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# UNRESTRICTED QUANTUM MODULI ALGEBRAS, II: NOETHERIANITY AND SIMPLE FRACTION RINGS AT ROOTS OF 1

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ABSTRACT. We prove that the unrestricted quantum moduli algebra of a punctured sphere and complex simple Lie algebra  $\mathfrak{g}$  is a finitely generated ring and a Noetherian ring, and that specializations at roots of unity of odd order  $l$  embed in a natural way in a central simple algebra of PI degree  $l^{(n-1)N-m}$ , where  $N$  is the number of positive roots of  $\mathfrak{g}$ ,  $m$  its rank, and  $n + 1 \geq 3$  the number of punctures.

*Keywords:* quantum groups, invariant theory, TQFT  
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## 1. INTRODUCTION

This paper is the second part of our work on the unrestricted quantum moduli algebras, that we initiated in [23]. These algebras, denoted by  $\mathcal{M}_{g,n}^A(\mathfrak{g})$  hereafter, are defined over the ground ring  $A = \mathbb{C}[q, q^{-1}]$  and associated to unrestricted quantum groups of complex simple Lie algebras  $\mathfrak{g}$ , and surfaces of genus  $g$  with  $n + 1$  punctures (thus,  $n = -1$  corresponds to closed surfaces). We are in particular interested in the specializations  $\mathcal{M}_{g,n}^{A,\epsilon}(\mathfrak{g})$  of  $\mathcal{M}_{g,n}^A(\mathfrak{g})$  at roots of unity  $q = \epsilon$ .

As in [23] we focus in this paper on the algebras  $\mathcal{M}_{0,n}^A(\mathfrak{g})$  associated to punctured spheres. From now on we fix a complex simple Lie algebra  $\mathfrak{g}$ , and when no confusion may arise we omit  $\mathfrak{g}$  from the notation of the various algebras.

The rational form  $\mathcal{M}_{g,n}$  of  $\mathcal{M}_{g,n}^A = \mathcal{M}_{g,n}^A(\mathfrak{g})$ , which is an algebra over  $\mathbb{C}(q)$ , has been introduced in the mid '90 by Alekseev-Grosse-Schomerus [2, 3] and Buffenoir-Roche [25, 26]. They defined  $\mathcal{M}_{g,n}$  by  $q$ -deforming the Fock-Rosly lattice models of the moduli spaces  $\mathcal{M}_{g,n}^{cl}$  of flat  $\mathfrak{g}$ -connections on surfaces of genus  $g$  with  $n + 1$  punctures. It is expected for long that the representation theory of  $\mathcal{M}_{g,n}$  at roots of unity recovers all known  $(2 + 1)$ -dimensional TQFTs based on quantum groups, and also provide new TQFTs for 3-manifolds endowed with flat  $\mathfrak{g}$ -connections.

For instance, representations of the semisimplification of  $\mathcal{M}_{g,n}^{A,\epsilon}$  have been constructed and classified in [4]; they involve only the irreducible representations of the so called finite quantum groups  $U_\epsilon^{fin}(\mathfrak{g})$  (in the notations of [30], Section 9.3). Moreover, by using their representations of  $\mathcal{M}_{g,n}^{A,\epsilon}$ , [4] deduced representations of the mapping class groups of surfaces, that are equivalent to those from which one can built the quantum invariants of 3-manifolds of Witten-Reshetikin-Turaev [71, 63].

Recently, representations of another quotient of  $\mathcal{M}_{g,n}^{A,\epsilon}$  have been constructed in [39]; in the  $sl(2)$  case they involve the irreducible and also the principal indecomposable representations of  $U_\epsilon^{fin}(sl(2))$ . The corresponding representations of the mapping class groups of surfaces are equivalent to those previously obtained by Lyubashenko-Majid [54]. The related link and 3-manifold invariants coincide with those of [55] and [17].

In general, the representation theory of  $\mathcal{M}_{g,n}^{A,\epsilon}$  is far from being completely understood. As mentioned above, it is expected to provide a good framework to construct and study quantum invariants of 3-manifolds equipped with flat  $\mathfrak{g}$ -connections. A family of such invariants, called *quantum hyperbolic invariants*, has already been defined for  $\mathfrak{g} = sl(2)$  by means of certain  $6j$ -symbols, *Deus ex machina* (see [10]–[16]). They are closely connected to classical Chern-Simons theory, provide generalized Volume Conjectures, and contain quantum Teichmüller theory. It is part of our present program, initiated in [7], to shed light on these quantum invariants and to generalize them to arbitrary  $\mathfrak{g}$  by developing the representation theory of  $\mathcal{M}_{g,n}^{A,\epsilon}$ .

Besides, the quantum moduli algebras are very interesting objects in themselves. They are now recognized as central objects from the viewpoints of factorization homology [18], (stated) skein theory [19, 41, 29] and, as already said, the mapping class group representations associated to topological quantum field theories [40].

We introduced the integral form  $\mathcal{M}_{0,n}^A$  and began its study in [23]. We presently prove new results about its algebra structure, especially when  $q$  is a root of unity, that hold for every complex simple Lie algebra  $\mathfrak{g}$ . We use a definition of  $\mathcal{M}_{0,n}^A$  that comes from the original combinatorial quantization method of [2, 3] and [25, 26], using also twists of module-algebras; this allows us to exploit fully the representation theory of quantum groups, by following ideas of classical invariant theory. Namely, as we shall describe more precisely below,  $\mathcal{M}_{0,n}^A$  can be regarded as the invariant subalgebra of a certain module-algebra  $\mathcal{L}_{0,n}^A$ , endowed with an action of the unrestricted (De Concini-Kac) integral form  $U_A = U_A(\mathfrak{g})$  of the quantum group  $U_q = U_q(\mathfrak{g})$ . We therefore study  $\mathcal{L}_{0,n}^A$  and its specializations  $\mathcal{L}_{0,n}^\epsilon$  at  $q = \epsilon$  a root of unity. It happens that under such a specialization,  $\mathcal{M}_{0,n}^A$  embeds in the invariant subalgebra  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  of  $\mathcal{L}_{0,n}^\epsilon$  under the action of the specialization  $U_\epsilon$  of  $U_A$  at  $q = \epsilon$ . Our results in this paper basically concern  $\mathcal{L}_{0,n}^A$ ,  $\mathcal{L}_{0,n}^\epsilon$  and  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ .

By using some standard tools of representation theory our results allow one to build a vector bundle of rank  $l^{2(N(n-1)-m)}$  ( $N$  being the number of positive roots of  $\mathfrak{g}$ , and  $m$  its rank) over a Zariski open subset of the maximal spectrum of the center  $\mathcal{Z}((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  of  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ , for  $n \geq 2$ . In [24] we describe the inclusion  $\mathcal{M}_{0,n}^{A,\epsilon} \subset (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  and the representations of  $\mathcal{M}_{0,n}^{A,\epsilon}$ ,

and we give applications to skein algebras (which is the  $sl(2)$  case). In [22] we consider the algebras  $\mathcal{M}_{g,n}^{A,\epsilon}$  for genus  $g \neq 0$ .

Let us now state our results. First we need to fix the terminology more precisely. Let  $U_q$  be the simply-connected quantum group of  $\mathfrak{g}$ , defined over the field  $\mathbb{C}(q)$ . From  $U_q$  one can define a  $U_q$ -module algebra  $\mathcal{L}_{0,n}$ , called *graph algebra*, where  $U_q$  acts by means of a right coadjoint action. The *quantum moduli algebra*  $\mathcal{M}_{0,n}$  is the subalgebra  $\mathcal{L}_{0,n}^{U_q}$  of invariant elements of  $\mathcal{L}_{0,n}$  for this action. The *unrestricted* quantum moduli algebra  $\mathcal{M}_{0,n}^A$  is an integral form of  $\mathcal{M}_{0,n}$  (thus, defined over  $A = \mathbb{C}[q, q^{-1}]$ ). As a  $\mathbb{C}(q)$ -module  $\mathcal{L}_{0,n}$  is just  $\mathcal{O}_q^{\otimes n}$ , where  $\mathcal{O}_q = \mathcal{O}_q(G)$  is the standard quantum function algebra of the connected and simply-connected Lie group  $G$  with Lie algebra  $\mathfrak{g}$ . The product of  $\mathcal{L}_{0,n}$  is obtained by twisting both the product of each factor  $\mathcal{O}_q$  and the product between them. It is equivariant with respect to a (right) coadjoint action of  $U_q$ , which defines the structure of  $U_q$ -module of  $\mathcal{L}_{0,n}$ . The module algebra  $\mathcal{L}_{0,n}$  has an integral form  $\mathcal{L}_{0,n}^A$ , defined over  $A$ , endowed with a coadjoint action of the *unrestricted* integral form  $U_A$  of  $U_q$  introduced by De Concini-Kac [33]. The algebra  $\mathcal{L}_{0,n}^A$  is obtained by replacing  $\mathcal{O}_q$  in the construction of  $\mathcal{L}_{0,n}$  with the restricted dual  $\mathcal{O}_A$  of the integral form  $U_A^{res}$  of  $U_q$  defined by Lusztig [52], or equivalently with the restricted dual of the integral form  $\Gamma$  of  $U_q$  defined by De Concini-Lyubashenko [36]. The unrestricted integral form  $\mathcal{M}_{0,n}^A$  of  $\mathcal{M}_{0,n}$  is defined as the subalgebra of invariant elements,

$$\mathcal{M}_{0,n}^A := (\mathcal{L}_{0,n}^A)^{U_A}.$$

A cornerstone of the theory of  $\mathcal{M}_{0,n}^A$  is a map originally due to Alekseev [1], building on works of Drinfeld [31] and Reshetikhin and Semenov-Tian-Shansky [61]. In [23] we showed that it eventually provides isomorphisms of module algebras and algebras respectively,

$$\Phi_n: \mathcal{L}_{0,n}^A \rightarrow (U_A^{\otimes n})^{lf}, \Phi_n: \mathcal{M}_{0,n}^A \rightarrow (U_A^{\otimes n})^{U_A}$$

where  $U_A^{\otimes n}$  is endowed with a right adjoint action of  $U_A$ , and  $(U_A^{\otimes n})^{lf}$  is the subalgebra of locally finite elements with respect to this action. When  $n = 1$  the algebra  $U_A^{lf}$  has been studied in great detail by Joseph-Letzter [44, 45, 43]; their results we use have been greatly simplified in [70].

All the material we need about the results discussed above is described in [23], and recalled in Section 2.1-2.2.

Our first result, proved in Section 3, is:

**Theorem 1.1.**  *$\mathcal{L}_{0,n}$ ,  $\mathcal{M}_{0,n}$  and their unrestricted integral forms and specializations at  $q \in \mathbb{C} \setminus \{0, 1\}$  are Noetherian rings, and finitely generated rings.*

In [23] we proved that these algebras have no non-trivial zero divisors. Also, we deduced Theorem 1.1 in the  $sl(2)$  case by using an isomorphism between  $\mathcal{M}_{0,n}(sl(2))$  and the skein algebra of a sphere with  $n + 1$  punctures, which by a result of [57] is Noetherian and finitely generated. Our approach here is completely different. For  $\mathcal{L}_{0,n}$  we adapt the proof given by Voigt-Yuncken [70] of a result of Joseph [43], which asserts that  $U_q^{lf}$  is a Noetherian ring (Theorem 3.1). For  $\mathcal{M}_{0,n}$  we deduce the result from the one for  $\mathcal{L}_{0,n}$ , by following a line of proof of the Hilbert-Nagata theorem in classical invariant theory (Theorem 3.2).

From Section 4 we consider the specializations  $\mathcal{L}_{0,n}^\epsilon$  of  $\mathcal{L}_{0,n}^A$  at  $q = \epsilon$ , a root of unity of odd order  $l$  coprime to 3 if  $\mathfrak{g}$  has  $G_2$  components. In [36], De Concini-Lyubashenko introduced a central subalgebra  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  of  $\mathcal{O}_\epsilon$  isomorphic to the coordinate ring  $\mathcal{O}(G)$ , and proved that the  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ -module  $\mathcal{O}_\epsilon$  is projective of rank  $l^{dim \mathfrak{g}}$ . As observed by Brown-Gordon-Stafford

[21], Bass' Cancellation theorem in  $K$ -theory and the fact that  $K_0(\mathcal{O}(G)) \cong \mathbb{Z}$ , proved by Marlin [59], imply that this module is free.

The section 4 proves the analogous property for  $\mathcal{L}_{0,n}^\epsilon$  and  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ , the subring of  $\mathcal{L}_{0,n}^\epsilon$  formed by the invariant elements of  $\mathcal{L}_{0,n}^\epsilon$  with respect to the right coadjoint action of  $U_\epsilon$ . Note that we trivially have an inclusion  $\mathcal{M}_{0,n}^{A,\epsilon} \subset (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ ; also the center  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  of  $\mathcal{L}_{0,n}^\epsilon$  is contained in  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  (this follows from [23], Proposition 6.17). We have (see Proposition 4.2 and Theorem 4.7):

**Theorem 1.2.**  $\mathcal{L}_{0,n}^\epsilon$  has a central subalgebra  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  isomorphic to  $\mathcal{O}(G)^{\otimes n}$ , and it is a free  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module of rank  $l^{n \cdot \dim \mathfrak{g}}$ , isomorphic to the  $\mathcal{O}(G)^{\otimes n}$ -module  $\mathcal{O}_\epsilon^{\otimes n}$ .

We give a direct and self-contained proof of the theorem by adapting to  $\mathcal{L}_{0,n}^\epsilon$  the arguments of Theorem 7.2 of De Concini-Lyubashenko [36]. In particular we study the coregular action of the braid group of  $\mathfrak{g}$  on  $\mathcal{L}_{0,1}^\epsilon$ ; by the way, in the Appendix we provide different proofs of some technical facts shown in [36].

However, Theorem 1.2 may be deduced from the results of [36, 21] recalled above once the required properties of  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  are settled. The most natural definition of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  is  $\Phi_1^{-1}(U_\epsilon^{lf} \cap \mathcal{Z}_0(U_\epsilon))$ , where  $\mathcal{Z}_0(U_\epsilon)$  is the De Concini-Kac-Procesi central subalgebra of  $U_\epsilon$ , and  $U_\epsilon^{lf}$  the specialization at  $q = \epsilon$  of the algebra  $U_A^{lf}$ . Thus it is not directly connected to  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ , and the algebra structures of  $\mathcal{L}_{0,1}^\epsilon$  and  $\mathcal{O}_\epsilon$  are completely different indeed. We show that nevertheless  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  and  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  coincide and give  $\mathcal{L}_{0,1}^\epsilon$  and  $\mathcal{O}_\epsilon$  the same module structures over these subalgebras. For arbitrary  $n$  we set  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) = \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$ , which we show is indeed central in  $\mathcal{L}_{0,n}^\epsilon$ . All this relies on results of De Concini-Kac [33], De Concini-Procesi [34, 35], and De Concini-Lyubashenko [36], that we recall in Section 2.3-2.4. Therefore  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{O}_\epsilon^{\otimes n}$  are the same modules over  $\mathcal{O}(G)^{\otimes n}$ , which proves Theorem 1.2 by using the results of [36, 21].

It is worth noticing that basis of  $\mathcal{L}_{0,n}^\epsilon$  over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  are complicated. For instance, the only basis we know in the case  $\mathfrak{g} = sl(2)$ , which is described in [37], is far from being obvious (see (43)).

In Section 5 we turn to fraction rings. As mentioned above  $\mathcal{L}_{0,n}^\epsilon$  has no non-trivial zero divisors. Therefore  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is an integral domain. Denote by  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  its fraction field. Consider the rings

$$Q(\mathcal{L}_{0,n}^\epsilon) = Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} \mathcal{L}_{0,n}^\epsilon$$

and

$$Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) = Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}.$$

Throughout the paper, unless we mention it explicitly we follow the conventions of Mc Connell-Robson [60] as regards the terminology of ring theory; in particular, for the notions of central simple algebras, their classical orders, maximal classical orders, PI degrees and trace ring see in [60] the sections 5.3 and 13.3.6-13.6.7.

Denote by  $m$  the rank of  $\mathfrak{g}$ , and by  $N$  the number of its positive roots. We prove:

**Theorem 1.3.** (1)  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a central simple algebra of PI degree  $l^{nN}$ , and  $\mathcal{L}_{0,n}^\epsilon$  is a maximal order of  $Q(\mathcal{L}_{0,n}^\epsilon)$ .

(2)  $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ ,  $n \geq 2$ , is a central simple algebra of PI degree  $l^{N(n-1)-m}$ .

The first claim means that  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a complex subalgebra of a full matrix algebra  $Mat_d(\mathbb{F})$ , where  $d = l^{nN}$  and  $\mathbb{F}$  is a finite extension of  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  such that

$$\mathbb{F} \otimes_{Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} Q(\mathcal{L}_{0,n}^\epsilon) = Mat_d(\mathbb{F}).$$

We deduce it from Theorem 1.2 and the computation of the degree of  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  as a field extension of  $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ . This computation uses  $\Phi_n$  and the computation of the degree of  $Q(\mathcal{Z}(U_\epsilon))$  over  $Q(\mathcal{Z}_0(U_\epsilon))$  by De Concini-Kac [33] (see Proposition 5.3).

The second claim implies in particular that  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is an integrally closed domain, and that it coincides with the trace ring of  $Q(\mathcal{L}_{0,n}^\epsilon)$ . It is proved in Theorem 5.6. More precisely we prove that  $\mathcal{L}_{0,n}^\epsilon$  is *integrally closed* in  $Q(\mathcal{L}_{0,n}^\epsilon)$ , in the sense of [33, 35]. So, before the theorem we show in Lemma 5.5 that a ring  $A$  with no non-trivial zero divisors, Noetherian center, and finite dimensional classical fraction algebra  $Q$ , which is the case of  $\mathcal{L}_{0,n}^\epsilon$  and  $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ , is integrally closed in  $Q$  if and only if it is maximal as a (classical) order. For the sake of clarity we have included a general discussion of all these notions before Theorem 5.6. The proof uses the facts that  $\mathcal{O}_\epsilon$  is a maximal order of its classical fraction algebra, which is Theorem 7.4 of [36], and that the twist which defines the algebra structure of  $\mathcal{L}_{0,n}^\epsilon$  from  $\mathcal{O}_\epsilon^{\otimes n}$  keeps the  $\mathcal{Z}_0$ -module structure unchanged. It seems rather difficult to prove that  $\mathcal{L}_{0,n}^\epsilon$  is a maximal order without this twist argument, essentially because it is not clear how to find two “independent” localizations which are maximal orders; however we can do it in the  $\mathfrak{sl}(2)$  case when  $n = 1$ .

At the end of the section we deduce Theorem 1.3 (2) from Theorem 1.3 (1), the centralizer theorem for central simple algebras, and a few results of [23] and [36].

**1.1. Basic notations.** Given a ring  $R$ , we denote by  $\mathcal{Z}(R)$  its center, by  $\text{Spec}(R)$  its spectrum, and by  $\text{SpecM}(R)$  its *maximal* spectrum. When  $R$  is commutative and has no non-trivial zero divisors,  $Q(R)$  denotes its fraction field.

Given a Hopf algebra  $H$  with product  $m$  and and coproduct  $\Delta$ , we denote by  $H^{\text{cop}}$  (resp.  $H_{\text{op}}$ ) the Hopf algebra with the same algebra (resp. coalgebra) structure as  $H$  but the opposite coproduct  $\sigma \circ \Delta$  (resp. opposite product  $m \circ \sigma$ ), where  $\sigma(x \otimes y) = y \otimes x$ , and the antipode  $S^{-1}$ . We use Sweedler’s coproduct notation,  $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ ,  $x \in H$ .

We let  $\mathfrak{g}$  be a finite dimensional complex simple Lie algebra of rank  $m$ , with Cartan matrix  $(a_{ij})$ . We fix a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  and a basis of simple roots  $\alpha_i \in \mathfrak{h}_{\mathbb{R}}^*$ ; we denote by  $d_1, \dots, d_m$  the unique coprime positive integers such that the matrix  $(d_i a_{ij})$  is symmetric, and  $(\cdot, \cdot)$  the unique inner product on  $\mathfrak{h}_{\mathbb{R}}^*$  such that  $d_i a_{ij} = (\alpha_i, \alpha_j)$ . For any root  $\alpha$  the coroot is  $\check{\alpha} = 2\alpha/(\alpha, \alpha)$ ; in particular  $\check{\alpha}_i = d_i^{-1} \alpha_i$ . The root lattice  $Q$  is the  $\mathbb{Z}$ -lattice in  $\mathfrak{h}_{\mathbb{R}}^*$  defined by  $Q = \sum_{i=1}^m \mathbb{Z} \alpha_i$ . The weight lattice  $P$  is the  $\mathbb{Z}$ -lattice formed by all  $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$  such that

$$(\lambda, \check{\alpha}_i) \in \mathbb{Z}$$

for every  $i = 1, \dots, m$ . So  $P = \sum_{i=1}^m \mathbb{Z} \varpi_i$ , where  $\varpi_i$  is the fundamental weight dual to the simple coroot  $\check{\alpha}_i$ , ie. satisfying  $(\varpi_i, \check{\alpha}_j) = \delta_{i,j}$ . We denote by  $P_+ := \sum_{i=1}^m \mathbb{Z}_{\geq 0} \varpi_i$  the cone of dominant integral weights, by  $N$  the number of positive roots of  $\mathfrak{g}$ , by  $\rho$  half the sum of the positive roots, and by  $D$  the smallest positive integer such that  $D(\lambda, \mu) \in \mathbb{Z}$  for every  $\lambda, \mu \in P$ . Note that  $(\lambda, \alpha) \in \mathbb{Z}$  for every  $\lambda \in P$ ,  $\alpha \in Q$ , and  $D$  is the smallest positive integer such that  $DP \subset Q$ . We denote by  $\mathcal{B}(\mathfrak{g})$  the braid group of  $\mathfrak{g}$ ; we recall its standard defining relations in the Appendix (Section 6.1).

We let  $G$  be the connected and simply-connected Lie group with Lie algebra  $\mathfrak{g}$ . We put  $T_G = \exp(\mathfrak{h})$ , the maximal torus of  $G$  generated by  $\mathfrak{h}$ ;  $N(T_G)$  is the normalizer of  $T_G$ ,  $W = N(T_G)/T_G$  is the Weyl group,  $B_{\pm}$  the unique Borel subgroups such that  $B_+ \cap B_- = T_G$ , and  $U_{\pm} \subset B_{\pm}$  their unipotent subgroups.

We let  $q$  be an indeterminate, set  $A = \mathbb{C}[q, q^{-1}]$ ,  $q_i = q^{d_i}$ , and given integers  $p, k$  with  $0 \leq k \leq p$  we put

$$[p]_q = \frac{q^p - q^{-p}}{q - q^{-1}}, \quad [0]_q! = 1, \quad [p]_q! = [1]_q [2]_q \cdots [p]_q, \quad \begin{bmatrix} p \\ k \end{bmatrix}_q = \frac{[p]_q!}{[p-k]_q! [k]_q!}$$

$$(p)_q = \frac{q^p - 1}{q - 1}, \quad (0)_q! = 1, \quad (p)_q! = (1)_q (2)_q \cdots (p)_q, \quad \binom{p}{k}_q = \frac{(p)_q!}{(p-k)_q! (k)_q!}.$$

We denote by  $\epsilon$  a primitive  $l$ -th root of unity such that  $\epsilon^{2d_i} \neq 1$  is also a primitive  $l$ -th root of unity for all  $i \in \{1, \dots, m\}$ . This means that  $l$  is odd, and coprime to 3 if  $\mathfrak{g}$  has  $G_2$ -components.

In this paper we use the definition of the unrestricted integral form  $U_A(\mathfrak{g})$  given in [35], [36]; in [23] we used the one of [33], [34]. The two are (trivially) isomorphic, and have the same specialization at  $q = \epsilon$ . Also, we denote here by  $L_i$  the generators of  $U_q(\mathfrak{g})$  we denoted by  $\ell_i$  in [23].

To facilitate the comparison with [36] we note that their generators, that we will denote by  $\tilde{K}_i, \tilde{E}_i$  and  $\tilde{F}_i$ , can be written respectively as  $K_i, K_i^{-1} E_i$  and  $F_i K_i$  in our notations. They satisfy the same algebra relations.

## 2. BACKGROUND RESULTS

**2.1. On  $U_q$ ,  $\mathcal{O}_q$ ,  $\mathcal{L}_{0,n}$ ,  $\mathcal{M}_{0,n}$ , and  $\Phi_n$ .** Except when stated differently, we refer to [23], Sections 2-4 and 6, and the references therein for details about the material of this section.

The *simply-connected* quantum group  $U_q = U_q(\mathfrak{g})$  is the Hopf algebra over  $\mathbb{C}(q)$  with generators  $E_i, F_i, L_i, L_i^{-1}$ ,  $1 \leq i \leq m$ , and defining relations

$$L_i L_j = L_j L_i, \quad L_i L_i^{-1} = L_i^{-1} L_i = 1, \quad L_i E_j L_i^{-1} = q_i^{\delta_{i,j}} E_j, \quad L_i F_j L_i^{-1} = q_i^{-\delta_{i,j}} F_j$$

$$E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r = 0 \quad \text{if } i \neq j$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{ij}-r} F_j F_i^r = 0 \quad \text{if } i \neq j.$$

where for  $\lambda = \sum_{i=1}^m m_i \varpi_i \in P$  we set  $K_\lambda = \prod_{i=1}^m L_i^{m_i}$ , and  $K_i = K_{\alpha_i} = \prod_{j=1}^m L_j^{a_{ji}}$ . The coproduct  $\Delta$ , antipode  $S$ , and counit  $\varepsilon$  of  $U_q$  are given by

$$\Delta(L_i) = L_i \otimes L_i, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = K_i^{-1} \otimes F_i + F_i \otimes 1$$

$$S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i, \quad S(L_i) = L_i^{-1}$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(L_i) = 1.$$

We fix a reduced expression  $s_{i_1} \dots s_{i_N}$  of the longest element  $w_0$  of the Weyl group of  $\mathfrak{g}$ . It induces a total ordering of the positive roots,

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \dots, \beta_N = s_{i_1} \dots s_{i_{N-1}}(\alpha_{i_N}).$$

The root vectors of  $U_q$  with respect to such an ordering are defined by

$$E_{\beta_k} = T_{i_1} \dots T_{i_{k-1}}(E_{i_k}), \quad F_{\beta_k} = T_{i_1} \dots T_{i_{k-1}}(F_{i_k})$$

where  $T_i$  is Lusztig's algebra automorphism of  $U_q$  associated to the simple root  $\alpha_i$  ([53, 52], see also [30], Ch. 8). In the Appendix we recall the relation between  $T_i$  and the generator

$\hat{w}_i$  of the quantum Weyl group, which we will mostly use. Let us just recall here that the monomials  $F_{\beta_1}^{r_1} \dots F_{\beta_N}^{r_N} K_\lambda E_{\beta_N}^{t_N} \dots E_{\beta_1}^{t_1}$  ( $r_i, t_i \in \mathbb{N}$ ,  $\lambda \in P$ ) form a basis of  $U_q$ .

$U_q$  is a *pivotal* Hopf algebra, with pivotal element

$$\ell := K_{2\rho} = \prod_{j=1}^m L_j^2.$$

So  $\ell$  is group-like, and  $S^2(x) = \ell x \ell^{-1}$  for every  $x \in U_q$ .

The *adjoint* quantum group  $U_q^{ad} = U_q^{ad}(\mathfrak{g})$  is the Hopf subalgebra of  $U_q$  generated by the elements  $E_i, F_i$  ( $i = 1, \dots, m$ ) and  $K_\alpha$  with  $\alpha \in Q$ ; so  $\ell \in U_q^{ad}$ . When  $\mathfrak{g} = sl(2)$ , we simply write the above generators  $E = E_1, F = F_1, L = L_1, K = K_1$ .

We denote by  $U_q(\mathfrak{n}_+)$ ,  $U_q(\mathfrak{n}_-)$  and  $U_q(\mathfrak{h})$  the subalgebras of  $U_q$  generated respectively by the  $E_i$ , the  $F_i$ , and the  $K_\lambda$  ( $\lambda \in P$ ), and by  $U_q(\mathfrak{b}_+)$  and  $U_q(\mathfrak{b}_-)$  the subalgebras generated by the  $E_i$  and the  $K_\lambda$ , and by the  $F_i$  and the  $K_\lambda$ , respectively (they are the two-sided ideals generated by  $U_q(\mathfrak{n}_\pm)$ ). We do similarly with  $U_q^{ad}$ .

$U_q^{ad}$  is not a braided Hopf algebra in a strict sense, but it has braided categorical completions. Namely, denote by  $\mathcal{C}$  the category of type 1 finite dimensional  $U_q^{ad}$ -modules, by  $Vect$  the category of finite dimensional  $\mathbb{C}(q)$ -vector spaces, and by  $F_{\mathcal{C}} : \mathcal{C} \rightarrow Vect$  the forgetful functor. The *categorical completion*  $\mathbb{U}_q^{ad}$  of  $U_q^{ad}$  is the set of natural transformations  $F_{\mathcal{C}} \rightarrow F_{\mathcal{C}}$ .

Let us recall briefly what this means and implies. For details we refer to the sections 2 and 3 of [23] (see also [70], Section 2.10, where  $\mathbb{U}_q$  below is formulated in terms of multiplier Hopf algebras). An element of  $\mathbb{U}_q^{ad}$  is a collection  $(a_V)_{V \in Ob(\mathcal{C})}$ , where  $a_V \in End_{\mathbb{C}(q)}(V)$  satisfies  $F_{\mathcal{C}}(f) \circ a_V = a_W \circ F_{\mathcal{C}}(f)$  for any objects  $V, W$  of  $\mathcal{C}$  and any arrow  $f \in Hom_{U_q^{ad}}(V, W)$ . It is not hard to see that  $\mathbb{U}_q^{ad}$  inherits from  $\mathcal{C}$  a natural structure of Hopf algebra such that the map

$$\begin{aligned} \iota : U_q^{ad} &\longrightarrow \mathbb{U}_q^{ad} \\ x &\longmapsto (\pi_V(x))_{V \in Ob(\mathcal{C})} \end{aligned}$$

is a morphism of Hopf algebras, where  $\pi_V : U_q^{ad} \rightarrow End(V)$  is the representation associated to a module  $V$  in  $\mathcal{C}$ . It is a theorem that this map is injective;  $\mathbb{U}_q^{ad}$  can be understood as a weak- $*$  completion of  $U_q^{ad}$  by means of the pairing  $\langle \cdot, \cdot \rangle$  introduced below. From now on, let us extend the coefficient ring of the modules and morphisms in  $\mathcal{C}$  to  $\mathbb{C}(q^{1/D})$ . Put

$$\mathbb{U}_q = \mathbb{U}_q^{ad} \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})$$

One can show that the map  $\iota$  above extends to an embedding of  $U_q \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})$  in  $\mathbb{U}_q$ . The category  $\mathcal{C}$ , with coefficients in  $\mathbb{C}(q^{1/D})$ , is braided and ribbon. We postpone a discussion of that fact to Section 2.3, where it will be developed. As a consequence,  $\mathbb{U}_q$  is a quasitriangular and ribbon Hopf algebra. The R-matrix of  $\mathbb{U}_q$  is the family of morphisms

$$R = ((R_h)_{V,W})_{V,W \in Ob(\mathcal{C})}$$

where  $q = e^h$ ,  $R_h$  is the universal R-matrix of the quantized universal enveloping algebra  $U_h(\mathfrak{g})$ , and  $(R_h)_{V,W} \in End(V \otimes W)$ , for every modules  $V, W$  in  $\mathcal{C}$ , is the endomorphism defined by the action of  $R_h$  on  $V \otimes W$  (which is well-defined). The ribbon element  $v_h$  of  $U_h(\mathfrak{g})$  defines similarly the ribbon element  $v = ((v_h)_V)_V$  of  $\mathbb{U}_q$ . One defines the categorical tensor product  $\mathbb{U}_q^{\hat{\otimes} 2}$  similarly as  $\mathbb{U}_q$ ; it contains all the infinite series of elements of  $\mathbb{U}_q^{\hat{\otimes} 2}$  having only a finite number of non-zero terms when evaluated on a given module  $V \otimes W$  of  $\mathcal{C}$ . The expansion of  $R_h$  as an infinite series in  $U_h(\mathfrak{g})^{\hat{\otimes} 2}$  induces an expansion of  $R$  as an infinite series in  $\mathbb{U}_q^{\hat{\otimes} 2}$ . Adapting Sweedler's coproduct notation  $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$  we find convenient



to write this series as

$$(1) \quad R = \sum_{(R)} R_{(1)} \otimes R_{(2)}.$$

We put  $R^+ := R$ ,  $R^- := (\sigma \circ R)^{-1}$  where  $\sigma$  is the flip map  $x \otimes y \mapsto y \otimes x$ .

The *quantum function Hopf algebra*  $\mathcal{O}_q = \mathcal{O}_q(G)$  is the restricted dual of  $U_q^{ad}$ , ie. the set of  $\mathbb{C}(q)$ -linear maps  $f: U_q^{ad} \rightarrow \mathbb{C}(q)$  such that  $\text{Ker}(f)$  contains a cofinite two sided ideal  $I$  (ie. such that  $I \oplus M = U_q$  for some finite dimensional vector space  $M$ ), and  $\prod_{s=-r}^r (K_i - q_i^s) \in I$  for some  $r \in \mathbb{N}$  and every  $i$ . The structure maps of  $\mathcal{O}_q$  are defined dually to those of  $U_q^{ad}$ . We denote by  $\star$  its product. The algebras  $\mathcal{O}_q(T_G)$ ,  $\mathcal{O}_q(U_\pm)$ ,  $\mathcal{O}_q(B_\pm)$  are defined similarly, by replacing  $U_q^{ad}$  with  $U_q^{ad}(\mathfrak{h})$ ,  $U_q^{ad}(\mathfrak{n}_\pm)$ ,  $U_q^{ad}(\mathfrak{b}_\pm)$  respectively.  $\mathcal{O}_q$  is generated as an algebra by the functionals  $x \mapsto w(\pi_V(x)v)$ ,  $x \in U_q^{ad}$ , for every object  $V \in \text{Ob}(\mathcal{C})$  and vectors  $v \in V$ ,  $w \in V^*$ . Such functionals are called *matrix coefficients*. We can uniquely extend the (non-degenerate) evaluation pairing  $\langle \cdot, \cdot \rangle: \mathcal{O}_q \otimes U_q^{ad} \rightarrow \mathbb{C}(q)$  to a bilinear pairing  $\langle \cdot, \cdot \rangle: \mathcal{O}_q \otimes \mathbb{U}_q \rightarrow \mathbb{C}(q^{1/D})$  such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}_q \otimes U_q^{ad} & \xrightarrow{\langle \cdot, \cdot \rangle} & \mathbb{C}(q) \\ id \otimes \iota \downarrow & \nearrow \langle \cdot, \cdot \rangle & \\ \mathcal{O}_q \otimes \mathbb{U}_q & & \end{array}$$

This pairing is defined by

$$\langle {}_Y\phi_v^w, (a_X)_X \rangle = w(a_Y v)$$

for every  $(a_X)_X \in \mathbb{U}_q$ ,  ${}_Y\phi_v^w \in \mathcal{O}_q$ . It is a perfect pairing, and reflects the properties of the R-matrix  $R \in \mathbb{U}_q^{\otimes 2}$  in a subtle way. In particular, these properties imply that the maps

$$(2) \quad \begin{aligned} \Phi^\pm : \mathcal{O}_q &\longrightarrow U_q^{cop} \\ \alpha &\longmapsto (\alpha \otimes id)(R^\pm) = \sum_{(R^\pm)} \langle \alpha, R_{(1)}^\pm \rangle R_{(2)}^\pm \end{aligned}$$

are well-defined morphisms of Hopf algebras. Here we stress that it is the simply-connected quantum group  $U_q^{cop}$  that is the range of  $\Phi^\pm$ . This will be explained in more details in Section 2.3.

The *quantum loop algebra*  $\mathcal{L}_{0,1} = \mathcal{L}_{0,1}(\mathfrak{g})$  is defined by twisting the product  $\star$  of  $\mathcal{O}_q$ , keeping the same underlying linear space. The new product is equivariant with respect to the right coadjoint action  $coad^r$  of  $U_q^{ad}$ ; noting that  $coad^r$  extends to an action of the simply-connected quantum group  $U_q$ , the new product thus gives  $\mathcal{L}_{0,1}$  a structure of  $U_q$ -module algebra. Recall that

$$coad^r(x)(\alpha) = \sum_{(x)} S(x_{(2)}) \triangleright \alpha \triangleleft x_{(1)}$$

for all  $x \in U_q$  and  $\alpha \in \mathcal{O}_q$ , where  $\triangleright, \triangleleft$  are the left and right coregular actions of  $U_q$  on  $\mathcal{O}_q$ , defined by

$$x \triangleright \alpha := \sum_{(\alpha)} \alpha_{(1)} \langle \alpha_{(2)}, x \rangle, \quad \alpha \triangleleft x := \sum_{(\alpha)} \langle \alpha_{(1)}, x \rangle \alpha_{(2)}.$$

Using the fact that  $U_q \otimes \mathbb{C}(q^{1/D})$  can be regarded as a subspace of  $\mathbb{U}_q$ , these actions extend naturally to actions of  $\mathbb{U}_q$ . The product of  $\mathcal{L}_{0,1}$  is expressed in terms of  $\star$  by the formula

([23], Proposition 4.1):

$$(3) \quad \alpha\beta = \sum_{(R),(R')} (R_{(2')}S(R_{(2)}) \triangleright \alpha) \star (R_{(1')} \triangleright \beta \triangleleft R_{(1)}),$$

where  $\sum_{(R)} R_{(1)} \otimes R_{(2)}$  and  $\sum_{(R')} R_{(1')} \otimes R_{(2')}$  are expansions of two copies of  $R \in \mathbb{U}_q^{\otimes 2}$ . Note that the sum in (3) has only a finite number of non zero terms. This product gives  $\mathcal{L}_{0,1}$  (like  $\mathcal{O}_q$ ) a structure of module algebra for the actions  $\triangleright, \triangleleft$ , and also for  $\text{coad}^r(x)$ . Spelling this out for  $\text{coad}^r$ , this means

$$\text{coad}^r(x)(\alpha\beta) = \sum_{(x)} \text{coad}^r(x_{(1)})(\alpha)\text{coad}^r(x_{(2)})(\beta).$$

The relations between  $\mathcal{O}_q$ ,  $\mathcal{L}_{0,1}$  and  $U_q$  (the simply-connected quantum group) are encoded by the map

$$(4) \quad \begin{array}{ccc} \Phi_1 : \mathcal{O}_q & \longrightarrow & \mathbb{U}_q \\ \alpha & \longmapsto & (\alpha \otimes id)(RR') \end{array}$$

where  $R' = \sigma \circ R$ , and as usual  $\sigma : x \otimes y \mapsto y \otimes x$ . Note that

$$\Phi_1 = m \circ (\Phi^+ \otimes (S^{-1} \circ \Phi^-)) \circ \Delta.$$

We call  $\Phi_1$  the *RSD* map, for Drinfeld, Reshetikhin and Semenov-Tian-Shansky introduced it first (see [31, 61],[58]). Recall that  $U_q$  embeds in  $\mathbb{U}_q$ . It is a fundamental result of the theory ([28, 43, 9]) that  $\Phi_1$  affords an isomorphism of  $U_q$ -modules

$$\Phi_1 : \mathcal{O}_q \rightarrow U_q^{\text{lf}}.$$

For full details on that result we refer to Section 2.12 of [70] (where different conventions are used). Here,  $U_q^{\text{lf}}$  is the set of *locally finite* elements of  $U_q$ , endowed with the right adjoint action  $\text{ad}^r$  of  $U_q$ . It is defined by

$$U_q^{\text{lf}} := \{x \in U_q \mid \text{rk}_{\mathbb{C}(q)}(\text{ad}^r(U_q)(x)) < \infty\}$$

and

$$\text{ad}^r(y)(x) = \sum_{(y)} S(y_{(1)})xy_{(2)}$$

for every  $x, y \in U_q$ . The action  $\text{ad}^r$  gives in fact  $U_q^{\text{lf}}$  a structure of right  $U_q$ -module algebra. Moreover,  $\Phi_1$  affords an isomorphism of  $U_q$ -module algebras

$$(5) \quad \Phi_1 : \mathcal{L}_{0,1} \rightarrow U_q^{\text{lf}}.$$

The centers  $\mathcal{Z}(\mathcal{L}_{0,1})$  of  $\mathcal{L}_{0,1}$ , and  $\mathcal{Z}(U_q)$  of  $U_q$ , coincide respectively with  $\mathcal{L}_{0,1}^{U_q}$  and  $U_q^{U_q}$ , the subsets of  $U_q$ -invariants elements of  $\mathcal{L}_{0,1}$  and  $U_q$ . As a consequence,  $\Phi_1$  affords an isomorphism between  $\mathcal{Z}(\mathcal{L}_{0,1})$  and  $\mathcal{Z}(U_q)$ .

The *quantum graph algebra*  $\mathcal{L}_{0,n} = \mathcal{L}_{0,n}(\mathfrak{g})$  is the braided tensor product of  $n$  copies of  $\mathcal{L}_{0,1}$  (considered as a  $\mathbb{U}_q$ -module algebra). Thus it coincides with  $\mathcal{L}_{0,1}^{\otimes n}$  as a linear space, and it is a right  $U_q$ -module algebra, the action of  $U_q$  (extending  $\text{coad}^r$  on  $\mathcal{L}_{0,1}$ ) being given by

$$\text{coad}_n^r(y)(\alpha^{(1)} \otimes \dots \otimes \alpha^{(n)}) = \sum_{(y)} \text{coad}^r(y_{(1)})(\alpha^{(1)}) \otimes \dots \otimes \text{coad}^r(y_{(n)})(\alpha^{(n)})$$

for all  $y \in U_q$  and  $\alpha^{(1)} \otimes \dots \otimes \alpha^{(n)} \in \mathcal{L}_{0,n}$ . The algebra structure can be explicited as follows. For every  $1 \leq a \leq n$  define  $\mathfrak{i}_a: \mathcal{L}_{0,1} \rightarrow \mathcal{L}_{0,n}$  by  $\mathfrak{i}_a(x) = 1^{\otimes(a-1)} \otimes x \otimes 1^{\otimes(n-a)}$ ;  $\mathfrak{i}_a$  is an embedding of  $U_q$ -module algebras. We will use the notations

$$\mathcal{L}_{0,n}^{(a)} := \text{Im}(\mathfrak{i}_a), \quad (\alpha)^{(a)} := \mathfrak{i}_a(\alpha).$$

Take  $(\alpha)^{(a)}, (\alpha')^{(a)} \in \mathcal{L}_{0,n}^{(a)}$  and  $(\beta)^{(b)}, (\beta')^{(b)} \in \mathcal{L}_{0,n}^{(b)}$  with  $a < b$ . Then the product of  $\mathcal{L}_{0,n}$  is given by the following formula (see in [23] the proposition 6.2-6.3 and the formulas (13)-(41)-(42)):

$$(6) \quad \begin{aligned} & \left( (\alpha)^{(a)} \otimes (\beta)^{(b)} \right) \left( (\alpha')^{(a)} \otimes (\beta')^{(b)} \right) \\ &= \sum_{(R^1), \dots, (R^4)} \left( \alpha \left( S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha' \triangleleft R_{(1)}^1 R_{(1)}^2 \right) \right)^{(a)} \\ & \quad \otimes \left( \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \right) \beta' \right)^{(b)} \end{aligned}$$

where  $R^i = \sum_{(R^i)} R_{(1)}^i \otimes R_{(2)}^i$ ,  $i \in \{1, 2, 3, 4\}$ , are expansions of four copies of  $R \in \mathbb{U}_q^{\hat{\otimes} 2}$ , and on the right-hand side the product is componentwise that of  $\mathcal{L}_{0,1}$ . Later we will use the fact that the product of  $\mathcal{L}_{0,n}$  is obtained from the standard (componentwise) product of  $\mathcal{L}_{0,1}^{\otimes n}$  by a process that may be inverted. Indeed, (6) can be rewritten as

$$(7) \quad \left( (\alpha)^{(a)} \otimes (\beta)^{(b)} \right) \left( (\alpha')^{(a)} \otimes (\beta')^{(b)} \right) = \sum_{(F)} (\alpha)^{(a)} \left( (\alpha')^{(a)} \cdot F_{(2)} \right) \otimes \left( (\beta)^{(b)} \cdot F_{(1)} \right) (\beta')^{(b)}$$

where  $F = \sum_{(F)} F_{(1)} \otimes F_{(2)} := (\Delta \otimes \Delta)(R')$ , and the symbol “ $\cdot$ ” stands for the right action of  $\mathbb{U}_q^{\hat{\otimes} 2}$  on  $\mathcal{L}_{0,1}$  that may be read from (6). The tensor  $F$  is known as a twist. Then, by replacing  $F$  with its inverse  $\bar{F} = (\Delta \otimes \Delta)(R'^{-1})$ , one can express the product of  $\mathcal{L}_{0,1}^{\otimes n}$  in terms of the product of  $\mathcal{L}_{0,n}$  by

$$(8) \quad (\alpha)^{(a)} (\alpha')^{(a)} \otimes (\beta)^{(b)} (\beta')^{(b)} = \sum_{(\bar{F})} \left( (\alpha)^{(a)} \otimes \left( (\beta)^{(b)} \cdot \bar{F}_{(1)} \right) \right) \left( \left( (\alpha')^{(a)} \cdot \bar{F}_{(2)} \right) \otimes (\beta')^{(b)} \right).$$

We call *quantum moduli algebra* and denote by

$$\mathcal{M}_{0,n} = \mathcal{M}_{0,n}(\mathfrak{g})$$

the subalgebra  $\mathcal{L}_{0,n}^{U_q}$  of  $\mathcal{L}_{0,n}$  formed by the  $U_q$ -invariant elements.

Consider the following action of  $U_q$  on the tensor product algebra  $U_q^{\otimes n}$ , which extends  $ad^r$  on  $U_q$ :

$$(9) \quad ad_n^r(y)(x) = \sum_{(y)} \Delta^{(n)}(S(y_{(1)})) x \Delta^{(n)}(y_{(2)})$$

for all  $y \in U_q$ ,  $x \in U_q^{\otimes n}$ . This action gives  $U_q^{\otimes n}$  a structure of right  $U_q$ -module algebra. In [1] Alekseev introduced a morphism of  $U_q$ -module algebras  $\Phi_n: \mathcal{L}_{0,n} \rightarrow U_q^{\otimes n}$  which extends  $\Phi_1$ . In Proposition 6.5 and Lemma 6.8 of [23] we showed that  $\Phi_n$  affords isomorphisms

$$(10) \quad \Phi_n: \mathcal{L}_{0,n} \rightarrow (U_q^{\otimes n})^{lf}, \quad \Phi_n: \mathcal{M}_{0,n} \rightarrow (U_q^{\otimes n})^{U_q}$$

where  $(U_q^{\otimes n})^{lf}$  is the set of  $ad_n^r$ -locally finite elements of  $U_q^{\otimes n}$ . We call  $\Phi_n$  the *Alekseev map*; we will not use the definition of  $\Phi_n$  in this paper.

It is a key argument of the proof of (10), to be used later, that the set of locally finite elements of  $U_q^{\otimes n}$  for  $(ad^r)^{\otimes n} \circ \Delta^{(n-1)}$  coincides with  $(U_q^{lf})^{\otimes n}$ ; this follows from the main result of [49]. Using that the map

$$(11) \quad \psi_n = \Phi_n \circ (\Phi_1^{-1})^{\otimes n}$$

extends to a linear automorphism of  $U_q^{\otimes n}$  which intertwines the actions  $(ad^r)^{\otimes n} \circ \Delta^{(n-1)}$  and  $ad_n^r$  of  $U_q$ , we deduced that  $\psi_n((U_q^{lf})^{\otimes n}) = (U_q^{\otimes n})^{lf}$ , whence  $\text{Im}(\Phi_n) = (U_q^{\otimes n})^{lf}$ .

**Remark 2.1.** We have  $(U_q^{lf})^{\otimes n} \neq (U_q^{\otimes n})^{lf}$ , and in fact there is not even an inclusion. Indeed let  $\Omega = (q - q^{-1})^2 FE + qK + q^{-1}K^{-1}$  be the standard Casimir element of  $U_q(sl(2))$ . We trivially have  $\Delta(\Omega) \in (U_q^{\otimes 2})^{lf}$  but

$$\Delta(\Omega) = (q - q^{-1})^2 (K^{-1}E \otimes FK + F \otimes E) + \Omega \otimes K + K^{-1} \otimes \Omega - (q + q^{-1})K^{-1} \otimes K$$

and therefore  $\Delta(\Omega) \notin (U_q^{lf})^{\otimes 2}$ , since  $K \notin U_q^{lf}$  (see eg. Theorem 2.2 (2)).

Let us point out here two important consequences of (10). First,  $\Phi_n$  yields isomorphisms between centers,  $\mathcal{Z}(\mathcal{L}_{0,n}) \cong \mathcal{Z}(U_q)^{\otimes n}$  and  $\mathcal{Z}(\mathcal{L}_{0,n}^{U_q}) \cong \mathcal{Z}((U_q^{\otimes n})^{U_q})$ , where one can show that ([23], Lemma 6.25)

$$\mathcal{Z}((U_q^{\otimes n})^{U_q}) \cong \Delta^{(n-1)}(\mathcal{Z}(U_q)) \otimes_{\mathbb{C}(q)} \mathcal{Z}(U_q)^{\otimes n}.$$

Second, we see that  $\mathcal{L}_{0,n}$  (and therefore  $\mathcal{M}_{0,n}$ ) has no non-trivial zero divisors, by using the isomorphisms  $\Phi_n: \mathcal{L}_{0,n} \rightarrow (U_q^{\otimes n})^{lf} \subset U_q^{\otimes n}$  and  $U_q^{\otimes n} \cong U_q(\mathfrak{g}^{\oplus n})$ , and the fact that  $U_q(\mathfrak{g}^{\oplus n})$  has no non-trivial zero divisors (proved eg. in [33]).

**2.2. Integral forms and specializations.** An integral form of a (Hopf)  $\mathbb{C}(q)$ -algebra is a (Hopf)  $A$ -subalgebra, where  $A = \mathbb{C}[q, q^{-1}]$ , that becomes isomorphic to the algebra after tensoring it with  $\mathbb{C}(q)$ . We consider three integral forms related by the pairing  $\langle \cdot, \cdot \rangle$ , one of  $U_q$ , one of  $U_q^{ad}$ , and one of  $\mathcal{O}_q$ .

The *unrestricted integral form* of  $U_q$  is the  $A$ -subalgebra  $U_A = U_A(\mathfrak{g})$  introduced by De Concini–Kac–Procesi in [35], Section 12 (and in a differently normalized form in [33] and [34]). It is generated by the elements  $(i = 1, \dots, m)$

$$\bar{E}_i = (q_i - q_i^{-1})E_i, \quad \bar{F}_i = (q_i - q_i^{-1})F_i, \quad L_i, \quad L_i^{-1}.$$

Clearly, the subalgebra of locally finite elements of  $U_A$  is  $U_A^{lf} = U_A \cap U_q^{lf}$ . Similarly, we define the *unrestricted integral form* of  $U_q^{ad}$  as the  $A$ -subalgebra  $U_A^{ad} \subset U_A$  generated by the elements  $\bar{E}_i, \bar{F}_i$  and  $K_i^{\pm 1}$ , for  $i = 1, \dots, m$ .

The *restricted integral form* of  $U_q^{ad}$  is the  $A$ -subalgebra  $\Gamma = \Gamma(\mathfrak{g})$  introduced by De Concini–Lyubashenko in [36], Sections 2-3. It is generated by the elements  $(i = 1, \dots, m)$

$$E_i^{(k)} = \frac{E_i^k}{[k]_{q_i}!}, \quad F_i^{(k)} = \frac{F_i^k}{[k]_{q_i}!}, \quad (K_i; t)_{q_i} = \prod_{s=1}^t \frac{K_i q_i^{-s+1} - 1}{q_i^s - 1}, \quad K_i^{-1}$$

where  $k \in \mathbb{N}$ ,  $t \in \mathbb{N}$  (setting  $(K_i; 0)_{q_i} = 1$  by convention).

Note that  $\Gamma$  contains the elements  $K_i$ , and the *unrestricted integral form*  $U_A^{ad}$ . It plays a fundamental rôle in relation with the integral pairings  $\pi_A^{\pm}$  considered in Section 2.3; it is by this rôle that  $\Gamma$  is more suited to our purposes than the more standard *restricted integral form*  $U_A^{res}$  defined by Lusztig, and discussed below.

The integral forms  $U_A(\mathfrak{h})$ ,  $U_A(\mathfrak{b}_{\pm})$  and  $\Gamma(\mathfrak{h})$ ,  $\Gamma(\mathfrak{b}_{\pm})$  associated to the subalgebras  $\mathfrak{h}$ ,  $\mathfrak{b}_{\pm} \subset \mathfrak{g}$  are the subalgebras of  $U_A$  and  $\Gamma$  defined in the obvious way. For instance the ‘‘Cartan’’ subalgebra  $\Gamma(\mathfrak{h})$  is generated by the elements  $(K_i; t)_{q_i}$  and  $K_i^{-1}$ .

Denote by  $\mathcal{C}_A$  the category of  $\Gamma$ -modules which are free  $A$ -modules of finite rank, and semisimple as  $\Gamma(\mathfrak{h})$ -modules; so they have a basis where  $K_i$  and  $(K_i; t)_{q_i}$  act diagonally with respective eigenvalues of the form

$$q_i^k, \left( \begin{array}{c} k \\ t \end{array} \right)_{q_i} \quad k \in \mathbb{Z}, t \in \mathbb{N}^*.$$

The *integral* quantum function Hopf algebra  $\mathcal{O}_A = \mathcal{O}_A(G)$  is the restricted dual of  $\Gamma$ , ie. the set of  $A$ -linear maps  $f: \Gamma \rightarrow A$  such that  $\text{Ker}(f)$  contains a cofinite two sided ideal  $I$ , and  $\prod_{s=-r}^r (K_i - q_i^s) \in I$  for some  $r \in \mathbb{N}$  and every  $i$ .  $\mathcal{O}_A$  is an integral form of  $\mathcal{O}_q$ . The algebras  $\mathcal{O}_A(T_G)$ ,  $\mathcal{O}_A(U_\pm)$ ,  $\mathcal{O}_A(B_\pm)$  are defined similarly, by replacing  $\Gamma$  with  $\Gamma(\mathfrak{h})$ ,  $\Gamma(\mathfrak{n}_\pm)$ ,  $\Gamma(\mathfrak{b}_\pm)$  respectively.  $\mathcal{O}_A$  is generated as an algebra by the matrix coefficients  $x \mapsto v^i(\pi_V(x)v_i)$ ,  $x \in \Gamma$ , for every module  $V$  in  $\mathcal{C}_A$  where  $(v_i)$  is an  $A$ -basis of  $V$  and  $(v^i)$  the dual  $A$ -basis of the dual module  $V^*$ .

It is immediate that the  $U_q$ -module structure of  $\mathcal{O}_q$  restricts to an  $U_A$ -module structure on  $\mathcal{O}_A$ .

We note that  $\mathcal{O}_A$  is also the restricted dual of  $U_A^{res}$ , the Lusztig integral form of  $U_q^{ad}$  [52, 53], defined as  $\Gamma$  except that the  $(K_i; t)_{q_i}$  ( $i = 1, \dots, m$ ), are replaced by the elements

$$[K_i; t]_{q_i} = \prod_{s=1}^t \frac{K_i q_i^{-s+1} - K_i^{-1} q_i^{s-1}}{q_i^s - q_i^{-s}}.$$

Indeed,  $\Gamma(\mathfrak{h})$  contains  $U_A^{res}(\mathfrak{h})$  strictly, but the restriction functor  $\mathcal{C}_A \rightarrow \mathcal{C}_A^{res}$  is an equivalence of categories, where  $\mathcal{C}_A^{res}$  is the category of  $U_A^{res}$ -modules defined as  $\mathcal{C}_A$  above, but replacing the condition on  $(K_i; t)_{q_i}$  by its analog for  $[K_i; t]_{q_i}$ , ie. that it acts diagonally with eigenvalues

$$\left[ \begin{array}{c} k \\ t \end{array} \right]_{q_i} \quad k \in \mathbb{Z}, t \in \mathbb{N}^*.$$

The *integral* form  $\mathcal{L}_{0,1}^A$  of  $\mathcal{L}_{0,1}$  is defined as the  $U_A$ -module  $\mathcal{O}_A$  endowed with the product of  $\mathcal{L}_{0,1}$ , and the *integral* form  $\mathcal{L}_{0,n}^A$  of  $\mathcal{L}_{0,n}$  is the braided tensor product of  $n$  copies of  $\mathcal{L}_{0,1}^A$ . That these two products are well-defined over  $A$  is elementary (see Definition 4.10 and 6.7 of [23] for the details). The *integral* quantum moduli algebra is

$$\mathcal{M}_{0,n}^A = (\mathcal{L}_{0,n}^A)^{U_A}.$$

Finally, given  $q = \epsilon' \in \mathbb{C}^\times$  we define  $U_{\epsilon'}$ ,  $\Gamma_{\epsilon'}$ ,  $\mathcal{O}_{\epsilon'}$ ,  $\mathcal{L}_{0,n}^{\epsilon'}$  and  $\mathcal{M}_{0,n}^{A,\epsilon'}$  as the  $\mathbb{C}$ -algebras obtained by tensoring  $U_A$ ,  $\Gamma$ ,  $\mathcal{O}_A$ ,  $\mathcal{L}_{0,n}^A$  and  $\mathcal{M}_{0,n}^A$  respectively with  $\mathbb{C}_{\epsilon'}$ , the  $A$ -module  $\mathbb{C}$  where  $q$  acts by multiplication by  $\epsilon'$ . They are the *specializations* of the latter algebras at  $q = \epsilon'$ ; they can also be defined as the quotients by the ideal generated by  $q - \epsilon'$ . We find convenient to use the notations

$$(12) \quad (U_A^{\otimes n})_{\epsilon'}^{U_A} := (U_A^{\otimes n})^{U_A} \otimes_A \mathbb{C}_{\epsilon'}, \quad (U^{\otimes n})_{\epsilon'}^{lf} := (U^{\otimes n})^{lf} \otimes_A \mathbb{C}_{\epsilon'}.$$

Let us stress here that when  $\epsilon'$  is a root of unity, taking the locally finite part and taking the specialization at  $\epsilon'$  are non commuting operations. Indeed, when  $\epsilon'$  has odd order, it follows from Theorem 2.14 below that  $U_{\epsilon'}$  is finite over  $\mathcal{Z}_0(U_{\epsilon'})$  and therefore has all its elements locally finite for  $ad^r$ ; on another hand  $U_A^{lf} \otimes_A \mathbb{C}_{\epsilon'}$ , ie.  $U_{\epsilon'}^{lf}$  in the notations above, does not contain the elements  $L_i$ .

In a similar manner, taking invariants and taking the specialization at  $\epsilon'$  are non commuting operations when  $\epsilon'$  is a root of unity: indeed, it is easily checked that in this case  $(U_A^{\otimes n})_{\epsilon'}^{U_A}$  and  $(U_{\epsilon'}^{\otimes n})^{U_{\epsilon'}}$ , or  $\mathcal{M}_{0,n}^{A,\epsilon'} = \mathcal{M}_{0,n}^A \otimes_A \mathbb{C}_{\epsilon'}$  and  $(\mathcal{L}_{0,n}^{\epsilon'})^{U_{\epsilon'}}$ , are distinct spaces. As explained in

the introduction, when  $\epsilon'$  is a root of unity, we will not consider the algebras  $\mathcal{M}_{0,n}^{A,\epsilon'}$  in this paper.

The morphism  $\Phi_n$  has also an integral form. In order to define it, we first consider the relations between  $U_A$  and  $U_A^{lf}$ . Denote by  $T \subset U_A$  the multiplicative Abelian group formed by the elements  $K_\lambda$ ,  $\lambda \in P$ , and by  $T_2 \subset T$  the subgroup formed by the  $K_\lambda$ ,  $\lambda \in 2P$ . Consider the subset  $T_{2-} \subset T_2$  formed by the elements  $K_{-\lambda}$ ,  $\lambda \in 2P_+$ . It is easily seen to be an Ore subset of  $U_A$ . Clearly  $T_2 = T_{2-}^{-1}T_{2-}$  and  $\text{Card}(T/T_2) = 2^m$ .

**Theorem 2.2.** (1)  $U_A^{lf} = \bigoplus_{\lambda \in 2P_+} ad^r(U_A)(K_{-\lambda})$ .  
 (2)  $U_A = T_{2-}^{-1}U_A^{lf}[T/T_2]$ , so  $U_A$  is free of rank  $2^m$  over  $T_{2-}^{-1}U_A^{lf}$ .  
 (3) The ring  $U_A^{lf}$  is (left and right) Noetherian.

*Proof.* These results are immediate adaptations to  $U_A^{lf}$  of those for  $U_q^{lf}$ , proved in Theorem 4.10 of [45], Theorem 6.4 of [44], and Theorem 7.4.8 of [43], respectively (see also the sections 7.1.6, 7.1.13 and 7.1.25 in [43]). For (1) and (3) we refer to Theorem 2.113 and 2.137 in [70], which provides simpler proofs.  $\square$

**Remark 2.3.** The summands in (1) are finite-dimensional  $U_A$ -modules (by eg. (14) below), so the action  $ad^r$  is completely reducible on  $U_A^{lf}$ . In fact,  $U_A^{lf}$  is the socle of  $ad^r$  on  $U_A$ , and by the theorem of separation of variables ([45, 43, 9], see also [70]),  $U_A^{lf}$  has an  $U_A$ -invariant subspace  $\mathbb{H}$  such that the multiplication in  $U_A$  affords an isomorphism of  $U_A$ -modules from  $\mathbb{H} \otimes_{\mathbb{C}(q)} \mathcal{Z}(U_A)$  onto  $U_A^{lf}$ . In particular,  $U_A^{lf}$  is free over  $\mathcal{Z}(U_A)$ . Moreover, any simple finite dimensional  $U_A$ -module has in  $\mathbb{H}$  a multiplicity equal to the dimension of its zero-weight subspace.

Recall the RSD map  $\Phi_1: \mathcal{O}_q \rightarrow U_q^{lf}$ . By construction  $\langle \cdot, \cdot \rangle$  induces a perfect pairing  $\langle \cdot, \cdot \rangle: \mathcal{O}_A \otimes \mathbb{U}_\Gamma \rightarrow A$ . Let  $V_{-\lambda}$  be the lowest weight  $\Gamma$ -module of lowest weight  $-\lambda \in -P_+$  (ie. the highest weight  $\Gamma$ -module  $V_{-w_0(\lambda)}$  of highest weight  $-w_0(\lambda)$ , where  $w_0$  is the longest element of the Weyl group; note that  $-w_0$  permutes the simple roots). Let  $v \in V_{-\lambda}$  be a lowest weight vector, and  $v^* \in V_{-\lambda}^*$  be such that  $v^*(v) = 1$  and  $v^*$  vanishes on a  $\Gamma(\mathfrak{h})$ -invariant complement of  $v$ . Define  $\psi_{-\lambda} \in \mathcal{O}_A$  by  $\langle \psi_{-\lambda}, x \rangle = v^*(xv)$ ,  $x \in \Gamma$ . From the definition (4) it is quite easy to see that

$$(13) \quad \Phi_1(\psi_{-\lambda}) = K_{-2\lambda}.$$

**Corollary 2.4.**  $\Phi_1$  restricts on  $\mathcal{O}_A$  to an isomorphism of  $U_A$ -modules  $\Phi_1: \mathcal{O}_A \rightarrow U_A^{lf}$  and an isomorphism of  $U_A$ -module algebras  $\Phi_1: \mathcal{L}_{0,1}^A \rightarrow U_A^{lf}$ .

*Proof.* An elementary computational proof of this result in the  $sl(2)$  case is given in Section 5 of [23]. A proof of the general case can be found in Lemma 4.11 of [23]. It uses Theorem 2.2 (1). We point out an alternative proof in Remark 2.13 (1).  $\square$

**Corollary 2.5.** Let us denote  $d = \psi_{-\rho} \in \mathcal{L}_{0,1}^A$ . We have:

(1) The set  $\{d^n\}_{n \in \mathbb{N}}$  is a left and right multiplicative Ore set in  $\mathcal{L}_{0,1}^A$ . We can therefore define the localization  $\mathcal{L}_{0,1}^A[d^{-1}]$ .

(2)  $\Phi_1$  extends to an isomorphism of  $U_A$ -module algebras  $\Phi_1: \mathcal{L}_{0,1}^A[d^{-1}] \rightarrow T_{2-}^{-1}U_A^{lf}$ .

*Proof.* (1) Because  $\mathcal{L}_{0,1}^A$  has no non-trivial zero divisors,  $d$  is a regular element. We have to show that for all  $x \in \mathcal{L}_{0,1}^A$  there exists elements  $y, y' \in \mathcal{L}_{0,1}^A$  such that  $xd = dy$  and  $dx = y'd$ . But  $\Phi_1(x)\Phi_1(d) = \Phi_1(x)K_{-2\rho} = K_{-2\rho}ad^r(K_{2\rho})(\Phi_1(x))$ , and  $ad^r(K_{2\rho})(\Phi_1(x)) = \Phi_1(\text{coad}^r(K_{2\rho})(x))$ . Therefore the Ore condition is satisfied with  $y = \text{coad}^r(K_{2\rho})(x)$ .

(2) It follows from the fact that  $\Phi_1(d) = K_{-2\rho} = \prod_{j=1}^m L_j^{-2}$ , so if we localize in  $d$  we obtain  $L_j^2 = \prod_{k \neq j} L_k^{-2} \Phi_1(d^{-1}) = \Phi_1(\prod_{k \neq j} \psi_{-\varpi_k} d^{-1}) \in \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$ . Therefore  $T_{2-}^{-1} \subset \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$ , which implies the assertion (2).  $\square$

**Remark 2.6.** When  $\mathfrak{g} = sl(2)$  the element  $d$  is the generator of  $\mathcal{L}_{0,1}(sl(2))$  appearing in (44) below. In this case we had already shown in [23] that  $\Phi_1: \mathcal{L}_{0,1}^A[d^{-1}] \rightarrow U_A^{ad} = T_{2-}^{-1} U_A^{lf}$  is an isomorphism of algebras.

Denote by  $C(\mu)$ ,  $\mu \in P^+$ , the linear subspace of  $\mathcal{L}_{0,1}$  generated by the matrix coefficients of  $V_\mu$ , the  $U_q$ -module of type 1 and highest weight  $\mu$ . The formula (13) can be used to prove (see Section 7.1.22 in [43], or page 112 of [70]) that  $\Phi_1$  yields the following linear isomorphism, which illuminates the claim (1) of Theorem 2.2:

$$(14) \quad \Phi_1: C(\mu) \rightarrow ad^r(U_q)(K_{-2w_0(\mu)}).$$

Working over the ground ring  $A$  one has to consider for  $V_\mu$  the highest weight  $\Gamma$ -module of highest weight  $\mu$ . In that situation  $\Phi_1$  affords an isomorphism from  $C(\mu)_A = End_A(V_\mu)^*$  to  $ad^r(U_A)(K_{-2w_0(\mu)})$ .

By (13) we have  $\Phi_1(\psi_{-\rho}) = \ell^{-1}$ , where as usual  $\ell$  is the pivotal element of  $U_A$ . Because the latter has the elementary factorization  $\ell = \prod_{j=1}^m L_j^2$ , this naturally raises the question of the factorization of  $\psi_{-\rho}$ . This question is considered in [46], where  $\mathcal{L}_{0,1}(\mathfrak{g})$  for  $\mathfrak{g} = gl(r+1)$  is analysed and quantum minors are extensively studied. Let us review here some of their results in relation with  $\psi_{-\rho}$ .

First note that for  $\mathfrak{g} = sl(r+1)$  the irreducible representation  $V_{-\rho}$  of lowest weight  $-\rho$  is isomorphic to the representation of highest weight  $V_\rho$  because  $-w_0(\rho) = \rho$ . By the Weyl formula the dimension of this representation is  $\prod_{\alpha > 0} \frac{(2\rho, \alpha)}{(\rho, \alpha)} = 2^N$ . In [50] a presentation of  $U_q(gl(r+1))$  is given, which differs from our presentation of  $U_q(sl(r+1))$  only by its subalgebra  $U_q(\mathfrak{h})$ , generated by  $r+1$  elements  $\mathbb{K}_1, \dots, \mathbb{K}_{r+1}$ . The inclusion  $U_q(sl(r+1)) \subset U_q(gl(r+1))$  is such that  $K_i = \mathbb{K}_i^2 \mathbb{K}_{i+1}^{-2}$ ,  $i = 1, \dots, r$ . The quantum minors, properly defined in [46], of the matrix of matrix elements of the natural representation of  $U_q(gl(r+1))$  are denoted  $det_q(A_{\geq k})$  for  $k = 1, \dots, r+1$ . We have  $det_q(A_{\geq 1}) = 1$  in the case of  $sl(r+1)$ . Then [46] proves that  $det_q(A_{\geq k}) = (\mathbb{K}_k \dots \mathbb{K}_{r+1})^2$ , and there exists an element  $\mathbb{K} \in U_q(gl(r+1))$  such that

$$\mathbb{K}^{-2\rho} = det_q(A_{\geq 1})^{-r} det_q(A_{\geq 2}) \dots det_q(A_{\geq r+1}).$$

This has to be interpreted in the  $sl(r+1)$  case as  $K_{-2\rho} = \Phi_1(det_q(A_{\geq 2}) \dots det_q(A_{\geq r+1}))$ . As a result this gives the equality

$$\psi_{-\rho} = det_q(A_{\geq 2}) \dots det_q(A_{\geq r+1}).$$

Corollary 2.4 can be extended as follows:

**Theorem 2.7.**  $\Phi_n$  restricts to an isomorphism of  $U_A$ -module algebras  $\Phi_n: \mathcal{L}_{0,n}^A \rightarrow (U_A^{\otimes n})^{lf}$ , and it restricts to an isomorphism of algebras  $\Phi_n: \mathcal{M}_{0,n}^A \rightarrow (U_A^{\otimes n})^{U_A}$ .

The proof relies on (10) and the expression of  $\Phi_n$  in terms of  $\Phi_1$  and  $R$ -matrices (see [23], Proposition 6.5 and Lemma 6.8).

In the case of  $\mathfrak{g} = sl(2)$  we proved in [25] the existence of elements  $\xi^{(i)} \in \mathcal{L}_{0,n}^A$  ( $i = 1, \dots, n$ ), and we defined an algebra  $loc \mathcal{L}_{0,n}^A$  generalizing  $\mathcal{L}_{0,1}^A[d^{-1}]$  above, containing  $\mathcal{L}_{0,n}^A$  as a subalgebra and the inverses of the elements  $\xi^{(i)}$ . We showed that  $\Phi_n$  extends to  $loc \mathcal{L}_{0,n}^A$ , and that  $\Phi_n(loc \mathcal{L}_{0,n}^A) = U_A^{ad}(sl(2))^{\otimes n}$ . The key property of  $\xi^{(i)}$  is

$$(15) \quad \Phi_n(\xi^{(i)}) = (K^{-1})^{(i)} \dots (K^{-1})^{(n)}.$$

For general  $\mathfrak{g}$  we now describe a partial generalization of this result. Define elements  $\xi_j^{(i)} \in \mathcal{L}_{0,n}^A$ , for  $i = 1, \dots, n$  and  $j = 1, \dots, m$ , by

$$(16) \quad \xi_j^{(i)} = v^*(M_j^{(i)} \cdots M_j^{(n)})(v)$$

where  $M_j^{(i)} \in \text{End}(V_{-\varpi_j}) \otimes \mathcal{L}_{0,n}^A$  is the matrix of matrix coefficients  $1^{\otimes(i-1)} \otimes_{V_{-\varpi_j}} \phi_{e_k}^{e_i} \otimes 1^{\otimes(n-i)}$ , where  $\{e_k\}$  is the canonical basis of weight vectors of  $V_{-\varpi_j}$ ,  $v$  is a lowest non-zero weight vector of  $V_{-\varpi_j}$ , and  $v^*$  the associated linear form, vanishing on a  $\Gamma(\mathfrak{h})$ -invariant complement of  $v$ . Similarly to (15) the elements  $\xi_j^{(i)}$  satisfy

$$(17) \quad \Phi_n(\xi_j^{(i)}) = (L_j^{-2})^{(i)} \cdots (L_j^{-2})^{(n)}.$$

These elements commute and the multiplicative sets  $\{\xi_j^{(1)k}\}_{k \in \mathbb{N}}$  are Ore set of  $\mathcal{L}_{0,n}^A$ , but for  $i \geq 2$  the multiplicative sets  $\{\xi_j^{(i)k}\}_{k \in \mathbb{N}}$  are not Ore sets of  $\mathcal{L}_{0,n}^A$ . In fact, as in the proof of Theorem 2.5 we see that  $\{\xi_j^{(i)k}\}_{k \in \mathbb{N}}$  is only an Ore set of the subalgebra  $\mathcal{L}_{0,n}^{(i \leq)}$  of  $\mathcal{L}_{0,n}^A$  generated by the subalgebras  $\mathcal{L}_{0,n}^{(a)}$ ,  $a \geq i$ . We therefore cannot localize  $\mathcal{L}_{0,n}^A$  with respect to the elements  $\xi_j^{(i)}$  as easily as in the case where  $n = 1$ .

In order to proceed let us explain the case  $n = 2$ . Since the elements  $\xi_j^{(1)}$ ,  $j \in \{1, \dots, m\}$ , are commuting regular Ore elements of  $\mathcal{L}_{0,2}^A$  we can define the localisation of  $\mathcal{L}_{0,2}^A$  with respect to the multiplicative sets  $\{\xi_1^{(1)k}\}_{k \in \mathbb{N}}, \dots, \{\xi_m^{(1)k}\}_{k \in \mathbb{N}}$ . Denote it  $\mathcal{L}_{0,2}^A[\{\xi_{j_1}^{(1)-1}\}]$ . Let us add new elements  $\nu_{j_1}^{(1)}$  such that  $(\nu_{j_1}^{(1)})^2 = \xi_{j_1}^{(1)}$  and  $\Phi_2(\nu_{j_1}^{(1)}) = (L_{j_1}^{-1})^{(1)}(L_{j_1}^{-1})^{(2)}$ . They are Ore elements, and we can define similarly the localisation  $\mathcal{L}_{0,2}^A[\{\nu_{j_1}^{(1)-1}\}]$  (see the following remark for an explanation of this additional construction). We want to define the inverses of the elements  $\xi_j^{(2)}$ ,  $j \in \{1, \dots, m\}$ , and a new algebra  $\mathcal{L}_{0,2}^A[\{\xi_{j_1}^{(1)-1}\}][\{\xi_{j_2}^{(2)-1}\}]$  such that  $\mathcal{L}_{0,2}^A[\{\xi_{j_1}^{(1)-1}\}] \subset \mathcal{L}_{0,2}^A[\{\xi_{j_1}^{(1)-1}\}][\{\xi_{j_2}^{(2)-1}\}]$  and  $\Phi_2$  extends naturally to an algebra homomorphism  $\Phi_2 : \mathcal{L}_{0,2}^A[\{\xi_{j_1}^{(1)-1}\}][\{\xi_{j_2}^{(2)-1}\}] \rightarrow U_A^{\otimes 2}$  such that  $\Phi_n(\xi_{j_2}^{(2)}) = (L_{j_2}^{-2})^{(2)}$  for all  $j_2 \in \{1, \dots, m\}$ . As in the  $sl(2)$  case described in [23], this can be done by writing explicitly, for every  $j_2 \in \{1, \dots, m\}$ , the exchange relations between the matrices  $M_{j_1}^{(1)}$  and  $M_{j_2}^{(2)}$  involving  $\xi_{j_2}^{(2)}$ , for every  $j_1 \in \{1, \dots, m\}$  (these matrices are defined in (16)). Such exchange relations have the form (31) in the graded algebra  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n}^A)$  defined in (33) below. Similarly, by replacing the elements  $\xi_{j_1}^{(1)}$ ,  $\xi_{j_2}^{(2)}$  with square roots  $\nu_{j_1}^{(1)}$ ,  $\nu_{j_2}^{(2)}$  we get a localization  $\mathcal{L}_{0,2}^A[\{\nu_{j_1}^{(1)-1}\}][\{\nu_{j_2}^{(2)-1}\}]$  such that  $\mathcal{L}_{0,2}^A[\{\nu_{j_1}^{(1)-1}\}] \subset \mathcal{L}_{0,2}^A[\{\nu_{j_1}^{(1)-1}\}][\{\nu_{j_2}^{(2)-1}\}]$  and  $\Phi_2$  extends to an algebra homomorphism  $\Phi_2 : \mathcal{L}_{0,2}^A[\{\nu_{j_1}^{(1)-1}\}][\{\nu_{j_2}^{(2)-1}\}] \rightarrow U_A^{\otimes 2}$  such that  $\Phi_n(\nu_{j_2}^{(2)}) = (L_{j_2}^{-1})^{(2)}$  for all  $j_2 \in \{1, \dots, m\}$ . This morphism of algebras will be shown to be an isomorphism.

For any  $n \geq 2$  we can proceed in the same way:

**Definition 2.8.** *By iterating the above construction we define:*

$$\begin{aligned} \text{loc} \mathcal{L}_{0,n}^A &= \mathcal{L}_{0,n}^A[\{\xi_{j_n}^{(n)-1}\}][\{\xi_{j_{n-1}}^{(n-1)-1}\}] \cdots [\{\xi_{j_1}^{(1)-1}\}], \\ \text{loc}' \mathcal{L}_{0,n}^A &= \mathcal{L}_{0,n}^A[\{\nu_{j_n}^{(n)-1}\}][\{\nu_{j_{n-1}}^{(n-1)-1}\}] \cdots [\{\nu_{j_1}^{(1)-1}\}]. \end{aligned}$$

In the sequel it will be convenient to define invertible elements  $\sqrt{\delta_j}^{(i)} \in \text{loc}' \mathcal{L}_{0,n}^A$ , for  $i = 1, \dots, n$  and  $j = 1, \dots, m$ , satisfying  $\nu_j^{(i)} = \sqrt{\delta_j}^{(i)} \cdots \sqrt{\delta_j}^{(n)}$ , i.e.  $\sqrt{\delta_j}^{(i)} = \nu_j^{(i)} / \nu_j^{(i+1)}$ .



The elements  $\sqrt{\delta_j^{(i)}}$  are invertible, commute and satisfy

$$\Phi_n(\sqrt{\delta_j^{(i)}}) = (L_j^{-1})^{(i)}.$$

**Theorem 2.9.**  $\Phi_n$  restricts to an isomorphism of  $U_A$ -module algebras  $\Phi_n : \text{loc}' \mathcal{L}_{0,n}^A \rightarrow U_A^{\otimes n}$ .

*Proof.* We know from Corollary 2.4 that  $\Phi_1 : \mathcal{L}_{0,n}^A \rightarrow U_A^{\text{lf}}$  is an isomorphism of algebra. Using  $U_A = T_{2-}^{-1} U_A^{\text{lf}} [T/T_2]$  and the fact that the image by  $\Phi_1$  of the elements  $(\sqrt{\delta_j^{(1)}})^{\pm 1}$  generates the group  $T$  we get the result for  $n = 1$ . The result for  $\Phi_n$  is obtained by induction. We have

$$\begin{aligned} (id \otimes \Phi_n)(M^{(n)}) &= R_{0n} R'_{0n} \\ (id \otimes \Phi_n)(M^{(a)}) &= (R_{0n} \dots R_{0a+1}) R_{0a} R'_{0a} (R_{0n} \dots R_{0a+1})^{-1}, 1 \leq a < n. \end{aligned}$$

Because the matrix elements of  $(id \otimes \Phi_n)(M^{(n)})$  generate  $1^{\otimes n-1} \otimes U_A^{\text{lf}}$  when  $V$  varies, the image of  $(\mathcal{L}_{0,n}^A)^{(n)}[\{\nu_{j_n}^{(n)-1}\}]$  by  $\Phi_n$  is  $1^{\otimes(n-1)} \otimes U_A$ . Since the matrix elements of  $R_{0n}$  and  $R_{0n}^{-1}$  are in  $1^{\otimes(n-1)} \otimes U_A$ , they belong to  $\Phi_n(\mathcal{L}_{0,n}^A[\{\nu_{j_n}^{(n)-1}\}])$  by the preceding remark. It follows that  $\Phi_n(\text{loc}' \mathcal{L}_{0,n}^A)$  contains the matrix elements of  $R_{0n}^{-1} (id \otimes \Phi_n)(M^{(n-1)}) R_{0n}$ , whence the matrix elements of  $R_{0n-1} R'_{0n-1}$ , and therefore the space  $1^{\otimes(n-2)} \otimes U_A^{\text{lf}} \otimes 1$ . It contains also the elements  $\Phi_n(\sqrt{\delta_j^{(n-1)}}) = (L_j^{-1})^{(n-1)}$ , so  $\Phi_n(\text{loc}' \mathcal{L}_{0,n}^A)$  contains  $1^{\otimes(n-2)} \otimes U_A \otimes 1$ . By a trivial induction we finally obtain that  $\Phi_n(\text{loc}' \mathcal{L}_{0,n}^A) = U_A^{\otimes n}$ .  $\square$

**Remark 2.10.** It is a natural problem to determine the image by  $\Phi_n$  of  $\text{loc}' \mathcal{L}_{0,n}^A$ , and it is natural to expect that it would be  $(T_{2-}^{-1} U_A^{\text{lf}})^{\otimes n}$ , because this is true for  $n = 1$ , as well as for any  $n$  in the  $sl(2)$  case, as shown in [23]. Unfortunately this is not so. This comes from the fact, eg. for  $n = 2$ , that the matrix elements of  $R_{02} R_{01} R'_{01} R_{02}^{-1}$  do not belong to  $(T_{2-}^{-1} U_A^{\text{lf}})^{\otimes 2}$  as can be shown by an explicit computation in the  $sl(3)$  case. This explains the reason why we had to introduce the square roots  $\nu_j^{(i)}$  in the previous theorem.

Arguments similar to those mentioned at the end of Section 2.1 imply that the algebras  $\mathcal{L}_{0,n}^A$ ,  $\mathcal{M}_{0,n}^A$  and  $\mathcal{L}_{0,n}^{\epsilon'}$ ,  $\mathcal{M}_{0,n}^{A,\epsilon'}$ ,  $\epsilon' \in \mathbb{C}^\times$ , have no non-trivial zero divisors (see [23], Proposition 7.1). By Theorem 2.7 the Alekseev map yields isomorphisms of  $U_{\epsilon'}$ -module algebras, and of algebras for the latter,

$$(18) \quad \Phi_n : \mathcal{L}_{0,n}^{\epsilon'} \rightarrow (U^{\otimes n})_{\epsilon'}^{\text{lf}}, \quad \Phi_n : \text{loc}' \mathcal{L}_{0,n}^{\epsilon'} \rightarrow U_{\epsilon'}^{\otimes n}, \quad \Phi_n : \mathcal{M}_{0,n}^{A,\epsilon'} \rightarrow (U_A^{\otimes n})_{\epsilon'}^{U_A} \subset (U_{\epsilon'}^{\otimes n})_{U_{\epsilon'}}$$

where we use the notations (12).

**2.3. Perfect pairings.** We will need restrictions on the integral forms  $\mathcal{O}_A(B_+)$ ,  $\mathcal{O}_A(B_-)$  of the morphisms  $\Phi^+$ ,  $\Phi^-$  in (2). We collect their properties in Theorem 2.11 and the discussion thereafter. In order to state it, we recall first a few facts about  $R$ -matrices and related pairings.

In [52, 53] Lusztig proved that the category of  $U_A^{\text{res}}$ -modules  $\mathcal{C}_A^{\text{res}} \otimes_A \mathbb{C}[q^{\pm 1/D}]$  (ie. with coefficients extended to  $\mathbb{C}[q^{\pm 1/D}]$ ) is braided and ribbon, with braiding given by the collection of endomorphisms

$$R_A = ((R_h)_{V,W})_{V,W \in \text{Ob}(\mathcal{C}_A^{\text{res}})}.$$

Actually,  $(R_h)_{V,W}$  is represented by a matrix with coefficients in  $q^{\pm 1/D} \mathbb{Z}[q^{\pm 1}]$  on the basis of  $V \otimes W$  formed by the tensor products of the canonical (Kashiwara-Lusztig) basis vectors

of  $V$  and  $W$ . The restriction functor  $\mathcal{C}_A \rightarrow \mathcal{C}_A^{res}$  is an equivalence of categories, so  $\mathcal{C}_A \otimes_{\mathbb{C}[q^{\pm 1/D}]}$  has the same braided and ribbon structure. This can be rephrased as follows in Hopf algebra terms. Denote by  $\mathbb{U}_\Gamma$  the categorical completion of  $\Gamma$ , ie. the Hopf algebra of natural transformations  $F_{\mathcal{C}_A} \rightarrow F_{\mathcal{C}_A}$ . Then  $\mathbb{U}_\Gamma \otimes_A \mathbb{C}[q^{\pm 1/D}]$  is quasi-triangular and ribbon with R-matrix

$$R_A \in \mathbb{U}_\Gamma^{\hat{\otimes} 2} \otimes_A \mathbb{C}[q^{\pm 1/D}].$$

As in (1), we can write

$$R_A^\pm = \sum_{(R)} R_{(1)}^\pm \otimes R_{(2)}^\pm.$$

There are pairings of Hopf algebras naturally related to the R-matrix  $R \in \mathbb{U}_q^{\hat{\otimes} 2}$ . What follows is standard (see eg. [47, 48, 51]), for details we refer to the results 2.73, 2.75, 2.92, 2.106 and 2.107 in [70]:

- There is a unique pairing of Hopf algebras  $\rho: U_q(\mathfrak{b}_-)^{cop} \otimes U_q(\mathfrak{b}_+) \rightarrow \mathbb{C}(q^{1/D})$  such that, for every  $\alpha, \lambda \in P$  and  $l, k \in U_q(\mathfrak{h})$ ,

$$\rho(K_\lambda, K_\alpha) = q^{(\lambda, \alpha)}, \quad \rho(F_i, E_j) = \delta_{i,j}(q_i - q_i^{-1})^{-1}, \quad \rho(l, E_j) = \rho(F_i, k) = 0.$$

- The *Drinfeld pairing*  $\tau: U_q(\mathfrak{b}_+)^{cop} \otimes U_q(\mathfrak{b}_-) \rightarrow \mathbb{C}(q^{1/D})$  is the bilinear map defined by  $\tau(X, Y) = \rho(S(Y), X)$ ; it satisfies

$$\tau(K_\lambda, K_\alpha) = q^{-(\lambda, \alpha)}, \quad \tau(E_j, F_i) = -\delta_{i,j}(q_i - q_i^{-1})^{-1}, \quad \tau(l, F_i) = \tau(E_j, k) = 0.$$

- $\rho$  and  $\tau$  are perfect pairings; this means that they yield *isomorphisms* of Hopf algebras  $i_\pm: U_q(\mathfrak{b}_\pm) \rightarrow \mathcal{O}_q(B_\mp)_{op}$  (with coefficients *a priori* extended to  $\mathbb{C}(q^{1/D})$ , but see below) defined by, for every  $X \in U_q(\mathfrak{b}_+)$ ,  $Y \in U_q(\mathfrak{b}_-)$ ,

$$\langle i_+(X), Y \rangle = \tau(S(X), Y), \quad \langle i_-(Y), X \rangle = \tau(X, Y).$$

Since  $\mathcal{O}_q(B_\mp)_{op}$  is equipped with the *inverse* of the antipode  $S_{\mathcal{O}_q}$  of  $\mathcal{O}_q(B_\mp)$ , it follows that  $i_\pm \circ S = S_{\mathcal{O}_q}^{-1} \circ i_\pm$ .

- Denote by  $p_\pm: \mathcal{O}_q(G) \rightarrow \mathcal{O}_q(B_\pm)$  the canonical projection map, ie. the Hopf algebra homomorphism dual to the inclusion map  $U_q(\mathfrak{b}_\pm) \hookrightarrow U_q(\mathfrak{g})$ . For every  $\alpha, \beta \in \mathcal{O}_q(G)$  we have

$$(19) \quad \langle \alpha \otimes \beta, R \rangle = \tau(i_+^{-1}(p_-(\beta)), i_-^{-1}(p_+(\alpha))).$$

Note that it is the use of weights  $\alpha, \lambda \in P$  that forces the pairings  $\rho, \tau$  to be defined over  $\mathbb{C}(q^{1/D})$ , instead of  $\mathbb{C}(q)$ . Then, let us consider the restrictions  $\pi_q^+$  of  $\rho$ , and  $\pi_q^-$  of  $\tau$ , obtained by taking  $\alpha \in Q$  and  $l \in U_q(\mathfrak{h})$ ,  $k \in U_q^{ad}(\mathfrak{h})$ . They take values in  $\mathbb{C}(q)$ , and define pairings

$$\pi_q^+: U_q(\mathfrak{b}_-)^{cop} \otimes U_q^{ad}(\mathfrak{b}_+) \rightarrow \mathbb{C}(q), \quad \pi_q^-: U_q(\mathfrak{b}_+)^{cop} \otimes U_q^{ad}(\mathfrak{b}_-) \rightarrow \mathbb{C}(q).$$

By the same arguments as for  $\rho$  and  $\tau$  (eg. in [70], Proposition 2.92), it follows that  $\pi_q^\pm$  are perfect pairings. Note also that  $\pi_q^- = \kappa \circ \pi_q^+ \circ (\kappa \otimes \kappa)$ , where  $\kappa$  is the conjugate-linear automorphism of  $U_q$ , viewed as a Hopf algebra over  $\mathbb{C}(q)$  with conjugation given by  $\kappa(q) = q^{-1}$ , defined by

$$(20) \quad \kappa(E_i) = F_i, \quad \kappa(F_i) = E_i, \quad \kappa(K_\lambda) = K_{-\lambda}, \quad \kappa(q) = q^{-1}.$$

In [36], De Concini-Lyubashenko described integral forms of  $\pi_q^\pm$  as follows. Denote by  $m^*: \mathcal{O}_A \rightarrow \mathcal{O}_A(B_+) \otimes \mathcal{O}_A(B_-)$  the map dual to the multiplication map  $\Gamma(\mathfrak{b}_+) \otimes \Gamma(\mathfrak{b}_-) \rightarrow \Gamma$ , so  $m^* = (p_+ \otimes p_-) \circ \Delta_{\mathcal{O}_A}$ . Let  $U_A(H)$  be the sub-Hopf algebra of  $U_A(\mathfrak{b}_-)^{cop} \otimes U_A(\mathfrak{b}_+)^{cop}$  generated by the elements  $(i \in \{1, \dots, m\})$

$$1 \otimes K_i^{-1} \bar{E}_i, \quad \bar{F}_i K_i \otimes 1, \quad L_i^{\pm 1} \otimes L_i^{\mp 1}.$$

Note that  $U_A(H)$  is free over  $A$ , and that a basis is given by the elements

$$\bar{F}_{\beta_1}^{n_1} \cdots \bar{F}_{\beta_N}^{n_N} K_{n_1\beta_1+\dots+n_N\beta_N} K_\lambda \otimes K_{-\lambda} K_{-p_1\beta_1 \dots -p_N\beta_N} \bar{E}_{\beta_1}^{p_1} \cdots \bar{E}_{\beta_N}^{p_N}$$

where  $\lambda \in P$  and  $n_1, \dots, n_N, p_1, \dots, p_N \in \mathbb{N}$ .

Recall the lowest weight  $\Gamma$ -module  $V_{-\lambda}$ ,  $\lambda \in P_+$ , the lowest weight vector  $v \in V_{-\lambda}$ , the dual vector  $v^* \in V_{-\lambda}^*$ , and  $\psi_{-\lambda} \in \mathcal{O}_A$  (see before Corollary 2.4). For every positive root  $\alpha$  define elements  $\psi_{-\lambda}^\alpha, \psi_{-\lambda}^{-\alpha} \in \mathcal{O}_A$  by the formulas (where  $x \in \Gamma$ , and we note that the root vectors  $E_\alpha, F_\alpha \in \Gamma$ ):

$$\langle \psi_{-\lambda}^\alpha, x \rangle = v^*(xE_\alpha v), \quad \langle \psi_{-\lambda}^{-\alpha}, x \rangle = v^*(F_\alpha x v).$$

Consider the maps  $j_q^\pm: \mathcal{O}_q(B_\pm) \rightarrow U_q(\mathfrak{b}_\mp)^{cop}$  defined by

$$\langle \alpha_+, X \rangle = \pi_q^+(j_q^+(\alpha_+), X), \quad \langle \alpha_-, Y \rangle = \pi_q^-(j_q^-(\alpha_-), Y)$$

where  $\alpha_\pm \in \mathcal{O}_q(B_\pm)$ ,  $X \in U_q^{ad}(\mathfrak{b}_+)$ ,  $Y \in U_q^{ad}(\mathfrak{b}_-)$ .

The following theorem summarizes results proved in the sections 3 and 4 of [36]. For the sake of clarity, let us spell out the correspondence between statements. First,  $\pi_q^+, \pi_q^-, U_q(\mathfrak{b}_\mp)^{cop}, U_A(\mathfrak{b}_\mp)^{cop}, \mathcal{O}_A(B_\pm), U_A(H)$  and  $J$  are denoted in [36] respectively by  $\pi'', \bar{\pi}'', U_q(\mathfrak{b}_\mp)_{op}, R_q[B_\pm]'', R_q[B_\pm], A''$  and  $\mu''$ . Also, the definition of  $j_A^\pm$  is implicit in the section 4.2 of [36], and the formulas in Theorem 2.11 (3) are related to those in Lemma 4.5 of [36] by observing that their generators  $\tilde{E}_i$  and  $\tilde{F}_i$  are respectively  $K_i^{-1}E_i$  and  $F_iK_i$  in our notations; this also explains the appearance of  $q_i, q_i^{-1}$  in the formulas in (3). Finally,  $\kappa$  in (20) maps  $\bar{E}_i, \bar{F}_i$  to  $-\bar{F}_i, -\bar{E}_i$ , whence the sign for the expression of  $J(\psi_{-\varpi_j}^{\alpha_i})$ .

**Theorem 2.11.** (1)  $\pi_q^\pm$  restricts to a perfect Hopf pairing between the unrestricted and restricted integral forms,  $\pi_A^\pm: U_A(\mathfrak{b}_\mp)^{cop} \otimes \Gamma(\mathfrak{b}_\pm) \rightarrow A$ .

(2)  $j_q^\pm$  yields an isomorphism of Hopf algebras  $j_A^\pm: \mathcal{O}_A(B_\pm) \rightarrow U_A(\mathfrak{b}_\mp)^{cop}$ , satisfying  $\langle \alpha_\pm, x_\pm \rangle = \pi_A^\pm(j_A^\pm(\alpha_\pm), x_\pm)$  for every  $\alpha_\pm \in \mathcal{O}_A(B_\pm)$ ,  $x_\pm \in \Gamma(\mathfrak{b}_\pm)$ .

(3) The map  $J = (j_A^+ \otimes j_A^-) \circ m^*: \mathcal{O}_A \rightarrow U_A(H) \subset U_A(\mathfrak{b}_-)^{cop} \otimes U_A(\mathfrak{b}_+)^{cop}$  is an embedding of Hopf algebras, and it extends to an isomorphism  $J: \mathcal{O}_A[\psi_{-\rho}^{-1}] \rightarrow U_A(H)$ . In particular it satisfies (where  $\lambda \in P_+$ ):

$$J(\psi_{-\lambda}) = K_{-\lambda} \otimes K_\lambda, \quad J(\psi_{-\varpi_j}^{\alpha_i}) = -\delta_{i,j} q_i L_i^{-1} \otimes L_i K_i^{-1} \bar{E}_i, \quad J(\psi_{-\varpi_j}^{-\alpha_i}) = \delta_{i,j} q_i^{-1} \bar{F}_i K_i L_i^{-1} \otimes L_i.$$

For our purposes it is necessary to reformulate this result. Consider the morphisms of Hopf algebras  $\Phi^\pm: \mathcal{O}_A(B_\pm) \rightarrow U_A(\mathfrak{b}_\mp)^{cop}$ ,  $\alpha \mapsto (\alpha \otimes id)(R_A^\pm)$ .

**Lemma 2.12.** We have  $\Phi^\pm = j_A^\pm$ .

Thus, the theorem above tells us that  $\Phi^\pm$  is an isomorphism of Hopf algebras, such that  $\langle \alpha_\pm, x_\pm \rangle = \pi_A^\pm(\Phi^\pm(\alpha_\pm), x_\pm)$  for every  $\alpha_\pm \in \mathcal{O}_A(B_\pm)$ ,  $x_\pm \in \Gamma(\mathfrak{b}_\pm)$ . Moreover, changing the notation  $J$  for  $\Phi$ ,

$$(21) \quad \Phi := (\Phi^+ \otimes \Phi^-) \circ m^*: \mathcal{O}_A \rightarrow U_A(H) \subset U_A(\mathfrak{b}_-)^{cop} \otimes U_A(\mathfrak{b}_+)^{cop}$$

is an embedding of Hopf algebras, and it extends to an isomorphism  $\Phi: \mathcal{O}_A[\psi_{-\rho}^{-1}] \rightarrow U_A(H)$  which in particular satisfies:

$$(22) \quad \Phi_1(\psi_{-\lambda}) = K_{-2\lambda}, \quad \Phi_1(\psi_{-\varpi_j}^{\alpha_i}) = \delta_{i,j} L_i^{-2} \bar{E}_i, \quad \Phi_1(\psi_{-\varpi_j}^{-\alpha_i}) = \delta_{i,j} q_i^{-1} \bar{F}_i K_i L_i^{-2}.$$

*Proof of Lemma 2.12.* By definitions, for every  $X \in U_q(\mathfrak{b}_+)^{cop}$ ,  $Y \in U_q^{ad}(\mathfrak{b}_-)$  we have  $\langle i_+(S^{-1}(X)), Y \rangle = \pi_q^-(X, Y)$ , and similarly for every  $X \in U_q^{ad}(\mathfrak{b}_+)$ ,  $Y \in U_q(\mathfrak{b}_-)^{cop}$  we have

$\langle i_-(S^{-1}(Y)), X \rangle = \pi_q^+(Y, X)$ . By keeping these respective notations for  $X$  and  $Y$ , we deduce  $j_q^-(i_+(S^{-1}(X))) = X$  and  $j_q^+(i_-(S^{-1}(Y))) = Y$ , ie.

$$(23) \quad j_q^\pm = S \circ i_\mp^{-1}.$$

Because  $S_{\mathcal{O}_q}^{-1} \circ i_\pm = i_\pm \circ S$ , it follows that

$$(24) \quad j_q^\pm \circ S_{\mathcal{O}_q} = S^{-1} \circ j_q^\pm.$$

Also, for every  $\alpha_- \in \mathcal{O}_q(B_-)$  we have

$$\langle \alpha_-, \Phi^+(i_-(Y)) \rangle = \langle i_-(Y) \otimes \alpha_-, R \rangle = \tau(i_+^{-1}(\alpha_-), Y) = \pi_q^-(j_q^-(S_{\mathcal{O}_q}(\alpha_-)), Y) = \langle \alpha_-, S(Y) \rangle$$

where the first equality is by definition of  $\Phi^+$  (see (2)), the second is (19), the third follows from (24), and the last from the definition of  $j_q^-$ . Similarly, for every  $\alpha_+ \in \mathcal{O}_q(B_+)$  we have

$$\begin{aligned} \langle \alpha_+, \Phi^-(i_+(X)) \rangle &= \langle i_+(X) \otimes \alpha_+, R^- \rangle \\ &= \langle \alpha_+ \otimes S_{\mathcal{O}_q}^{-1} \circ i_+(X), R \rangle \\ &= \langle \alpha_+ \otimes i_+(S(X)), R \rangle \\ &= \tau(S(X), i_-^{-1}(\alpha_+)) \\ &= \pi_q^+(S(i_-^{-1}(\alpha_+)), S(X)) = \pi_q^+(j_q^+(\alpha_+), S(X)) = \langle \alpha_+, S(X) \rangle. \end{aligned}$$

These computations imply  $\Phi^\pm = S \circ i_\mp^{-1} = j_q^\pm$ , and the result follows by taking integral forms.  $\square$

**Remark 2.13.** (1) Since  $\Phi_1 = m \circ (id \otimes S^{-1}) \circ \Phi$  and  $\text{Im}(\Phi) \subset U_A(\mathfrak{b}_-)^{cop} \otimes U_A(\mathfrak{b}_+)^{cop}$ ,  $\Phi_1(\mathcal{O}_A) \subset U_A$ . Because  $\Phi_1(\mathcal{O}_q) = U_q^{lf}$ , we have also  $\Phi_1(\mathcal{O}_A) \subset U_A^{lf}$ . The converse inclusion  $\Phi_1(\mathcal{O}_A) \supset U_A^{lf}$  holds true as well, since  $\Phi_1(\mathcal{O}_q) = U_q^{lf}$  and  $\mathcal{O}_A$  is an  $A$ -lattice of  $\mathcal{O}_q$ .

(2) The components of  $R_A^\pm$  may be described explicitly: if  $\{\xi_i\}_i$  is a basis of  $\Gamma(\mathfrak{b}_+)$  (say, as obtained in section 3 of [36]), one can determine the dual basis  $\{\xi_i^*\}_i$  of  $U_A(\mathfrak{b}_-)$  by using the perfect pairing  $\pi_A^+$ ; then  $R_A^+ = \sum_i \xi_i \otimes \xi_i^*$ . Note that, like  $U_A^{ad}$  is contained in  $\Gamma$ ,  $U_A$  is contained in the restricted integral form of  $U_q$ , whose categorical completion is  $\mathbb{U}_\Gamma \otimes \mathbb{C}[q^{\pm 1/D}]$ . Therefore the components  $\xi_i^*$  of  $R_A^+$  can be viewed as elements of  $\mathbb{U}_\Gamma \otimes \mathbb{C}[q^{\pm 1/D}]$ . This is compatible with the fact that  $R_A^+$  is an element of  $\mathbb{U}_\Gamma^{\otimes 2} \otimes \mathbb{C}[q^{\pm 1/D}]$ .

(3) The dualities of Theorem 2.11 (2) afford a refinement defined over  $A$  of the quantum Killing form  $\kappa: U_q \otimes_{\mathbb{C}(q)} U_q \rightarrow \mathbb{C}(q^{1/D})$  (studied eg. in [70], Section 2.8). This form is the duality realizing the isomorphism  $ad^r(U_A)(K_{-2w_0(\mu)}) \cong \text{End}_A({}_A V_\mu)^*$  stated after (14).

**2.4. Structure theorems for  $U_\epsilon$  and  $\mathcal{O}_\epsilon$ .** As usual we denote by  $\epsilon$  a primitive  $l$ -th root of unity, where  $l$  is odd, and coprime to 3 if  $\mathfrak{g}$  has  $G_2$ -components.

Let  $G^0 = B_+ B_-$  (the *big cell* of  $G$ ), and define the group

$$H = \{(u_+ t, u_- t^{-1}), t \in T_G, u_\pm \in U_\pm\}.$$

Consider the map

$$\begin{aligned} \sigma: B_+ \times B_- &\longrightarrow G^0 \\ (b_+, b_-) &\longmapsto b_+ b_-^{-1}. \end{aligned}$$

The restriction of  $\sigma$  to  $H$  is an unramified covering of degree  $2^m$ . It can be seen as the classical analog of the map  $m \circ (id \otimes S^{-1}): \mathcal{O}_\epsilon(B_+) \otimes \mathcal{O}_\epsilon(B_-) \rightarrow \mathcal{O}_\epsilon(G)$ .

Denote by  $\mathcal{Z}_1(U_\epsilon)$  the image of  $\mathcal{Z}(U_q)$  in  $\mathcal{Z}(U_\epsilon)$  under the specialization map  $U_q \rightarrow U_\epsilon$ , and by  $\mathcal{Z}_0(U_\epsilon) \subset U_\epsilon$  the subalgebra generated by  $E_{\beta_k}^l, F_{\beta_k}^l, L_i^{\pm l}$ , for  $k \in \{1, \dots, N\}$  and

$i \in \{1, \dots, m\}$ . In [33], Section 1.8-3.3-3.8, and [35], Theorem 14.1-21.5, the following results are proved:

**Theorem 2.14.** (1)  $U_\epsilon$  has no non-trivial zero divisors,  $\mathcal{Z}_0(U_\epsilon)$  is a central Hopf subalgebra of  $U_\epsilon$ , and  $U_\epsilon$  is a free  $\mathcal{Z}_0(U_\epsilon)$ -module of rank  $l^{\dim \mathfrak{g}}$ . Moreover  $U_\epsilon$  is a maximal order of its classical fraction algebra  $Q(U_\epsilon) = Q(\mathcal{Z}(U_\epsilon)) \otimes_{\mathcal{Z}(U_\epsilon)} U_\epsilon$ , and  $Q(U_\epsilon)$  is a central simple algebra of PI degree  $l^N$ .

(2)  $\text{SpecM}(\mathcal{Z}_0(U_\epsilon))$  is a group isomorphic to  $H$  above, and the multiplication map yields an isomorphism  $\mathcal{Z}_0(U_\epsilon) \otimes_{\mathcal{Z}_0 \cap \mathcal{Z}_1} \mathcal{Z}_1(U_\epsilon) \rightarrow \mathcal{Z}(U_\epsilon)$ .

It follows from (1) and  $\dim \mathfrak{g} = m + 2N$  that the field  $Q(\mathcal{Z}(U_\epsilon))$  is an extension of  $Q(\mathcal{Z}_0(U_\epsilon))$  of degree  $l^m$ . Conversely, this degree and the rank of  $U_\epsilon$  over  $\mathcal{Z}_0(U_\epsilon)$  imply that  $Q(U_\epsilon)$  has PI degree  $l^N$ .

As for (2), note that  $\mathcal{Z}_0(U_\epsilon)$  being an affine and commutative algebra, the set  $\text{SpecM}(\mathcal{Z}_0(U_\epsilon))$ , viewed as the set of characters of  $\mathcal{Z}_0(U_\epsilon)$ , acquires by duality a structure of affine algebraic group. Thus, the first claim means precisely the identification of this group with  $H$ .

In addition to (2),  $\text{SpecM}(\mathcal{Z}_0(U_\epsilon))$  and  $H$  have natural Poisson structures, that the isomorphism identifies. Moreover we have the following identifications (see [35], Section 21.2). Consider the  $l^m$ -fold covering  $\tilde{T}_G \rightarrow T_G$ . Recall that  $T$  is the group formed by the elements  $K_\lambda \in U_A$ ,  $\lambda \in P$ . We can identify  $T$  with the additive group  $P$ ,  $U_A(\mathfrak{h}) = \mathbb{C}[T] = \mathbb{C}[P]$  with  $\mathcal{O}(\tilde{T}_G)$ , and therefore  $\mathcal{Z}_0(U_\epsilon) \cap U_\epsilon(\mathfrak{h}) = \mathbb{C}[lP]$  with  $\mathcal{O}(T_G)$ . The quantum Harish-Chandra isomorphism then identifies  $\mathcal{Z}_1(U_\epsilon)$  with  $\mathbb{C}[2P]^W \cong \mathcal{O}(\tilde{T}_G/(2))^W$ , where we denote by (2) the subgroup of 2-torsion elements in  $\tilde{T}_G$ . Composing  $\sigma: H \rightarrow G^0$  with the quotient map under conjugation,  $G^0 \hookrightarrow G \rightarrow G//G$ , we get dually an embedding of  $\mathcal{O}(G//G) = \mathcal{O}(G)^G$  in  $\mathcal{O}(H)$ . The isomorphism of Theorem 2.14 (2) then affords identifications

$$\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon) \cong \mathcal{O}(G)^G$$

as a subalgebra of  $\mathcal{Z}_0(U_\epsilon) \cong \mathcal{O}(H)$ , and

$$\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon) = \mathbb{C}[2lP]^W \cong \mathcal{O}(\tilde{T}_G/(2l))^W \cong \mathcal{O}(T_G/(2))^W$$

as a subalgebra of  $\mathcal{Z}_1(U_\epsilon) \cong \mathcal{O}(\tilde{T}_G/(2))^W$ .

A result similar to Theorem 2.14 holds true for  $\mathcal{O}_\epsilon$ . Namely, take the specializations at  $q = \epsilon$  in Theorem 2.11. Denote by  $\mathcal{Z}_0(U_\epsilon(H))$  the subalgebra of  $U_\epsilon(H)$  generated by the elements  $(k \in \{1, \dots, N\}, i \in \{1, \dots, m\})$

$$1 \otimes K_{-l\beta_k} E_{\beta_k}^l, F_{\beta_k}^l K_{l\beta_k} \otimes 1, L_i^{\pm l} \otimes L_i^{\mp l}.$$

It is a central Hopf subalgebra. Recall that  $\mathcal{O}(G)$  can be realized as a Hopf subalgebra of  $U(\mathfrak{g})^\circ$ , the restricted dual of the envelopping algebra  $U(\mathfrak{g})$  over  $\mathbb{C}$ . In [36] De Concini-Lyubashenko introduced an epimorphism of Hopf algebras  $\eta: \Gamma_\epsilon \rightarrow U(\mathfrak{g})$  (essentially a version of Lusztig's "Frobenius" epimorphism in [52]). Let us put

$$(25) \quad \mathcal{Z}_0(\mathcal{O}_\epsilon) := \eta^*(\mathcal{O}(G))$$

where  $\eta^*: U(\mathfrak{g})^\circ \rightarrow \Gamma_\epsilon^\circ$  is the monomorphism dual to  $\eta$ .

**Theorem 2.15.** (1)  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  is a central Hopf subalgebra of  $\mathcal{O}_\epsilon \subset \Gamma_\epsilon^\circ$ , and  $Q(\mathcal{Z}_0(\mathcal{O}_\epsilon))$  is an extension of  $Q(\mathcal{Z}_0(\mathcal{O}_\epsilon))$  of degree  $l^m$  if  $l$  is coprime to the coefficients of the Cartan matrix of  $\mathfrak{g}$ .

(2)  $\psi_{-l\rho} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ , and  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  is generated by the matrix coefficients of the irreducible  $\Gamma$ -modules of highest weight  $l\lambda$ ,  $\lambda \in P_+$ . Moreover, the map  $\Phi$  in (21) affords an algebra embedding  $\mathcal{Z}_0(\mathcal{O}_\epsilon) \rightarrow \mathcal{Z}_0(U_\epsilon(H))$  and algebra isomorphisms  $\mathcal{Z}_0(\mathcal{O}_\epsilon)[\psi_{-l\rho}^{-1}] \rightarrow \mathcal{Z}_0(U_\epsilon(H))$ ,  $\mathcal{O}_\epsilon[\psi_{-l\rho}^{-1}] \rightarrow U_\epsilon(H)$ .

(3)  $\mathcal{O}_\epsilon$  has no non-trivial zero divisors, and it is a free  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ -module of rank  $l^{\dim \mathfrak{g}}$ . Moreover  $\mathcal{O}_\epsilon$  is a maximal order of its classical fraction algebra  $Q(\mathcal{O}_\epsilon) = Q(\mathcal{Z}(\mathcal{O}_\epsilon)) \otimes_{\mathcal{Z}(\mathcal{O}_\epsilon)} \mathcal{O}_\epsilon$ , and  $Q(\mathcal{O}_\epsilon)$  is a central simple algebra of PI degree  $l^N$ .

For the proof, see in [36]: the proposition 6.4 for the first claim of (1) (where  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  and  $\mathcal{Z}_0(U_\epsilon(H))$  are denoted  $F_0$  and  $A_0$  respectively), the appendix of Enriquez and [38] for the second claim of (1), the propositions 6.4-6.5 for (2), and for (3) the theorems 7.2-7.4 (where  $\mathcal{O}_\epsilon$  is shown to be projective over  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ ) and [21] (which provides the additional K-theoretic arguments to deduce that  $\mathcal{O}_\epsilon$  is free).

As above for  $U_\epsilon$ , it follows from (3) that  $Q(\mathcal{Z}(\mathcal{O}_\epsilon))$  has degree  $l^m$  over  $Q(\mathcal{Z}_0(\mathcal{O}_\epsilon))$ . Generators of  $\mathcal{Z}(\mathcal{O}_\epsilon)$  are described in Enriquez' Appendix in [36] under the assumptions on  $l$  stated in (1). We do not know a presentation by generators and relations for general  $G$ , nor a basis of  $\mathcal{O}_\epsilon$  over  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ , but see [37] for the case of  $SL_2$ .

We will recall the known results in this case of  $SL_2$  before Lemma 4.3.

There is a natural action of the braid group  $\mathcal{B}(\mathfrak{g})$  on  $\mathcal{O}_\epsilon$ , that we will use. Namely, let  $n_i \in N(T_G)$  be a representative of the reflection  $s_i \in W = N(T_G)/T_G$  associated to the simple root  $\alpha_i$ . In [67, 66] Soibelman-Vaksman introduced functionals  $t_i : \mathcal{O}_A \rightarrow A$  which quantize the elements  $n_i$ . They correspond dually to generators of the quantum Weyl group of  $\mathfrak{g}$ ; in the Appendix we recall their main properties (see also [30], Section 8.2, and [47, 67, 51, 48, 36]). Denote by  $\triangleleft$  the natural right action of functionals on  $\mathcal{O}_A$ , namely (using Sweedler's notation)

$$\alpha \triangleleft h = \sum_{(\alpha)} h(\alpha_{(1)}) \alpha_{(2)}$$

for every  $\alpha \in \mathcal{O}_A$  and  $h \in \mathcal{O}_A \rightarrow A$ . Let us identify  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  with  $\mathcal{O}(G)$  by means of (25). We have ([36], Proposition 7.1):

**Proposition 2.16.** *The maps  $\triangleleft t_i$  on  $\mathcal{O}_\epsilon$  preserve  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ , and satisfy  $(f \triangleleft t_i)(a) = f(n_i a)$  and  $(f \star \alpha) \triangleleft t_i = (f \triangleleft t_i)(\alpha \triangleleft t_i)$  for every  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ ,  $a \in G$ ,  $\alpha \in \mathcal{O}_\epsilon$ .*

We provide an alternative, non computational, proof of this result in the Appendix (Section 6.2).

### 3. NOETHERIANITY AND FINITENESS

In this section we prove Theorem 1.1. Recall that by Noetherian we mean right and left Noetherian.

**Theorem 3.1.** *The algebras  $\mathcal{L}_{0,n}$ ,  $\mathcal{L}_{0,n}^A$  and  $\mathcal{L}_{0,n}^{\epsilon'}$ ,  $\epsilon' \in \mathbb{C}^\times$ , are Noetherian.*

Let us note that the algebras in this theorem are generated by a finite number of elements over their respective ground rings  $\mathbb{C}(q)$ ,  $A$  and  $\mathbb{C}$ . Indeed, by the formula (6) it is enough to verify this for  $\mathcal{L}_{0,1}^A$ , but  $\mathcal{L}_{0,1}^A = \mathcal{O}_A$  as a vector space, and  $\mathcal{O}_A$  with its product  $\star$  is well-known to be finitely generated by the matrix coefficients of the fundamental  $\Gamma$ -modules  $AV_{\varpi_k}$ ,  $k \in \{1, \dots, m\}$ . Then the claim follows from the formula inverse to (3), expressing the product  $\star$  in terms of the product of  $\mathcal{L}_{0,1}$  (see (18) in [23]).

*Proof of Theorem 3.1.* The result for  $\mathcal{L}_{0,1}$  and  $\mathcal{L}_{0,1}^A$  follows immediately from Theorem 2.2 (3) by identifying  $\mathcal{L}_{0,1}^A$  with  $U_A^{lf}$  via  $\Phi_1$ . Assume now that  $n > 1$ . We are going to develop the proof for  $\mathcal{L}_{0,n}$ ; the arguments can be repeated verbatim for  $\mathcal{L}_{0,n}^A$ , and the result for  $\mathcal{L}_{0,n}^{\epsilon'}$  will then follow immediately by lifting ideals by the quotient map  $\mathcal{L}_{0,n}^A \rightarrow \mathcal{L}_{0,n}^{\epsilon'} = \mathcal{L}_{0,n}^A / (q - \epsilon') \mathcal{L}_{0,n}^A$ .

Recall the isomorphism of  $U_q$ -modules (see (11)):

$$(26) \quad \mathcal{L}_{0,n} \xrightarrow{\Phi_n} (U_q(\mathfrak{g})^{\otimes n})^{lf} \xrightarrow{\psi_n^{-1}} U_q^{lf}(\mathfrak{g})^{\otimes n} = U_q^{lf}(\mathfrak{g}^{\oplus n})$$

where  $lf$  means respectively locally finite for the action  $ad_n^r$  of  $U_q(\mathfrak{g})$  on  $U_q(\mathfrak{g})^{\otimes n}$ , locally finite for the action  $ad^r$  of  $U_q(\mathfrak{g})$  on  $U_q(\mathfrak{g})$ , and locally finite for the action  $ad^r$  of  $U_q^{lf}(\mathfrak{g}^{\oplus n})$  on itself. It is a fact that Theorem 2.2 (3) holds true by replacing  $U_q^{lf}(\mathfrak{g})$  with  $U_q^{lf}(\mathfrak{g}^{\oplus n})$ , but one cannot use this to deduce the result because  $\psi_n$  is not a morphism of algebras. However, one can adapt the arguments of the proof of Theorem 2.2 (3) given in Theorem 2.137 of [70]. Let us begin by recalling these arguments.

As usual let  $C(\mu)$  be the vector space generated by the matrix coefficients of  $V_\mu$ , the simple  $U_q$ -module of type 1 and highest weight  $\mu \in P_+$ . Denote by  $C(\mu)_\lambda \subset C(\mu)$  the subspace of weight  $\lambda$  for the left coregular action of  $U_q(\mathfrak{h})$ ; so  $\alpha \in C(\mu)_\lambda$  if

$$K_\nu \triangleright \alpha = q^{(\nu, \lambda)} \alpha, \nu \in P.$$

Consider the ordered semigroup

$$\Lambda = \{(\mu, \lambda) \in P_+ \times P, \lambda \text{ is a weight of } V_\mu\}$$

with the partial order  $(\mu, \lambda) \leq (\mu', \lambda')$  if and only if  $\mu' - \mu \in P_+, \lambda' - \lambda \in P_+$ . Since  $\mathcal{L}_{0,1}$  and  $\mathcal{O}_q$  are isomorphic vector spaces we have  $\mathcal{L}_{0,1} = \bigoplus_{\mu \in P_+} C(\mu) = \bigoplus_{(\mu, \lambda) \in \Lambda} C(\mu)_\lambda$ . Consider the filtration  $\mathcal{F}_2$  of the vector space  $\mathcal{L}_{0,1}$  given by the family of subspaces

$$\mathcal{F}_2^{\mu, \lambda} = \bigoplus_{(\mu', \lambda') \leq (\mu, \lambda)} C(\mu')_{\lambda'}, (\mu, \lambda) \in \Lambda.$$

Denote by  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  the associated graded vector space. The standard vector space isomorphism  $\mathcal{L}_{0,1} \rightarrow Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$ , assigning to  $x \in C(\mu)_\lambda$  its coset  $\bar{x} \in \mathcal{F}_2^{\mu, \lambda} / (\bigoplus_{(\mu', \lambda') < (\mu, \lambda)} C(\mu')_{\lambda'})$ , implies

$$Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1}) = \bigoplus_{(\mu, \lambda) \in \Lambda} C(\mu)_\lambda.$$

Now, one has the following facts:

(i) First, taking the product in  $\mathcal{L}_{0,1}$  we have

$$(27) \quad \alpha\beta \in \mathcal{F}_2^{\mu_1 + \mu_2, \lambda_1 + \lambda_2} \quad \text{for } \alpha \in C(\mu_1)_{\lambda_1}, \beta \in C(\mu_2)_{\lambda_2}.$$

Therefore  $\mathcal{F}_2$  is an algebra filtration of  $\mathcal{L}_{0,1}$ , and  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  a graded algebra. Denote by  $\alpha \circ \beta$  the product in  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  of  $\alpha, \beta \in \mathcal{L}_{0,1}$ ; by definition, if  $\alpha \in C(\mu_1)_{\lambda_1}, \beta \in C(\mu_2)_{\lambda_2}$  then  $\alpha \circ \beta$  is the projection of  $\alpha\beta$  onto  $C(\mu_1 + \mu_2)_{\lambda_1 + \lambda_2}$ .

(ii) Second, denote by  $\bar{\star}$  the product  $\star$  of  $\mathcal{O}_q$  followed by the projection onto the component  $C(\mu + \nu)$ . Then we have

$$(28) \quad C(\mu) \circ C(\nu) = C(\mu) \bar{\star} C(\nu) = C(\mu + \nu).$$

(iii) Finally, for every  $\mu \in P_+$  fix a basis of weight vectors  $e_1^\mu, \dots, e_m^\mu$  of  $V_\mu$ . Denote by  $e_1^\mu, \dots, e_m^\mu \in V_\mu^*$  the dual basis, and by  $w(e_i^\mu)$  the weight of  $e_i^\mu$ . One can assume that the ordering of  $e_1^\mu, \dots, e_m^\mu$  is such that  $w(e_i^\mu) > w(e_j^\mu)$  implies  $i < j$ ; indeed,  $e_1^\mu$  generates the subspace of weight  $\mu$ , then come (in any order) the  $e_i^\mu$  such that  $w(e_i^\mu) = \mu - \alpha_s$  for some  $s$ , then those such that  $w(e_i^\mu) = \mu - \alpha_s - \alpha_t$  for some  $s$  and  $t$ , etc. Consider the matrix coefficients  ${}_\mu \phi_i^j(x) := e_i^\mu(\pi_V(x)(e_j^\mu))$ ,  $x \in U_q$ . By (3), using the explicit form of the  $R$ -matrix

it can be shown that

$$(29) \quad \begin{aligned} \nu\phi_k^l \circ \mu\phi_i^j - q_{ijkl} \mu\phi_i^j \circ \nu\phi_k^l &= \sum_{r=i}^m \sum_{s=1}^k \sum_{u=1}^{l-1} \sum_{v=j+1}^m \delta_{rsuv}^{ijkl} \mu\phi_r^v \circ \nu\phi_s^u \\ &\quad - \sum_{r=i+1}^m \sum_{s=1}^{k-1} q_{ijkl} \gamma_{rs}^{ijkl} \mu\phi_r^j \circ \nu\phi_s^l \end{aligned}$$

where  $q_{ijkl} = q^{(w(e_j^\mu) + w(e_i^\mu), w(e_k^\nu) - w(e_l^\nu))}$ , and  $\gamma_{rs}^{ijkl}, \delta_{rsuv}^{ijkl} \in \mathbb{C}(q^{1/D})$  are such that  $\gamma_{rs}^{ijkl} = 0$  unless  $w(e_r^\mu) < w(e_i^\mu)$  and  $w(e_s^\nu) > w(e_k^\nu)$ , and  $\delta_{rsuv}^{ijkl} = 0$  unless  $w(e_u^\nu) > w(e_l^\nu)$ ,  $w(e_v^\mu) < w(e_j^\mu)$ ,  $w(e_r^\mu) \leq w(e_i^\mu)$  and  $w(e_s^\nu) \geq w(e_k^\nu)$ .

By (28) (or more simply by using (3), as observed before the proof),  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  is generated by the matrix coefficients  $\varpi_k \phi_i^j$  of the fundamental representations  $V_{\varpi_k}$ . One can list these matrix coefficients, say  $M$  in number, in an ordered sequence  $u_1, \dots, u_M$  such that the following condition holds: if  $w(e_k^{\varpi_s}) < w(e_i^{\varpi_r})$ , or  $w(e_k^{\varpi_s}) = w(e_i^{\varpi_r})$  and  $w(e_l^{\varpi_s}) < w(e_j^{\varpi_r})$ , then  $u_a := \varpi_r \phi_i^j$  and  $u_b := \varpi_s \phi_k^l$  satisfy  $b < a$ . Then denoting  $\mu\phi_i^j, \nu\phi_k^l$  in (29) by  $u_j, u_i$  respectively, and assuming  $u_j < u_i$ , one finds that all terms  $u_s := \mu\phi_r^v, \mu\phi_r^j$  in the sums are  $< u_j$ . Therefore, for all  $1 \leq j < i \leq M$  it takes the form:

$$(30) \quad u_i \circ u_j - q_{ij} u_j \circ u_i = \sum_{s=1}^{j-1} \sum_{t=1}^M \alpha_{ij}^{st} u_s \circ u_t$$

for some  $q_{ij} \in \mathbb{C}(q^{1/D})^\times, \alpha_{ij}^{st} \in \mathbb{C}(q^{1/D})$ . By Proposition I.8.17 of [20] (see also Proposition 2.133 of [70]) an algebra  $A$  over a field  $\mathbb{K}$  generated by elements  $u_1, \dots, u_M$  such that

$$(31) \quad u_i \circ u_j - q_{ij} u_j \circ u_i = \sum_{s=1}^{j-1} \sum_{t=1}^M \alpha_{ij}^{st} u_s \circ u_t + \beta_{ij}^{st} u_t \circ u_s$$

for all  $1 \leq j < i \leq M$  and some  $q_{ij} \in \mathbb{K}^\times$  and  $\alpha_{ij}^{st}, \beta_{ij}^{st} \in \mathbb{K}$ , is Noetherian. In fact  $A$  has an algebra filtration, say  $\mathcal{F}_3$ , such that  $Gr_{\mathcal{F}_3}(A)$  is a quotient of a skew-polynomial algebra, and thus is Noetherian. Moreover, it is classical that a filtered algebra which graded algebra is Noetherian is Noetherian too (see eg. [60], 1.6.9-1.6.11). Applying this to  $A = Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})$  and going up the filtration  $\mathcal{F}_2$  it follows that  $\mathcal{L}_{0,1}$  is Noetherian too.

We are going to extend all these facts to  $\mathcal{L}_{0,n}$ . The main point is to generalize the filtration  $\mathcal{F}_2$ , which we do first. Consider the semigroup

$$[\Lambda] = \{([\mu], [\lambda]) \in P_+^n \times P_+^n \mid (\mu_i, \lambda_i) \in \Lambda \text{ where } [\mu] = (\mu_i)_{i=1}^n, [\lambda] = (\lambda_i)_{i=1}^n\}.$$

Put the lexicographic partial order on  $[\Lambda]$ , starting from the tail: so  $([\mu'], [\lambda']) \leq ([\mu], [\lambda])$  if  $\mu_n - \mu'_n \in P_+ \setminus \{0\}$ , or  $\mu_n = \mu'_n$  and  $\lambda_n - \lambda'_n \in P_+ \setminus \{0\}$ , or there is  $k \in \{n, \dots, 2\}$  such that  $\mu_i = \mu'_i, \lambda_i = \lambda'_i$  for  $i \in \{n, \dots, k\}$  and  $\mu_{k-1} - \mu'_{k-1} \in P_+ \setminus \{0\}$ , or  $\mu_{k-1} = \mu'_{k-1}$  and  $\lambda_{k-1} - \lambda'_{k-1} \in P_+ \setminus \{0\}$ , replacing this last condition by  $\lambda_1 - \lambda'_1 \in P_+$  when  $k = 2$ . Now recall that  $\mathcal{L}_{0,n} = \mathcal{L}_{0,1}^{\otimes n} = \mathcal{O}_q^{\otimes n}$  as vector spaces. For every  $([\mu], [\lambda]) \in [\Lambda]$  consider the subspaces  $C([\mu]_{[\lambda]}] \subset C([\mu]) \subset \mathcal{L}_{0,n}$  defined by

$$\begin{aligned} C([\mu]) &= C(\mu_1) \otimes \dots \otimes C(\mu_n) \\ C([\mu]_{[\lambda]}) &= C(\mu_1)_{\lambda_1} \otimes \dots \otimes C(\mu_n)_{\lambda_n}. \end{aligned}$$



Then  $\mathcal{L}_{0,n} = \bigoplus_{[\mu] \in P_+^n} C([\mu])$  and  $C([\mu]) = \bigoplus_{([\mu], [\lambda]) \in [\Lambda]} C([\mu])_{[\lambda]}$ . For every  $([\mu], [\lambda]) \in [\Lambda]$  define

$$(32) \quad \mathcal{F}_2^{[\mu], [\lambda]} = \bigoplus_{([\mu'], [\lambda']) \leq ([\mu], [\lambda])} \bigotimes_{j=1}^n C(\mu'_j)_{\lambda'_j}.$$

Clearly  $\mathcal{F}_2^{[\mu'], [\lambda']} \subset \mathcal{F}_2^{[\mu], [\lambda]}$  for  $([\mu'], [\lambda']) \leq ([\mu], [\lambda])$ , and the vector space  $\mathcal{L}_{0,n}$  is the union of the subspaces  $\mathcal{F}_2^{[\mu], [\lambda]}$  over all  $([\mu], [\lambda]) \in [\Lambda]$ , so these form a filtration of  $\mathcal{L}_{0,n}$ . Let us denote it  $\mathcal{F}_2$ , as when  $n = 1$ . As usual, write  $([\mu'], [\lambda']) < ([\mu], [\lambda])$  for  $([\mu'], [\lambda']) \leq ([\mu], [\lambda])$  and  $([\mu'], [\lambda']) \neq ([\mu], [\lambda])$ , and put

$$\mathcal{F}_2^{<[\mu], [\lambda]} = \sum_{([\mu'], [\lambda']) < ([\mu], [\lambda])} \mathcal{F}_2^{[\mu'], [\lambda']}.$$

Then define

$$Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})_{[\mu], [\lambda]} = \mathcal{F}_2^{[\mu], [\lambda]} / \mathcal{F}_2^{<[\mu], [\lambda]}.$$

This space is canonically identified with  $C([\mu])_{[\lambda]}$ , so the graded vector space associated to  $\mathcal{F}_2$  is

$$(33) \quad Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n}) = \bigoplus_{([\mu], [\lambda]) \in [\Lambda]} Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})_{[\mu], [\lambda]} = \bigoplus_{([\mu], [\lambda]) \in [\Lambda]} C([\mu])_{[\lambda]}.$$

We claim that  $\mathcal{F}_2$  is an *algebra filtration* with respect to the product of  $\mathcal{L}_{0,n}$ , and therefore  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})$  is a graded algebra.

For notational simplicity let us prove it for  $n = 2$ , the general case being strictly similar. Recall that the product of  $\mathcal{L}_{0,n}$  is given by the formula (6). Take  $([\mu], [\lambda]), ([\mu'], [\lambda']) \in [\Lambda]$ , and elements  $\alpha \otimes \beta \in C(\mu_1)_{\lambda_1} \otimes C(\mu_2)_{\lambda_2}$  and  $\alpha' \otimes \beta' \in C(\mu'_1)_{\lambda'_1} \otimes C(\mu'_2)_{\lambda'_2}$ . The  $R$ -matrix expands as  $R = \Theta \hat{R}$ , where  $\Theta = q^{\sum_{i,j=1}^m (B^{-1})_{ij} H_i \otimes H_j} \in \mathbb{U}_q^{\otimes 2}$ , with  $B \in M_m(\mathbb{Q})$  the matrix with entries  $B_{ij} := d_j^{-1} a_{ij}$ , and  $\hat{R} = \sum_{(\hat{R})} \hat{R}_{(1)} \otimes \hat{R}_{(2)} \in \mathbb{U}_q(\mathfrak{n}_+) \otimes \mathbb{U}_q(\mathfrak{n}_-)$  (see eg. [30], Theorem 8.3.9, or [70], Theorem 2.108). If  $x, y$  are weight vectors of weights  $\mu, \nu$  respectively, then  $\Theta(x \otimes y) = q^{(\mu, \nu)} x \otimes y$ . Moreover,  $\hat{R}$  has weight 0 for the adjoint action of  $U_q(\mathfrak{h})$ ; that is, complementary components  $\hat{R}_{(1)}$  and  $\hat{R}_{(2)}$  have opposite weights. Note also that the coregular actions  $\triangleright, \triangleleft$  fix globally each component  $C(\mu)$ ,  $\mu \in P_+$ . Then, for every  $\nu \in P$  and any of the components  $R_{(2)}^1, \dots, R_{(2)}^4$  we have

$$\begin{aligned} K_\nu \triangleright \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \right) &= \sum_{(\beta), (\beta)} \beta_{(1)}(R_{(2)}^2 R_{(2)}^4) \left( K_\nu S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta_{(2)} \right) \\ &= q^{-(\nu, \gamma)} \sum_{(\beta), (\beta)} \beta_{(1)}(R_{(2)}^2 R_{(2)}^4) \left( S(R_{(2)}^1 R_{(2)}^3) K_\nu \triangleright \beta_{(2)} \right) \\ &= q^{(\nu, \lambda_2 - \gamma)} \sum_{(\beta), (\beta)} \beta_{(1)}(R_{(2)}^2 R_{(2)}^4) \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta_{(2)} \right) \\ &= q^{(\nu, \lambda_2 - \gamma)} \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \right). \end{aligned}$$

for some positive root  $\gamma \in Q_+$ . Therefore  $S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \in C(\mu_2)_{\lambda_2 - \gamma}$ ; by a similar computation we find that  $S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha' \triangleleft R_{(1)}^1 R_{(1)}^2 \in C(\mu'_1)_{\lambda'_1 + \gamma}$  for the complementary components  $R_{(1)}^1, \dots, R_{(1)}^4$ . Then we always have  $S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \in \mathcal{F}_2^{\mu_2, \lambda_2}$ , and if

$S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \in C(\mu_2)_{\lambda_2}$  then  $S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha' \triangleleft R_{(1)}^1 R_{(1)}^2 \in C(\mu'_1)_{\lambda'_1}$ . Since the product of  $\mathcal{L}_{0,n}$  is componentwise that of  $\mathcal{L}_{0,1}$ , by (27) we have

$$(34) \quad \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \right) \beta' \in \mathcal{F}_2^{\mu_2 + \mu'_2, \lambda_2 + \lambda'_2}$$

and if  $(S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4) \beta' \in C(\mu_2 + \mu'_2)_{\lambda_2 + \lambda'_2}$  then

$$(35) \quad \alpha \left( S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha' \triangleleft R_{(1)}^1 R_{(1)}^2 \right) \in C(\mu_1)_{\lambda_1} C(\mu'_1)_{\lambda'_1} \subset \mathcal{F}_2^{\mu_1 + \mu'_1, \lambda_1 + \lambda'_1}.$$

In conclusion

$$(\alpha \otimes \beta)(\alpha' \otimes \beta') \in \mathcal{F}_2^{[\mu], [\lambda]}.$$

Similar arguments work for any  $n \geq 2$ . This proves that  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})$  is a graded algebra with the product inherited from  $\mathcal{L}_{0,n}$ , which we denote by  $\circ_n$ . Recall that it is defined on homogeneous elements  $\overline{\alpha \otimes \beta} \in Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})_{[\mu], [\lambda]}$ ,  $\overline{\alpha' \otimes \beta'} \in Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})_{[\mu'], [\lambda']}$  by

$$\overline{\alpha \otimes \beta} \circ_n \overline{\alpha' \otimes \beta'} = (\alpha \otimes \beta)(\alpha' \otimes \beta') + \mathcal{F}_2^{<[\mu + \mu'], [\lambda + \lambda']]}.$$

Next we show that (28) implies the same property for the product  $\circ$  of  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})$ . First it gives  $(C(\mu_1) \circ C(\mu'_1)) \otimes (C(\mu_2) \circ C(\mu'_2)) = C([\mu + \mu'])$ . Now, by the previous remark and (34), (35) we have

$$C([\mu]) \circ_n C([\mu']) \subset (C(\mu_1) \circ C(\mu'_1)) \otimes (C(\mu_2) \circ C(\mu'_2)).$$

The converse inclusion holds true as well, as one can see by reasoning as above, starting with the standard (componentwise) product of  $\mathcal{L}_{0,1}^{\otimes n}$  expressed in terms of the product of  $\mathcal{L}_{0,n}$  by the formula (8). In conclusion

$$(36) \quad C([\mu]) \circ_n C([\mu']) = C([\mu + \mu']).$$

We are left to show that (29) generalizes to  $\mathcal{L}_{0,n}$ . Again we note that this cannot be deduced from the case  $n = 1$ , because for the vector space isomorphism  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n}) \rightarrow Gr_{\mathcal{F}_2}(\mathcal{L}_{0,1})^{\otimes n}$  induced from the equality  $\mathcal{L}_{0,n} = \mathcal{L}_{0,1}^{\otimes n}$  is not a morphism of algebras with respect to the products  $\circ_n$  and  $\circ^{\otimes n}$ . Therefore one cannot take the filtration on  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})$  which is componentwise  $\mathcal{F}_3$ , and that we will denote again by  $\mathcal{F}_3$ , to deduce that  $Gr_{\mathcal{F}_3}(Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n}))$  is a quotient of a quasi-polynomial algebra, whence Noetherian. However, we can proceed in essentially the same way. We give the details when  $n = 2$ , the general case being similar. Let us write the twist  $F$  in (7) as

$$F = \sum_{(F)} F_{(1)} \otimes F_{(2)} = \sum_{(F)} F_{(1)1} \otimes F_{(1)2} \otimes F_{(2)1} \otimes F_{(2)2}$$

that is, setting  $F_{(1)1} := R_{(2)}^2 R_{(2)}^4$ ,  $F_{(1)2} := R_{(2)}^1 R_{(2)}^3$ ,  $F_{(2)1} := R_{(1)}^1 R_{(1)}^2$ ,  $F_{(2)2} := R_{(1)}^3 R_{(1)}^4$ . Keep the notations of (29), and put  $d(\mu) := \dim(V_\mu)$ ,  $\mu \in P_+$ , and

$$\Delta^{(2)}(\mu_2 \phi_{k_2}^{l_2}) = \sum_{p,s=1}^{d(\mu_2)} \mu_2 \phi_{k_2}^p \otimes \mu_2 \phi_p^s \otimes \mu_2 \phi_s^{l_2}, \quad \Delta^{(2)}(\mu'_1 \phi_{k'_1}^{l'_1}) = \sum_{p',s'=1}^{d(\mu'_1)} \mu'_1 \phi_{k'_1}^{p'} \otimes \mu'_1 \phi_{p'}^{s'} \otimes \mu'_1 \phi_{s'}^{l'_1}.$$

Assume that  $\mu'_2 \phi_{k'_2}^{l'_2} < \mu_2 \phi_{k_2}^{l_2}$  and  $\mu'_1 \phi_{k'_1}^{l'_1} < \mu_1 \phi_{k_1}^{l_1}$ . From (6) and then (29)-(30) in the second equality, one obtains

$$\begin{aligned} & \left( \mu_1 \phi_{k_1}^{l_1} \otimes \mu_2 \phi_{k_2}^{l_2} \right) \circ_2 \left( \mu'_1 \phi_{k'_1}^{l'_1} \otimes \mu'_2 \phi_{k'_2}^{l'_2} \right) \\ &= \sum_{(F)} \sum_{p,s=1}^{d(\mu_2)} \sum_{p',s'=1}^{d(\mu'_1)} \left( \mu_1 \phi_{k_1}^{l_1} \circ \mu'_1 \phi_{p'}^{s'} \left( \mu'_1 \phi_{k'_1}^{p'}(F_{(2)1}) \mu'_1 \phi_{s'}^{l'_1}(S(F_{(2)2})) \right) \right) \\ & \quad \otimes \left( \mu_2 \phi_p^s \circ \mu'_2 \phi_{k'_2}^{l'_2} \left( \mu_2 \phi_{k_2}^p(F_{(1)1}) \mu_2 \phi_s^{l_2}(S(F_{(1)2})) \right) \right) \\ &= \sum_{(F)} \sum_{p,s=1}^{d(\mu_2)} \sum_{p',s'=1}^{d(\mu'_1)} q_{p's'k_1l_1} q_{k'_2l'_2ps} \left( \mu'_1 \phi_{p'}^{s'} \circ \mu_1 \phi_{k_1}^{l_1} \left( \mu'_1 \phi_{k'_1}^{p'}(F_{(2)1}) \mu'_1 \phi_{s'}^{l'_1}(S(F_{(2)2})) \right) \right) \\ & \quad \otimes \left( \mu'_2 \phi_{k'_2}^{l'_2} \circ \mu_2 \phi_p^s \left( \mu_2 \phi_{k_2}^p(F_{(1)1}) \mu_2 \phi_s^{l_2}(S(F_{(1)2})) \right) \right) + \dots \end{aligned}$$

Here the dots are sums of tensors of the form  $(x_1 \circ x_2) \otimes (y_1 \circ y_2)$  where  $x_1 < \mu'_1 \phi_{k'_1}^{l'_1}$  and  $y_1 < \mu'_2 \phi_{k'_2}^{l'_2}$ . In fact, by the expression of  $R = \Theta \hat{R}$  we have  $\mu_2 \phi_{k_2}^p(F_{(1)1}) \mu_2 \phi_s^{l_2}(S(F_{(1)2})) = 0$  unless  $k_2 \geq p$  and  $s \geq l_2$ , and  $\mu'_1 \phi_{k'_1}^{p'}(F_{(2)1}) \mu'_1 \phi_{s'}^{l'_1}(S(F_{(2)2})) = 0$  unless  $k'_1 \leq p'$  and  $s' \leq l'_1$ . It is immediate that  $\mu_2 \phi_p^s \in C(\mu_2)_{w(e_s^{\mu_2})}$ . By definition  $s > l_2$  implies  $w(e_s^{\mu_2}) \leq w(e_{l_2}^{\mu_2})$ , and by (27) if  $w(e_s^{\mu_2}) < w(e_{l_2}^{\mu_2})$  then  $\mu_2 \phi_{k'_2}^{l'_2} \circ \mu_2 \phi_p^s \in \mathcal{F}_2^{<\mu''_2, \lambda''_2}$ , where  $\mu''_2 := \mu_2 + \mu'_2$ ,  $\lambda''_2 := w(e_{l_2}^{\mu_2}) + w(e_{l'_2}^{\mu'_2})$ . In that case the term  $(\mu'_1 \phi_{p'}^{s'} \circ \mu_1 \phi_{k_1}^{l_1}) \otimes (\mu'_2 \phi_{k'_2}^{l'_2} \circ \mu_2 \phi_p^s)$  in the sum above vanishes in  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,2})$ . Moreover, the computations before (35) show that such a summand achieves the maximal weight  $w(e_{l_1}^{\mu_1}) + w(e_{l'_1}^{\mu'_1}) + w(e_{l_2}^{\mu_2}) + w(e_{l'_2}^{\mu'_2})$  only if  $w(e_{s'}^{\mu'_1}) = w(e_{l'_1}^{\mu'_1})$  and  $w(e_s^{\mu_2}) = w(e_{l_2}^{\mu_2})$ , which occurs when  $F_{(1)2}, F_{(2)2}$  have no component of  $\hat{R}$  but only of  $\Theta$ . Also, if  $p = k_2, s = l_2, p' = k'_1, s' = l'_1$  then

$$\begin{aligned} & \sum_{(F)} \mu_2 \phi_{k_2}^{k_2}(F_{(1)1}) \mu_2 \phi_{l_2}^{l_2}(S(F_{(1)2})) \mu'_1 \phi_{k'_1}^{k'_1}(F_{(2)1}) \mu'_1 \phi_{l'_1}^{l'_1}(S(F_{(2)2})) \\ &= \left\langle \mu_2 \phi_{k_2}^{k_2} \otimes \mu_2 \phi_{l_2}^{l_2} \otimes \mu'_1 \phi_{k'_1}^{k'_1} \otimes \mu'_1 \phi_{l'_1}^{l'_1}, \Theta_{13} \Theta_{14}^{-1} \Theta_{24} \Theta_{23}^{-1} \right\rangle \\ &= q \left( w(e_{k_2}^{\mu_2}) - w(e_{l_2}^{\mu_2}), w(e_{k'_1}^{\mu'_1}) - w(e_{l'_1}^{\mu'_1}) \right). \end{aligned}$$

Denoting by  $q'_{k_2 l_2 k'_1 l'_1}$  this scalar, it follows

$$\begin{aligned} & \left( \mu_1 \phi_{k_1}^{l_1} \otimes \mu_2 \phi_{k_2}^{l_2} \right) \circ_2 \left( \mu'_1 \phi_{k'_1}^{l'_1} \otimes \mu'_2 \phi_{k'_2}^{l'_2} \right) \\ &= q_{k'_1 l'_1 k_1 l_1} q_{k'_2 l'_2 k_2 l_2} q'_{k_2 l_2 k'_1 l'_1} \left( \left( \mu'_1 \phi_{k'_1}^{l'_1} \circ \mu_1 \phi_{k_1}^{l_1} \right) \otimes \left( \mu'_2 \phi_{k'_2}^{l'_2} \circ \mu_2 \phi_{k_2}^{l_2} \right) \right) \\ & \quad + \sum_{p=1}^{k_2-1} \sum_{p'=k'_1+1}^{d(\mu'_1)} \alpha_{p'k_1 k'_2 p}^{l'_1 l_1 l'_2 l_2} \left( \left( \mu'_1 \phi_{p'}^{l'_1} \circ \mu_1 \phi_{k_1}^{l_1} \right) \otimes \left( \mu'_2 \phi_{k'_2}^{l'_2} \circ \mu_2 \phi_p^{l_2} \right) \right) + \dots \end{aligned}$$

for some scalars  $\alpha_{p'k_1 k'_2 p}^{l'_1 l_1 l'_2 l_2} \in \mathbb{C}(q^{1/D})$ , with the dots as above. Moreover  $\alpha_{p'k_1 k'_2 p}^{l'_1 l_1 l'_2 l_2} = 0$  unless  $w(e_p^{\mu_2}) > w(e_{k_2}^{\mu_2})$  and  $w(e_{p'}^{\mu'_1}) < w(e_{k'_1}^{\mu'_1})$ . Now, recall (8). In a similar way we find for all

$p \in \{1, \dots, k_2\}$ ,  $p' \in \{k'_1, \dots, d(\mu'_1)\}$  that

$$\begin{aligned}
& \left( \mu'_1 \phi_{p'}^{l'_1} \circ \mu_1 \phi_{k_1}^{l_1} \right) \otimes \left( \mu'_2 \phi_{k'_2}^{l'_2} \circ \mu_2 \phi_p^{l_2} \right) \\
&= \sum_{(\bar{F})} \sum_{r,t=1}^{d(\mu_1)} \sum_{r',t'=1}^{d(\mu'_2)} \left( \mu'_1 \phi_{p'}^{l'_1} \otimes \mu'_2 \phi_{r'}^{t'} \left( \mu'_2 \phi_{k'_2}^{r'} (\bar{F}(2)1)_{\mu'_2} \phi_{t'}^{l'_2} (S(\bar{F}(2)2)) \right) \right) \\
&\quad \circ_2 \left( \mu_1 \phi_r^t \left( \mu_1 \phi_{k_1}^r (\bar{F}(1)1)_{\mu_1} \phi_t^{l_1} (S(\bar{F}(1)2)) \right) \right) \otimes \mu_2 \phi_p^{l_2} \\
&= q'_{k_1 l_1 k'_2 l'_2}^{-1} \left( \left( \mu'_1 \phi_{p'}^{l'_1} \otimes \mu'_2 \phi_{k'_2}^{l'_2} \right) \circ_2 \left( \mu_1 \phi_{k_1}^{l_1} \otimes \mu_2 \phi_p^{l_2} \right) \right) \\
&\quad + \sum_{r=1}^{k_1-1} \sum_{r'=k'_2+1}^{d(\mu'_2)} \beta'_{p'r'r'p}{}^{l'_1 l'_2 l_1 l_2} \left( \left( \mu'_1 \phi_{p'}^{l'_1} \otimes \mu'_2 \phi_{r'}^{l'_2} \right) \circ_2 \left( \mu_1 \phi_r^{l_1} \otimes \mu_2 \phi_p^{l_2} \right) \right)
\end{aligned}$$

for some scalars  $\beta'_{p'r'r'p}{}^{l'_1 l'_2 l_1 l_2} \in \mathbb{C}(q^{1/D})$  such that  $\beta'_{p'r'r'p}{}^{l'_1 l'_2 l_1 l_2} = 0$  unless  $w(e_r^{\mu_1}) > w(e_{k_1}^{\mu_1})$  and  $w(e_{r'}^{\mu'_2}) < w(e_{k'_2}^{\mu'_2})$ . Summing up we obtain

$$\begin{aligned}
& \left( \mu_1 \phi_{k_1}^{l_1} \otimes \mu_2 \phi_p^{l_2} \right) \circ_2 \left( \mu'_1 \phi_{k'_1}^{l'_1} \otimes \mu'_2 \phi_{k'_2}^{l'_2} \right) \\
&= q_{k'_1 l'_1 k_1 l_1} q_{k'_2 l'_2 k_2 l_2} q'_{k_2 l_2 k'_1 l'_1} q'_{k_1 l_1 k'_2 l'_2}^{-1} \left( \left( \mu'_1 \phi_{k'_1}^{l'_1} \otimes \mu'_2 \phi_{k'_2}^{l'_2} \right) \circ_2 \left( \mu_1 \phi_{k_1}^{l_1} \otimes \mu_2 \phi_p^{l_2} \right) \right) \\
&\quad + \sum_{p=1}^{k_2} \sum_{p'=k'_1}^{d(\mu'_1)} \sum_{r=1}^{k_1-1} \sum_{r'=k'_2+1}^{d(\mu'_2)} \alpha'_{p'k_1 k'_2 p}{}^{l'_1 l'_2 l_1 l_2} \beta'_{p'r'r'p}{}^{l'_1 l'_2 l_1 l_2} \left( \left( \mu'_1 \phi_{p'}^{l'_1} \otimes \mu'_2 \phi_{r'}^{l'_2} \right) \circ_2 \left( \mu_1 \phi_r^{l_1} \otimes \mu_2 \phi_p^{l_2} \right) \right) + \dots
\end{aligned}$$

where at  $p = k_2$ ,  $p' = k'_1$  we set  $\alpha'_{k'_1 k_1 k'_2 k_2}{}^{l'_1 l'_2 l_1 l_2} := q_{k'_1 l'_1 k_1 l_1} q_{k'_2 l'_2 k_2 l_2} q'_{k_2 l_2 k'_1 l'_1}$ , and the dots are sums of tensors of the form  $(x_1 \otimes y_1) \circ_2 (x_2 \otimes y_2)$  where  $y_1 < \mu'_2 \phi_{k'_2}^{l'_2}$ . Recall that in (30) we denoted by  $u_1, \dots, u_M$  the ordered list of matrix coefficients  $\varpi_k \phi_i^j$ . Let us order in a lexicographic way the elements  $u_i \otimes u_j$ , ie. as a sequence  $u_1^{(2)}, \dots, u_{M^2}^{(2)}$  such that the following condition holds: if  $\varpi_{i'} \phi_{s'}^{t'} < \varpi_{k'} \phi_{i'}^{j'}$ , or  $\varpi_{i'} \phi_{s'}^{t'} = \varpi_{k'} \phi_{i'}^{j'}$  and  $\varpi_{l'} \phi_s^t < \varpi_k \phi_i^j$ , then  $u_a^{(2)} := \varpi_k \phi_i^j \otimes \varpi_{k'} \phi_{i'}^{j'}$  and  $u_b^{(2)} := \varpi_{i'} \phi_{s'}^{t'} \otimes \varpi_{l'} \phi_s^t$  satisfy  $u_b^{(2)} < u_a^{(2)}$ . Then, by the conditions ensuring when  $\alpha'_{p'k_1 k'_2 p}{}^{l'_1 l'_2 l_1 l_2}$  and  $\beta'_{p'r'r'p}{}^{l'_1 l'_2 l_1 l_2}$  are non zero, the last identity takes the form of (30) by replacing  $u_i$ ,  $u_j$  with  $u_i^{(2)} := \mu_1 \phi_{k_1}^{l_1} \otimes \mu_2 \phi_p^{l_2}$ ,  $u_j^{(2)} := \mu'_1 \phi_{k'_1}^{l'_1} \otimes \mu'_2 \phi_{k'_2}^{l'_2}$ . At the beginning of this computation we assumed  $\mu'_2 \phi_{k'_2}^{l'_2} < \mu_2 \phi_p^{l_2}$  and  $\mu'_1 \phi_{k'_1}^{l'_1} < \mu_1 \phi_{k_1}^{l_1}$ , but the same result occurs (in a simpler fashion) if  $\mu'_2 \phi_{k'_2}^{l'_2} = \mu_2 \phi_p^{l_2}$  and  $\mu'_1 \phi_{k'_1}^{l'_1} < \mu_1 \phi_{k_1}^{l_1}$ , so eventually we find that (30) holds true for all cases  $1 \leq u_j^{(2)} < u_i^{(2)} \leq M^2$ .

As in the case of  $\mathcal{L}_{0,1}$ , by using Proposition I.8.17 of [20] one can therefore conclude that there is a filtration  $\mathcal{F}_3$  of  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})$  such that  $Gr_{\mathcal{F}_3}(Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n}))$  is a quotient of a quasi-polynomial algebra, and finally that  $\mathcal{L}_{0,n}$  is Noetherian.  $\square$

**Theorem 3.2.** *The algebra  $\mathcal{M}_{0,n} = \mathcal{L}_{0,n}^{U_q}$  (respectively  $\mathcal{M}_{0,n}^A$ , and  $\mathcal{M}_{0,n}^{A,\epsilon'}$ ,  $\epsilon' \in \mathbb{C}^\times$ ) is Noetherian, and generated over  $\mathbb{C}(q)$  (resp.  $A$ ,  $\mathbb{C}$ ) by a finite number of elements.*

Our method of proof follows closely that of the Hilbert-Nagata theorem (see [32]). Let us recall one version of this theorem, which is enough for our purposes. Let  $A = K[a_1, \dots, a_n]$  be

a finitely generated commutative algebra over an arbitrary field  $K$ , and  $G$  a group of algebra automorphisms of  $A$ .

**Theorem 3.3.** *If the action of  $G$  on  $A$  is completely reducible on finite dimensional representations, then the ring  $A^G$  of invariants of  $A$  with respect to  $G$  is Noetherian and a finitely generated algebra over  $K$ .*

We recall here the main steps of the proof that we will adapt in order to prove Theorem 3.2:

(a) From the complete reducibility of the action of  $G$  on  $A$  one can define a linear map

$$R : A \rightarrow A^G$$

namely the projection onto the space of invariants along the space of non-trivial isotypical components of  $A$ . This linear map is called the Reynolds operator; it satisfies

$$(37) \quad R(hf) = hR(f)$$

for every  $f \in A$ ,  $h \in A^G$ .

(b) Let  $I$  be an ideal of  $A^G$ . Then  $I = R(AI) = AI \cap A^G$ . Because  $AI$  is an ideal of  $A$ , and  $A$  is Noetherian, there exist elements  $b_1, \dots, b_s$ , that can be chosen in  $I \subset A^G$ , such that  $AI = Ab_1 + \dots + Ab_s$ . Since  $I = R(AI) = R(Ab_1 + \dots + Ab_s) = A^G b_1 + \dots + A^G b_s$ ,  $I$  is finitely generated over  $A^G$ . Therefore  $A^G$  is Noetherian.

(c) Let  $B$  be an algebra graded over  $\mathbb{N}$  (for simplicity of notations):  $B = \bigoplus_{i=0}^{+\infty} B_n$ , with  $B_m \cdot B_n \subset B_{m+n}$ . The augmentation ideal of  $B$  is  $B^+ = \bigoplus_{i=1}^{+\infty} B_n$ . If  $B^+$  is a Noetherian ideal of  $B$ , then  $B$  is a finitely generated algebra over  $B_0$ . This is Lemma 2.4.5 of [65] (in that statement  $B$  is commutative, but this hypothesis is not necessary for the proof).

(d) Assume that  $A^G$  is graded over  $\mathbb{N}$  (for simplicity of notations):  $A^G = \bigoplus_{i=0}^{+\infty} A_n^G$  with  $A_0^G = K$ . Then  $A^{G+} = \bigoplus_{i=1}^{+\infty} A_n^G$  is an ideal of  $A^G$ , which is Noetherian by (b) above. Applying (c) we deduce that  $A^G$  is a finitely generated algebra over  $K$ .

*Proof of Theorem 3.2.* As for Theorem 3.1 the result for  $\mathcal{M}_{0,n}^{A,\epsilon'}$  follows from that for  $\mathcal{M}_{0,n}^A$  and  $\mathcal{M}_{0,n}$ , which are proved in the same way. Let us consider  $\mathcal{M}_{0,n}$ . Consider the filtration  $\mathcal{F}$  of  $\mathcal{L}_{0,n}$  by the subspaces

$$\mathcal{F}^{[\mu]} = \bigoplus_{[\mu'] \leq [\mu]} C([\mu']), \mu \in P_+^n$$

where  $P_+^n$  is given the lexicographic partial order induced from  $[\Lambda]$ . It is easily seen that  $\mathcal{F}$  is an algebra filtration: the coregular actions  $\triangleright, \triangleleft$  fix globally each component  $C(\mu)$  of  $\mathcal{L}_{0,1}$ , so the claim follows from (3), (6) and the fact that  $C(\mu) \star C(\nu) \subset C(\mu + \nu)$  for all  $\mu, \nu \in P_+$ . Denote by  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})$  the corresponding graded algebra. Again

$$(38) \quad Gr_{\mathcal{F}}(\mathcal{L}_{0,n}) = \mathcal{L}_{0,n} = \bigoplus_{[\mu] \in P_+^n} C([\mu]).$$

Because each space  $C([\mu])$  is stabilized by the coadjoint action of  $U_q$ , the decomposition (38) has a key advantage on (33). Indeed, since  $\mathcal{L}_{0,n}$  is a  $U_q$ -module algebra, the action of  $U_q$  is well-defined on  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})$  and it gives it a structure of  $U_q$ -module algebra. As vector spaces we have

$$(39) \quad Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q} = \bigoplus_{[\mu] \in P_+^n} C([\mu])^{U_q} = \mathcal{L}_{0,n}^{U_q}.$$

Now we can adapt the different steps (a)–(d) recalled above:

(a') The action of  $U_q$  on  $\mathcal{L}_{0,n}$  is completely reducible. This follows from Theorem 2.2 (1) (noting that the summands, being isomorphic by (14) to spaces  $C(\mu)$ , are finite-dimensional and thus completely reducible  $U_q$ -modules), and the isomorphism of  $U_q$ -modules (see (11)):

$$(40) \quad \mathcal{L}_{0,n} \xrightarrow{\Phi_n} (U_q(\mathfrak{g})^{\otimes n})^{lf} \xrightarrow{\psi_n^{-1}} U_q^{lf}(\mathfrak{g})^{\otimes n}$$

where  $lf$  means respectively locally finite for the action  $ad_n^r$  of  $U_q(\mathfrak{g})$  on  $U_q(\mathfrak{g})^{\otimes n}$ , and locally finite for the action  $ad^r$  of  $U_q(\mathfrak{g})$  on  $U_q(\mathfrak{g})$ . By (38) it follows that  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})$  is also completely reducible. We can therefore define the Reynolds operator  $R: Gr_{\mathcal{F}}(\mathcal{L}_{0,n}) \rightarrow Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$  as in (a).

(b') In the proof of Theorem 3.1 we showed that  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})$  is Noetherian, and then deduced that  $\mathcal{L}_{0,n}$  is Noetherian by a classical argument (see eg. [60], 1.6.9). This same argument implies that  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})$  is Noetherian, because (38) shows it is filtered by  $\mathcal{F}_2$ , and  $Gr_{\mathcal{F}_2}(Gr_{\mathcal{F}}(\mathcal{L}_{0,n})) = Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n})$  is Noetherian. As in (b) we deduce that  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$  is Noetherian. But  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q} = Gr_{\mathcal{F}}(\mathcal{L}_{0,n}^{U_q})$ , which implies that  $\mathcal{L}_{0,n}^{U_q}$  is Noetherian.

(c'-d') Then we can apply the steps (c)-(d). As a result  $Gr_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$  is finitely generated, say by  $k$  homogeneous elements  $\bar{x}_i \in \mathcal{F}^{[\mu_i]} / (\oplus_{[\mu'] < [\mu_i]} C([\mu']))$ .

(e') From (39) we deduce that  $\mathcal{L}_{0,n}^{U_q}$  is generated by the  $x_i \in C([\mu_i])$  with leading terms  $\bar{x}_1, \dots, \bar{x}_k$ . This follows from the following elementary fact: if  $A$  is a filtered  $\mathbb{K}$ -algebra ( $\mathbb{K}$  a field) which graded algebra  $Gr(A)$  is finitely generated, then  $A$  is finitely generated by elements which leading terms generate  $Gr(A)$ . Indeed, let  $A$  have the algebra filtration  $(A_i)_{i \in \mathbb{N}}$  (we take a filtration over  $\mathbb{N}$  to simplify notations). Put  $Gr(A) = \oplus_{i \in \mathbb{N}} A_{(i)}$ ,  $A_{(i+1)} := A_{i+1}/A_i$ . We have  $\bar{a} + \bar{b} = \overline{a+b}$  and  $\bar{a}\bar{b} = \overline{ab} + A_{n+m-1} \in A_{(m+n)}$ , so  $\bar{a}\bar{b} = 0$  if  $ab \in A_{n+m-1}$ , and  $\bar{a}\bar{b} = \overline{ab}$  otherwise. Now assume that  $Gr(A)$  is finitely generated over  $\mathbb{K}$ . Denote by  $\bar{f}_1, \dots, \bar{f}_n$  a finite set of generators. Let  $a \in A$ , with  $a \in A_k \setminus A_{k-1}$ . Then  $\bar{a} \in A_{(k)}$  can be written as  $\bar{a} = \sum_{(i_1, \dots, i_n) \in I} \lambda_{(i_1, \dots, i_n)} \bar{f}_1^{i_1} \dots \bar{f}_n^{i_n}$  for some finite set  $I \subset \mathbb{N}^n$  and scalars  $\lambda_{(i_1, \dots, i_n)} \in \mathbb{K}$ . Either  $\bar{f}_1^{i_1} \dots \bar{f}_n^{i_n}$  is zero, or  $\bar{f}_1^{i_1} \dots \bar{f}_n^{i_n} = \overline{f_1^{i_1} \dots f_n^{i_n}}$ . We therefore have  $a - \sum_{(i_1, \dots, i_n) \in I} \lambda_{(i_1, \dots, i_n)} f_1^{i_1} \dots f_n^{i_n} = 0 \in A_{(k)}$ , that is,  $a - \sum_{(i_1, \dots, i_n) \in I} \lambda_{(i_1, \dots, i_n)} f_1^{i_1} \dots f_n^{i_n} \in A_{k-1}$ . The conclusion follows by an easy induction.  $\square$

**Remark 3.4.** (1) Because  $\mathcal{L}_{0,1}^{U_q}$  is the center of  $\mathcal{L}_{0,1}$ , (e') proves it is finitely generated. Of course this follows also from the isomorphism  $\mathcal{L}_{0,1} \cong U_q^{lf}$  and the fact that the center of  $U_q^{lf}$  is the center of  $U_q$  (by Theorem 2.2), plus the well-known description of the latter. But the argument here is elementary and it applies to  $\mathcal{L}_{0,n}^{U_q}$  for any  $n \geq 1$ .

(2) In spite of the isomorphism  $\Phi_n: \mathcal{M}_{0,n} \rightarrow (U_q^{\otimes n})^{U_q}$ , in order to prove Theorem 3.2 one cannot bypass the hard study of  $(U_q^{\otimes n})^{lf}$ , whence of  $\mathcal{L}_{0,1} \cong U_q^{lf}$ , by working directly with  $U_q$ . Indeed the adjoint action is not completely reducible thereon. In fact,  $U_q^{lf}$  is exactly the socle of this action (see [43], Lemma 7.1.24).

(3) In the  $sl(2)$  case the filtration  $\mathcal{F}$  on  $\mathcal{L}_{0,n}^{U_q}$  should correspond via the Wilson loop isomorphism (defined in [23], Section 8.2) to the filtration of skein algebras of spheres with  $n+1$  punctures used in [57].

#### 4. PROOF OF THEOREM 1.2

As usual we let  $\epsilon$  be a primitive  $l$ -th root of unity with  $l$  odd and  $l > d_i$  for all  $i \in \{1, \dots, m\}$ .

Recall that  $\mathcal{Z}_0(U_\epsilon) \subset U_\epsilon$  is the central polynomial subalgebra generated by  $E_{\beta_k}^l, F_{\beta_k}^l, L_i^{\pm l}$ , for  $k \in \{1, \dots, N\}$  and  $i \in \{1, \dots, m\}$ . Define

$$\mathcal{Z}_0(U_\epsilon^{lf}) = U_\epsilon^{lf} \cap \mathcal{Z}_0(U_\epsilon).$$

Examples show that generating sets of  $\mathcal{Z}_0(U_\epsilon^{lf})$  have complicated expressions in general. Nevertheless, specializing  $q$  at  $\epsilon$  in Theorem 2.2 (2) we get

$$(41) \quad \mathcal{Z}_0(U_\epsilon) = T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{lf}) [T^{(l)}/T_2^{(l)}]$$

where  $T^{(l)}, T_{2-}^{(l)}$  and  $T_2^{(l)}$  are the subsets of  $T, T_{2-}$  and  $T_2$  formed by the elements  $K_\lambda$  with  $\lambda \in P, \lambda \in -2P_+$  and  $\lambda \in 2P$  respectively. Define

$$\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \Phi_1^{-1}(\mathcal{Z}_0(U_\epsilon^{lf})).$$

Recall the isomorphism  $\eta^*: \mathcal{O}(G) \rightarrow \mathcal{Z}_0(\mathcal{O}_\epsilon)$  (see Theorem 2.15 (1)).

**Proposition 4.1.**  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$ , and therefore  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  is isomorphic to  $\mathcal{O}(G)$ .

*Proof.* The claim follows from the fact that  $\Phi_1: \mathcal{L}_{0,1}^\epsilon \rightarrow U_\epsilon^{lf}$  is an isomorphism (see (18)), the identity  $\Phi_1 = m \circ (id \otimes S^{-1}) \circ \Phi$ , and Theorem 2.15 (2).

Here is an alternative proof of the isomorphism  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \cong \mathcal{O}(G)$ , not using  $\eta^*$ . Recall the notations introduced before Theorem 2.14. As varieties  $H = U_+ T_G U_- = G^0$ , so the map  $\sigma$  yields identifications  $\mathcal{O}(H) = \mathcal{O}(U_+) \mathcal{O}(T_G) \mathcal{O}(U_-)$  and  $\mathcal{O}(G^0) = \mathcal{O}(U_+) \mathcal{O}(T_G/(2)) \mathcal{O}(U_-)$ ; we can identify  $\mathcal{O}(G^0)$  with the subalgebra  $\sigma_H^*(\mathcal{O}(G^0))$  of  $\mathcal{O}(H)$ . Consider the space  $V = \wedge^N \mathfrak{g}$ , endowed with the action of  $G$  given on each factor by the adjoint representation. Put on  $\mathfrak{g}$  a basis consisting of one element  $e_\alpha$  per root space  $\mathfrak{g}_\alpha$ , along with a basis of  $\mathfrak{h}$ . Let  $v \in V$  be the exterior power of the  $e_\alpha$ 's for  $\alpha$  negative, and  $v^*$  a dual vector such that  $v^*(v) = 1$  and  $v^*$  vanishes on a  $T_G$ -invariant complement of  $v$ . It is classical that  $G \setminus G^0$  has defining equation  $\delta(g) = 0$ , where  $\delta$  is the matrix coefficient  $\delta(g) = v^*(\pi_V(g)v)$  (see eg. [42], page 174). Hence  $\mathcal{O}(G^0) = \mathcal{O}(G)[\delta^{-1}]$ . On  $G^0$  we have  $\delta(u_+ t u_-) = \chi_{-\rho}(t)$ , where  $\chi_{-\rho}$  is the character of  $T_G$  associated to the root  $-\rho$ . Now we can make the connection with  $U_\epsilon$ . The isomorphism  $\mathcal{Z}_0(U_\epsilon) \cong \mathcal{O}(H)$  of Theorem 2.14 (2) identifies  $\mathcal{Z}_0(U_\epsilon) \cap U_\epsilon(\mathfrak{h}) = \mathbb{C}[T^{(l)}]$  with  $\mathcal{O}(T_G)$  by mapping  $K_\lambda$  to the character of  $T_G$  associated to  $\lambda$ . Therefore it maps  $\mathbb{C}[T_2^{(l)}]$  to  $\mathcal{O}(T_G/(2))$ ,  $\ell^l = K_{l\rho}$  to  $\chi_\rho$ , and  $T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{lf}) = \mathcal{Z}_0(U_\epsilon^{lf})[\ell^l]$  to  $\mathcal{O}(G^0)$  by (41). Since  $\mathcal{O}(G^0) = \mathcal{O}(G)[\delta^{-1}]$  and  $\delta^{-1} = \chi_\rho$  on  $G^0$ , it follows that  $\mathcal{Z}_0(U_\epsilon^{lf}) \cong \mathcal{O}(G)$ . Then  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \cong \mathcal{O}(G)$  by injectivity of  $\Phi_1$ .  $\square$

Consider the linear subspace of  $\mathcal{L}_{0,n}^\epsilon$  defined by

$$\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) = \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}.$$

By Proposition 4.1 we have an isomorphism of algebras  $(\eta^{*-1})^{\otimes n}: \mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) \rightarrow \mathcal{O}(G)^{\otimes n}$ .

**Proposition 4.2.** (1)  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a central subalgebra of  $\mathcal{L}_{0,n}^\epsilon$ , and  $\mathcal{L}_{0,n}^\epsilon = \mathcal{O}_\epsilon^{\otimes n}$  as modules over  $\mathcal{O}(G)^{\otimes n}$ . Moreover  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a Noetherian ring.

(2) The  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module  $\mathcal{L}_{0,n}^\epsilon$  is generated by the elements of the spaces  $C([\mu])$  where  $[\mu] = (\mu_1, \dots, \mu_n) \in P_+^n$  satisfies  $0 \leq (\mu_i, \check{\alpha}_j) < l$  for every  $i \in \{1, \dots, n\}, j \in \{1, \dots, m\}$ , where as usual  $\check{\alpha}_j := 2\alpha_j/(\alpha_j, \alpha_j)$ . In particular it is a finite Noetherian  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module, and therefore a Noetherian ring.

*Proof.* (1) Recall the formula (6). It is enough to consider the case where  $\alpha = \beta' = 1$ , and show that  $(1 \otimes (\beta)^{(b)})((\alpha)^{(a)} \otimes 1) = (\alpha)^{(a)} \otimes (\beta)^{(b)}$  whenever  $\alpha \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , for this is trivially

equal to  $((\alpha)^{(a)} \otimes 1)(1 \otimes (\beta)^{(b)})$  (similar arguments apply if instead  $\beta \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  and  $\alpha$  is arbitrary). We have (denoting  $\sum_{(\alpha),(\alpha),(\alpha),(\alpha)}$  by  $\sum_{(\alpha)^4}$ ,  $\Delta(\alpha_{(1)}) = \sum_{(\alpha)} \alpha_{(1)(1)} \otimes \alpha_{(1)(2)}$  etc.):

$$\begin{aligned} (1 \otimes (\beta)^{(b)})((\alpha)^{(a)} \otimes 1) &= \sum_{(R^i)} \left( S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha \triangleleft R_{(1)}^1 R_{(1)}^2 \right)^{(a)} \\ &\quad \otimes \left( S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4 \right)^{(b)} \\ &= \sum_{(R^i), (\alpha)^4, (\beta)^2} (\alpha_{(2)})^{(a)} \otimes (\beta_{(2)})^{(b)} \\ &\quad \times \beta_{(1)} \left( \alpha_{(1)(2)}(R_{(1)}^2) R_{(2)}^2 \alpha_{(3)(1)}(S(R_{(1)}^4)) R_{(2)}^4 \right) \\ &\quad \times \beta_{(3)} \left( \alpha_{(3)(2)}(R_{(1)}^3) R_{(2)}^3 \alpha_{(1)(1)}(R_{(1)}^1) S(R_{(2)}^1) \right). \end{aligned}$$

By Theorem 2.15 (2) it follows that  $\alpha_{(1)(2)}(R_{(1)}^2) R_{(2)}^2 = \Phi^+(\alpha_{(1)(2)}) \in \mathcal{Z}_0(U_\epsilon)$ , and similarly  $\alpha_{(3)(1)}(S(R_{(1)}^4)) R_{(2)}^4, \alpha_{(3)(2)}(R_{(1)}^3) R_{(2)}^3, \alpha_{(1)(1)}(R_{(1)}^1) S(R_{(2)}^1) \in \mathcal{Z}_0(U_\epsilon)$ . Denote by  $z$  any such element. Note that  $\mathcal{Z}_0(U_\epsilon)$  acts by the trivial character on  $\Gamma$ -modules, and that the expression of  $z$  in terms of the corresponding  $\alpha_{(i)(j)}$  implies  $\epsilon(z) = \epsilon(\alpha_{(i)(j)})$ . Then

$$\begin{aligned} \beta_{(1)} \left( \alpha_{(1)(2)}(R_{(1)}^2) R_{(2)}^2 \alpha_{(3)(1)}(S(R_{(1)}^4)) R_{(2)}^4 \right) &= \epsilon(\alpha_{(1)(2)} \alpha_{(3)(1)}) \beta_{(1)}(1) \\ &= \epsilon(\alpha_{(1)(2)}) \epsilon(\alpha_{(3)(1)}) \epsilon(\beta_{(1)}) \\ \beta_{(3)} \left( \alpha_{(3)(2)}(R_{(1)}^3) R_{(2)}^3 \alpha_{(1)(1)}(R_{(1)}^1) S(R_{(2)}^1) \right) &= \epsilon(\alpha_{(3)(2)}) \epsilon(\alpha_{(1)(1)}) \epsilon(\beta_{(3)}) \end{aligned}$$

and finally  $(1 \otimes (\beta)^{(b)})((\alpha)^{(a)} \otimes 1) = (\alpha)^{(a)} \otimes (\beta)^{(b)}$ . Therefore  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{(a)}$  is central in  $\mathcal{L}_{0,n}^\epsilon$  for all  $a = 1, \dots, n$ . These algebras generate  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  in  $(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$ , and hence in  $\mathcal{L}_{0,n}^\epsilon$  (this follows from the comment before (7)). Therefore  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is central in  $\mathcal{L}_{0,n}^\epsilon$ . A computation similar to the above one, based on the formula (3) instead of (6), shows that  $\alpha\beta = \alpha \star \beta$  whenever  $\alpha \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ . Hence  $\mathcal{L}_{0,1}^\epsilon$  and  $\mathcal{O}_\epsilon$  coincide as modules over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$ . The second claim follows immediately; for instance when  $n = 2$ , given  $\alpha', \beta' \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  we have  $(\alpha' \otimes \beta')(\alpha \otimes \beta) = (\alpha' \otimes 1)(1 \otimes \beta')(\alpha \otimes 1)(1 \otimes \beta)$  immediately by (6), and  $(1 \otimes \beta')(\alpha \otimes 1) = \alpha \otimes \beta' = (\alpha \otimes 1)(1 \otimes \beta')$  as above. Then  $(\alpha' \otimes \beta')(\alpha \otimes \beta) = \alpha' \alpha \otimes \beta' \beta$ . Finally  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a Noetherian ring as it is isomorphic to  $\mathcal{O}(G^n)$  (Proposition 4.1), the coordinate ring of an affine algebraic variety.

(2) Let  $[\lambda] = (\lambda_1, \dots, \lambda_n) \in P_+^n$ . For all  $i \in \{1, \dots, n\}$  there are unique  $\lambda_{i0}, \lambda_{i1} \in P_+$  such that  $\lambda_i = \lambda_{i0} + l\lambda_{i1}$  and  $0 \leq (\lambda_{i0}, \check{\alpha}_j) < l$  for every  $j \in \{1, \dots, m\}$ . Then set  $[\lambda_0] = (\lambda_{10}, \dots, \lambda_{n0})$ ,  $[\lambda_1] = (\lambda_{11}, \dots, \lambda_{n1})$ . We have  $C([\lambda]) = C([\lambda_0]) \circ C([l\lambda_1])$  by (36), and  $C(l\lambda_{i1}) \subset \mathcal{Z}_0(\mathcal{O}_\epsilon) = \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  for all  $i \in \{1, \dots, n\}$  by Theorem 2.15 (2) and Proposition 4.1, so  $C([l\lambda_1]) \subset \mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ . It follows that  $Gr_{\mathcal{F}_2}(\mathcal{L}_{0,n}^\epsilon) = \bigoplus_{[\lambda] \in P_+^n} C([\lambda])$  is generated over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  by the elements of the spaces  $C([\lambda_0])$ ,  $[\lambda] \in P_+^n$ . By step (e') of the proof of Theorem 3.2, formulated for modules instead of algebras, it follows that  $\mathcal{L}_{0,n}^\epsilon$  is generated over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  by the same elements. The last claims follow in a standard way: since  $\mathcal{L}_{0,n}^\epsilon$  is a finitely generated  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module and  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is Noetherian,  $\mathcal{L}_{0,n}^\epsilon$  is a Noetherian  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module (eg. by [5] Proposition 6.5). Moreover  $\mathcal{L}_{0,n}^\epsilon$  is a Noetherian ring (by eg. [60], 1.3).  $\square$

Note that we had already obtained independently the Noetherianity of the ring  $\mathcal{L}_{0,n}^\epsilon$  as a consequence of the Noetherianity of  $\mathcal{L}_{0,n}^A$  (Proposition 3.1).



We need below explicit descriptions of the  $\mathcal{Z}$ - and  $\mathcal{Z}_0$ -centers for  $\mathfrak{g} = \mathfrak{sl}(2)$ . Let us recall a few facts in this case. Denote by  $a, b, c, d$  the standard generators of  $\mathcal{O}_q(SL_2)$ , ie. the matrix coefficients in the basis of weight vectors  $v_0, v_1 = F.v_0$  of the 2-dimensional irreducible representation  $V_2$  of  $U_q(\mathfrak{sl}(2))$ . Denote by  $x^{\star k}$ ,  $k \in \mathbb{N}$ , the  $k$ -th power of an element  $x \in \mathcal{O}_A(SL_2)$ . The algebra  $\mathcal{O}_A(SL_2)$  is generated by  $a, b, c, d$ ; the monomials  $a^{\star i} \star b^{\star j} \star d^{\star r}$  and  $a^{\star i} \star c^{\star k} \star d^{\star r}$ ,  $i, j, k, r \in \mathbb{N}, k > 0$ , form an  $A$ -basis of  $\mathcal{O}_A(SL_2)$ . The algebra  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$  is generated by  $a^{\star l}, b^{\star l}, c^{\star l}, d^{\star l}$ ; the monomials  $a^{\star il} \star b^{\star jl} \star d^{\star rl}$  and  $a^{\star il} \star c^{\star kl} \star d^{\star rl}$  form a basis of  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ , and  $\mathcal{Z}(\mathcal{O}_\epsilon(SL_2))$  is generated by  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$  and the elements  $b^{\star(l-k)} \star c^{\star k}$ ,  $k = 0, \dots, l$  (see [36], Proposition 1.4 and the Appendix). We have the relation

$$(42) \quad a^{\star l} \star d^{\star l} - b^{\star l} \star c^{\star l} = 1$$

and the Frobenius isomorphism of Parshall-Wang (see [56], Chapter 7) coincides with the map

$$Fr_{PW}: \mathcal{O}(SL_2) \rightarrow \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$$

induced by  $\eta^*$ ; it sends the standard generators  $\underline{a}, \underline{b}, \underline{c}, \underline{d}$  of  $\mathcal{O}(SL_2) = \mathcal{O}_1(SL_2)$  respectively to  $a^{\star l}, b^{\star l}, c^{\star l}, d^{\star l}$ . Finally, let us quote from [37] that a basis of the rank  $l^3$  free  $\mathcal{Z}_0(\mathcal{O}(SL_2))$ -module  $\mathcal{O}_\epsilon(SL_2)$  (see Theorem 2.15 (3)) is formed by the monomials  $a^m b^n c^{s'} d^r$ , with the integers  $m, n, r, s', s''$  in the range

$$(43) \quad 1 \leq m \leq l-1, 0 \leq n, r \leq l-1, m \leq s' \leq l-1, 0 \leq s'' \leq l-r-1.$$

Now consider  $\mathcal{L}_{0,1}^A(\mathfrak{sl}(2))$ . Recall that  $\mathcal{L}_{0,1}^A = \mathcal{O}_A$  as  $U_A$ -modules. The algebra  $\mathcal{L}_{0,1}^A(\mathfrak{sl}(2))$  is also generated by  $a, b, c, d$ ; a set of defining relations is (see [23], Section 5):

$$(44) \quad \begin{aligned} ad &= da & , & \quad ab - ba = -(1 - q^{-2})bd \\ db &= q^2 bd & , & \quad cb - bc = (1 - q^{-2})(da - d^2) \\ cd &= q^2 dc & , & \quad ac - ca = (1 - q^{-2})dc \\ & & & \quad ad - q^2 bc = 1. \end{aligned}$$

The element  $\omega := qa + q^{-1}d$  is central. Let  $T_k$ ,  $k \in \mathbb{N}$ , be such that  $T_k(x)/2$  is the  $k$ -th Chebyshev polynomial of the first type in the variable  $x/2$ . We have (see [23], Proposition 7.3, for the generalization to  $\mathcal{L}_{0,n}^\epsilon(\mathfrak{sl}(2))$ ):

**Lemma 4.3.**  $\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}(2))) = \mathbb{C}[\omega, b^l, c^l, d^l]/\mathcal{I}$  and  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}(2))) = \mathbb{C}[(T_l(\omega), b^l, c^l, d^l)/\mathcal{I}$ , where  $\mathcal{I}$  is the ideal of  $\mathbb{C}[\omega, b^l, c^l, d^l]$  generated by  $(T_l(\omega) - d^l)d^l - b^l c^l - 1$ .

Here  $b^l, c^l, d^l$  are the  $l$ -th powers of  $b, c, d$  computed using the product of  $\mathcal{L}_{0,1}^A(\mathfrak{sl}(2))$ , not the product  $\star$  of  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ . The above generator of  $\mathcal{I}$  can be interpreted as a determinant, and  $\omega$  as a quantum trace on  $V_2$ .

**Lemma 4.4.** Viewed as element of  $\mathcal{O}_A(SL_2)$ ,  $T_l(\omega) - d^l = a^{\star l}$  and  $x^l = x^{\star l}$ ,  $x \in \{b, c, d\}$ .

*Proof.* Let  $\alpha$  and  $\varpi$  be the simple root and fundamental weight of  $\mathfrak{sl}(2)$ . In the notations of (22) we have  $b = \psi_{-\varpi}^{-\alpha}$ ,  $c = \psi_{-\varpi}^{\alpha}$ ,  $d = \psi_{-\varpi}$ ; the formulas give  $\Phi_1(b^{\star l}) = (q - q^{-1})^l F^l$ ,  $\Phi_1(c^{\star l}) = (q - q^{-1})^l E^l K^{-l}$ ,  $\Phi_1(d^{\star l}) = K^{-l}$ . These coincide respectively with  $\Phi_1(b^l)$ ,  $\Phi_1(c^l)$ ,  $\Phi_1(d^l)$  (see (32) in [23]). By passing to the localization  $\mathcal{O}_A(SL_2)[d^{-1}]$ , and using Parshall-Wang's relation (42), one deduces easily  $\Phi_1(a^{\star l}) = K^l + (q - q^{-1})^{2l} F^l E^l = T_l(\Omega) - K^{-l}$ , where  $\Omega$  is  $(q - q^{-1})^{2l}$  times the Casimir element of  $U_q(\mathfrak{sl}(2))$ , and  $T_l(x)/2$  is the  $l$ -th Chebyshev polynomial of the first type in the variable  $x/2$ . We have  $\Phi_1(\omega) = \Omega$ , so  $\Phi_1(a^{\star l}) = T_l(\omega) - d^l$ . The conclusion follows from the injectivity of  $\Phi_1$ .  $\square$

This lemma proves that we have a commutative diagram

$$\begin{array}{ccccc}
\mathcal{O}(SL_2) & \xrightarrow{Fr_{PW}} & \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2)) & \hookrightarrow & \mathcal{O}_\epsilon(SL_2) \\
& \searrow Fr & \downarrow & & \downarrow \\
& & \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2))) & \hookrightarrow & \mathcal{L}_{0,1}^\epsilon(sl(2))
\end{array}$$

where  $Fr_{PW}$  is Parshall-Wang's Frobenius isomorphism recalled above,  $Fr$  is the Frobenius isomorphism introduced in [23], Definition 7.2, and the vertical arrows are the identifications as vector spaces (the middle one proved by Proposition 4.2).

**Remark 4.5.** By Lemma 4.3 the monomials  $T_l(\omega)^i b^{jl} d^{rl}$  and  $T_l(\omega)^i c^{kl} d^{rl}$ , for  $i, j, k, r \in \mathbb{N}$  and  $k > 0$ , form an  $A$ -basis of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$ . It is straightforward (though cumbersome) to express these basis elements in terms of the basis elements  $a^{*il} \star b^{*jl} \star d^{*rl}$  and  $a^{*il} \star c^{*kl} \star d^{*rl}$  of  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ , and conversely; this can be done by using Lemma 4.4, the formula (3) or the inverse one (expressing  $\star$  in terms of the product of  $\mathcal{L}_{0,1}$ , see (18) in [23]), and the formula of the coproduct  $\Delta: \mathcal{L}_{0,1}^\epsilon(sl(2)) \rightarrow \mathcal{L}_{0,2}^\epsilon(sl(2))$  restricted to  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$  (given in Proposition 6.15 and Lemma 7.7 of [23]).

Since  $\mathcal{L}_{0,1}^A = \mathcal{O}_A$  as an  $A$ -module, the functionals  $t_i$  in Proposition 2.16 can be seen as maps  $t_i: \mathcal{L}_{0,1}^A \rightarrow A$ . Though the algebra structures of  $\mathcal{O}_\epsilon$  and  $\mathcal{L}_{0,1}^\epsilon$  are very different, we have the analogous result:

**Proposition 4.6.** *The maps  $\triangleleft t_i$  preserve  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , and they satisfy  $(f \triangleleft t_i)(a) = f(n_i a)$  and  $(f \alpha) \triangleleft t_i = (f \triangleleft t_i)(\alpha \triangleleft t_i)$  for every  $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ ,  $a \in G$ ,  $\alpha \in \mathcal{L}_{0,1}^\epsilon$ .*

*Proof.* The first two claims follow from Proposition 2.16 and the equality  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$  in Proposition 4.1.

The last claim follows from the case  $\mathfrak{g} = sl(2)$ , as in the proof of Proposition 7.1 of [36]. In fact it is enough to show that  $t(fg) = t(f)t(g)$  for every  $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$ ,  $g \in \mathcal{L}_{0,1}^\epsilon(sl(2))$ ; for completeness we explain this in the Appendix, see (64). A word of caution is needed: the proof of (64) uses that  $\Delta: \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon \otimes \mathcal{O}_\epsilon$  is a morphism of algebras. The analogous property for  $\mathcal{L}_{0,1}^\epsilon$  is that  $\Delta$  yields a morphism of algebras  $\Delta: \mathcal{L}_{0,1}^\epsilon \rightarrow \mathcal{L}_{0,2}^\epsilon$ . Since the algebra structure of  $\mathcal{L}_{0,2}^\epsilon$  is not the product one on  $\mathcal{L}_{0,1}^\epsilon \otimes \mathcal{L}_{0,1}^\epsilon$ , it is not true in general that  $\sum_{(f),(g)} (f_{(1)} \otimes f_{(2)})(g_{(1)} \otimes g_{(2)}) = \sum_{(f),(g)} f_{(1)} g_{(1)} \otimes f_{(2)} g_{(2)}$  for every  $f, g \in \mathcal{L}_{0,1}^\epsilon$ . However it holds whenever  $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , since  $\Delta(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)) \subset \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \otimes \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  and therefore  $f_{(2)} \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$  commutes in  $\mathcal{L}_{0,2}^\epsilon$  with any  $g_{(1)} \in \mathcal{L}_{0,1}^\epsilon = \mathcal{O}_\epsilon$ .

It is enough to prove the identity  $t(fg) = t(f)t(g)$  when  $f$  ranges in a set of generators of the algebra  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$ . So one can take  $f$  among, say,  $T_l(\omega) - d^l = a^{*l}$  and  $x^l = x^{*l}$ ,  $x \in \{b, c, d\}$  (using Lemma 4.3). By (3) and Proposition 6.1 in the Appendix we have

$$t(fg) = \sum_{(R),(R')} t(R_{(2')} S(R_{(2)}) \triangleright f) t(R_{(1')} \triangleright g \triangleleft R_{(1)}).$$

Expanding coproducts and using that  $R^{-1} = (S \otimes id)(R)$  we deduce

$$\begin{aligned}
t(fg) &= \sum_{(f),(R),(R')} t(f_{(1)}) \langle f_{(2)}, R_{(2')} S(R_{(2)}) \rangle t(R_{(1')} \triangleright g \triangleleft R_{(1)}) \\
&= \sum_{(f),(R),(R')} t(f_{(1)}) t \left( \langle f_{(2)}, R_{(2')} \rangle R_{(1')} \triangleright g \triangleleft \langle f_{(3)}, S(R_{(2)}) \rangle R_{(1)} \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{(f)} t(f_{(1)}) t\left(S^{-1}(\Phi^-(f_{(2)})) \triangleright g \triangleleft S^{-2}(\Phi^-(f_{(3)}))\right) \\
&= \sum_{(f)} t(f_{(1)}) \left\langle g, S^{-2}(\Phi^-(f_{(3)})) \underline{w} S^{-1}(\Phi^-(f_{(2)})) \right\rangle \\
&= \sum_{(f)} t(f_{(1)}) \varepsilon\left(S^{-2}(\Phi^-(f_{(3)}))\right) \varepsilon\left(S^{-1}(\Phi^-(f_{(2)}))\right) t(g)
\end{aligned}$$

where  $\underline{w} \in \mathbb{U}_\Gamma$  is the quantum Weyl group element dual to  $t$  (see Section 6.1), and in the last equality we used that  $\Phi^-$  maps  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  into  $\mathcal{Z}_0(U_\epsilon)$  (see Theorem 2.15 (2)), which acts on  $\Gamma$ -modules by the trivial character (the counit)  $\varepsilon: U_\epsilon \rightarrow \mathbb{C}$ . By (56)-(57) in the Appendix we have  $t(a^{*l}) = t(d^{*l}) = 0$  and  $t(b^{*l}) = 1$ ,  $t(c^{*l}) = -1$ . Now the computation of  $t(fg)$  follows easily. For instance, taking  $f = b^l = b^{*l}$ , by using  $\Delta(b^{*l}) = a^{*l} \otimes b^{*l} + b^{*l} \otimes d^{*l}$  and  $\Delta(d^{*l}) = c^{*l} \otimes b^{*l} + d^{*l} \otimes d^{*l}$  we get

$$t(b^l g) = \varepsilon\left(S^{-2}(\Phi^-(b^{*l}))\right) \varepsilon\left(S^{-1}(\Phi^-(c^{*l}))\right) t(g) + \varepsilon\left(S^{-2}(\Phi^-(d^{*l}))\right) \varepsilon\left(S^{-1}(\Phi^-(d^{*l}))\right) t(g)$$

Since  $b^{*l} \in \mathcal{O}_\epsilon(U_+)$ ,  $\Phi^-(b^{*l}) = 0$ . Also, it is immediate from the definition of  $\Phi^-$  that  $\Phi^-(d^{*l}) = \Phi^-(d^l) = L^l$ ; alternatively, one can bypass this computation by observing that  $\Phi^-$  sets an isomorphism from  $\mathcal{O}_\epsilon(T_G) = \mathcal{O}_\epsilon(B_+) \cap \mathcal{O}_\epsilon(B_-)$  to  $\mathbb{C}[L^{\pm 1}] = U_\epsilon(\mathfrak{b}_+) \cap U_\epsilon(\mathfrak{b}_-)$ , mapping a generator  $d$  to  $L$  or  $L^{-1}$ . We have  $\varepsilon(L^l) = 1$ , and therefore

$$t(b^l g) = t(g) = t(b^l) t(g).$$

The other cases  $f = T_l(\omega) - d^l, c^l, d^l$  are similar.  $\square$

**Theorem 4.7.**  $\mathcal{L}_{0,n}^\epsilon$  is a free  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module of rank  $l^{n \cdot \dim \mathfrak{g}}$ .

*Proof.* By Proposition 4.2 (1),  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{O}_\epsilon^{\otimes n}$  coincide as modules over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon^{\otimes n})$ , so the claim follows immediately from Theorem 7.2 of [36], which shows that  $\mathcal{O}_\epsilon$  is a finitely generated projective module of rank  $l^{\dim \mathfrak{g}}$  over  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ , and the arguments of [21], implying that this module is free.

For the sake of clarity let us provide a self-contained proof of the first claim, not invoking directly [36, 21]. Since  $\mathcal{L}_{0,n}^\epsilon$  and  $\mathcal{L}_{0,1}^{\otimes n}$  coincide as modules over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) = \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$ , the result follows from the case  $n = 1$ . Then we argue in four steps. First, using Theorem 2.2 we show that a certain localization of  $\mathcal{L}_{0,1}^\epsilon$  is a free module of rank  $l^{\dim \mathfrak{g}}$ . Then, assuming that  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective, we explain why it has constant rank  $l^{\dim \mathfrak{g}}$  (this is very classical). Thirdly, we prove that  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective as in Theorem 7.2 of [36]. Finally we obtain that it is a free module as in [21].

We have  $T_2^{(l)-1} U_\epsilon^{lf} = T_2^{-1} U_\epsilon^{lf} = U_\epsilon^{lf}[\ell] = U_\epsilon^{lf}[\ell^l]$ , where  $\ell$  is the pivotal element. Then Theorem 2.14 (1) and (41) imply that  $U_\epsilon$  is a free  $\mathcal{Z}_0(U_\epsilon^{lf})[\ell^l]$ -module of rank  $2^m l^{\dim \mathfrak{g}}$ , and Theorem 2.2 (2) says that it is also the direct sum of  $2^m$  copies of the (free)  $\mathcal{Z}_0(U_\epsilon^{lf})[\ell^l]$ -module  $U_\epsilon^{lf}[\ell^l]$ . The decomposition being unique, it follows that  $U_\epsilon^{lf}[\ell^l]$  is free of rank  $l^{\dim \mathfrak{g}}$  over  $\mathcal{Z}_0(U_\epsilon^{lf})[\ell^l]$ . Pulling this back via  $\Phi_1$ , this proves  $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$  is free of rank  $l^{\dim \mathfrak{g}}$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$ , where  $d = \psi_{-\rho} = \Phi_1^{-1}(\ell)$  (see Corollary 2.4, Theorem 2.7).

Here we note that, by the first formula of Theorem 2.11 (3), taking powers with respect to the product of  $\mathcal{L}_{0,1}$  we have

$$\psi_{-\rho}^l = \psi_{-l\rho}.$$

Assume that  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective. Let us show that its rank is  $l^{dim\mathfrak{g}}$ . The localization  $(\mathcal{L}_{0,1}^\epsilon)_P$  of  $\mathcal{L}_{0,1}^\epsilon$  at any prime ideal  $P$  of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  is a free module over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)_P$  ([62], Proposition 2.12.15); the ranks of such modules are finite in number ([62], Proposition 2.12.20). If these ranks are all equal, then, by definition, it is the rank of  $\mathcal{L}_{0,1}^\epsilon$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ . This happens if  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  has no non-trivial (ie.  $\neq 1$ ) idempotent ([62], Corollary 2.12.23), which is the case since it has non non-trivial zero divisors. To compute the rank, suppose  $P$  does not contain  $d^l$ . Such ideals  $P$  are in 1-1 correspondence with the prime ideals of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$  by the natural ring monomorphism  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \rightarrow \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$ . The set  $S = \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \setminus P$  is multiplicatively closed, and we have also a ring morphism  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}] \rightarrow S^{-1}\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , which is also an injection (there are no zero divisors in  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , whence in  $S$ ). Then

$$(45) \quad (\mathcal{L}_{0,1}^\epsilon)_P = S^{-1}\mathcal{L}_{0,1}^\epsilon = \mathcal{L}_{0,1}^\epsilon[d^{-l}] \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]} S^{-1}\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$$

shows that  $(\mathcal{L}_{0,1}^\epsilon)_P$  has over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)_P = S^{-1}\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  the same rank  $l^{dim\mathfrak{g}}$  as  $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$ . This proves our claim.

In order to show that  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  it is enough to show it is finite locally free, ie. there are elements  $d_i \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  such that the localizations  $\mathcal{L}_{0,1}^\epsilon[d_i^{-1}]$  are finite free  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d_i^{-1}]$ -modules, and  $\text{SpecM}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))$  is covered by the open sets  $U(d_i)$  made of the ideals not containing  $d_i$  (see Lemma 77.2 of [68]).

We have seen above that  $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$  is free of rank  $l^{dim\mathfrak{g}}$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$ . In the proof of Proposition 4.1 we saw that there are isomorphisms  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}] \cong \mathcal{Z}_0(U_\epsilon^{lf})[\ell^l] \cong \mathcal{O}(G^0)$ , and  $\mathcal{O}(G^0) = \mathcal{O}(G)[\delta^{-1}]$ . Now, given  $w \in W$  with a reduced expression  $s_{i_1} \dots s_{i_k}$ , put  $t_w = t_{i_1} \dots t_{i_k}$ . Let  $w$  be represented by  $n_w = n_{i_1} \dots n_{i_k}$  in  $N(T_G)$ . By Proposition 4.6 we have  $(f \triangleleft t_w)(x) = f(n_w x)$  for every  $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ ,  $x \in G$ . Then

$$\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}] \triangleleft t_w \cong \mathcal{O}(n_w^{-1}G^0) \cong \mathcal{O}(G)[(\delta \triangleleft t_w)^{-1}].$$

If  $b_1, \dots, b_r$  ( $r := l^{dim\mathfrak{g}}$ ) is a basis of  $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$  over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$ , then  $\mathcal{L}_{0,1}^\epsilon[d^{-l}] \triangleleft t_w$  is free over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[(\delta \triangleleft t_w)^{-1}] \cong \mathcal{O}(n_w^{-1}G^0)$  with basis  $b_1 \triangleleft t_w, \dots, b_r \triangleleft t_w$ . Consider the Bruhat decomposition of  $G$ : any  $g \in G$  can be written in the form  $g = b_1 n b_2$ , where  $b_1, b_2 \in B_-$ ,  $n \in W$ . Hence we have  $g = n n^{-1} b_1 n b_2 \in n B_+ B_- = n G^0$ , and therefore

$$G = \cup_{w \in W} (B_- n_w B_-) = \cup_{w \in W} (n_w G^0).$$

For every  $w \in W$  put

$$d_w^l := d^l \triangleleft t_w.$$

Under the isomorphism of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  with  $G$ , we thus get that  $\text{SpecM}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))$  is covered by the open sets  $U(d_w^l) \cong n_w G^0$ , and  $\mathcal{L}_{0,1}^\epsilon[d_w^l]$  is finite free over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d_w^l]$ . Therefore  $\mathcal{L}_{0,1}^\epsilon$  is finitely generated and projective over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ .

Finally, let us explain why  $\mathcal{L}_{0,1}^\epsilon$  is free over  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ , following the arguments of [21]. Let  $R$  be a commutative Noetherian ring, put  $X = \text{SpecM}(R)$ , and let  $P$  be an  $R$ -module. Denote by  $R_I, P_I$  the localizations of  $R, P$  at a maximal ideal  $I \in X$ . Define the  $f$ -rank of  $P$  as  $f\text{-rank}(P) = \inf_{I \in X} \{ f\text{-rank}_{R_I}(P_I) \}$ , where  $f\text{-rank}_{R_I}(P_I) = \sup \{ r \in \mathbb{N}, R_I^{\otimes r} \subset P_I \} \in \mathbb{N} \cup \{+\infty\}$  (ie. the maximal dimension of a free summand of  $P_I$ ). Bass' Cancellation theorem asserts that if  $P$  is projective and  $f\text{-rank}(P) > \dim(X)$ , and  $P \oplus Q \cong M \oplus Q$  for some  $R$ -modules  $Q$  and  $M$  such that  $Q$  is finitely generated and projective, then  $P \cong M$  (see [8], IV.3.5 and pages 167 and 170, taking  $A = R$ , or [60], section 11.7.13). Let us apply this to  $R = \mathcal{O}(G)$  and  $P = \mathcal{L}_{0,1}^\epsilon$ . We proved above that  $f\text{-rank}_{R_I}(P_I) = l^{dim\mathfrak{g}}$ , a constant, and we have  $l^{dim\mathfrak{g}} > dim\mathfrak{g} = \dim(G)$ . By a result of Marlin [59], the Grothendieck ring  $K_0(\mathcal{O}(G))$  is isomorphic to  $\mathbb{Z}$ . Therefore

$\mathcal{L}_{0,1}^\epsilon \oplus Q \cong \mathcal{O}(G)^r$  for some free  $\mathcal{O}(G)$ -module  $Q$  and  $r \in \mathbb{N}$ . Then Bass' Cancellation implies  $\mathcal{L}_{0,1}^\epsilon$  is free over  $\mathcal{Z}_0(\mathcal{L}_{0,1}) \cong \mathcal{O}(G)$ .  $\square$

### 5. PROOF OF THEOREM 1.3

We begin with two lemmas, interesting in themselves.

**Lemma 5.1.**  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is a finite  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module and a Noetherian ring. Therefore the ring  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is integral over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ .

*Proof.* We know that  $\mathcal{L}_{0,n}^\epsilon$  is finite over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  (Theorem 4.7), and  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  is a Noetherian ring (Proposition 4.2). Therefore  $\mathcal{L}_{0,n}^\epsilon$  is a Noetherian  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module. This implies that the submodule  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is finitely generated. But being finite over the Noetherian ring  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ , it is a Noetherian ring (by eg. Proposition 7.2 of [5]).

Let  $x \in \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ . The  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -submodule  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)[x]$  of  $\mathcal{L}_{0,n}^\epsilon$  is finitely generated by the same argument. Using the fact that an element  $x$  is integral over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  if and only if  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)[x]$  is a finitely generated  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module (by eg. Proposition 5.1 of [5]), this proves the last claim.  $\square$

As usual, denote by  $Q(Z)$  the quotient field of a commutative integral domain  $Z$ . Then, consider the fields  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  and  $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ . Since  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is finite over  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$  and has no non-trivial zero divisors, the ring  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon) \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)} Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$  is a field. Therefore it is equal to  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ .

**Lemma 5.2.**  $[Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = l^{mn}$ .

*Proof.* First consider the case  $n = 1$ . Note that  $Q(\mathcal{Z}(U_\epsilon)) = Q(\mathcal{Z}(U_\epsilon^{lf}))(T^{(l)}/T_2^{(l)})$  by (41), and similarly by replacing  $\mathcal{Z}$  with  $\mathcal{Z}_0$ . Then, applying  $\Phi_1$  and using that  $Q(\mathcal{Z}(U_\epsilon))$  has degree  $l^m$  over  $Q(\mathcal{Z}_0(U_\epsilon))$  (see the comment after Theorem 2.14) we deduce

$$(46) \quad [Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))] = [Q(\mathcal{Z}(U_\epsilon^{lf})) : Q(\mathcal{Z}_0(U_\epsilon^{lf}))] = l^m.$$

Next let  $n \geq 1$  be arbitrary. Recall from (16) the matrices  $M_j^{(i)}$ , for  $i = 1, \dots, n$  and  $j = 1, \dots, m$ . More generally, for every  $\lambda \in P_+$  and  $i = 1, \dots, n$  consider the matrix of matrix coefficients  $M_\lambda^{(i)} := (v_\lambda \phi_{e_k}^{e_l(i)})_{k,l} \in \text{End}(V_\lambda) \otimes \mathcal{L}_{0,n}^A$ , where  $v_\lambda \phi_{e_k}^{e_l(i)} := 1^{\otimes(i-1)} \otimes v_\lambda \phi_{e_k}^{e_l} \otimes 1^{\otimes(n-i)}$ , and  $\{e_k\}$  is the canonical basis of weight vectors of  $V_\lambda$ . In [23], Proposition 6.22, we showed that the elements  ${}_\lambda \omega^{(i)} := \text{Tr}(\pi_{V_\lambda}(\ell) M_\lambda^{(i)})$  are central in  $\mathcal{L}_{0,n}^A$ , where  $\text{Tr}$  is the standard trace on  $\text{End}(V_\lambda)$ ; moreover, the family  $\{{}_\lambda \omega^{(i)}, \lambda \in P_+\}$  is a basis of the center of the  $i$ -th factor  $\mathcal{L}_{0,1}^A$  of  $\mathcal{L}_{0,n}^A$ . Now, recall the graded algebra  $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})$  in (38). Take  $\lambda = \varpi_j$ . By (36), for every  $i \in \{1, \dots, n\}$ ,  $j \in \{1, \dots, m\}$  and  $r \in \mathbb{N}$  the leading coefficient of  $(\varpi_j \omega^{(i)})^r$  in  $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})$  belongs to the space  $C([0, \dots, 0, r\varpi_j, 0, \dots, 0])$  (with  $\varpi_j$  in the  $i$ -th entry). Take the specialization at  $q = \epsilon$  and denote by  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{(i)}$  the  $i$ -th factor of  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ . Recall that  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{(i)} = \bigoplus_{\mu \in P_+} C([0, \dots, 0, l\mu, 0, \dots, 0])$ . Then we get that the minimal polynomial of  $\varpi_j \omega^{(i)}$  over  $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)^{(i)})$  has degree  $l$ . On another hand, denoting by  $\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)^{(i)}$  the center of the  $i$ -th factor of  $\mathcal{L}_{0,n}^\epsilon$ , we have  $[Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)^{(i)}) : Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{(i)})] = l^m$  as in the  $n = 1$  case above. It follows that  $Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)^{(i)})$  is generated over  $Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{(i)})$  by the elements  ${}_\lambda \omega^{(i)}$ ,  $\lambda \in P_+$ . The fields  $Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)^{(i)})$  are supported by distinct tensor factors, so they are linearly disjoint subfields of  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ . The same is true of the fields  $Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{(i)})$ . Therefore, the compositum of the fields  $Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)^{(i)})$ , say  $Q'$ , has degree  $(l^m)^n$  over the compositum of the fields  $Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{(i)})$ , which is  $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ . By the same argument,  $Q(\mathcal{Z}(U_\epsilon^{\otimes n}))$  has degree

$l^{mn}$  over  $Q(\mathcal{Z}_0(U_\epsilon^{\otimes n}))$ . Now, recall from (18) the isomorphism of algebras  $\Phi_n: \text{loc}'\mathcal{L}_{0,n}^\epsilon \rightarrow U_\epsilon^{\otimes n}$ . Consider the induced isomorphism of fields  $\Phi_n: Q(\mathcal{Z}(\text{loc}'\mathcal{L}_{0,n}^\epsilon)) \rightarrow Q(\mathcal{Z}(U_\epsilon^{\otimes n}))$ , and the field extensions in  $Q(\mathcal{Z}(\text{loc}'\mathcal{L}_{0,n}^\epsilon))$ ,

$$\begin{aligned} \text{loc}'Q' &= Q'[\{\nu_{j_n}^{(n)-l}\}][\{\nu_{j_{n-1}}^{(n-1)-l}\}] \cdots [\{\nu_{j_1}^{(1)-l}\}] \\ \text{loc}'Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)) &= Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))[\{\nu_{j_n}^{(n)-l}\}][\{\nu_{j_{n-1}}^{(n-1)-l}\}] \cdots [\{\nu_{j_1}^{(1)-l}\}]. \end{aligned}$$

Compose  $\Phi_n$  with the linear automorphism of  $Q(\mathcal{Z}(U_\epsilon^{\otimes n}))$  induced by  $\psi_n^{-1} = \Phi_1^{\otimes n} \circ \Phi_n^{-1}$  (see (11) for the latter). The image of  $\text{loc}'Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$  by  $\psi_n^{-1} \circ \Phi_n$  is  $Q(\mathcal{Z}_0(U_\epsilon^{\otimes n}))$ . Since  $l^{mn} = [Q' : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = [\text{loc}'Q' : \text{loc}'Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = [\psi_n^{-1} \circ \Phi_n(\text{loc}'Q') : Q(\mathcal{Z}_0(U_\epsilon^{\otimes n}))]$  and  $[Q(\mathcal{Z}(U_\epsilon^{\otimes n})) : Q(\mathcal{Z}_0(U_\epsilon^{\otimes n}))] = l^{mn}$ , it follows that  $\psi_n^{-1} \circ \Phi_n(\text{loc}'Q') = Q(\mathcal{Z}(U_\epsilon^{\otimes n}))$ , whence  $\text{loc}'Q' = Q(\mathcal{Z}(\text{loc}'\mathcal{L}_{0,n}^\epsilon)) = \text{loc}'Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , and then  $Q' = Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ . This eventually proves  $l^{mn} = [Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))]$ .  $\square$

Recall the ring

$$Q(\mathcal{L}_{0,n}^\epsilon) = Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} \mathcal{L}_{0,n}^\epsilon.$$

The center of  $Q(\mathcal{L}_{0,n}^\epsilon)$  is  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ . By the comment before Lemma 5.2 we have another description of  $Q(\mathcal{L}_{0,n}^\epsilon)$ , namely

$$(47) \quad Q(\mathcal{L}_{0,n}^\epsilon) = Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)} \mathcal{L}_{0,n}^\epsilon.$$

Recall that we denote by  $N$  the number of positive roots of  $\mathfrak{g}$ .

**Proposition 5.3.**  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a central simple algebra of dimension  $l^{2Nn}$  over  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ .

*Proof.* From its definition  $Q(\mathcal{L}_{0,n}^\epsilon)$  is a vector space over  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ . Because  $\mathcal{L}_{0,n}^\epsilon$  has no non-trivial divisors and  $Q(\mathcal{L}_{0,n}^\epsilon)$  is finite-dimensional over  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , it is a division algebra over  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , whence a simple algebra. Its center being  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , this proves the first part of the statement. By classical theory (see eg. Section 13.3.5 of [60], or [62], Corollary 2.3.25), it then follows that there is a finite extension (a *splitting field*)  $\mathbb{F}$  of  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  such that

$$\mathbb{F} \otimes_{Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} Q(\mathcal{L}_{0,n}^\epsilon) = M_d(\mathbb{F})$$

where  $d \in \mathbb{N}$ , the PI degree of  $Q(\mathcal{L}_{0,n}^\epsilon)$ , is given by  $d^2 = [Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]$ . Therefore

$$d^2 = \frac{[Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))]}{[Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))]} = l^{2Nn}$$

where we use Theorem 4.7 and Lemma 5.2, and we recall that  $\dim \mathfrak{g} = m + 2N$ .  $\square$

Let us recall for the sake of clarity different notions of ring theory, bearing in mind that we will apply them to the case where  $A = \mathcal{L}_{0,n}^\epsilon$ . Let  $A$  be a ring with no non-trivial zero divisors. The center  $Z = Z(A)$  is a commutative integral domain. Denote by  $Q(Z)$  its field of fractions, and let

$$Q(A) = Q(Z) \otimes_Z A.$$

It is an algebra over its center  $Q(Z)$ .

An element  $a \in A$  is *integral over  $Z$*  if  $Z[a]$  is a finitely generated  $Z$ -module.  $A$  is *integral over  $Z$*  if every element of  $A$  is integral over  $Z$ . An element  $a \in A$  is *c-integral over  $Z$*  if  $Z[a]$  is contained in a finitely generated  $Z$ -module.  $A$  is *c-integral over  $Z$*  if every element of  $A$  is c-integral over  $Z$ . When  $Z$  is a Noetherian ring these two notions are equivalent ([60], Lemma 5.3.2).

If  $Q(A)$  has finite dimension over  $Q(Z)$ , it is a division algebra and therefore a central simple algebra. Moreover  $Q(A)$  is integral over  $Q(Z)$ .

There are different notions of orders, that are equivalent in our context. Let  $B \subset Q(A)$  be a subring.  $B$  is said to be an *order* of  $Q(A)$  ([60], 3.1.2) if every element  $q$  of  $Q(A)$  can be written as  $q = rs^{-1} = s'^{-1}r'$ , where  $r, s, r', s' \in B$ . In particular,  $A$  is an order of  $Q(A)$  and  $s, s'$  can in this case be chosen in  $Z(A)$ .  $B$  is a *classical order* of  $Q(A)$  ([60], 5.3.5) if  $Z \subset B$ ,  $Q(Z)B = Q(A)$ , and  $B$  is finitely generated as a  $Z$ -module. In particular, if  $A$  is a finitely generated  $Z$ -module then  $A$  is a classical order of  $Q(A)$ . Finally,  $B$  is a  *$Z$ -order* of  $Q(A)$  ([60], 5.3.6) if  $Z \subset B$ ,  $Q(Z)B = Q(A)$  and  $B$  is  $c$ -integral over  $Z$ .

We now assume that  $Z$  is Noetherian and  $Q(A)$  is of finite dimension over  $Q(Z)$ . Let  $B$  be a subring of  $Q(A)$  having center  $Z(B) = Z(A) = Z$ . Then the following assertions are equivalent:

- (1)  $B$  is an order of  $Q(A)$ ;
- (2)  $B$  is a classical order of  $Q(A)$ ;
- (3)  $B$  is a  $Z$ -order of  $Q(A)$ .

Indeed, from the definitions we trivially have (2)  $\Rightarrow$  (3). From the Noetherianity of  $Z$ , Proposition 5.3.14 of [60] gives (3)  $\Rightarrow$  (2). Because  $Q(A)$  is a central simple algebra, the equivalence (1)  $\Leftrightarrow$  (3) is part of Proposition 5.3.10 of [60].

In particular, because  $A$  is always an order of  $Q(A)$  these equivalences imply that  $A$  is a classical order, whence a finitely generated  $Z$ -module. Moreover  $A$  is Noetherian by Proposition 5.3.14 of [60].

There are two standard notions of maximality for  $Z$ -orders  $B$  of  $Q(A)$ . One applies strictly to  $Z$ -orders ([60], 5.3.13): namely, a  $Z$ -order  $B$  of  $Q(A)$  is a *maximal  $Z$ -order* if  $B \subset C$  with  $C$  a  $Z$ -order implies  $B = C$ . The other notion of maximality applies to arbitrary orders of  $Q(A)$  (see [60], 5.1.1); when  $B$  is a  $Z$ -order with center  $Z$ ,  $B$  is a maximal  $Z$ -order if and only if it is maximal in this latter sense.

If  $A$  is a maximal order, then  $Z$  contains all the  $c$ -integral elements over it, ie.  $Z$  is  $c$ -integrally closed ([60], 5.3.13). Since  $Z$  is Noetherian, it is then integrally closed (in the usual sense), and therefore coincides with the *trace ring* of  $A$  (ie. the subring of  $Q(Z)$  generated over  $Z$  by the coefficients of the characteristic polynomials of elements of  $A$ , represented as endomorphisms by left multiplication on  $Q(A)$ , see [60], 13.9.2-13.9.8).

Finally, we say that  $A$  is *DCK-integrally closed* if the following condition holds: for every subring  $R$  of  $Q(A)$  such that  $A \subset R \subset z^{-1}A$  for some non zero  $z \in Z(A)$ , we have  $R = A$ . We borrow this notion from [33]; it is closely related to that of fractional ideal of  $Q(A)$  (see [60], 3.1.11-3.1.12 and 5.1.4), but simpler. Its relevance comes from the following lemma, which shows that in the commutative and Noetherian case it is equivalent to the usual definition of integrally closure.

**Lemma 5.4.** *Let  $B$  be a commutative Noetherian ring with no non-trivial zero divisors. Then  $B$  is integrally closed in  $Q(B)$  if and only if  $B$  is DCK-integrally closed.*

*Proof.* Assume  $B$  is DCK-integrally closed. Let  $x = b/c \in Q(B)$  be integral over  $B$ . Denote by  $n$  the degree of its minimal polynomial over  $B$ . The ring  $R = B[x]$  is contained in  $c^{-n}B$ , so  $R = B$  and  $x \in B$ . (Note that this does not need  $B$  Noetherian.)

Conversely, let  $R$  be a subring of  $Q(B)$  such that  $B \subset R \subset z^{-1}B$ , and let  $x \in R$ . Then  $M = B[x]$  is a  $B$ -submodule of  $Q(B)$  such that  $xM \subset M$ . It is also a  $B$ -submodule of  $z^{-1}B$ , which is free with basis  $z^{-1}$  over  $B$ . Since  $B$  is Noetherian,  $z^{-1}B$  is a Noetherian  $B$ -module, and therefore  $M$  is a finite  $B$ -module. It follows that  $x$  is integral over  $B$  (Proposition 5.1 of [5]), whence  $x \in B$ .  $\square$

**Lemma 5.5.** *Assume that  $A$  is a ring with no non-trivial zero divisors, with center  $Z$  Noetherian and such that  $Q(A)$  has finite dimension over  $Q(Z)$ . Then  $A$  is DCK-integrally closed if and only if  $A$  is a maximal order.*

*Proof.* Assume that  $A$  is DCK-integrally closed. Let  $B$  a  $Z$ -order of  $Q(A)$  such that  $A \subset B$ , and let  $b \in B$ . Since  $b$  is  $c$ -integral over  $Z$  and  $Z$  is Noetherian,  $b$  is integral and thus  $A[b]$  is a finitely generated  $A$ -module. Let  $e_i = a_i/z_i \in Q(A)$ ,  $z_i \in Z$  and  $i = 1, \dots, n$ , be the generators of  $A[b]$ . We have  $A[b] \subset z^{-1}A$  with  $z = \prod_{i=1}^n z_i$ . Therefore  $A[b] = A$ , whence  $b \in A$ . This proves that  $A$  is a maximal order.

Conversely, assume that  $A$  is maximal order, and let  $B$  be a subring of  $Q(A)$  such that  $A \subset B \subset z^{-1}A$ . Since  $A$  is a finitely generated  $Z$ -module,  $B$  is contained in a finitely generated  $Z$ -module, which is necessarily a Noetherian  $Z$ -module because  $Z$  is Noetherian. Therefore  $B$  is a finitely generated  $Z$ -module. As clearly  $Z \subset B$  and  $Q(Z)B = Q(A)$ , it is in fact a classical order of  $Q(A)$ . Because  $A$  is maximal, we have  $A = B$ , which proves  $A$  is DCK-integrally closed.  $\square$

**Theorem 5.6.**  $\mathcal{L}_{0,n}^\epsilon$  is a maximal order of  $Q(\mathcal{L}_{0,n}^\epsilon)$ .

Note that, by the discussion before Lemma 5.4 it follows that  $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$  is integrally closed and coincides with the trace ring of  $Q(\mathcal{L}_{0,n}^\epsilon)$ .

*Proof of Theorem 5.6.* We derive the result by “twisting” the analogous statement for  $\mathcal{O}_\epsilon$ , obtained in Theorem 7.4 of [36]. We have already proved that  $\mathcal{L}_{0,n}^\epsilon$  satisfies the hypothesis on  $A$  in Lemma 5.5. So let  $R \subset Q(\mathcal{L}_{0,n}^\epsilon)$  be a subring such that  $\mathcal{L}_{0,n}^\epsilon \subset R \subset x^{-1}\mathcal{L}_{0,n}^\epsilon$  for some non zero  $x \in \mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ . We have to show that  $\mathcal{L}_{0,n}^\epsilon = R$ . We know  $\mathcal{L}_{0,n}^\epsilon = \mathcal{O}_\epsilon^{\otimes n}$  as  $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -modules (Proposition 4.2 (1)), so  $x^{-1}\mathcal{L}_{0,n}^\epsilon = x^{-1} \star \mathcal{O}_\epsilon^{\otimes n}$ . Also, the product of  $R$  is inherited from that of  $\mathcal{L}_{0,n}^\epsilon$  (for, given  $r_1, r_2 \in R$  we have  $r_1 r_2 = x^{-2}(x r_1)(x r_2)$ ), and the latter is defined from the one of  $\mathcal{O}_\epsilon^{\otimes n}$  by two consecutive twists (see the formulas (3) and (6)). Therefore, by applying the inverse twists on the inclusions  $\mathcal{L}_{0,n}^\epsilon \subset R \subset x^{-1}\mathcal{L}_{0,n}^\epsilon$  we get  $\mathcal{O}_\epsilon^{\otimes n} \subset R' \subset x^{-1} \star \mathcal{O}_\epsilon^{\otimes n}$  where  $R'$  is the vector space  $R$  endowed with the ring structure inherited from  $\mathcal{O}_\epsilon^{\otimes n}$ . Now  $\mathcal{O}_\epsilon^{\otimes n} = \mathcal{O}_\epsilon(G^m)$  is a maximal order of  $Q(\mathcal{O}_\epsilon(G^m))$  by Theorem 7.4 in [36]. Therefore  $\mathcal{O}_\epsilon^{\otimes n} = R'$ , and finally  $\mathcal{L}_{0,n}^\epsilon = R$ .  $\square$

A proof of Theorem 5.6 independent of Theorem 7.4 of [36] seems to be difficult for arbitrary  $\mathfrak{g}$ , even for  $n = 1$ . In the case of  $\mathcal{L}_{0,1}(sl(2))$  we can however apply a similar reasoning. Let us explain the details.

Identify  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$  with  $\mathcal{O}(SL_2)$  as described in Lemma 4.4 and before Lemma 4.3. We proceed in two steps : (a) we show that  $\mathcal{L}_{0,1}^\epsilon(sl(2))[d^{-l}]$  and  $\mathcal{L}_{0,1}^\epsilon(sl(2))[b^{-l}]$  are maximal orders, (b) from this we deduce the result.

As for (a), recall that  $\mathcal{L}_{0,1}^\epsilon(sl(2))[d^{-l}] \cong U_\epsilon^{lf}(sl(2))[d^l]$ . We have  $U_\epsilon^{lf}(sl(2))[d^l] = U_\epsilon^{ad}(sl(2))$ , which is a maximal order by Theorem 1.8 of [33]. Using (44) one computes easily that  $\mathcal{L}_{0,1}^\epsilon(sl(2))[b^{-l}]$  is generated by  $\omega$ ,  $b^{\pm 1}$  and  $d$ , with defining relations:  $\omega$  is central,  $db = q^2bd$ . Therefore  $\mathcal{L}_{0,1}^\epsilon(sl(2))[b^{-l}]$  is the tensor product of  $\mathbb{C}[\omega]$  with a quasi-polynomial algebra. The  $l$ -th powers of  $d$ ,  $b$  are central, and  $\mathbb{C}[\omega]$  is integrally closed, so by a direct application of Proposition 1.8 of [33] it follows that  $\mathcal{L}_{0,1}^\epsilon(sl(2))[b^{-l}]$  is a maximal order.

Let us now deduce that  $\mathcal{L}_{0,1}^\epsilon(sl(2))$  is a maximal order. Let  $x \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$  and  $R$  a subring of  $Q(\mathcal{L}_{0,1}^\epsilon(sl(2)))$  such that  $\mathcal{L}_{0,1}^\epsilon(sl(2)) \subset R \subset x^{-1}\mathcal{L}_{0,1}^\epsilon(sl(2))$ . Let  $y \in R$ . Because  $\mathcal{L}_{0,1}^\epsilon(sl(2))[d^{-l}]$  is maximal, we have  $\mathcal{L}_{0,1}^\epsilon(sl(2))[d^{-l}] = R[d^{-l}]$ . So  $yd^{l.n_d} \in \mathcal{L}_{0,1}^\epsilon(sl(2))$  for some non negative integer  $n_d$ . Similarly,  $yb^{l.n_b} \in \mathcal{L}_{0,1}^\epsilon(sl(2))$  for some non negative integer  $n_b$ . Now, note that  $d^l \triangleleft_{t_w} b^l$ . Consider the open subsets  $U(d^l)$  and  $U(b^l)$  of  $SL_2$  consisting of matrices with non vanishing lower right entry and non vanishing upper right entry, respectively. As in the proof of Theorem 4.7 we have  $\text{SpecM}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))) = SL_2 = U(d^l) \cup U(b^l)$ . Since  $U(d^l) = U(d^{l.n_d})$ ,  $U(b^l) = U(b^{l.n_b})$ , likewise  $\text{SpecM}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))) = U(d^{l.n_d}) \cup U(b^{l.n_b})$ . This



implies that the ideal of  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$  generated by  $d^{l.n_d}$  and  $b^{l.n_b}$  is  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$  itself; hence there are elements  $u_d, u_b \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(sl(2)))$  such that  $u_d d^{l.n_d} + u_b b^{l.n_b} = 1$ , which proves  $y = u_d d^{l.n_d} y + u_b b^{l.n_b} y \in \mathcal{L}_{0,1}^\epsilon(sl(2))$ . This concludes the proof that  $\mathcal{L}_{0,1}^\epsilon(sl(2))$  is a maximal order.

**Proof of Theorem 1.3 (2).** Denote by  $\Delta^{(n)}: \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon^{\otimes n}$ ,  $n \geq 2$ , the  $n$ -fold co-product, that is  $\Delta^{(n)} := (id \otimes \Delta) \circ \Delta^{(n-1)}$  for  $n \geq 3$ , and  $\Delta^{(2)} := \Delta$ , the standard co-product of  $\mathcal{O}_\epsilon$ . Identifying  $\mathcal{L}_{0,n}^\epsilon$  with  $\mathcal{O}_\epsilon^{\otimes n}$  as a vector space, we consider  $\Delta^{(n)}$  as a map  $\Delta^{(n)}: \mathcal{L}_{0,1}^\epsilon \rightarrow \mathcal{L}_{0,n}^\epsilon$ . It is an algebra morphism ([23], Proposition 6.16), injective because  $(\varepsilon^{\otimes(n-1)} \otimes id)\Delta^{(n)} = id$ . Then it extends uniquely to the fraction algebra  $Q(\mathcal{L}_{0,1}^\epsilon)$ . As noted above  $Q(\mathcal{L}_{0,1}^\epsilon) = Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)) \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)} \mathcal{L}_{0,1}^\epsilon$ . Since  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$  is a Hopf subalgebra of  $\mathcal{O}_\epsilon$  ([36], Proposition 6.4),  $\Delta^{(n)}$  maps  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$  to  $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$ . Extending the scalars of  $Q(\mathcal{L}_{0,1}^\epsilon)$  by the field  $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ , consider

$$\begin{aligned} \tilde{Q}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) &:= Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)))} \Delta^{(n)}(Q(\mathcal{L}_{0,1}^\epsilon)) \\ &= Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))} \Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon). \end{aligned}$$

It is a subalgebra of  $Q(\mathcal{L}_{0,n}^\epsilon)$ , and a simple algebra. The center of  $\tilde{Q}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$  is the field  $\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) := Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))} \Delta^{(n)}(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))$ . We proved in Proposition 6.17 of [23] that the ring  $(\mathcal{L}_{0,n}^A)^{U_A}$  is the centralizer of  $\Delta^{(n)}(\mathcal{L}_{0,1}^A)$  in  $\mathcal{L}_{0,n}^A$ ; the arguments of the proof apply as well to  $\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)$  in  $\mathcal{L}_{0,n}^\epsilon$ , so  $\tilde{Q}((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) := Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$  is the centralizer of  $\tilde{Q}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$  in  $Q(\mathcal{L}_{0,n}^\epsilon)$ . Then the centralizer theorem (see eg. [69], Theorem 7.1) implies that  $\tilde{Q}((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is a simple algebra, and

$$[\tilde{Q}((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))] = \frac{[Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]}{[\tilde{Q}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]} = l^{2nN - (2N+m)}.$$

Moreover the centralizer of  $\tilde{Q}((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$  is  $\tilde{Q}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$ , so  $\tilde{Q}((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) \cap \tilde{Q}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) = \tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$  is its center. It then follows

$$\begin{aligned} [\tilde{Q}((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) : \tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))] &= \frac{[\tilde{Q}((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]}{[\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]} \\ &= l^{2nN - (2N+m)} l^{-m} = l^{2(N(n-1) - m)}. \end{aligned}$$

This concludes the proof.  $\square$

## 6. APPENDIX

**6.1. Quantum Weyl group.** We recall some of the formulas of [27]. Let  $e_q(z)$  be the formal power series in  $z$  with coefficients in  $\mathbb{C}(q)$  defined by:

$$(48) \quad e_q(z) = \sum_{n=0}^{+\infty} \frac{z^n}{(n)_q}.$$

We first consider the case of  $\mathfrak{g} = sl(2)$ . As explained in [23], Section 3, the Cartan element  $H \in \mathfrak{g}$  defines an element of  $\mathbb{U}_q(sl(2))$ . Viewed as elements of  $\mathbb{U}_q$  we have  $L = q^{H/2}$ . The series  $\Theta = q^{H \otimes H/2}$  defines an element of  $\mathbb{U}_q(sl(2))^{\otimes 2}$ , its image under multiplication being  $q^{H^2/2}$ . The  $R$ -matrix can be expressed as  $R = \Theta \hat{R}$  where  $\hat{R} = e_{q^{-1}}((q - q^{-1})E \otimes F)$  is a well

defined element of  $\mathbb{U}_q^{\hat{\otimes} 2}$ . Consider the Lusztig [52] braid group automorphism of  $U_q(sl(2))$ , defined by

$$(49) \quad T(L) = L^{-1}, T(E) = -FK^{-1}, T(F) = -KE.$$

For every  $x \in U_q(sl(2))$  it satisfies:

$$(50) \quad \Delta(T(x)) = \hat{R}^{-1}(T \otimes T)(\Delta(x))\hat{R}$$

Define the quantum Weyl group element  $\hat{w} \in \mathbb{U}_q(sl(2))$  by Saito's formula [64]:

$$(51) \quad \hat{w} = e_{q^{-1}}(F)q^{-H^2/4}e_{q^{-1}}(-E)q^{-H^2/4}e_{q^{-1}}(F)q^{-H/2}.$$

For every  $x \in U_q(sl(2))$  it satisfies:

$$(52) \quad T(x) = \hat{w}x\hat{w}^{-1},$$

$$(53) \quad \Delta(\hat{w}) = \hat{R}^{-1}(\hat{w} \otimes \hat{w}),$$

$$(54) \quad \hat{w}^2 = q^{H^2/2}\xi\theta,$$

where  $\theta \in \mathbb{U}_q(sl(2))$  is the ribbon element, and  $\xi \in \mathbb{U}_q(sl(2))$  is the central group element whose value on the type 1 simple module  $X$  of  $U_q(sl(2))$  of dimension  $k+1$  is the scalar endomorphism  $(-1)^k id_X$ .

In order to compare our setting to the one of [36] we need an explicit formula of  $\hat{w}$ . Consider the type 1 simple module  $V_{r+1}$  of dimension  $r+1$ , and its basis vectors  $v_0, \dots, v_r$  such that:

$$\begin{aligned} K.v_j &= \epsilon^{r-2j}v_j, \\ F.v_j &= v_{j+1} \text{ if } j < r, \quad F.v_r = 0, \\ E.v_j &= [j]_\epsilon[r-j+1]_\epsilon v_{j-1} \text{ if } j > 0, \quad E.v_0 = 0. \end{aligned}$$

Setting  $v'_j = v_j/[j]!$  and using (49), (52) and (54), we obtain:

$$(55) \quad \hat{w}v'_j = (-1)^j q^{-j(k-j-1)-k} v'_{k-j}.$$

In [36] another quantum Weyl group element  $\underline{w}$  is defined. It is dual to the Vaksman-Soibelman functional  $t: \mathcal{O}_q(SL_2) \rightarrow \mathbb{C}(q)$  of [67, 66], so  $t(\alpha) = \langle \alpha, \underline{w} \rangle$ ,  $\alpha \in \mathcal{O}_q(SL_2)$ . By comparing (55) with the formulas defining the action of  $t$  in Section 1.7 of [36], we find

$$\underline{w} = \xi \hat{w} K$$

and the basis vectors  $w_r^p$  of [36] are related to the vectors  $v'_j$  above as follows:

$$v'_j = \lambda_j w_r^p$$

where  $k = 2p$ ,  $j = p - r$ ,  $\lambda_0 = 1$ ,  $\lambda_1 = [k]q^{-k}$ , and

$$\lambda_j = \frac{[k]!}{[j]![k-(j-2)]!} q^{j(j+1)-j(k+2)}, \quad j \geq 2.$$

Explicit formulas of the evaluation of  $t$  on basis vectors of  $\mathcal{O}_q(SL_2)$  can be computed. We get:

$$(56) \quad t(\tilde{a}^{\star m} \star \tilde{b}^{\star n} \star \tilde{d}^{\star p}) = \delta_{m,p} q^{-np} \prod_{i=1}^p (1 - q^{-2i}),$$

$$(57) \quad t(\tilde{a}^{\star m} \star \tilde{c}^{\star n} \star \tilde{d}^{\star p}) = (-1)^n \delta_{m,p} q^{-n(p+1)} \prod_{i=1}^p (1 - q^{-2i})$$

where

$$(58) \quad \tilde{a} = a, \quad \tilde{b} = qb, \quad \tilde{c} = q^{-1}c, \quad \tilde{d} = d$$

and as usual  $a, b, c, d$  are the standard generators of  $\mathcal{O}_q(SL_2)$ , ie. the matrix coefficients in the basis of weight vectors  $v_0, v_1 = F.v_0$  of the 2-dimensional irreducible representation  $V_2$  of  $U_q(sl(2))$ . Here we have introduced the generators  $\tilde{a}, \dots, \tilde{d}$  to facilitate the comparison with the formulas in [36]; these generators come naturally in their setup because they use different generators  $E_i$  and  $F_i$  of  $U_q(\mathfrak{g})$ , which in our notations can be written respectively as  $K_i^{-1}E_i$  and  $F_iK_i$ .

The formulas (56)-(57) can be shown by two independent methods. The first uses a definition of  $t$  as a *GNS* state associated to an infinite dimensional representation of  $\mathcal{O}_q(SL_2)$ , as recalled in Section 1.6 of [36]. The second is to write eg.

$$(59) \quad t(\tilde{a}^{\star m} \star \tilde{b}^{\star n} \star \tilde{d}^{\star p}) = \left\langle \tilde{a}^{\otimes m} \otimes \tilde{b}^{\otimes n} \otimes \tilde{d}^{\otimes p}, \Delta^{(m+n+p-1)}(\underline{w}) \right\rangle$$

and to use explicit expressions of  $\Delta^{(m+n+p-1)}(\underline{w})$  when represented on  $V_2^{\otimes(m+n+p)}$ . In general one can check that

$$(60) \quad \Delta^{(n)}(\hat{\omega}) = \left( \Delta^{(n-1)} \otimes id \right) \left( \hat{R}^{-1} \right) \left( \left( \Delta^{(n-2)} \otimes id \right) \left( \hat{R}^{-1} \right) \otimes id \right) \\ \dots \left( \left( \Delta \otimes id \right) \left( \hat{R}^{-1} \right) \otimes id^{\otimes(n-3)} \right) \left( \hat{R}^{-1} \otimes id^{\otimes(n-2)} \right) \hat{\omega}^{\otimes n}.$$

By (55) or (56)-(57) we see that  $\hat{w}$  (or  $\underline{w}$ ) and  $t$  are well-defined on the integral forms,

$$\hat{w} \in \mathbb{U}_\Gamma, \quad t: \mathcal{O}_A(SL_2) \rightarrow A.$$

We now consider the case where  $\mathfrak{g}$  is of rank  $m \geq 2$ . To each simple root  $\alpha_i, 1 \leq i \leq m$ , is associated the subalgebra of  $U_q$  generated by  $E_i, F_i, L_i, L_i^{-1}$ . It is a copy of  $U_{q_i}(sl(2))$ , where  $q_i = q^{d_i}$ . Let  $\hat{w}_i$  be the corresponding quantum Weyl group element in  $\mathbb{U}_q = \mathbb{U}_q(\mathfrak{g})$ , defined by Saito's formula (51), replacing  $H, E, F$  by  $H_i, E_i$  and  $F_i$ . Also, denote by  $\nu_i: \mathcal{O}_q \rightarrow \mathcal{O}_{q_i}(SL_2)$  the projection map dual to the inclusion  $U_{q_i}(sl(2)) \otimes_{\mathbb{C}(q_i)} \mathbb{C}(q) \hookrightarrow U_q$  associated to  $\alpha_i$ , and put  $t_i = t \circ \nu_i$ . Let  $\underline{w}_i$  be the corresponding quantum Weyl group element in  $\mathbb{U}_q$ , ie.  $t_i(\alpha) = \langle \alpha, \underline{w}_i \rangle$  for all  $\alpha \in \mathcal{O}_q$ . On integral forms they yield well-defined elements  $\hat{w}_i, \underline{w}_i \in \mathbb{U}_\Gamma$  and  $t_i: \mathcal{O}_A \rightarrow A$  (see [36], Proposition 5.1). They satisfy the defining relations of the braid group  $\mathcal{B}(\mathfrak{g})$  of  $\mathfrak{g}$  [47]:

$$\hat{w}_i \hat{w}_j \hat{w}_i = \hat{w}_j \hat{w}_i \hat{w}_j \quad \text{if } a_{ij} a_{ji} = 1 \\ (\hat{w}_i \hat{w}_j)^k = (\hat{w}_j \hat{w}_i)^k \quad \text{for } k = 1, 2, 3 \text{ if } a_{ij} a_{ji} = 0, 2, 3$$

and similarly by replacing  $\hat{w}_i$  with  $\underline{w}_i$ , or with  $t_i$  (see [66] for the latter). The Weyl group  $W = W(\mathfrak{g}) = N(T_G)/T_G$  is generated by the reflexions  $s_i$  associated to the simple roots  $\alpha_i$ . Denote by  $n_i \in N(T_G)$  a representative of  $s_i$ . Let  $w \in W$  and denote by  $w = s_{i_1} \dots s_{i_k}$  a reduced expression. Because of the braid group relations the elements  $\hat{w} = \hat{w}_{i_1} \dots \hat{w}_{i_k}$ ,  $\underline{w} = \underline{w}_{i_1} \dots \underline{w}_{i_k}$  and the functional  $t_w = t_{i_1} \dots t_{i_k}$  do not depend on the choice of reduced expression. The Lusztig [52] braid group automorphism  $T_w: \Gamma \rightarrow \Gamma$  associated to  $w$  satisfies (see [36]):

$$T_w(x) = \hat{w} x \hat{w}^{-1}, \quad x \in \Gamma.$$

Let  $w_0$  be the longest element in  $W$ . We have

$$(61) \quad \Delta(\hat{w}_0) = \hat{R}^{-1}(\hat{w}_0 \otimes \hat{w}_0)$$

where as usual  $R = \Theta \hat{R}$ .

**6.2. Regular action on  $\mathcal{O}_\epsilon$ .** The following result is proved in Section 1.10 of [36]. For completeness let us give a (different) proof. Recall from (25) that we may identify  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  with  $\mathcal{O}(G)$ .

**Proposition 6.1.** *For every  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ ,  $g \in \mathcal{O}_\epsilon$  we have*

$$(62) \quad t_i(f) = f(n_i)$$

$$(63) \quad t_i(f \star g) = t_i(f)t_i(g).$$

*Proof.* It is sufficient to prove the results for  $SL_2$  because  $\nu_i : \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon(SL_2)$  is a morphism of Hopf algebras and  $\nu_i(\mathcal{Z}_0(\mathcal{O}_\epsilon)) \subset \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ . In this case (62) can be proved by using (56)-(57), evaluating  $t$  on basis elements of  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$  as is done in Lemma 1.5 (a) of [36]. Such a basis is formed by monomials like in (56)-(57), with all exponents divisible by  $l$ ; then for instance

$$t(\tilde{a}^{\star ml} \star \tilde{b}^{\star nl} \star \tilde{d}^{\star pl}) = \delta_{p,0} \delta_{m,0} = \underline{a}^m \underline{b}^n \underline{d}^p(n)$$

where  $\underline{a}, \dots, \underline{d}$  are the generators of  $\mathcal{O}(G) = \mathcal{O}_1(G)$  corresponding to  $a, \dots, d$ , and we take

$$n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

as representative of the reflexion  $s$  generating the Weyl group  $W(sl(2))$ . Here is an alternative proof of (62): (63) shows that  $t$  is a homomorphism on  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ , so by proving (63) at first one is reduced to check (62) on the generators  $a^{\star l}, \dots, d^{\star l}$ , which is easy by means of (59) and (61).

We provide a proof of (63) that we find more conceptual than the one in Lemma 1.5 (b) of [36] (which uses again (56)-(57)). As above let us denote  $\underline{w} = \xi \hat{w} K$ . For any  $f, g \in \mathcal{O}_\epsilon$  we have

$$\begin{aligned} t(f \star g) &= (f \otimes g)(\Delta(\underline{w})) \\ &= (f \otimes g) \left( \widehat{R}^{-1}(\underline{w} \otimes \underline{w}) \right) \\ &= \sum_{(\widehat{R}^{-1})} f \left( (\widehat{R}^{-1})_{(1)} \underline{w} \right) g \left( (\widehat{R}^{-1})_{(2)} \underline{w} \right) \\ &= \sum_{(\widehat{R}^{-1}), (f)} f_{(1)} \left( (\widehat{R}^{-1})_{(1)} \right) f_{(2)}(\underline{w}) g \left( (\widehat{R}^{-1})_{(2)} \underline{w} \right) \\ &= \sum_{(f)} f_{(2)}(\underline{w}) g \left( (f_{(1)} \otimes id)(\widehat{R}^{-1}) \underline{w} \right). \end{aligned}$$

Assume now  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ . Since  $\mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$  is a Hopf subalgebra of  $\mathcal{O}_\epsilon(SL_2)$  we have  $f_{(1)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon(SL_2))$ . From Theorem 2.15 (2) we deduce

$$(f_{(1)} \otimes id)(\widehat{R}^{-1}) \in U_\epsilon(\mathfrak{n}_-) \cap \mathcal{Z}_0(U_\epsilon).$$

Denote by  $z$  this element. Note that from its expression we have  $\epsilon(z) = \epsilon(f_{(1)})$ . Now  $g(z \underline{w}) = \sum_{(g)} g_{(1)}(z) g_{(2)}(\underline{w})$ , but  $g_{(1)}$  is a linear combination of matrix elements of  $\Gamma$ -modules, on which  $\mathcal{Z}_0(U_\epsilon)$  acts by the trivial character. Therefore

$$g(z \underline{w}) = \sum_{(g)} \epsilon(z) g_{(1)}(1) g_{(2)}(\underline{w}) = \epsilon(z) g(\underline{w}) = \epsilon(f_{(1)}) g(\underline{w})$$

and eventually

$$t(f \star g) = \sum_{(f)} f_{(2)}(\underline{w}) \epsilon(f_{(1)}) g(\underline{w}) = t(f)t(g).$$

This concludes the proof.  $\square$

For the sake of completeness, let us show how this result implies:

*Proof of Proposition 2.16 (ie. Proposition 7.1 of [36]).* We have  $f \triangleleft t_i = \sum_{(f)} t_i(f_{(1)})f_{(2)}$ ,  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ . Since  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$  is a Hopf subalgebra of  $\mathcal{O}_\epsilon$ ,  $f_{(2)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$  and therefore the maps  $\triangleleft t_i: \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon$  preserve  $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ . Moreover,  $(f \triangleleft t_i)(a) = \sum_{(f)} f_{(1)}(n_i)f_{(2)}(a) = f(n_i a)$ ,  $a \in G$ , by (62).

It remains to show that  $(f \star \alpha) \triangleleft t_i = (f \triangleleft t_i)(\alpha \triangleleft t_i)$  for every  $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ ,  $\alpha \in \mathcal{O}_\epsilon$ . We have

$$\begin{aligned}
 (f \star g) \triangleleft t_i &= \sum_{(f \star g)} t_i((f \star g)_{(1)}) (f \star g)_{(2)} = \sum_{(f),(g)} t_i(f_{(1)} \star g_{(1)}) f_{(2)} \star g_{(2)} \\
 &= \sum_{(f),(g)} t(\nu_i(f_{(1)})\nu_i(g_{(1)})) f_{(2)} \star g_{(2)} \\
 (64) \qquad &= \sum_{(f),(g)} t(\nu_i(f_{(1)})) t(\nu_i(g_{(1)})) f_{(2)} \star g_{(2)}
 \end{aligned}$$

using that  $\nu_i$  is a homomorphism in the third equality, and (63) in the last one. The result is just  $(f \triangleleft t_i)(g \triangleleft t_i)$ .  $\square$

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