{0,2}^A[\{\xi_{j_1}^{(1)-1}\}][\{\xi_{j_2}^{(2)-1}\}]$  and  $\Phi_2$  extends naturally to an algebra homomorphism  $\Phi_2 : \mathcal{L}_{0,2}^A[\{\xi_{j_1}^{(1)-1}\}][\{\xi_{j_2}^{(2)-1}\}] \rightarrow U_A^{\otimes 2}$  such that  $\Phi_n(\xi_{j_2}^{(2)}) = (L_{j_2}^{-2})^{(2)}$  for all  $j_2 \in \{1, \dots, m\}$ . As in the  $sl(2)$  case described in [28], this can be done by writing explicitly, for every  $j_2 \in \{1, \dots, m\}$ , the exchange relations between the matrices  $M_{j_1}^{(1)}$  and  $M_{j_2}^{(2)}$  involving  $\xi_{j_2}^{(2)}$ , for every  $j_1 \in \{1, \dots, m\}$  (these matrices are defined in (16)). Similarly, by replacing the elements  $\xi_{j_1}^{(1)}$ ,  $\xi_{j_2}^{(2)}$  with square roots  $\nu_{j_1}^{(1)}$ ,  $\nu_{j_2}^{(2)}$  we get a localization  $\mathcal{L}_{0,2}^A[\{\nu_{j_1}^{(1)-1}\}][\{\nu_{j_2}^{(2)-1}\}]$  such that  $\mathcal{L}_{0,2}^A[\{\nu_{j_1}^{(1)-1}\}] \subset \mathcal{L}_{0,2}^A[\{\nu_{j_1}^{(1)-1}\}][\{\nu_{j_2}^{(2)-1}\}]$  and  $\Phi_2$  extends to an algebra homomorphism  $\Phi_2 : \mathcal{L}_{0,2}^A[\{\nu_{j_1}^{(1)-1}\}][\{\nu_{j_2}^{(2)-1}\}] \rightarrow U_A^{\otimes 2}$  such that  $\Phi_n(\nu_{j_2}^{(2)}) = (L_{j_2}^{-1})^{(2)}$  for all  $j_2 \in \{1, \dots, m\}$ . This morphism of algebras will be shown to be an isomorphism.

For any  $n \geq 2$  we can proceed in the same way:

**Definition 2.8.** *By iterating the above construction we define:*

$$\begin{aligned} \text{loc} \mathcal{L}_{0,n}^A &= \mathcal{L}_{0,n}^A[\{\xi_{j_n}^{(n)-1}\}][\{\xi_{j_{n-1}}^{(n-1)-1}\}] \cdots [\{\xi_{j_1}^{(1)-1}\}], \\ \text{loc}' \mathcal{L}_{0,n}^A &= \mathcal{L}_{0,n}^A[\{\nu_{j_n}^{(n)-1}\}][\{\nu_{j_{n-1}}^{(n-1)-1}\}] \cdots [\{\nu_{j_1}^{(1)-1}\}]. \end{aligned}$$

In the sequel it will be convenient to define invertible elements  $\sqrt{\delta_j^{(i)}} \in \text{loc}' \mathcal{L}_{0,n}^A$ , for  $i = 1, \dots, n$  and  $j = 1, \dots, m$ , satisfying  $\nu_j^{(i)} = \sqrt{\delta_j^{(i)}} \cdots \sqrt{\delta_j^{(n)}}$ , i.e  $\sqrt{\delta_j^{(i)}} = \nu_j^{(i)} / \nu_j^{(i+1)}$ .

The elements  $\sqrt{\delta_j^{(i)}}$  are invertible, commute and satisfy

$$\Phi_n(\sqrt{\delta_j^{(i)}}) = (L_j^{-1})^{(i)}.$$

**Theorem 2.9.**  $\Phi_n$  restricts to an isomorphism of  $U_A$ -module algebras  $\Phi_n : \text{loc}' \mathcal{L}_{0,n}^A \rightarrow U_A^{\otimes n}$ .

*Proof.* We know from Corollary 2.4 that  $\Phi_1 : \mathcal{L}_{0,n}^A \rightarrow U_A^{lf}$  is an isomorphism of algebra. Using  $U_A = T_2^{-1} U_A^{lf} [T/T_2]$  and the fact that the image by  $\Phi_1$  of the elements  $(\sqrt{\delta_j^{(1)}})^{\pm 1}$  generates the group  $T$  we get the result for  $n = 1$ . The result for  $\Phi_n$  is obtained by induction. We have

$$\begin{aligned} (id \otimes \Phi_n)(M^{(n)}) &= R_{0n} R'_{0n} \\ (id \otimes \Phi_n)(M^{(a)}) &= (R_{0n} \cdots R_{0a+1}) R_{0a} R'_{0a} (R_{0n} \cdots R_{0a+1})^{-1}, 1 \leq a < n. \end{aligned}$$

Because the matrix elements of  $(id \otimes \Phi_n)(M^{(n)})$  generate  $1^{\otimes n-1} \otimes U_A^{lf}$  when  $V$  varies, the image of  $(\mathcal{L}_{0,n}^A)^{(n)}[\{\nu_{j_n}^{(n)-1}\}]$  by  $\Phi_n$  is  $1^{\otimes(n-1)} \otimes U_A$ . Since the matrix elements of  $R_{0n}$  and  $R_{0n}^{-1}$  are in  $1^{\otimes(n-1)} \otimes U_A$ , they belong to  $\Phi_n(\mathcal{L}_{0,n}^A[\{\nu_{j_n}^{(n)-1}\}])$  by the preceding remark. It

follows that  $\Phi_n(\text{loc}'\mathcal{L}_{0,n}^A)$  contains the matrix elements of  $R_{0n}^{-1}(\text{id} \otimes \Phi_n)(M^{(n-1)})R_{0n}$ , whence the matrix elements of  $R_{0n-1}R'_{0n-1}$ , and therefore the space  $1^{\otimes(n-2)} \otimes U_A^{lf} \otimes 1$ . It contains also the elements  $\Phi_n(\sqrt{\delta_j^{(n-1)}}) = (L_j^{-1})^{(n-1)}$ , so  $\Phi_n(\text{loc}'\mathcal{L}_{0,n}^A)$  contains  $1^{\otimes(n-2)} \otimes U_A \otimes 1$ . By a trivial induction we finally obtain that  $\Phi_n(\text{loc}'\mathcal{L}_{0,n}^A) = U_A^{\otimes n}$ .  $\square$

**Remark 2.10.** It is a natural problem to determine the image by  $\Phi_n$  of  $\text{loc}'\mathcal{L}_{0,n}^A$ , and it is natural to expect that it would be  $(T_2^{-1}U_A^{lf})^{\otimes n}$ , because this is true for  $n = 1$ , as well as for any  $n$  in the  $sl(2)$  case, as shown in [28]. Unfortunately this is not so. This comes from the fact, eg. for  $n = 2$ , that the matrix elements of  $R_{02}R_{01}R'_{01}R_{02}^{-1}$  do not belong to  $(T_2^{-1}U_A^{lf})^{\otimes 2}$  as can be shown by an explicit computation in the  $sl(3)$  case. This explains the reason why we had to introduce the square roots  $\nu_j^{(i)}$  in the previous theorem.

Arguments similar to those mentioned at the end of Section 2.1 imply that the algebras  $\mathcal{L}_{0,n}^A$ ,  $\mathcal{M}_{0,n}^A$  and  $\mathcal{L}_{0,n}^{\epsilon'}$ ,  $\mathcal{M}_{0,n}^{A,\epsilon'}$ ,  $\epsilon' \in \mathbb{C}^\times$ , have no non-trivial zero divisors (see [28], Proposition 7.1). By Theorem 2.7 the Alekseev map yields isomorphisms of  $U_{\epsilon'}$ -module algebras, and of algebras for the latter,

$$(18) \quad \Phi_n: \mathcal{L}_{0,n}^{\epsilon'} \rightarrow (U^{\otimes n})_{\epsilon'}^{lf}, \quad \Phi_n: \text{loc}'\mathcal{L}_{0,n}^{\epsilon'} \rightarrow U_{\epsilon'}^{\otimes n}, \quad \Phi_n: \mathcal{M}_{0,n}^{A,\epsilon'} \rightarrow (U_A^{\otimes n})_{\epsilon'}^{U_A} \subset (U_{\epsilon'}^{\otimes n})^{U_{\epsilon'}}$$

where we use the notations (12).

**2.3. Perfect pairings.** We will need restrictions on the integral forms  $\mathcal{O}_A(B_+)$ ,  $\mathcal{O}_A(B_-)$  of the morphisms  $\Phi^+$ ,  $\Phi^-$  in (2). We collect their properties in Theorem 2.11 and the discussion thereafter. In order to state it, we recall first a few facts about  $R$ -matrices and related pairings.

In [60, 61] Lusztig proved that the category of  $U_A^{res}$ -modules  $\mathcal{C}_A^{res} \otimes_A \mathbb{C}[q^{\pm 1/D}]$  (ie. with coefficients extended to  $\mathbb{C}[q^{\pm 1/D}]$ ) is braided and ribbon, with braiding given by the collection of endomorphisms

$$R_A = ((R_h)_{V,W})_{V,W \in \text{Ob}(\mathcal{C}_A^{res})}.$$

Actually,  $(R_h)_{V,W}$  is represented by a matrix with coefficients in  $q^{\pm 1/D}\mathbb{Z}[q^{\pm 1}]$  on the basis of  $V \otimes W$  formed by the tensor products of the canonical (Kashiwara-Lusztig) basis vectors of  $V$  and  $W$ . The restriction functor  $\mathcal{C}_A \rightarrow \mathcal{C}_A^{res}$  is an equivalence of categories, so  $\mathcal{C}_A \otimes_A \mathbb{C}[q^{\pm 1/D}]$  has the same braided and ribbon structure. This can be rephrased as follows in Hopf algebra terms. Denote by  $\mathbb{U}_\Gamma$  the categorical completion of  $\Gamma$ , ie. the Hopf algebra of natural transformations  $F_{\mathcal{C}_A} \rightarrow F_{\mathcal{C}_A}$ . Then  $\mathbb{U}_\Gamma \otimes_A \mathbb{C}[q^{\pm 1/D}]$  is quasi-triangular and ribbon with R-matrix

$$R_A \in \mathbb{U}_\Gamma^{\hat{\otimes} 2} \otimes_A \mathbb{C}[q^{\pm 1/D}].$$

As in (1), we can write

$$R_A^\pm = \sum_{(R)} R_{(1)}^\pm \otimes R_{(2)}^\pm.$$

There are pairings of Hopf algebras naturally related to the R-matrix  $R \in \mathbb{U}_q^{\hat{\otimes} 2}$ . What follows is standard (see eg. [55, 56, 59]), for details we refer to the results 2.73, 2.75, 2.92, 2.106 and 2.107 in [80]:

- There is a unique pairing of Hopf algebras  $\rho: U_q(\mathfrak{b}_-)^{cop} \otimes U_q(\mathfrak{b}_+) \rightarrow \mathbb{C}(q^{1/D})$  such that, for every  $\alpha, \lambda \in P$  and  $l, k \in U_q(\mathfrak{h})$ ,

$$\rho(K_\lambda, K_\alpha) = q^{(\lambda, \alpha)}, \quad \rho(F_i, E_j) = \delta_{i,j}(q_i - q_i^{-1})^{-1}, \quad \rho(l, E_j) = \rho(F_i, k) = 0.$$

- The *Drinfeld pairing*  $\tau: U_q(\mathfrak{b}_+)^{cop} \otimes U_q(\mathfrak{b}_-) \rightarrow \mathbb{C}(q^{1/D})$  is the bilinear map defined by  $\tau(X, Y) = \rho(S(Y), X)$ ; it satisfies

$$\tau(K_\lambda, K_\alpha) = q^{-(\lambda, \alpha)}, \quad \tau(E_j, F_i) = -\delta_{i,j}(q_i - q_i^{-1})^{-1}, \quad \tau(l, F_i) = \tau(E_j, k) = 0.$$

- $\rho$  and  $\tau$  are perfect pairings; this means that they yield *isomorphisms* of Hopf algebras  $i_\pm: U_q(\mathfrak{b}_\pm) \rightarrow \mathcal{O}_q(B_\mp)_{op}$  (with coefficients *a priori* extended to  $\mathbb{C}(q^{1/D})$ , but see below) defined by, for every  $X \in U_q(\mathfrak{b}_+)$ ,  $Y \in U_q(\mathfrak{b}_-)$ ,

$$\langle i_+(X), Y \rangle = \tau(S(X), Y), \quad \langle i_-(Y), X \rangle = \tau(X, Y).$$

Since  $\mathcal{O}_q(B_\mp)_{op}$  is equipped with the *inverse* of the antipode  $S_{\mathcal{O}_q}$  of  $\mathcal{O}_q(B_\mp)$ , it follows that  $i_\pm \circ S = S_{\mathcal{O}_q}^{-1} \circ i_\pm$ .

- Denote by  $p_\pm: \mathcal{O}_q(G) \rightarrow \mathcal{O}_q(B_\pm)$  the canonical projection map, ie. the Hopf algebra homomorphism dual to the inclusion map  $U_q(\mathfrak{b}_\pm) \hookrightarrow U_q(\mathfrak{g})$ . For every  $\alpha, \beta \in \mathcal{O}_q(G)$  we have

$$(19) \quad \langle \alpha \otimes \beta, R \rangle = \tau(i_+^{-1}(p_-(\beta)), i_-^{-1}(p_+(\alpha))).$$

Note that it is the use of weights  $\alpha, \lambda \in P$  that forces the pairings  $\rho, \tau$  to be defined over  $\mathbb{C}(q^{1/D})$ , instead of  $\mathbb{C}(q)$ . Then, let us consider the restrictions  $\pi_q^+$  of  $\rho$ , and  $\pi_q^-$  of  $\tau$ , obtained by taking  $\alpha \in Q$  and  $l \in U_q(\mathfrak{h})$ ,  $k \in U_q^{ad}(\mathfrak{h})$ . They take values in  $\mathbb{C}(q)$ , and define pairings

$$\pi_q^+: U_q(\mathfrak{b}_-)^{cop} \otimes U_q^{ad}(\mathfrak{b}_+) \rightarrow \mathbb{C}(q), \quad \pi_q^-: U_q(\mathfrak{b}_+)^{cop} \otimes U_q^{ad}(\mathfrak{b}_-) \rightarrow \mathbb{C}(q).$$

By the same arguments as for  $\rho$  and  $\tau$  (eg. in [80], Proposition 2.92), it follows that  $\pi_q^\pm$  are perfect pairings. Note also that  $\pi_q^- = \kappa \circ \pi_q^+ \circ (\kappa \otimes \kappa)$ , where  $\kappa$  is the conjugate-linear automorphism of  $U_q$ , viewed as a Hopf algebra over  $\mathbb{C}(q)$  with conjugation given by  $\kappa(q) = q^{-1}$ , defined by

$$(20) \quad \kappa(E_i) = F_i, \quad \kappa(F_i) = E_i, \quad \kappa(K_\lambda) = K_{-\lambda}, \quad \kappa(q) = q^{-1}.$$

In [42], De Concini-Lyubashenko described integral forms of  $\pi_q^\pm$  as follows. Denote by  $m^*: \mathcal{O}_A \rightarrow \mathcal{O}_A(B_+) \otimes \mathcal{O}_A(B_-)$  the map dual to the multiplication map  $\Gamma(\mathfrak{b}_+) \otimes \Gamma(\mathfrak{b}_-) \rightarrow \Gamma$ , so  $m^* = (p_+ \otimes p_-) \circ \Delta_{\mathcal{O}_A}$ . Let  $U_A(H)$  be the sub-Hopf algebra of  $U_A(\mathfrak{b}_-)^{cop} \otimes U_A(\mathfrak{b}_+)^{cop}$  generated by the elements  $(i \in \{1, \dots, m\})$

$$1 \otimes K_i^{-1} \bar{E}_i, \quad \bar{F}_i K_i \otimes 1, \quad L_i^{\pm 1} \otimes L_i^{\mp 1}.$$

Note that  $U_A(H)$  is free over  $A$ , and that a basis is given by the elements

$$\bar{F}_{\beta_1}^{n_1} \dots \bar{F}_{\beta_N}^{n_N} K_{n_1 \beta_1 + \dots + n_N \beta_N} K_\lambda \otimes K_{-\lambda} K_{-p_1 \beta_1 \dots - p_N \beta_N} \bar{E}_{\beta_1}^{p_1} \dots \bar{E}_{\beta_N}^{p_N}$$

where  $\lambda \in P$  and  $n_1, \dots, n_N, p_1, \dots, p_N \in \mathbb{N}$ .

Recall the lowest weight  $\Gamma$ -module  $V_{-\lambda}$ ,  $\lambda \in P_+$ , the lowest weight vector  $v \in V_{-\lambda}$ , the dual vector  $v^* \in V_{-\lambda}^*$ , and  $\psi_{-\lambda} \in \mathcal{O}_A$  (see before Corollary 2.4). For every positive root  $\alpha$  define elements  $\psi_{-\lambda}^\alpha, \psi_{-\lambda}^{-\alpha} \in \mathcal{O}_A$  by the formulas (where  $x \in \Gamma$ , and we note that the root vectors  $E_\alpha, F_\alpha \in \Gamma$ ):

$$\langle \psi_{-\lambda}^\alpha, x \rangle = v^*(xE_\alpha v), \quad \langle \psi_{-\lambda}^{-\alpha}, x \rangle = v^*(F_\alpha x v).$$

Consider the maps  $j_q^\pm: \mathcal{O}_q(B_\pm) \rightarrow U_q(\mathfrak{b}_\mp)^{cop}$  defined by

$$\langle \alpha_+, X \rangle = \pi_q^+(j_q^+(\alpha_+), X), \quad \langle \alpha_-, Y \rangle = \pi_q^-(j_q^-(\alpha_-), Y)$$

where  $\alpha_\pm \in \mathcal{O}_q(B_\pm)$ ,  $X \in U_q^{ad}(\mathfrak{b}_+)$ ,  $Y \in U_q^{ad}(\mathfrak{b}_-)$ .

The following theorem summarizes results proved in the sections 3 and 4 of [42]. For the sake of clarity, let us spell out the correspondence between statements. First,  $\pi_q^+, \pi_q^-$ ,

$U_q(\mathfrak{b}_\mp)^{cop}$ ,  $U_A(\mathfrak{b}_\mp)^{cop}$ ,  $\mathcal{O}_A(B_\pm)$ ,  $U_A(H)$  and  $J$  are denoted in [42] respectively by  $\pi''$ ,  $\bar{\pi}''$ ,  $U_q(\mathfrak{b}_\mp)_{op}$ ,  $R_q[B_\pm]''$ ,  $R_q[B_\pm]$ ,  $A''$  and  $\mu''$ . Also, the definition of  $j_A^\pm$  is implicit in the section 4.2 of [42], and the formulas in Theorem 2.11 (3) are related to those in Lemma 4.5 of [42] by observing that their generators  $\tilde{E}_i$  and  $\tilde{F}_i$  are respectively  $K_i^{-1}E_i$  and  $F_iK_i$  in our notations; this also explains the appearance of  $q_i, q_i^{-1}$  in the formulas in (3). Finally,  $\kappa$  in (20) maps  $\tilde{E}_i, \tilde{F}_i$  to  $-\tilde{F}_i, -\tilde{E}_i$ , whence the sign for the expression of  $J(\psi_{-\varpi_j}^{\alpha_i})$ .

**Theorem 2.11.** (1)  $\pi_q^\pm$  restricts to a perfect Hopf pairing between the unrestricted and restricted integral forms,  $\pi_A^\pm: U_A(\mathfrak{b}_\mp)^{cop} \otimes \Gamma(\mathfrak{b}_\pm) \rightarrow A$ .

(2)  $j_q^\pm$  yields an isomorphism of Hopf algebras  $j_A^\pm: \mathcal{O}_A(B_\pm) \rightarrow U_A(\mathfrak{b}_\mp)^{cop}$ , satisfying  $\langle \alpha_\pm, x_\pm \rangle = \pi_A^\pm(j_A^\pm(\alpha_\pm), x_\pm)$  for every  $\alpha_\pm \in \mathcal{O}_A(B_\pm)$ ,  $x_\pm \in \Gamma(\mathfrak{b}_\pm)$ .

(3) The map  $J = (j_A^+ \otimes j_A^-) \circ m^*: \mathcal{O}_A \rightarrow U_A(H) \subset U_A(\mathfrak{b}_-)^{cop} \otimes U_A(\mathfrak{b}_+)^{cop}$  is an embedding of Hopf algebras, and it extends to an isomorphism  $J: \mathcal{O}_A[\psi_{-\rho}^{-1}] \rightarrow U_A(H)$ . In particular it satisfies (where  $\lambda \in P_+$ ):

$$J(\psi_{-\lambda}) = K_{-\lambda} \otimes K_\lambda, \quad J(\psi_{-\varpi_j}^{\alpha_i}) = -\delta_{i,j} q_i L_i^{-1} \otimes L_i K_i^{-1} \bar{E}_i, \quad J(\psi_{-\varpi_j}^{-\alpha_i}) = \delta_{i,j} q_i^{-1} \bar{F}_i K_i L_i^{-1} \otimes L_i.$$

For our purposes it is necessary to reformulate this result. Consider the morphisms of Hopf algebras  $\Phi^\pm: \mathcal{O}_A(B_\pm) \rightarrow U_A(\mathfrak{b}_\mp)^{cop}$ ,  $\alpha \mapsto (\alpha \otimes id)(R_A^\pm)$ .

**Lemma 2.12.** We have  $\Phi^\pm = j_A^\pm$ .

Thus, the theorem above tells us that  $\Phi^\pm$  is an isomorphism of Hopf algebras, such that  $\langle \alpha_\pm, x_\pm \rangle = \pi_A^\pm(\Phi^\pm(\alpha_\pm), x_\pm)$  for every  $\alpha_\pm \in \mathcal{O}_A(B_\pm)$ ,  $x_\pm \in \Gamma(\mathfrak{b}_\pm)$ . Moreover, changing the notation  $J$  for  $\Phi$ ,

$$(21) \quad \Phi := (\Phi^+ \otimes \Phi^-) \circ m^*: \mathcal{O}_A \rightarrow U_A(H) \subset U_A(\mathfrak{b}_-)^{cop} \otimes U_A(\mathfrak{b}_+)^{cop}$$

is an embedding of Hopf algebras, and it extends to an isomorphism  $\Phi: \mathcal{O}_A[\psi_{-\rho}^{-1}] \rightarrow U_A(H)$  which in particular satisfies:

$$(22) \quad \Phi_1(\psi_{-\lambda}) = K_{-2\lambda}, \quad \Phi_1(\psi_{-\varpi_j}^{\alpha_i}) = \delta_{i,j} L_i^{-2} \bar{E}_i, \quad \Phi_1(\psi_{-\varpi_j}^{-\alpha_i}) = \delta_{i,j} q_i^{-1} \bar{F}_i K_i L_i^{-2}.$$

*Proof of Lemma 2.12.* By definitions, for every  $X \in U_q(\mathfrak{b}_+)^{cop}$ ,  $Y \in U_q^{ad}(\mathfrak{b}_-)$  we have  $\langle i_+(S^{-1}(X)), Y \rangle = \pi_q^-(X, Y)$ , and similarly for every  $X \in U_q^{ad}(\mathfrak{b}_+)$ ,  $Y \in U_q(\mathfrak{b}_-)^{cop}$  we have  $\langle i_-(S^{-1}(Y)), X \rangle = \pi_q^+(Y, X)$ . By keeping these respective notations for  $X$  and  $Y$ , we deduce  $j_q^-(i_+(S^{-1}(X))) = X$  and  $j_q^+(i_-(S^{-1}(Y))) = Y$ , ie.

$$(23) \quad j_q^\pm = S \circ i_\mp^{-1}.$$

Because  $S_{\mathcal{O}_q}^{-1} \circ i_\pm = i_\pm \circ S$ , it follows that

$$(24) \quad j_q^\pm \circ S_{\mathcal{O}_q} = S^{-1} \circ j_q^\pm.$$

Also, for every  $\alpha_- \in \mathcal{O}_q(B_-)$  we have

$$\langle \alpha_-, \Phi^+(i_-(Y)) \rangle = \langle i_-(Y) \otimes \alpha_-, R \rangle = \tau(i_+^{-1}(\alpha_-), Y) = \pi_q^-(j_q^-(S_{\mathcal{O}_q}(\alpha_-)), Y) = \langle \alpha_-, S(Y) \rangle$$

where the first equality is by definition of  $\Phi^+$  (see (2)), the second is (19), the third follows from (24), and the last from the definition of  $j_q^-$ . Similarly, for every  $\alpha_+ \in \mathcal{O}_q(B_+)$  we have

$$\begin{aligned} \langle \alpha_+, \Phi^-(i_+(X)) \rangle &= \langle i_+(X) \otimes \alpha_+, R^- \rangle \\ &= \langle \alpha_+ \otimes S_{\mathcal{O}_q}^{-1} \circ i_+(X), R \rangle \\ &= \langle \alpha_+ \otimes i_+(S(X)), R \rangle \\ &= \tau(S(X), i_-^{-1}(\alpha_+)) \\ &= \pi_q^+(S(i_-^{-1}(\alpha_+)), S(X)) = \pi_q^+(j_q^+(\alpha_+), S(X)) = \langle \alpha_+, S(X) \rangle. \end{aligned}$$

These computations imply  $\Phi^\pm = S \circ i_\mp^{-1} = j_q^\pm$ , and the result follows by taking integral forms.  $\square$

**Remark 2.13.** (1) Since  $\Phi_1 = m \circ (id \otimes S^{-1}) \circ \Phi$  and  $\text{Im}(\Phi) \subset U_A(\mathfrak{b}_-)^{cop} \otimes U_A(\mathfrak{b}_+)^{cop}$ ,  $\Phi_1(\mathcal{O}_A) \subset U_A$ . Because  $\Phi_1(\mathcal{O}_q) = U_q^{lf}$ , we have also  $\Phi_1(\mathcal{O}_A) \subset U_A^{lf}$ . The converse inclusion  $\Phi_1(\mathcal{O}_A) \supset U_A^{lf}$  holds true as well, since  $\Phi_1(\mathcal{O}_q) = U_q^{lf}$  and  $\mathcal{O}_A$  is an  $A$ -lattice of  $\mathcal{O}_q$ .

(2) The components of  $R_A^\pm$  may be described explicitly: if  $\{\xi_i\}_i$  is a basis of  $\Gamma(\mathfrak{b}_+)$  (say, as obtained in section 3 of [42]), one can determine the dual basis  $\{\xi_i^*\}_i$  of  $U_A(\mathfrak{b}_-)$  by using the perfect pairing  $\pi_A^+$ ; then  $R_A^+ = \sum_i \xi_i \otimes \xi_i^*$ . Note that, like  $U_A^{ad}$  is contained in  $\Gamma$ ,  $U_A$  is contained in the restricted integral form of  $U_q$ , whose categorical completion is  $\mathbb{U}_\Gamma \otimes \mathbb{C}[q^{\pm 1/D}]$ . Therefore the components  $\xi_i^*$  of  $R_A^+$  can be viewed as elements of  $\mathbb{U}_\Gamma \otimes \mathbb{C}[q^{\pm 1/D}]$ . This is compatible with the fact that  $R_A^+$  is an element of  $\mathbb{U}_\Gamma^{\otimes 2} \otimes \mathbb{C}[q^{\pm 1/D}]$ .

(3) The dualities of Theorem 2.11 (2) afford a refinement defined over  $A$  of the quantum Killing form  $\kappa: U_q \otimes_{\mathbb{C}(q)} U_q \rightarrow \mathbb{C}(q^{1/D})$  (studied eg. in [80], Section 2.8). This form is the duality realizing the isomorphism  $ad^r(U_A)(K_{-2w_0(\mu)}) \cong \text{End}_A({}_A V_\mu)^*$  stated after (14).

**2.4. Structure theorems for  $U_\epsilon$  and  $\mathcal{O}_\epsilon$ .** As usual we denote by  $\epsilon$  a primitive  $l$ -th root of unity, where  $l$  is odd, and coprime to 3 if  $\mathfrak{g}$  has  $G_2$ -components.

Let  $G^0 = B_+ B_-$  (the *big cell* of  $G$ ), and define the group

$$H = \{(u_+ t, u_- t^{-1}), t \in T_G, u_\pm \in U_\pm\}.$$

Consider the map

$$\begin{aligned} \sigma: B_+ \times B_- &\longrightarrow G^0 \\ (b_+, b_-) &\longmapsto b_+ b_-^{-1}. \end{aligned}$$

The restriction of  $\sigma$  to  $H$  is an unramified covering of degree  $2^m$ . It can be seen as the classical analog of the map  $m \circ (id \otimes S^{-1}): \mathcal{O}_\epsilon(B_+) \otimes \mathcal{O}_\epsilon(B_-) \rightarrow \mathcal{O}_\epsilon(G)$ .

Denote by  $\mathcal{Z}_1(U_\epsilon)$  the image of  $\mathcal{Z}(U_q)$  in  $\mathcal{Z}(U_\epsilon)$  under the specialization map  $U_q \rightarrow U_\epsilon$ , and by  $\mathcal{Z}_0(U_\epsilon) \subset U_\epsilon$  the subalgebra generated by  $E_{\beta_k}^l, F_{\beta_k}^l, L_i^{\pm l}$ , for  $k \in \{1, \dots, N\}$  and  $i \in \{1, \dots, m\}$ . In [38], Section 1.8-3.3-3.8, and [40], Theorem 14.1 and Section 20-21, the following results are proved:

**Theorem 2.14.** (1)  $U_\epsilon$  has no non-trivial zero divisors,  $\mathcal{Z}_0(U_\epsilon)$  is a central Hopf subalgebra of  $U_\epsilon$ , and  $U_\epsilon$  is a free  $\mathcal{Z}_0(U_\epsilon)$ -module of rank  $l^{\dim \mathfrak{g}}$ . Moreover  $U_\epsilon$  is a maximal order of its classical fraction algebra  $Q(U_\epsilon) = Q(\mathcal{Z}(U_\epsilon)) \otimes_{\mathcal{Z}(U_\epsilon)} U_\epsilon$ , and  $Q(U_\epsilon)$  is a central simple algebra of PI degree  $l^N$ .

(2)  $\text{Maxspec}(\mathcal{Z}_0(U_\epsilon))$  is a group isomorphic to  $H$  above, and the multiplication map yields an isomorphism  $\mathcal{Z}_0(U_\epsilon) \otimes_{\mathcal{Z}_0 \cap \mathcal{Z}_1} \mathcal{Z}_1(U_\epsilon) \rightarrow \mathcal{Z}(U_\epsilon)$ .

It follows from (1) and  $\dim \mathfrak{g} = m + 2N$  that the field  $Q(\mathcal{Z}(U_\epsilon))$  is an extension of  $Q(\mathcal{Z}_0(U_\epsilon))$  of degree  $l^m$ . Conversely, this degree and the rank of  $U_\epsilon$  over  $\mathcal{Z}_0(U_\epsilon)$  imply that  $Q(U_\epsilon)$  has PI degree  $l^N$ .

As for (2), note that  $\mathcal{Z}_0(U_\epsilon)$  being an affine and commutative algebra,  $\text{Maxspec}(\mathcal{Z}_0(U_\epsilon))$ , viewed as the set of characters of  $\mathcal{Z}_0(U_\epsilon)$ , acquires by duality a structure of affine algebraic group. Thus, the first claim means precisely the identification of this group with  $H$ .

In addition to (2),  $\text{Maxspec}(\mathcal{Z}_0(U_\epsilon))$  and  $H$  have natural Poisson structures, that the isomorphism identifies. Moreover we have the following identifications (see [40], Section 21.2). Consider the  $l^m$ -fold covering  $\tilde{T}_G \rightarrow T_G$ . Recall that  $T$  is the group formed by the elements  $K_\lambda \in U_A$ ,  $\lambda \in P$ . We can identify  $T$  with the additive group  $P$ ,  $U_A(\mathfrak{h}) = \mathbb{C}[T] = \mathbb{C}[P]$  with  $\mathcal{O}(\tilde{T}_G)$ , and therefore  $\mathcal{Z}_0(U_\epsilon) \cap U_\epsilon(\mathfrak{h}) = \mathbb{C}[lP]$  with  $\mathcal{O}(T_G)$ . The quantum Harish-Chandra isomorphism then identifies  $\mathcal{Z}_1(U_\epsilon)$  with  $\mathbb{C}[2P]^W \cong \mathcal{O}(\tilde{T}_G/(2))^W$ , where we denote by (2) the subgroup of 2-torsion elements in  $\tilde{T}_G$ . Composing  $\sigma: H \rightarrow G^0$  with the quotient map under conjugation,  $G^0 \hookrightarrow G \rightarrow G//G$ , we get dually an embedding of  $\mathcal{O}(G//G) = \mathcal{O}(G)^G$  in  $\mathcal{O}(H)$ . The isomorphism of Theorem 2.14 (2) then affords identifications

$$\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon) \cong \mathcal{O}(G)^G$$

as a subalgebra of  $\mathcal{Z}_0(U_\epsilon) \cong \mathcal{O}(H)$ , and

$$\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon) = \mathbb{C}[2lP]^W \cong \mathcal{O}(\tilde{T}_G/(2l))^W \cong \mathcal{O}(T_G/(2))^W$$

as a subalgebra of  $\mathcal{Z}_1(U_\epsilon) \cong \mathcal{O}(\tilde{T}_G/(2))^W$ .

A result similar to Theorem 2.14 holds true for  $\mathcal{O}_\epsilon$ . Namely, take the specializations at  $q = \epsilon$  in Theorem 2.11. Denote by  $\mathcal{Z}_0(U_\epsilon(H))$  the subalgebra of  $U_\epsilon(H)$  generated by the elements  $(k \in \{1, \dots, N\}, i \in \{1, \dots, m\})$

$$1 \otimes K_{-l\beta_k} E_{\beta_k}^l, F_{\beta_k}^l K_{l\beta_k} \otimes 1, L_i^{\pm l} \otimes L_i^{\mp l}.$$

It is a central Hopf subalgebra. Recall that  $\mathcal{O}(G)$  can be realized as a Hopf subalgebra of  $U(\mathfrak{g})^\circ$ , the restricted dual of the envelopping algebra  $U(\mathfrak{g})$  over  $\mathbb{C}$ . In [42] De Concini-Lyubashenko introduced an epimorphism of Hopf algebras  $\eta: \Gamma_\epsilon \rightarrow U(\mathfrak{g})$  (essentially a version of Lusztig's ‘‘Frobenius’’ epimorphism in [60]). Let us put

$$(25) \quad \mathcal{Z}_0(\mathcal{O}_\epsilon) := \eta^*(\mathcal{O}(G))$$

where  $\eta^*: U(\mathfrak{g})^\circ \rightarrow \Gamma_\epsilon^\circ$  is the monomorphism dual to  $\eta$ .

**Theorem 2.15.** (1)  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  is a central Hopf subalgebra of  $\mathcal{O}_\epsilon \subset \Gamma_\epsilon^\circ$ , and  $Q(\mathcal{Z}(\mathcal{O}_\epsilon))$  is an extension of  $Q(\mathcal{Z}_0(\mathcal{O}_\epsilon))$  of degree  $l^m$  if  $l$  is coprime to the coefficients of the Cartan matrix of  $\mathfrak{g}$ .

(2)  $\psi_{-l\rho} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ , and  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  is generated by the matrix coefficients of the irreducible  $\Gamma$ -modules of highest weight  $l\lambda$ ,  $\lambda \in P_+$ . Moreover, the map  $\Phi$  in (21) affords an algebra embedding  $\mathcal{Z}_0(\mathcal{O}_\epsilon) \rightarrow \mathcal{Z}_0(U_\epsilon(H))$  and algebra isomorphisms  $\mathcal{Z}_0(\mathcal{O}_\epsilon)[\psi_{-l\rho}^{-1}] \rightarrow \mathcal{Z}_0(U_\epsilon(H))$ ,  $\mathcal{O}_\epsilon[\psi_{-l\rho}^{-1}] \rightarrow U_\epsilon(H)$ .

(3)  $\mathcal{O}_\epsilon$  has no non-trivial zero divisors, and it is a free  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ -module of rank  $l^{\dim \mathfrak{g}}$ . Moreover  $\mathcal{O}_\epsilon$  is a maximal order of its classical fraction algebra  $Q(\mathcal{O}_\epsilon) = Q(\mathcal{Z}(\mathcal{O}_\epsilon)) \otimes_{\mathcal{Z}(\mathcal{O}_\epsilon)} \mathcal{O}_\epsilon$ , and  $Q(\mathcal{O}_\epsilon)$  is a central simple algebra of PI degree  $l^N$ .

For the proof, see in [42]: the proposition 6.4 for the first claim of (1) (where  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  and  $\mathcal{Z}_0(U_\epsilon(H))$  are denoted  $F_0$  and  $A_0$  respectively), the appendix of Enriquez and [46] for the second claim of (1), the propositions 6.4-6.5 for (2), the theorem 7.2 (where  $\mathcal{O}_\epsilon$  is shown to be projective over  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ ) and [25] (which provides the additional K-theoretic arguments to

deduce that  $\mathcal{O}_\epsilon$  is free), or Remark 2.18(b) of [6], for the first claim of (3), and the theorem 7.4 for the second claim.

As above for  $U_\epsilon$ , it follows directly from (3) that  $Q(\mathcal{Z}(\mathcal{O}_\epsilon))$  has degree  $l^m$  over  $Q(\mathcal{Z}_0(\mathcal{O}_\epsilon))$ . A complete description of  $\mathcal{Z}(\mathcal{O}_\epsilon)$  is obtained in [46] and Enriquez' Appendix in [42]. We do not know a basis of  $\mathcal{O}_\epsilon$  over  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  for general  $G$ , but see [43] for the case of  $SL_2$ . We will recall the known results in this case of  $SL_2$  before Lemma 4.3.

There is a natural action of the braid group  $\mathcal{B}(\mathfrak{g})$  on  $\mathcal{O}_\epsilon$ , that we will use. Namely, let  $n_i \in N(T_G)$  be a representative of the reflection  $s_i \in W = N(T_G)/T_G$  associated to the simple root  $\alpha_i$ . In [77, 76] Soibelman-Vaksman introduced functionals  $t_i : \mathcal{O}_A \rightarrow A$  which quantize the elements  $n_i$ . They correspond dually to generators of the quantum Weyl group of  $\mathfrak{g}$ ; in the Appendix we recall their main properties (see also [35], Section 8.2, and [55, 77, 59, 56, 42]). Denote by  $\triangleleft$  the natural right action of functionals on  $\mathcal{O}_A$ , namely (using Sweedler's notation)

$$\alpha \triangleleft h = \sum_{(\alpha)} h(\alpha_{(1)})\alpha_{(2)}$$

for every  $\alpha \in \mathcal{O}_A$  and  $h \in \mathcal{O}_A \rightarrow A$ . Let us identify  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  with  $\mathcal{O}(G)$  by means of (25). We have ([42], Proposition 7.1):

**Proposition 2.16.** *The maps  $\triangleleft t_i$  on  $\mathcal{O}_\epsilon$  preserve  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ , and satisfy  $(f \triangleleft t_i)(a) = f(n_i a)$  and  $(f \star \alpha) \triangleleft t_i = (f \triangleleft t_i)(\alpha \triangleleft t_i)$  for every  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ ,  $a \in G$ ,  $\alpha \in \mathcal{O}_\epsilon$ .*

We provide an alternative, non computational, proof of this result in the Appendix (Section 6.2).

### 3. NOETHERIANITY AND FINITENESS

In this section we prove Theorem 1.1. Recall that by Noetherian we mean right and left Noetherian.

**Theorem 3.1.** *The algebras  $\mathcal{L}_{0,n}$ ,  $\mathcal{L}_{0,n}^A$  and  $\mathcal{L}_{0,n}^{\epsilon'}$ ,  $\epsilon' \in \mathbb{C}^\times$ , are Noetherian.*

Let us note that the algebras in this theorem are generated by a finite number of elements over their respective ground rings  $\mathbb{C}(q)$ ,  $A$  and  $\mathbb{C}$ . Indeed, by the formula (6) it is enough to verify this for  $\mathcal{L}_{0,1}^A$ , but  $\mathcal{L}_{0,1}^A = \mathcal{O}_A$  as a vector space, and  $\mathcal{O}_A$  with its product  $\star$  is well-known to be finitely generated by the matrix coefficients of the fundamental  $\Gamma$ -modules  $AV_{\varpi_k}$ ,  $k \in \{1, \dots, m\}$ . Then the claim follows from the formula inverse to (3), expressing the product  $\star$  in terms of the product of  $\mathcal{L}_{0,1}$  (see (18) in [28]).

*Proof of Theorem 3.1.* The result for  $\mathcal{L}_{0,1}$  and  $\mathcal{L}_{0,1}^A$  follows immediately from Theorem 2.2 (3) by identifying  $\mathcal{L}_{0,1}^A$  with  $U_A^{lf}$  via  $\Phi_1$ . Assume now that  $n > 1$ . We are going to develop the proof for  $\mathcal{L}_{0,n}$ ; the arguments can be repeated verbatim for  $\mathcal{L}_{0,n}^A$ , and the result for  $\mathcal{L}_{0,n}^{\epsilon'}$  will then follow immediately by lifting ideals by the quotient map  $\mathcal{L}_{0,n}^A \rightarrow \mathcal{L}_{0,n}^{\epsilon'} = \mathcal{L}_{0,n}^A / (q - \epsilon')\mathcal{L}_{0,n}^A$ .

Recall the isomorphism of  $U_q$ -modules (see (11)):

$$(26) \quad \mathcal{L}_{0,n} \xrightarrow{\Phi_n} (U_q(\mathfrak{g})^{\otimes n})^{lf} \xrightarrow{\psi_n^{-1}} U_q^{lf}(\mathfrak{g})^{\otimes n} = U_q^{lf}(\mathfrak{g}^{\oplus n})$$

where  $lf$  means respectively locally finite for the action  $ad_n^r$  of  $U_q(\mathfrak{g})$  on  $U_q(\mathfrak{g})^{\otimes n}$ , locally finite for the action  $ad^r$  of  $U_q(\mathfrak{g})$  on  $U_q(\mathfrak{g})$ , and locally finite for the action  $ad^r$  of  $U_q^{lf}(\mathfrak{g}^{\oplus n})$  on itself. It is a fact that Theorem 2.2 (3) holds true by replacing  $U_q^{lf}(\mathfrak{g})$  with  $U_q^{lf}(\mathfrak{g}^{\oplus n})$ , but one cannot use this to deduce the result because  $\psi_n$  is not a morphism of algebras. However, one can adapt the arguments of the proof of Theorem 2.2 (3) given in Theorem 2.137 of [80]. Let us begin by recalling these arguments.



As usual let  $C(\mu)$  be the vector space generated by the matrix coefficients of  $V_\mu$ , the simple  $U_q$ -module of type 1 and highest weight  $\mu \in P_+$ . Denote by  $C(\mu)_\lambda \subset C(\mu)$  the subspace of weight  $\lambda$  for the left coregular action of  $U_q(\mathfrak{h})$ ; so  $\alpha \in C(\mu)_\lambda$  if

$$K_\nu \triangleright \alpha = q^{(\nu, \lambda)} \alpha, \nu \in P.$$

Consider the ordered semigroup

$$\Lambda = \{(\mu, \lambda) \in P_+ \times P, \lambda \text{ is a weight of } V_\mu\}$$

with the partial order  $(\mu, \lambda) \leq (\mu', \lambda')$  if and only if  $\mu' - \mu \in P_+, \lambda' - \lambda \in P_+$ . Since  $\mathcal{L}_{0,1}$  and  $\mathcal{O}_q$  are isomorphic vector spaces we have  $\mathcal{L}_{0,1} = \bigoplus_{\mu \in P_+} C(\mu) = \bigoplus_{(\mu, \lambda) \in \Lambda} C(\mu)_\lambda$ . Consider the filtration  $\mathcal{F}_2$  of the vector space  $\mathcal{L}_{0,1}$  given by the family of subspaces

$$\mathcal{F}_2^{\mu, \lambda} = \bigoplus_{(\mu', \lambda') \leq (\mu, \lambda)} C(\mu')_{\lambda'}, (\mu, \lambda) \in \Lambda.$$

Denote by  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  the associated graded vector space. The standard vector space isomorphism  $\mathcal{L}_{0,1} \rightarrow Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$ , assigning to  $x \in C(\mu)_\lambda$  its coset  $\bar{x} \in \mathcal{F}_2^{\mu, \lambda} / (\bigoplus_{(\mu', \lambda') < (\mu, \lambda)} C(\mu')_{\lambda'})$ , implies

$$Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1}) = \bigoplus_{(\mu, \lambda) \in \Lambda} C(\mu)_\lambda.$$

Now, one has the following facts:

(i) First, taking the product in  $\mathcal{L}_{0,1}$  we have

$$(27) \quad \alpha\beta \in \mathcal{F}_2^{\mu_1 + \mu_2, \lambda_1 + \lambda_2} \quad \text{for } \alpha \in C(\mu_1)_{\lambda_1}, \beta \in C(\mu_2)_{\lambda_2}.$$

Therefore  $\mathcal{F}_2$  is an algebra filtration of  $\mathcal{L}_{0,1}$ , and  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  a graded algebra. Denote by  $\alpha \circ \beta$  the product in  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  of  $\alpha, \beta \in \mathcal{L}_{0,1}$ ; by definition, if  $\alpha \in C(\mu_1)_{\lambda_1}, \beta \in C(\mu_2)_{\lambda_2}$  then  $\alpha \circ \beta$  is the projection of  $\alpha\beta$  onto  $C(\mu_1 + \mu_2)_{\lambda_1 + \lambda_2}$ .

(ii) Second, denote by  $\bar{\star}$  the product  $\star$  of  $\mathcal{O}_q$  followed by the projection onto the component  $C(\mu + \nu)$ . Then we have

$$(28) \quad C(\mu) \circ C(\nu) = C(\mu) \bar{\star} C(\nu) = C(\mu + \nu).$$

(iii) Finally, for every  $\mu \in P_+$  fix a basis of weight vectors  $e_1^\mu, \dots, e_m^\mu$  of  $V_\mu$ . Denote by  $e_1^1, \dots, e_m^m \in V_\mu^*$  the dual basis, and by  $w(e_i^\mu)$  the weight of  $e_i^\mu$ . One can assume that the ordering of  $e_1^\mu, \dots, e_m^\mu$  is such that  $w(e_i^\mu) > w(e_j^\mu)$  implies  $i < j$ ; indeed,  $e_1^\mu$  generates the subspace of weight  $\mu$ , then come (in any order) the  $e_i^\mu$  such that  $w(e_i^\mu) = \mu - \alpha_s$  for some  $s$ , then those such that  $w(e_i^\mu) = \mu - \alpha_s - \alpha_t$  for some  $s$  and  $t$ , etc. Consider the matrix coefficients  $\mu\phi_i^j(x) := e_i^j(\pi_V(x)(e_j^\mu))$ ,  $x \in U_q$ . By (3), using the explicit form of the  $R$ -matrix it can be shown that

$$(29) \quad \begin{aligned} \nu\phi_k^l \circ \mu\phi_i^j - q_{ijkl} \mu\phi_i^j \circ \nu\phi_k^l &= \sum_{r=i}^m \sum_{s=1}^k \sum_{u=1}^{l-1} \sum_{v=j+1}^m \delta_{rsuv}^{ijkl} \mu\phi_r^v \circ \nu\phi_s^u \\ &\quad - \sum_{r=i+1}^m \sum_{s=1}^{k-1} q_{ijkl} \gamma_{rs}^{ijkl} \mu\phi_r^j \circ \nu\phi_s^l \end{aligned}$$

where  $q_{ijkl} = q^{(w(e_j^\mu) + w(e_i^\mu), w(e_k^\nu) - w(e_l^\nu))}$ , and  $\gamma_{rs}^{ijkl}, \delta_{rsuv}^{ijkl} \in \mathbb{C}(q^{1/D})$  are such that  $\gamma_{rs}^{ijkl} = 0$  unless  $w(e_r^\mu) < w(e_i^\mu)$  and  $w(e_s^\nu) > w(e_k^\nu)$ , and  $\delta_{rsuv}^{ijkl} = 0$  unless  $w(e_u^\nu) > w(e_l^\nu)$ ,  $w(e_v^\mu) < w(e_j^\mu)$ ,  $w(e_r^\mu) \leq w(e_i^\mu)$  and  $w(e_s^\nu) \geq w(e_k^\nu)$ .

By (28) (or more simply by using (3), as observed before the proof),  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  is generated by the matrix coefficients  $\varpi_k \phi_i^j$  of the fundamental representations  $V_{\varpi_k}$ . One can list

these matrix coefficients, say  $M$  in number, in an ordered sequence  $u_1, \dots, u_M$  such that the following condition holds: if  $w(e_k^{\overline{\varpi}^s}) < w(e_i^{\overline{\varpi}^r})$ , or  $w(e_k^{\overline{\varpi}^s}) = w(e_i^{\overline{\varpi}^r})$  and  $w(e_l^{\overline{\varpi}^s}) < w(e_j^{\overline{\varpi}^r})$ , then  $u_a := \varpi_r \phi_i^j$  and  $u_b := \varpi_s \phi_k^l$  satisfy  $b < a$ . Then denoting  ${}_\mu \phi_i^j$ ,  ${}_\nu \phi_k^l$  in (29) by  $u_j$ ,  $u_i$  respectively, and assuming  $u_j < u_i$ , one finds that all terms  $u_s := {}_\mu \phi_r^v$ ,  ${}_\mu \phi_r^j$  in the sums are  $< u_j$ . Therefore, for all  $1 \leq j < i \leq M$  it takes the form:

$$(30) \quad u_i \circ u_j - q_{ij} u_j \circ u_i = \sum_{s=1}^{j-1} \sum_{t=1}^M \alpha_{ij}^{st} u_s \circ u_t$$

for some  $q_{ij} \in \mathbb{C}(q^{1/D})^\times$ ,  $\alpha_{ij}^{st} \in \mathbb{C}(q^{1/D})$ . By Proposition I.8.17 of [23] (see also Proposition 2.133 of [80]) an algebra  $A$  over a field  $\mathbb{K}$  generated by elements  $u_1, \dots, u_M$  such that

$$(31) \quad u_i \circ u_j - q_{ij} u_j \circ u_i = \sum_{s=1}^{j-1} \sum_{t=1}^M \alpha_{ij}^{st} u_s \circ u_t + \beta_{ij}^{st} u_t \circ u_s$$

for all  $1 \leq j < i \leq M$  and some  $q_{ij} \in \mathbb{K}^\times$  and  $\alpha_{ij}^{st}, \beta_{ij}^{st} \in \mathbb{K}$ , is Noetherian. In fact  $A$  has an algebra filtration, say  $\mathcal{F}_3$ , such that  $Gr_{\mathcal{F}_3}(A)$  is a quotient of a skew-polynomial algebra, and thus is Noetherian. Moreover, it is classical that a filtered algebra which graded algebra is Noetherian is Noetherian too (see eg. [69], 1.6.9-1.6.11). Applying this to  $A = Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  and going up the filtration  $\mathcal{F}_2$  it follows that  $\mathcal{L}_{0,1}$  is Noetherian too.

We are going to extend all these facts to  $\mathcal{L}_{0,n}$ . The main point is to generalize the filtration  $\mathcal{F}_2$ , which we do first. Consider the semigroup

$$[\Lambda] = \{([\mu], [\lambda]) \in P_+^n \times P_+^n \mid (\mu_i, \lambda_i) \in \Lambda \text{ where } [\mu] = (\mu_i)_{i=1}^n, [\lambda] = (\lambda_i)_{i=1}^n\}.$$

Put the lexicographic partial order on  $[\Lambda]$ , starting from the tail: so  $([\mu'], [\lambda']) \leq ([\mu], [\lambda])$  if  $\mu_n - \mu'_n \in P_+ \setminus \{0\}$ , or  $\mu_n = \mu'_n$  and  $\lambda_n - \lambda'_n \in P_+ \setminus \{0\}$ , or there is  $k \in \{n, \dots, 2\}$  such that  $\mu_i = \mu'_i, \lambda_i = \lambda'_i$  for  $i \in \{n, \dots, k\}$  and  $\mu_{k-1} - \mu'_{k-1} \in P_+ \setminus \{0\}$ , or  $\mu_{k-1} = \mu'_{k-1}$  and  $\lambda_{k-1} - \lambda'_{k-1} \in P_+ \setminus \{0\}$ , replacing this last condition by  $\lambda_1 - \lambda'_1 \in P_+$  when  $k = 2$ . Now recall that  $\mathcal{L}_{0,n} = \mathcal{L}_{0,1}^{\otimes n} = \mathcal{O}_q^{\otimes n}$  as vector spaces. For every  $([\mu], [\lambda]) \in [\Lambda]$  consider the subspaces  $C([\mu]_{[\lambda]}) \subset C([\mu]) \subset \mathcal{L}_{0,n}$  defined by

$$\begin{aligned} C([\mu]) &= C(\mu_1) \otimes \dots \otimes C(\mu_n) \\ C([\mu]_{[\lambda]}) &= C(\mu_1)_{\lambda_1} \otimes \dots \otimes C(\mu_n)_{\lambda_n}. \end{aligned}$$

Then  $\mathcal{L}_{0,n} = \bigoplus_{[\mu] \in P_+^n} C([\mu])$  and  $C([\mu]) = \bigoplus_{([\mu'], [\lambda']) \in [\Lambda]} C([\mu]_{[\lambda']})$ . For every  $([\mu], [\lambda]) \in [\Lambda]$  define

$$(32) \quad \mathcal{F}_2^{[\mu], [\lambda]} = \bigoplus_{([\mu'], [\lambda']) \leq ([\mu], [\lambda])} \bigotimes_{j=1}^n C(\mu'_j)_{\lambda'_j}.$$

Clearly  $\mathcal{F}_2^{[\mu'], [\lambda']} \subset \mathcal{F}_2^{[\mu], [\lambda]}$  for  $([\mu'], [\lambda']) \leq ([\mu], [\lambda])$ , and the vector space  $\mathcal{L}_{0,n}$  is the union of the subspaces  $\mathcal{F}_2^{[\mu], [\lambda]}$  over all  $([\mu], [\lambda]) \in [\Lambda]$ , so these form a filtration of  $\mathcal{L}_{0,n}$ . Let us denote it  $\mathcal{F}_2$ , as when  $n = 1$ . As usual, write  $([\mu'], [\lambda']) < ([\mu], [\lambda])$  for  $([\mu'], [\lambda']) \leq ([\mu], [\lambda])$  and  $([\mu'], [\lambda']) \neq ([\mu], [\lambda])$ , and put

$$\mathcal{F}_2^{<[\mu], [\lambda]} = \sum_{([\mu'], [\lambda']) < ([\mu], [\lambda])} \mathcal{F}_2^{[\mu'], [\lambda']}.$$

Then define

$$Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})_{[\mu], [\lambda]} = \mathcal{F}_2^{[\mu], [\lambda]} / \mathcal{F}_2^{<[\mu], [\lambda]}.$$

This space is canonically identified with  $C([\mu])_{[\lambda]}$ , so the graded vector space associated to  $\mathcal{F}_2$  is

$$(33) \quad Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n}) = \bigoplus_{([\mu],[\lambda]) \in [\Lambda]} Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})_{([\mu],[\lambda])} = \bigoplus_{([\mu],[\lambda]) \in [\Lambda]} C([\mu])_{[\lambda]}.$$

We claim that  $\mathcal{F}_2$  is an *algebra filtration* with respect to the product of  $\mathcal{L}_{0,n}$ , and therefore  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})$  is a graded algebra.

For notational simplicity let us prove it for  $n = 2$ , the general case being strictly similar. Recall that the product of  $\mathcal{L}_{0,n}$  is given by the formula (6). Take  $([\mu], [\lambda]), ([\mu'], [\lambda']) \in [\Lambda]$ , and elements  $\alpha \otimes \beta \in C(\mu_1)_{\lambda_1} \otimes C(\mu_2)_{\lambda_2}$  and  $\alpha' \otimes \beta' \in C(\mu'_1)_{\lambda'_1} \otimes C(\mu'_2)_{\lambda'_2}$ . The  $R$ -matrix expands as  $R = \Theta \hat{R}$ , where  $\Theta = q^{\sum_{i,j=1}^m (B^{-1})_{ij} H_i \otimes H_j} \in \mathbb{U}_q^{\otimes 2}$ , with  $B \in M_m(\mathbb{Q})$  the matrix with entries  $B_{ij} := d_j^{-1} a_{ij}$ , and  $\hat{R} = \sum_{(\hat{R})} \hat{R}_{(1)} \otimes \hat{R}_{(2)} \in \mathbb{U}_q(\mathfrak{n}_+) \otimes \mathbb{U}_q(\mathfrak{n}_-)$  (see eg. [35], Theorem 8.3.9, or [80], Theorem 2.108). If  $x, y$  are weight vectors of weights  $\mu, \nu$  respectively, then  $\Theta(x \otimes y) = q^{(\mu, \nu)} x \otimes y$ . Moreover,  $\hat{R}$  has weight 0 for the adjoint action of  $U_q(\mathfrak{h})$ ; that is, complementary components  $\hat{R}_{(1)}$  and  $\hat{R}_{(2)}$  have opposite weights. Note also that the coregular actions  $\triangleright, \triangleleft$  fix globally each component  $C(\mu)$ ,  $\mu \in P_+$ . Then, for every  $\nu \in P$  and any of the components  $R_{(2)}^1, \dots, R_{(2)}^4$  we have

$$\begin{aligned} K_\nu \triangleright \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \right) &= \sum_{(\beta), (\beta)} \beta_{(1)}(R_{(2)}^2 R_{(2)}^4) \left( K_\nu S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta_{(2)} \right) \\ &= q^{-(\nu, \gamma)} \sum_{(\beta), (\beta)} \beta_{(1)}(R_{(2)}^2 R_{(2)}^4) \left( S(R_{(2)}^1 R_{(2)}^3) K_\nu \triangleright \beta_{(2)} \right) \\ &= q^{(\nu, \lambda_2 - \gamma)} \sum_{(\beta), (\beta)} \beta_{(1)}(R_{(2)}^2 R_{(2)}^4) \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta_{(2)} \right) \\ &= q^{(\nu, \lambda_2 - \gamma)} \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \right). \end{aligned}$$

for some positive root  $\gamma \in Q_+$ . Therefore  $S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \in C(\mu_2)_{\lambda_2 - \gamma}$ ; by a similar computation we find that  $S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha' \triangleleft R_{(1)}^1 R_{(1)}^2 \in C(\mu'_1)_{\lambda'_1 + \gamma}$  for the complementary components  $R_{(1)}^1, \dots, R_{(1)}^4$ . Then we always have  $S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \in \mathcal{F}_2^{\mu_2, \lambda_2}$ , and if  $S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \in C(\mu_2)_{\lambda_2}$  then  $S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha' \triangleleft R_{(1)}^1 R_{(1)}^2 \in C(\mu'_1)_{\lambda'_1}$ . Since the product of  $\mathcal{L}_{0,n}$  is componentwise that of  $\mathcal{L}_{0,1}$ , by (27) we have

$$(34) \quad \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \right) \beta' \in \mathcal{F}_2^{\mu_2 + \mu'_2, \lambda_2 + \lambda'_2}$$

and if  $(S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4) \beta' \in C(\mu_2 + \mu'_2)_{\lambda_2 + \lambda'_2}$  then

$$(35) \quad \alpha \left( S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha' \triangleleft R_{(1)}^1 R_{(1)}^2 \right) \in C(\mu_1)_{\lambda_1} C(\mu'_1)_{\lambda'_1} \subset \mathcal{F}_2^{\mu_1 + \mu'_1, \lambda_1 + \lambda'_1}.$$

In conclusion

$$(\alpha \otimes \beta)(\alpha' \otimes \beta') \in \mathcal{F}_2^{[\mu], [\lambda]}.$$

Similar arguments work for any  $n \geq 2$ . This proves that  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})$  is a graded algebra with the product inherited from  $\mathcal{L}_{0,n}$ , which we denote by  $\circ_n$ . Recall that it is defined on homogeneous elements  $\overline{\alpha \otimes \beta} \in Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})_{([\mu],[\lambda])}$ ,  $\overline{\alpha' \otimes \beta'} \in Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})_{([\mu'], [\lambda'])}$  by

$$\overline{\alpha \otimes \beta} \circ_n \overline{\alpha' \otimes \beta'} = (\alpha \otimes \beta)(\alpha' \otimes \beta') + \mathcal{F}_2^{<[\mu + \mu'], [\lambda + \lambda']}.$$

Next we show that (28) implies the same property for the product  $\circ$  of  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})$ . First it gives  $(C(\mu_1) \circ C(\mu'_1)) \otimes (C(\mu_2) \circ C(\mu'_2)) = C([\mu + \mu'])$ . Now, by the previous remark and (34), (35) we have

$$C([\mu]) \circ_n C([\mu']) \subset (C(\mu_1) \circ C(\mu'_1)) \otimes (C(\mu_2) \circ C(\mu'_2)).$$

The converse inclusion holds true as well, as one can see by reasoning as above, starting with the standard (componentwise) product of  $\mathcal{L}_{0,1}^{\otimes n}$  expressed in terms of the product of  $\mathcal{L}_{0,n}$  by the formula (8). In conclusion

$$(36) \quad C([\mu]) \circ_n C([\mu']) = C([\mu + \mu']).$$

We are left to show that (29) generalizes to  $\mathcal{L}_{0,n}$ . Again we note that this cannot be deduced from the case  $n = 1$ , because for the vector space isomorphism  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n}) \rightarrow Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})^{\otimes n}$  induced from the equality  $\mathcal{L}_{0,n} = \mathcal{L}_{0,1}^{\otimes n}$  is not a morphism of algebras with respect to the products  $\circ_n$  and  $\circ^{\otimes n}$ . Therefore one cannot take the filtration on  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})$  which is componentwise  $\mathcal{F}_3$ , and that we will denote again by  $\mathcal{F}_3$ , to deduce that  $Gr_{\mathcal{F}_3}(Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n}))$  is a quotient of a quasi-polynomial algebra, whence Noetherian. However, we can proceed in essentially the same way. We give the details when  $n = 2$ , the general case being similar. Let us write the twist  $F$  in (7) as

$$F = \sum_{(F)} F_{(1)} \otimes F_{(2)} = \sum_{(F)} F_{(1)1} \otimes F_{(1)2} \otimes F_{(2)1} \otimes F_{(2)2}$$

that is, setting  $F_{(1)1} := R_{(2)}^2 R_{(2)}^4$ ,  $F_{(1)2} := R_{(2)}^1 R_{(2)}^3$ ,  $F_{(2)1} := R_{(1)}^1 R_{(1)}^2$ ,  $F_{(2)2} := R_{(1)}^3 R_{(1)}^4$ . Keep the notations of (29), and put  $d(\mu) := \dim(V_\mu)$ ,  $\mu \in P_+$ , and

$$\Delta^{(2)}(\mu_2 \phi_{k_2}^{l_2}) = \sum_{p,s=1}^{d(\mu_2)} \mu_2 \phi_{k_2}^p \otimes \mu_2 \phi_p^s \otimes \mu_2 \phi_s^{l_2}, \quad \Delta^{(2)}(\mu'_1 \phi_{k'_1}^{l'_1}) = \sum_{p',s'=1}^{d(\mu'_1)} \mu'_1 \phi_{k'_1}^{p'} \otimes \mu'_1 \phi_{p'}^{s'} \otimes \mu'_1 \phi_{s'}^{l'_1}.$$

Assume that  $\mu'_2 \phi_{k'_2}^{l'_2} < \mu_2 \phi_{k_2}^{l_2}$  and  $\mu'_1 \phi_{k'_1}^{l'_1} < \mu_1 \phi_{k_1}^{l_1}$ . From (6) and then (29)-(30) in the second equality, one obtains

$$\begin{aligned} & \left( \mu_1 \phi_{k_1}^{l_1} \otimes \mu_2 \phi_{k_2}^{l_2} \right) \circ_2 \left( \mu'_1 \phi_{k'_1}^{l'_1} \otimes \mu'_2 \phi_{k'_2}^{l'_2} \right) \\ &= \sum_{(F)} \sum_{p,s=1}^{d(\mu_2)} \sum_{p',s'=1}^{d(\mu'_1)} \left( \mu_1 \phi_{k_1}^{l_1} \circ \mu'_1 \phi_{p'}^{s'} \left( \mu'_1 \phi_{k'_1}^{p'} (F_{(2)1}) \mu'_1 \phi_{s'}^{l'_1} (S(F_{(2)2})) \right) \right) \\ & \quad \otimes \left( \mu_2 \phi_p^s \circ \mu'_2 \phi_{k'_2}^{l'_2} \left( \mu_2 \phi_{k_2}^p (F_{(1)1}) \mu_2 \phi_s^{l_2} (S(F_{(1)2})) \right) \right) \\ &= \sum_{(F)} \sum_{p,s=1}^{d(\mu_2)} \sum_{p',s'=1}^{d(\mu'_1)} q_{p's'k_1 l_1} q_{k'_2 l'_2 p s} \left( \mu'_1 \phi_{p'}^{s'} \circ \mu_1 \phi_{k_1}^{l_1} \left( \mu'_1 \phi_{k'_1}^{p'} (F_{(2)1}) \mu'_1 \phi_{s'}^{l'_1} (S(F_{(2)2})) \right) \right) \\ & \quad \otimes \left( \mu'_2 \phi_{k'_2}^{l'_2} \circ \mu_2 \phi_p^s \left( \mu_2 \phi_{k_2}^p (F_{(1)1}) \mu_2 \phi_s^{l_2} (S(F_{(1)2})) \right) \right) + \dots \end{aligned}$$

Here the dots are sums of tensors of the form  $(x_1 \circ x_2) \otimes (y_1 \circ y_2)$  where  $x_1 < \mu'_1 \phi_{k'_1}^{l'_1}$  and  $y_1 < \mu'_2 \phi_{k'_2}^{l'_2}$ . In fact, by the expression of  $R = \Theta \hat{R}$  we have  $\mu_2 \phi_{k_2}^p (F_{(1)1}) \mu_2 \phi_s^{l_2} (S(F_{(1)2})) = 0$  unless  $k_2 \geq p$  and  $s \geq l_2$ , and  $\mu'_1 \phi_{k'_1}^{p'} (F_{(2)1}) \mu'_1 \phi_{s'}^{l'_1} (S(F_{(2)2})) = 0$  unless  $k'_1 \leq p'$  and  $s' \leq l'_1$ . It is immediate that  $\mu_2 \phi_p^s \in C(\mu_2)_{w(e_s^{\mu_2})}$ . By definition  $s > l_2$  implies  $w(e_s^{\mu_2}) \leq w(e_{l_2}^{\mu_2})$ , and by (27) if  $w(e_s^{\mu_2}) < w(e_{l_2}^{\mu_2})$  then  $\mu'_2 \phi_{k'_2}^{l'_2} \circ \mu_2 \phi_p^s \in \mathcal{F}_2^{< \mu_2'', \lambda_2''}$ , where  $\mu_2'' := \mu_2 + \mu'_2$ ,  $\lambda_2'' := w(e_{l_2}^{\mu_2}) + w(e_{l_2}^{\mu'_2})$ .

In that case the term  $(\mu'_1 \phi_{p'}^{s'} \circ \mu_1 \phi_{k_1}^{l_1}) \otimes (\mu'_2 \phi_{k'_2}^{l'_2} \circ \mu_2 \phi_p^s)$  in the sum above vanishes in  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,2})$ . Moreover, the computations before (35) show that such a summand achieves the maximal weight  $w(e_{l_1}^{\mu_1}) + w(e_{l'_1}^{\mu'_1}) + w(e_{l_2}^{\mu_2}) + w(e_{l'_2}^{\mu'_2})$  only if  $w(e_{s'}^{\mu'_1}) = w(e_{l'_1}^{\mu'_1})$  and  $w(e_{s'}^{\mu_2}) = w(e_{l_2}^{\mu_2})$ , which occurs when  $F_{(1)2}, F_{(2)2}$  have no component of  $\hat{R}$  but only of  $\Theta$ . Also, if  $p = k_2, s = l_2, p' = k'_1, s' = l'_1$  then

$$\begin{aligned} \sum_{(F)} \mu_2 \phi_{k_2}^{k_2} (F_{(1)1}) \mu_2 \phi_{l_2}^{l_2} (S(F_{(1)2})) \mu'_1 \phi_{k'_1}^{k'_1} (F_{(2)1}) \mu'_1 \phi_{l'_1}^{l'_1} (S(F_{(2)2})) \\ = \left\langle \mu_2 \phi_{k_2}^{k_2} \otimes \mu_2 \phi_{l_2}^{l_2} \otimes \mu'_1 \phi_{k'_1}^{k'_1} \otimes \mu'_1 \phi_{l'_1}^{l'_1}, \Theta_{13} \Theta_{14}^{-1} \Theta_{24} \Theta_{23}^{-1} \right\rangle \\ = q \left( w(e_{k_2}^{\mu_2}) - w(e_{l_2}^{\mu_2}), w(e_{k'_1}^{\mu'_1}) - w(e_{l'_1}^{\mu'_1}) \right). \end{aligned}$$

Denoting by  $q'_{k_2 l_2 k'_1 l'_1}$  this scalar, it follows

$$\begin{aligned} \left( \mu_1 \phi_{k_1}^{l_1} \otimes \mu_2 \phi_{k_2}^{l_2} \right) \circ_2 \left( \mu'_1 \phi_{k'_1}^{l'_1} \otimes \mu'_2 \phi_{k'_2}^{l'_2} \right) \\ = q'_{k'_1 l'_1 k_1 l_1} q'_{k'_2 l'_2 k_2 l_2} q'_{k_2 l_2 k'_1 l'_1} \left( \left( \mu'_1 \phi_{k'_1}^{l'_1} \circ \mu_1 \phi_{k_1}^{l_1} \right) \otimes \left( \mu'_2 \phi_{k'_2}^{l'_2} \circ \mu_2 \phi_{k_2}^{l_2} \right) \right) \\ + \sum_{p=1}^{k_2-1} \sum_{p'=k'_1+1}^{d(\mu'_1)} \alpha'_{p' k_1 k'_2 p}{}^{l'_1 l_1 l'_2 l_2} \left( \left( \mu'_1 \phi_{p'}^{l'_1} \circ \mu_1 \phi_{k_1}^{l_1} \right) \otimes \left( \mu'_2 \phi_{k'_2}^{l'_2} \circ \mu_2 \phi_p^{l_2} \right) \right) + \dots \end{aligned}$$

for some scalars  $\alpha'_{p' k_1 k'_2 p}{}^{l'_1 l_1 l'_2 l_2} \in \mathbb{C}(q^{1/D})$ , with the dots as above. Moreover  $\alpha'_{p' k_1 k'_2 p}{}^{l'_1 l_1 l'_2 l_2} = 0$  unless  $w(e_p^{\mu_2}) > w(e_{k_2}^{\mu_2})$  and  $w(e_{p'}^{\mu'_1}) < w(e_{k'_1}^{\mu'_1})$ . Now, recall (8). In a similar way we find for all  $p \in \{1, \dots, k_2\}, p' \in \{k'_1, \dots, d(\mu'_1)\}$  that

$$\begin{aligned} \left( \mu'_1 \phi_{p'}^{l'_1} \circ \mu_1 \phi_{k_1}^{l_1} \right) \otimes \left( \mu'_2 \phi_{k'_2}^{l'_2} \circ \mu_2 \phi_p^{l_2} \right) \\ = \sum_{(\bar{F})} \sum_{r,t=1}^{d(\mu_1)} \sum_{r',t'=1}^{d(\mu'_2)} \left( \mu'_1 \phi_{p'}^{l'_1} \otimes \mu'_2 \phi_{r'}^{t'} \left( \mu'_2 \phi_{k'_2}^{r'} (\bar{F}_{(2)1}) \mu'_2 \phi_{t'}^{l'_2} (S(\bar{F}_{(2)2})) \right) \right) \\ \circ_2 \left( \mu_1 \phi_r^t \left( \mu_1 \phi_{k_1}^r (\bar{F}_{(1)1}) \mu_1 \phi_t^{l_1} (S(\bar{F}_{(1)2})) \right) \right) \otimes \mu_2 \phi_p^{l_2} \\ = q'_{k_1 l_1 k'_2 l'_2}{}^{-1} \left( \left( \mu'_1 \phi_{p'}^{l'_1} \otimes \mu'_2 \phi_{k'_2}^{l'_2} \right) \circ_2 \left( \mu_1 \phi_{k_1}^{l_1} \otimes \mu_2 \phi_p^{l_2} \right) \right) \\ + \sum_{r=1}^{k_1-1} \sum_{r'=k'_2+1}^{d(\mu'_2)} \beta'_{p' r' r p}{}^{l'_1 l'_2 l_1 l_2} \left( \left( \mu'_1 \phi_{p'}^{l'_1} \otimes \mu'_2 \phi_{r'}^{l'_2} \right) \circ_2 \left( \mu_1 \phi_r^{l_1} \otimes \mu_2 \phi_p^{l_2} \right) \right) \end{aligned}$$

for some scalars  $\beta'_{p' r' r p}{}^{l'_1 l'_2 l_1 l_2} \in \mathbb{C}(q^{1/D})$  such that  $\beta'_{p' r' r p}{}^{l'_1 l'_2 l_1 l_2} = 0$  unless  $w(e_r^{\mu_1}) > w(e_{k_1}^{\mu_1})$  and  $w(e_{r'}^{\mu'_2}) < w(e_{k'_2}^{\mu'_2})$ . Summing up we obtain

$$\begin{aligned} \left( \mu_1 \phi_{k_1}^{l_1} \otimes \mu_2 \phi_{k_2}^{l_2} \right) \circ_2 \left( \mu'_1 \phi_{k'_1}^{l'_1} \otimes \mu'_2 \phi_{k'_2}^{l'_2} \right) \\ = q'_{k'_1 l'_1 k_1 l_1} q'_{k'_2 l'_2 k_2 l_2} q'_{k_2 l_2 k'_1 l'_1} q'_{k_1 l_1 k'_2 l'_2}{}^{-1} \left( \left( \mu'_1 \phi_{k'_1}^{l'_1} \otimes \mu'_2 \phi_{k'_2}^{l'_2} \right) \circ_2 \left( \mu_1 \phi_{k_1}^{l_1} \otimes \mu_2 \phi_{k_2}^{l_2} \right) \right) \\ + \sum_{p=1}^{k_2} \sum_{p'=k'_1}^{d(\mu'_1)} \sum_{r=1}^{k_1-1} \sum_{r'=k'_2+1}^{d(\mu'_2)} \alpha'_{p' k_1 k'_2 p}{}^{l'_1 l_1 l'_2 l_2} \beta'_{p' r' r p}{}^{l'_1 l'_2 l_1 l_2} \left( \left( \mu'_1 \phi_{p'}^{l'_1} \otimes \mu'_2 \phi_{r'}^{l'_2} \right) \circ_2 \left( \mu_1 \phi_r^{l_1} \otimes \mu_2 \phi_p^{l_2} \right) \right) + \dots \end{aligned}$$

where at  $p = k_2$ ,  $p' = k'_1$  we set  $\alpha_{k'_1 k_1 k'_2 k_2}^{l'_1 l_1 l'_2 l_2} := q_{k'_1 l'_1 k_1 l_1} q_{k'_2 l'_2 k_2 l_2} q'_{k_2 l_2 k'_1 l'_1}$ , and the dots are sums of tensors of the form  $(x_1 \otimes y_1) \circ_2 (x_2 \otimes y_2)$  where  $y_1 < \mu'_2 \phi_{k'_2}^{l'_2}$ . Recall that in (30) we denoted by  $u_1, \dots, u_M$  the ordered list of matrix coefficients  $\varpi_k \phi_i^j$ . Let us order in a lexicographic way the elements  $u_i \otimes u_j$ , ie. as a sequence  $u_1^{(2)}, \dots, u_M^{(2)}$  such that the following condition holds: if  $\varpi_{i'} \phi_{s'}^{t'} < \varpi_{k'} \phi_{i'}^{j'}$ , or  $\varpi_{i'} \phi_{s'}^{t'} = \varpi_{k'} \phi_{i'}^{j'}$  and  $\varpi_l \phi_s^t < \varpi_k \phi_i^j$ , then  $u_a^{(2)} := \varpi_k \phi_i^j \otimes \varpi_{k'} \phi_{i'}^{j'}$  and  $u_b^{(2)} := \varpi_l \phi_s^t \otimes \varpi_{i'} \phi_{s'}^{t'}$  satisfy  $u_b^{(2)} < u_a^{(2)}$ . Then, by the conditions ensuring when  $\alpha_{p' k_1 k_2 p}^{l'_1 l_1 l'_2 l_2}$  and  $\beta_{p' r' r p}^{l'_1 l'_2 l_1 l_2}$  are non zero, the last identity takes the form of (30) by replacing  $u_i, u_j$  with  $u_i^{(2)} := \mu_1 \phi_{k_1}^{l_1} \otimes \mu_2 \phi_{k_2}^{l_2}$ ,  $u_j^{(2)} := \mu'_1 \phi_{k'_1}^{l'_1} \otimes \mu'_2 \phi_{k'_2}^{l'_2}$ . At the beginning of this computation we assumed  $\mu'_2 \phi_{k'_2}^{l'_2} < \mu_2 \phi_{k_2}^{l_2}$  and  $\mu'_1 \phi_{k'_1}^{l'_1} < \mu_1 \phi_{k_1}^{l_1}$ , but the same result occurs (in a simpler fashion) if  $\mu'_2 \phi_{k'_2}^{l'_2} = \mu_2 \phi_{k_2}^{l_2}$  and  $\mu'_1 \phi_{k'_1}^{l'_1} < \mu_1 \phi_{k_1}^{l_1}$ , so eventually we find that (30) holds true for all cases  $1 \leq u_j^{(2)} < u_i^{(2)} \leq M^2$ .

As in the case of  $\mathcal{L}_{0,1}$ , by using Proposition I.8.17 of [23] one can therefore conclude that there is a filtration  $\mathcal{F}_3$  of  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})$  such that  $Gr_{\mathcal{F}_3}(Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n}))$  is a quotient of a quasi-polynomial algebra, and finally that  $\mathcal{L}_{0,n}$  is Noetherian.  $\square$

**Theorem 3.2.** *The algebra  $\mathcal{M}_{0,n} = \mathcal{L}_{0,n}^{U_q}$  (respectively  $\mathcal{M}_{0,n}^A$ , and  $\mathcal{M}_{0,n}^{A,\epsilon'}$ ,  $\epsilon' \in \mathbb{C}^\times$ ) is Noetherian, and generated over  $\mathbb{C}(q)$  (resp.  $A, \mathbb{C}$ ) by a finite number of elements.*

Our method of proof follows closely that of the Hilbert-Nagata theorem (see [37]). Let us recall one version of this theorem, which is enough for our purposes. Let  $A = K[a_1, \dots, a_n]$  be a finitely generated commutative algebra over an arbitrary field  $K$ , and  $G$  a group of algebra automorphisms of  $A$ .

**Theorem 3.3.** *If the action of  $G$  on  $A$  is completely reducible on finite dimensional representations, then the ring  $A^G$  of invariants of  $A$  with respect to  $G$  is Noetherian and a finitely generated algebra over  $K$ .*

We recall here the main steps of the proof that we will adapt in order to prove Theorem 3.2:

(a) From the complete reducibility of the action of  $G$  on  $A$  one can define a linear map

$$R : A \rightarrow A^G$$

namely the projection onto the space of invariants along the space of non-trivial isotypical components of  $A$ . This linear map is called the Reynolds operator; it satisfies

$$(37) \quad R(hf) = hR(f)$$

for every  $f \in A$ ,  $h \in A^G$ .

(b) Let  $I$  be an ideal of  $A^G$ . Then  $I = R(AI) = AI \cap A^G$ . Because  $AI$  is an ideal of  $A$ , and  $A$  is Noetherian, there exist elements  $b_1, \dots, b_s$ , that can be chosen in  $I \subset A^G$ , such that  $AI = Ab_1 + \dots + Ab_s$ . Since  $I = R(AI) = R(Ab_1 + \dots + Ab_s) = A^G b_1 + \dots + A^G b_s$ ,  $I$  is finitely generated over  $A^G$ . Therefore  $A^G$  is Noetherian.

(c) Let  $B$  be an algebra graded over  $\mathbb{N}$  (for simplicity of notations):  $B = \bigoplus_{i=0}^{+\infty} B_n$ , with  $B_m \cdot B_n \subset B_{m+n}$ . The augmentation ideal of  $B$  is  $B^+ = \bigoplus_{i=1}^{+\infty} B_n$ . If  $B^+$  is a Noetherian ideal of  $B$ , then  $B$  is a finitely generated algebra over  $B_0$ . This is Lemma 2.4.5 of [75] (in that statement  $B$  is commutative, but this hypothesis is not necessary for the proof).

(d) Assume that  $A^G$  is graded over  $\mathbb{N}$  (for simplicity of notations):  $A^G = \bigoplus_{i=0}^{+\infty} A_n^G$  with  $A_0^G = K$ . Then  $A^{G+} = \bigoplus_{i=1}^{+\infty} A_n^G$  is an ideal of  $A^G$ , which is Noetherian by (b) above. Applying (c) we deduce that  $A^G$  is a finitely generated algebra over  $K$ .

*Proof of Theorem 3.2.* As for Theorem 3.1 the result for  $\mathcal{M}_{0,n}^{A,\epsilon'}$  follows from that for  $\mathcal{M}_{0,n}^A$  and  $\mathcal{M}_{0,n}$ , which are proved in the same way. Let us consider  $\mathcal{M}_{0,n}$ . Consider the filtration  $\mathcal{F}$  of  $\mathcal{L}_{0,n}$  by the subspaces

$$\mathcal{F}^{[\mu]} = \bigoplus_{[\mu'] \leq [\mu]} C([\mu']), \mu \in P_+^n$$

where  $P_+^n$  is given the lexicographic partial order induced from  $[\Lambda]$ . It is easily seen that  $\mathcal{F}$  is an algebra filtration: the coregular actions  $\triangleright, \triangleleft$  fix globally each component  $C(\mu)$  of  $\mathcal{L}_{0,1}$ , so the claim follows from (3), (6) and the fact that  $C(\mu) \star C(\nu) \subset C(\mu + \nu)$  for all  $\mu, \nu \in P_+$ . Denote by  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})$  the corresponding graded algebra. Again

$$(38) \quad Gr_{\mathcal{F}}(\mathcal{L}_{0,n}) = \mathcal{L}_{0,n} = \bigoplus_{[\mu] \in P_+^n} C([\mu]).$$

Because each space  $C([\mu])$  is stabilized by the coadjoint action of  $U_q$ , the decomposition (38) has a key advantage on (33). Indeed, since  $\mathcal{L}_{0,n}$  is a  $U_q$ -module algebra, the action of  $U_q$  is well-defined on  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})$  and it gives it a structure of  $U_q$ -module algebra. As vector spaces we have

$$(39) \quad Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q} = \bigoplus_{[\mu] \in P_+^n} C([\mu])^{U_q} = \mathcal{L}_{0,n}^{U_q}.$$

Now we can adapt the different steps (a)–(d) recalled above:

(a') The action of  $U_q$  on  $\mathcal{L}_{0,n}$  is completely reducible. This follows from Theorem 2.2 (1) (noting that the summands, being isomorphic by (14) to spaces  $C(\mu)$ , are finite-dimensional and thus completely reducible  $U_q$ -modules), and the isomorphism of  $U_q$ -modules (see (11)):

$$(40) \quad \mathcal{L}_{0,n} \xrightarrow{\Phi_n} (U_q(\mathfrak{g})^{\otimes n})^{lf} \xrightarrow{\psi_n^{-1}} U_q^{lf}(\mathfrak{g})^{\otimes n}$$

where  $lf$  means respectively locally finite for the action  $ad_n^r$  of  $U_q(\mathfrak{g})$  on  $U_q(\mathfrak{g})^{\otimes n}$ , and locally finite for the action  $ad^r$  of  $U_q(\mathfrak{g})$  on  $U_q(\mathfrak{g})$ . By (38) it follows that  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})$  is also completely reducible. We can therefore define the Reynolds operator  $R: Gr_{\mathcal{F}}(\mathcal{L}_{0,n}) \rightarrow Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$  as in (a).

(b') In the proof of Theorem 3.1 we showed that  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})$  is Noetherian, and then deduced that  $\mathcal{L}_{0,n}$  is Noetherian by a classical argument (see eg. [69], 1.6.9). This same argument implies that  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})$  is Noetherian, because (38) shows it is filtered by  $\mathcal{F}_2$ , and  $Gr_{\mathcal{F}_2}(Gr_{\mathcal{F}}(\mathcal{L}_{0,n})) = Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})$  is Noetherian. As in (b) we deduce that  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$  is Noetherian. But  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q} = Gr_{\mathcal{F}}(\mathcal{L}_{0,n}^{U_q})$ , which implies that  $\mathcal{L}_{0,n}^{U_q}$  is Noetherian.

(c'-d') Then we can apply the steps (c)–(d). As a result  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$  is finitely generated, say by  $k$  homogeneous elements  $\bar{x}_i \in \mathcal{F}^{[\mu_i]} / (\bigoplus_{[\mu'] < [\mu_i]} C([\mu']))$ .

(e') From (39) we deduce that  $\mathcal{L}_{0,n}^{U_q}$  is generated by the  $x_i \in C([\mu_i])$  with leading terms  $\bar{x}_1, \dots, \bar{x}_k$ . This follows from the following elementary fact: if  $A$  is a filtered  $\mathbb{K}$ -algebra ( $\mathbb{K}$  a field) which graded algebra  $Gr(A)$  is finitely generated, then  $A$  is finitely generated by elements which leading terms generate  $Gr(A)$ . Indeed, let  $A$  have the algebra filtration  $(A_i)_{i \in \mathbb{N}}$  (we take a filtration over  $\mathbb{N}$  to simplify notations). Put  $Gr(A) = \bigoplus_{i \in \mathbb{N}} A_{(i)}$ ,  $A_{(i+1)} := A_{i+1}/A_i$ . We have  $\bar{a} + \bar{b} = \overline{a+b}$  and  $\bar{a}\bar{b} = ab + A_{n+m-1} \in A_{(m+n)}$ , so  $\bar{a}\bar{b} = 0$  if  $ab \in A_{n+m-1}$ , and  $\bar{a}\bar{b} = \overline{ab}$  otherwise. Now assume that  $Gr(A)$  is finitely generated over  $\mathbb{K}$ .

Denote by  $\bar{f}_1, \dots, \bar{f}_n$  a finite set of generators. Let  $a \in A$ , with  $a \in A_k \setminus A_{k-1}$ . Then  $\bar{a} \in A_{(k)}$  can be written as  $\bar{a} = \sum_{(i_1, \dots, i_n) \in I} \lambda_{(i_1, \dots, i_n)} \bar{f}_1^{i_1} \dots \bar{f}_n^{i_n}$  for some finite set  $I \subset \mathbb{N}^n$  and scalars  $\lambda_{(i_1, \dots, i_n)} \in \mathbb{K}$ . Either  $\bar{f}_1^{i_1} \dots \bar{f}_n^{i_n}$  is zero, or  $\bar{f}_1^{i_1} \dots \bar{f}_n^{i_n} = \overline{f_1^{i_1} \dots f_n^{i_n}}$ . We therefore have  $a - \sum_{(i_1, \dots, i_n) \in I} \lambda_{(i_1, \dots, i_n)} f_1^{i_1} \dots f_n^{i_n} = 0 \in A_{(k)}$ , that is,  $a - \sum_{(i_1, \dots, i_n) \in I} \lambda_{(i_1, \dots, i_n)} f_1^{i_1} \dots f_n^{i_n} \in A_{k-1}$ . The conclusion follows by an easy induction.  $\square$

**Remark 3.4.** (1) Because  $\mathcal{L}_{0,1}^{U_q}$  is the center of  $\mathcal{L}_{0,1}$ , (e') proves it is finitely generated. Of course this follows also from the isomorphism  $\mathcal{L}_{0,1} \cong U_q^{lf}$  and the fact that the center of  $U_q^{lf}$  is the center of  $U_q$  (by Theorem 2.2), plus the well-known description of the latter. But the argument here is elementary and it applies to  $\mathcal{L}_{0,n}^{U_q}$  for any  $n \geq 1$ .

(2) In spite of the isomorphism  $\Phi_n: \mathcal{M}_{0,n} \rightarrow (U_q^{\otimes n})^{U_q}$ , in order to prove Theorem 3.2 one cannot bypass the hard study of  $(U_q^{\otimes n})^{lf}$ , whence of  $\mathcal{L}_{0,1} \cong U_q^{lf}$ , by working directly with  $U_q$ . Indeed the adjoint action is not completely reducible thereon. In fact,  $U_q^{lf}$  is exactly the socle of this action (see [51], Lemma 7.1.24).

(3) In the  $sl(2)$  case the filtration  $\mathcal{F}$  on  $\mathcal{L}_{0,n}^{U_q}$  should correspond via the Wilson loop isomorphism (defined in [28], Section 8.2) to the filtration of skein algebras of spheres with  $n+1$  punctures used in [66].

#### 4. PROOF OF THEOREM 1.2

As usual we let  $\epsilon$  be a primitive  $l$ -th root of unity with  $l$  odd and  $l > d_i$  for all  $i \in \{1, \dots, m\}$ .

Recall that  $\mathcal{Z}_0(U_\epsilon) \subset U_\epsilon$  is the central polynomial subalgebra generated by  $E_{\beta_k}^l, F_{\beta_k}^l, L_i^{\pm l}$ , for  $k \in \{1, \dots, N\}$  and  $i \in \{1, \dots, m\}$ . Define

$$\mathcal{Z}_0(U_\epsilon^{lf}) = U_\epsilon^{lf} \cap \mathcal{Z}_0(U_\epsilon).$$

Examples show that generating sets of  $\mathcal{Z}_0(U_\epsilon^{lf})$  have complicated expressions in general. Nevertheless, specializing  $q$  at  $\epsilon$  in Theorem 2.2 (2) we get

$$(41) \quad \mathcal{Z}_0(U_\epsilon) = T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{lf}) [T^{(l)} / T_2^{(l)}]$$

where  $T^{(l)}$ ,  $T_{2-}^{(l)}$  and  $T_2^{(l)}$  are the subsets of  $T$ ,  $T_{2-}$  and  $T_2$  formed by the elements  $K_{\lambda l}$  with  $\lambda \in P$ ,  $\lambda \in -2P_+$  and  $\lambda \in 2P$  respectively. Define

$$\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \Phi_1^{-1}(\mathcal{Z}_0(U_\epsilon^{lf})).$$

Recall the isomorphism  $\eta^*: \mathcal{O}(G) \rightarrow \mathcal{Z}_0(\mathcal{O}_\epsilon)$  (see Theorem 2.15 (1)).

**Proposition 4.1.**  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$ , and therefore  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  is isomorphic to  $\mathcal{O}(G)$ .

*Proof.* The claim follows from the fact that  $\Phi_1: \mathcal{L}_{0,1}^\epsilon \rightarrow U_\epsilon^{lf}$  is an isomorphism (see (18)), the identity  $\Phi_1 = m \circ (id \otimes S^{-1}) \circ \Phi$ , and Theorem 2.15 (2).

Here is an alternative proof of the isomorphism  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \cong \mathcal{O}(G)$ , not using  $\eta^*$ . Recall the notations introduced before Theorem 2.14. As varieties  $H = U_+ T_G U_- = G^0$ , so the map  $\sigma$  yields identifications  $\mathcal{O}(H) = \mathcal{O}(U_+) \mathcal{O}(T_G) \mathcal{O}(U_-)$  and  $\mathcal{O}(G^0) = \mathcal{O}(U_+) \mathcal{O}(T_G / (2)) \mathcal{O}(U_-)$ ; we can identify  $\mathcal{O}(G^0)$  with the subalgebra  $\sigma_H^*(\mathcal{O}(G^0))$  of  $\mathcal{O}(H)$ . Consider the space  $V = \wedge^N \mathfrak{g}$ , endowed with the action of  $G$  given on each factor by the adjoint representation. Put on  $\mathfrak{g}$  a basis consisting of one element  $e_\alpha$  per root space  $\mathfrak{g}_\alpha$ , along with a basis of  $\mathfrak{h}$ . Let  $v \in V$  be the exterior power of the  $e_\alpha$ 's for  $\alpha$  negative, and  $v^*$  a dual vector such that  $v^*(v) = 1$  and  $v^*$  vanishes on a  $T_G$ -invariant complement of  $v$ . It is classical that  $G \setminus G^0$  has defining equation



$\delta(g) = 0$ , where  $\delta$  is the matrix coefficient  $\delta(g) = v^*(\pi_V(g)v)$  (see eg. [50], page 174). Hence  $\mathcal{O}(G^0) = \mathcal{O}(G)[\delta^{-1}]$ . On  $G^0$  we have  $\delta(u_+tu_-) = \chi_{-\rho}(t)$ , where  $\chi_{-\rho}$  is the character of  $T_G$  associated to the root  $-\rho$ . Now we can make the connection with  $U_\epsilon$ . The isomorphism  $\mathcal{Z}_0(U_\epsilon) \cong \mathcal{O}(H)$  of Theorem 2.14 (2) identifies  $\mathcal{Z}_0(U_\epsilon) \cap U_\epsilon(\mathfrak{h}) = \mathbb{C}[T^{(l)}]$  with  $\mathcal{O}(T_G)$  by mapping  $K_\lambda$  to the character of  $T_G$  associated to  $\lambda$ . Therefore it maps  $\mathbb{C}[T_2^{(l)}]$  to  $\mathcal{O}(T_G/(2))$ ,  $\ell^l = K_{l\rho}$  to  $\chi_\rho$ , and  $T_2^{(l)-1}\mathcal{Z}_0(U_\epsilon^{lf}) = \mathcal{Z}_0(U_\epsilon^{lf})[\ell^l]$  to  $\mathcal{O}(G^0)$  by (41). Since  $\mathcal{O}(G^0) = \mathcal{O}(G)[\delta^{-1}]$  and  $\delta^{-1} = \chi_\rho$  on  $G^0$ , it follows that  $\mathcal{Z}_0(U_\epsilon^{lf}) \cong \mathcal{O}(G)$ . Then  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \cong \mathcal{O}(G)$  by injectivity of  $\Phi_1$ .  $\square$

Consider the linear subspace of  $\mathcal{L}_{0,n}^\epsilon$  defined by

$$\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) = \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}.$$

By Proposition 4.1 we have an isomorphism of algebras  $(\eta^{*-1})^{\otimes n}: \mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) \rightarrow \mathcal{O}(G)^{\otimes n}$ .

**Proposition 4.2.** (1)  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a central subalgebra of  $\mathcal{L}_{0,n}^\epsilon$ , and  $\mathcal{L}_{0,n}^\epsilon = \mathcal{O}_\epsilon^{\otimes n}$  as modules over  $\mathcal{O}(G)^{\otimes n}$ . Moreover  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a Noetherian ring.

(2) The  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module  $\mathcal{L}_{0,n}^\epsilon$  is generated by the elements of the spaces  $C([\mu])$  where  $[\mu] = (\mu_1, \dots, \mu_n) \in P_+^n$  satisfies  $0 \leq (\mu_i, \check{\alpha}_j) < l$  for every  $i \in \{1, \dots, n\}$ ,  $j \in \{1, \dots, m\}$ , where as usual  $\check{\alpha}_j := 2\alpha_j/(\alpha_j, \alpha_j)$ . In particular it is a finite Noetherian  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module, and therefore a Noetherian ring.

*Proof.* (1) Recall the formula (6). It is enough to consider the case where  $\alpha = \beta' = 1$ , and show that  $(1 \otimes (\beta)^{(b)})((\alpha)^{(a)} \otimes 1) = (\alpha)^{(a)} \otimes (\beta)^{(b)}$  whenever  $\alpha \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , for this is trivially equal to  $((\alpha)^{(a)} \otimes 1)(1 \otimes (\beta)^{(b)})$  (similar arguments apply if instead  $\beta \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  and  $\alpha$  is arbitrary). We have (denoting  $\sum_{(\alpha), (\alpha), (\alpha), (\alpha)}$  by  $\sum_{(\alpha)^4}$ ,  $\Delta(\alpha_{(1)}) = \sum_{(\alpha)} \alpha_{(1)(1)} \otimes \alpha_{(1)(2)}$  etc.):

$$\begin{aligned} (1 \otimes (\beta)^{(b)})((\alpha)^{(a)} \otimes 1) &= \sum_{(R^i)} \left( S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha \triangleleft R_{(1)}^1 R_{(1)}^2 \right)^{(a)} \\ &\quad \otimes \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \right)^{(b)} \\ &= \sum_{(R^i), (\alpha)^4, (\beta)^2} (\alpha_{(2)})^{(a)} \otimes (\beta_{(2)})^{(b)} \\ &\quad \times \beta_{(1)} \left( \alpha_{(1)(2)}(R_{(1)}^2) R_{(2)}^2 \alpha_{(3)(1)}(S(R_{(1)}^4)) R_{(2)}^4 \right) \\ &\quad \times \beta_{(3)} \left( \alpha_{(3)(2)}(R_{(1)}^3) R_{(2)}^3 \alpha_{(1)(1)}(R_{(1)}^1) S(R_{(2)}^1) \right). \end{aligned}$$

By Theorem 2.15 (2) it follows that  $\alpha_{(1)(2)}(R_{(1)}^2) R_{(2)}^2 = \Phi^+(\alpha_{(1)(2)}) \in \mathcal{Z}_0(U_\epsilon)$ , and similarly  $\alpha_{(3)(1)}(S(R_{(1)}^4)) R_{(2)}^4$ ,  $\alpha_{(3)(2)}(R_{(1)}^3) R_{(2)}^3$ ,  $\alpha_{(1)(1)}(R_{(1)}^1) S(R_{(2)}^1) \in \mathcal{Z}_0(U_\epsilon)$ . Denote by  $z$  any such element. Note that  $\mathcal{Z}_0(U_\epsilon)$  acts by the trivial character on  $\Gamma$ -modules, and that the expression of  $z$  in terms of the corresponding  $\alpha_{(i)(j)}$  implies  $\epsilon(z) = \epsilon(\alpha_{(i)(j)})$ . Then

$$\begin{aligned} \beta_{(1)} \left( \alpha_{(1)(2)}(R_{(1)}^2) R_{(2)}^2 \alpha_{(3)(1)}(S(R_{(1)}^4)) R_{(2)}^4 \right) &= \epsilon(\alpha_{(1)(2)} \alpha_{(3)(1)}) \beta_{(1)}(1) \\ &= \epsilon(\alpha_{(1)(2)}) \epsilon(\alpha_{(3)(1)}) \epsilon(\beta_{(1)}) \\ \beta_{(3)} \left( \alpha_{(3)(2)}(R_{(1)}^3) R_{(2)}^3 \alpha_{(1)(1)}(R_{(1)}^1) S(R_{(2)}^1) \right) &= \epsilon(\alpha_{(3)(2)}) \epsilon(\alpha_{(1)(1)}) \epsilon(\beta_{(3)}) \end{aligned}$$

and finally  $(1 \otimes (\beta)^{(b)})((\alpha)^{(a)} \otimes 1) = (\alpha)^{(a)} \otimes (\beta)^{(b)}$ . Therefore  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{(a)}$  is central in  $\mathcal{L}_{0,n}^\epsilon$  for all  $a = 1, \dots, n$ . These algebras generate  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  in  $(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$ , and hence in  $\mathcal{L}_{0,n}^\epsilon$  (this follows from the comment before (7)). Therefore  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is central in  $\mathcal{L}_{0,n}^\epsilon$ . A computation

similar to the above one, based on the formula (3) instead of (6), shows that  $\alpha\beta = \alpha \star \beta$  whenever  $\alpha \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ . Hence  $\mathcal{L}_{0,1}^\epsilon$  and  $\mathcal{O}_\epsilon$  coincide as modules over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$ . The second claim follows immediately; for instance when  $n = 2$ , given  $\alpha', \beta' \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  we have  $(\alpha' \otimes \beta')(\alpha \otimes \beta) = (\alpha' \otimes 1)(1 \otimes \beta')(\alpha \otimes 1)(1 \otimes \beta)$  immediately by (6), and  $(1 \otimes \beta')(\alpha \otimes 1) = \alpha \otimes \beta' = (\alpha \otimes 1)(1 \otimes \beta')$  as above. Then  $(\alpha' \otimes \beta')(\alpha \otimes \beta) = \alpha' \alpha \otimes \beta' \beta$ . Finally  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a Noetherian ring as it is isomorphic to  $\mathcal{O}(G^n)$  (Proposition 4.1), the coordinate ring of an affine algebraic variety.

(2) Let  $[\lambda] = (\lambda_1, \dots, \lambda_n) \in P_+^n$ . For all  $i \in \{1, \dots, n\}$  there are unique  $\lambda_{i0}, \lambda_{i1} \in P_+$  such that  $\lambda_i = \lambda_{i0} + l\lambda_{i1}$  and  $0 \leq (\lambda_{i0}, \alpha_j) < l$  for every  $j \in \{1, \dots, m\}$ . Then set  $[\lambda_0] = (\lambda_{10}, \dots, \lambda_{n0})$ ,  $[\lambda_1] = (\lambda_{11}, \dots, \lambda_{n1})$ . We have  $C([\lambda]) = C([\lambda_0]) \circ C([\lambda_1])$  by (36), and  $C(l\lambda_{i1}) \subset \mathcal{Z}_0(\mathcal{O}_\epsilon) = \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  for all  $i \in \{1, \dots, n\}$  by Theorem 2.15 (2) and Proposition 4.1, so  $C([\lambda_1]) \subset \mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ . It follows that  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n}^\epsilon) = \bigoplus_{[\lambda] \in P_+^n} C([\lambda])$  is generated over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  by the elements of the spaces  $C([\lambda_0])$ ,  $[\lambda] \in P_+^n$ . By step (e') of the proof of Theorem 3.2, formulated for modules instead of algebras, it follows that  $\mathcal{L}_{0,n}^\epsilon$  is generated over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  by the same elements. The last claims follow in a standard way: since  $\mathcal{L}_{0,n}^\epsilon$  is a finitely generated  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module and  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is Noetherian,  $\mathcal{L}_{0,n}^\epsilon$  is a Noetherian  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module (eg. by [7] Proposition 6.5). Moreover  $\mathcal{L}_{0,n}^\epsilon$  is a Noetherian ring (by eg. [69], 1.3).  $\square$

Note that we had already obtained independently the Noetherianity of the ring  $\mathcal{L}_{0,n}^\epsilon$  as a consequence of the Noetherianity of  $\mathcal{L}_{0,n}^A$  (Proposition 3.1).

We need below explicit descriptions of the  $\mathcal{Z}$ - and  $\mathcal{Z}_0$ -centers for  $\mathfrak{g} = sl(2)$ . Let us recall a few facts in this case. Denote by  $a, b, c, d$  the standard generators of  $\mathcal{O}_q(SL_2)$ , ie. the matrix coefficients in the basis of weight vectors  $v_0, v_1 = F.v_0$  of the 2-dimensional irreducible representation  $V_2$  of  $U_q(sl(2))$ . Denote by  $x^{*k}$ ,  $k \in \mathbb{N}$ , the  $k$ -th power of an element  $x \in \mathcal{O}_A(SL_2)$ . The algebra  $\mathcal{O}_A(SL_2)$  is generated by  $a, b, c, d$ ; the monomials  $a^{*i} \star b^{*j} \star d^{*r}$  and  $a^{*i} \star c^{*k} \star d^{*r}$ ,  $i, j, k, r \in \mathbb{N}, k > 0$ , form an  $A$ -basis of  $\mathcal{O}_A(SL_2)$ . The algebra  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$  is generated by  $a^{*l}, b^{*l}, c^{*l}, d^{*l}$ ; the monomials  $a^{*il} \star b^{*jl} \star d^{*rl}$  and  $a^{*il} \star c^{*kl} \star d^{*rl}$  form a basis of  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ , and  $\mathcal{Z}(\mathcal{O}_\epsilon(SL_2))$  is generated by  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$  and the elements  $b^{*(l-k)} \star c^{*k}$ ,  $k = 0, \dots, l$  (see [42], Proposition 1.4 and the Appendix). We have the relation

$$(42) \quad a^{*l} \star d^{*l} - b^{*l} \star c^{*l} = 1$$

and the Frobenius isomorphism of Parshall-Wang (see [65], Chapter 7) coincides with the map

$$Fr_{PW}: \mathcal{O}(SL_2) \rightarrow \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$$

induced by  $\eta^*$ ; it sends the standard generators  $\underline{a}, \underline{b}, \underline{c}, \underline{d}$  of  $\mathcal{O}(SL_2) = \mathcal{O}_1(SL_2)$  respectively to  $a^{*l}, b^{*l}, c^{*l}, d^{*l}$ . Finally, let us quote from [43] that a basis of the rank  $l^3$  free  $\mathcal{Z}_0(\mathcal{O}(SL_2))$ -module  $\mathcal{O}_\epsilon(SL_2)$  (see Theorem 2.15 (3)) is formed by the monomials  $a^m b^n c^{s'}$  and  $b^n c^{s''} d^r$ , with the integers  $m, n, r, s', s''$  in the range

$$(43) \quad 1 \leq m \leq l-1, 0 \leq n, r \leq l-1, m \leq s' \leq l-1, 0 \leq s'' \leq l-r-1.$$

Now consider  $\mathcal{L}_{0,1}^A(sl(2))$ . Recall that  $\mathcal{L}_{0,1}^A = \mathcal{O}_A$  as  $U_A$ -modules. The algebra  $\mathcal{L}_{0,1}^A(sl(2))$  is also generated by  $a, b, c, d$ ; a set of defining relations is (see [28], Section 5):

$$(44) \quad \begin{aligned} ad &= da & , & \quad ab - ba = -(1 - q^{-2})bd \\ db &= q^2bd & , & \quad cb - bc = (1 - q^{-2})(da - d^2) \\ cd &= q^2dc & , & \quad ac - ca = (1 - q^{-2})dc \\ & & & \quad ad - q^2bc = 1. \end{aligned}$$

The element  $\omega := qa + q^{-1}d$  is central. Let  $T_k$ ,  $k \in \mathbb{N}$ , be such that  $T_k(x)/2$  is the  $k$ -th Chebyshev polynomial of the first type in the variable  $x/2$ . We have (see [28], Proposition 7.3, for the generalization to  $\mathcal{L}_{0,n}^\epsilon(sl(2))$ ):

**Lemma 4.3.**  $\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon(sl(2))) = \mathbb{C}[\omega, b^l, c^l, d^l]/\mathcal{I}$  and  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2))) = \mathbb{C}[(T_l(\omega), b^l, c^l, d^l)/\mathcal{I}]$ , where  $\mathcal{I}$  is the ideal of  $\mathbb{C}[\omega, b^l, c^l, d^l]$  generated by  $(T_l(\omega) - d^l)d^l - b^l c^l - 1$ .

Here  $b^l, c^l, d^l$  are the  $l$ -th powers of  $b, c, d$  computed using the product of  $\mathcal{L}_{0,1}^A(sl(2))$ , not the product  $\star$  of  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ . The above generator of  $\mathcal{I}$  can be interpreted as a determinant, and  $\omega$  as a quantum trace on  $V_2$ .

**Lemma 4.4.** Viewed as element of  $\mathcal{O}_A(SL_2)$ ,  $T_l(\omega) - d^l = a^{*l}$  and  $x^l = x^{*l}$ ,  $x \in \{b, c, d\}$ .

*Proof.* Let  $\alpha$  and  $\varpi$  be the simple root and fundamental weight of  $sl(2)$ . In the notations of (22) we have  $b = \psi_{-\varpi}^{-\alpha}$ ,  $c = \psi_{-\varpi}^{\alpha}$ ,  $d = \psi_{-\varpi}$ ; the formulas give  $\Phi_1(b^{*l}) = (q - q^{-1})^l F^l$ ,  $\Phi_1(c^{*l}) = (q - q^{-1})^l E^l K^{-l}$ ,  $\Phi_1(d^{*l}) = K^{-l}$ . These coincide respectively with  $\Phi_1(b^l)$ ,  $\Phi_1(c^l)$ ,  $\Phi_1(d^l)$  (see (32) in [28]). By passing to the localization  $\mathcal{O}_A(SL_2)[d^{-1}]$ , and using Parshall-Wang's relation (42), one deduces easily  $\Phi_1(a^{*l}) = K^l + (q - q^{-1})^{2l} F^l E^l = T_l(\Omega) - K^{-l}$ , where  $\Omega$  is  $(q - q^{-1})^{2l}$  times the Casimir element of  $U_q(sl(2))$ , and  $T_l(x)/2$  is the  $l$ -th Chebyshev polynomial of the first type in the variable  $x/2$ . We have  $\Phi_1(\omega) = \Omega$ , so  $\Phi_1(a^{*l}) = T_l(\omega) - d^l$ . The conclusion follows from the injectivity of  $\Phi_1$ .  $\square$

This lemma proves that we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}(SL_2) & \xrightarrow{Fr_{PW}} & \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2)) & \hookrightarrow & \mathcal{O}_\epsilon(SL_2) \\ & \searrow^{Fr} & \downarrow & & \downarrow \\ & & \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2))) & \hookrightarrow & \mathcal{L}_{0,1}^\epsilon(sl(2)) \end{array}$$

where  $Fr_{PW}$  is Parshall-Wang's Frobenius isomorphism recalled above,  $Fr$  is the Frobenius isomorphism introduced in [28], Definition 7.2, and the vertical arrows are the identifications as vector spaces (the middle one proved by Proposition 4.2).

**Remark 4.5.** By Lemma 4.3 the monomials  $T_l(\omega)^i b^{jl} d^{rl}$  and  $T_l(\omega)^i c^{kl} d^{rl}$ , for  $i, j, k, r \in \mathbb{N}$  and  $k > 0$ , form an  $A$ -basis of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$ . It is straightforward (though cumbersome) to express these basis elements in terms of the basis elements  $a^{*il} \star b^{*jl} \star d^{*rl}$  and  $a^{*il} \star c^{*kl} \star d^{*rl}$  of  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ , and conversely; this can be done by using Lemma 4.4, the formula (3) or the inverse one (expressing  $\star$  in terms of the product of  $\mathcal{L}_{0,1}$ , see (18) in [28]), and the formula of the coproduct  $\Delta: \mathcal{L}_{0,1}^\epsilon(sl(2)) \rightarrow \mathcal{L}_{0,2}^\epsilon(sl(2))$  restricted to  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$  (given in Proposition 6.15 and Lemma 7.7 of [28]).

Since  $\mathcal{L}_{0,1}^A = \mathcal{O}_A$  as an  $A$ -module, the functionals  $t_i$  in Proposition 2.16 can be seen as maps  $t_i: \mathcal{L}_{0,1}^A \rightarrow A$ . Though the algebra structures of  $\mathcal{O}_\epsilon$  and  $\mathcal{L}_{0,1}^\epsilon$  are very different, we have the analogous result:

**Proposition 4.6.** The maps  $\triangleleft t_i$  preserve  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , and they satisfy  $(f \triangleleft t_i)(a) = f(n_i a)$  and  $(f \alpha) \triangleleft t_i = (f \triangleleft t_i)(\alpha \triangleleft t_i)$  for every  $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ ,  $a \in G$ ,  $\alpha \in \mathcal{L}_{0,1}^\epsilon$ .

*Proof.* The first two claims follow from Proposition 2.16 and the equality  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$  in Proposition 4.1.

The last claim follows from the case  $\mathfrak{g} = sl(2)$ , as in the proof of Proposition 7.1 of [42]. In fact it is enough to show that  $t(fg) = t(f)t(g)$  for every  $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$ ,  $g \in \mathcal{L}_{0,1}^\epsilon(sl(2))$ ; for completeness we explain this in the Appendix, see (66). A word of caution

is needed: the proof of (66) uses that  $\Delta: \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon \otimes \mathcal{O}_\epsilon$  is a morphism of algebras. The analogous property for  $\mathcal{L}_{0,1}^\epsilon$  is that  $\Delta$  yields a morphism of algebras  $\Delta: \mathcal{L}_{0,1}^\epsilon \rightarrow \mathcal{L}_{0,2}^\epsilon$ . Since the algebra structure of  $\mathcal{L}_{0,2}^\epsilon$  is not the product one on  $\mathcal{L}_{0,1}^\epsilon \otimes \mathcal{L}_{0,1}^\epsilon$ , it is not true in general that  $\sum_{(f),(g)} (f_{(1)} \otimes f_{(2)})(g_{(1)} \otimes g_{(2)}) = \sum_{(f),(g)} f_{(1)}g_{(1)} \otimes f_{(2)}g_{(2)}$  for every  $f, g \in \mathcal{L}_{0,1}^\epsilon$ . However it holds whenever  $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , since  $\Delta(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)) \subset \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \otimes \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  and therefore  $f_{(2)} \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$  commutes in  $\mathcal{L}_{0,2}^\epsilon$  with any  $g_{(1)} \in \mathcal{L}_{0,1}^\epsilon = \mathcal{O}_\epsilon$ .

It is enough to prove the identity  $t(fg) = t(f)t(g)$  when  $f$  ranges in a set of generators of the algebra  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$ . So one can take  $f$  among, say,  $T_l(\omega) - d^l = a^{*l}$  and  $x^l = x^{*l}$ ,  $x \in \{b, c, d\}$  (using Lemma 4.3). By (3) and Proposition 6.1 in the Appendix we have

$$t(fg) = \sum_{(R),(R')} t(R_{(2')}S(R_{(2)}) \triangleright f) t(R_{(1')} \triangleright g \triangleleft R_{(1)}).$$

Expanding coproducts and using that  $R^{-1} = (S \otimes id)(R)$  we deduce

$$\begin{aligned} t(fg) &= \sum_{(f),(R),(R')} t(f_{(1)}) \langle f_{(2)}, R_{(2')}S(R_{(2)}) \rangle t(R_{(1')} \triangleright g \triangleleft R_{(1)}) \\ &= \sum_{(f),(R),(R')} t(f_{(1)}) t\left(\langle f_{(2)}, R_{(2')} \rangle R_{(1')} \triangleright g \triangleleft \langle f_{(3)}, S(R_{(2)}) \rangle R_{(1)}\right) \\ &= \sum_{(f)} t(f_{(1)}) t\left(S^{-1}(\Phi^-(f_{(2)})) \triangleright g \triangleleft S^{-2}(\Phi^-(f_{(3)}))\right) \\ &= \sum_{(f)} t(f_{(1)}) \left\langle g, S^{-2}(\Phi^-(f_{(3)})) \underline{w} S^{-1}(\Phi^-(f_{(2)})) \right\rangle \\ &= \sum_{(f)} t(f_{(1)}) \varepsilon\left(S^{-2}(\Phi^-(f_{(3)}))\right) \varepsilon\left(S^{-1}(\Phi^-(f_{(2)}))\right) t(g) \end{aligned}$$

where  $\underline{w} \in \mathbb{U}_\Gamma$  is the quantum Weyl group element dual to  $t$  (see Section 6.1), and in the last equality we used that  $\Phi^-$  maps  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  into  $\mathcal{Z}_0(U_\epsilon)$  (see Theorem 2.15 (2)), which acts on  $\Gamma$ -modules by the trivial character (the counit)  $\varepsilon: U_\epsilon \rightarrow \mathbb{C}$ . By (58)-(59) in the Appendix we have  $t(a^{*l}) = t(d^{*l}) = 0$  and  $t(b^{*l}) = 1$ ,  $t(c^{*l}) = -1$ . Now the computation of  $t(fg)$  follows easily. For instance, taking  $f = b^l = b^{*l}$ , by using  $\Delta(b^{*l}) = a^{*l} \otimes b^{*l} + b^{*l} \otimes d^{*l}$  and  $\Delta(d^{*l}) = c^{*l} \otimes b^{*l} + d^{*l} \otimes d^{*l}$  we get

$$t(b^l g) = \varepsilon\left(S^{-2}(\Phi^-(b^{*l}))\right) \varepsilon\left(S^{-1}(\Phi^-(c^{*l}))\right) t(g) + \varepsilon\left(S^{-2}(\Phi^-(d^{*l}))\right) \varepsilon\left(S^{-1}(\Phi^-(d^{*l}))\right) t(g)$$

Since  $b^{*l} \in \mathcal{O}_\epsilon(U_+)$ ,  $\Phi^-(b^{*l}) = 0$ . Also, it is immediate from the definition of  $\Phi^-$  that  $\Phi^-(d^{*l}) = \Phi^-(d^l) = L^l$ ; alternatively, one can bypass this computation by observing that  $\Phi^-$  sets an isomorphism from  $\mathcal{O}_\epsilon(T_G) = \mathcal{O}_\epsilon(B_+) \cap \mathcal{O}_\epsilon(B_-)$  to  $\mathbb{C}[L^{\pm 1}] = U_\epsilon(\mathfrak{b}_+) \cap U_\epsilon(\mathfrak{b}_-)$ , mapping a generator  $d$  to  $L$  or  $L^{-1}$ . We have  $\varepsilon(L^l) = 1$ , and therefore

$$t(b^l g) = t(g) = t(b^l)t(g).$$

The other cases  $f = T_l(\omega) - d^l, c^l, d^l$  are similar.  $\square$

**Theorem 4.7.**  $\mathcal{L}_{0,n}^\epsilon$  is a free  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module of rank  $l^{n \cdot \dim \mathfrak{g}}$ , and  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is a Noetherian ring and a finite, whence Noetherian,  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module.

We will see in Section 5 (proof of Theorem 1.3 (2) and (3)) that in fact  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is finite free of rank  $l^{(n-1)\dim\mathfrak{g}+m}$  over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ .

*Proof of Theorem 4.7.* By Proposition 4.2 (1),  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{O}_\epsilon^{\otimes n}$  coincide as modules over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon^{\otimes n})$ , so the first claim follows immediately from Theorem 7.2 of [42], which shows that  $\mathcal{O}_\epsilon$  is a finitely generated projective module of rank  $l^{\dim\mathfrak{g}}$  over  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ , and the arguments of [68] and [25], which imply that this module is free. Alternatively, it follows from the fact that  $\mathcal{O}_\epsilon$  is a cleft extension of  $\mathcal{O}(G)$  (see [6] and [21]). For the second claim, since  $\mathcal{L}_{0,n}^\epsilon$  is a Noetherian  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module, the  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -submodule  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is necessarily finitely generated. But  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  being Noetherian,  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is then a Noetherian  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module and a Noetherian ring.

For the sake of clarity let us provide a self-contained proof of the first claim, not invoking directly [42, 25] or [6, 21]. Since  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{L}_{0,1}^{\otimes n}$  coincide as modules over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) = \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$ , the result follows from the case  $n = 1$ . Then we argue in four steps. First, using Theorem 2.2 we show that a certain localization of  $\mathcal{L}_{0,1}^\epsilon$  is a free module of rank  $l^{\dim\mathfrak{g}}$ . Then, assuming that  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective, we explain why it has constant rank  $l^{\dim\mathfrak{g}}$  (this is very classical). Thirdly, we prove that  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective as in Theorem 7.2 of [42]. Finally we obtain that it is a free module as in [25].

We have  $T_2^{(l)-1}U_\epsilon^{lf} = T_2^{-1}U_\epsilon^{lf} = U_\epsilon^{lf}[\ell] = U_\epsilon^{lf}[\ell^l]$ , where  $\ell$  is the pivotal element. Then Theorem 2.14 (1) and (41) imply that  $U_\epsilon$  is a free  $\mathcal{Z}_0(U_\epsilon^{lf})[\ell^l]$ -module of rank  $2^m l^{\dim\mathfrak{g}}$ , and Theorem 2.2 (2) says that it is also the direct sum of  $2^m$  copies of the (free)  $\mathcal{Z}_0(U_\epsilon^{lf})[\ell^l]$ -module  $U_\epsilon^{lf}[\ell^l]$ . The decomposition being unique, it follows that  $U_\epsilon^{lf}[\ell^l]$  is free of rank  $l^{\dim\mathfrak{g}}$  over  $\mathcal{Z}_0(U_\epsilon^{lf})[\ell^l]$ . Pulling this back via  $\Phi_1$ , this proves  $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$  is free of rank  $l^{\dim\mathfrak{g}}$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$ , where  $d = \psi_{-\rho} = \Phi_1^{-1}(\ell)$  (see Corollary 2.4, Theorem 2.7).

Here we note that, by the first formula of Theorem 2.11 (3), taking powers with respect to the product of  $\mathcal{L}_{0,1}$  we have

$$\psi_{-\rho}^l = \psi_{-l\rho}.$$

Assume that  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective. Let us show that its rank is  $l^{\dim\mathfrak{g}}$ . The localization  $(\mathcal{L}_{0,1}^\epsilon)_P$  of  $\mathcal{L}_{0,1}^\epsilon$  at any prime ideal  $P$  of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  is a free module over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)_P$  ([72], Proposition 2.12.15); the ranks of such modules are finite in number ([72], Proposition 2.12.20). If these ranks are all equal, then, by definition, it is the rank of  $\mathcal{L}_{0,1}^\epsilon$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ . This happens if  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  has no non-trivial (ie.  $\neq 1$ ) idempotent ([72], Corollary 2.12.23), which is the case since it has non non-trivial zero divisors. To compute the rank, suppose  $P$  does not contain  $d^l$ . Such ideals  $P$  are in 1-1 correspondence with the prime ideals of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$  by the natural ring monomorphism  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \rightarrow \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$ . The set  $S = \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \setminus P$  is multiplicatively closed, and we have also a ring morphism  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}] \rightarrow S^{-1}\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , which is also an injection (there are no zero divisors in  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , whence in  $S$ ). Then

$$(45) \quad (\mathcal{L}_{0,1}^\epsilon)_P = S^{-1}\mathcal{L}_{0,1}^\epsilon = \mathcal{L}_{0,1}^\epsilon[d^{-l}] \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]} S^{-1}\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$$

shows that  $(\mathcal{L}_{0,1}^\epsilon)_P$  has over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)_P = S^{-1}\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  the same rank  $l^{\dim\mathfrak{g}}$  as  $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$ . This proves our claim.

In order to show that  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  it is enough to show it is finite locally free, ie. there are elements  $d_i \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  such that the localizations  $\mathcal{L}_{0,1}^\epsilon[d_i^{-1}]$  are finite free  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d_i^{-1}]$ -modules, and  $\text{Maxspec}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))$  is covered by the open sets  $U(d_i)$  made of the ideals not containing  $d_i$  (see Lemma 77.2 of [78]).

We have seen above that  $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$  is free of rank  $l^{\dim \mathfrak{g}}$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$ . In the proof of Proposition 4.1 we saw that there are isomorphisms  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}] \cong \mathcal{Z}_0(U_\epsilon^{lf})[l^l] \cong \mathcal{O}(G^0)$ , and  $\mathcal{O}(G^0) = \mathcal{O}(G)[\delta^{-1}]$ . Now, given  $w \in W$  with a reduced expression  $s_{i_1} \dots s_{i_k}$ , put  $t_w = t_{i_1} \dots t_{i_k}$ . Let  $w$  be represented by  $n_w = n_{i_1} \dots n_{i_k}$  in  $N(T_G)$ . By Proposition 4.6 we have  $(f \triangleleft t_w)(x) = f(n_w x)$  for every  $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ ,  $x \in G$ . Then

$$\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}] \triangleleft t_w \cong \mathcal{O}(n_w^{-1}G^0) \cong \mathcal{O}(G)[(\delta \triangleleft t_w)^{-1}].$$

If  $b_1, \dots, b_r$  ( $r := l^{\dim \mathfrak{g}}$ ) is a basis of  $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$ , then  $\mathcal{L}_{0,1}^\epsilon[d^{-l}] \triangleleft t_w$  is free over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[(\delta \triangleleft t_w)^{-1}] \cong \mathcal{O}(n_w^{-1}G^0)$  with basis  $b_1 \triangleleft t_w, \dots, b_r \triangleleft t_w$ . Consider the Bruhat decomposition of  $G$ : any  $g \in G$  can be written in the form  $g = b_1 n b_2$ , where  $b_1, b_2 \in B_-$ ,  $n \in W$ . Hence we have  $g = n n^{-1} b_1 n b_2 \in n B_+ B_- = n G^0$ , and therefore

$$G = \cup_{w \in W} (B_- n_w B_-) = \cup_{w \in W} (n_w G^0).$$

For every  $w \in W$  put

$$d_w^l := d^l \triangleleft t_w.$$

Under the isomorphism of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  with  $G$ , we thus get that  $\text{Maxspec}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))$  is covered by the open sets  $U(d_w^l) \cong n_w G^0$ , and  $\mathcal{L}_{0,1}^\epsilon[d_w^{-l}]$  is finite free over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d_w^{-l}]$ . Therefore  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ .

Finally, let us explain why  $\mathcal{L}_{0,1}^\epsilon$  is free over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , following the arguments of [25]. Let  $R$  be a commutative Noetherian ring, put  $X = \text{Maxspec}(R)$ , and let  $P$  be an  $R$ -module. Denote by  $R_I, P_I$  the localizations of  $R, P$  at a maximal ideal  $I \in X$ . Define the  $f$ -rank of  $P$  as  $\text{f-rank}(P) = \inf_{I \in X} \{ \text{f-rank}_{R_I}(P_I) \}$ , where  $\text{f-rank}_{R_I}(P_I) = \sup \{ r \in \mathbb{N}, R_I^{\otimes r} \subset P_I \} \in \mathbb{N} \cup \{+\infty\}$  (ie. the maximal dimension of a free summand of  $P_I$ ). Bass' Cancellation theorem asserts that if  $P$  is projective and  $\text{f-rank}(P) > \dim(X)$ , and  $P \oplus Q \cong M \oplus Q$  for some  $R$ -modules  $Q$  and  $M$  such that  $Q$  is finitely generated and projective, then  $P \cong M$  (see [10], IV.3.5 and pages 167 and 170, taking  $A = R$ , or [69], section 11.7.13). Let us apply this to  $R = \mathcal{O}(G)$  and  $P = \mathcal{L}_{0,1}^\epsilon$ . We proved above that  $\text{f-rank}_{R_I}(P_I) = l^{\dim \mathfrak{g}}$ , a constant, and we have  $l^{\dim \mathfrak{g}} > \dim(G) = \dim(X)$ . By a result of Marlin [68], the Grothendieck ring  $K_0(\mathcal{O}(G))$  is isomorphic to  $\mathbb{Z}$ . Therefore  $\mathcal{L}_{0,1}^\epsilon \oplus Q \cong \mathcal{O}(G)^r$  for some free  $\mathcal{O}(G)$ -module  $Q$  and  $r \in \mathbb{N}$ . Then Bass' Cancellation implies  $\mathcal{L}_{0,1}^\epsilon$  is free over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \cong \mathcal{O}(G)$ .  $\square$

## 5. PROOF OF THEOREM 1.3

We begin with two lemmas, interesting in themselves.

**Lemma 5.1.**  *$\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is a finite  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module and a Noetherian ring. Therefore the ring  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is integral over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ .*

*Proof.* We know that  $\mathcal{L}_{0,n}^\epsilon$  is finite over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  (Theorem 4.7), and  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a Noetherian ring (Proposition 4.2). Therefore  $\mathcal{L}_{0,n}^\epsilon$  is a Noetherian  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module. This implies that the submodule  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is finitely generated. But being finite over the Noetherian ring  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ , it is a Noetherian ring (by eg. Proposition 7.2 of [7]).

Let  $x \in \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ . The  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -submodule  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)[x]$  of  $\mathcal{L}_{0,n}^\epsilon$  is finitely generated by the same argument. Using the fact that an element  $x$  is integral over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  if and only if  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)[x]$  is a finitely generated  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module (by eg. Proposition 5.1 of [7]), this proves the last claim.  $\square$

As usual, denote by  $Q(Z)$  the quotient field of a commutative integral domain  $Z$ . Then, consider the fields  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  and  $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ . Since  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is finite over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  and

has no non-trivial zero divisors, the ring  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon) \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)} Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$  is a field. Therefore it is equal to  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ .

**Lemma 5.2.**  $[Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = l^{mn}$ .

*Proof.* First consider the case  $n = 1$ . Note that  $Q(\mathcal{Z}(U_\epsilon)) = Q(\mathcal{Z}(U_\epsilon^{lf}))(T^{(l)}/T_2^{(l)})$  by (41), and similarly by replacing  $\mathcal{Z}$  with  $\mathcal{Z}_0$ . Then, applying  $\Phi_1$  and using that  $Q(\mathcal{Z}(U_\epsilon))$  has degree  $l^m$  over  $Q(\mathcal{Z}_0(U_\epsilon))$  (see the comment after Theorem 2.14) we deduce

$$(46) \quad [Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))] = [Q(\mathcal{Z}(U_\epsilon^{lf})) : Q(\mathcal{Z}_0(U_\epsilon^{lf}))] = l^m.$$

Next let  $n \geq 1$  be arbitrary. Recall from (16) the matrices  $M_j^{(i)}$ , for  $i = 1, \dots, n$  and  $j = 1, \dots, m$ . More generally, for every  $\lambda \in P_+$  and  $i = 1, \dots, n$  consider the matrix of matrix coefficients  $M_\lambda^{(i)} := (v_\lambda \phi_{e_k}^{e_i(i)})_{k,l} \in \text{End}(V_\lambda) \otimes \mathcal{L}_{0,n}^A$ , where  $v_\lambda \phi_{e_k}^{e_i(i)} := 1^{\otimes(i-1)} \otimes v_\lambda \phi_{e_k}^{e_i} \otimes 1^{\otimes(n-i)}$ , and  $\{e_k\}$  is the canonical basis of weight vectors of  $V_\lambda$ . In [28], Proposition 6.22, we showed that the elements  ${}_\lambda \omega^{(i)} := \text{Tr}(\pi_{V_\lambda}(\ell) M_\lambda^{(i)})$  are central in  $\mathcal{L}_{0,n}^A$ , where  $\text{Tr}$  is the standard trace on  $\text{End}(V_\lambda)$ ; moreover, the family  $\{{}_\lambda \omega^{(i)}, \lambda \in P_+\}$  is a basis of the center of the  $i$ -th factor  $\mathcal{L}_{0,1}^A$  of  $\mathcal{L}_{0,n}^A$ . Now, recall the graded algebra  $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})$  in (38). Take  $\lambda = \varpi_j$ . By (36), for every  $i \in \{1, \dots, n\}$ ,  $j \in \{1, \dots, m\}$  and  $r \in \mathbb{N}$  the leading coefficient of  $(\varpi_j \omega^{(i)})^r$  in  $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})$  belongs to the space  $C([0, \dots, 0, r\varpi_j, 0, \dots, 0])$  (with  $\varpi_j$  in the  $i$ -th entry). Take the specialization at  $q = \epsilon$  and denote by  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{(i)}$  the  $i$ -th factor of  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ . Recall that  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{(i)} = \bigoplus_{\mu \in P_+} C([0, \dots, 0, l\mu, 0, \dots, 0])$ . Then we get that the minimal polynomial of  $\varpi_j \omega^{(i)}$  over  $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)^{(i)})$  has degree  $l$ . On another hand, denoting by  $\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)^{(i)}$  the center of the  $i$ -th factor of  $\mathcal{L}_{0,n}^\epsilon$ , we have  $[Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)^{(i)}) : Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{(i)})] = l^m$  as in the  $n = 1$  case above. It follows that  $Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)^{(i)})$  is generated over  $Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{(i)})$  by the elements  ${}_\lambda \omega^{(i)}$ ,  $\lambda \in P_+$ . The fields  $Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)^{(i)})$  are supported by distinct tensor factors, so they are linearly disjoint subfields of  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ . The same is true of the fields  $Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{(i)})$ . Therefore, the compositum of the fields  $Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)^{(i)})$ , say  $Q'$ , has degree  $(l^m)^n$  over the compositum of the fields  $Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{(i)})$ , which is  $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ . By the same argument,  $Q(\mathcal{Z}(U_\epsilon^{\otimes n}))$  has degree  $l^{mn}$  over  $Q(\mathcal{Z}_0(U_\epsilon^{\otimes n}))$ . Now, recall from (18) the isomorphism of algebras  $\Phi_n : \text{loc}' \mathcal{L}_{0,n}^\epsilon \rightarrow U_\epsilon^{\otimes n}$ . Consider the induced isomorphism of fields  $\Phi_n : Q(\mathcal{Z}(\text{loc}' \mathcal{L}_{0,n}^\epsilon)) \rightarrow Q(\mathcal{Z}(U_\epsilon^{\otimes n}))$ , and the field extensions in  $Q(\mathcal{Z}(\text{loc}' \mathcal{L}_{0,n}^\epsilon))$ ,

$$\begin{aligned} \text{loc}' Q' &= Q'[\{\nu_{j_n}^{(n)-l}\}][\{\nu_{j_{n-1}}^{(n-1)-l}\}] \cdots [\{\nu_{j_1}^{(1)-l}\}] \\ \text{loc}' Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)) &= Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))[\{\nu_{j_n}^{(n)-l}\}][\{\nu_{j_{n-1}}^{(n-1)-l}\}] \cdots [\{\nu_{j_1}^{(1)-l}\}]. \end{aligned}$$

Compose  $\Phi_n$  with the linear automorphism of  $Q(\mathcal{Z}(U_\epsilon^{\otimes n}))$  induced by  $\psi_n^{-1} = \Phi_1^{\otimes n} \circ \Phi_n^{-1}$  (see (11) for the latter). The image of  $\text{loc}' Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$  by  $\psi_n^{-1} \circ \Phi_n$  is  $Q(\mathcal{Z}_0(U_\epsilon^{\otimes n}))$ . Since  $l^{mn} = [Q' : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = [\text{loc}' Q' : \text{loc}' Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = [\psi_n^{-1} \circ \Phi_n(\text{loc}' Q') : Q(\mathcal{Z}_0(U_\epsilon^{\otimes n}))]$  and  $[Q(\mathcal{Z}(U_\epsilon^{\otimes n})) : Q(\mathcal{Z}_0(U_\epsilon^{\otimes n}))] = l^{mn}$ , it follows that  $\psi_n^{-1} \circ \Phi_n(\text{loc}' Q') = Q(\mathcal{Z}(U_\epsilon^{\otimes n}))$ , whence  $\text{loc}' Q' = Q(\mathcal{Z}(\text{loc}' \mathcal{L}_{0,n}^\epsilon)) = \text{loc}' Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , and then  $Q' = Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ . This eventually proves  $l^{mn} = [Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))]$ .  $\square$

Recall the ring

$$Q(\mathcal{L}_{0,n}^\epsilon) = Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} \mathcal{L}_{0,n}^\epsilon.$$

The center of  $Q(\mathcal{L}_{0,n}^\epsilon)$  is  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ . By the comment before Lemma 5.2 we have another description of  $Q(\mathcal{L}_{0,n}^\epsilon)$ , namely

$$(47) \quad Q(\mathcal{L}_{0,n}^\epsilon) = Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)) \otimes_{Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))} \mathcal{L}_{0,n}^\epsilon.$$

Recall that we denote by  $N$  the number of positive roots of  $\mathfrak{g}$ .

**Proposition 5.3.**  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a central simple algebra of dimension  $l^{2Nn}$  over  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ .

*Proof.* From its definition  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a vector space over  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ . Because  $\mathcal{L}_{0,n}^\epsilon$  has no non-trivial divisors and  $Q(\mathcal{L}_{0,n}^\epsilon)$  is finite-dimensional over  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , it is a division algebra over  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , whence a simple algebra. Its center being  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , this proves the first part of the statement. By classical theory (see eg. Section 13.3.5 of [69], or [72], Corollary 2.3.25), it then follows that there is a finite extension (a *splitting field*)  $\mathbb{F}$  of  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  such that

$$\mathbb{F} \otimes_{Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} Q(\mathcal{L}_{0,n}^\epsilon) = M_d(\mathbb{F})$$

where  $d \in \mathbb{N}$ , the PI degree of  $Q(\mathcal{L}_{0,n}^\epsilon)$ , is given by  $d^2 = [Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]$ . Therefore

$$d^2 = \frac{[Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))]}{[Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))]} = l^{2Nn}$$

where we use Theorem 4.7 and Lemma 5.2, and we recall that  $\dim \mathfrak{g} = m + 2N$ .  $\square$

Let us recall for the sake of clarity different notions of ring theory, bearing in mind that we will apply them to the case where  $A = \mathcal{L}_{0,n}^\epsilon$ . Let  $A$  be a ring with no non-trivial zero divisors. The center  $Z = Z(A)$  is a commutative integral domain. Denote by  $Q(Z)$  its field of fractions, and let

$$Q(A) = Q(Z) \otimes_Z A.$$

It is an algebra over its center  $Q(Z)$ .

An element  $a \in A$  is *integral over  $Z$*  if  $Z[a]$  is a finitely generated  $Z$ -module.  $A$  is *integral over  $Z$*  if every element of  $A$  is integral over  $Z$ . An element  $a \in A$  is *c-integral over  $Z$*  if  $Z[a]$  is contained in a finitely generated  $Z$ -module.  $A$  is *c-integral over  $Z$*  if every element of  $A$  is c-integral over  $Z$ . When  $Z$  is a Noetherian ring these two notions are equivalent ([69], Lemma 5.3.2).

If  $Q(A)$  has finite dimension over  $Q(Z)$ , it is a division algebra and therefore a central simple algebra. Moreover  $Q(A)$  is integral over  $Q(Z)$ .

There are different notions of orders, that are equivalent in our context. Let  $B \subset Q(A)$  be a subring.  $B$  is said to be an *order* of  $Q(A)$  ([69], 3.1.2) if every element  $q$  of  $Q(A)$  can be written as  $q = rs^{-1} = s'^{-1}r'$ , where  $r, s, r', s' \in B$ . In particular,  $A$  is an order of  $Q(A)$  and  $s, s'$  can in this case be chosen in  $Z$ .  $B$  is a *classical order* of  $Q(A)$  ([69], 5.3.5) if  $Z \subset B$ ,  $Q(Z)B = Q(A)$ , and  $B$  is finitely generated as a  $Z$ -module. In particular, if  $A$  is a finitely generated  $Z$ -module then  $A$  is a classical order of  $Q(A)$ . Finally,  $B$  is a  *$Z$ -order* of  $Q(A)$  ([69], 5.3.6) if  $Z \subset B$ ,  $Q(Z)B = Q(A)$  and  $B$  is c-integral over  $Z$ .

We now assume that  $Z$  is Noetherian and  $Q(A)$  is of finite dimension over  $Q(Z)$ . Let  $B$  be a subring of  $Q(A)$  having center  $Z(B) = Z(A) = Z$ . Then the following assertions are equivalent:

- (1)  $B$  is an order of  $Q(A)$ ;
- (2)  $B$  is a classical order of  $Q(A)$ ;
- (3)  $B$  is a  $Z$ -order of  $Q(A)$ .

Indeed, from the definitions we trivially have (2)  $\Rightarrow$  (3). From the Noetherianity of  $Z$ , Proposition 5.3.14 of [69] gives (3)  $\Rightarrow$  (2). Because  $Q(A)$  is a central simple algebra, the equivalence (1)  $\Leftrightarrow$  (3) is part of Proposition 5.3.10 of [69].



In particular, because  $A$  is always an order of  $Q(A)$  these equivalences imply that  $A$  is a classical order, whence a finitely generated  $Z$ -module. Moreover  $A$  is Noetherian by Proposition 5.3.14 of [69].

There are two standard notions of maximality for  $Z$ -orders  $B$  of  $Q(A)$ . One applies strictly to  $Z$ -orders ([69], 5.3.13): namely, a  $Z$ -order  $B$  of  $Q(A)$  is a *maximal  $Z$ -order* if  $B \subset C$  with  $C$  a  $Z$ -order implies  $B = C$ . The other notion of maximality applies to arbitrary orders of  $Q(A)$  (see [69], 5.1.1); when  $B$  is a  $Z$ -order with center  $Z$ ,  $B$  is a maximal  $Z$ -order if and only if it is maximal in this latter sense.

If  $A$  is a maximal order, then  $Z$  contains all the  $c$ -integral elements over it, ie.  $Z$  is  *$c$ -integrally closed* ([69], 5.3.13). Since  $Z$  is Noetherian, it is then integrally closed (in the usual sense, see [69], Lemma 5.3.2), and by [71], Theorem 10.1, it therefore coincides with the *trace ring* of  $A$  (ie. the subring of  $Q(Z)$  generated over  $Z$  by the coefficients of the characteristic polynomials of elements of  $A$ , represented by left multiplication as elements of the matrix algebra  $Q(A) \otimes_Z \mathbb{F}$ , where  $\mathbb{F}$  is a splitting field of  $Q(A)$ ).

Finally, we say that  $A$  is *DCK-integrally closed* if the following condition holds: for every subring  $R$  of  $Q(A)$  such that  $A \subset R \subset z^{-1}A$  for some non zero  $z \in Z(A)$ , we have  $R = A$ . We borrow this notion from [38]; it is closely related to that of fractional ideal of  $Q(A)$  (see [69], 3.1.11-3.1.12 and 5.1.4), but simpler. Its relevance comes from the following lemma, which shows that in the commutative and Noetherian case it is equivalent to the usual definition of integrally closure.

**Lemma 5.4.** *Let  $B$  be a commutative Noetherian ring with no non-trivial zero divisors. Then  $B$  is integrally closed in  $Q(B)$  if and only if  $B$  is DCK-integrally closed.*

*Proof.* Assume  $B$  is DCK-integrally closed. Let  $x = b/c \in Q(B)$  be integral over  $B$ . Denote by  $n$  the degree of its minimal polynomial over  $B$ . The ring  $R = B[x]$  is contained in  $c^{-n}B$ , so  $R = B$  and  $x \in B$ . (Note that this does not need  $B$  Noetherian.)

Conversely, let  $R$  be a subring of  $Q(B)$  such that  $B \subset R \subset z^{-1}B$ , and let  $x \in R$ . Then  $M = B[x]$  is a  $B$ -submodule of  $Q(B)$  such that  $xM \subset M$ . It is also a  $B$ -submodule of  $z^{-1}B$ , which is free with basis  $z^{-1}$  over  $B$ . Since  $B$  is Noetherian,  $z^{-1}B$  is a Noetherian  $B$ -module, and therefore  $M$  is a finite  $B$ -module. It follows that  $x$  is integral over  $B$  (Proposition 5.1 of [7]), whence  $x \in B$ .  $\square$

**Lemma 5.5.** *Assume that  $A$  is a ring with no non-trivial zero divisors, with center  $Z$  Noetherian and such that  $Q(A)$  has finite dimension over  $Q(Z)$ . Then  $A$  is DCK-integrally closed if and only if  $A$  is a maximal order.*

*Proof.* Assume that  $A$  is DCK-integrally closed. Let  $B$  a  $Z$ -order of  $Q(A)$  such that  $A \subset B$ , and let  $b \in B$ . Since  $b$  is  $c$ -integral over  $Z$  and  $Z$  is Noetherian,  $b$  is integral and thus  $A[b]$  is a finitely generated  $A$ -module. Let  $e_i = a_i/z_i \in Q(A)$ ,  $z_i \in Z$  and  $i = 1, \dots, n$ , be the generators of  $A[b]$ . We have  $A[b] \subset z^{-1}A$  with  $z = \prod_{i=1}^n z_i$ . Therefore  $A[b] = A$ , whence  $b \in A$ . This proves that  $A$  is a maximal order.

Conversely, assume that  $A$  is maximal order, and let  $B$  be a subring of  $Q(A)$  such that  $A \subset B \subset z^{-1}A$ . Since  $A$  is a finitely generated  $Z$ -module,  $B$  is contained in a finitely generated  $Z$ -module, which is necessarily a Noetherian  $Z$ -module because  $Z$  is Noetherian. Therefore  $B$  is a finitely generated  $Z$ -module. As clearly  $Z \subset B$  and  $Q(Z)B = Q(A)$ , it is in fact a classical order of  $Q(A)$ . Because  $A$  is maximal, we have  $A = B$ , which proves  $A$  is DCK-integrally closed.  $\square$

**Theorem 5.6.**  $\mathcal{L}_{0,n}^\epsilon$  is a maximal order of  $Q(\mathcal{L}_{0,n}^\epsilon)$ .

*Proof.* We derive the result by “twisting” the analogous statement for  $\mathcal{O}_\epsilon$ , obtained in Theorem 7.4 of [42]. We have already proved that  $\mathcal{L}_{0,n}^\epsilon$  satisfies the hypothesis on  $A$  in

Lemma 5.5. So let  $R \subset Q(\mathcal{L}_{0,n}^\epsilon)$  be a subring such that  $\mathcal{L}_{0,n}^\epsilon \subset R \subset x^{-1}\mathcal{L}_{0,n}^\epsilon$  for some non zero  $x \in \mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ . We have to show that  $\mathcal{L}_{0,n}^\epsilon = R$ . We know  $\mathcal{L}_{0,n}^\epsilon = \mathcal{O}_\epsilon^{\otimes n}$  as  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -modules (Proposition 4.2 (1)), so  $x^{-1}\mathcal{L}_{0,n}^\epsilon = x^{-1} \star \mathcal{O}_\epsilon^{\otimes n}$ . Also, the product of  $R$  is inherited from that of  $\mathcal{L}_{0,n}^\epsilon$  (for, given  $r_1, r_2 \in R$  we have  $r_1 r_2 = x^{-2}(x r_1)(x r_2)$ ), and the latter is defined from the one of  $\mathcal{O}_\epsilon^{\otimes n}$  by two consecutive twists (see the formulas (3) and (6)). Therefore, by applying the inverse twists on the inclusions  $\mathcal{L}_{0,n}^\epsilon \subset R \subset x^{-1}\mathcal{L}_{0,n}^\epsilon$  we get  $\mathcal{O}_\epsilon^{\otimes n} \subset R' \subset x^{-1} \star \mathcal{O}_\epsilon^{\otimes n}$  where  $R'$  is the vector space  $R$  endowed with the ring structure inherited from  $\mathcal{O}_\epsilon^{\otimes n}$ . Now  $\mathcal{O}_\epsilon^{\otimes n} = \mathcal{O}_\epsilon(G^n)$  is a maximal order of  $Q(\mathcal{O}_\epsilon(G^n))$  by Theorem 7.4 in [42]. Therefore  $\mathcal{O}_\epsilon^{\otimes n} = R'$ , and finally  $\mathcal{L}_{0,n}^\epsilon = R$ .  $\square$

**Corollary 5.7.** *The ring  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is integrally closed and coincides with the trace ring of  $\mathcal{L}_{0,n}^\epsilon$ . Moreover  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon) = \mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$ , and it is a free  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module of rank  $l^{mn}$ .*

*Proof.* The first two claims follow from the last theorem and the discussion before Lemma 5.4. The third follows from the proof of Proposition 5.2 (ie. the inclusion  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon) \supset \mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$ , and the fact that both rings have quotient fields of the same degree  $l^{mn}$  over  $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ ). Finally, it is enough to show the last claim for  $n = 1$ . Denote by  $t_{red}: Q(\mathcal{L}_{0,1}^\epsilon) \rightarrow Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))$  the reduced trace map of the central simple algebra  $Q(\mathcal{L}_{0,1}^\epsilon)$  (see eg. [71], Section 9). Because  $\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)$  is the trace ring of  $\mathcal{L}_{0,1}^\epsilon$ , we have  $\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon) = t_{red}(\mathcal{L}_{0,1}^\epsilon)$ . Trivially the inclusion map  $i: \mathcal{Z}(\mathcal{L}_{0,1}^\epsilon) \rightarrow Q(\mathcal{L}_{0,1}^\epsilon)$  satisfies  $t_{red} \circ i = id$ , so  $\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)$  is a direct summand of  $\mathcal{L}_{0,1}^\epsilon$  as a  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ -module. But  $\mathcal{L}_{0,1}^\epsilon$  is free over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , so  $\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)$  is a projective  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ -module. Arguing as in Theorem 4.7, one deduces that the module is free. The rank is, again, given by Proposition 5.2.  $\square$

A proof of Theorem 5.6 independent of Theorem 7.4 of [42] seems to be difficult for arbitrary  $\mathfrak{g}$ , even for  $n = 1$ . In the case of  $\mathcal{L}_{0,1}(sl(2))$  we can however apply a similar reasoning. Let us explain the details.

Identify  $\mathcal{Z}_0(\mathcal{L}_{0,1}(sl(2)))$  with  $\mathcal{O}(SL_2)$  as described in Lemma 4.4 and before Lemma 4.3. We proceed in two steps : (a) we show that  $\mathcal{L}_{0,1}^\epsilon(sl(2))[d^{-l}]$  and  $\mathcal{L}_{0,1}^\epsilon(sl(2))[b^{-l}]$  are maximal orders, (b) from this we deduce the result.

As for (a), recall that  $\mathcal{L}_{0,1}^\epsilon(sl(2))[d^{-l}] \cong U_\epsilon^{lf}(sl(2))[d^l]$ . We have  $U_\epsilon^{lf}(sl(2))[d^l] = U_\epsilon^{ad}(sl(2))$ , which is a maximal order by Theorem 1.8 of [38]. Using (44) one computes easily that  $\mathcal{L}_{0,1}^\epsilon(sl(2))[b^{-l}]$  is generated by  $\omega, b^{\pm 1}$  and  $d$ , with defining relations:  $\omega$  is central,  $db = q^2bd$ . Therefore  $\mathcal{L}_{0,1}^\epsilon(sl(2))[b^{-l}]$  is the tensor product of  $\mathbb{C}[\omega]$  with a quasi-polynomial algebra. The  $l$ -th powers of  $d, b$  are central, and  $\mathbb{C}[\omega]$  is integrally closed, so by a direct application of Proposition 1.8 of [38] it follows that  $\mathcal{L}_{0,1}^\epsilon(sl(2))[b^{-l}]$  is a maximal order.

Let us now deduce that  $\mathcal{L}_{0,1}^\epsilon(sl(2))$  is a maximal order. Let  $x \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$  and  $R$  a subring of  $Q(\mathcal{L}_{0,1}^\epsilon(sl(2)))$  such that  $\mathcal{L}_{0,1}^\epsilon(sl(2)) \subset R \subset x^{-1}\mathcal{L}_{0,1}^\epsilon(sl(2))$ . Let  $y \in R$ . Because  $\mathcal{L}_{0,1}^\epsilon(sl(2))[d^{-l}]$  is maximal, we have  $\mathcal{L}_{0,1}^\epsilon(sl(2))[d^{-l}] = R[d^{-l}]$ . So  $yd^{l.n_d} \in \mathcal{L}_{0,1}^\epsilon(sl(2))$  for some non negative integer  $n_d$ . Similarly,  $yb^{l.n_b} \in \mathcal{L}_{0,1}^\epsilon(sl(2))$  for some non negative integer  $n_b$ . Now, note that  $d^l \triangleleft t_w = b^l$ . Consider the open subsets  $U(d^l)$  and  $U(b^l)$  of  $SL_2$  consisting of matrices with non vanishing lower right entry and non vanishing upper right entry, respectively. As in the proof of Theorem 4.7 we have  $\text{Maxspec}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))) = SL_2 = U(d^l) \cup U(b^l)$ . Since  $U(d^l) = U(d^{l.n_d})$ ,  $U(b^l) = U(b^{l.n_b})$ , likewise  $\text{Maxspec}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))) = U(d^{l.n_d}) \cup U(b^{l.n_b})$ . This implies that the ideal of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$  generated by  $d^{l.n_d}$  and  $b^{l.n_b}$  is  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$  itself; hence there are elements  $u_d, u_b \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$  such that  $u_d d^{l.n_d} + u_b b^{l.n_b} = 1$ , which proves  $y = u_d d^{l.n_d} y + u_b b^{l.n_b} y \in \mathcal{L}_{0,1}^\epsilon(sl(2))$ . This concludes the proof that  $\mathcal{L}_{0,1}^\epsilon(sl(2))$  is a maximal order.

**Proof of Theorem 1.3 (2) and (3).** Denote by  $\Delta^{(n)}: \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon^{\otimes n}$ ,  $n \geq 2$ , the  $n$ -fold coproduct, that is  $\Delta^{(n)} := (id \otimes \Delta) \circ \Delta^{(n-1)}$  for  $n \geq 3$ , and  $\Delta^{(2)} := \Delta$ , the standard coproduct of  $\mathcal{O}_\epsilon$ . Identifying  $\mathcal{L}_{0,n}^\epsilon$  with  $\mathcal{O}_\epsilon^{\otimes n}$  as a vector space, we consider  $\Delta^{(n)}$  as a map  $\Delta^{(n)}: \mathcal{L}_{0,1}^\epsilon \rightarrow \mathcal{L}_{0,n}^\epsilon$ . It is an algebra morphism ([28], Proposition 6.16), injective because  $(\epsilon^{\otimes(n-1)} \otimes id)\Delta^{(n)} = id$ . Then it extends uniquely to the fraction algebra  $Q(\mathcal{L}_{0,1}^\epsilon)$ . As noted above  $Q(\mathcal{L}_{0,1}^\epsilon) = Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)) \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)} \mathcal{L}_{0,1}^\epsilon$ . Since  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$  is a Hopf subalgebra of  $\mathcal{O}_\epsilon$  ([42], Proposition 6.4),  $\Delta^{(n)}$  maps  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  to  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$ . Then, extending the scalars of  $\Delta^{(n)}(Q(\mathcal{L}_{0,1}^\epsilon))$  by the field  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , consider

$$\begin{aligned} Q(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) &:= Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))} \Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon) \\ &= Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)))} \Delta^{(n)}(Q(\mathcal{L}_{0,1}^\epsilon)) \\ &= \left( Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)))} \Delta^{(n)}(Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))) \right) \\ &\quad \otimes_{\Delta^{(n)}(Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)))} \Delta^{(n)}(Q(\mathcal{L}_{0,1}^\epsilon)). \end{aligned}$$

By Proposition 5.3 the right factor is a  $\Delta^{(n)}(Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)))$ -central simple algebra. The left factor is a field and is equal to

$$\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) := Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))} \Delta^{(n)}(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))$$

for it contains  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , it is contained in its fraction field, and  $\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  is a field because  $\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)$  is finite over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  and has no non trivial zero divisors. Therefore  $Q(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$  is a central simple algebra over  $\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  (see eg. [72], Theorem 1.7.27, or [79], Lemma 4.9). We proved in Proposition 6.17 of [28] that the ring  $(\mathcal{L}_{0,n}^A)^{U_A}$  is the centralizer of  $\Delta^{(n)}(\mathcal{L}_{0,1}^A)$  in  $\mathcal{L}_{0,n}^A$ ; the same arguments show that  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is the centralizer of  $\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)$  in  $\mathcal{L}_{0,n}^\epsilon$ , so  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) := Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is the centralizer of  $Q(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$  in  $Q(\mathcal{L}_{0,n}^\epsilon)$ . Then the double centralizer theorem (see eg. [72], Theorem 7.1.9, or [79], Theorem 7.1) implies that  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is a simple algebra, with dimension

$$[Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))] = \frac{[Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]}{[Q(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]} = l^{2nN-(2N+m)},$$

and the centralizer of  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is  $Q(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$ . In particular  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  has center  $\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) = Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) \cap Q(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$ . It then follows

$$\begin{aligned} [Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) : \tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))] &= \frac{[Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]}{[\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]} \\ &= l^{2nN-(2N+m)} \cdot l^{-m} = l^{2(N(n-1)-m)}. \end{aligned}$$

Therefore  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is a central simple algebra of PI degree  $l^{N(n-1)-m}$ . Next, consider the centralizer of  $\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  in  $Q(\mathcal{L}_{0,n}^\epsilon)$ . It is a  $\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ -central simple algebra of dimension  $l^{2nN-2m}$  over  $\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , whence  $l^{2nN-2m} \cdot l^m = l^{2nN-m}$  over  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , and it may be identified with  $Q(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) \otimes_{\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  by using the product map  $m: Q(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) \otimes_{\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) \rightarrow Q(\mathcal{L}_{0,n}^\epsilon)$  (see the proof of [72], Theorem 7.1.9 (i)). Being a simple algebra, it has a unique simple (left) module up to isomorphisms. Being a division algebra, it has no non trivial primitive idempotent, so this simple module is

$Q(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) \otimes_{\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  with its left regular action. Moreover, the left regular action on  $Q(\mathcal{L}_{0,n}^\epsilon)$  is completely reducible. The above computation of dimension over  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  then yields the following decomposition into simple left  $Q(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) \otimes_{\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ -modules,

$$(48) \quad Q(\mathcal{L}_{0,n}^\epsilon) \cong \left( Q(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) \otimes_{\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) \right)^{\oplus l^m}.$$

The product map  $m: Q(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) \otimes_{\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) \rightarrow Q(\mathcal{L}_{0,n}^\epsilon)$  gives one summand, and it is the localisation by  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  of the inclusion  $\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon) \otimes_{\Delta^{(n)}(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon} \rightarrow \mathcal{L}_{0,n}^\epsilon$ , for we have

$$\begin{aligned} & Q(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) \otimes_{\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) \\ &= \left( Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))} \Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon) \right) \otimes_{\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} \left( Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon} \right) \\ &= Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} \left( \Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon) \otimes_{\Delta^{(n)}(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon} \right). \end{aligned}$$

Now, by the Skolem-Noether theorem (eg. in the form of [72], Theorem 7.1.10, or [5], Theorem III.3.1) any summand of (48) is related to  $\text{Im}(m)$  by an inner automorphism of  $Q(\mathcal{L}_{0,n}^\epsilon)$ ,  $x \mapsto uxu^{-1}$ , where clearly one may take  $u$  in  $\mathcal{L}_{0,n}^\epsilon$ . Then we have an embedding of left  $(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon) \otimes_{\Delta^{(n)}(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ -modules,

$$(49) \quad \tilde{m}: \left( \Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon) \otimes_{\Delta^{(n)}(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon} \right)^{\oplus l^m} \rightarrow \mathcal{L}_{0,n}^\epsilon$$

made componentwise of  $m$  and these inner automorphisms. We claim  $\tilde{m}$  is an isomorphism. Indeed, consider the source and target as  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ -modules, and the quotient module  $\mathcal{L}_{0,n}^\epsilon/\text{Im}(\tilde{m})$ . We have

$$Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} (\mathcal{L}_{0,n}^\epsilon/\text{Im}(\tilde{m})) = Q(\mathcal{L}_{0,n}^\epsilon)/(Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} \text{Im}(\tilde{m})) = 0$$

by (48), and therefore  $\mathcal{L}_{0,n}^\epsilon/\text{Im}(\tilde{m}) = 0$  (see eg. [7], Corollary 3.4 and Proposition 3.5 and 3.8). Because  $\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)$  is a direct summand of  $\mathcal{L}_{0,1}^\epsilon$  (see the proof of Corollary 5.7), we can decompose  $\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon) \otimes_{\Delta^{(n)}(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon} = 1 \otimes (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon} \oplus \Delta^{(n)}(M) \otimes (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  for some  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ -submodule  $M$  of  $\mathcal{L}_{0,1}^\epsilon$ . This, (49) and Theorem 4.7 ( $\mathcal{L}_{0,n}^\epsilon$  is free over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ ) then imply that  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is a projective  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module, and the arguments in the proof (ie. [10] plus [68]) eventually show that  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is a free  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module. Denoting by  $r$  the rank of  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ , we deduce  $l^{n \cdot \dim \mathfrak{g}} = l^m \cdot l^{2N} \cdot r$ , whence  $r = l^{(n-1) \cdot \dim \mathfrak{g}}$ .

Finally we prove that  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is a maximal order of  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ . We use Lemma 5.5. Let  $B \subset Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  a subring such that  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon} \subset B \subset z^{-1}(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  for some non zero element  $z \in \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ . We have to show  $B \subset (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ . Because of (47) we can as well assume that  $z \in \mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ . Then (49), thought as an identification, yields inclusions (in  $Q(\mathcal{L}_{0,n}^\epsilon)$ ):

$$\mathcal{L}_{0,n}^\epsilon \subset \left( \Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon) \otimes_{\Delta^{(n)}(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))} B \right)^{\oplus l^m} \subset z^{-1} \mathcal{L}_{0,n}^\epsilon.$$

By Theorem 5.6, in such a situation we have  $\mathcal{L}_{0,n}^\epsilon = (\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon) \otimes_{\Delta^{(n)}(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))} B)^{\oplus l^m}$ . The summands must be the simple components of  $\mathcal{L}_{0,n}^\epsilon$ , so necessarily  $\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon) \otimes_{\Delta^{(n)}(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))} B = \Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon) \otimes_{\Delta^{(n)}(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ , and therefore  $B \subset (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ . This concludes the proof.  $\square$

## 6. APPENDIX

**6.1. Quantum Weyl group.** We recall some of the formulas of [32]. Let  $e_q(z)$  be the formal power series in  $z$  with coefficients in  $\mathbb{C}(q)$  defined by:

$$(50) \quad e_q(z) = \sum_{n=0}^{+\infty} \frac{z^n}{(n)_q}.$$

We first consider the case of  $\mathfrak{g} = sl(2)$ . As explained in [28], Section 3, the Cartan element  $H \in \mathfrak{g}$  defines an element of  $\mathbb{U}_q(sl(2))$ . Viewed as elements of  $\mathbb{U}_q$  we have  $L = q^{H/2}$ . The series  $\Theta = q^{H \otimes H/2}$  defines an element of  $\mathbb{U}_q(sl(2))^{\hat{\otimes} 2}$ , its image under multiplication being  $q^{H^2/2}$ . The  $R$ -matrix can be expressed as  $R = \Theta \hat{R}$  where  $\hat{R} = e_{q^{-1}}((q - q^{-1})E \otimes F)$  is a well defined element of  $\mathbb{U}_q^{\hat{\otimes} 2}$ . Consider the Lusztig [60] braid group automorphism of  $\mathbb{U}_q(sl(2))$ , defined by

$$(51) \quad T(L) = L^{-1}, T(E) = -FK^{-1}, T(F) = -KE.$$

For every  $x \in U_q(sl(2))$  it satisfies:

$$(52) \quad \Delta(T(x)) = \hat{R}^{-1}(T \otimes T)(\Delta(x))\hat{R}$$

Define the quantum Weyl group element  $\hat{w} \in \mathbb{U}_q(sl(2))$  by Saito's formula [74]:

$$(53) \quad \hat{w} = e_{q^{-1}}(F)q^{-H^2/4}e_{q^{-1}}(-E)q^{-H^2/4}e_{q^{-1}}(F)q^{-H/2}.$$

For every  $x \in U_q(sl(2))$  it satisfies:

$$(54) \quad T(x) = \hat{w}x\hat{w}^{-1},$$

$$(55) \quad \Delta(\hat{w}) = \hat{R}^{-1}(\hat{w} \otimes \hat{w}),$$

$$(56) \quad \hat{w}^2 = q^{H^2/2}\xi\theta,$$

where  $\theta \in \mathbb{U}_q(sl(2))$  is the ribbon element, and  $\xi \in \mathbb{U}_q(sl(2))$  is the central group element whose value on the type 1 simple module  $X$  of  $U_q(sl(2))$  of dimension  $k + 1$  is the scalar endomorphism  $(-1)^k id_X$ .

In order to compare our setting to the one of [42] we need an explicit formula of  $\hat{w}$ . Consider the type 1 simple module  $V_{r+1}$  of dimension  $r + 1$ , and its basis vectors  $v_0, \dots, v_r$  such that:

$$\begin{aligned} K.v_j &= \epsilon^{r-2j}v_j, \\ F.v_j &= v_{j+1} \text{ if } j < r, \quad F.v_r = 0, \\ E.v_j &= [j]_\epsilon[r-j+1]_\epsilon v_{j-1} \text{ if } j > 0, \quad E.v_0 = 0. \end{aligned}$$

Setting  $v'_j = v_j/[j]!$  and using (51), (54) and (56), we obtain:

$$(57) \quad \hat{w}v'_j = (-1)^j q^{-j(k-j-1)-k} v'_{k-j}.$$

In [42] another quantum Weyl group element  $\underline{w}$  is defined. It is dual to the Vaksman-Soibelman functional  $t: \mathcal{O}_q(SL_2) \rightarrow \mathbb{C}(q)$  of [77, 76], so  $t(\alpha) = \langle \alpha, \underline{w} \rangle$ ,  $\alpha \in \mathcal{O}_q(SL_2)$ . By comparing (57) with the formulas defining the action of  $t$  in Section 1.7 of [42], we find

$$\underline{w} = \xi \hat{w} K$$

and the basis vectors  $w_r^p$  of [42] are related to the vectors  $v'_j$  above as follows:

$$v'_j = \lambda_j w_r^p$$

where  $k = 2p$ ,  $j = p - r$ ,  $\lambda_0 = 1$ ,  $\lambda_1 = [k]q^{-k}$ , and

$$\lambda_j = \frac{[k]!}{[j]![k - (j - 2)]!} q^{j(j+1) - j(k+2)}, \quad j \geq 2.$$

Explicit formulas of the evaluation of  $t$  on basis vectors of  $\mathcal{O}_q(SL_2)$  can be computed. We get:

$$(58) \quad t(\tilde{a}^{\star m} \star \tilde{b}^{\star n} \star \tilde{d}^{\star p}) = \delta_{m,p} q^{-np} \prod_{i=1}^p (1 - q^{-2i}),$$

$$(59) \quad t(\tilde{a}^{\star m} \star \tilde{c}^{\star n} \star \tilde{d}^{\star p}) = (-1)^n \delta_{m,p} q^{-n(p+1)} \prod_{i=1}^p (1 - q^{-2i})$$

where

$$(60) \quad \tilde{a} = a, \quad \tilde{b} = qb, \quad \tilde{c} = q^{-1}c, \quad \tilde{d} = d$$

and as usual  $a, b, c, d$  are the standard generators of  $\mathcal{O}_q(SL_2)$ , ie. the matrix coefficients in the basis of weight vectors  $v_0, v_1 = F.v_0$  of the 2-dimensional irreducible representation  $V_2$  of  $U_q(\mathfrak{sl}(2))$ . Here we have introduced the generators  $\tilde{a}, \dots, \tilde{d}$  to facilitate the comparison with the formulas in [42]; these generators come naturally in their setup because they use different generators  $E_i$  and  $F_i$  of  $U_q(\mathfrak{g})$ , which in our notations can be written respectively as  $K_i^{-1}E_i$  and  $F_iK_i$ .

The formulas (58)-(59) can be shown by two independent methods. The first uses a definition of  $t$  as a *GNS* state associated to an infinite dimensional representation of  $\mathcal{O}_q(SL_2)$ , as recalled in Section 1.6 of [42]. The second is to write eg.

$$(61) \quad t(\tilde{a}^{\star m} \star \tilde{b}^{\star n} \star \tilde{d}^{\star p}) = \left\langle \tilde{a}^{\otimes m} \otimes \tilde{b}^{\otimes n} \otimes \tilde{d}^{\otimes p}, \Delta^{(m+n+p-1)}(\underline{w}) \right\rangle$$

and to use explicit expressions of  $\Delta^{(m+n+p-1)}(\underline{w})$  when represented on  $V_2^{\otimes(m+n+p)}$ . In general one can check that

$$(62) \quad \Delta^{(n)}(\hat{w}) = \left( \Delta^{(n-1)} \otimes id \right) \left( \hat{R}^{-1} \right) \left( \left( \Delta^{(n-2)} \otimes id \right) \left( \hat{R}^{-1} \right) \otimes id \right) \\ \dots \left( \left( \Delta \otimes id \right) \left( \hat{R}^{-1} \right) \otimes id^{\otimes(n-3)} \right) \left( \hat{R}^{-1} \otimes id^{\otimes(n-2)} \right) \hat{w}^{\otimes n}.$$

By (57) or (58)-(59) we see that  $\hat{w}$  (or  $\underline{w}$ ) and  $t$  are well-defined on the integral forms,

$$\hat{w} \in \mathbb{U}_\Gamma, \quad t: \mathcal{O}_A(SL_2) \rightarrow A.$$

We now consider the case where  $\mathfrak{g}$  is of rank  $m \geq 2$ . To each simple root  $\alpha_i$ ,  $1 \leq i \leq m$ , it is associated the subalgebra of  $U_q$  generated by  $E_i, F_i, L_i, L_i^{-1}$ . It is a copy of  $U_{q_i}(\mathfrak{sl}(2))$ , where  $q_i = q^{d_i}$ . Let  $\hat{w}_i$  be the corresponding quantum Weyl group element in  $\mathbb{U}_q = \mathbb{U}_q(\mathfrak{g})$ , defined by Saito's formula (53), replacing  $H, E, F$  by  $H_i, E_i$  and  $F_i$ . Also, denote by  $\nu_i: \mathcal{O}_q \rightarrow \mathcal{O}_{q_i}(SL_2)$  the projection map dual to the inclusion  $U_{q_i}(\mathfrak{sl}(2)) \otimes_{\mathbb{C}(q_i)} \mathbb{C}(q) \hookrightarrow U_q$  associated to  $\alpha_i$ , and put  $t_i = t \circ \nu_i$ . Let  $\underline{w}_i$  be the corresponding quantum Weyl group element in  $\mathbb{U}_q$ , ie.  $t_i(\alpha) = \langle \alpha, \underline{w}_i \rangle$  for all  $\alpha \in \mathcal{O}_q$ . On integral forms they yield well-defined elements  $\hat{w}_i, \underline{w}_i \in \mathbb{U}_\Gamma$  and  $t_i: \mathcal{O}_A \rightarrow A$  (see [42], Proposition 5.1). They satisfy the defining relations of the braid group  $\mathcal{B}(\mathfrak{g})$  of  $\mathfrak{g}$  [55]:

$$\hat{w}_i \hat{w}_j \hat{w}_i = \hat{w}_j \hat{w}_i \hat{w}_j \quad \text{if } a_{ij} a_{ji} = 1 \\ (\hat{w}_i \hat{w}_j)^k = (\hat{w}_j \hat{w}_i)^k \quad \text{for } k = 1, 2, 3 \text{ if } a_{ij} a_{ji} = 0, 2, 3$$

and similarly by replacing  $\hat{w}_i$  with  $\underline{w}_i$ , or with  $t_i$  (see [76] for the latter). The Weyl group  $W = W(\mathfrak{g}) = N(T_G)/T_G$  is generated by the reflexions  $s_i$  associated to the simple roots  $\alpha_i$ .

Denote by  $n_i \in N(T_G)$  a representative of  $s_i$ . Let  $w \in W$  and denote by  $w = s_{i_1} \dots s_{i_k}$  a reduced expression. Because of the braid group relations the elements  $\hat{w} = \hat{w}_{i_1} \dots \hat{w}_{i_k}$ ,  $\underline{w} = \underline{w}_{i_1} \dots \underline{w}_{i_k}$  and the functional  $t_w = t_{i_1} \dots t_{i_k}$  do not depend on the choice of reduced expression. The Lusztig [60] braid group automorphism  $T_w: \Gamma \rightarrow \Gamma$  associated to  $w$  satisfies (see [42]):

$$T_w(x) = \hat{w}x\hat{w}^{-1}, \quad x \in \Gamma.$$

Let  $w_0$  be the longest element in  $W$ . We have

$$(63) \quad \Delta(\hat{w}_0) = \hat{R}^{-1}(\hat{w}_0 \otimes \hat{w}_0)$$

where as usual  $R = \Theta\hat{R}$ .

**6.2. Regular action on  $\mathcal{O}_\epsilon$ .** The following result is proved in Section 1.10 of [42]. For completeness let us give a (different) proof. Recall from (25) that we may identify  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  with  $\mathcal{O}(G)$ .

**Proposition 6.1.** *For every  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ ,  $g \in \mathcal{O}_\epsilon$  we have*

$$(64) \quad t_i(f) = f(n_i)$$

$$(65) \quad t_i(f \star g) = t_i(f)t_i(g).$$

*Proof.* It is sufficient to prove the results for  $SL_2$  because  $\nu_i: \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon(SL_2)$  is a morphism of Hopf algebras and  $\nu_i(\mathcal{Z}_0(\mathcal{O}_\epsilon)) \subset \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ . In this case (64) can be proved by using (58)-(59), evaluating  $t$  on basis elements of  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$  as is done in Lemma 1.5 (a) of [42]. Such a basis is formed by monomials like in (58)-(59), with all exponents divisible by  $l$ ; then for instance

$$t(\tilde{a}^{\star ml} \star \tilde{b}^{\star nl} \star \tilde{d}^{\star pl}) = \delta_{p,0}\delta_{m,0} = \underline{a}^m \underline{b}^n \underline{d}^p(n)$$

where  $\underline{a}, \dots, \underline{d}$  are the generators of  $\mathcal{O}(G) = \mathcal{O}_1(G)$  corresponding to  $a, \dots, d$ , and we take

$$n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

as representative of the reflexion  $s$  generating the Weyl group  $W(sl(2))$ . Here is an alternative proof of (64): (65) shows that  $t$  is a homomorphism on  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ , so by proving (65) at first one is reduced to check (64) on the generators  $a^{\star l}, \dots, d^{\star l}$ , which is easy by means of (61) and (63).

We provide a proof of (65) that we find more conceptual than the one in Lemma 1.5 (b) of [42] (which uses again (58)-(59)). As above let us denote  $\underline{w} = \xi\hat{w}K$ . For any  $f, g \in \mathcal{O}_\epsilon$  we have

$$\begin{aligned} t(f \star g) &= (f \otimes g)(\Delta(\underline{w})) \\ &= (f \otimes g)\left(\hat{R}^{-1}(\underline{w} \otimes \underline{w})\right) \\ &= \sum_{(\hat{R}^{-1})} f\left((\hat{R}^{-1})_{(1)}\underline{w}\right) g\left((\hat{R}^{-1})_{(2)}\underline{w}\right) \\ &= \sum_{(\hat{R}^{-1}), (f)} f_{(1)}\left((\hat{R}^{-1})_{(1)}\right) f_{(2)}(\underline{w}) g\left((\hat{R}^{-1})_{(2)}\underline{w}\right) \\ &= \sum_{(f)} f_{(2)}(\underline{w}) g\left((f_{(1)} \otimes id)(\hat{R}^{-1})\underline{w}\right). \end{aligned}$$

Assume now  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ . Since  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$  is a Hopf subalgebra of  $\mathcal{O}_\epsilon(SL_2)$  we have  $f_{(1)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ . From Theorem 2.15 (2) we deduce

$$(f_{(1)} \otimes id)(\widehat{R}^{-1}) \in U_\epsilon(\mathfrak{n}_-) \cap \mathcal{Z}_0(U_\epsilon).$$

Denote by  $z$  this element. Note that from its expression we have  $\epsilon(z) = \epsilon(f_{(1)})$ . Now  $g(z\underline{w}) = \sum_{(g)} g_{(1)}(z)g_{(2)}(\underline{w})$ , but  $g_{(1)}$  is a linear combination of matrix elements of  $\Gamma$ -modules, on which  $\mathcal{Z}_0(U_\epsilon)$  acts by the trivial character. Therefore

$$g(z\underline{w}) = \sum_{(g)} \epsilon(z)g_{(1)}(1)g_{(2)}(\underline{w}) = \epsilon(z)g(\underline{w}) = \epsilon(f_{(1)})g(\underline{w})$$

and eventually

$$t(f \star g) = \sum_{(f)} f_{(2)}(\underline{w})\epsilon(f_{(1)})g(\underline{w}) = t(f)t(g).$$

This concludes the proof.  $\square$

For the sake of completeness, let us show how this result implies:

*Proof of Proposition 2.16 (ie. Proposition 7.1 of [42]).* We have  $f \triangleleft t_i = \sum_{(f)} t_i(f_{(1)})f_{(2)}$ ,  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ . Since  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  is a Hopf subalgebra of  $\mathcal{O}_\epsilon$ ,  $f_{(2)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$  and therefore the maps  $\triangleleft t_i: \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon$  preserve  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ . Moreover,  $(f \triangleleft t_i)(a) = \sum_{(f)} f_{(1)}(n_i)f_{(2)}(a) = f(n_i a)$ ,  $a \in G$ , by (64).

It remains to show that  $(f \star \alpha) \triangleleft t_i = (f \triangleleft t_i)(\alpha \triangleleft t_i)$  for every  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ ,  $\alpha \in \mathcal{O}_\epsilon$ . We have

$$\begin{aligned} (f \star g) \triangleleft t_i &= \sum_{(f \star g)} t_i((f \star g)_{(1)})(f \star g)_{(2)} = \sum_{(f),(g)} t_i(f_{(1)} \star g_{(1)})f_{(2)} \star g_{(2)} \\ &= \sum_{(f),(g)} t(\nu_i(f_{(1)})\nu_i(g_{(1)}))f_{(2)} \star g_{(2)} \\ (66) \qquad &= \sum_{(f),(g)} t(\nu_i(f_{(1)}))t(\nu_i(g_{(1)}))f_{(2)} \star g_{(2)} \end{aligned}$$

using that  $\nu_i$  is a homomorphism in the third equality, and (65) in the last one. The result is just  $(f \triangleleft t_i)(g \triangleleft t_i)$ .  $\square$

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