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Unrestricted Quantum Moduli Algebras, II: Noetherianity and Simple Fraction Rings at Roots of 1

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Abstract. We prove that the quantum graph algebra and the quantum moduli algebra associated to a punctured sphere and complex semisimple Lie algebra \mathfrak{g} are Noetherian rings and finitely generated rings over $\mathbb{C}(q)$. Moreover, we show that these two properties still hold on $\mathbb{C}[q, q^{-1}]$ for the integral version of the quantum graph algebra. We also study the specializations $\mathcal{L}_{0,n}^\epsilon$ of the quantum graph algebra at a root of unity ϵ of odd order, and show that $\mathcal{L}_{0,n}^\epsilon$ and its invariant algebra under the quantum group $U_\epsilon(\mathfrak{g})$ have classical fraction algebras which are central simple algebras of PI degrees that we compute.

Key words: quantum groups; invariant theory; character varieties; skein algebras; TQFT

2020 Mathematics Subject Classification: 16R30; 17B37; 20G42; 57M27; 57R56; 81R50

1 Introduction

This paper is the second part of our work, initiated in [18], on the quantum graph algebra $\mathcal{L}_{g,n}(\mathfrak{g})$ and the quantum moduli algebra $\mathcal{M}_{g,n}(\mathfrak{g})$, which are associated to a surface $\Sigma_{g,n+1}$ of genus g with $n+1$ punctures and a complex semisimple Lie algebra \mathfrak{g} . As in [18], we focus in this paper on punctured spheres ($g=0$, $n \geq 1$). From now on we fix \mathfrak{g} , and when no confusion may arise we omit it from the notations of the various algebras.

The algebras $\mathcal{L}_{g,n}$ and $\mathcal{M}_{g,n}$ are defined over the field $\mathbb{C}(q)$. They were introduced in the mid 90's by Alekseev–Grosse–Schomerus [2, 3] and Buffenoi–Roche [29, 30] by a method called *combinatorial quantization*. By this method, the pair formed by $\mathcal{L}_{g,n}$ and $\mathcal{M}_{g,n}$ appear naturally as a q -deformation of the Fock–Rosly [55] lattice model of the algebra of functions on the “classical” moduli space $\mathcal{M}_{g,n}^{cl}$ of flat \mathfrak{g} -connections on the surface $\Sigma_{g,n+1}$.

In [18], we showed that both $\mathcal{L}_{0,n}$ and $\mathcal{M}_{0,n}$ have integral forms $\mathcal{L}_{0,n}^A$ and $\mathcal{M}_{0,n}^A$ defined over the ring $A = \mathbb{C}[q, q^{-1}]$ (in fact we could have taken $\mathbb{Q}[q, q^{-1}]$ or $\mathbb{Z}[q, q^{-1}]$ as ground ring, see Section 1.1). One can thus consider the specializations of these algebras at $q = \epsilon \in \mathbb{C}^\times$, which we denote by $\mathcal{L}_{0,n}^\epsilon$ and $\mathcal{M}_{0,n}^{A,\epsilon}$ respectively. The algebra $\mathcal{L}_{0,n}^A$ is endowed with an action of the Lusztig integral form $U_A^{\text{res}} = U_A^{\text{res}}(\mathfrak{g})$ of the quantum group $U_q = U_q(\mathfrak{g})$, and $\mathcal{M}_{0,n}^A$ is the subalgebra of invariant elements under this action. Therefore,

$$\mathcal{M}_{0,n}^A := (\mathcal{L}_{0,n}^A)^{U_A^{\text{res}}}, \quad \mathcal{M}_{0,n} := \mathcal{L}_{0,n}^{U_q} = \mathcal{M}_{0,n}^A \bigotimes_A \mathbb{C}(q).$$

The definition of $\mathcal{L}_{0,n}^A$ is based on the original combinatorial quantization method, together with twists of module-algebras and Lusztig’s theory of canonical bases of quantum groups [83]. This allows us to address the structure and representation theory of $\mathcal{L}_{0,n}^A$ and $\mathcal{M}_{0,n}^A$ by means of quantum groups, following ideas of classical invariant theory. In particular, we obtained that $\mathcal{L}_{0,n}$ and $\mathcal{L}_{0,n}^\epsilon$ have no nontrivial zero divisors (and therefore do as well the subalgebras $\mathcal{M}_{0,n}$, $\mathcal{L}_{0,n}^A$, $\mathcal{M}_{0,n}^A$, and $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon^{\text{res}}}$, where U_ϵ^{res} is the specialization of U_A^{res} at $q = \epsilon$).

Also, by extending the quantum coadjoint action of De Concini–Kac–Procesi [39, 40, 42], we described in the \mathfrak{sl}_2 case an action by derivations of the center $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ of $\mathcal{L}_{0,n}^\epsilon$ on $\mathcal{L}_{0,n}^\epsilon$, and we defined a subalgebra $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}} \subset \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$, which is a finite extension of the ring of regular functions on the character variety of the sphere with $(n+1)$ punctures (see [18, Corollary 7.20 and Theorem 8.8]). Moreover, from these results we derived an action by derivation of $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}}$ on $\mathcal{M}_{0,n}^{A,\epsilon}(\mathfrak{sl}_2)$.

Representations of a quotient (the semisimplification) of $\mathcal{M}_{g,n}^{A,\epsilon}$ were already constructed and classified in [4]; they involve only the irreducible representations of the finite-dimensional “small” quantum group $u_\epsilon(\mathfrak{g})$. Moreover, [4] deduced from these representations of $\mathcal{M}_{g,n}^{A,\epsilon}$ a family of representations of the mapping class groups of surfaces, that is equivalent to the one associated to the Witten–Reshetikhin–Turaev TQFT [95, 106]. Recently, representations of another, larger quotient of $\mathcal{M}_{g,n}^{A,\epsilon}$, and the corresponding representations of the mapping class groups of surfaces, were constructed in [52, 53]. These representations are equivalent to those previously obtained by Lyubashenko–Majid [85], and are associated to the TQFT defined in [44, 45]. In the \mathfrak{sl}_2 case, they involve the irreducible and also the principal indecomposable representations of the small quantum group $u_\epsilon(\mathfrak{sl}_2)$. The related link and 3-manifold invariants coincide with those of [21, 90].

In general, the representation theory of $\mathcal{M}_{g,n}^{A,\epsilon}$ is by now far from being understood. Because $\mathcal{M}_{g,n}^{A,\epsilon}$ deforms the classical moduli space $\mathcal{M}_{g,n}^{cl}$, it is natural to expect that its representation theory provides $(2+1)$ -dimensional TQFTs for 3-manifolds endowed with general flat \mathfrak{g} -connections, extending the known TQFTs based on quantum groups (where purely topological ones correspond to the trivial connection). A family of such invariants, called *quantum hyperbolic invariants*, has already been defined for $\mathfrak{g} = \mathfrak{sl}_2$ by means of certain 6j-symbols, *Deus ex machina*; they are closely connected to classical Chern–Simons theory, provide generalized volume conjectures, and contain quantum Teichmüller theory (see [9, 10, 11, 12, 13, 14, 15]). It is part of our present program, initiated in [8], to shed light on these invariants and to generalize them to arbitrary \mathfrak{g} by developing the representation theory of $\mathcal{M}_{g,n}^{A,\epsilon}$.

The quantum moduli algebras have also been recognized as central objects from the viewpoints of factorization homology [22], multiplicative quiver varieties [58] and (stated) skein theory [16, 33, 36, 54]. In another direction, one may expect that the equivalence proved in [89] between combinatorial quantisation for the Drinfeld double $D(H)$ of a finite-dimensional semisimple Hopf algebra H , and Kitaev’s lattice model in topological quantum computation, can be extended to the setup of quantum moduli algebras.

In the present paper, we study $\mathcal{L}_{0,n}$, its integral form $\mathcal{L}_{0,n}^A$, and the specialization $\mathcal{L}_{0,n}^\epsilon$ of $\mathcal{L}_{0,n}^A$ at $q = \epsilon$ a primitive root of unity of odd order. We study also the subalgebras of invariant elements $\mathcal{M}_{0,n} = \mathcal{L}_{0,n}^{U_q}$ and $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$. Here, U_ϵ is the specialization of U_A at $q = \epsilon$, where U_A is the De Concini–Kac integral form of U_q (see Section 1.1). Our results hold for every complex semisimple Lie algebra \mathfrak{g} . The main ones are proofs that $\mathcal{L}_{0,n}$, $\mathcal{L}_{0,n}^A$ and $\mathcal{M}_{0,n}$ are Noetherian and finitely generated rings (see Theorem 1.1), and that the classical fraction algebras of $\mathcal{L}_{0,n}^\epsilon$ and $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ are central simple algebras of PI degrees l^{mN} and $l^{N(n-1)-m}$ respectively (see Theorem 1.3). Here, m and N are the rank and the number of positive roots of \mathfrak{g} .

In the sequel [16] to this paper, in collaboration with M. Faitg, we extend Theorem 1.1 to the algebras $\mathcal{L}_{g,n}$ and $\mathcal{M}_{g,n}$, associated to arbitrary finite type surfaces (arbitrary genus and number of punctures). Also, we show that $\mathcal{M}_{g,n}$ is isomorphic to the \mathfrak{g} -skein algebra of $\Sigma_{g,n+1}$, and $\mathcal{L}_{g,n}$ to the stated skein algebra of the compact surface $\bar{\Sigma}_{g,n+1}$ with one boundary component and one marked point on the boundary component. This was proved for $\mathfrak{g} = \mathfrak{sl}_2$ in [54]. In this specific case $\mathfrak{g} = \mathfrak{sl}_2$, the fact that the stated skein algebra of any finite type surface is Noetherian and finitely generated was proved in [80]. Still in the \mathfrak{sl}_2 case, for related results, e.g., on non-zero divisors and computation of PI degrees, see [23, 24, 57, 64, 73, 74, 75, 78]. For recent results on $\mathfrak{g} = \mathfrak{sl}_n$, see [79, 105].

By using the analysis developed in the present paper for $\mathcal{L}_{0,n}^A$, one can define the integral form $\mathcal{L}_{g,n}^A$ as well, and show that it is a Noetherian and finitely generated ring. We do not have a proof yet of these properties for the algebra $\mathcal{M}_{0,n}^A$, which seems to be much more difficult to handle. We note that there is a strict inclusion $\mathcal{M}_{0,n}^{A,\epsilon} \subset (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$. This is discussed after Theorem 1.2. In [17], we study further properties of $(\mathcal{L}_{g,n}^\epsilon)^{U_\epsilon}$, and we consider also the subalgebra $\mathcal{M}_{g,n}^{A,\epsilon}$.

1.1 Statement of results

Let us recall a few notations and facts from [18]. Let U_q be the simply-connected quantum group of \mathfrak{g} , defined over the field $\mathbb{C}(q)$. From U_q one can define a U_q -module algebra $\mathcal{L}_{0,n}$, called (quantum, daisy) *graph algebra*, where U_q acts by means of a right coadjoint action. The set of invariant elements of $\mathcal{L}_{0,n}$ for this action is an algebra; we denote it $\mathcal{M}_{0,n} := \mathcal{L}_{0,n}^{U_q}$ and call it *quantum moduli algebra*. As a $\mathbb{C}(q)$ -module $\mathcal{L}_{0,n}$ is just $\mathcal{O}_q^{\otimes n}$, where $\mathcal{O}_q = \mathcal{O}_q(G)$ is the standard quantum function algebra of the connected and simply-connected Lie group G with Lie algebra \mathfrak{g} . The product of $\mathcal{L}_{0,n}$ is obtained by twisting both the product of each factor \mathcal{O}_q and the product between them. It is equivariant with respect to a (right) coadjoint action of U_q , which defines the structure of U_q -module of $\mathcal{L}_{0,n}$.

The module algebra $\mathcal{L}_{0,n}$ has an integral form $\mathcal{L}_{0,n}^A$, which is defined over $A = \mathbb{C}[q, q^{-1}]$, and endowed with an (coadjoint) action of the Lusztig [82] integral form U_A^{res} of U_q . It is obtained by replacing \mathcal{O}_q in the construction of $\mathcal{L}_{0,n}$ with the restricted dual \mathcal{O}_A of the integral form U_A^{res} , or equivalently with the restricted dual of the integral form Γ of U_q defined by De Concini–Lyubashenko [41]. Since U_A^{res} contains the De Concini–Kac [39] integral form U_A , and U_A has the same set of invariant elements in $\mathcal{L}_{0,n}^A$, we systematically denote the latter

$$\mathcal{M}_{0,n}^A := (\mathcal{L}_{0,n}^A)^{U_A} \quad (= (\mathcal{L}_{0,n}^A)^{U_A^{\text{res}}}).$$

We call $\mathcal{M}_{0,n}^A$ the *integral quantum moduli algebra*.

A cornerstone of the theory of $\mathcal{M}_{0,n}$ is a map Φ_n originally due to Alekseev [1], building on works of Drinfeld [48] and Reshetikhin and Semenov-Tian-Shansky [94]. In [18], we showed that Φ_n eventually provides isomorphisms of module algebras and algebras respectively,

$$\Phi_n: \mathcal{L}_{0,n} \rightarrow (U_q^{\otimes n})^{\text{lf}}, \quad \Phi_n: \mathcal{M}_{0,n} \rightarrow (U_q^{\otimes n})^{U_q},$$

where $U_q^{\otimes n}$ is endowed with a right adjoint action of U_q , and $(U_q^{\otimes n})^{\text{lf}}$ is the subalgebra of locally finite elements with respect to this action. When $n = 1$ the algebra U_q^{lf} has been studied in great detail by Joseph–Letzter [61, 62, 63]; we will use simplified proofs of their results, obtained in [104].

All the material we need about the results discussed above is described in [18], and recalled in Sections 2.1 and 2.2.

Our first result, proved in Section 3, is the following.

Theorem 1.1. *$\mathcal{L}_{0,n}$, $\mathcal{M}_{0,n}$ and the integral form $\mathcal{L}_{0,n}^A$ are Noetherian rings, and finitely generated rings.*

It follows immediately from the theorem that the specializations $\mathcal{L}_{0,n}^\epsilon$, $\epsilon \in \mathbb{C}^\times$, are Noetherian and finitely generated rings as well. In [18] we proved that all these algebras (and therefore $\mathcal{M}_{0,n}^A$ and $\mathcal{M}_{0,n}^{A,\epsilon}$) have no nontrivial zero divisors.

Because the construction of the integral form $\mathcal{L}_{0,n}^A$ is based on the Kashiwara–Lusztig theory of canonical bases, we could have defined $\mathcal{L}_{0,n}^A$ over the ground ring $\mathbb{Z}[q, q^{-1}]$, and Theorem 1.1 for $\mathcal{L}_{0,n}^A$ holds true as well in this generality. Since we are mainly interested in the representation theory of the specializations $\mathcal{L}_{0,n}^\epsilon$ and $\mathcal{M}_{0,n}^{A,\epsilon}$, which will be addressed in [17], the choice

of $A = \mathbb{C}[q, q^{-1}]$ is natural. Note however that the proof of Proposition 2.18 uses that $\mathbb{C}[q, q^{-1}]$ is a PID.

We describe the background material on canonical bases in Section 2.2.2; we have tried to make the exposition pedestrian and self-contained, so as to be more accessible to non experts.

After we finished this work, we discovered that [47] already proved that $\mathcal{L}_{0,1}(\mathfrak{gl}(n))$ and $\mathcal{L}_{0,n}(\mathfrak{gl}(2))$ are Noetherian and finitely generated rings. Our approach here is completely different. For $\mathcal{L}_{0,n}$, we adapt the proof given by Voigt–Yuncken [104] of a result of Joseph [61], which asserts that U_q^{lf} is a Noetherian ring (see Theorem 3.1). For $\mathcal{M}_{0,n}$, we deduce the result from the one for $\mathcal{L}_{0,n}$, by following a line of proof of the Hilbert–Nagata theorem in classical invariant theory (see Theorem 3.4).

At present, we do not have a proof that $\mathcal{M}_{0,n}^A$ is a Noetherian and finitely generated ring for arbitrary \mathfrak{g} and $n \geq 1$, though it is natural to expect it is the case. Indeed, when $\mathfrak{g} = \mathfrak{sl}_2$, $\mathcal{M}_{0,n}^A(\mathfrak{sl}_2)$ is isomorphic to the skein algebra of a sphere with $n + 1$ punctures (see [18, Theorem 8.6]), which is finitely generated and Noetherian by results of [32] and [93]. In our general situation, key arguments in the proof of Theorem 1.1 for $\mathcal{M}_{0,n}$ depend on the existence of a Reynolds operator on the U_q -module $\mathcal{L}_{0,n}$, and one can easily show there is no Reynolds operator on $\mathcal{L}_{0,n}^A$. This follows from the corresponding fact for the integral quantum coordinate ring \mathcal{O}_A (see Remark 2.19).

From Section 4, we consider the specializations $\mathcal{L}_{0,n}^\epsilon$ of $\mathcal{L}_{0,n}^A$ at $q = \epsilon$, a primitive root of unity of odd order l (and coprime to 3 if \mathfrak{g} has G_2 components). In [41], De Concini–Lyubashenko introduced a central subalgebra $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ of \mathcal{O}_ϵ isomorphic to the coordinate ring $\mathcal{O}(G)$, and proved that the $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ -module \mathcal{O}_ϵ is projective of rank $l^{\dim \mathfrak{g}}$. As observed by Brown–Gordon–Stafford [28], Bass’ cancellation theorem in K -theory and the fact that $K_0(\mathcal{O}(G)) \cong \mathbb{Z}$, proved by Marlin [87], imply that this module is free. Alternatively, this follows also from the fact that \mathcal{O}_ϵ is a cleft extension of $\mathcal{O}(G)$ by the dual of the Frobenius–Lusztig kernel $u_\epsilon(\mathfrak{g})$, as proved by Andruskiewitsch–Garcia (see [6, Remark 2.18 (b)], and also [25, Section 3.2]; this argument was explained to us by K.A. Brown).

The Section 4 proves the analogous property for $\mathcal{L}_{0,n}^\epsilon$. Namely:

Theorem 1.2. *$\mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$ is a central subalgebra of $\mathcal{L}_{0,n}^\epsilon$, and $\mathcal{L}_{0,n}^\epsilon$ is a free $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$ -module of rank $l^{n \cdot \dim \mathfrak{g}}$, isomorphic to the $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$ -module $\mathcal{O}_\epsilon^{\otimes n}$.*

In the sequel we systematically denote $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) := \mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$. We prove the first and third claims of Theorem 1.2 in Proposition 4.1. The arguments use results of De Concini–Kac [39], De Concini–Procesi [40, 42], and De Concini–Lyubashenko [41], that we recall in Sections 2.3–2.5. Let us stress that the algebra structures of $\mathcal{L}_{0,n}^\epsilon$ and $\mathcal{O}_\epsilon^{\otimes n}$ are completely different.

Since $\mathcal{Z}_0(\mathcal{O}_\epsilon) \cong \mathcal{O}(G)$, we may deduce the second claim of Theorem 1.2 from the first and third claims together with the results of [41, 87], or [6], recalled above. Nevertheless, we give a self-contained proof that $\mathcal{L}_{0,1}^\epsilon$ is finite projective of rank $l^{\dim \mathfrak{g}}$ over $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$, by adapting the original arguments of De Concini–Lyubashenko [41, Theorem 7.2]. In particular, we study the coregular action of the braid group of \mathfrak{g} on $\mathcal{L}_{0,1}^\epsilon$; by the way, in the appendix, we provide different proofs of some technical facts shown in [41]. Of course, it remains an exciting problem to describe the centralizing extension $\mathcal{O}(G)^{\otimes n} \subset \mathcal{L}_{0,n}^\epsilon$ (and similarly $\mathcal{O}(G)^{\otimes n} \subset (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ below), aiming at generalizing the results of [6] and finding a direct, more structural proof of freeness in Theorem 1.2. Also, we note that bases of $\mathcal{L}_{0,n}^\epsilon$ over $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ are complicated. The only case we know is for $\mathcal{O}_\epsilon(\mathfrak{sl}_2)$, described in [38], and it is far from being obvious (see (4.4)).

In Section 5, we turn to fraction rings. As mentioned above $\mathcal{L}_{0,n}^\epsilon$ has no nontrivial zero divisors. Therefore, its center $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ is an integral domain. Denote by $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ its fraction field. Denote by $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ the subring of $\mathcal{L}_{0,n}^\epsilon$ formed by the invariant elements of $\mathcal{L}_{0,n}^\epsilon$ with respect to the right coadjoint action of U_ϵ . The center $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ of $\mathcal{L}_{0,n}^\epsilon$ is contained in $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ (this follows from [18, Proposition 6.19]). Note also that we trivially have an inclusion

$\mathcal{M}_{0,n}^{A,\epsilon} \subset (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$, and these two algebras are distinct in general. For instance, when $n = 1$, we have $(\mathcal{L}_{0,1}^\epsilon)^{U_\epsilon} = \mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)$, which is a finite extension of $\mathcal{Z}_0(\mathcal{O}_\epsilon) \cong \mathcal{O}(G)$ (see Lemma 5.1). On another hand, $\mathcal{M}_{0,1}^{A,\epsilon}$ is the specialization at $q = \epsilon$ of $\mathcal{Z}(\mathcal{L}_{0,1}^A)$, a polynomial algebra in $\text{rk}(\mathfrak{g})$ variables, which may be identified via Φ_1 with the center $\mathcal{Z}(U_A)$ of the integral form U_A .

Consider the rings

$$Q(\mathcal{L}_{0,n}^\epsilon) = Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \bigotimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} \mathcal{L}_{0,n}^\epsilon, \quad Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) = Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \bigotimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}.$$

In general, given a ring A with center $\mathcal{Z}(A)$ an integral domain we reserve the notation $Q(A)$ to the central localization of A , i.e., $Q(A) := Q(\mathcal{Z}(A)) \bigotimes_{\mathcal{Z}(A)} A$. Though the center $\mathcal{Z}((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ of $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ is larger than $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$, the notation $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ is valid, for $\mathcal{Z}((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ is an integral domain finite over $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$, and hence the central localization of $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ coincides with $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ as defined above. Throughout the paper, unless we mention it explicitly, we follow the conventions of McConnell–Robson [88] as regards the terminology of ring theory; in particular, for the notions of central simple algebras and PI degrees, see in [88, Sections 5.3 and 13.3.6–13.6.7].

Denote by m the rank of \mathfrak{g} , and by N the number of its positive roots. In Section 5, we prove the following.

Theorem 1.3.

- (1) $Q(\mathcal{L}_{0,n}^\epsilon)$ is a division algebra and a central simple algebra of PI degree l^{nN} .
- (2) $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$, $n \geq 2$, is a division algebra and a central simple algebra of PI degree $l^{N(n-1)-m}$.

The second claim of (1) means that $Q(\mathcal{L}_{0,n}^\epsilon)$ is a complex subalgebra of a full matrix algebra $\text{Mat}_d(\mathbb{F})$, where $d = l^{nN}$ and \mathbb{F} is a finite extension of $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ such that

$$\mathbb{F} \bigotimes_{Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} Q(\mathcal{L}_{0,n}^\epsilon) = \text{Mat}_d(\mathbb{F}).$$

That $Q(\mathcal{L}_{0,n}^\epsilon)$ is a division algebra and a central simple algebra follows from Theorem 1.2 and the fact that $\mathcal{L}_{0,n}^\epsilon$ has no nontrivial zero divisors (proved in [18]). The computation of $d = l^{nN}$ uses a lower bound coming from the representation theory of U_ϵ , and a lower bound for the degree of $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ as a field extension of $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$, obtained by using specializations to $q = \epsilon$ of certain central elements in $\mathcal{L}_{0,n}$ (for q generic). In this computation a main role is played by results of De Concini–Kac [39].

We deduce (2) from (1), the double centralizer theorem for central simple algebras, a few results of [18, 41], and Theorem 1.2 again.

1.2 Basic notations

Given a ring R , we denote by $\mathcal{Z}(R)$ its center. When R is commutative and has no nontrivial zero divisors, $Q(R)$ denotes its fraction field.

Given a Hopf algebra H with product m and coproduct Δ , we denote by H^{cop} (resp. H_{op}) the Hopf algebra with the same algebra (resp. coalgebra) structure as H but the opposite coproduct $\Delta^{\text{cop}} := \sigma \circ \Delta$ (resp. opposite product $m \circ \sigma$), where $\sigma(x \otimes y) = y \otimes x$, and the antipode S^{-1} . We use Sweedler's coproduct notation, $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$, $x \in H$, and we set $\Delta^{(1)} := \text{id}$, $\Delta^{(2)} := \Delta$, and $\Delta^{(n)} := (\Delta \otimes \text{id})\Delta^{(n-1)}$ for $n \geq 3$ (this is not the convention used in [18]).

The results of this paper hold true for any finite-dimensional complex semisimple Lie algebra \mathfrak{g} , but unless we state it differently, we will assume \mathfrak{g} is simple. We will denote its rank

by m , and its Cartan matrix by (a_{ij}) . We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a basis of simple roots $\alpha_i \in \mathfrak{h}_{\mathbb{R}}^*$, and denote by \mathfrak{b}_{\pm} the Borel subalgebras associated to it. We denote by N the number of positive roots of \mathfrak{g} , and by ρ half the sum of the positive roots.

We denote by d_1, \dots, d_m the unique coprime positive integers such that the matrix $(d_i a_{ij})$ is symmetric, and $(\ , \)$ the unique inner product on $\mathfrak{h}_{\mathbb{R}}^*$ such that $d_i a_{ij} = (\alpha_i, \alpha_j)$. For any root α , the coroot is $\check{\alpha} = 2\alpha/(\alpha, \alpha)$; in particular $\check{\alpha}_i = d_i^{-1}\alpha_i$. The root lattice Q is the \mathbb{Z} -lattice in $\mathfrak{h}_{\mathbb{R}}^*$ defined by $Q = \sum_{i=1}^m \mathbb{Z}\alpha_i$. The weight lattice P is the \mathbb{Z} -lattice formed by all $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ such that $(\lambda, \check{\alpha}_i) \in \mathbb{Z}$ for every $i = 1, \dots, m$. So $P = \sum_{i=1}^m \mathbb{Z}\varpi_i$, where ϖ_i is the fundamental weight dual to the simple coroot $\check{\alpha}_i$, which satisfies $(\varpi_i, \check{\alpha}_j) = \delta_{i,j}$. Note that $(\lambda, \alpha) \in \mathbb{Z}$ for every $\lambda \in P$, $\alpha \in Q$. We denote by D the cardinality of the quotient lattice P/Q . Then D is the smallest positive integer such that $D(\lambda, \mu) \in \mathbb{Z}$ for every $\lambda, \mu \in P$, that is, such that $DP \subset Q$.

We denote by

$$P_+ := \sum_{i=1}^m \mathbb{Z}_{\geq 0} \varpi_i$$

the cone of dominant integral weights, and we put

$$Q_+ := \sum_{i=1}^m \mathbb{Z}_{\geq 0} \alpha_i.$$

Though $Q \subset P$, it is not true that $Q_+ \subset P_+$, but we have $DP_+ \subset Q_+$. This last property is not trivial, and follows from the classical fact that the inverse of the Cartan matrix (a_{ij}) has coefficients in $D^{-1}\mathbb{N}$.

We will use the standard partial order relation \leq on P , defined by: $\lambda, \mu \in P$ satisfy $\lambda \leq \mu$ if $\mu - \lambda \in Q_+$. In Section 3, we will also use the partial order relation \preceq on P defined by: $\lambda \preceq \mu$ if $\mu - \lambda \in D^{-1}Q_+$.

We denote by $\mathcal{B}(\mathfrak{g})$ the braid group of \mathfrak{g} ; we recall its standard defining relations in Appendix B.

We denote by G the connected and simply-connected algebraic group with Lie algebra \mathfrak{g} , and by T_G the maximal torus of G with Lie algebra \mathfrak{h} ; $N(T_G)$ is the normalizer of T_G , $W = N(T_G)/T_G$ is the Weyl group, B_{\pm} are the Borel subgroups of G with Lie algebra \mathfrak{b}_{\pm} , and $U_{\pm} \subset B_{\pm}$ are their unipotent subgroups.

We denote by $\mathcal{O}(G)$ the coordinate ring of G . It is a commutative Hopf algebra, which can be identified with the restricted dual of the universal enveloping algebra $U(\mathfrak{g})$ (see [76, 84]). Similarly we denote by $\mathcal{O}(B_{\pm})$ the coordinate ring of B_{\pm} .

Let q be an indeterminate, let $q^{1/D}$ be such that $(q^{1/D})^D = q$, set $A = \mathbb{C}[q, q^{-1}]$, $q_i = q^{d_i}$, $q_{\beta} = q^{(\beta, \beta)/2}$ for $\beta \in Q$, and given integers p, k with $0 \leq k \leq p$, we put

$$\begin{aligned} [p]_q &= \frac{q^p - q^{-p}}{q - q^{-1}}, & [0]_q! &= 1, & [p]_q! &= [1]_q [2]_q \cdots [p]_q, & \begin{bmatrix} p \\ k \end{bmatrix}_q &= \frac{[p]_q!}{[p-k]_q! [k]_q!}, \\ (p)_q &= \frac{q^p - 1}{q - 1}, & (0)_q! &= 1, & (p)_q! &= (1)_q (2)_q \cdots (p)_q, & \binom{p}{k}_q &= \frac{(p)_q!}{(p-k)_q! (k)_q!}. \end{aligned}$$

We denote by $\mathcal{A}_0 \subset \mathbb{C}(q)$ the ring of functions regular at $q = 0$; this ring is used only in Section 2.2.2.

We denote by ϵ a primitive l -th root of unity such that $\epsilon^{2d_i} \neq 1$ is also a primitive l -th root of unity for all $i \in \{1, \dots, m\}$. This means that l is odd, and coprime to 3 if \mathfrak{g} is G_2 . We put $\epsilon_i := \epsilon^{d_i}$.

In this paper, we use the definition of the unrestricted integral form $U_A(\mathfrak{g})$ given in [41, 42]; in [18] we used the one of [39, 40]. The two are (trivially) isomorphic, and have the same

specialization at $q = \epsilon$. Also, we denote here by L_i the generators of $U_q(\mathfrak{g})$ we denoted by ℓ_i in [18].

In order to facilitate the comparison with the results of [41], we note that their generators denoted K_i , E_i and F_i , that we will denote by \tilde{K}_i , \tilde{E}_i and \tilde{F}_i , can be written as K_i , $K_i^{-1}E_i$ and F_iK_i in our notations. They satisfy the same algebra relations.

2 Background results

2.1 On U_q , \mathcal{O}_q , $\mathcal{L}_{0,n}$, $\mathcal{M}_{0,n}$, and Φ_n

Except when stated differently, we refer to [18, Sections 2–4 and 6], and the references therein for details about the material of this section. We stress that the simply-connected quantum group, that we denote U_q below, was denoted \tilde{U}_q in [18]. Also, the adjoint quantum group U_q^{ad} was denoted U_q .

The *simply-connected* quantum group $U_q = U_q(\mathfrak{g})$ is the Hopf algebra over $\mathbb{C}(q)$ with generators E_i , F_i , L_i , L_i^{-1} , $1 \leq i \leq m$, and defining relations

$$\begin{aligned} L_i L_j &= L_j L_i, & L_i L_i^{-1} &= L_i^{-1} L_i = 1, & L_i E_j L_i^{-1} &= q_i^{\delta_{i,j}} E_j, & L_i F_j L_i^{-1} &= q_i^{-\delta_{i,j}} F_j, \\ E_i F_j - F_j E_i &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r &= 0 & \text{if } i \neq j, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{ij}-r} F_j F_i^r &= 0 & \text{if } i \neq j, \end{aligned}$$

where for $\lambda = \sum_{i=1}^m m_i \varpi_i \in P$ we set $K_\lambda = \prod_{i=1}^m L_i^{m_i}$, and $K_i = K_{\alpha_i} = \prod_{j=1}^m L_j^{\alpha_{ji}}$. The coproduct Δ , antipode S , and counit ε of U_q are given by

$$\begin{aligned} \Delta(L_i) &= L_i \otimes L_i, & \Delta(E_i) &= E_i \otimes K_i + 1 \otimes E_i, & \Delta(F_i) &= F_i \otimes 1 + K_i^{-1} \otimes F_i, \\ S(E_i) &= -E_i K_i^{-1}, & S(F_i) &= -K_i F_i, & S(L_i) &= L_i^{-1}, \\ \varepsilon(E_i) &= \varepsilon(F_i) = 0, & \varepsilon(L_i) &= 1. \end{aligned}$$

We fix a reduced expression $s_{i_1} \cdots s_{i_N}$ of the longest element w_0 of the Weyl group of \mathfrak{g} . It induces a total ordering of the positive roots,

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \dots, \quad \beta_N = s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N}).$$

The root vectors of U_q with respect to such an ordering are defined by

$$E_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(E_{i_k}), \quad F_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k}), \quad (2.1)$$

where T_i is the Lusztig algebra automorphism of U_q associated to the simple root α_i [82, 83] (see also [35, Chapter 8]). The braid group $\mathcal{B}(\mathfrak{g})$ acts on U_q by means of the Lusztig automorphisms. In the appendix, we recall the relation between T_i and the generator \hat{w}_i of the quantum Weyl group, which we will mostly use. Let us just recall here that the monomials $F_{\beta_1}^{r_1} \cdots F_{\beta_N}^{r_N} K_\lambda E_{\beta_N}^{t_N} \cdots E_{\beta_1}^{t_1}$ ($r_i, t_i \in \mathbb{N}$, $\lambda \in P$) form a basis of U_q , the *PBW basis*.

U_q is a *pivotal* Hopf algebra, with pivotal element $\ell := K_{2\rho} = \prod_{j=1}^m L_j^2$. So ℓ is group-like, and $S^2(x) = \ell x \ell^{-1}$ for every $x \in U_q$.

The *adjoint* quantum group $U_q^{\text{ad}} = U_q^{\text{ad}}(\mathfrak{g})$ is the Hopf subalgebra of U_q generated by the elements E_i, F_i ($i = 1, \dots, m$) and K_α with $\alpha \in Q$; so $\ell \in U_q^{\text{ad}}$. When $\mathfrak{g} = \mathfrak{sl}_2$, we simply write the above generators $E = E_1, F = F_1, L = L_1, K = K_1$.

We denote by $U_q(\mathfrak{n}_+)$, $U_q(\mathfrak{n}_-)$ and $U_q(\mathfrak{h})$ the subalgebras of U_q generated respectively by the E_i , the F_i , and the K_λ ($\lambda \in P$), and by $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$ the subalgebras generated by the E_i and the K_λ , and by the F_i and the K_λ , respectively. We do similarly with U_q^{ad} , where now $U_q^{\text{ad}}(\mathfrak{h})$ is generated by the K_λ with $\lambda \in Q$.

The Hopf algebra U_q^{ad} is not braided in a strict sense, but it has braided categorical completions. Let us recall briefly what this means and implies. For details, we refer to [18, Sections 2 and 3] (see also [104, Section 3.10], where \mathbb{U}_q below is formulated in terms of multiplier Hopf algebras).

A U_q^{ad} -module V is said of *type 1* if it has finite dimension and the generators K_i are diagonalizable on V with eigenvalues in $q_i^{\mathbb{Z}}$. We denote by \mathcal{C} the category of U_q^{ad} -modules of type 1, by Vect the category of finite-dimensional $\mathbb{C}(q)$ -vector spaces, and by $F_{\mathcal{C}}: \mathcal{C} \rightarrow \text{Vect}$ the forgetful functor. The category \mathcal{C} is semisimple. The simple objects are highest weight U_q^{ad} -modules; we denote by V_μ the simple module with highest weight $\mu \in P_+$. In the case $\mathfrak{g} = \mathfrak{sl}_2$, we identify P_+ with \mathbb{N} , and denote by V_n the simple module of dimension $n + 1$. Note that V_μ is canonically endowed with a structure of U_q -module of type 1, the generators L_i being diagonalizable with eigenvalues in $q_i^{\mathbb{Z}/D}$. The *categorical completion* \mathbb{U}_q^{ad} of U_q^{ad} is the set of natural transformations $F_{\mathcal{C}} \rightarrow F_{\mathcal{C}}$. An element of \mathbb{U}_q^{ad} is a collection $(a_V)_{V \in \text{Ob}(\mathcal{C})}$, where $a_V \in \text{End}_{\mathbb{C}(q)}(V)$ satisfies $F_{\mathcal{C}}(f) \circ a_V = a_W \circ F_{\mathcal{C}}(f)$ for any objects V, W of \mathcal{C} and any arrow $f \in \text{Hom}_{U_q^{\text{ad}}}(V, W)$. It is not hard to see that \mathbb{U}_q^{ad} inherits from \mathcal{C} a natural structure of (completed) Hopf algebra such that the map

$$\iota: U_q^{\text{ad}} \longrightarrow \mathbb{U}_q^{\text{ad}}, \quad x \longmapsto (\pi_V(x))_{V \in \text{Ob}(\mathcal{C})} \quad (2.2)$$

is a morphism of Hopf algebras, where $\pi_V: U_q^{\text{ad}} \rightarrow \text{End}(V)$ is the representation associated to a module V in \mathcal{C} . It is a theorem that this map is injective. From now on, let us extend the coefficient ring of the modules and morphisms in \mathcal{C} to $\mathbb{C}(q^{1/D})$. Put $\mathbb{U}_q = \mathbb{U}_q^{\text{ad}} \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})$. The map ι above extends to an embedding of U_q in \mathbb{U}_q . The category \mathcal{C} , with coefficients extended to $\mathbb{C}(q^{1/D})$, is braided and ribbon; we postpone a discussion of that fact to Section 2.3, where it will be developed. As a consequence, we can regard \mathbb{U}_q as a quasitriangular and ribbon Hopf algebra in a generalized sense (see [18]). The R -matrix of \mathbb{U}_q is the family of morphisms

$$R = (R_{V,W})_{V,W \in \text{Ob}(\mathcal{C})},$$

where $R_{V,W} \in \text{End}(V \otimes W)$ is the endomorphism defined by the action of Drinfeld's universal R -matrix on $V \otimes W$. The ribbon element of \mathbb{U}_q is defined similarly by Drinfeld's universal ribbon element. One defines the *categorical tensor product* $\mathbb{U}_q^{\hat{\otimes} 2}$ similarly as \mathbb{U}_q ; in particular it contains all the infinite series of elements of $\mathbb{U}_q^{\hat{\otimes} 2}$ having only a finite number of non-zero terms when evaluated on a given module $V \otimes W$ of \mathcal{C} . There is an expansion of R as an infinite series in $\mathbb{U}_q^{\hat{\otimes} 2}$. Adapting Sweedler's coproduct notation $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$, we find convenient to write this series as

$$R = \sum_{(R)} R_{(1)} \otimes R_{(2)}. \quad (2.3)$$

We put $R^+ := R$, $R^- := (\sigma \circ R)^{-1}$ where σ is the flip map $x \otimes y \mapsto y \otimes x$. We will not use any explicit formula of R , but the following factorization formula

$$R = \Theta \hat{R}, \quad (2.4)$$

where

$$\Theta = q^{\sum_{i,j=1}^m (B^{-1})_{ij} H_i \otimes H_j} \in \mathbb{U}_q^{\hat{\otimes} 2},$$

with $B \in M_m(\mathbb{Q})$ the matrix with entries $B_{ij} := d_j^{-1} a_{ij}$, and

$$\hat{R} = \sum_{(\hat{R})} \hat{R}_{(1)} \otimes \hat{R}_{(2)} \in \mathbb{U}_q(\mathfrak{n}_+) \hat{\otimes} \mathbb{U}_q(\mathfrak{n}_-)$$

(see [18, Section 3.2], and for details, e.g., [35, Theorem 8.3.9], or [104, Theorem 3.108]). If x, y are weight vectors of weights μ, ν respectively, then $\Theta(x \otimes y) = q^{(\mu, \nu)} x \otimes y$. Moreover, \hat{R} has weight 0 for the adjoint action of $U_q(\mathfrak{h})$; that is, complementary components $\hat{R}_{(1)}$ and $\hat{R}_{(2)}$ have opposite weights.

Recall that we denote by G the connected and simply-connected algebraic group with Lie algebra \mathfrak{g} . The *quantum function Hopf algebra* $\mathcal{O}_q = \mathcal{O}_q(G)$ is defined as the restricted dual of U_q^{ad} with respect to the category \mathcal{C} , that is, the set of $\mathbb{C}(q)$ -linear maps $f: U_q^{\text{ad}} \rightarrow \mathbb{C}(q)$ such that $\text{Ker}(f)$ contains a cofinite two sided ideal I (i.e., such that $I \oplus M = U_q$ for some finite-dimensional vector space M), and $\prod_{s=-r}^r (K_i - q_i^s) \in I$ for some $r \in \mathbb{N}$ and every i (see, e.g., [26, Chapter I.7]).

The space \mathcal{O}_q is a Hopf algebra, with structure maps defined dually to those of U_q^{ad} . We denote by \star its product. The algebras $\mathcal{O}_q(T_G), \mathcal{O}_q(U_{\pm}), \mathcal{O}_q(B_{\pm})$ are defined similarly, by replacing U_q^{ad} with $U_q^{\text{ad}}(\mathfrak{h}), U_q^{\text{ad}}(\mathfrak{n}_{\pm}), U_q^{\text{ad}}(\mathfrak{b}_{\pm})$, respectively. As a vector space, \mathcal{O}_q is generated by the functionals $x \mapsto w(\pi_V(x)v), x \in U_q^{\text{ad}}$, for every object $V \in \text{Ob}(\mathcal{C})$ and vectors $v \in V, w \in V^*$. Such functionals are called *matrix coefficients*. Because the morphism $\iota: U_q^{\text{ad}} \rightarrow \mathbb{U}_q$ is injective (see (2.2)), the Hopf duality pairing $\langle \cdot, \cdot \rangle: \mathcal{O}_q \times U_q^{\text{ad}} \rightarrow \mathbb{C}(q)$ is non degenerate. By extending the coefficient ring from $\mathbb{C}(q)$ to $\mathbb{C}(q^{1/D})$, we can uniquely extend it to a bilinear pairing

$$\langle \cdot, \cdot \rangle: (\mathcal{O}_q \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})) \times \mathbb{U}_q \rightarrow \mathbb{C}(q^{1/D})$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}_q \otimes U_q^{\text{ad}} & \xrightarrow{\langle \cdot, \cdot \rangle} & \mathbb{C}(q) \\ \text{id} \otimes \iota \downarrow & & \downarrow \\ (\mathcal{O}_q \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})) \otimes \mathbb{U}_q & \xrightarrow{\langle \cdot, \cdot \rangle} & \mathbb{C}(q^{1/D}). \end{array}$$

This pairing is defined by $\langle Y\phi_v^w, (a_X) \rangle = w(a_Y v)$ for every $(a_X) \in \mathbb{U}_q$ and $Y\phi_v^w \in \mathcal{O}_q$. It is non degenerate.

The maps

$$\Phi^{\pm}: \mathcal{O}_q \longrightarrow U_q^{\text{cop}}, \quad \alpha \longmapsto (\alpha \otimes \text{id})(R^{\pm}) = \sum_{(R^{\pm})} \langle \alpha, R_{(1)}^{\pm} \rangle R_{(2)}^{\pm} \quad (2.5)$$

are well-defined morphisms of Hopf algebras. Here we stress that it is the simply-connected quantum group U_q^{cop} that is the range of Φ^{\pm} . This will be explained with more details in Section 2.3.

Let us make two simple observations, for future reference. Firstly, because \mathcal{O}_q is spanned by the matrix coefficients of the objects of \mathcal{C} , and \mathcal{C} is semisimple with simple objects the U_q^{ad} -modules $V_{\mu}, \mu \in P_+$, there is a decomposition of U_q -bimodule

$$\mathcal{O}_q = \bigoplus_{\mu \in P_+} C(\mu), \quad (2.6)$$

where $C(\mu) = V_\mu^* \otimes V_\mu$, the space of matrix coefficients of V_μ , is endowed with the left action on the factor V_μ and the right action on V_μ^* , and \mathcal{O}_q has the left and right coregular actions \triangleleft and \triangleright , defined by

$$x \triangleright \alpha := \sum_{(\alpha)} \alpha_{(1)} \langle \alpha_{(2)}, x \rangle, \quad \alpha \triangleleft x := \sum_{(\alpha)} \langle \alpha_{(1)}, x \rangle \alpha_{(2)}$$

for all $x \in U_q$ and $\alpha \in \mathcal{O}_q$. Here we recall that each U_q^{ad} -module V_μ can be regarded as a U_q -module, so the above expressions make sense. The decomposition (2.6) is the *Peter–Weyl* decomposition of \mathcal{O}_q . It will be refined in Section 2.2.2.

Moreover, the algebra \mathcal{O}_q is generated by the matrix coefficients of the simple U_q^{ad} -modules V_{ϖ_k} with highest weights the fundamental weights ϖ_k , $k = 1, \dots, m$; see, e.g., [26, Proposition I.7.8] for a proof. This relies on the standard fact that, for any $\mu, \nu \in P_+$ we have a direct sum decomposition of modules (where $m(\lambda) \in \mathbb{N}$)

$$V_\mu \otimes V_\nu = V_{\mu+\nu} \oplus \bigoplus_{\lambda < \mu+\nu} V_\lambda^{\oplus m(\lambda)}. \quad (2.7)$$

In particular, this decomposition implies that, up to scalar multiples, there is a unique non-zero morphism $V_{\mu+\nu} \rightarrow V_\mu \otimes V_\nu$, which is injective and splits. Dually, this means that, applying the product in \mathcal{O}_q followed by the projection onto the subspace $C(\mu + \nu)$ we get a canonical projection map

$$C(\mu) \otimes C(\nu) \rightarrow C(\mu + \nu). \quad (2.8)$$

The *loop algebra* $\mathcal{L}_{0,1} = \mathcal{L}_{0,1}(\mathfrak{g})$ is defined by twisting the product \star of \mathcal{O}_q , keeping the same underlying linear space. The new product is equivariant with respect to the right coadjoint action coad^r of U_q , defined by

$$\text{coad}^r(x)(\alpha) = \sum_{(x)} S(x_{(2)}) \triangleright \alpha \triangleleft x_{(1)}$$

for all $x \in U_q$ and $\alpha \in \mathcal{O}_q$. By equivariant we mean that $\mathcal{L}_{0,1}$ is a U_q -module algebra. Let us spell out its product and equivariance property. Using the fact that U_q can be regarded as a subspace of \mathbb{U}_q , the actions \triangleleft and \triangleright extend naturally to actions of \mathbb{U}_q , and the product of $\mathcal{L}_{0,1}$ is expressed in terms of \star by the formula (see [18, Proposition 4.1]):

$$\alpha\beta = \sum_{(R),(R)} (R_{(2)'} S(R_{(2)}) \triangleright \alpha) \star (R_{(1)'} \triangleright \beta \triangleleft R_{(1)}), \quad (2.9)$$

where $\sum_{(R)} R_{(1)} \otimes R_{(2)}$ and $\sum_{(R)} R_{(1)'} \otimes R_{(2)'}$ are expansions of two copies of $R \in \mathbb{U}_q^{\otimes 2}$. Note that the sum in (2.9) has only a finite number of non-zero terms. By using that

$$R\Delta = \Delta^{\text{cop}}R,$$

this product can equivalently be expressed as

$$\alpha\beta = \sum_{(R),(R)} (\beta \triangleleft R_{(1)} R_{(1)'}) \star (S(R_{(2)}) \triangleright \alpha \triangleleft R_{(2)'}). \quad (2.10)$$

This product gives $\mathcal{L}_{0,1}$ (like \mathcal{O}_q) a structure of U_q -module algebra for the actions \triangleright , \triangleleft , but also for coad^r (which is not the case of \mathcal{O}_q). Spelling this out for coad^r , this means

$$\text{coad}^r(x)(\alpha\beta) = \sum_{(x)} \text{coad}^r(x_{(1)})(\alpha) \text{coad}^r(x_{(2)})(\beta).$$

The relations between \mathcal{O}_q , $\mathcal{L}_{0,1}$ and U_q are encoded by the map

$$\Phi_1: \mathcal{O}_q \longrightarrow \mathbb{U}_q, \quad \alpha \longmapsto (\alpha \otimes \text{id})(RR'), \quad (2.11)$$

where $R' = \sigma \circ R$, and as usual $\sigma: x \otimes y \mapsto y \otimes x$. Note that

$$\Phi_1 = m \circ (\Phi^+ \otimes (S^{-1} \circ \Phi^-)) \circ \Delta. \quad (2.12)$$

We call Φ_1 the *RSD* map, for Drinfeld, Reshetikhin and Semenov-Tian-Shansky introduced it first (see [48, 86, 94]). It is a fundamental result of the theory (see [20, 34, 61]) that Φ_1 affords an isomorphism of U_q -modules $\Phi_1: \mathcal{O}_q \rightarrow U_q^{\text{lf}}$. For full details on that result we refer to [104, Section 3.12]. Here, U_q^{lf} is the set of *locally finite* elements of U_q , endowed with the right adjoint action ad^r of U_q . It is defined by

$$U_q^{\text{lf}} := \{x \in U_q \mid \text{rk}_{\mathbb{C}(q)}(\text{ad}^r(U_q)(x)) < \infty\}$$

and

$$\text{ad}^r(y)(x) = \sum_{(y)} S(y_{(1)})xy_{(2)}$$

for every $x, y \in U_q$. The action ad^r gives in fact U_q^{lf} a structure of right U_q -module algebra. It is also a right coideal, that is $\Delta(U_q^{\text{lf}}) \subset U_q^{\text{lf}} \otimes U_q$. Moreover, Φ_1 affords an isomorphism of U_q -module algebras $\Phi_1: \mathcal{L}_{0,1} \rightarrow U_q^{\text{lf}}$. It is a fact that Φ_1 affords an isomorphism between the centers $\mathcal{Z}(\mathcal{L}_{0,1})$ of $\mathcal{L}_{0,1}$ and $\mathcal{Z}(U_q)$ of U_q [18, Proposition 6.24]. Since Φ_1 is an isomorphism of U_q -modules and $\mathcal{Z}(U_q) = U_q^{U_q}$, it follows that $\mathcal{Z}(\mathcal{L}_{0,1}) = \mathcal{L}_{0,1}^{U_q}$.

Let us recall a few fundamental results about U_q^{lf} that we will meet again later. Denote by $T \subset U_q$ the multiplicative Abelian group formed by the elements K_λ , $\lambda \in P$, and by $T_2 \subset T$ the subgroup formed by the elements K_λ , $\lambda \in 2P$. Consider the subset $T_{2-} \subset T_2$ formed by the elements $K_{-\lambda}$, $\lambda \in 2P_+$. Clearly, $T_2 = T_{2-}^{-1}T_{2-}$ and $\text{Card}(T/T_2) = 2^m$.

Theorem 2.1.

- (1) $U_q^{\text{lf}} = \bigoplus_{\lambda \in 2P_+} \text{ad}^r(U_q)(K_{-\lambda})$.
- (2) $U_q = T_{2-}^{-1}U_q^{\text{lf}}[T/T_2]$, so U_q is a free $T_{2-}^{-1}U_q^{\text{lf}}$ -module of rank 2^m .
- (3) The ring U_q^{lf} is (left and right) Noetherian.

These results were proved by Joseph–Letzter in [63, Theorem 4.10], [62, Theorem 6.4], and [61, Theorem 7.4.8], respectively (see also [61, Sections 7.1.6, 7.1.13 and 7.1.25]). For (1) and (3), we refer also to [104, Theorems 3.113 and 3.137], which provides simpler proofs. For instance, in the \mathfrak{sl}_2 case, we have

$$U_q(\mathfrak{sl}_2) = U_q(\mathfrak{sl}_2)^{\text{lf}}[K] \oplus U_q(\mathfrak{sl}_2)^{\text{lf}}[K].L.$$

The actual values of Φ_1 are complicated in general, however, there is a simple important one, that we describe now. Let $V_{-\lambda}$ be the type 1 simple U_q^{ad} -module of lowest weight $-\lambda \in -P_+$ (i.e., the highest weight U_q^{ad} -module $V_{-w_0(\lambda)}$ of highest weight $-w_0(\lambda)$, where w_0 is the longest element of the Weyl group; note that $-w_0$ permutes the simple roots). Let $v \in V_{-\lambda}$ be a lowest weight vector, and $v^* \in V_{-\lambda}^*$ be such that $v^*(v) = 1$ and v^* vanishes on a $U_q^{\text{ad}}(\mathfrak{h})$ -invariant complement of v . Define $\psi_{-\lambda} \in \mathcal{O}_q$ by $\langle \psi_{-\lambda}, x \rangle = v^*(xv)$, $x \in U_q$. From the definition (2.11), it is quite easy to see that

$$\Phi_1(\psi_{-\lambda}) = K_{-2\lambda}. \quad (2.13)$$

In particular, $\Phi_1(\psi_{-\rho}) = \ell^{-1}$, where as usual ℓ is the pivotal element of U_q .

Remark 2.2. Since $\mathcal{L}_{0,1} = \mathcal{O}_q$ as a vector space, we still denote by $C(\mu)$, $\mu \in P^+$, the linear subspace generated by the matrix coefficients of V_μ , the U_q^{ad} -module of type 1 and highest weight μ . It can be proved (see [61, Section 7.1.22], or [104, p. 156], where different conventions are used) that Φ_1 yields an isomorphism of U_q -modules

$$\Phi_1: C(-w_0(\mu)) \rightarrow \text{ad}^r(U_q)(K_{-2\mu}). \quad (2.14)$$

Therefore, the summands in (1) are finite-dimensional U_q -modules, and the action ad^r is completely reducible on U_q^{lf} . In fact, U_q^{lf} is the socle of ad^r on U_q .

Remark 2.3. Because $\ell = \prod_{j=1}^m L_j^2$ and $\Phi_1(\psi_{-\rho}) = \ell^{-1}$, a natural question is the factorization of $\psi_{-\rho}$ in $\mathcal{L}_{0,1}$ (see Corollary 2.23). This question is considered in [60], where $\mathcal{L}_{0,1}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{gl}(r+1)$ is analysed and quantum minors are extensively studied. Let us review here some of their results in relation with $\psi_{-\rho}$.

First note that for $\mathfrak{g} = \mathfrak{sl}(r+1)$ the irreducible representation $V_{-\rho}$ of lowest weight $-\rho$ is isomorphic to the representation of highest weight V_ρ because $-w_0(\rho) = \rho$. By the Weyl formula, the dimension of this representation is

$$\prod_{\alpha > 0} \frac{(2\rho, \alpha)}{(\rho, \alpha)} = 2^N.$$

In [71], a presentation of $U_q(\mathfrak{gl}(r+1))$ is given, which differs from our presentation of $U_q(\mathfrak{sl}(r+1))$ only by its subalgebra $U_q(\mathfrak{h})$, generated by $r+1$ elements $\mathbb{K}_1, \dots, \mathbb{K}_{r+1}$. The inclusion

$$U_q(\mathfrak{sl}(r+1)) \subset U_q(\mathfrak{gl}(r+1))$$

is such that $K_i = \mathbb{K}_i^2 \mathbb{K}_{i+1}^{-2}$, $i = 1, \dots, r$. The quantum minors, properly defined in [60], of the matrix of matrix elements of the natural representation of $U_q(\mathfrak{gl}(r+1))$ are denoted $\det_q(A_{\geq k})$ for $k = 1, \dots, r+1$. We have $\det_q(A_{\geq 1}) = 1$ in the case of $\mathfrak{sl}(r+1)$. Then [60] proves that $\det_q(A_{\geq k}) = (\mathbb{K}_k \cdots \mathbb{K}_{r+1})^2$, and there exists an element $\mathbb{K} \in U_q(\mathfrak{gl}(r+1))$ such that

$$\mathbb{K}^{-2\rho} = \det_q(A_{\geq 1})^{-r} \det_q(A_{\geq 2}) \cdots \det_q(A_{\geq r+1}).$$

This has to be interpreted as $K_{-2\rho} = \Phi_1(\det_q(A_{\geq 2}) \cdots \det_q(A_{\geq r+1}))$ in the case of $\mathfrak{sl}(r+1)$. As a result, this gives the equality

$$\psi_{-\rho} = \det_q(A_{\geq 2}) \cdots \det_q(A_{\geq r+1}).$$

The (*quantum*) *graph algebra* $\mathcal{L}_{0,n} = \mathcal{L}_{0,n}(\mathfrak{g})$ is the braided tensor product of n copies of $\mathcal{L}_{0,1}$ (considered as a U_q -module algebra). As a linear space and U_q -bimodule with actions \triangleleft and \triangleangleright , it coincides with $\mathcal{L}_{0,1}^{\otimes n}$, and thus with $\mathcal{O}_q^{\otimes n}$. It is also a right U_q -module algebra, with the following action of U_q (extending coad^r on $\mathcal{L}_{0,1}$):

$$\text{coad}_n^r(y)(\alpha^{(1)} \otimes \cdots \otimes \alpha^{(n)}) = \sum_{(y)} \text{coad}^r(y_{(1)})(\alpha^{(1)}) \otimes \cdots \otimes \text{coad}^r(y_{(n)})(\alpha^{(n)}) \quad (2.15)$$

for all $y \in U_q$ and $\alpha^{(1)} \otimes \cdots \otimes \alpha^{(n)} \in \mathcal{L}_{0,n}$. The product of $\mathcal{L}_{0,n}$ can be expressed as follows. For every $1 \leq a \leq n$, define $\mathbf{i}_a: \mathcal{L}_{0,1} \rightarrow \mathcal{L}_{0,n}$ by $\mathbf{i}_a(x) = 1^{\otimes(a-1)} \otimes x \otimes 1^{\otimes(n-a)}$; \mathbf{i}_a is an embedding of U_q -module algebras. We will use the notations

$$\mathcal{L}_{0,n}^{(a)} := \text{Im}(\mathbf{i}_a), \quad (\alpha)^{(a)} := \mathbf{i}_a(\alpha). \quad (2.16)$$

Take $(\alpha)^{(a)}, (\alpha')^{(a)} \in \mathcal{L}_{0,n}^{(a)}$ and $(\beta)^{(b)}, (\beta')^{(b)} \in \mathcal{L}_{0,n}^{(b)}$ with $a < b$. Then the product of $\mathcal{L}_{0,n}$ is given by the following formula (see [18, Section 6]):

$$\begin{aligned} ((\alpha)^{(a)} \otimes (\beta)^{(b)})((\alpha')^{(a)} \otimes (\beta')^{(b)}) &= \sum_{(R^1), \dots, (R^4)} (\alpha(S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha' \triangleleft R_{(1)}^1 R_{(1)}^2))^{(a)} \\ &\quad \otimes ((S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4) \beta')^{(b)}, \end{aligned} \quad (2.17)$$

where $R^i = \sum_{(R^i)} R_{(1)}^i \otimes R_{(2)}^i$, $i \in \{1, 2, 3, 4\}$, are expansions of four copies of $R \in \mathbb{U}_q^{\otimes 2}$, and on the right-hand side the product is componentwise that of $\mathcal{L}_{0,1}$. Later we will use the fact that the product of $\mathcal{L}_{0,n}$ is obtained from the standard (componentwise) product of $\mathcal{L}_{0,1}^{\otimes n}$ by a process that may be inverted. Indeed, (2.17) can be rewritten as

$$((\alpha)^{(a)} \otimes (\beta)^{(b)})((\alpha')^{(a)} \otimes (\beta')^{(b)}) = \sum_{(F)} (\alpha)^{(a)} ((\alpha')^{(a)} \cdot F_{(2)}) \otimes ((\beta)^{(b)} \cdot F_{(1)}) (\beta')^{(b)}, \quad (2.18)$$

where $F = \sum_{(F)} F_{(1)} \otimes F_{(2)} := (\Delta \otimes \Delta)(R')$, and the symbol “ \cdot ” stands for the right action of $\mathbb{U}_q^{\otimes 2}$ on $\mathcal{L}_{0,1}$ that may be read from (2.17). The tensor F is known as a twist. Then, by replacing F with its inverse $\bar{F} = (\Delta \otimes \Delta)(R'^{-1})$, one can express the product of $\mathcal{L}_{0,1}^{\otimes n}$ in terms of the product of $\mathcal{L}_{0,n}$ by

$$(\alpha)^{(a)} (\alpha')^{(a)} \otimes (\beta)^{(b)} (\beta')^{(b)} = \sum_{(\bar{F})} ((\alpha)^{(a)} \otimes ((\beta)^{(b)} \cdot \bar{F}_{(1)})) ((\alpha')^{(a)} \cdot \bar{F}_{(2)}) \otimes (\beta')^{(b)}. \quad (2.19)$$

We call *quantum moduli algebra* and denote by $\mathcal{M}_{0,n} = \mathcal{L}_{0,n}^{U_q}$ the subalgebra of $\mathcal{L}_{0,n}$ formed by the U_q -invariant elements.

The map Φ_1 can be extended to $\mathcal{L}_{0,n}$ as follows. Consider the following action of U_q on the tensor product algebra $U_q^{\otimes n}$, which extends ad^r on U_q :

$$\text{ad}_n^r(y)(x) = \sum_{(y)} \Delta^{(n)}(S(y_{(1)})) x \Delta^{(n)}(y_{(2)})$$

for all $y \in U_q$, $x \in U_q^{\otimes n}$. This action gives $U_q^{\otimes n}$ a structure of right U_q -module algebra. In [1], Alekseev introduced a morphism of U_q -module algebras $\Phi_n: \mathcal{L}_{0,n} \rightarrow U_q^{\otimes n}$ which extends Φ_1 . In [18, Proposition 6.7], we showed that Φ_n affords isomorphisms

$$\Phi_n: \mathcal{L}_{0,n} \rightarrow (U_q^{\otimes n})^{\text{lf}}, \quad \Phi_n: \mathcal{M}_{0,n} \rightarrow (U_q^{\otimes n})^{U_q}, \quad (2.20)$$

where $(U_q^{\otimes n})^{\text{lf}}$ is the set of ad_n^r -locally finite elements of $U_q^{\otimes n}$. We call Φ_n the *Alekseev map*; we do not recall here the definition of Φ_n , for we will not use it. It is a key argument of the proof of (2.20) that the set of locally finite elements of $U_q^{\otimes n}$ for $(\text{ad}^r)^{\otimes n} \circ \Delta^{(n)}$ coincides with $(U_q^{\text{lf}})^{\otimes n}$; this follows from the main result of [72]. Using that the map

$$\psi_n = \Phi_n \circ (\Phi_1^{-1})^{\otimes n}: (U_q^{\text{lf}})^{\otimes n} \rightarrow (U_q^{\otimes n})^{\text{lf}} \quad (2.21)$$

intertwines the actions $(\text{ad}^r)^{\otimes n} \circ \Delta^{(n-1)}$ and ad_n^r , we deduced that $\text{Im}(\Phi_n) = (U_q^{\otimes n})^{\text{lf}}$.

Remark 2.4. We have $(U_q^{\text{lf}})^{\otimes n} \neq (U_q^{\otimes n})^{\text{lf}}$ and in fact there is not even an inclusion. Indeed, let $\Omega = (q - q^{-1})^2 FE + qK + q^{-1}K^{-1}$ be the Casimir element of $U_q(\mathfrak{sl}_2)$. We trivially have $\Delta(\Omega) \in (U_q^{\otimes 2})^{\text{lf}}$ but

$$\Delta(\Omega) = (q - q^{-1})^2 (K^{-1}E \otimes FK + F \otimes E) + \Omega \otimes K + K^{-1} \otimes \Omega - (q + q^{-1})K^{-1} \otimes K$$

and therefore $\Delta(\Omega) \notin (U_q^{\text{lf}})^{\otimes 2}$, since $K \notin U_q^{\text{lf}}$ (see, e.g., Theorem 2.1(2)). This reflects the fact that U_q^{lf} is only a right coideal of U_q (and not a subcoalgebra).

As in Remark 2.2, denote by $C(\mu)$, $\mu \in P^+$, the linear subspace of $\mathcal{L}_{0,1}$ generated by the matrix coefficients of V_μ . For every tuple $[\mu] = (\mu_1, \dots, \mu_n) \in P_+^n$ put

$$C([\mu]) = C(\mu_1) \otimes \cdots \otimes C(\mu_n). \quad (2.22)$$

Then $\mathcal{L}_{0,n} = \bigoplus_{[\mu] \in P_+^n} C([\mu])$. Each space $C([\mu])$ is a finite-dimensional U_q -module under the action coad_n^r , whence it is completely reducible. Therefore, $\mathcal{L}_{0,n} = \mathcal{M}_{0,n} \oplus I$ as U_q -modules, where I is the sum of nontrivial isotypical components of $\mathcal{L}_{0,n}$. The $\mathbb{C}(q)$ -linear projection map

$$\mathcal{R}: \mathcal{L}_{0,n} \rightarrow \mathcal{M}_{0,n}, \quad \text{Ker}(\mathcal{R}) = I \quad (2.23)$$

is called the *Reynolds operator*. For all $\alpha \in \mathcal{M}_{0,n}$, $\beta \in \mathcal{L}_{0,n}$ it satisfies $\mathcal{R}(\alpha\beta) = \alpha\mathcal{R}(\beta)$. This property will be crucial in the sequel, so let us recall a (classical) proof of it. We can write $\beta = \mathcal{R}(\beta) + \gamma$ with $\gamma \in I$, and then we have to show $\alpha\gamma \in I$. We can reduce to the case where γ is contained in a simple summand V of I . Multiplication by the invariant element α yields a surjective map $V \rightarrow \alpha V$, which is a morphism of U_q -modules. Since V is simple, it is either the 0 map, or an isomorphism. In either cases it follows $\alpha V \subset I$ (in fact the first case cannot happen, for $\mathcal{L}_{0,n}$ has no nontrivial zero divisors, as explained after (2.25)).

We can formulate the Reynolds operator in the following way. Recall that \mathcal{O}_q has a unique left (or right, or 2-sided) Haar integral, that is a linear map $h: \mathcal{O}_q \rightarrow \mathbb{C}(q)$ such that

$$h(1) = 1 \quad \text{and} \quad (\text{id} \otimes h)\Delta(\alpha) = h(\alpha)1, \quad \forall \alpha \in \mathcal{O}_q.$$

(See, e.g., [35, Proposition 13.3.6].) It vanishes on all matrix coefficients except the one of the trivial representation, to which it gives the value 1. Denote by $\Delta_{\mathcal{L}}: \mathcal{L}_{0,n} \rightarrow \mathcal{L}_{0,n} \otimes \mathcal{O}_q$ the right coaction dual to the action coad_n^r of U_q on $\mathcal{L}_{0,n}$. Then, in analogy with the formula of the averaging operator $\mathcal{C}^\infty(G) \rightarrow \mathcal{C}^\infty(G)^G$, $f \rightarrow [f] = \int_G f(g^{-1} \cdot g) d\mu(g)$, for a locally compact group G with Haar measure $d\mu(g)$, it is straightforward that

$$\mathcal{R} = (\text{id} \otimes h)\Delta_{\mathcal{L}}. \quad (2.24)$$

Note that the complete reducibility of $\mathcal{L}_{0,n}$ discussed after (2.22) follows also from Theorem 2.1 (1), since by (2.21) we have an isomorphism of U_q -modules

$$\mathcal{L}_{0,n} \xrightarrow{\Phi_n} (U_q(\mathfrak{g})^{\otimes n})^{\text{lf}} \xrightarrow{\psi_n^{-1}} U_q^{\text{lf}}(\mathfrak{g})^{\otimes n},$$

where lf means respectively locally finite for the action ad_n^r of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})^{\otimes n}$, and locally finite for the action ad^r of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$. An explicit basis of $\mathcal{M}_{0,n}$ is described in [18, Proposition 6.22].

Finally, let us point out here two important consequences of (2.20). First, Φ_n yields isomorphisms between centers, $\mathcal{Z}(\mathcal{L}_{0,n}) \cong \mathcal{Z}(U_q)^{\otimes n}$ and $\mathcal{Z}(\mathcal{L}_{0,n}^{U_q}) \cong \mathcal{Z}((U_q^{\otimes n})^{U_q})$, where one can show that [18, Lemma 6.29]

$$\mathcal{Z}((U_q^{\otimes n})^{U_q}) \cong \Delta^{(n)}(\mathcal{Z}(U_q)) \bigotimes_{\mathbb{C}(q)} \mathcal{Z}(U_q)^{\otimes n}. \quad (2.25)$$

Second, $\mathcal{L}_{0,n}$ (and therefore $\mathcal{M}_{0,n}$) has no nontrivial zero divisors because of the isomorphisms $\Phi_n: \mathcal{L}_{0,n} \rightarrow (U_q^{\otimes n})^{\text{lf}} \subset U_q^{\otimes n}$ and $U_q^{\otimes n} \cong U_q(\mathfrak{g}^{\oplus n})$, and the fact that $U_q(\mathfrak{g}^{\oplus n})$ has no nontrivial zero divisors (proved, e.g., in [39]).

2.2 Integral forms and specializations

Let $A = \mathbb{C}[q, q^{-1}]$. We call integral form of a (Hopf) $\mathbb{C}(q)$ -algebra H a (Hopf) A -subalgebra ${}_A H$ such that the canonical map ${}_A H \otimes_A \mathbb{C}(q) \rightarrow H$ is an isomorphism. Note that the standard notion of integral form of $\mathbb{C}(q)$ -algebra uses $\mathbb{Z}[q, q^{-1}]$ instead of $\mathbb{C}[q, q^{-1}]$; our choice is made for simplicity ($\mathbb{C}[q, q^{-1}]$ is a principal ideal domain, whereas $\mathbb{Z}[q, q^{-1}]$ is not).

2.2.1 Definitions

The *unrestricted* integral form of U_q is the A -subalgebra $U_A = U_A(\mathfrak{g})$ introduced by De Concini–Kac–Procesi in [42, Section 12] (and in a differently normalized form in [39, 40]). It is the smallest A -subalgebra of U_q which contains the elements $(i = 1, \dots, m)$

$$\bar{E}_i = (q_i - q_i^{-1})E_i, \quad \bar{F}_i = (q_i - q_i^{-1})F_i, \quad L_i, \quad L_i^{-1} \quad (2.26)$$

and is stable under the action of $\mathcal{B}(\mathfrak{g})$ given by the Lusztig automorphisms (see (2.1)). Recall the root vectors E_{β_k}, F_{β_k} defined in (2.1). Let us put $q_{\beta} := q^{(\beta, \beta)/2}$. The algebra U_A is a free A -module with basis the monomials $\bar{E}_{\beta_1}^{p_1} \dots \bar{E}_{\beta_N}^{p_N} K_{\lambda} \bar{F}_{\beta_N}^{n_N} \dots \bar{F}_{\beta_1}^{n_1}$, where $\lambda \in P$ and we set

$$\bar{E}_{\beta_k} = (q_{\beta_k} - q_{\beta_k}^{-1})E_{\beta_k}, \quad \bar{F}_{\beta_k} = (q_{\beta_k} - q_{\beta_k}^{-1})F_{\beta_k}.$$

We denote $U_A^{\text{lf}} := U_A \cap U_q^{\text{lf}}$. The unrestricted integral form of U_q^{ad} is defined similarly, as the smallest A -subalgebra $U_A^{\text{ad}} \subset U_A$ which contains the elements \bar{E}_i, \bar{F}_i and $K_i^{\pm 1}$, for $i = 1, \dots, m$, and is stable under the Lusztig action of $\mathcal{B}(\mathfrak{g})$.

For β a positive root, we define the divided powers

$$E_{\beta}^{(k)} = \frac{E_{\beta}^k}{[k]_{q_{\beta}}!}, \quad F_{\beta}^{(k)} = \frac{F_{\beta}^k}{[k]_{q_{\beta}}!}, \quad k \in \mathbb{N}.$$

The Lusztig *restricted* integral form of U_q^{ad} [82, 83] (see also [35, Chapter 9.3]) is the A -subalgebra U_A^{res} generated by the elements $(i = 1, \dots, m, k \in \mathbb{N}^*)$

$$E_i^{(k)} = \frac{E_i^k}{[k]_{q_i}!}, \quad F_i^{(k)} = \frac{F_i^k}{[k]_{q_i}!}, \quad K_i, \quad K_i^{-1}.$$

The algebra U_A^{res} is a free A -module with Poincaré–Birkhoff–Witt (PBW) basis

$$E_{\beta_1}^{(p_1)} \dots E_{\beta_N}^{(p_N)} \prod_{i=1}^m K_i^{\sigma_i} [K_i; t_i]_{q_i} F_{\beta_N}^{(n_N)} \dots F_{\beta_1}^{(n_1)},$$

where $\sigma_i \in \{0, 1\}$, $n_i, p_i, t_i \in \mathbb{N}$, and we set $[K_i; 0]_{q_i} := 1$ and

$$[K_i; t]_{q_i} = \prod_{s=1}^t \frac{K_i q_i^{-s+1} - K_i^{-1} q_i^{s-1}}{q_i^s - q_i^{-s}}.$$

The integral forms $U_A(\mathfrak{h}), U_A(\mathfrak{b}_{\pm})$ and $U_A^{\text{res}}(\mathfrak{h}), U_A^{\text{res}}(\mathfrak{b}_{\pm})$ associated to the subalgebras $\mathfrak{h}, \mathfrak{b}_{\pm} \subset \mathfrak{g}$ are the subalgebras of U_A and U_A^{res} , respectively, defined in the obvious way. For instance, the ‘‘Cartan’’ subalgebra $U_A^{\text{res}}(\mathfrak{h}) = U_q(\mathfrak{h}) \cap U_A^{\text{res}}$ is generated as a A -module by the elements $\prod_{i=1}^m K_i^{\sigma_i} [K_i; t_i]_{q_i}$.

Denote by \mathcal{C}_A the category of U_A^{res} -modules of *type 1*, i.e., free A -modules of finite rank which have a basis where the elements K_i act diagonally with eigenvalues of the form q_i^k , $k \in \mathbb{Z}$ (in general, finiteness of the rank imposes eigenvalues of the form $\pm q_i^k$, $k \in \mathbb{Z}$). The category \mathcal{C}_A is a rigid and tensor category. It is not semisimple, and this makes the study of \mathcal{C}_A a complicated task; for this, see [18], and Section 2.2.2 below. Every type 1 finite-dimensional simple U_q -module V_{μ} , $\mu \in P_+$, has a U_A^{res} -invariant full A -sublattice, that we denote by ${}_A V_{\mu}$. These U_A^{res} -modules form the simple objects of \mathcal{C}_A . Moreover, $\mathcal{C}_A \otimes \mathbb{C}[q^{1/D}, q^{-1/D}]$ is a ribbon category (see Section 2.3).

The *integral* quantum function Hopf algebra $\mathcal{O}_A = \mathcal{O}_A(G)$ is the (type 1) restricted dual of U_A^{res} , that is, the A -span of the matrix coefficients $x \mapsto v^i(\pi_V(x)v_i)$, $x \in U_A^{\text{res}}$, for every module V in \mathcal{C}_A , where (v_i) is an A -basis of V and (v^i) the dual A -basis of the dual module V^* (compare with the definition of \mathcal{O}_q). We can also regard \mathcal{O}_A as the set of A -linear maps $f: U_A^{\text{res}} \rightarrow A$

such that $\text{Ker}(f)$ contains a cofinite two sided ideal I , and $\prod_{s=-r}^r (K_i - q_i^s) \in I$ for some $r \in \mathbb{N}$ and every i . Because of the inclusions of $U_A^{\text{res}}(\mathfrak{h})$, $U_A^{\text{res}}(\mathfrak{n}_{\pm})$, $U_A^{\text{res}}(\mathfrak{b}_{\pm})$ in U_A^{res} , there are Hopf epimorphisms from \mathcal{O}_A to the A -duals of these subalgebras, that we denote by $\mathcal{O}_A(T_G)$, $\mathcal{O}_A(U_{\pm})$ and $\mathcal{O}_A(B_{\pm})$, respectively.

The algebra \mathcal{O}_A has been introduced by Lusztig in [82, 83]. It is an integral form of \mathcal{O}_q , so $\mathcal{O}_q = \mathcal{O}_A \otimes_A \mathbb{C}(q)$.

\mathcal{O}_A is also the restricted dual of the integral form $\Gamma = \Gamma(\mathfrak{g})$ of U_q^{ad} introduced by De Concini–Lyubashenko in [41, Sections 2 and 3]; Γ is the A -subalgebra of U_q^{ad} generated by the elements $(i = 1, \dots, m)$

$$E_i^{(k)} = \frac{E_i^k}{[k]_{q_i}!}, \quad F_i^{(k)} = \frac{F_i^k}{[k]_{q_i}!}, \quad (K_i; t)_{q_i} = \prod_{s=1}^t \frac{K_i q_i^{-s+1} - 1}{q_i^s - 1}, \quad K_i^{-1},$$

where $k \in \mathbb{N}$, $t \in \mathbb{N}$ (setting $(K_i; 0)_{q_i} = 1$ by convention). Note that the definition of Γ is less symmetric than that of U_A^{res} . However, Γ contains the elements K_i , and the commutation relations between the generators $E_i^{(k)}$, $F_i^{(k)}$ imply that the symmetrized elements $[K_i; t]_{q_i}$ belong to Γ . In fact, let us denote $\Gamma(\mathfrak{h}) = U_q(\mathfrak{h}) \cap \Gamma$ and $\Gamma(\mathfrak{b}_{\pm}) = U_q(\mathfrak{b}_{\pm}) \cap \Gamma$. It is proved in [41, Theorem 3.1] that $\Gamma(\mathfrak{h})$ contains $U_A^{\text{res}}(\mathfrak{h})$ and that the elements $\prod_{i=1}^m K_i^{-\sigma(t_i)} (K_i; t_i)_{q_i}$, $t_i \in \mathbb{N}$, where $\sigma(t)$ is the integer part of $t/2$, is an A -basis of $\Gamma(\mathfrak{h})$. A PBW basis of Γ is formed by the monomials

$$E_{\beta_1}^{(p_1)} \dots E_{\beta_N}^{(p_N)} \prod_{i=1}^m K_i^{-\sigma(t_i)} (K_i; t_i)_{q_i} F_{\beta_N}^{(n_N)} \dots F_{\beta_1}^{(n_1)}.$$

The inclusion $U_A^{\text{res}} \subset \Gamma$ is strict, for the elements $(K_i; t)_{q_i}$, $t \neq 0$, do not belong to U_A^{res} . However, the restriction functor $\mathcal{C}_{\Gamma} \rightarrow \mathcal{C}_A$ is obviously an equivalence, where \mathcal{C}_{Γ} is the category of Γ -modules of *type 1*, i.e., free A -modules of finite rank which have a basis where the elements K_i act diagonally with eigenvalues of the form q_i^k , $k \in \mathbb{Z}$. Therefore, we can identify the two categories, and \mathcal{O}_A with the (type 1) restricted dual of Γ . We will thus consider the U_A^{res} -modules ${}_A V_{\mu}$, $\mu \in P_+$, equally as Γ -modules. We will sometimes use Γ instead of U_A^{res} in order to make direct the connection with results of De Concini–Lyubashenko about the integral pairings π_A^{\pm} considered in Section 2.3.

The *integral* form $\mathcal{L}_{0,1}^A$ of $\mathcal{L}_{0,1}$ is defined as the U_A^{res} -module \mathcal{O}_A endowed with the product of $\mathcal{L}_{0,1}$. The *integral* form $\mathcal{L}_{0,n}^A$ of $\mathcal{L}_{0,n}$ is the braided tensor product of n copies of $\mathcal{L}_{0,1}^A$; in particular, $\mathcal{L}_{0,n}^A = \mathcal{O}_A^{\otimes n}$ as U_A^{res} -modules. That the products of $\mathcal{L}_{0,1}$ and $\mathcal{L}_{0,n}$ are well defined over A was shown in [18, Proposition 6.9].

The *integral* quantum moduli algebra is

$$\mathcal{M}_{0,n}^A := (\mathcal{L}_{0,n}^A)^{U_A^{\text{res}}} = (\mathcal{L}_{0,n}^A)^{U_A}.$$

Finally, given $q = \epsilon \in \mathbb{C}^{\times}$ we define the *specializations* U_{ϵ} , Γ_{ϵ} , \mathcal{O}_{ϵ} , $\mathcal{L}_{0,n}^{\epsilon}$ and $\mathcal{M}_{0,n}^{A,\epsilon}$ as the \mathbb{C} -algebras obtained by tensoring U_A , Γ , \mathcal{O}_A , $\mathcal{L}_{0,n}^A$ and $\mathcal{M}_{0,n}^A$ respectively with \mathbb{C}_{ϵ} , the A -module \mathbb{C} where q acts by multiplication by ϵ . Each one can also be defined as the quotient by the ideal generated by $q - \epsilon$. We find convenient to use the notations

$$(U_A^{\otimes n})_{\epsilon}^{U_A} := (U_A^{\otimes n})^{U_A} \otimes_A \mathbb{C}_{\epsilon}, \quad (U^{\otimes n})_{\epsilon}^{\text{lf}} := (U_A^{\otimes n})^{\text{lf}} \otimes_A \mathbb{C}_{\epsilon}. \quad (2.27)$$

Let us stress here that when ϵ is a root of unity, taking the locally finite part and taking the specialization at ϵ are non commuting operations. Indeed, as shown by Theorem 2.27 below, U_{ϵ} is finite over $\mathcal{Z}_0(U_{\epsilon})$ and therefore all its elements are locally finite for ad^r ; on another hand $U_{\epsilon}^{\text{lf}} = U_A^{\text{lf}} \otimes_A \mathbb{C}_{\epsilon}$ does not contain the elements L_i .

Similarly, taking invariants and taking the specialization at ϵ are non commuting operations when ϵ is a root of unity: indeed, it is easily checked that in this case $(U_A^{\otimes n})_\epsilon^{U_A}$ and $(U_\epsilon^{\otimes n})^{U_\epsilon}$, or $\mathcal{M}_{0,n}^{A,\epsilon} = \mathcal{M}_{0,n}^A \otimes_A \mathbb{C}_\epsilon$ and $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$, are distinct spaces. When ϵ is a root of unity, we will not consider the algebras $\mathcal{M}_{0,n}^{A,\epsilon}$ in this paper.

Arguments similar to those mentioned at the end of Section 2.1 imply that the algebras $\mathcal{L}_{0,n}^A$, $\mathcal{M}_{0,n}^A$ and $\mathcal{L}_{0,n}^{\epsilon'}$, $\mathcal{M}_{0,n}^{A,\epsilon'}$, $\epsilon' \in \mathbb{C}^\times$, have no nontrivial zero divisors (see [18, Propositions 6.11 and 6.30]).

2.2.2 Canonical bases and modified quantum groups

Because the category \mathcal{C}_A is not semisimple, it is not clear from the above definition of \mathcal{O}_A whether or not it is a finitely generated algebra, if $\mathcal{M}_{0,n}^A$ is a direct summand of the A -module $\mathcal{L}_{0,n}^A$, or if the projection map (2.8) may be refined to a morphism between underlying A -modules.

Such properties, using the appropriate formalism developed by Kashiwara–Lusztig, indeed hold true, and will play a key role later. We state them precisely in Proposition 2.10, Theorem 2.15 and Proposition 2.12. These results are consequences of the existence of an A -basis of \mathcal{O}_A with favourable properties, which implies in particular that \mathcal{O}_A is a free A -module. In order to introduce this A -basis it is necessary to consider a variant of U_q^{ad} introduced by Lusztig [83], called *modified quantum group*, and use the Kashiwara–Lusztig theory of canonical bases [65, 66, 67, 83]. We are going to recall the background material step by step.

The Lusztig *modified quantum group* is the $\mathbb{C}(q)$ -algebra $\dot{\mathbf{U}}$ obtained by replacing $U_q^{\text{ad}}(\mathfrak{h})$ with the direct sum of infinitely many one-dimensional algebras, generated by orthogonal idempotents 1_λ indexed by the elements λ of the weight lattice P [83, Chapter 23]. Namely, as a vector space $\dot{\mathbf{U}} = \bigoplus_{\lambda', \lambda'' \in P} \lambda' \dot{\mathbf{U}}_{\lambda''}$, where

$$\lambda' \dot{\mathbf{U}}_{\lambda''} = U_q^{\text{ad}} \left/ \left(\sum_{\alpha \in Q} (K_\alpha - q^{(\alpha, \lambda')}) U_q^{\text{ad}} + \sum_{\alpha \in Q} U_q^{\text{ad}} (K_\alpha - q^{(\alpha, \lambda'')}) \right) \right.$$

Denote by $\pi_{\lambda', \lambda''}: U_q^{\text{ad}} \rightarrow \lambda' \dot{\mathbf{U}}_{\lambda''}$ the canonical projection. The product of $\dot{\mathbf{U}}$ is given by $\pi_{\lambda'_1, \lambda''_1}(s) \pi_{\lambda'_2, \lambda''_2}(t) = \pi_{\lambda'_1, \lambda''_2}(st)$ if $\lambda''_1 = \lambda'_2$ and zero otherwise. Set $1_\lambda := \pi_{\lambda, \lambda}(1)$. The algebra $\dot{\mathbf{U}}$ has not unit, but the family $(1_\lambda)_{\lambda \in P}$ can be regarded as a substitute of it. Denote by Δ the collection of maps

$$\Delta_{\lambda'_1, \lambda'_2, \lambda''_1, \lambda''_2}: \lambda'_1 + \lambda'_2 \dot{\mathbf{U}}_{\lambda''_1 + \lambda''_2} \rightarrow \lambda'_1 \dot{\mathbf{U}}_{\lambda''_1} \otimes \lambda'_2 \dot{\mathbf{U}}_{\lambda''_2}$$

such that

$$\Delta_{\lambda'_1, \lambda'_2, \lambda''_1, \lambda''_2} \pi_{\lambda'_1 + \lambda'_2, \lambda''_1 + \lambda''_2} = (\pi_{\lambda'_1, \lambda''_1} \otimes \pi_{\lambda'_2, \lambda''_2}) \Delta_{U_q^{\text{ad}}}, \quad (2.28)$$

where $\Delta_{U_q^{\text{ad}}}$ is the coproduct of U_q^{ad} . We can regard Δ as a (categorically completed) coproduct $\Delta: \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}^{\otimes 2}$. There is a natural structure of U_q^{ad} -bimodule on $\dot{\mathbf{U}}$, defined by

$$t' \pi_{\lambda', \lambda''}(s) t'' = \pi_{\lambda' + \nu', \lambda'' - \nu''}(t' s t'') \quad (2.29)$$

for all $s \in U_q^{\text{ad}}$ and all elements $t', t'' \in U_q^{\text{ad}}$ of respective weights ν', ν'' . This structure affords a triangular decomposition of $\dot{\mathbf{U}}$: given bases $\{b^\pm\}$ of $U_q^{\text{ad}}(\mathfrak{n}_\pm)$, the set of elements $b^+ 1_\lambda b^-$ (or $b^- 1_\lambda b^+$, or $b^+ b^- 1_\lambda$), where $\lambda \in P$, is a basis of $\dot{\mathbf{U}}$.

Given any U_q^{ad} -module X of type 1, and any weight subspace $X^\lambda \subset X$ of weight $\lambda \in P$, one can define the action of an element $u 1_\lambda \in \dot{\mathbf{U}}$, $u \in U_q^{\text{ad}}$, on X as the projection onto X^λ followed by the action of u . This way, one can identify the category \mathcal{C} with the one of finite-dimensional

unital $\dot{\mathbf{U}}$ -modules, where unital means that all elements 1_λ act as 0 but a finite number of them, and $\sum_{\lambda \in P} 1_\lambda$ acts as the identity. It is proved in [83, Section 29.5.1], that

$$\mathcal{O}_q = \left\{ f: \dot{\mathbf{U}} \rightarrow \mathbb{C}(q) \mid \begin{array}{l} f \text{ is } \mathbb{C}(q)\text{-linear and vanishes on some} \\ \text{two-sided ideal of finite codimension of } \dot{\mathbf{U}} \end{array} \right\}.$$

There is an analogous realization of \mathcal{O}_A , of the form (see [83, Sections 23.2 and 29.5.2], and [84])

$$\mathcal{O}_A = \left\{ f: \dot{\mathbf{U}}_A \rightarrow A \mid \begin{array}{l} f \text{ is } A\text{-linear and vanishes on some} \\ \text{two-sided ideal of finite corank of } \dot{\mathbf{U}}_A \end{array} \right\},$$

where $\dot{\mathbf{U}}_A$ is the A -subalgebra of $\dot{\mathbf{U}}$ generated by the elements $E_i^{(k)} 1_\lambda$ and $F_i^{(k)} 1_\lambda$, for all $i \in \{1, \dots, m\}$, $k \in \mathbb{N}$ and $\lambda \in P$. It is a U_A^{res} -subbimodule of $\dot{\mathbf{U}}$, and the coproduct restricts to a map $\Delta: \dot{\mathbf{U}}_A \rightarrow \dot{\mathbf{U}}_A^{\otimes 2}$. The above identification of the category \mathcal{C} with the one of finite-dimensional unital $\dot{\mathbf{U}}$ -modules yields an identification of the category \mathcal{C}_A of U_A^{res} -modules of type 1 with the category of $\dot{\mathbf{U}}_A$ -modules of finite rank.

The key advantage of this realization of \mathcal{O}_A is that $\dot{\mathbf{U}}_A$ can be equipped with a canonical A -basis $\dot{\mathbf{B}}$. The construction of $\dot{\mathbf{B}}$ is described in [83, Chapter 25]. It relies on the Kashiwara–Lusztig *canonical basis* of $U_A^{\text{res}}(\mathfrak{n}_-)$. This last basis, denoted by \mathbf{B}^- , is defined in [83, Chapter 14], and [65] (a review can be found in [35, Chapter 14]). It enjoys the following nice properties. Denote by $\bar{\cdot}: \mathbb{C}(q) \rightarrow \mathbb{C}(q)$ the field involution such that $\bar{q} = q^{-1}$, and by $\bar{\cdot}: U_q^{\text{ad}} \rightarrow U_q^{\text{ad}}$ the homomorphism of \mathbb{C} -algebras such that

$$\bar{E}_i = E_i, \quad \bar{F}_i = F_i, \quad \bar{K}_\lambda = K_{-\lambda}, \quad \overline{f\bar{x}} = \bar{f}x$$

for all $f \in \mathbb{C}(q)$, $x \in U_q^{\text{ad}}$ (\bar{E}_i and \bar{F}_i above, which will not appear elsewhere, should not be confused with the normalized elements in (2.26)). The map $\bar{\cdot}$ yields a \mathbb{C} -algebra homomorphism $\bar{\cdot}: \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. Then, we have

- (1) the elements of \mathbf{B}^- are weight vectors under the adjoint action of $U_q^{\text{ad}}(\mathfrak{h})$;
- (2) for every $b \in \mathbf{B}^-$, $\bar{b} = b$;
- (3) for every $b, b' \in \mathbf{B}^-$, $bb' = \sum_{b'' \in \mathbf{B}^-} N_{b''}^{bb'} b''$ where $N_{b''}^{bb'} \in \mathbb{Z}[q, q^{-1}]$;
- (4) for every $b, b' \in \mathbf{B}^-$, $\Delta(b) = \sum_{b', b'' \in \mathbf{B}^-} C_{b'b''}^b b' \otimes b''$ where $C_{b'b''}^b \in \mathbb{Z}[q, q^{-1}]$;
- (5) for every $\mu \in P^+$, denoting by v_μ the highest weight vector of the U_A^{res} -module ${}_A V_\mu$, the elements bv_μ which are non-zero, where $b \in \mathbf{B}^-$, form an A -basis of ${}_A V_\mu$.

When \mathfrak{g} is simply laced, the coefficients $N_{b''}^{bb'}$ and $C_{b'b''}^b$ belong to $\mathbb{N}[q, q^{-1}]$ [83, Theorem 14.3.13]. In the case of $\mathfrak{g} = \mathfrak{sl}_2$, the elements of \mathbf{B}^- are just the divided powers $F^{(k)}$, $k \in \mathbb{N}$. Formulas in terms of PBW basis elements are known also for $\mathfrak{g} = \mathfrak{sl}_3$ and \mathfrak{sl}_4 , and an algorithm exists in the general case (see [43] and the references therein).

Correspondingly to \mathbf{B}^- , the set $\mathbf{B}^+ = \omega(\mathbf{B}^-)$ is a basis of $U_A^{\text{res}}(\mathfrak{n}_+)$, where $\omega: U_q^{\text{ad}} \rightarrow U_q^{\text{ad}}$ is the ($\mathbb{C}(q)$ -linear) *Cartan automorphism*, defined by

$$\omega(E_i) = F_i, \quad \omega(F_i) = E_i, \quad \omega(K_i) = K_i^{-1}$$

for $i = 1, \dots, m$. The triangular decomposition of $\dot{\mathbf{U}}$ implies that the elements $b^+ 1_\lambda b'^-$, where $b^+ \in \mathbf{B}^+$, $b'^- \in \mathbf{B}^-$ and $\lambda \in P$, form a basis of $\dot{\mathbf{U}}$. They form in fact an A -basis of $\dot{\mathbf{U}}_A$, and its elements are fixed by the involution $\bar{\cdot}: \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$.

Lusztig has constructed another A -basis of $\dot{\mathbf{U}}_A$, denoted $\dot{\mathbf{B}}$, and called the *canonical basis* of $\dot{\mathbf{U}}_A$. It satisfies numerous properties that we now review. Its elements are denoted by $b \diamond_\lambda b'$,

where $b, b' \in \mathbf{B}^-$ and $\lambda \in P$, and are related to the elements $b^+b'^-1_\lambda$, where $b^+ := \omega(b)$ and $b'^- := b'$, by a specific trigonal change of basis with coefficients in A . Although we always have $b^+1_\lambda, b'^-1_\lambda \in \mathbf{B}$, to our knowledge explicit formulas of the elements of \mathbf{B} as linear combinations of elements $b^+1_\lambda b'^-$ or $b'^-1_\lambda b^+$ are known only for $\mathfrak{g} = \mathfrak{sl}_2$ or \mathfrak{sl}_3 (see [83, Section 25.3] and [37]). In the former case, identifying P with \mathbb{Z} and Q with $2\mathbb{Z}$ the canonical basis \mathbf{B} is formed by the elements

$$E^{(k)}1_{-n}F^{(l)} \quad \text{and} \quad F^{(l)}1_nE^{(k)}, \quad k, l, n \in \mathbb{N}, \quad n \geq k + l,$$

where $E^{(k)}1_{-n}F^{(l)} = F^{(l)}1_nE^{(k)}$ for $n = k + l$.

We are going to review Lusztig's construction of \mathbf{B} , its canonical partition $\dot{\mathbf{B}} = \bigcup_{\lambda \in P_+} \dot{\mathbf{B}}[\lambda]$, the dual basis \mathbf{B}^* , and Kashiwara's approach to \mathbf{B}^* [66, 67]. The latter is stated in Theorem 2.6 below. At first we need to recall the notions of based module and balanced triple; for details on these notions we refer to [83, Chapter 27] and [66] (see also [68], [104, Sections 3.15 and 3.16], or [35, Chapter 14] for overviews).

Denote by $\mathcal{A}_0 \subset \mathbb{C}(q)$ the ring of rational functions regular at $q = 0$. By applying the involution $\bar{}$, put $\mathcal{A}_\infty = \overline{\mathcal{A}_0}$. Since \mathcal{A}_0 is the localization of $\mathbb{C}[q]$ at $q = 0$, we may regard \mathcal{A}_∞ as the localization of $\mathbb{C}[q^{-1}]$ at $q = \infty$.

Let us recall briefly the definition of crystal basis (see [65]). Denote by $U_q^{\text{ad}}(\mathfrak{g})_i$ the subalgebra of $U_q^{\text{ad}}(\mathfrak{g})$ generated by E_i, F_i and $K_i^{\pm 1}$; thus $U_q^{\text{ad}}(\mathfrak{g})_i$ is isomorphic to $U_{q_i}(\mathfrak{sl}_2)$. Let M be a U_q^{ad} -module of type 1. Denote M^ζ the subspace of M of weight $\zeta \in P$. For every $i = 1, \dots, m$, we can regard M as a $U_q^{\text{ad}}(\mathfrak{g})_i$ -module, so $M \cong \bigoplus_j V_{\lambda_j}$ for some simple $U_q^{\text{ad}}(\mathfrak{g})_i$ -modules V_{λ_j} . These being generated by primitive weight vectors, the PBW basis of $U_q^{\text{ad}}(\mathfrak{g})_i$ yields

$$M = \bigoplus_{\zeta \in P} \bigoplus_{0 \leq n \leq (\check{\alpha}_i, \zeta)} F_i^{(n)}(\text{Ker}(E_i) \cap M^\zeta).$$

The *Kashiwara operators* \tilde{e}_i, \tilde{f}_i are the endomorphisms of M defined by, for every $v \in \text{Ker}(E_i) \cap M^\zeta$ and $0 \leq n \leq (\check{\alpha}_i, \zeta)$,

$$\tilde{f}_i(F_i^{(n)}v) = F_i^{(n+1)}v, \quad \tilde{e}_i(F_i^{(n)}v) = F_i^{(n-1)}v.$$

A *crystal basis* of M at $q = 0$ consists of a pair $(\mathcal{L}, \mathcal{B})$, where

- \mathcal{L} is a free \mathcal{A}_0 -sublattice of M such that the canonical map $\mathcal{L} \otimes_{\mathcal{A}_0} \mathbb{C}(q) \rightarrow M$ is an isomorphism;
- \mathcal{B} is a basis of the \mathbb{C} -vector space $\mathcal{L}/q\mathcal{L}$;
- $\mathcal{L} = \bigoplus_{\zeta \in P} \mathcal{L}^\zeta$ and $\mathcal{B} = \coprod_{\zeta \in P} (\mathcal{B} \cap \mathcal{L}^\zeta / q\mathcal{L}^\zeta)$, where $\mathcal{L}^\zeta = \mathcal{L} \cap M^\zeta$;
- for every $i = 1, \dots, m$ the Kashiwara operators \tilde{e}_i, \tilde{f}_i preserve \mathcal{L} , and the induced maps on $\mathcal{L}/q\mathcal{L}$ send \mathcal{B} into $\mathcal{B} \cup \{0\}$, and satisfy $b' = \tilde{f}_i(b)$ if and only if $b = \tilde{e}_i(b')$ for every $b, b' \in \mathcal{B}$.

Crystal bases at $q = \infty$ are defined similarly, by replacing \mathcal{A}_0 with \mathcal{A}_∞ and q with q^{-1} .

A *based module* consists of a pair (M, B) where M is a U_q^{ad} -module of type 1 endowed with a $\mathbb{C}(q)$ -basis B such that the following conditions hold:

- (i) For every weight $\zeta \in P$, the set $B \cap M^\zeta$ is a basis of the weight subspace $M^\zeta \subset M$.
- (ii) The A -module ${}_A M$ generated by B is stable under U_A^{res} .

We will denote by \mathcal{L}_M the \mathcal{A}_0 -submodule of M generated by B , and by $\bar{\mathcal{L}}_M$ the \mathcal{A}_∞ -submodule of M generated by B .

- (iii) The \mathbb{C} -linear involution $\bar{} : M \rightarrow M$ defined by $\overline{fb} = \bar{f}b$ for all $f \in \mathbb{C}(q)$ and $b \in B$ is compatible with the action of U_q^{ad} in the sense that $\overline{x\bar{m}} = \bar{x}m$ for all $x \in U_q^{\text{ad}}, m \in M$.

- (iv) The \mathcal{A}_∞ -submodule $\bar{\mathcal{L}}_M$ of M together with the image of B in $\bar{\mathcal{L}}_M/q^{-1}\bar{\mathcal{L}}_M$ forms a crystal basis of M at $q = \infty$.

If (M, B) is a based module, we will denote by \bar{B} the image of B in $\bar{\mathcal{L}}_M/q^{-1}\bar{\mathcal{L}}_M$. From the notion of balanced triple that we recall now, denoting by \mathcal{B} the image of B in $\mathcal{L}_M/q\mathcal{L}_M$, we see that $(\mathcal{L}_M, \mathcal{B})$ is a crystal basis at $q = 0$.

Indeed, consider more generally a $\mathbb{C}(q)$ -vector space V , finite-dimensional or not, a sub- A -module ${}_A V$, a sub- \mathcal{A}_0 -module ${}_{\mathcal{A}_0} V$ and a sub- \mathcal{A}_∞ -module ${}_{\mathcal{A}_\infty} V$ satisfying the conditions (all isomorphisms being the canonical maps)

$$V \cong \mathbb{C}(q) \bigotimes_A {}_A V, \quad V \cong \mathbb{C}(q) \bigotimes_{\mathcal{A}_0} {}_{\mathcal{A}_0} V, \quad V \cong \mathbb{C}(q) \bigotimes_{\mathcal{A}_\infty} {}_{\mathcal{A}_\infty} V.$$

Consider the \mathbb{C} -vector space $E := {}_A V \cap {}_{\mathcal{A}_0} V \cap {}_{\mathcal{A}_\infty} V$. Then $({}_A V, {}_{\mathcal{A}_0} V, {}_{\mathcal{A}_\infty} V)$ is a *balanced triple* [65, 66] if the canonical maps

$$A \bigotimes_{\mathbb{C}} E \rightarrow {}_A V, \quad \mathcal{A}_0 \bigotimes_{\mathbb{C}} E \rightarrow {}_{\mathcal{A}_0} V, \quad \mathcal{A}_\infty \bigotimes_{\mathbb{C}} E \rightarrow {}_{\mathcal{A}_\infty} V \quad (2.30)$$

are isomorphisms. Equivalently, $({}_A V, {}_{\mathcal{A}_0} V, {}_{\mathcal{A}_\infty} V)$ is balanced if and only if the canonical map $E \rightarrow {}_{\mathcal{A}_0} V/q{}_{\mathcal{A}_0} V$ is an isomorphism, if and only if the canonical map $E \rightarrow {}_{\mathcal{A}_\infty} V/q^{-1}{}_{\mathcal{A}_\infty} V$ is an isomorphism [66, Lemma 2.1.1].

Given a based module (M, B) , the elements of B are weight vectors and $\bar{b} = b$ for every $b \in B$. Also, if an element $m \in {}_A M$ satisfies $\bar{m} = m$ and $m \in B + q^{-1}\bar{\mathcal{L}}_M$, then $m \in B$ (see [83, Section 27.1.5] for details on this fact). It follows that the canonical quotient map

$${}_A M \cap \mathcal{L}_M \cap \bar{\mathcal{L}}_M \rightarrow \bar{\mathcal{L}}_M/q^{-1}\bar{\mathcal{L}}_M \quad (2.31)$$

is an isomorphism of \mathbb{C} -vector spaces. This provides another way of viewing based modules: by (2.31), $({}_A M, \mathcal{L}_M, \bar{\mathcal{L}}_M)$ is a balanced triple, and by (2.30) the A -lattice ${}_A M$ is completely determined by the crystal base $(\bar{\mathcal{L}}_M, \bar{B})$. We will say that $(\bar{\mathcal{L}}_M, \bar{B})$ (or the corresponding crystal base $(\mathcal{L}_M, \mathcal{B})$ at $q = 0$) is *melted into* the based module (M, B) .

We will indifferently apply the notion of based module to finite-dimensional unital \dot{U} -modules, since they are equivalent to U_q^{ad} -modules of type 1.

Every module V_μ , $\mu \in P^+$, supports a structure of based module (see [83, Section 14.4.10] and [65]); the corresponding basis, called *canonical basis* and that we will denote by $\underline{\mathbf{B}}_\mu$, is formed by the elements $bv_\mu \in {}_A V_\mu$ which are non-zero, where $b \in \mathbf{B}^-$ and v_μ is the canonical highest weight vector of V_μ , corresponding to the coset of $1 \in U_q^{\text{ad}}(\mathfrak{n}_-)$ in the Verma module construction of V_μ . Note that the involution $\bar{\cdot} : V_\mu \rightarrow V_\mu$ defined by (iii) above is indeed an automorphism, for the space V_μ with action of U_q^{ad} defined by $x \cdot v := \bar{x}v$, for all $x \in U_q^{\text{ad}}$, $v \in V_\mu$, has highest weight μ , and is thus isomorphic to V_μ via the map $\bar{\cdot}$. The crystal base $(\mathcal{L}_\mu^{\text{low}}, \mathcal{B}_\mu^{\text{low}})$ at $q = 0$ is formed by the \mathcal{A}_0 -sublattice $\mathcal{L}_\mu^{\text{low}}$ of V_μ generated by $\underline{\mathbf{B}}_\mu$ (which is eventually the same as the \mathcal{A}_0 -sublattice generated by the vectors of the form $\tilde{f}_{i_1} \circ \cdots \circ \tilde{f}_{i_k}(v_\mu)$, where $i_1, \dots, i_k \in \{1, \dots, m\}$), and $\mathcal{B}_\mu^{\text{low}}$ is the set of non-zero images of these vectors in $\mathcal{L}_\mu^{\text{low}}/q\mathcal{L}_\mu^{\text{low}}$.

There is the following uniqueness result [65, Theorem 3].

Theorem 2.5. *Let M be a U_q^{ad} -module of type 1, and $(\mathcal{L}, \mathcal{B})$ a crystal base at $q = 0$ of M . Then there exists a $\mathbb{C}(q)$ -isomorphism $M \rightarrow \bigoplus_j V_{\lambda_j}$ by which $(\mathcal{L}, \mathcal{B})$ is \mathcal{A}_0 -isomorphic to $\bigoplus_j (\mathcal{L}_{\lambda_j}^{\text{low}}, \mathcal{B}_{\lambda_j}^{\text{low}})$.*

The based modules form a category. Given based modules (M, B) and (M', B') , a morphism of U_q^{ad} -modules $f : M \rightarrow M'$ is a morphism of based modules if

- (a) $f(b) \in B' \cup \{0\}$ for any $b \in B$;
- (b) $B \cap \text{Ker}(f)$ is a basis of $\text{Ker}(f)$.

The direct sum of based modules (M, B) and (M', B') is a based module $(M \oplus M', B \cup B')$; and a submodule M' of a based module (M, B) spanned over $\mathbb{C}(q)$ by a subset B' of B forms a based module (M', B') . The quotient module M/M' together with the image of $B \setminus B'$ is then a based module.

The tensor product of based modules (M, B) , (M', B') is also defined. Namely, consider the \mathbb{C} -linear map $\Psi: M \otimes M' \rightarrow M \otimes M'$ defined by

$$\Psi(m \otimes m') = \hat{R}^{-1}(\bar{m} \otimes \bar{m}'),$$

where $\hat{R} = \Theta^{-1}R$, see (2.4) (note that, as we use the coproduct opposite to [83] our quasi- R -matrix is \hat{R}^{-1}). It can be checked that Ψ is an involution compatible with the action of $\dot{\mathbf{U}}$ in the sense of (iii) above in the definition of based module. Moreover, denote by $\mathcal{L}_{M, M'}$ the $\mathbb{C}[q^{-1}]$ -submodule of $M \otimes M'$ spanned by the basis elements $b \otimes b'$, where $b \in B$, $b' \in B'$. It is shown in [83, Section 27.3], that for every pair $(b, b') \in B \times B'$ there is a unique element $b \diamond b' \in \mathcal{L}_{M, M'}$ such that

- (a) $\Psi(b \diamond b') = b \otimes b'$,
- (b) $b \diamond b' - b \otimes b' \in q^{-1}\mathcal{L}_{M, M'}$.

Moreover, $B_\diamond = \{b \diamond b', b \in B, b' \in B'\}$ is a basis of $M \otimes M'$, a $\mathbb{C}[q^{-1}]$ -basis of $\mathcal{L}_{M, M'}$, a $\mathbb{C}[q, q^{-1}]$ -basis of the $\mathbb{C}[q, q^{-1}]$ -module ${}_A\mathcal{L}_{M, M'}$ of $M \otimes M'$ generated by the elements $b \otimes b'$, where $b \in B$, $b' \in B'$, and $(M \otimes M', B_\diamond)$ is a based module.

This construction of B_\diamond is associative. Since (V_μ, \mathbf{B}_μ) is for every $\mu \in P_+$ a based module, it follows that any tensor product M of a finite number of the simple modules V_μ is naturally a based module. The corresponding basis is called *the canonical basis* of M . These canonical basis have been computed explicitly in [56] in the case $\mathfrak{g} = \mathfrak{sl}_2$.

Consider now the U_q^{ad} -module ${}^\omega V_\mu$ with underlying space V_μ , $\mu \in P_+$, and action defined by $x \cdot {}^\omega v := \omega(x)v$, for every $x \in U_q^{\text{ad}}$ and $v \in V_\mu$ (as usual $\omega: U_q^{\text{ad}} \rightarrow U_q^{\text{ad}}$ is the Cartan automorphism). Note that there are isomorphisms ${}^\omega V_\mu \cong V_{-w_0(\mu)} \cong V_\mu^*$ (endowed with the standard left action of U_q^{ad}). Let us denote by ${}^\omega v_\mu$ the vector v_μ regarded in ${}^\omega V_\mu$ (i.e., its canonical lowest weight vector), and by ${}^\omega \mathbf{B}_\mu := \{b \cdot {}^\omega v_\mu, b \in \mathbf{B}^+\} \setminus \{0\}$ its canonical basis; note that ${}^\omega \mathbf{B}_\mu = \{\omega(b)v_\mu, b \in \omega(\mathbf{B}^-)\} \setminus \{0\} = \{bv_\mu, b \in \mathbf{B}^-\} \setminus \{0\} = \mathbf{B}_\mu$. Then ${}^\omega V_{\mu'} \otimes V_{\mu''}$ has the canonical basis $\mathbf{B}_{\mu', \mu''} := \{\underline{b}' \diamond \underline{b}'', \underline{b}' \in {}^\omega \mathbf{B}_{\mu'}, \underline{b}'' \in \mathbf{B}_{\mu''}\}$. Since $\underline{b}' \diamond \underline{b}''$ is canonically determined by the elements $\underline{b}', \underline{b}'' \in \mathbf{B}^-$ such that $\underline{b}' = \omega(b') \cdot {}^\omega v_{\mu'}$, $\underline{b}'' = b'' v_{\mu''}$, following Lusztig we denote it by $(b' \diamond b'')_{\mu', \mu''}$.

Denote by $v_{w_0(\mu)}$ the canonical lowest weight vector of V_μ , and by ${}^\omega v_{w_0(\mu)}$ the vector $v_{w_0(\mu)}$ regarded in ${}^\omega V_\mu$. It is a crucial observation that ${}^\omega v_{w_0(\mu')} \otimes v_{w_0(\mu'')}$ is a cyclic vector of ${}^\omega V_{\mu'} \otimes V_{\mu''}$ (see, e.g., [83, Proposition 23.3.6]; note that ${}^\omega v_{w_0(\mu')} \otimes v_{w_0(\mu'')}$ plays the role of $\xi_{-\mu'} \otimes \eta_{\mu''} := {}^\omega v_{\mu'} \otimes v_{\mu''}$ in [83], because we use opposite coproducts on U_q^{ad} but the factors ${}^\omega V_{\mu'}$ and $V_{\mu''}$ are ordered in the same way).

We can now state the definition of the canonical basis $\dot{\mathbf{B}}$ of $\dot{\mathbf{U}}$: each element u of $\dot{\mathbf{B}}$ belongs to $\dot{\mathbf{U}}_A 1_\zeta$ for some (unique) $\zeta \in P$, and it is then uniquely determined by the property that, for every $\mu', \mu'' \in P^+$ such that $w_0(\mu'' - \mu') = \zeta$, we have

$$u({}^\omega v_{w_0(\mu')} \otimes v_{w_0(\mu'')}) = (b' \diamond b'')_{\mu', \mu''} \tag{2.32}$$

for some $(b' \diamond b'')_{\mu', \mu''} \in \mathbf{B}_{\mu', \mu''}$ [83, Section 25.2]. We write $u = b' \diamond_\zeta b''$, and as in [84] we denote by $\dot{\mathbf{B}}_{\mu', \mu''}$ the finite subset of \mathbf{B} which is in bijection with $\mathbf{B}_{\mu', \mu''}$ under the map $u \mapsto u({}^\omega v_{w_0(\mu')} \otimes v_{w_0(\mu'')})$. So

$$\dot{\mathbf{B}} = \bigcup_{\mu', \mu'' \in P_+} \dot{\mathbf{B}}_{\mu', \mu''}. \tag{2.33}$$

Note in particular that $\dot{\mathbf{B}}$ is formed by weight vectors for the left and right action of $U_q^{\text{ad}}(\mathfrak{h})$ (defined as usual by (2.29)).

In a sense, one can view $\dot{\mathbf{U}}$ as the projective limit of an inverse system formed by the $(U_q^{\text{ad}} \otimes U_q^{\text{ad}})$ -modules ${}^\omega V_{\mu'} \otimes V_{\mu''}$, where $\mu', \mu'' \in P^+$; then $\dot{\mathbf{B}}$ is the basis resulting from the corresponding inverse system of basis $\{\dot{\mathbf{B}}_{\mu', \mu''}\}_{\mu', \mu''}$.

Lusztig has produced a partition of $\dot{\mathbf{B}}$ as follows. First, consider the situation of a based module (M, B) . For every $\lambda \in P_+$ denote by $M[\lambda]$ the sum of the simple submodules of M isomorphic to V_λ (i.e., its isotypical component). Set

$$M[\geq \lambda] = \bigoplus_{\lambda' \geq \lambda} M[\lambda']. \quad (2.34)$$

Then, for every base element $b \in B$ there is a unique $\lambda \in P_+$ such that $b \in M[\geq \lambda]$ and λ is maximal with this property [83, Section 27.2]. Denote by $B[\lambda]$ the set of all $b \in B$ that give rise to $\lambda \in P_+$ in this way. Clearly, the sets $B[\lambda]$, $\lambda \in P_+$, form a partition of B .

Now, given $b \in \dot{\mathbf{B}}$, let $\zeta \in P$ be the unique weight such that $b \in \dot{\mathbf{U}}_A 1_\zeta$, and let $\mu', \mu'' \in P^+$ be such that $w_0(\mu'' - \mu') = \zeta$, and $(\check{\alpha}_i, \mu')$ is large enough for all $i = 1, \dots, m$ so that $u({}^\omega v_{w_0(\mu')} \otimes v_{w_0(\mu'')})$ is non-zero. This element belongs to the canonical basis $\underline{\mathbf{B}}_{\mu', \mu''}$ of ${}^\omega V_{\mu'} \otimes V_{\mu''}$, and therefore to one of the subsets $\underline{\mathbf{B}}_{\mu', \mu''}[\lambda]$, for a unique $\lambda \in P_+$. It is a result that λ does not depend on the choice of μ', μ'' (see [83, Section 29.1.1]). Hence there is a well-defined map $\dot{\mathbf{B}} \rightarrow P_+$, $b \mapsto \lambda$. Denoting by $\dot{\mathbf{B}}[\lambda]$ the fiber of this map, we thus obtain a partition

$$\dot{\mathbf{B}} = \coprod_{\lambda \in P_+} \dot{\mathbf{B}}[\lambda]. \quad (2.35)$$

The sets $\dot{\mathbf{B}}[\lambda]$ are called *2-sided cells*. They are finite sets and have the following remarkable properties. For every $\lambda \in P_+$ denote by $\dot{\mathbf{U}}[\geq \lambda]$ and $\dot{\mathbf{U}}[> \lambda]$ the subspaces of $\dot{\mathbf{U}}$ spanned by $\coprod_{\lambda' \geq \lambda} \dot{\mathbf{B}}[\lambda']$ and $\coprod_{\lambda' > \lambda} \dot{\mathbf{B}}[\lambda']$ respectively. Then $\dot{\mathbf{U}}[\geq \lambda]$ (respectively $\dot{\mathbf{U}}[> \lambda]$) consists of the elements $u \in \dot{\mathbf{U}}$ such that if u acts on V_μ by a non-zero linear map, then $\mu \geq \lambda$ (respectively $\mu > \lambda$) [83, Lemmas 29.1.3 and 29.1.4]. Both $\dot{\mathbf{U}}[\geq \lambda]$ and $\dot{\mathbf{U}}[> \lambda]$ are two-sided ideals of $\dot{\mathbf{U}}$. Moreover, the algebra homomorphism $\pi_\lambda: \dot{\mathbf{U}}[\geq \lambda] \rightarrow \text{End}(V_\lambda)$ given by the $\dot{\mathbf{U}}$ -module structure on V_λ descends to an algebra and U_q^{ad} -bimodule isomorphism (keeping the same notation) [83, Proposition 29.2.2]

$$\bar{\pi}_\lambda: \dot{\mathbf{U}}[\geq \lambda] / \dot{\mathbf{U}}[> \lambda] \rightarrow \text{End}(V_\lambda). \quad (2.36)$$

For instance, when $\mathfrak{g} = \mathfrak{sl}_2$ the 2-sided cell $\dot{\mathbf{B}}[n]$ associated to the simple $U_q^{\text{ad}}(\mathfrak{sl}_2)$ -module of type 1 and dimension $n + 1$ is the set of cardinality $(n + 1)^2$ given by [83, Section 29.4.3]

$$\dot{\mathbf{B}}[n] = \{E^{(k)} 1_{-n} F^{(l)}, n \geq k + l\} \cup \{F^{(l)} 1_n E^{(k)}, n \geq k + l\}, \quad (2.37)$$

with the identification $E^{(k)} 1_{-n} F^{(l)} = F^{(l)} 1_n E^{(k)}$ when $n = k + l$. As we are mainly interested in \mathcal{O}_A , we are going to describe the dual partition of $\dot{\mathbf{B}}^*$, see Theorem 2.6. The duality with (2.35) is discussed after that theorem.

First, we follow the approach of Kashiwara [66, 67]. For every $\lambda \in P_+$, denote by V_λ^r the dual space of V_λ endowed with its natural structure of right U_q^{ad} -module, defined by $(fx)(v) = f(xv)$ for every $f \in V_\lambda^r$, $x \in U_q^{\text{ad}}$, $v \in V_\lambda$. Clearly, V_λ^r is a simple module of highest weight λ . Let $\varphi: U_q^{\text{ad}} \rightarrow U_q^{\text{ad}}$ be the anti-automorphism of $\mathbb{C}(q)$ -algebra given by $\varphi(E_i) = F_i$, $\varphi(F_i) = E_i$, $\varphi(K_\lambda) = K_\lambda$. By using φ , any right U_q^{ad} -module can be considered as a left U_q^{ad} -module. In particular, by the Verma module construction of V_λ it follows

$$V_\lambda^r \cong U_q^{\text{ad}} \left/ \left(\sum_{\mu \in P_+} (K_\mu - q^{(\lambda, \mu)}) U_q^{\text{ad}} + \sum_{i=1}^m E_i^{1+(\check{\alpha}_i, \lambda)} U_q^{\text{ad}} \right) \right.,$$

and φ affords an isomorphism of the right module V_λ^r with the *left* module V_λ . We will denote by f_λ the unique highest weight vector of V_λ^r satisfying $\langle f_\lambda, v_\lambda \rangle = 1$.

The space $V_\lambda^r \otimes V_\lambda$ can be identified with $\text{End}(V_\lambda)^*$, and thus acquires by duality a natural structure of U_q^{ad} -bimodule (or equivalently left $U_q^{\text{ad}} \otimes (U_q^{\text{ad}})^{\text{op}}$ -module); the left and right actions are given by

$$x(f \otimes v)y = fy \otimes xv \quad (2.38)$$

for every $x, y \in U_q^{\text{ad}}$, $f \in V_\lambda^r$, $v \in V_\lambda$. The space $V_\lambda^r \otimes V_\lambda$ also acquires by duality a natural ‘‘upper’’ crystal structure over $U_q^{\text{ad}} \otimes (U_q^{\text{ad}})^{\text{op}}$, as we explain now. Denote by $\langle \cdot, \cdot \rangle_\lambda: V_\lambda \times V_\lambda \rightarrow \mathbb{C}(q)$ the unique symmetric bilinear form such that

$$\langle v_\lambda, v_\lambda \rangle_\lambda = 1 \quad \text{and} \quad \langle \varphi(x)u, v \rangle_\lambda = \langle u, xv \rangle_\lambda \quad (2.39)$$

for every $u, v \in V_\lambda$ and $x \in U_q^{\text{ad}}$. Recall the crystal base $(\mathcal{L}_\mu^{\text{low}}, \mathcal{B}_\mu^{\text{low}})$ at $q = 0$ introduced before Theorem 2.5. In Kashiwara’s terminology [65, 66], the pair $(\mathcal{L}_\lambda^{\text{low}}, \mathcal{B}_\lambda^{\text{low}})$ is the *lower crystal base of V_λ* at $q = 0$. Applying the involution $\bar{\cdot}: V_\lambda \rightarrow V_\lambda$, one obtains the lower crystal base $(\overline{\mathcal{L}_\lambda^{\text{low}}}, \overline{\mathcal{B}_\lambda^{\text{low}}})$ at $q = \infty$. Because the canonical bases are determined by the crystal bases (see the discussion about (2.31)), we call $(V_\lambda, \underline{\mathbf{B}}_\lambda)$ the *lower based module of V_λ* , and $\underline{\mathbf{B}}_\lambda$ the *lower canonical basis* of V_λ .

Put

$$\begin{aligned} {}_A V_\lambda^{\text{up}} &:= \{v \in V_\lambda, \langle v, {}_A V_\lambda \rangle_\lambda \subset A\}, & \mathcal{L}_\lambda^{\text{up}} &:= \{v \in V_\lambda, \langle v, \mathcal{L}_\lambda^{\text{low}} \rangle_\lambda \subset \mathcal{A}_0\}, \\ \overline{\mathcal{L}_\lambda^{\text{up}}} &:= \{v \in V_\lambda, \langle v, \overline{\mathcal{L}_\lambda^{\text{low}}} \rangle_\lambda \subset \mathcal{A}_\infty\}. \end{aligned} \quad (2.40)$$

Then $({}_A V_\lambda^{\text{up}}, \mathcal{L}_\lambda^{\text{up}}, \overline{\mathcal{L}_\lambda^{\text{up}}})$ is a balanced triple [66, Lemma 4.2.1]. Denote by $\mathcal{B}_\lambda^{\text{up}}$ the basis of $\mathcal{L}_\lambda^{\text{up}}/q\mathcal{L}_\lambda^{\text{up}}$ dual to $\mathcal{B}_\lambda^{\text{low}}$ by the induced pairing $\langle \cdot, \cdot \rangle_\lambda: \mathcal{L}_\lambda^{\text{up}}/q\mathcal{L}_\lambda^{\text{up}} \times \mathcal{L}_\lambda^{\text{low}}/q\mathcal{L}_\lambda^{\text{low}} \rightarrow \mathbb{C}$. The pair $(\mathcal{L}_\lambda^{\text{up}}, \mathcal{B}_\lambda^{\text{up}})$ is the *upper crystal base* of V_λ at $q = 0$. The weight spaces of the \mathcal{A}_0 -modules $\mathcal{L}_\lambda^{\text{low}}$ and $\mathcal{L}_\lambda^{\text{up}}$ are related by

$$(\mathcal{L}_\lambda^{\text{up}})^\mu = q^{\frac{(\lambda, \lambda)}{2} - \frac{(\mu, \mu)}{2}} (\mathcal{L}_\lambda^{\text{low}})^\mu, \quad \mu \in P. \quad (2.41)$$

Correspondingly, denoting $(\mathcal{B}_\lambda^{\text{up}})^\mu := \mathcal{B}_\lambda^{\text{up}} \cap (\mathcal{L}_\lambda^{\text{up}})^\mu$ and $(\mathcal{B}_\lambda^{\text{low}})^\mu := \mathcal{B}_\lambda^{\text{low}} \cap (\mathcal{L}_\lambda^{\text{low}})^\mu$, we have (see [65] and [66, equation (4.2.9)])

$$(\mathcal{B}_\lambda^{\text{up}})^\mu = q^{\frac{(\lambda, \lambda)}{2} - \frac{(\mu, \mu)}{2}} (\mathcal{B}_\lambda^{\text{low}})^\mu.$$

The A -module ${}_A V_\lambda^{\text{up}}$ is characterized by the following two properties [66, equations (4.2.10)–(4.2.12)]:

$$({}_A V_\lambda^{\text{up}})^\lambda = \mathbb{C}[q, q^{-1}]v_\lambda, \quad ({}_A V_\lambda^{\text{up}})^\mu = \{v \in V_\lambda \mid U_A^{\text{res}}(\mathfrak{n}^+)^{\lambda-\mu}v \in \mathbb{C}[q, q^{-1}]v_\lambda\},$$

where $U_A^{\text{res}}(\mathfrak{n}^+)^\gamma = \{u \in U_A^{\text{res}}(\mathfrak{n}^+) \mid \forall \nu \in P, K_\nu u K_\nu^{-1} = q^{(\nu, \gamma)}u\}$. Denote by $\underline{\mathbf{B}}_\lambda^{\text{up}}$ the inverse image of $\mathcal{B}_\lambda^{\text{up}}$ by the isomorphism ${}_A V_\lambda^{\text{up}} \cap \mathcal{L}_\lambda^{\text{up}} \cap \overline{\mathcal{L}_\lambda^{\text{up}}} \rightarrow \mathcal{L}_\lambda^{\text{up}}/q\mathcal{L}_\lambda^{\text{up}}$. By (2.30), the set $\underline{\mathbf{B}}_\lambda^{\text{up}}$ is a basis of ${}_A V_\lambda^{\text{up}}$; we call it the *upper canonical basis* of V_λ . In the appendix, we describe in details the \mathfrak{sl}_2 case.

Similarly, the right module V_λ^r with its canonical basis $\underline{\mathbf{B}}_\lambda^r = \{f_\lambda b, b \in \mathbf{B}^+\} \setminus \{0\}$ has the lower crystal base $(\mathcal{L}_\lambda^{r, \text{low}}, \mathcal{B}_\lambda^{r, \text{low}})$, and it supports a balanced triple $({}_A V_\lambda^{\text{up}}, \mathcal{L}_\lambda^{r, \text{up}}, \overline{\mathcal{L}_\lambda^{r, \text{up}}})$ defined again by duality. We denote by $(\mathcal{L}_\lambda^{r, \text{up}}, \mathcal{B}_\lambda^{r, \text{up}})$ and $\underline{\mathbf{B}}_\lambda^{r, \text{up}}$ the corresponding crystal base and upper canonical basis of V_λ^r , respectively.

It follows that $({}_A V_\lambda^{r, \text{up}} \otimes_{{}_A A} V_\lambda^{\text{up}}, \mathcal{L}_\lambda^{r, \text{up}} \otimes_{\mathcal{A}_0} \mathcal{L}_\lambda^{\text{up}}, \overline{\mathcal{L}_\lambda^{r, \text{up}}} \otimes_{\mathcal{A}_\infty} \overline{\mathcal{L}_\lambda^{\text{up}}})$ is a balanced triple; equivalently $V_\lambda^r \otimes V_\lambda$ with the bimodule structure (2.38) and the basis $\underline{\mathbf{B}}_\lambda^{r, \text{up}} \otimes \underline{\mathbf{B}}_\lambda^{\text{up}}$ is a based $(U_q^{\text{ad}} \otimes (U_q^{\text{ad}})^{\text{op}})$ -module.

Denote again by $\langle \cdot, \cdot \rangle: \mathcal{O}_q \times \dot{\mathbf{U}} \rightarrow \mathbb{C}(q)$ the pairing of U_q^{ad} -bimodules induced by the canonical pairing $\langle \cdot, \cdot \rangle: \mathcal{O}_q \times U_q^{\text{ad}} \rightarrow \mathbb{C}(q)$, and let $\Phi_\lambda: V_\lambda^r \otimes V_\lambda \rightarrow \mathcal{O}_q$, $\lambda \in P_+$, be the ‘‘matrix coefficient’’ map, i.e.,

$$\langle \Phi_\lambda(f \otimes v), x \rangle = \langle f, xv \rangle_\lambda \quad (2.42)$$

for every $f \in V_\lambda^r$, $x \in U_q^{\text{ad}}$, $v \in V_\lambda$. The map $\Phi := \bigoplus_{\lambda \in P_+} \Phi_\lambda$ is an isomorphism of U_q^{ad} -bimodules, so let us use it to identify \mathcal{O}_q with $\bigoplus_{\lambda \in P_+} V_\lambda^r \otimes V_\lambda$ (which is the content of the Peter–Weyl decomposition (2.6)). Define

$$\begin{aligned} \mathcal{L}(\mathcal{O}_q) &= \bigoplus_{\lambda \in P_+} \left(\mathcal{L}_\lambda^{r \text{ up}} \otimes_{\mathcal{A}_0} \mathcal{L}_\lambda^{\text{up}} \right), & \mathcal{B}(\mathcal{O}_q) &:= \prod_{\lambda \in P_+} \mathcal{B}_\lambda^{r \text{ up}} \otimes \mathcal{B}_\lambda^{\text{up}}, \\ \overline{\mathcal{L}}(\mathcal{O}_q) &= \bigoplus_{\lambda \in P_+} \left(\overline{\mathcal{L}_\lambda^{r \text{ up}}} \otimes_{\mathcal{A}_\infty} \overline{\mathcal{L}_\lambda^{\text{up}}} \right), & \overline{\mathcal{B}}(\mathcal{O}_q) &:= \prod_{\lambda \in P_+} \overline{\mathcal{B}_\lambda^{r \text{ up}}} \otimes \overline{\mathcal{B}_\lambda^{\text{up}}}. \end{aligned}$$

Theorem 2.6.

- (i) *The triple $(\mathcal{O}_A, \mathcal{L}(\mathcal{O}_q), \overline{\mathcal{L}}(\mathcal{O}_q))$ is balanced. Therefore, denoting by G the inverse of the canonical map $\mathcal{O}_A \cap \mathcal{L}(\mathcal{O}_q) \cap \overline{\mathcal{L}}(\mathcal{O}_q) \rightarrow \mathcal{L}(\mathcal{O}_q)/q\mathcal{L}(\mathcal{O}_q)$, we have*

$$\mathcal{O}_A = \bigoplus_{b \in \mathcal{B}(\mathcal{O}_q)} AG(b).$$

- (ii) *The basis $G(\mathcal{B}(\mathcal{O}_q)) := \{G(b), b \in \mathcal{B}(\mathcal{O}_q)\}$ coincides with the dual canonical basis $\dot{\mathbf{B}}^*$, i.e., the elements $a^* \in \mathcal{O}_A$, for every $a \in \dot{\mathbf{B}}$, defined by $a^*(a') = \delta_{a,a'}$ for every $a' \in \dot{\mathbf{B}}$. Therefore,*

$$\mathcal{O}_A = \bigoplus_{b \in \dot{\mathbf{B}}} Ab^*.$$

The statement (i) is [66, Theorem 1], and (ii) is [67, Theorem 10.1 and Proposition 10.2.2] and [83, Section 29.5]. The basis $G(\mathcal{B}(\mathcal{O}_q)) = \dot{\mathbf{B}}^*$ is called the *global basis*, or *canonical basis*, of \mathcal{O}_q . The proof of Theorem 2.6 (ii) in [67] (see also [68]) exhibits an isomorphism of crystals over $U_q^{\text{ad}} \otimes (U_q^{\text{ad}})^{\text{op}}$,

$$\psi: \mathcal{B}(\mathcal{O}_q) \rightarrow \mathcal{B}(\dot{\mathbf{U}}), \quad (2.43)$$

where $(\mathcal{L}(\dot{\mathbf{U}}), \mathcal{B}(\dot{\mathbf{U}}))$ is the crystal base of $\dot{\mathbf{U}}$ associated to the canonical basis $\dot{\mathbf{B}}$. The isomorphism ψ satisfies $\langle G(b), G(b') \rangle = \delta_{\psi(b), \psi(b')}$ for every $b \in \mathcal{B}(\mathcal{O}_q)$, $b' \in \mathcal{B}(\dot{\mathbf{U}})$. The unit 1 of \mathcal{O}_A is $(1_0)^*$; the constant structures of \mathcal{O}_A are studied in [83, 84].

The canonical basis of \mathcal{O}_A when $\mathfrak{g} = \mathfrak{sl}_2$. Denote by a, b, c, d the matrix coefficients in the canonical basis $(v_+, v_- := Fv_+)$ of V_1 , the simple $U_q^{\text{ad}}(\mathfrak{sl}_2)$ -module of type 1 and dimension two, read from the top left to the bottom right. In that case of V_1 the upper canonical basis $\underline{\mathbf{B}}_1^{r \text{ up}}$ and $\underline{\mathbf{B}}_1^{\text{up}}$ coincide with the lower ones (this is not true in general, see Example 2.17). The basis $\dot{\mathbf{B}}^*(\mathfrak{sl}_2)$ is formed by the monomials $c^s a^p b^r$ where $p, r, s \in \mathbb{N}$, and $c^s d^p b^r$ where $p, r, s \in \mathbb{N}$ and $p > 0$; this is stated in [66, Proposition 9.1.1] (in [41, Proposition 1.3], similar monomials are shown to form an A -basis of $\mathcal{O}_A(\text{SL}_2)$, but without reference to the canonical basis; see the comments before (4.3) below). More precisely, recall the 2-sided cells (2.37). We verified by a tedious though straightforward computation that we have the duality pairing

$$\begin{aligned} \langle c^s d^p b^r, E^{(i)} 1_{-k} F^{(j)} \rangle &= \delta_{p+r+s, k} \delta_{r, i} \delta_{s, j}, & \langle c^s d^p b^r, F^{(j)} 1_k E^{(i)} \rangle &= 0, \\ \langle c^s a^p b^r, E^{(i)} 1_{-k} F^{(j)} \rangle &= 0, & \langle c^s a^p b^r, F^{(j)} 1_k E^{(i)} \rangle &= \delta_{p+r+s, k} \delta_{r, i} \delta_{s, j}. \end{aligned}$$

Therefore,

$$\begin{aligned} \dot{\mathbf{B}}[n]^* &:= \{c^s a^p b^r, p, r, s \in \mathbb{N}, p + r + s = n\} \\ &\cup \{c^s d^p b^r, p, r, s \in \mathbb{N}, p > 0, p + r + s = n\}. \end{aligned}$$

A description of $\dot{\mathbf{B}}^*$ in the case of $\mathfrak{g} = \mathfrak{sl}_n$ can be found in [49]. Moreover, denote by V_n the simple $U_q^{\text{ad}}(\mathfrak{sl}_2)$ -module of type 1 and dimension $n + 1$, by (v_k) the canonical basis of V_n , by (v^k) the dual basis, and by $\pi_n: \dot{\mathbf{U}}(\mathfrak{sl}_2) \rightarrow \text{End}(V_n)$ the representation morphism. By using the above pairing, it is readily checked that for every $0 \leq l, m \leq n$, we have

$$\begin{aligned} &v^l(\pi_n(\cdot) v_m) \\ &= \sum_{\substack{0 \leq i, j, k \\ i+j \leq k \leq n \\ j-i=l-m}} \delta_{-k, n-2(m+j)} \begin{bmatrix} m+j \\ j \end{bmatrix}_q \begin{bmatrix} n-m+i-j \\ i \end{bmatrix}_q (E^{(i)} 1_{-k} F^{(j)})^* \\ &+ \sum_{\substack{0 \leq i, j, k \\ i+j < k \leq n \\ j-i=l-m}} \delta_{k, n-2(m-i)} \begin{bmatrix} m-i+j \\ j \end{bmatrix}_q \begin{bmatrix} n-m+i \\ i \end{bmatrix}_q (F^{(j)} 1_{+k} E^{(i)})^*. \end{aligned} \quad (2.44)$$

In particular, we see in this case of $\mathfrak{g} = \mathfrak{sl}_2$ that in general the matrix coefficients of simple U_A^{res} -modules of type 1 are not elements of the dual canonical basis $\dot{\mathbf{B}}^*$. Moreover, these matrix coefficients do not form a basis of \mathcal{O}_A . For instance, it follows from (2.44) that the matrix of matrix coefficients of V_2 has the following form:

$$\begin{pmatrix} a^2 & [2]_q ab & b^2 \\ ca & [2]_q bc + 1 & db \\ c^2 & [2]_q cd & d^2 \end{pmatrix}. \quad (2.45)$$

The matrix coefficient $v_0^* \otimes v_0$ being equal to $[2]_q bc + 1$, this shows bc cannot be expressed as a linear combination over A of matrix coefficients of simple modules.

The refined Peter–Weyl theorem. Let us discuss the U_A^{res} -bimodule structure of \mathcal{O}_A , and its relation with the partition (2.35). For every $\lambda \in P_+$, put

$${}_A \dot{C}(\lambda) := \bigoplus_{b \in \dot{\mathbf{B}}[\lambda]} Ab^* \quad (2.46)$$

and

$$\mathcal{O}_{A(\leq \lambda)} := \bigoplus_{\lambda' \leq \lambda} {}_A \dot{C}(\lambda'), \quad \mathcal{O}_{A(< \lambda)} := \bigoplus_{\lambda' < \lambda} {}_A \dot{C}(\lambda').$$

In particular, in the \mathfrak{sl}_2 case the A -module ${}_A \dot{C}(n\varpi_1)$ has basis $\dot{\mathbf{B}}[n]^*$ given above, of cardinality $(n + 1)^2$.

Recall that $\dot{\mathbf{U}}[\geq \lambda]$ and $\dot{\mathbf{U}}[> \lambda]$ are two-sided ideals of $\dot{\mathbf{U}}$, and the algebra (whence U_q^{ad} -bimodule) isomorphism $\bar{\pi}_\lambda: \dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda] \rightarrow \text{End}(V_\lambda)$ (see (2.36)). In [83, Section 29.3], Lusztig groups this isomorphism and its properties under the general term of *refined Peter–Weyl theorem*. We are going to reinterpret it in terms of \mathcal{O}_A . First observe that

Lemma 2.7. *The A -modules $\mathcal{O}_{A(\leq \lambda)}$ and $\mathcal{O}_{A(< \lambda)}$ are U_A^{res} -bimodules, and the surjective map*

$$d_\lambda: \mathcal{O}_{A(\leq \lambda)} \longrightarrow \text{Hom}(\dot{\mathbf{U}}_A[\geq \lambda]/\dot{\mathbf{U}}_A[> \lambda], A), \quad \alpha \longmapsto \langle \alpha, \cdot \rangle \quad (2.47)$$

descends to an isomorphism of U_A^{res} -bimodules \bar{d}_λ on $\mathcal{O}_{A(\leq \lambda)}/\mathcal{O}_{A(< \lambda)}$.

Proof. For every $\alpha \in \mathcal{O}_A(\leq \lambda)$, $x, y \in U_A^{\text{res}}$, and $b \in \dot{\mathbf{B}}[\mu]$ with $\mu \not\leq \lambda$, we have $xb y \in \dot{\mathbf{U}}_A[\geq \mu]$. Since $\dot{\mathbf{U}}_A[\geq \mu] = \bigoplus_{\eta \geq \mu} A\dot{\mathbf{B}}[\eta]$ and $\eta \geq \mu$ implies $\eta \not\leq \lambda$, it follows that $\langle xby, \alpha \rangle = 0$, i.e., $(x \triangleright \alpha \triangleleft y)(b) = 0$. This shows $x \triangleright \alpha \triangleleft y \in \mathcal{O}_A(\leq \lambda)$. The same proof applies as well to $\mathcal{O}_A(< \lambda)$, whence the first claim. Since $\dot{\mathbf{U}}[\geq \lambda]$ and $\dot{\mathbf{U}}[> \lambda]$ are two-sided ideals of $\dot{\mathbf{U}}$, $\dot{\mathbf{B}}$ is a basis of $\dot{\mathbf{U}}_A$, and the A -modules $\dot{\mathbf{U}}_A[\geq \lambda]$ and $\dot{\mathbf{U}}_A[> \lambda]$ are spanned by $\prod_{\lambda' \geq \lambda} \dot{\mathbf{B}}[\lambda']$ and $\prod_{\lambda' > \lambda} \dot{\mathbf{B}}[\lambda']$, both are two-sided ideals of $\dot{\mathbf{U}}_A$, and $\dot{\mathbf{U}}_A[\geq \lambda]/\dot{\mathbf{U}}_A[> \lambda]$ inherits the quotient U_A^{res} -bimodule structure. Clearly, the map d_λ is well defined, it is a morphism of U_A^{res} -bimodules, and its kernel contains $\mathcal{O}_A(< \lambda)$. Bijectivity of \bar{d}_λ comes by comparing the cardinality of canonical bases: $\mathcal{O}_A(\leq \lambda)/\mathcal{O}_A(< \lambda)$ has the basis formed by the cosets of the elements of the basis $(\dot{\mathbf{B}}[\lambda])^*$ of ${}_A\dot{C}(\lambda)$, and $\dot{\mathbf{U}}_A[\geq \lambda]/\dot{\mathbf{U}}_A[> \lambda]$ the basis formed by the cosets of the elements of $\dot{\mathbf{B}}[\lambda]$, all cosets being non-zero and pairwise distinct. \blacksquare

Since $\dot{\mathbf{U}}_A$ preserves the canonical basis \mathbf{B}_λ of ${}_AV_\lambda$, $\bar{\pi}_\lambda$ descends to an isomorphism of U_A^{res} -bimodules $\bar{\pi}_\lambda: \dot{\mathbf{U}}_A[\geq \lambda]/\dot{\mathbf{U}}_A[> \lambda] \rightarrow \text{End}({}_AV_\lambda)$. We thus get exact sequences of U_A^{res} -bimodules

$$0 \longrightarrow \dot{\mathbf{U}}_A[> \lambda] \longrightarrow \dot{\mathbf{U}}_A[\geq \lambda] \xrightarrow{\bar{\pi}_\lambda} \text{End}({}_AV_\lambda) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_A(< \lambda) \longrightarrow \mathcal{O}_A(\leq \lambda) \xrightarrow{(\bar{\pi}_\lambda^{-1})^* \circ d_\lambda} (\text{End}({}_AV_\lambda))^* \longrightarrow 0. \quad (2.48)$$

They split as sequences of A -modules but not as sequences of bimodules. In fact,

$$\begin{aligned} (\text{End}({}_AV_\lambda))^* &:= \text{Hom}(\text{End}({}_AV_\lambda), A) \\ &\cong \text{Hom}({}_A^\omega V_\lambda \otimes_A {}_AV_\lambda, A) = {}_AV_\lambda^{\text{up}} \otimes_A ({}^\omega V_\lambda)^{\text{up}}, \end{aligned} \quad (2.49)$$

with the “up” structure defined in (2.40), and corresponding basis $\mathbf{B}_\lambda^{\text{up}} \otimes ({}^\omega \mathbf{B}_\lambda)^{\text{up}}$. Moreover, the exact sequence (2.48) shows that this A -module of matrix coefficients, regarded as an A -submodule of \mathcal{O}_A by means of the coefficient map $\Phi := \bigoplus_{\lambda \in P_+} \Phi_\lambda$ (see (2.42)), is contained in $\mathcal{O}_A(\leq \lambda)$. This for all $\lambda' \leq \lambda$ yields $\bigoplus_{\lambda' \leq \lambda} (\text{End}({}_AV_{\lambda'}))^* \subset \mathcal{O}_A(\leq \lambda)$. Now, using the isomorphism $\bar{\pi}_\lambda$, we get

$$\text{rank}_A(\mathcal{O}_A(\leq \lambda)) = \sum_{\lambda' \leq \lambda} \text{Card}(\dot{\mathbf{B}}[\lambda']) = \sum_{\lambda' \leq \lambda} \text{rank}({}_AV_{\lambda'})^2$$

and therefore

$$\dim_{\mathbb{C}(q)}(\mathcal{O}_A(\leq \lambda) \otimes_A \mathbb{C}(q)) = \sum_{\lambda' \leq \lambda} \dim(V_{\lambda'})^2 = \sum_{\lambda' \leq \lambda} \dim(C(\lambda')), \quad (2.50)$$

where as usual $C(\lambda')$ denotes the space of matrix coefficients of $V_{\lambda'}$ (see (2.22)). It follows

$$\mathcal{O}_A(\leq \lambda) \otimes_A \mathbb{C}(q) = \bigoplus_{\lambda' \leq \lambda} C(\lambda'), \quad \mathcal{O}_A(< \lambda) \otimes_A \mathbb{C}(q) = \bigoplus_{\lambda' < \lambda} C(\lambda'). \quad (2.51)$$

However, in general ${}_A\dot{C}(\lambda) \otimes_A \mathbb{C}(q)$ is not equal to $C(\lambda)$, ${}_A\dot{C}(\lambda)$ is not an A -sublattice of $C(\lambda)$, and ${}_A\dot{C}(\lambda)$ is not a U_A^{res} -bimodule (it is because of this discrepancy that we have introduced the dot notation “ $\dot{\cdot}$ ”). For instance, we can see the first two facts in the case of $\mathfrak{g} = \mathfrak{sl}_2$, by inverting the system of identities (2.44) for all $0 \leq l, m \leq n$ (or more simply by considering the identity $v_0^* \otimes v_0 = [2]_q bc + 1$ from (2.45)). For the third fact, we have $1_2 E \in \dot{\mathbf{B}}[2]$ (see (2.37)), so $\langle (1_2 E)^* \triangleleft E \rangle(1_0) = \langle \Delta((1_2 E)^*), E \otimes 1_0 \rangle = \langle (1_2 E)^*, E 1_0 \rangle = \langle (1_2 E)^*, 1_2 E \rangle = 1$ since $E 1_0 = 1_2 E$. Therefore, $(1_2 E)^* \triangleleft E \notin {}_A\dot{C}(2)$.

From the formulas (2.44) and Appendix A, we can observe the isomorphism (2.49) in the case of $\mathfrak{g} = \mathfrak{sl}_2$. More simply, by projecting the matrix (2.45) onto $(\text{End}({}_A V_2))^*$ the entries are unchanged except the (1, 1) entry, which becomes $[2]_q bc$. All factors $[2]_q$ in the middle column disappear if one uses matrix coefficients in the upper canonical basis of V_2 , which is $v_0^{\text{up}} := v_0$, $v_1^{\text{up}} := [2]_q^{-1} v_1$, $v_2^{\text{up}} := v_2$ in the notations of (2.44), since we have $v^l(\pi_2(\cdot) v_m) = [\delta_{m,1} + 1]_q \langle v_l^{\text{up}}, \cdot v_m^{\text{up}} \rangle$ for $l, m \in \{0, 1, 2\}$, where $\langle \cdot, \cdot \rangle$ is the pairing (2.39). Thus, in this particular example of $(\text{End}({}_A V_2))^*$ we see explicitly the identification of the basis $(\bar{\pi}_2^*)^{-1} \circ d_2(\dot{\mathbf{B}}[2]^*)$ and $\underline{\mathbf{B}}_2^{\text{up}} \otimes ({}^\omega \underline{\mathbf{B}}_2)^{\text{up}}$.

Summing up this discussion, the Lusztig refined Peter–Weyl theorem of [83, Section 29.3], implies the following.

Theorem 2.8. *As an A -module we have a direct sum decomposition*

$$\mathcal{O}_A = \bigoplus_{\lambda \in P_+} {}_A \dot{C}(\lambda), \quad (2.52)$$

as U_A^{res} -bimodules we have a (directed by inclusion, and non direct) sum

$$\mathcal{O}_A = \sum_{\lambda \in P_+} \mathcal{O}_A(\leq \lambda), \quad (2.53)$$

and the composition factors of \mathcal{O}_A are the bimodules

$$(\text{End}({}_A V_\lambda))^* \cong ({}^\omega V_\lambda \otimes {}_A V_\lambda)^* \quad (2.54)$$

for every $\lambda \in P_+$, each of multiplicity 1.

Remark 2.9. The above filtration and its composition factors appear in disguised manner as *good filtration* in [5] and [91] (see also [103]).

Because $\dot{\mathbf{B}}$ is formed by weight vectors for the left and right action of $U_q^{\text{ad}}(\mathfrak{h})$ (see (2.33)), the same is true of $\dot{\mathbf{B}}^*$ and (2.52) can thus be refined into a weight space decomposition

$$\mathcal{O}_A = \bigoplus_{\mu, \nu \in P} \bigoplus_{\lambda \in P_+} ({}_A \dot{C}(\lambda))_{\mu, \nu}. \quad (2.55)$$

Now recall the property (2.33). Consider in particular the finite subsets $\dot{\mathbf{B}}_{0, \varpi_i}$ and $\dot{\mathbf{B}}_{\varpi_i, 0}$ associated to the fundamental weights ϖ_i , $i = 1, \dots, m$. The map $u \mapsto u({}^\omega v_0 \otimes v_{w_0(\varpi_i)})$, $u \in \dot{\mathbf{U}}$, allows one to identify $\dot{\mathbf{B}}_{0, \varpi_i}$ with the canonical basis $\underline{\mathbf{B}}_{\varpi_i}$ of ${}^\omega V_0 \otimes V_{\varpi_i} \cong V_{\varpi_i}$, and therefore with a uniquely determined finite subset \mathbf{B}_{ϖ_i} of the canonical basis \mathbf{B}^- of $U_q^{\text{ad}}(\mathfrak{n}_-)$; similarly, one can identify $\dot{\mathbf{B}}_{\varpi_i, 0}$ with a uniquely determined finite subset ${}^\omega \mathbf{B}_{\varpi_i}$ of the canonical basis \mathbf{B}^+ of $U_q^{\text{ad}}(\mathfrak{n}_+)$. The elements of $\dot{\mathbf{B}}_{0, \varpi_i}$ and $\dot{\mathbf{B}}_{\varpi_i, 0}$ are respectively of the form $b^- 1_{\varpi_i}$ and $b^+ 1_{-\varpi_i}$, where $b^- \in \mathbf{B}_{\varpi_i}$ and $b^+ \in {}^\omega \mathbf{B}_{\varpi_i}$, and we have (see [84, Proposition 3.3 and Section 3.4]):

Proposition 2.10. *The algebra \mathcal{O}_A is finitely generated. A system of generators is provided by the elements $a^* \in \dot{\mathbf{B}}^*$, where $a \in \bigcup_{i=1}^m (\dot{\mathbf{B}}_{0, \varpi_i} \cup \dot{\mathbf{B}}_{\varpi_i, 0})$.*

Note that the above system of generators of \mathcal{O}_A has $2 \sum_{i=1}^m \dim(V_{\varpi_i})$ elements. In fact, recall that $\varphi: U_q^{\text{ad}} \rightarrow U_q^{\text{ad}}$ is the anti-automorphism given by $\varphi(E_i) = F_i$, $\varphi(F_i) = E_i$, $\varphi(K_\lambda) = K_\lambda$. Denote by $v_{-\varpi_i}$ and $f_{-\varpi_i}$ the canonical lowest-weight vectors of the highest weight modules $V_{-w_0(\varpi_i)}$ and $V_{-w_0(\varpi_i)}^r$, respectively, and put the superscript “up” for the upper canonical basis vectors.

Lemma 2.11. *For every $b^- \in \mathbf{B}_{\varpi_i}$ and $b^+ \in {}^\omega \mathbf{B}_{\varpi_i}$, we have*

$$(b^- 1_{\varpi_i})^* = \Phi_{\varpi_i}((f_{\varpi_i} \varphi(b^-))^{\text{up}} \otimes v_{\varpi_i}), \quad (2.56)$$

$$(b^+ 1_{-\varpi_i})^* = \Phi_{-w_0(\varpi_i)}((f_{-\varpi_i} \varphi(b^+))^{\text{up}} \otimes v_{-\varpi_i}). \quad (2.57)$$

In other words, $(b^- 1_{\varpi_i})^*$ and $(b^+ 1_{-\varpi_i})^*$ are the matrix coefficients lying on the first and last columns of the matrix representations in the upper canonical bases of the spaces V_{ϖ_i} , $i = 1, \dots, m$.

Proof. This can be checked by using the isomorphism (2.43). The key observation is that

$$\langle \Phi_\lambda(f_\lambda \otimes v_\lambda), 1_\mu \rangle = \langle f_\lambda, 1_\mu v_\lambda \rangle_\lambda = \delta_{\lambda, \mu}$$

for every $\lambda \in P_+$, $\mu \in P$, and therefore $\Phi_\lambda(f_\lambda \otimes v_\lambda) = 1_\lambda^*$. Then the computation proceeds by using the equivariance of Φ under the action of $U_q^{\text{ad}} \otimes (U_q^{\text{ad}})^{\text{op}}$, the fact that $\langle \cdot, \cdot \rangle$ dualizes the bimodules structures on \mathcal{O}_q and $\dot{\mathbf{U}}$, and the description of the associated Kashiwara operators on $\mathcal{B}(\mathcal{O}_q)$ and $\mathcal{B}(\dot{\mathbf{U}})$. Here is an alternative argument. By the very definition of the sets $\dot{\mathbf{B}}[\lambda]$ we have $b^- 1_{\varpi_i} \in \dot{\mathbf{B}}[\varpi_i]$, $b^+ 1_{-\varpi_i} \in \dot{\mathbf{B}}[-w_0(\varpi_i)]$. We wish to check if their duals $(b^- 1_{\varpi_i})^*$, $(b^+ 1_{-\varpi_i})^*$ coincide with the elements of \mathcal{O}_A on the right sides of (2.56) and (2.57). As already noticed after (2.48), by the isomorphism $\mathcal{O}_A(\leq \lambda)/\mathcal{O}_A(< \lambda) \cong \text{End}({}_A V_\lambda)^*$ every matrix coefficient of ${}_A V_\lambda$ belongs to $\mathcal{O}_A(\leq \lambda)$. Now, the A -modules $\mathcal{O}_A(\leq \varpi_i)$ and $\mathcal{O}_A(\leq -w_0(\varpi_i))$ are generated by $\dot{\mathbf{B}}[\varpi_i]^*$ and $\dot{\mathbf{B}}[-w_0(\varpi_i)]^*$, respectively. Because $((\bar{\pi}_\lambda^*)^{-1} \circ d_\lambda)(\dot{\mathbf{B}}[\lambda]^*)$ coincides with $\underline{\mathbf{B}}_\lambda^{\text{up}} \otimes ({}^\omega \underline{\mathbf{B}}_\lambda)^{\text{up}}$, the conclusion follows. \blacksquare

Note that the same argument implies that, for every $\lambda \in P_+$, any matrix coefficient of V_λ in the upper canonical basis and vanishing on the elements of $\dot{\mathbf{B}}[\lambda']$ for $\lambda' < \lambda$ must belong to $\dot{\mathbf{B}}[\lambda]^*$. For instance, in the \mathfrak{sl}_2 case, $\mathcal{O}_A(\leq 2)$ has canonical basis $\dot{\mathbf{B}}[0]^* \llbracket \dot{\mathbf{B}}[2]^*$, so the matrix coefficients of V_2 vanishing on 1_0 belong to $\dot{\mathbf{B}}[2]^*$. This can be observed in (2.45), using the comments in the paragraph before (2.52).

Though the A -module ${}_A V_\mu \otimes_A {}_A V_\nu$ has no decomposition like (2.7), we can refine the map $C(\mu) \otimes C(\nu) \rightarrow C(\mu + \nu)$ in (2.8) to an A -linear map defined on ${}_A C(\mu) \otimes_A {}_A C(\nu)$. Indeed, there is a unique injective morphism of U_A^{res} -modules $\mathfrak{T}_{\mu, \nu}: {}_A V_{\mu+\nu} \rightarrow {}_A V_\mu \otimes_A {}_A V_\nu$, which is given by $\mathfrak{T}_{\mu, \nu}(v_{\mu+\nu}) = v_\mu \otimes v_\nu$ [83, Proposition 25.1.2 (a)–(b)]. It defines a morphism of based modules

$$(V_{\mu+\nu}, \underline{\mathbf{B}}_{\mu+\nu}) \rightarrow (V_\mu \otimes V_\nu, \underline{\mathbf{B}}_\mu \diamond \underline{\mathbf{B}}_\nu),$$

where $\underline{\mathbf{B}}_\mu \diamond \underline{\mathbf{B}}_\nu := \{b \diamond b', b \in \underline{\mathbf{B}}_\mu, b' \in \underline{\mathbf{B}}_\nu\}$ [83, Proposition 27.1.7]. Hence, $\mathfrak{T}_{\mu, \nu}$ is a split A -linear map, i.e., there exists a A -linear map $\mathfrak{S}_{\mu, \nu}: {}_A V_\mu \otimes_A {}_A V_\nu \rightarrow {}_A V_{\mu+\nu}$ such that $\mathfrak{S}_{\mu, \nu} \circ \mathfrak{T}_{\mu, \nu} = \text{id}$. Note that $\mathfrak{S}_{\mu, \nu}$ is not a U_A^{res} -morphism. Similarly, the unique morphism of U_A^{res} -modules ${}^\omega \mathfrak{T}_{\mu, \nu}: {}^\omega V_{\mu+\nu} \rightarrow {}^\omega V_\mu \otimes_A {}^\omega V_\nu$ is a split injection. Define $\rho_{\mu', \mu''}: \dot{\mathbf{U}}_A \rightarrow {}^\omega V_{\mu'} \otimes_A {}^\omega V_{\mu''}$ by

$$\rho_{\mu', \mu''}(u) = u \left({}^\omega v_{w_0(\mu')} \otimes_A v_{w_0(\mu'')} \right),$$

and $\rho_{\mu', \mu'', \nu', \nu''}: \dot{\mathbf{U}}_A^{\otimes 2} \rightarrow {}^\omega V_{\mu'} \otimes_A {}^\omega V_{\mu''} \otimes_A {}^\omega V_{\nu'} \otimes_A {}^\omega V_{\nu''}$ by

$$\rho_{\mu', \mu'', \nu', \nu''}(u) = u \left({}^\omega v_{w_0(\mu')} \otimes_A v_{w_0(\mu'')} \otimes_A {}^\omega v_{w_0(\nu')} \otimes_A v_{w_0(\nu'')} \right).$$

Define $\tau_{\mu', \mu'', \nu', \nu''}: {}^\omega V_{\mu'+\nu'} \otimes_A {}^\omega V_{\mu''+\nu''} \rightarrow {}^\omega V_{\mu'} \otimes_A {}^\omega V_{\mu''} \otimes_A {}^\omega V_{\nu'} \otimes_A {}^\omega V_{\nu''}$ by

$$\tau_{\mu', \mu'', \nu', \nu''} = (1 \otimes \hat{R}^{-1} \otimes 1) ({}^\omega \mathfrak{T}_{\mu', \nu'} \otimes {}^\omega \mathfrak{T}_{\mu'', \nu''}).$$

It is an injective morphism of U_A^{res} -modules. In [84, Section 1.13], Lusztig proved that $\tau_{\mu', \mu'', \nu', \nu''}$ is a split A -linear map ([84] uses \hat{R} instead of \hat{R}^{-1} , since our coproducts on U_q^{ad} are opposite), and that it satisfies

$$\tau_{\mu', \mu'', \nu', \nu''} \rho_{\mu' + \mu'', \nu' + \nu''} = \rho_{\mu', \mu'', \nu', \nu''} \Delta, \quad (2.58)$$

where Δ is the coproduct of \dot{U}_A , see (2.28).

Now take $\mu := \mu' = \mu''$, $\nu := \nu' = \nu'' \in P_+$, and put $\tau_{\mu, \nu} := \tau_{\mu, \mu, \nu, \nu}$. It follows from the classical decomposition (2.7) over $\mathbb{C}(q)$, and (2.8) and (2.51), that the product of \mathcal{O}_A yields a map $m: \mathcal{O}_A(\leq \mu) \otimes_A \mathcal{O}_A(\leq \nu) \rightarrow \mathcal{O}_A(\leq \mu + \nu)$.

Denote the projection map $p_{\mu + \nu}: \mathcal{O}_A(\leq \mu + \nu) \rightarrow {}_A\dot{C}(\mu + \nu)$, define ${}_A\dot{\tau}_{\mu, \nu} := p_{\mu + \nu} \circ m$, and put

$$\pi'_\lambda: \mathcal{O}_A(\leq \lambda) \longrightarrow \mathcal{O}_A(\leq \lambda) / \mathcal{O}_A(< \lambda) \xrightarrow{(\bar{\pi}'_\lambda)^{-1} \circ \bar{d}_\lambda} (\text{End}({}_A V_\lambda))^*,$$

where the first map is the quotient map. Consider the diagram

$$\begin{array}{ccc} {}_A\dot{C}(\mu) \otimes {}_A\dot{C}(\nu) & \xrightarrow{{}_A\dot{\tau}_{\mu, \nu}} & {}_A\dot{C}(\mu + \nu) \\ \downarrow \pi'_\mu \otimes \pi'_\nu & & \downarrow \pi'_{\mu + \nu} \\ (\text{End}({}_A V_\mu))^* \otimes (\text{End}({}_A V_\nu))^* & \xrightarrow{\tau_{\mu, \nu}^t} & (\text{End}({}_A V_{\mu + \nu}))^*, \end{array}$$

where $\tau_{\mu, \nu}^t$ is the transpose of Lusztig's map $\tau_{\mu, \nu}$.

Proposition 2.12. *The map ${}_A\dot{\tau}_{\mu, \nu}: {}_A\dot{C}(\mu) \otimes_A {}_A\dot{C}(\nu) \rightarrow {}_A\dot{C}(\mu + \nu)$ is split as an A -linear map and the above diagram is commutative.*

Proof. The commutativity of the diagram comes from equation (2.58). The epimorphism π'_λ is injective on ${}_A\dot{C}(\lambda)$, and maps the canonical basis elements to the elements of the upper canonical basis $\mathbf{B}_\lambda^{\text{up}} \otimes (\omega \mathbf{B}_\lambda)^{\text{up}}$. By Lusztig's results recalled above, the epimorphism $\tau_{\mu, \nu}^t$ splits as an A -linear map. Therefore, the same is true of ${}_A\dot{\tau}_{\mu, \nu}$. \blacksquare

We stress that ${}_A\dot{\tau}_{\mu, \nu}$ plays for \mathcal{O}_A the same role as the map (2.8) for \mathcal{O}_q .

Finally, we consider for any $n \geq 1$ the invariant elements of $\mathcal{O}_A^{\otimes n}$ endowed with the action coad_n^r of U_A^{res} , see (2.15) (recall that $\mathcal{L}_{0, n} = \mathcal{O}_q^{\otimes n}$ as U_q^{ad} -module).

First note that, by definition, $\mathcal{O}_A(G^n)$ is the restricted dual of the Hopf algebra $U_A^{\text{res}}(\mathfrak{g}^{\oplus n})$, associated to its category of type 1 modules. By ordering the summands of $\mathfrak{g}^{\oplus n}$ we get an isomorphism $U_A^{\text{res}}(\mathfrak{g}^{\oplus n}) \cong U_A^{\text{res}}(\mathfrak{g})^{\otimes n}$, and any type 1 simple $U_A^{\text{res}}(\mathfrak{g})^{\otimes n}$ -module is isomorphic to $V_{[\lambda]} := \bigotimes_{i=1}^n V_{\lambda_i}$ endowed with the componentwise action, for some $[\lambda] := (\lambda_1, \dots, \lambda_n) \in P_+^n$ (this is a classical fact; see, e.g., [51, Theorem 3.10.2]). Therefore, we have an isomorphism $\mathcal{O}_A(G^n) \cong \mathcal{O}_A^{\otimes n}$. With the same notation $[\lambda] := (\lambda_1, \dots, \lambda_n) \in P_+^n$, let us put

$$\begin{aligned} {}_A\dot{C}([\lambda]) &:= \bigotimes_{i=1}^n {}_A\dot{C}(\lambda_i) = \bigoplus_{b \in \bigotimes_{i=1}^n \dot{\mathbf{B}}[\lambda_i]^*} Ab, \\ \mathcal{O}_A(\leq [\lambda]) &:= \bigotimes_{i=1}^n \mathcal{O}_A(\leq \lambda_i) = \bigoplus_{[\lambda'] \in P_+^n, \lambda'_i \leq \lambda_i} {}_A\dot{C}([\lambda']). \end{aligned}$$

We thus obtain a decomposition into based $(U_A^{\text{res}} \otimes (U_A^{\text{res}})^{\text{op}})^{\otimes n}$ -modules

$$\mathcal{O}_A^{\otimes n} = \sum_{[\lambda] \in P_+^n} \mathcal{O}_A(\leq [\lambda]).$$

Now $\text{coad}_n^r = (\text{coad}^r)^{\otimes n} \circ \Delta^{(n-1)}$ gives structures of U_A^{res} -modules to $\mathcal{O}_A^{\otimes n}$ and $\mathcal{O}_A(\leq [\lambda])$. In order to make it a based module, we give it the “ \diamond ” product of the canonical bases of the factors $\mathcal{O}_A(\leq \lambda_i)$, i.e.,

$$\dot{\mathbf{B}}[[\lambda]]^* := \diamond_{i=1}^n \left(\prod_{\lambda'_i \leq \lambda_i} \dot{\mathbf{B}}[\lambda'_i]^* \right).$$

We thus obtain a decomposition into based U_A^{res} -modules

$$\mathcal{O}_A^{\otimes n} = \sum_{[\lambda] \in P_+^n} (\mathcal{O}_A(\leq [\lambda]), \dot{\mathbf{B}}[[\lambda]]^*), \quad (2.59)$$

with composition factors $\otimes_{i=1}^n (\text{End}({}_A V_{\lambda_i}))^*$. By the properties of “ \diamond ” products of bases of based modules, the underlying A -module is

$$\mathcal{O}_A^{\otimes n} = \bigoplus_{[\lambda] \in P_+^n} {}_A \dot{C}([\lambda]). \quad (2.60)$$

Finally, we state the last property of based modules we need. Let (M, B) be a based module. Recall the notations introduced around (2.34). It is proved in [83, Proposition 27.1.8] that for every $\lambda \in P_+$ the submodule $M[\geq \lambda]$ is a sub-based module of M , and that it has the basis

$$B \cap M[\geq \lambda] = \bigcup_{\lambda' \geq \lambda} B[\lambda']. \quad (2.61)$$

Consider $M[\neq 0] := \bigoplus_{\lambda \neq 0} M[\lambda]$, the largest proper submodule of M that contains no non-zero invariant element. Recall that the space of *coinvariants* of M is

$$M_{U_q^{\text{ad}}} = M/M[\neq 0] = M/\mathbb{C}(q)\{um - \varepsilon(u)m, m \in M, u \in U_q^{\text{ad}}\}$$

that is, the largest quotient of M with trivial action, where $\varepsilon: U_q^{\text{ad}} \rightarrow \mathbb{C}(q)$ is the counit. It follows from (2.61) that $M[\neq 0]$ is a sub-based module of M , with the basis $\bigcup_{\lambda \neq 0} B[\lambda]$, and we have (this is, [83, Proposition 27.2.6]):

Proposition 2.13. *The quotient map $\pi: M \rightarrow M_{U_q^{\text{ad}}}$ is a morphism of based modules, where $M_{U_q^{\text{ad}}}$ is endowed with the basis $B_{U_q^{\text{ad}}} := \pi(B[0])$.*

Keeping the same notations, let ${}_A M \subset M$ be the A -module generated by B , and let ${}_A M^* \subset M^*$ be the A -module generated by B^* . They are U_A^{res} -modules. Denote by $({}_A M^*)^{U_A^{\text{res}}}$ the submodule of U_A^{res} -invariant elements of ${}_A M^*$, regarded as a right module in the natural way.

Lemma 2.14. *We have a direct sum decomposition of A -modules*

$${}_A M^* = ({}_A M^*)^{U_A^{\text{res}}} \bigoplus_A {}_A N, \quad (2.62)$$

where ${}_A N \subset {}_A M^*$ is the A -submodule generated by $\bigcup_{\lambda \neq 0} B[\lambda]^*$.

Proof. By Proposition 2.13, the transpose map $\pi^t: (M_{U_q^{\text{ad}}})^* \rightarrow M^*$ is a monomorphism mapping the dual basis $B_{U_q^{\text{ad}}}^*$ to the subset $B[0]^*$ of B^* . The image of π^t is $(M^*)^{U_q^{\text{ad}}}$. If we set ${}_A M_{U_A^{\text{res}}} = \pi({}_A M)$, then $\pi^t({}_A M_{U_A^{\text{res}}})^* = ({}_A M^*)^{U_A^{\text{res}}}$ is generated by $B[0]^*$, which concludes the proof. \blacksquare

Note that, since $B[0]$ is in general not invariant under the action of U_A^{res} , ${}_A N$ need not be stable under this action.

We are now ready to draw consequences of this discussion and the previous results. As usual denote by $(\mathcal{O}_A^{\otimes n})^{U_A^{\text{res}}}$ the subspace of invariant elements of $\mathcal{O}_A^{\otimes n}$ for the action coad_n^r . In the case $n = 1$, it is just the center $\mathcal{Z}(\mathcal{O}_A)$.

Theorem 2.15. $(\mathcal{O}_A^{\otimes n})^{U_A^{\text{res}}}$ is a direct summand of the A -module $\mathcal{O}_A^{\otimes n}$ for any $n \geq 1$.

Proof. By equation (2.59), it is enough to show that for every $[\lambda] \in P_+^n$ the invariant elements of $\mathcal{O}_A(\leq [\lambda])$ form a direct summand, and these summands are compatible with non-empty intersections $\mathcal{O}_A(\leq [\lambda]) \cap \mathcal{O}_A(\leq [\lambda'])$. Using that $\mathcal{O}_A(G^n) \cong \mathcal{O}_A^{\otimes n}$ and viewing P_+^n as the weight lattice of G^n , it is enough to prove these claims for $n = 1$. Given $\lambda \in P_+$ put

$$P_\lambda = \{\lambda' \in P_+, \lambda' \not\leq \lambda\},$$

and denote by $\dot{\mathbf{U}}_A[P_\lambda]$ the A -submodule of $\dot{\mathbf{U}}_A$ generated by $\coprod_{\lambda' \in P_\lambda} \dot{\mathbf{B}}[\lambda']$. Also, let us put $\dot{\mathbf{U}}[P_\lambda] = \dot{\mathbf{U}}_A[P_\lambda] \otimes_A \mathbb{C}(q)$. The complement $P_+ \setminus P_\lambda$ is finite, and if $\lambda' \in P_\lambda$ and $\lambda'' \geq \lambda'$, then $\lambda'' \in P_\lambda$. By the results of [83, Section 29.2], $\dot{\mathbf{U}}[P_\lambda]$ is a two-sided ideal, and the quotient algebra $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]$ is finite-dimensional with unit the coset of $\sum_{\lambda' \leq \lambda} 1_{\lambda'}$, and it is semisimple, isomorphic to $\bigoplus_{\lambda' \leq \lambda} \text{End}(V_{\lambda'})$ (whereas $\dot{\mathbf{U}}_A/\dot{\mathbf{U}}_A[P_\lambda]$ has indecomposable modules, see Example 2.17). It inherits from $\dot{\mathbf{U}}$ a canonical basis, formed by the non-zero cosets of elements of $\dot{\mathbf{B}}$, and with this basis $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]$ is a based module for the right adjoint action ad^r . Similarly as for (2.47), we have a morphism of U_A^{res} -modules

$$\tilde{d}_\lambda: \mathcal{O}_A(\leq \lambda) \longrightarrow \text{Hom}(\dot{\mathbf{U}}_A/\dot{\mathbf{U}}_A[P_\lambda], A), \quad \alpha \longmapsto \langle \alpha, \cdot \rangle,$$

which is an isomorphism by (2.50) and the computation $\dim(\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]) = \sum_{\lambda' \leq \lambda} \dim(V_{\lambda'})^2$ in [83, Section 29.2]. Applying Proposition 2.13 and (2.62) to the based module $M = \dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]$, we obtain that the invariant elements of $\mathcal{O}_A(\leq \lambda)$ form a direct summand. Finally, for any $\lambda, \lambda' \in P_+$ we have $\mathcal{O}_A(\leq \lambda) \cap \mathcal{O}_A(\leq \lambda') \cong \text{Hom}(\dot{\mathbf{U}}_A/(\dot{\mathbf{U}}_A[P_\lambda] + \dot{\mathbf{U}}_A[P_{\lambda'}]), A)$. Applying Proposition 2.13 and (2.62) to the based module $M := \dot{\mathbf{U}}/(\dot{\mathbf{U}}[P_\lambda] + \dot{\mathbf{U}}[P_{\lambda'}])$, we obtain that the invariant elements $({}_A M^*)^{U_A^{\text{res}}}$ of $\mathcal{O}_A(\leq \lambda) \cap \mathcal{O}_A(\leq \lambda')$ form a direct A -summand. Since the latter is a based U_A^{res} -submodule of $\mathcal{O}_A(\leq \lambda)$ and $\mathcal{O}_A(\leq \lambda')$, this summand is also a direct A -summand of $\mathcal{O}_A(\leq \lambda)^{U_A^{\text{res}}}$ and $\mathcal{O}_A(\leq \lambda')^{U_A^{\text{res}}}$. This shows the A -modules $\mathcal{O}_A(\leq \lambda)^{U_A^{\text{res}}}$ for all $\lambda \in P_+$ match to form the A -summand $(\mathcal{O}_A)^{U_A^{\text{res}}}$ of \mathcal{O}_A , and thus concludes the proof. \blacksquare

Remark 2.16. Let $(M, B), (M', B')$ be based modules, with tensor product $(M \otimes M', B_\diamond)$, and $B_\diamond[0] \subset B_\diamond$ the subset in bijection with the canonical basis of the space of coinvariants $(M \otimes M')_{U^{\text{ad}}}$ (see Proposition 2.13). This subset is described in [83, Proposition 27.3.8] in terms of B and B' . Since $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P_\lambda]$ is semisimple with known summands, and the construction of the “ \diamond ” product of canonical bases is associative, one can recursively compute the subset of the canonical basis of $\bigotimes_{i=1}^n \dot{\mathbf{U}}/\dot{\mathbf{U}}[P_{\lambda_i}]$ (endowed with the action dual to coad_n^r) which is in bijection with the canonical basis of the space of coinvariants. Therefore, a complete (though highly nontrivial) characterization of the basis of $(\mathcal{O}_A^{\otimes n})^{U_A^{\text{res}}}$ can be obtained. Examples can be found in [83, Section 27.3.10]. In the case $\mathfrak{g} = \mathfrak{sl}_2$, the canonical basis of the dual space $\text{End}(V_1^{\otimes n})^*$ has been identified in [56] with the canonical basis of the Temperley–Lieb algebra $TL_n(q)$.

Example 2.17. The simplest case is already instructive. Namely, consider V_1 and V_2 , the simple $U_q^{\text{ad}}(\mathfrak{sl}_2)$ -modules of type 1 and dimension two and three.

On V_1 , we have the lower canonical basis vectors v_+ and v_- , such that $Kv_+ = qv_+, Ev_+ = 0, v_- = Fv_+$. The canonical lower and upper bases of V_1 are both $\{v_+, v_-\}$. Using the relation (2.32), we see that the elements of $\dot{\mathbf{B}}_{0,1}$ and $\dot{\mathbf{B}}_{1,0}$ are $1_1, F1_1$ and $1_{-1}, E1_{-1}$, respectively;

the dual linear forms generate $\mathcal{O}_A(\mathrm{SL}_2)$, they are the matrix coefficients a, c, d and b respectively. By (2.37), we have $\hat{\mathbf{B}}[1] = \hat{\mathbf{B}}_{0,1} \amalg \hat{\mathbf{B}}_{1,0}$.

Next consider V_2 . On V_2 , we have the canonical highest weight vector v_0 of weight 2, and lower canonical basis $\mathbf{B}_2 = \{v_0, v_1, v_2\}$, where $v_1 = Fv_0$ and $v_2 = F^{(2)}v_0$. We have $\mathbf{B}_2^{\mathrm{up}} = \{v_0, [2]_q^{-1}v_1, v_2\}$ (see Appendix A). We can identify the ambient space of the right module V_2^r with that of V_2 ; its highest weight vector is then v_0 , and its canonical lower and upper bases are $\mathbf{B}_2^r = \{v_0, v_1, v_2\}$ and $\mathbf{B}_2^{r,\mathrm{up}} = \{v_0, [2]_q^{-1}v_1, v_2\}$.

Consider now the module ${}^\omega V_1 \otimes V_1$. We have

$$\hat{R} = \sum_{n=0}^{\infty} \frac{(q - q^{-1})^n}{[n]_q!} q^{n(n-1)/2} E^n \otimes F^n,$$

so the matrix of the involution $\Psi = \hat{R}^{-1} \circ^-$ in the basis $v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-$ is

$$(\hat{R}^{-1} \circ^-)_{\omega V_1, V_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q^{-1} - q & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, the canonical basis $\mathbf{B}_{1,1}$ is formed by the vectors $v_+ \diamond v_+ = v_+ \otimes v_+ + q^{-1}v_- \otimes v_-$ and $v_+ \diamond v_- = v_+ \otimes v_-, v_- \diamond v_+ = v_- \otimes v_+, v_- \diamond v_- = v_- \otimes v_-$. Consider the partition $\mathbf{B}_{1,1} = \mathbf{B}_{1,1}[2] \cup \mathbf{B}_{1,1}[0]$. We have $\mathbf{B}_{1,1}[2] = \{v_- \diamond v_+, v_+ \diamond v_+, v_+ \diamond v_-\}$, which is a basis of the three-dimensional submodule W_2 of $V_1 \otimes V_1$. Since $\mathbf{B}_{1,1}$ is an A -basis of ${}^\omega V_1 \otimes_A V_1$, it follows that the epimorphism $\tau_{1,1}^t: {}_A \dot{C}(1) \otimes_A {}_A \dot{C}(1) \rightarrow {}_A \dot{C}(2)$ splits (see Proposition 2.12). The vector $v_- \diamond v_-$ is cyclic, so $\mathbf{B}_{1,1}[0] = \{v_- \diamond v_-\}$. By the definitions, we have $v_+ \diamond v_+ = (1 \diamond_0 1)_{1,1}$, $v_+ \diamond v_- = (1 \diamond_0 F)_{1,1}$, $v_- \diamond v_+ = (F \diamond_0 1)_{1,1}$, $v_- \diamond v_- = (F \diamond_0 F)_{1,1}$, so the corresponding elements of $\hat{\mathbf{B}}_{1,1} \subset \hat{\mathbf{B}}$ are respectively $1_0, 1_{-2}F, 1_2E$, and $F1_2E = E1_{-2}F$.

The invariant submodule W_0 of ${}^\omega V_1 \otimes V_1$ is generated by $v' = v_- \otimes v_- - q^{-1}v_+ \otimes v_+$. The U_A^{res} -modules ${}^\omega V_1 \otimes_A V_1$ and $W_2 \oplus W_0$ are not equal, though they are by extending scalars to $\mathbb{C}(q)$. Indeed, we have

$$v_+ \otimes v_+ = [2]_q^{-1}(qv_+ \diamond v_+ - v') \notin W_2 \oplus W_0.$$

The module of coinvariants is $({}^\omega V_1 \otimes V_1)_{U_q^{\mathrm{ad}}} = \mathbb{C}(q)\{\pi(v_- \otimes v_-)\}$, where as usual $\pi: {}^\omega V_1 \otimes V_1 \rightarrow ({}^\omega V_1 \otimes V_1)_{U_q^{\mathrm{ad}}}$ is the quotient map. The transpose map $\pi^t: ({}^\omega V_1 \otimes V_1)_{U_q^{\mathrm{ad}}}^* \rightarrow ({}^\omega V_1 \otimes V_1)^*$ sends $(v_- \diamond v_-)^*$ to the unique U_q^{ad} -invariant linear map

$$\mathrm{ev}_1: {}^\omega V_1 \otimes V_1 \rightarrow \mathbb{C}(q)$$

such that $\mathrm{ev}_1(v_- \otimes v_-) = 1$.

Note that, since elements of $\dot{\mathbf{U}}_A[\lambda > 2]$ act trivially on modules with all isotypical components of highest weight ≤ 2 , ${}^\omega V_1 \otimes_A V_1$ is an indecomposable module over $\dot{\mathbf{U}}_A/\dot{\mathbf{U}}_A[\lambda > 2]$ (that is, $\dot{\mathbf{U}}_A/\dot{\mathbf{U}}_A[P_2]$ in the notations of Theorem 2.15).

2.2.3 Some consequences on $\mathcal{L}_{0,n}^A$ and $\mathcal{M}_{0,n}^A$

Recall from Section 2.2.1 the definition of the integral forms $\mathcal{L}_{0,n}^A$ and $\mathcal{M}_{0,n}^A$.

Proposition 2.18. $\mathcal{L}_{0,n}^A$ and $\mathcal{M}_{0,n}^A$ are free A -modules, and $\mathcal{M}_{0,n}^A$ is a direct summand of the A -module $\mathcal{L}_{0,n}^A$. Moreover, $\mathcal{L}_{0,n}^A$ is a finitely generated ring.

Proof. Since $\mathcal{L}_{0,n}^A = \mathcal{O}_A^{\otimes n}$ as U_A^{res} -modules, by (2.60) it has the basis $\bigcup_{[\lambda] \in P_+^n} \dot{\mathbf{B}}[[\lambda]]^*$. Therefore, $\mathcal{L}_{0,n}^A$ is a free A -module. Since A is a principal ideal domain, it follows that $\mathcal{M}_{0,n}^A$ is a free A -submodule [77, Appendix 2.2]. By Theorem 2.15, there is a direct sum decomposition as A -module

$$\mathcal{L}_{0,n}^A = \mathcal{M}_{0,n}^A \oplus {}_A N, \quad (2.63)$$

and the proof identifies a basis of $\mathcal{M}_{0,n}^A$ as a subset of $\bigcup_{[\lambda] \in P_+^n} \dot{\mathbf{B}}[[\lambda]]^*$.

Next, consider the question of finite generation. By the formula (2.17), it is enough to verify this for $\mathcal{L}_{0,1}^A$, but $\mathcal{L}_{0,1}^A = \mathcal{O}_A$ as an A -module, and \mathcal{O}_A is finitely generated by the matrix coefficients of the fundamental U_A^{res} -modules ${}_A V_{\varpi_k}$, $k \in \{1, \dots, m\}$ (see (2.56) and (2.57)). Any monomials in these generators can be written as a A -linear combination of monomials in the same generators but with the product of $\mathcal{L}_{0,1}^A$, instead of the product \star . This follows from the integrality properties of the R -matrix, and the formula inverse to (2.9) (see in [18, Section 3.3 and the formulas (4.6)–(4.8)]). ■

Remark 2.19.

- (a) As noted in (2.62), the A -module ${}_A N$ in the decomposition (2.63) is in general not a U_A^{res} -module. Therefore, the A -linear projection map $\mathcal{R}_A: \mathcal{L}_{0,n}^A \rightarrow \mathcal{M}_{0,n}^A$ such that $\text{Ker}(\mathcal{R}_A) = {}_A N$ is not a Reynolds operator, for it does not satisfy the identity $\mathcal{R}_A(\alpha\beta) = \alpha\mathcal{R}_A(\beta)$ for all $\alpha \in \mathcal{M}_{0,n}^A$, $\beta \in \mathcal{L}_{0,n}^A$.
- (b) Recall (2.24). In coherence with (a) above, there is no normalized Haar measure on \mathcal{O}_A taking values in A . Indeed, by extending scalars over $\mathbb{C}(q)$ it should otherwise coincide with the Haar measure $h: \mathcal{O}_q \rightarrow \mathbb{C}(q)$, but in the notations of Example 2.17 (see also the comments after (2.44)), since $h(v_0^* \otimes v_0) = 0$ we have $h(bc) = -1/(q + q^{-1})$, whence h cannot be defined on \mathcal{O}_A .
- (c) The Haar measure yields a well-defined \mathcal{A}_0 -linear map $h: \mathcal{L}(\mathcal{O}_q) \rightarrow \mathcal{A}_0$ (and analogously \mathcal{A}_0 -linear and \mathcal{A}_∞ -linear maps $h: \mathcal{L}_\diamond(\mathcal{O}_q^{\otimes n}) \rightarrow \mathcal{A}_0$ and $\bar{h}: \bar{\mathcal{L}}_\diamond(\mathcal{O}_q^{\otimes n}) \rightarrow \mathcal{A}_\infty$ for any $n \geq 1$, where $(\mathcal{L}_\diamond(\mathcal{O}_q^{\otimes n}), \mathcal{B}[[\lambda]]^*)$ is the crystal basis at $q = 0$ underlying the based U_q^{ad} -module (2.59)). Indeed, by (2.41) the lattice $\mathcal{L}_\lambda^{\text{up}} \otimes_{\mathcal{A}_0} \mathcal{L}_\lambda^{\text{up}}$ is generated by the matrix coefficients in the canonical bases of V_λ^r and V_λ . Since the normalisation by powers of q is vacuous on the trivial module $V_0^* \otimes V_0$, and h vanishes on $V_\lambda^* \otimes V_\lambda$ for $\lambda \in P_+ \setminus \{0\}$, the claim follows.

2.3 Perfect pairings

We will need to restrict the morphisms Φ^+ , Φ^- in (2.5) on the integral forms $\mathcal{O}_A(B_+)$, $\mathcal{O}_A(B_-)$. We collect their properties in Theorem 2.20 and the discussion thereafter. In order to state it, we recall first a few facts about R -matrices and related pairings.

Recall that \mathcal{C}_A is the category of U_A^{res} -modules of type 1. In [82, 83], Lusztig proved that $\mathcal{C}_A \otimes_A \mathbb{C}[q^{\pm 1/D}]$ is braided and ribbon, with braiding given by the collection of endomorphisms

$$R = (R_{V,W})_{V,W \in \text{Ob}(\mathcal{C}_A)}.$$

Actually, $R_{V,W}$ is represented by a matrix with coefficients in $q^{\mathbb{Z}/D} \mathbb{C}[q^{\pm 1}]$ on the tensor product of the lower canonical bases of V and W (see [83, Corollary 24.1.5]).

This can be rephrased as follows in Hopf algebra terms. Denote by \mathbb{U}_Γ the categorical completion of Γ , i.e., the Hopf algebra of natural transformations $F_{\mathcal{C}_A} \rightarrow F_{\mathcal{C}_A}$, where $F_{\mathcal{C}_A}: \mathcal{C}_A \rightarrow A\text{-Mod}_f$ is the forgetful functor towards the category $A\text{-Mod}_f$ of finite rank A -modules. Then

$\mathbb{U}_\Gamma \otimes_A \mathbb{C}[q^{\pm 1/D}]$ is quasi-triangular and ribbon with R -matrix

$$R \in \mathbb{U}_\Gamma^{\hat{\otimes} 2} \bigotimes_A \mathbb{C}[q^{\pm 1/D}].$$

As in (2.3), we can write

$$R^\pm = \sum_{(R)} R_{(1)}^\pm \otimes R_{(2)}^\pm.$$

There are pairings of Hopf algebras naturally related to the R -matrix R , considered as an element of $\mathbb{U}_q^{\hat{\otimes} 2}$. What follows is standard (see, e.g., [69, 70, 81]), for details we refer to [104, Proposition 3.73, Lemma 3.75, Theorem 3.92, Propositions 3.106 and 3.107]:

- There is a unique pairing of Hopf algebras $\rho: U_q(\mathfrak{b}_-)^{\text{cop}} \otimes U_q(\mathfrak{b}_+) \rightarrow \mathbb{C}(q^{1/D})$ such that, for every $\alpha, \lambda \in P$ and $l, k \in U_q(\mathfrak{h})$,

$$\begin{aligned} \rho(K_\lambda, K_\alpha) &= q^{(\lambda, \alpha)}, & \rho(F_i, E_j) &= \delta_{i,j} (q_i - q_i^{-1})^{-1}, \\ \rho(l, E_j) &= \rho(F_i, k) = 0. \end{aligned} \tag{2.64}$$

- The *Drinfeld pairing* $\tau: U_q(\mathfrak{b}_+)^{\text{cop}} \otimes U_q(\mathfrak{b}_-) \rightarrow \mathbb{C}(q^{1/D})$ is the bilinear map defined by $\tau(X, Y) = \rho(S(Y), X)$; it satisfies

$$\begin{aligned} \tau(K_\lambda, K_\alpha) &= q^{-(\lambda, \alpha)}, & \tau(E_j, F_i) &= -\delta_{i,j} (q_i - q_i^{-1})^{-1}, \\ \tau(l, F_i) &= \tau(E_j, k) = 0. \end{aligned} \tag{2.65}$$

- ρ and τ are perfect pairings; this means that they yield *isomorphisms* of Hopf algebras $i_\pm: U_q(\mathfrak{b}_\pm) \rightarrow \mathcal{O}_q(B_\mp)_{\text{op}}$ (with coefficients *a priori* extended to $\mathbb{C}(q^{1/D})$, but see below) defined by, for every $X \in U_q(\mathfrak{b}_+)$, $Y \in U_q(\mathfrak{b}_-)$,

$$\langle i_+(X), Y \rangle = \tau(S(X), Y), \quad \langle i_-(Y), X \rangle = \tau(X, Y).$$

Since $\mathcal{O}_q(B_\mp)_{\text{op}}$ is equipped with the *inverse* of the antipode of $\mathcal{O}_q(B_\mp)$, which is induced by the antipode $S_{\mathcal{O}_q}$ of \mathcal{O}_q , it follows that $i_\pm \circ S = S_{\mathcal{O}_q}^{-1} \circ i_\pm$.

- Denote by $p_\pm: \mathcal{O}_q(G) \rightarrow \mathcal{O}_q(B_\pm)$ the canonical projection map, i.e., the Hopf algebra homomorphism dual to the inclusion map $U_q(\mathfrak{b}_\pm) \hookrightarrow U_q(\mathfrak{g})$. For every $\alpha, \beta \in \mathcal{O}_q(G)$, we have

$$\langle \alpha \otimes \beta, R \rangle = \tau(i_+^{-1}(p_-(\beta)), i_-^{-1}(p_+(\alpha))). \tag{2.66}$$

Note that it is the use of weights $\alpha, \lambda \in P$ that forces the pairings ρ, τ to be defined over $\mathbb{C}(q^{1/D})$, instead of $\mathbb{C}(q)$. Then, let us consider the restrictions π_q^+ of ρ , and π_q^- of τ defined by the formulas (2.64) and (2.65), where now $\alpha \in Q$ and $k \in U_q^{\text{ad}}(\mathfrak{h})$. They take values in $\mathbb{C}(q)$, and define pairings

$$\pi_q^+: U_q(\mathfrak{b}_-)^{\text{cop}} \otimes U_q^{\text{ad}}(\mathfrak{b}_+) \rightarrow \mathbb{C}(q), \quad \pi_q^-: U_q(\mathfrak{b}_+)^{\text{cop}} \otimes U_q^{\text{ad}}(\mathfrak{b}_-) \rightarrow \mathbb{C}(q).$$

By the same arguments as for ρ and τ (e.g., in [104, Proposition 3.92]), it follows that π_q^\pm are perfect pairings. Note also that $\pi_q^- = \kappa \circ \pi_q^+ \circ (\kappa \otimes \kappa)$, where $\kappa: U_q \rightarrow U_q$ is the \mathbb{C} -linear automorphism extending $-: U_q^{\text{ad}} \rightarrow U_q^{\text{ad}}$ in Section 2.2.2, so defined by

$$\kappa(E_i) = F_i, \quad \kappa(F_i) = E_i, \quad \kappa(K_\lambda) = K_{-\lambda}, \quad \kappa(q) = q^{-1}. \tag{2.67}$$

In [41], De Concini–Lyubashenko described integral forms of π_q^\pm as follows. Denote by $m^*: \mathcal{O}_A \rightarrow \mathcal{O}_A(B_+) \otimes \mathcal{O}_A(B_-)$ the map dual to the multiplication map $\Gamma(\mathfrak{b}_+) \otimes \Gamma(\mathfrak{b}_-) \rightarrow \Gamma$, so $m^* = (p_+ \otimes p_-) \circ \Delta_{\mathcal{O}_A}$. Let $U_A(G^*)$ be the smallest A -subalgebra of $U_A(\mathfrak{b}_-)^{\text{cop}} \otimes U_A(\mathfrak{b}_+)^{\text{cop}}$ which contains the elements

$$1 \otimes K_i^{-1} \bar{E}_i, \quad \bar{F}_i K_i \otimes 1, \quad L_i^{\pm 1} \otimes L_i^{\mp 1}, \quad i = 1, \dots, m,$$

and is stable under the diagonal action of $\mathcal{B}(\mathfrak{g})$. The reason for the notation $U_A(G^*)$ will be explained at the beginning of Section 2.5. Note that $U_A(G^*)$ is free over A , a Hopf subalgebra, and that a basis is given by the elements

$$\bar{F}_{\beta_1}^{n_1} \cdots \bar{F}_{\beta_N}^{n_N} K_{n_1 \beta_1 + \cdots + n_N \beta_N} K_\lambda \otimes K_{-\lambda} K_{-p_1 \beta_1 - \cdots - p_N \beta_N} \bar{E}_{\beta_1}^{p_1} \cdots \bar{E}_{\beta_N}^{p_N}, \quad (2.68)$$

where $\lambda \in P$ and $n_1, \dots, n_N, p_1, \dots, p_N \in \mathbb{N}$.

Now, let v be a lowest weight vector of the lowest weight Γ -module ${}_A V_{-\lambda}$, $\lambda \in P_+$. As after Theorem 2.1, denote by $v^* \in {}_A V_{-\lambda}^*$ the dual vector, and by $\psi_{-\lambda} \in \mathcal{O}_A$ the matrix coefficient defined by $\langle \psi_{-\lambda}, x \rangle = v^*(xv)$ for every $x \in \Gamma$. Consider the maps $j_q^\pm: \mathcal{O}_q(B_\pm) \rightarrow U_q(\mathfrak{b}_\mp)^{\text{cop}}$ defined by

$$\langle \alpha_+, X \rangle = \pi_q^+(j_q^+(\alpha_+), X), \quad \langle \alpha_-, Y \rangle = \pi_q^-(j_q^-(\alpha_-), Y),$$

where $\alpha_\pm \in \mathcal{O}_q(B_\pm)$, $X \in U_q^{\text{ad}}(\mathfrak{b}_+)$, and $Y \in U_q^{\text{ad}}(\mathfrak{b}_-)$.

The following theorem summarizes results proved in [41, Sections 3 and 4]. Denote by $\mathcal{O}_A[\psi_{-\rho}^{-1}]$ the localization of \mathcal{O}_A by the element $\psi_{-\rho}$; this localization is well defined, for the set $\{\psi_{-\rho}^n\}_{n \in \mathbb{N}}$ is a left and right multiplicative Ore subset of \mathcal{O}_A (see Corollary 2.23 below for an analogous statement for $\mathcal{L}_{0,1}^A$). For the sake of clarity, let us spell out the correspondence of notations between statements: π_q^+ , π_q^- , $U_q(\mathfrak{b}_\mp)^{\text{cop}}$, $U_A(\mathfrak{b}_\mp)^{\text{cop}}$, $\mathcal{O}_A(B_\pm)$, $U_A(G^*)$ and Φ are denoted in [41] respectively by π'' , $\bar{\pi}''$, $U_q(\mathfrak{b}_\mp)_{\text{op}}$, $R_q[B_\pm]''$, $R_q[B_\pm]$, A'' and μ'' (the definition of j_A^\pm is implicit in [41, Section 4.2]).

Theorem 2.20.

- (1) π_q^\pm restricts to a perfect Hopf pairing between the unrestricted and restricted integral forms, $\pi_A^\pm: U_A(\mathfrak{b}_\mp)^{\text{cop}} \otimes \Gamma(\mathfrak{b}_\pm) \rightarrow A$.
- (2) j_q^\pm yields an isomorphism of Hopf algebras $j_A^\pm: \mathcal{O}_A(B_\pm) \rightarrow U_A(\mathfrak{b}_\mp)^{\text{cop}}$, satisfying $\langle \alpha_\pm, x_\pm \rangle = \pi_A^\pm(j_A^\pm(\alpha_\pm), x_\pm)$ for every $\alpha_\pm \in \mathcal{O}_A(B_\pm)$, $x_\pm \in \Gamma(\mathfrak{b}_\pm)$.
- (3) The map $\Phi := (j_A^+ \otimes j_A^-) \circ m^*: \mathcal{O}_A \rightarrow U_A(G^*) \subset U_A(\mathfrak{b}_-)^{\text{cop}} \otimes U_A(\mathfrak{b}_+)^{\text{cop}}$ is an embedding of Hopf algebras, and it extends to an isomorphism $\Phi: \mathcal{O}_A[\psi_{-\rho}^{-1}] \rightarrow U_A(G^*)$.

For our purposes, it is necessary to reformulate this result. Consider the morphisms of Hopf algebras $\Phi^\pm: \mathcal{O}_A(B_\pm) \rightarrow U_A(\mathfrak{b}_\mp)^{\text{cop}}$, $\alpha \mapsto (\alpha \otimes \text{id})(R^\pm)$.

Lemma 2.21. *We have $\Phi^\pm = j_A^\pm$.*

Proof. By definitions, for every $X \in U_q(\mathfrak{b}_+)^{\text{cop}}$, $Y \in U_q^{\text{ad}}(\mathfrak{b}_-)$, we have $\langle i_+(S^{-1}(X)), Y \rangle = \pi_q^-(X, Y)$, and similarly for every $X \in U_q^{\text{ad}}(\mathfrak{b}_+)$, $Y \in U_q(\mathfrak{b}_-)^{\text{cop}}$, we have $\langle i_-(S^{-1}(Y)), X \rangle = \pi_q^+(Y, X)$. By keeping these notations for X and Y , we deduce $j_q^-(i_+(S^{-1}(X))) = X$ and $j_q^+(i_-(S^{-1}(Y))) = Y$, i.e., $j_q^\pm = S \circ i_\mp^{-1}$. Because $S_{\mathcal{O}_q}^{-1} \circ i_\pm = i_\pm \circ S$, it follows that

$$j_q^\pm \circ S_{\mathcal{O}_q} = S^{-1} \circ j_q^\pm. \quad (2.69)$$

Also, for every $\alpha_- \in \mathcal{O}_q(B_-)$, we have

$$\begin{aligned} \langle \alpha_-, \Phi^+(i_-(Y)) \rangle &= \langle i_-(Y) \otimes \alpha_-, R \rangle = \tau(i_+^{-1}(\alpha_-), Y) \\ &= \pi_q^-(j_q^-(S_{\mathcal{O}_q}(\alpha_-)), Y) = \langle \alpha_-, S(Y) \rangle, \end{aligned}$$

where the first equality is by definition of Φ^+ (see (2.5)), the second is (2.66), the third follows from (2.69), and the last from the definition of j_q^- . Similarly, for every $\alpha_+ \in \mathcal{O}_q(B_+)$, we have

$$\begin{aligned} \langle \alpha_+, \Phi^-(i_+(X)) \rangle &= \langle i_+(X) \otimes \alpha_+, R^- \rangle = \langle \alpha_+ \otimes S_{\mathcal{O}_q}^{-1} \circ i_+(X), R \rangle = \langle \alpha_+ \otimes i_+(S(X)), R \rangle \\ &= \tau(S(X), i_-^{-1}(\alpha_+)) = \pi_q^+(S(i_-^{-1}(\alpha_+)), S(X)) \\ &= \pi_q^+(j_q^+(\alpha_+), S(X)) = \langle \alpha_+, S(X) \rangle. \end{aligned}$$

These computations imply $\Phi^\pm = S \circ i_{\mp}^{-1} = j_q^\pm$, and the result follows by taking integral forms. \blacksquare

2.4 Integral form and specialization of Φ_n

Recall the isomorphism of U_q -module algebras $\Phi_1: \mathcal{L}_{0,1} \rightarrow U_q^{\text{lf}}$, and that $U_A^{\text{lf}} = U_A \cap U_q^{\text{lf}}$. We have:

Corollary 2.22. *The map Φ_1 affords an embedding of U_A^{res} -module algebras $\Phi_1: \mathcal{L}_{0,1}^A \rightarrow U_A^{\text{lf}}$.*

Proof. The only thing to be proved is that $\Phi_1(\mathcal{O}_A) \subset U_A^{\text{lf}}$, since $\mathcal{L}_{0,1}^A = \mathcal{O}_A$ as A -module. But Lemma 2.21 and (2.12) imply $\Phi_1 = m \circ (\text{id} \otimes S^{-1}) \circ \Phi$, and Φ maps \mathcal{O}_A into $U_A(\mathfrak{b}_-)^{\text{cop}} \otimes U_A(\mathfrak{b}_+)^{\text{cop}}$ by Theorem 2.20. The conclusion follows. \blacksquare

Let us denote

$$d = \psi_{-\rho} \in \mathcal{L}_{0,1}^A.$$

(The linear forms $\psi_{-\lambda}$ have been introduced before Theorem 2.20.) When $\mathfrak{g} = \mathfrak{sl}_2$ the element d is one of the ‘‘standard’’ generators of $\mathcal{L}_{0,1}(\mathfrak{sl}_2)$ (see (4.5) below). In this case we have shown in [18, Lemma 5.7] that $\mathcal{L}_{0,1}^A$ has a well-defined localization $\mathcal{L}_{0,1}^A[d^{-1}]$, and that $\Phi_1: \mathcal{L}_{0,1}^A[d^{-1}] \rightarrow U_A^{\text{ad}} = T_{2^-}^{-1}U_A^{\text{lf}}$ is an isomorphism of algebras. A generalization of these facts to any \mathfrak{g} is provided by the following statement. As usual $\ell = K_{2\rho}$, the pivotal element.

Corollary 2.23.

- (1) *The set $\{d^n\}_{n \in \mathbb{N}}$ is a left and right multiplicative Ore set in $\mathcal{L}_{0,1}^A$. We can therefore define the localization $\mathcal{L}_{0,1}^A[d^{-1}]$.*
- (2) *Φ_1 extends to an embedding of U_A^{res} -module algebras $\Phi_1: \mathcal{L}_{0,1}^A[d^{-1}] \rightarrow U_A^{\text{lf}}[\ell]$, and $U_A^{\text{lf}}[\ell] = T_{2^-}^{-1}U_A^{\text{lf}}$.*

Proof. (1) Because $\mathcal{L}_{0,1}^A$ has no nontrivial zero divisors, d is a regular element. We have to show that for all $x \in \mathcal{L}_{0,1}^A$ there exists elements $y, y' \in \mathcal{L}_{0,1}^A$ and $d', d'' \in \{d^n\}_{n \in \mathbb{N}}$ such that $xd' = dy$ and $d''x = y'd$. In fact, $d' = d'' = d$ in the present situation. Indeed by (2.13), we have $\Phi_1(x)\Phi_1(d) = \Phi_1(x)K_{-2\rho} = K_{-2\rho}\text{ad}^r(K_{2\rho})(\Phi_1(x))$, and $\text{ad}^r(K_{2\rho})(\Phi_1(x)) = \Phi_1(\text{coad}^r(K_{2\rho})(x))$. Therefore, the left Ore condition is satisfied with $y = \text{coad}^r(K_{2\rho})(x)$. Similarly, one finds y' .

(2) The first claim follows immediately from Corollary 2.22 and $\Phi_1(d) = \ell^{-1}$, which is a regular element of U_A . For the second claim, since $K_{-2\rho} = \prod_{j=1}^m L_j^{-2}$, localizing in d we obtain

$$L_j^2 = \prod_{k \neq j} L_k^{-2} \Phi_1(d^{-1}) = \Phi_1 \left(\prod_{k \neq j} \psi_{-\varpi_k} d^{-1} \right) \in \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}]).$$

Therefore, $T_{2^-}^{-1} \subset \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$, which implies the assertion (2). \blacksquare

We expect that the inclusion $\Phi_1(\mathcal{O}_A) \subset U_A^{\text{lf}}$ is an equality, but have no proof yet. However, recall Joseph–Letzter’s Theorem 2.1 (1) and (2).

Proposition 2.24. *We have*

$$U_A = T_2^{-1}U_A^{\text{lf}}[T/T_2] = \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2],$$

and therefore $\Phi_1: \mathcal{L}_{0,1}^A[d^{-1}] \rightarrow T_2^{-1}U_A^{\text{lf}}$ is an isomorphism. Moreover,

$$\Phi_1(\mathcal{O}_A) = \bigoplus_{\lambda \in 2P_+} \text{ad}^r(U_A^{\text{res}})(K_{-\lambda}).$$

Proof. The inclusions $T \subset U_A$, $U_A^{\text{lf}} \subset U_A$ and $\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}]) \subset T_2^{-1}U_A^{\text{lf}}$ imply

$$\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2] \subset T_2^{-1}U_A^{\text{lf}}[T/T_2] \subset U_A.$$

For the inverse inclusion, it is enough to show that any PBW basis vector of U_A lies in $\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2]$. This will follow at once if this is true of all root vectors $\bar{E}_{\beta_k}, \bar{F}_{\beta_k}$. Let us show this explicitly for the simple root vectors \bar{E}_i and \bar{F}_i . For every positive root α , define elements $\psi_{-\lambda}^\alpha, \psi_{-\lambda}^{-\alpha} \in \mathcal{O}_A$ by the formulas

$$\langle \psi_{-\lambda}^\alpha, x \rangle = v^*(xE_\alpha v), \quad \langle \psi_{-\lambda}^{-\alpha}, x \rangle = v^*(F_\alpha x v),$$

where $x \in \Gamma$. It is shown in [41, Lemma 4.5] that

$$\begin{aligned} \Phi(\psi_{-\lambda}^\alpha) &= K_{-\lambda} \otimes K_\lambda, & \Phi(\psi_{-\varpi_j}^{\alpha_i}) &= -\delta_{i,j} q_i L_i^{-1} \otimes L_i K_i^{-1} \bar{E}_i, \\ \Phi(\psi_{-\varpi_j}^{-\alpha_i}) &= \delta_{i,j} q_i^{-1} \bar{F}_i K_i L_i^{-1} \otimes L_i. \end{aligned}$$

(Note that the generators denoted by E_i and F_i in [41] are respectively $K_i^{-1}E_i$ and $F_i K_i$ in our notations, which explains the factors q_i, q_i^{-1} in the formulas below; also κ in (2.67) maps \bar{E}_i, \bar{F}_i to $-\bar{F}_i, -\bar{E}_i$, whence the sign for the expression of $\Phi(\psi_{-\varpi_j}^{\alpha_i})$.) Since $\Phi_1 = m \circ (\text{id} \otimes S^{-1}) \circ \Phi$, we have

$$\Phi_1(\psi_{-\lambda}^\alpha) = K_{-2\lambda}, \quad \Phi_1(\psi_{-\varpi_j}^{\alpha_i}) = \delta_{i,j} L_i^{-2} \bar{E}_i, \quad \Phi_1(\psi_{-\varpi_j}^{-\alpha_i}) = \delta_{i,j} q_i^{-1} \bar{F}_i K_i L_i^{-2}. \quad (2.70)$$

Therefore,

$$\bar{E}_i, \bar{F}_i, L_i^{\pm 1} \in T_2^{-1}\Phi_1(\mathcal{L}_{0,1}^A)[T/T_2] = \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2].$$

These elements do not generate U_A ; it is necessary to consider general root vectors. By the stability of $U_A(G^*)$ under $\mathcal{B}(\mathfrak{g})$ and the isomorphism $\mathcal{O}_A[\psi_{-\rho}^{-1}] \rightarrow U_A(G^*)$ of Theorem 2.20 (3), for every positive root β_k , we have $1 \otimes K_{\beta_k}^{-1} \bar{E}_{\beta_k}, \bar{F}_{\beta_k} K_{\beta_k} \otimes 1 \in \Phi(\mathcal{O}_A[\psi_{-\rho}^{-1}]) = \Phi(\mathcal{L}_{0,1}^A[d^{-1}])$. Therefore, $\bar{F}_{\beta_k} K_{\beta_k}, S^{-1}(\bar{E}_{\beta_k}) K_{\beta_k} \in \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$, and $\bar{F}_{\beta_k}, S^{-1}(\bar{E}_{\beta_k}) \in \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2]$. The sets $S^{-1}(\bar{E}_{\beta_k})U_A(\mathfrak{h})$ generate the subalgebra $U_A(\mathfrak{b}_+)$ of U_A (in fact, let us quote that a formula of $S^{-1}(\bar{E}_{\beta_k})$ is given in [107]). From the triangular decomposition $U_A = U_A(\mathfrak{n}_-)U_A(\mathfrak{h})U_A(\mathfrak{n}_+)$, the inclusion $U_A \subset \Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])[T/T_2]$ follows, whence the equality too. In particular, U_A is a free $\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}])$ -module with a basis formed by representatives of the cosets in T/T_2 . By the uniqueness of this free decomposition, we find $\Phi_1(\mathcal{L}_{0,1}^A[d^{-1}]) = T_2^{-1}U_A^{\text{lf}}$. Therefore, Φ_1 in Corollary 2.23 (2) is surjective.

For the third claim, recall the isomorphism $\Phi_1: C(-w_0(\mu)) \rightarrow \text{ad}^r(U_q)(K_{-2\mu})$ (see (2.14)), and that $\psi_{-\mu}$ is the matrix coefficient dual to the vector ${}^\omega v_{-\mu} \otimes v_{-\mu} \in \text{End}_A(V_{-w_0(\mu)})$. This vector is cyclic by (2.32), so by equivariance $\Phi_1: {}_A C(-w_0(\mu)) \rightarrow \text{ad}^r(U_A^{\text{res}})(K_{-2\mu})$ is an isomorphism of U_A^{res} -modules. The second claim follows from this and (2.60) for $n = 1$. \blacksquare

Recall from (2.20) the isomorphisms of U_q -module algebras $\Phi_n: \mathcal{L}_{0,n} \rightarrow (U_q^{\otimes n})^{\text{lf}}$ and of algebras $\Phi_n: \mathcal{M}_{0,n} \rightarrow (U_q^{\otimes n})^{U_q}$, and from (2.27) the notations for specializations. Corollary 2.22 can be extended to Φ_n as follows:

Corollary 2.25. *The map Φ_n affords embeddings of module algebras $\Phi_n: \mathcal{L}_{0,n}^A \rightarrow (U_A^{\otimes n})^{\text{lf}}$ and $\Phi_n: \mathcal{L}_{0,n}^{\epsilon'} \rightarrow (U^{\otimes n})_{\epsilon'}^{\text{lf}}$, $q = \epsilon' \in \mathbb{C}^\times$.*

Proof. For the first claim, the only thing to prove is the inclusion $\Phi_n(\mathcal{L}_{0,n}^A) \subset U_A^{\otimes n}$. It follows from Corollary 2.22 and the expression of Φ_n in terms of Φ_1 and R -matrices (in particular, the fact that they preserve integrality, see [18, Lemma 6.10]). For the specialization at $q = \epsilon' \in \mathbb{C}^\times$, we have to justify that Φ_n is injective. One uses the fact, to be developed in Theorem 2.29 below, that $\Phi: \mathcal{O}_\epsilon \rightarrow U_\epsilon(G^*)$ is an embedding. The algebra $U_\epsilon(G^*)$ has the basis elements (2.68), and the map $m \circ (\text{id} \otimes S^{-1})$ sends this basis to a free family of U_ϵ . Therefore, $\Phi_1: \mathcal{L}_{0,1}^\epsilon \rightarrow U_\epsilon$ is injective. Since Φ_n differs from $\Phi_1^{\otimes n}$ by a linear isomorphism (induced by the conjugation action of R -matrices on the components ${}_A C([\lambda])$ of $\mathcal{L}_{0,n}^A$ in (2.60), see [18, equation (6.10)]), $\Phi_n: \mathcal{L}_{0,n}^\epsilon \rightarrow U_\epsilon^{\otimes n}$ is an embedding as well. ■

Remark 2.26.

- (1) It is a natural problem to determine the image of Φ_n . One may expect that it would be $(T_{2-}^{-1}U_A^{\text{lf}})^{\otimes n}$, because this is true for $n = 1$, as well as for any n in the \mathfrak{sl}_2 case, as shown in [18]. Unfortunately, this is not so. This comes from the fact, e.g., for $n = 2$, that the matrix elements of $R_{02}R_{01}R'_{01}R_{02}^{-1}$ do not belong to $(T_{2-}^{-1}U_A^{\text{lf}})^{\otimes 2}$ as can be shown by an explicit computation in the $\mathfrak{sl}(3)$ case.
- (2) In the case of $\mathfrak{g} = \mathfrak{sl}_2$, we defined in [18] an algebra ${}_{\text{loc}}\mathcal{L}_{0,n}^A$ generalizing $\mathcal{L}_{0,1}^A[d^{-1}]$ above, containing $\mathcal{L}_{0,n}^A$ as a subalgebra, and such that Φ_n extends to ${}_{\text{loc}}\mathcal{L}_{0,n}^A$ and yields an isomorphism $\Phi_n: {}_{\text{loc}}\mathcal{L}_{0,n}^A \rightarrow U_A^{\text{ad}}(\mathfrak{sl}_2)^{\otimes n}$. The definition of ${}_{\text{loc}}\mathcal{L}_{0,n}^A$ involves elements $\xi^{(i)} \in \mathcal{L}_{0,n}^A$ ($i = 1, \dots, n$) such that $\Phi_n(\xi^{(i)}) = (K^{-1})^{(i)} \dots (K^{-1})^{(n)}$. It may be of interest to study a similar extension of Φ_n for general \mathfrak{g} .

2.5 Structure theorems for U_ϵ and \mathcal{O}_ϵ

As usual, we denote by ϵ a primitive l -th root of unity, where l is odd, and coprime to 3 if \mathfrak{g} has G_2 -components.

Recall the subgroups T_G , U_\pm and B_\pm of G . Let $G^0 = B_+B_-$ (the *big cell* of G), and define the subgroup

$$G^* = \{(u_+t, u_-t^{-1}), t \in T_G, u_\pm \in U_\pm\} \subset B_+^{\text{op}} \times B_-^{\text{op}},$$

where B_\pm^{op} is the group B_\pm with opposite multiplication. The group G^* can be naturally identified with the Poisson–Lie dual of G with its standard structure.

Recall also that there is an injective homomorphism $\gamma_q^{-1} \circ h_q: \mathcal{Z}(U_q) \rightarrow U_q(\mathfrak{h})$, defined by means of the quantum Harish-Chandra homomorphism (see, e.g., [35, Section 9.1.C], or [104, Section 3.13]). The image of $\gamma_q^{-1} \circ h_q$ is the set $U_q(\mathfrak{h})^{\tilde{W}}$ of invariant elements under \tilde{W} , the subgroup of $W \times P_2^*$ generated by the conjugates $\sigma W \sigma$ of W by elements $\sigma \in P_2^*$. Here, P_2^* is the group of homomorphisms $P \rightarrow \mathbb{Z}/2\mathbb{Z}$, and the semidirect product $W \times P_2^*$ acts on $U_q(\mathfrak{h})$ by the standard action of the Weyl group W , and by the action of P_2^* given by $\sigma \cdot K_\lambda := \sigma(\lambda)K_\lambda$.

Consider the inverse map $h_q^{-1} \circ \gamma_q: U_q(\mathfrak{h})^{\tilde{W}} \rightarrow \mathcal{Z}(U_q)$. The elements of the domain and target, when expanded in the PBW basis, have coefficients in $\mathbb{C}(q)$. It was shown in [42, Section 21.1] that if an element of $U_q(\mathfrak{h})^{\tilde{W}}$ has no coefficient with a pole at $q = \epsilon$, then its image by $h_q^{-1} \circ \gamma_q$ has no coefficient with a pole at $q = \epsilon$. We therefore have a well-defined injection

$$U_\epsilon(\mathfrak{h})^{\tilde{W}} \rightarrow \mathcal{Z}(U_\epsilon).$$

We denote its image by $\mathcal{Z}_1(U_\epsilon)$. For instance, when $U_\epsilon = U_\epsilon(\mathfrak{sl}_2)$, $\mathcal{Z}_1(U_\epsilon)$ is the polynomial algebra generated by the Casimir element $\Omega = (\epsilon - \epsilon^{-1})^2 FE + \epsilon K + \epsilon^{-1} K^{-1}$.

Denote by $\mathcal{Z}_0(U_\epsilon) \subset U_\epsilon$ the smallest subalgebra containing the elements E_i^l, F_i^l, K_α^l , for $i \in \{1, \dots, m\}$, $\alpha \in P$, and stable under $\mathcal{B}(\mathfrak{g})$; it is also the subalgebra generated by $E_{\beta_k}^l, F_{\beta_k}^l, L_i^{\pm l}$, for $k \in \{1, \dots, N\}$ and $i \in \{1, \dots, m\}$ [42, Section 18]. We will denote by $\mathcal{Z}_0(U_\epsilon(\mathfrak{n}_-))$, $\mathcal{Z}_0(U_\epsilon(\mathfrak{h}))$ and $\mathcal{Z}_0(U_\epsilon(\mathfrak{n}_+))$ the subalgebras of $\mathcal{Z}_0(U_\epsilon)$ generated by the elements $F_{\beta_k}^l, K_\lambda^l$ ($\lambda \in P$), and $E_{\beta_k}^l$, respectively. In [39, Sections 1.8, 3.3 and 3.8] and [42, Theorem 14.1 and Sections 20–21], the following results are proved:

Theorem 2.27.

- (1) U_ϵ has no nontrivial zero divisors, $\mathcal{Z}_0(U_\epsilon)$ is a central Hopf subalgebra of U_ϵ , and U_ϵ is a free $\mathcal{Z}_0(U_\epsilon)$ -module of rank $l^{\dim \mathfrak{g}}$. Moreover, the classical fraction algebra $Q(U_\epsilon) = Q(\mathcal{Z}(U_\epsilon)) \otimes_{\mathcal{Z}(U_\epsilon)} U_\epsilon$ is a central simple algebra of PI degree l^N , and U_ϵ is a maximal order of $Q(U_\epsilon)$.
- (2) $\text{Maxspec}(\mathcal{Z}_0(U_\epsilon))$ is a group isomorphic to G^* above, and the multiplication map yields an isomorphism $\mathcal{Z}_0(U_\epsilon) \otimes_{\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon)} \mathcal{Z}_1(U_\epsilon) \rightarrow \mathcal{Z}(U_\epsilon)$.

By this theorem, the dimension of $Q(U_\epsilon)$ over its center $Q(\mathcal{Z}(U_\epsilon))$ is l^{2N} , and its dimension over $Q(\mathcal{Z}_0(U_\epsilon))$ is $l^{\dim \mathfrak{g}} = l^{m+2N}$. Therefore, the field $Q(\mathcal{Z}(U_\epsilon))$ is an extension of $Q(\mathcal{Z}_0(U_\epsilon))$ of degree l^m .

Note that, because $\mathcal{Z}_0(U_\epsilon)$ is an affine and commutative algebra, the maximal spectrum $\text{Maxspec}(\mathcal{Z}_0(U_\epsilon))$, viewed as the set of characters of $\mathcal{Z}_0(U_\epsilon)$, acquires by duality a structure of affine algebraic group. Thus, the first claim of (2) in the theorem means precisely that this group can be identified with G^* . See, for instance, [18, Section 7.2.1] for an explicit description in the \mathfrak{sl}_2 case.

In addition, $\text{Maxspec}(\mathcal{Z}_0(U_\epsilon))$ and G^* have natural Poisson structures which correspond one to the other under the isomorphism of (2), and we have the following identifications (see [42, Section 21.2]). The dual isomorphism $\mathcal{O}(G^*) \rightarrow \mathcal{Z}_0(U_\epsilon)$ identifies $\mathcal{O}(T_G)$ with $\mathcal{Z}_0(U_\epsilon) \cap U_\epsilon(\mathfrak{h}) = \mathbb{C}[lP]$, where as usual $U_\epsilon(\mathfrak{h}) = U_A(\mathfrak{h}) \otimes_A \mathbb{C}_\epsilon$. Therefore, we can identify $\mathbb{C}[P]$ with $\mathcal{O}(\tilde{T}_G)$, the coordinate ring of the l^m -fold covering space $\tilde{T}_G \rightarrow T_G$. The quantum Harish-Chandra isomorphism identifies $\mathcal{Z}_1(U_\epsilon)$ with $\mathbb{C}[2P]^W \cong \mathcal{O}(\tilde{T}_G/(2))^W$, where we denote by (2) the subgroup of 2-torsion elements in T_G . Consider the map

$$\sigma: B_+ \times B_- \longrightarrow G^0, \quad (b_+, b_-) \longmapsto b_+ b_-^{-1}.$$

The restriction of σ to G^* is an unramified covering map of degree 2^m . Composing $\sigma: G^* \rightarrow G^0$ with the quotient map under conjugation, $G^0 \hookrightarrow G \rightarrow G//G$, we get dually an embedding of $\mathcal{O}(G//G) = \mathcal{O}(G)^G$ in $\mathcal{O}(G^*)$. Collecting these observations, we see that the isomorphism of Theorem 2.27 (2) affords identifications

$$\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon) \cong \mathcal{O}(G)^G$$

as a subalgebra of $\mathcal{Z}_0(U_\epsilon) \cong \mathcal{O}(G^*)$, and

$$\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon) = \mathbb{C}[2lP]^W \cong \mathcal{O}(\tilde{T}_G/(2l))^W \cong \mathcal{O}(T_G/(2))^W$$

as a subalgebra of $\mathcal{Z}_1(U_\epsilon) \cong \mathcal{O}(\tilde{T}_G/(2))^W$.

We will use the following obvious though crucial fact. Note that U_A^{ad} is naturally a subalgebra of U_A^{res} , and therefore acts on U_ϵ^{res} -modules. Denote by $\mathcal{Z}_0(U_A^{\text{ad}}) \subset U_A^{\text{ad}}$ the subalgebra generated by the elements $\bar{E}_{\beta_k}^l, \bar{F}_{\beta_k}^l, K_i^{\pm l}$, for $k \in \{1, \dots, N\}$ and $i \in \{1, \dots, m\}$.

Lemma 2.28. *For every U_A^{res} -module V of type 1, the action of $\mathcal{Z}_0(U_A^{\text{ad}})$ on the specialization $V_\epsilon := V \otimes_A \mathbb{C}_\epsilon$ is trivial.*

Proof. This comes from $E_i^l = [l]_{q_i}! E_i^{(l)}$, $F_i^l = [l]_{q_i}! F_i^{(l)}$ and the fact that K_i acts on V by powers of q_i . Specializing to $q = \epsilon$ ends the proof. \blacksquare

A result similar to Theorem 2.27 holds true for \mathcal{O}_ϵ . Namely, take the specializations at $q = \epsilon$ in Theorem 2.20. Denote by $\mathcal{Z}_0(U_\epsilon(G^*))$ the subalgebra of $U_\epsilon(G^*)$ generated by the elements $(k \in \{1, \dots, N\}, i \in \{1, \dots, m\})$

$$1 \otimes K_{-l\beta_k} E_{\beta_k}^l, \quad F_{\beta_k}^l K_{l\beta_k} \otimes 1, \quad L_i^{\pm l} \otimes L_i^{\mp l}.$$

It is a central Hopf subalgebra. Recall that the coordinate ring $\mathcal{O}(G)$ can be identified as a Hopf algebra with $U(\mathfrak{g})^\circ$, where as usual $U(\mathfrak{g})^\circ$ denotes the restricted dual of the enveloping algebra $U(\mathfrak{g})$ over \mathbb{C} . In [41, Section 6], De Concini–Lyubashenko introduced an epimorphism of Hopf algebras $\eta: \Gamma_\epsilon \rightarrow U(\mathfrak{g})$ (essentially a version of Lusztig’s “Frobenius” epimorphism in [82]), defined by

$$\begin{aligned} \eta(E_i^{(p)}) &= \begin{cases} \frac{e_i^{p/l}}{(p/l)!} & \text{if } l \text{ divides } p, \\ 0 & \text{otherwise,} \end{cases} & \eta(F_i^{(p)}) &= \begin{cases} \frac{f_i^{p/l}}{(p/l)!} & \text{if } l \text{ divides } p, \\ 0 & \text{otherwise,} \end{cases} \\ \eta(K_i) &= 1, & \eta((K_i; p)_{q_i}) &= \begin{cases} \frac{h_i(h_i - 1) \cdots (h_i - (p/l) + 1)}{(p/l)!} & \text{if } l \text{ divides } p, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (2.71)$$

where $p \in \mathbb{N}$, and e_i, f_i and $h_i, i \in \{1, \dots, m\}$, denote the standard generators of $U(\mathfrak{g})$. The kernel of η is generated by the elements $E_i, F_i, K_i - 1$, and $(K_i; p)_{q_i}$ where l does not divide p . Put

$$\mathcal{Z}_0(\mathcal{O}_\epsilon) := \eta^*(\mathcal{O}(G)), \quad (2.72)$$

where $\eta^*: U(\mathfrak{g})^\circ \rightarrow \Gamma_\epsilon^\circ$ is the monomorphism dual to η . Let us define special matrix coefficients, analogous to those introduced in Theorem 2.20. Denote by v_{ϖ_i} and $v_{w_0(\varpi_i)}$ a highest weight vector and a lowest weight vector of the Γ -module ${}_A V_{\varpi_i}$. Denote also by $v_{w_0(\varpi_i)}^*$ and $v_{\varpi_i}^*$ a highest and lowest weight vector of the dual module Γ -module ${}_A V_{\varpi_i}^* \cong {}_A V_{-w_0(\varpi_i)}$. Define the matrix coefficients $b_{\varpi_i}, c_{\varpi_i} \in \mathcal{O}_A$ by

$$b_{\varpi_i}(x) = v_{\varpi_i}^*(xv_{w_0(\varpi_i)}), \quad c_{\varpi_i}(x) = v_{w_0(\varpi_i)}^*(xv_{\varpi_i})$$

for all $x \in \Gamma$. We consider them as elements of \mathcal{O}_ϵ . Denote by $\mathcal{Z}_1(\mathcal{O}_\epsilon)$ the subalgebra of \mathcal{O}_ϵ generated by the elements $b_{\varpi_i}^k c_{\varpi_i}^{l-k}$ for $1 \leq i \leq m$ and $0 \leq k \leq l$.

Theorem 2.29.

- (1) $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ is a central Hopf subalgebra of $\mathcal{O}_\epsilon \subset \Gamma_\epsilon^\circ$, and $Q(\mathcal{Z}(\mathcal{O}_\epsilon))$ is an extension of $Q(\mathcal{Z}_0(\mathcal{O}_\epsilon))$ of degree l^m .
- (2) $\psi_{-l\rho} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$, and $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ is generated by matrix coefficients of irreducible Γ -modules of highest weight $l\lambda$, $\lambda \in P_+$. Moreover, the multiplication map yields an isomorphism

$$\mathcal{Z}_0(\mathcal{O}_\epsilon) \otimes_{\mathcal{Z}_0(\mathcal{O}_\epsilon) \cap \mathcal{Z}_1(\mathcal{O}_\epsilon)} \mathcal{Z}_1(\mathcal{O}_\epsilon) \rightarrow \mathcal{Z}(\mathcal{O}_\epsilon),$$

and the map Φ in Theorem 2.20 affords an algebra embedding $\mathcal{Z}_0(\mathcal{O}_\epsilon) \rightarrow \mathcal{Z}_0(U_\epsilon(G^*))$ and algebra isomorphisms $\mathcal{Z}_0(\mathcal{O}_\epsilon)[\psi_{-l\rho}^{-1}] \rightarrow \mathcal{Z}_0(U_\epsilon(G^*))$ and $\mathcal{O}_\epsilon[\psi_{-l\rho}^{-1}] \rightarrow U_\epsilon(G^*)$.

- (3) \mathcal{O}_ϵ has no nontrivial zero divisors, and it is a free $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ -module of rank $l^{\dim \mathfrak{g}}$. Moreover, the classical fraction algebra $Q(\mathcal{O}_\epsilon) = Q(\mathcal{Z}(\mathcal{O}_\epsilon)) \otimes_{\mathcal{Z}(\mathcal{O}_\epsilon)} \mathcal{O}_\epsilon$ is a central simple algebra of PI degree l^N , and \mathcal{O}_ϵ is a maximal order of $Q(\mathcal{O}_\epsilon)$.

For the proof, see [41]: Proposition 6.4 for the first claim of (1) (where $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ and $\mathcal{Z}_0(U_\epsilon(G^*))$ are denoted F_0 and A_0 respectively), the appendix of Enriquez and [50] for the second claim of (1) and (2), Propositions 6.4 and 6.5 for the other claims of (2), Theorem 7.2 (where \mathcal{O}_ϵ is shown to be projective over $\mathcal{Z}_0(\mathcal{O}_\epsilon)$) and [28] (which provides the additional K-theoretic arguments to deduce that \mathcal{O}_ϵ is free), or [6, Remark 2.18 (b)], for the second claim of (3), and Corollary 7.3 and Theorem 7.4 for the third claim. The fact that \mathcal{O}_ϵ has no nontrivial zero divisors follows from the embedding $\mathcal{O}_\epsilon \rightarrow U_\epsilon(G^*)$ via Φ .

As above for U_ϵ , it follows directly from (3) that $Q(\mathcal{Z}(\mathcal{O}_\epsilon))$ has degree l^m over $Q(\mathcal{Z}_0(\mathcal{O}_\epsilon))$. For a more complete description of $\mathcal{Z}(\mathcal{O}_\epsilon)$ we refer to [50] and Enriquez' appendix in [41], as well as [27].

We do not know a basis of \mathcal{O}_ϵ over $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ for general G , but see [38] for the case of SL_2 . We will recall the known results in this case of SL_2 before Lemma 4.5.

Finally, there is a natural action of the braid group $\mathcal{B}(\mathfrak{g})$ on \mathcal{O}_ϵ , that we will use. Namely, let $n_i \in N(T_G)$ be a representative of the reflection $s_i \in W = N(T_G)/T_G$ associated to the simple root α_i . In [98, 102], Soibelman–Vaksman introduced functionals $t_i: \mathcal{O}_q \rightarrow \mathbb{C}(q)$ which quantize the elements n_i . They correspond dually to generators of the quantum Weyl group of \mathfrak{g} ; in the appendix, we recall their main properties, in particular, they map \mathcal{O}_A to A (see also [35, Section 8.2], and [41, 69, 70, 81, 102]). Denote by \triangleleft the natural right action of functionals on \mathcal{O}_A , namely (using Sweedler's notation)

$$\alpha \triangleleft h = \sum_{(\alpha)} h(\alpha_{(1)})\alpha_{(2)}$$

for every $\alpha \in \mathcal{O}_A$ and $h \in \mathcal{O}_A \rightarrow A$. Let us identify $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ with $\mathcal{O}(G)$ by means of (2.72). We have [41, Proposition 7.1]:

Proposition 2.30. *The maps $\triangleleft t_i$ on \mathcal{O}_ϵ preserve $\mathcal{Z}_0(\mathcal{O}_\epsilon)$, and satisfy $(f \triangleleft t_i)(a) = f(n_i a)$ and $(f \star \alpha) \triangleleft t_i = (f \triangleleft t_i)(\alpha \triangleleft t_i)$ for every $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$, $a \in G$, $\alpha \in \mathcal{O}_\epsilon$.*

We provide an alternative, non computational, proof of this result in Appendix C.

3 Noetherianity and finiteness

In this section, we prove Theorem 1.1. Recall that by Noetherian we mean right and left Noetherian. We begin with

Theorem 3.1. *The algebras $\mathcal{L}_{0,n}$, $\mathcal{L}_{0,n}^A$ and $\mathcal{L}_{0,n}^{\epsilon'}$, $\epsilon' \in \mathbb{C}^\times$, are Noetherian.*

By Proposition 2.18, each of the algebras in this theorem is finitely generated.

Theorem 3.1 for $\mathcal{L}_{0,1}$ and any \mathfrak{g} follows immediately from Joseph–Letzter's Theorem 2.1, claim (3), by identifying $\mathcal{L}_{0,1}$ with U_q^{lf} via Φ_1 . The method of proof uses filtration arguments. An alternative proof in the case of $\mathfrak{sl}(n)$, which works also for $\mathcal{L}_{0,1}^A$, was obtained by Domokos–Lenagan in [47], by exhibiting special sequences of generators of $\mathcal{L}_{0,1}^A$ satisfying *polynomial* relations, as we define now.

Definition 3.2 (see [104, Proposition 3.133]). Let R be a Noetherian Abelian ring, and B a finitely generated R -algebra with product \circ . We call *polynomial* a set of relations between generators u_1, \dots, u_M of B , of the form

$$u_i \circ u_j - q_{ij} u_j \circ u_i = \sum_{s=1}^{j-1} \sum_{t=1}^M (\alpha_{ij}^{st} u_s \circ u_t + \beta_{ij}^{st} u_t \circ u_s) \quad (3.1)$$

for all $1 \leq j < i \leq M$, where $\alpha_{ij}^{st}, \beta_{ij}^{st} \in R$, and the elements $q_{ij} \in R$ are invertible.

Note that this definition is more restrictive than the more standard one, e.g., in [26, Definition II.4.1]. If such a set of relations exists in B , then B can be endowed with an algebra filtration such that the associated graded algebra is a quotient of a skew-polynomial algebra [26, Proposition I.8.17]. By classical results, we have (see, e.g., [88, Theorems 1.2.9, 1.6.9 and Examples 1.6.11], or [104, Lemmas 3.130–3.131]):

Theorem 3.3. *If the algebra filtration is well founded, then B is a Noetherian ring.*

In [47], Theorem 3.1 is also proved for any $n \geq 1$ in the case of $\mathfrak{g} = \mathfrak{sl}_2$ by considering $\mathcal{L}_{0,n}^A(\mathfrak{sl}_2)$ as an iterated overring of $\mathcal{L}_{0,1}(\mathfrak{sl}_2)$.

The proof of Theorem 3.1 that we develop for any \mathfrak{g} and $n \geq 1$ is also based on polynomial relations. In our proof, the generating set of $\mathcal{L}_{0,n}$ that we will consider is evident, as they are matrix coefficients in the modules V_{ϖ_k} , $k \in \{1, \dots, m\}$; the task is then to exhibit a set of polynomial relations between them, that hold in a certain graded algebra associated to $\mathcal{L}_{0,n}$. Indeed, as explained above this will imply that the graded algebra is Noetherian, and that $\mathcal{L}_{0,n}$ is Noetherian as well. In the case of $\mathcal{L}_{0,n}^A$, the proof is formally similar, but it needs the use of canonical bases discussed in Section 2.2.2.

Proof of Theorem 3.1. First, we develop the proof for $\mathcal{L}_{0,n}$, and then for $\mathcal{L}_{0,n}^A$; the result for

$$\mathcal{L}_{0,n}^{\epsilon'} = \mathcal{L}_{0,n}^A / (q - \epsilon') \mathcal{L}_{0,n}^A$$

follows immediately by lifting ideals by the quotient map $\mathcal{L}_{0,n}^A \rightarrow \mathcal{L}_{0,n}^{\epsilon'}$.

We adapt the proof of Theorem 2.1 (3) given in [104, Theorem 3.137]. Let us begin by recalling these arguments. In doing this, let us stress that [104] takes on \mathcal{O}_q and $\mathcal{L}_{0,1}$ the product opposite to ours, and below in (3.7) and (3.8) we respect their convention.

As usual, let $C(\mu)$ be the vector space generated by the matrix coefficients of V_μ , the simple U_q^{ad} -module of highest weight $\mu \in P_+$. Denote by $C(\mu)_\lambda \subset C(\mu)$ the subspace of weight λ for the left coregular action of $U_q(\mathfrak{h})$; so $\alpha \in C(\mu)_\lambda$ if $K_\nu \triangleright \alpha = q^{(\nu, \lambda)} \alpha$, $\nu \in P$. Consider the semigroup

$$\Lambda = \{(\mu, \lambda) \in P_+ \times P, \lambda \text{ is a weight of } V_\mu\}.$$

Recall that the partial order \preceq on P is defined by $\mu \preceq \mu'$ if and only if $\mu' - \mu \in D^{-1}Q_+$. Define \preceq on Λ by: $(\mu, \lambda) \preceq (\mu', \lambda')$ if and only if $\mu' - \mu \in D^{-1}Q_+$ and $\lambda' - \lambda \in D^{-1}Q_+$. If $(\mu, \lambda) \preceq (\mu', \lambda')$ and $(\mu, \lambda) \neq (\mu', \lambda')$, we write $(\mu, \lambda) \prec (\mu', \lambda')$. Since $\mathcal{L}_{0,1}$ and \mathcal{O}_q are isomorphic vector spaces, we have $\mathcal{L}_{0,1} = \bigoplus_{\mu \in P_+} C(\mu) = \bigoplus_{(\mu, \lambda) \in \Lambda} C(\mu)_\lambda$. Consider the family of subspaces

$$\mathcal{F}_2^{\mu, \lambda} := \bigoplus_{(\mu', \lambda') \preceq (\mu, \lambda)} C(\mu')_{\lambda'}, \quad \mathcal{F}_2^{\prec \mu, \lambda} := \bigoplus_{(\mu', \lambda') \prec (\mu, \lambda)} C(\mu')_{\lambda'}, \quad (\mu, \lambda) \in \Lambda.$$

We have

$$\mathcal{L}_{0,1} = \bigcup_{(\mu, \lambda) \in \Lambda} \mathcal{F}_2^{\mu, \lambda}. \quad (3.2)$$

Indeed, clearly

$$\mathcal{L}_{0,1} = \sum_{(\mu,\lambda) \in \Lambda} \mathcal{F}_2^{\mu,\lambda},$$

so (3.2) follows from the following fact: for every $(\mu, \lambda), (\mu', \lambda') \in \Lambda$, the element $(\mu'', \lambda'') := (\mu + \mu', \lambda + \lambda')$ is such that

$$\mathcal{F}_2^{\mu,\lambda} + \mathcal{F}_2^{\mu',\lambda'} \subset \mathcal{F}_2^{\mu'',\lambda''}.$$

Note that in general, since $Q_+ \not\subset P_+$ (but $P_+ \subset D^{-1}Q_+$), it is not true that there exists an element (μ'', λ'') satisfying such an inclusion if one replaces \preceq with the standard ‘‘product’’ partial order \leq on Λ , defined by $(\mu, \lambda) \leq (\mu', \lambda')$ if and only if $\mu' - \mu \in Q_+$ and $\lambda' - \lambda \in Q_+$. Note also that \preceq is finer than \leq , in the sense that if $\mu \leq \mu'$, then $\mu \preceq \mu'$. Again, this would not be true if we had replaced $D^{-1}Q_+$ by P_+ in the definition of \preceq .

The family $\mathcal{F}_2 := \{\mathcal{F}_2^{\mu,\lambda}\}_{(\mu,\lambda) \in \Lambda}$ is a filtration of the vector space $\mathcal{L}_{0,1}$, which is clearly well founded (i.e., every subset of Λ contains a minimal element, or equivalently any decreasing infinite sequence of elements in Λ is eventually constant).

Consider the associated graded vector space $\text{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1}) := \bigoplus_{(\mu,\lambda)} \mathcal{F}_2^{\mu,\lambda} / \mathcal{F}_2^{\prec \mu,\lambda}$. By identifying an element $x \in C(\mu)_\lambda$ with its coset $\bar{x} \in \mathcal{F}_2^{\mu,\lambda} / \mathcal{F}_2^{\prec \mu,\lambda}$, we get an equality of vector spaces $\text{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1}) = \bigoplus_{(\mu,\lambda) \in \Lambda} C(\mu)_\lambda$. Now, one has the following facts:

(i) Taking the product in $\mathcal{L}_{0,1}$, we have

$$\alpha\beta \in \mathcal{F}_2^{\mu_1 + \mu_2, \lambda_1 + \lambda_2} \quad \text{for } \alpha \in C(\mu_1)_{\lambda_1}, \quad \beta \in C(\mu_2)_{\lambda_2}. \quad (3.3)$$

This follows from (2.7) and the fact that, for every $\nu \in P_+$ and every summand of the formula (2.9), denoting by $-r \in -Q_+$ the weight of the R -matrix component $R_{(2)}$ we have

$$\begin{aligned} K_\nu \triangleright ((R_{(2)'} S(R_{(2)})) \triangleright \alpha) \star (R_{(1)'} \triangleright \beta \triangleleft R_{(1)}) \\ = q^{(\nu, \lambda_1 + \lambda_2 - r)} (R_{(2)'} S(R_{(2)}) \triangleright \alpha) \star (R_{(1)'} \triangleright \beta \triangleleft R_{(1)}). \end{aligned}$$

(Details of a similar computation are given below (3.12).) It follows from (3.3) that \mathcal{F}_2 is an algebra filtration of $\mathcal{L}_{0,1}$, and then $\text{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1})$ is a graded algebra.

(ii) Denote by $\alpha \circ \beta$ the product in $\text{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1})$ of $\alpha, \beta \in \mathcal{L}_{0,1}$. The space $C(\mu_1 + \mu_2)$ has multiplicity one in $C(\mu_1) \otimes C(\mu_2)$ (again by (2.7)), therefore if $\alpha \in C(\mu_1)_{\lambda_1}$ and $\beta \in C(\mu_2)_{\lambda_2}$, then $\alpha \circ \beta$ is the projection of $\alpha\beta$ onto $C(\mu_1 + \mu_2)_{\lambda_1 + \lambda_2}$. Denote by $\bar{\star}$ the product \star of \mathcal{O}_q followed by the projection onto the component $C(\mu + \nu)$. Then, we have

$$C(\mu) \circ C(\nu) = C(\mu) \bar{\star} C(\nu) = C(\mu + \nu). \quad (3.4)$$

This follows from the formula (2.9), and the fact that it is given by an invertible twist of the product \star .

(iii) For every $\mu \in P_+$, fix a basis of weight vectors $e_1^\mu, \dots, e_{d(\mu)}^\mu$ of V_μ . Denote by $e_\mu^1, \dots, e_\mu^{d(\mu)} \in V_\mu^*$ the dual basis, and by $w(e_i^\mu)$ the weight of e_i^μ . Consider the matrix coefficients ${}_\mu \phi_j^i(x) := e_\mu^i(\pi_V(x)(e_j^\mu))$, $x \in U_q$. By using the formula (2.9) and the explicit form of the R -matrix, one can check that

$$\begin{aligned} {}_\mu \phi_j^i \circ {}_\nu \phi_l^k &= \sum_{j', l'} c_{j', l'}^{ikjl} {}_\mu \phi_{j'}^i \bar{\star} {}_\nu \phi_{l'}^k \\ &= q^{(w(e_j^\mu), w(e_{l'}^\nu) - w(e_{k'}^\nu))} {}_\mu \phi_j^i \bar{\star} {}_\nu \phi_{l'}^k + \sum_{\substack{j', l' \\ j' \neq j, l' \neq l}} d_{j', l'}^{ikjl} {}_\mu \phi_{j'}^i \circ {}_\nu \phi_{l'}^k, \end{aligned} \quad (3.5)$$

where $\sum'_{j',l'}$ is the sum over indices with weights satisfying

$$w(e_j^\mu) + w(e_l^\nu) = w(e_{j'}^\mu) + w(e_{l'}^\nu), \quad w(e_{j'}^\mu) \leq w(e_j^\mu) \quad \text{and} \quad w(e_{l'}^\nu) \geq w(e_l^\nu),$$

and the coefficient $c_{j,l}^{ikjl}$, equal to $q^{(w(e_j^\mu), w(e_l^\nu) - w(e_k^\nu))}$, is computed from the term Θ in the R -matrix factorization (2.4). In general, all the coefficients $c_{j',l'}^{ikjl}$ and $d_{j',l'}^{ikjl}$ belong to $\mathbb{C}(q)$ (see [18, Proposition 4.1]); in particular $q^{(w(e_j^\mu), w(e_l^\nu) - w(e_k^\nu))} \in q^{\mathbb{Z}}$ since $w(e_l^\nu) - w(e_k^\nu) \in Q$. The second equality follows by repeated use of the first and (3.4). Similarly, by using (2.10) one gets

$$\begin{aligned} \nu\phi_l^k \circ \mu\phi_j^i &= \sum'_{i',k'} e_{i',k'}^{kilj} \mu\phi_j^{i'} \bar{\kappa} \nu\phi_l^{k'} \\ &= q^{(w(e_i^\mu), w(e_k^\nu) - w(e_{l'}^\nu))} \mu\phi_j^i \bar{\kappa} \nu\phi_l^k + \sum'_{\substack{i',k' \\ i' \neq i, k' \neq k}} e_{i',k'}^{kilj} \mu\phi_j^{i'} \bar{\kappa} \nu\phi_l^{k'} \\ &= q^{(w(e_i^\mu), w(e_k^\nu) - w(e_{l'}^\nu))} \mu\phi_j^i \bar{\kappa} \nu\phi_l^k + \sum'_{\substack{i',k',j',l' \\ i' \neq i, k' \neq k}} f_{i',k'}^{kilj} \mu\phi_{j'}^{i'} \circ \nu\phi_{l'}^{k'}, \end{aligned}$$

where $e_{i',k'}^{kilj}, f_{i',k'}^{kilj} \in \mathbb{C}(q)$, and $\sum'_{i',k'}$ is the sum over indices with weights satisfying

$$\begin{aligned} w(e_i^\mu) + w(e_k^\nu) &= w(e_{i'}^\mu) + w(e_{k'}^\nu), & w(e_{i'}^\mu) &\leq w(e_i^\mu), \\ w(e_{k'}^\nu) &\geq w(e_k^\nu), & e_{i,k}^{kilj} &= q^{(w(e_i^\mu), w(e_k^\nu) - w(e_{l'}^\nu))}. \end{aligned}$$

The third equality comes from the second and (3.5); the sum is over indices with weights satisfying

$$\begin{aligned} w(e_i^\mu) + w(e_k^\nu) &= w(e_{i'}^\mu) + w(e_{k'}^\nu), \\ w(e_{i'}^\mu) &< w(e_i^\mu), & w(e_{k'}^\nu) &> w(e_k^\nu), & w(e_{j'}^\mu) &\leq w(e_j^\mu), & w(e_{l'}^\nu) &\geq w(e_l^\nu). \end{aligned}$$

By eliminating the leading term $\mu\phi_j^i \bar{\kappa} \nu\phi_l^k$, one deduces

$$\nu\phi_l^k \circ \mu\phi_j^i - q_{ijkl} \mu\phi_j^i \circ \nu\phi_l^k = \sum'_{\substack{i',k',j',l' \\ i' \neq i, k' \neq k}} f_{i',k'}^{kilj} \mu\phi_{j'}^{i'} \circ \nu\phi_{l'}^{k'} - \sum'_{\substack{j',l' \\ j' \neq j, l' \neq l}} q_{ijkl} d_{j',l'}^{ikjl} \mu\phi_{j'}^i \circ \nu\phi_{l'}^k, \quad (3.6)$$

where $q_{ijkl} = q^{(w(e_j^\mu) + w(e_i^\mu), w(e_k^\nu) - w(e_{l'}^\nu))}$.

(iv) We can always reorder the weight vectors $e_1^\mu, \dots, e_{d(\mu)}^\mu$ so that $w(e_i^\mu) > w(e_j^\mu)$ implies $i < j$; then (3.6) reads

$$\begin{aligned} \nu\phi_l^k \circ \mu\phi_j^i - q_{ijkl} \mu\phi_j^i \circ \nu\phi_l^k &= \sum_{r=i}^{d(\mu)} \sum_{s=1}^k \sum_{u=1}^{l-1} \sum_{v=j+1}^{d(\mu)} \delta_{rsuv}^{ijkl} \mu\phi_v^r \circ \nu\phi_u^s \\ &\quad - \sum_{r=i+1}^{d(\mu)} \sum_{s=1}^{k-1} q_{ijkl} \gamma_{rs}^{ijkl} \mu\phi_j^r \circ \nu\phi_l^s, \end{aligned} \quad (3.7)$$

where $\gamma_{rs}^{ijkl}, \delta_{rsuv}^{ijkl} \in \mathbb{C}(q)$ are such that $\gamma_{rs}^{ijkl} = 0$ unless $w(e_r^\mu) < w(e_i^\mu)$ and $w(e_s^\nu) > w(e_k^\nu)$, and $\delta_{rsuv}^{ijkl} = 0$ unless $w(e_u^\nu) > w(e_l^\nu)$, $w(e_v^\mu) < w(e_j^\mu)$, $w(e_r^\mu) \leq w(e_i^\mu)$ and $w(e_s^\nu) \geq w(e_k^\nu)$. Now, from (3.7) one can extract a defining set of polynomial relations for $\text{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1})$, as in (3.1). Indeed, like $\mathcal{L}_{0,1}$ the algebra $\text{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1})$ is generated by the matrix coefficients $\varpi_k \phi_i^j$ of the fundamental representations V_{ϖ_k} . One can list these matrix coefficients, say M in number, in an ordered sequence u_1, \dots, u_M such that the following condition holds: if $w(e_k^{\varpi_s}) < w(e_i^{\varpi_r})$,

or $w(e_k^{\overline{rs}}) = w(e_i^{\overline{rs}})$ and $w(e_l^{\overline{rs}}) < w(e_j^{\overline{rs}})$, then $u_a := \varpi_r \phi_j^i$ and $u_b := \varpi_s \phi_l^k$ satisfy $b < a$. Then denoting $\mu \phi_j^i$, $\nu \phi_l^k$ in (3.7) by u_j , u_i , respectively, and assuming $u_j < u_i$, one finds that all terms $u_s := \mu \phi_v^r$, $\mu \phi_j^r$ in the sums are $< u_j$. Therefore, for all $1 \leq j < i \leq M$ it takes the form

$$u_i \circ u_j - q_{ij} u_j \circ u_i = \sum_{s=1}^{j-1} \sum_{t=1}^M \alpha_{ij}^{st} u_s \circ u_t \quad (3.8)$$

for some $q_{ij} \in q^{\mathbb{Z}}$ and $\alpha_{ij}^{st} \in \mathbb{C}(q)$. As explained after (3.1), it follows that $\text{Gr}_{\mathcal{F}_2}(\mathcal{L}_{0,1})$ is a Noetherian ring, and since the filtration \mathcal{F}_2 is well founded, it implies that $\mathcal{L}_{0,1}$ is Noetherian too.

We are going to extend all these facts to $\mathcal{L}_{0,n}$, $n > 1$. First, we need to refine the filtration \mathcal{F}_2 on $\mathcal{L}_{0,1}$. Consider the action of $U_q(\mathfrak{h})$ on $C(\mu)_\lambda$ given by

$$K_\nu \cdot \alpha := \text{coad}(K_\nu^{-1})(\alpha), \quad \nu \in P, \quad \alpha \in C(\mu)_\lambda. \quad (3.9)$$

Denote by $C(\mu)_{\lambda,\gamma} \subset C(\mu)_\lambda$ the subspace of weight γ for this action; so $\alpha \in C(\mu)_{\lambda,\gamma}$ if $K_\nu \cdot \alpha = q^{(\nu,\gamma)} \alpha$. Consider the semigroup

$$\Lambda_P = \{(\mu, \lambda, \gamma) \in P_+ \times P^2, \lambda \text{ is a weight of } V_\mu \text{ for } \triangleright, \gamma \text{ is a weight of } V_\mu \text{ for } \cdot\}$$

with the partial order $(\mu, \lambda, \gamma) \preceq (\mu', \lambda', \gamma')$ if and only if $\mu' - \mu, \lambda' - \lambda, \gamma' - \gamma \in D^{-1}Q_+$. Define

$$\begin{aligned} [\Lambda_P] &= \{([\mu], [\lambda], [\gamma]) \in P_+^n \times P^n \times P^n \\ &\quad | (\mu_i, \lambda_i, \gamma_i) \in \Lambda_P, [\mu] = (\mu_i)_{i=1}^n, [\lambda] = (\lambda_i)_{i=1}^n, [\gamma] = (\gamma_i)_{i=1}^n\}. \end{aligned}$$

Let us put the following lexicographic order on $[\Lambda_P]$, starting from the tail: $([\mu'], [\lambda'], [\gamma']) \preceq ([\mu], [\lambda], [\gamma])$ if $(\mu'_n, \lambda'_n, \gamma'_n) \prec (\mu_n, \lambda_n, \gamma_n)$, or $(\mu_n, \lambda_n, \gamma_n) = (\mu'_n, \lambda'_n, \gamma'_n)$ and $(\mu'_{n-1}, \lambda'_{n-1}, \gamma'_{n-1}) \prec (\mu_{n-1}, \lambda_{n-1}, \gamma_{n-1}), \dots$, or $(\mu_k, \lambda_k, \gamma_k) = (\mu'_k, \lambda'_k, \gamma'_k)$ for all $1 < k \leq n$ and $(\mu'_1, \lambda'_1, \gamma'_1) \preceq (\mu_1, \lambda_1, \gamma_1)$. (As usual, we write $([\mu'], [\lambda'], [\gamma']) \prec ([\mu], [\lambda], [\gamma])$ for $([\mu'], [\lambda'], [\gamma']) \preceq ([\mu], [\lambda], [\gamma])$ and $([\mu'], [\lambda'], [\gamma']) \neq ([\mu], [\lambda], [\gamma])$.)

Now recall that $\mathcal{L}_{0,n} = \mathcal{L}_{0,1}^{\otimes n} = \mathcal{O}_q^{\otimes n}$ as vector spaces. For every $([\mu], [\lambda], [\gamma]) \in [\Lambda_P]$, consider the subspace $C([\mu])_{[\lambda],[\gamma]} \subset \mathcal{L}_{0,n}$ defined by

$$C([\mu]) = C(\mu_1) \otimes \cdots \otimes C(\mu_n), \quad C([\mu])_{[\lambda],[\gamma]} = C(\mu_1)_{\lambda_1, \gamma_1} \otimes \cdots \otimes C(\mu_n)_{\lambda_n, \gamma_n}.$$

Then $\mathcal{L}_{0,n} = \bigoplus_{[\mu] \in P_+^n} C([\mu])$ and $C([\mu]) = \bigoplus_{([\lambda],[\gamma])} C([\mu])_{[\lambda],[\gamma]}$. For every $([\mu], [\lambda], [\gamma]) \in [\Lambda_P]$ define

$$\begin{aligned} \mathcal{F}_3^{[\mu],[\lambda],[\gamma]} &= \bigoplus_{([\mu'], [\lambda'], [\gamma']) \preceq ([\mu],[\lambda],[\gamma])} C([\mu'])_{[\lambda'], [\gamma']}, \\ \mathcal{F}_3^{\prec [\mu],[\lambda],[\gamma]} &= \bigoplus_{([\mu'], [\lambda'], [\gamma']) \prec ([\mu],[\lambda],[\gamma])} C([\mu'])_{[\lambda'], [\gamma']}. \end{aligned} \quad (3.10)$$

Clearly, $\mathcal{L}_{0,n}$ is the union of the subspaces $\mathcal{F}_3^{[\mu],[\lambda],[\gamma]}$ over all $([\mu], [\lambda], [\gamma]) \in [\Lambda_P]$, so these form a vector space filtration of $\mathcal{L}_{0,n}$. Let us denote it \mathcal{F}_3 , and define

$$\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu],[\lambda],[\gamma]} = \mathcal{F}_3^{[\mu],[\lambda],[\gamma]} / \mathcal{F}_3^{\prec [\mu],[\lambda],[\gamma]}.$$

This space is canonically identified with $C([\mu])_{[\lambda],[\gamma]}$, so the graded vector space associated to \mathcal{F}_3 is

$$\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n}) = \bigoplus_{([\mu],[\lambda],[\gamma]) \in [\Lambda_P]} \text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu],[\lambda],[\gamma]} = \bigoplus_{([\mu],[\lambda],[\gamma]) \in [\Lambda_P]} C([\mu])_{[\lambda],[\gamma]}. \quad (3.11)$$

We claim that \mathcal{F}_3 is an algebra filtration with respect to the product of $\mathcal{L}_{0,n}$, and therefore $\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})$ is a graded algebra.

For notational simplicity, let us prove it for $n = 2$, the general case being strictly similar. Recall the R -matrix factorization (2.4). Take tuples $([\mu], [\lambda], [\gamma]) = ((\mu_1, \mu_2), (\lambda_1, \lambda_2), (\gamma_1, \gamma_2))$ and $([\mu'], [\lambda'], [\gamma']) = ((\mu'_1, \mu'_2), (\lambda'_1, \lambda'_2), (\gamma'_1, \gamma'_2))$ in $[\Lambda_P]$, and elements $\alpha \otimes \beta \in C([\mu])_{[\lambda], [\gamma]}$ and $\alpha' \otimes \beta' \in C([\mu'])_{[\lambda'], [\gamma']}$. Recall from (2.17) that the product of $\mathcal{L}_{0,2}$ is given by the formula

$$\begin{aligned} & (\alpha \otimes \beta)(\alpha' \otimes \beta') \\ &= \sum_{(R^1), \dots, (R^4)} \alpha(S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha' \triangleleft R_{(1)}^1 R_{(1)}^2) \otimes (S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4) \beta'. \end{aligned} \quad (3.12)$$

For every $\nu \in P$ and any of the components $R_{(2)}^1, \dots, R_{(2)}^4$, denoting by $-r_j \in -Q_+$ the weight of $R_{(2)}^j$, we have

$$\begin{aligned} & K_\nu \triangleright (S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4) \\ &= \sum_{(\beta)} \beta_{(1)}(R_{(2)}^2 R_{(2)}^4) (K_\nu S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta_{(2)}) \\ &= q^{-(\nu, r_1 + r_3)} \sum_{(\beta)} \beta_{(1)}(R_{(2)}^2 R_{(2)}^4) (S(R_{(2)}^1 R_{(2)}^3) K_\nu \triangleright \beta_{(2)}) \\ &= q^{(\nu, \lambda_2 - r_1 - r_3)} \sum_{(\beta)} \beta_{(1)}(R_{(2)}^2 R_{(2)}^4) (S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta_{(2)}) \\ &= q^{(\nu, \lambda_2 - r_1 - r_3)} (S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4). \end{aligned}$$

By similar computations for the action $\text{coad}(K_\nu^{-1})$, and for all terms in the right-hand side of (3.12), and using (3.3) componentwisely, we find that

$$\alpha(S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha' \triangleleft R_{(1)}^1 R_{(1)}^2) \otimes (S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4) \beta' \in \mathcal{F}_3^{[\mu] + [\mu'], [\lambda'] + [\lambda], [\gamma'] + [\gamma]},$$

where

$$\begin{aligned} \lambda'' &= (\lambda_1 + \lambda'_1 + r_3 + r_4, \lambda_2 + \lambda'_2 - r_1 - r_3), \\ \gamma'' &= (\gamma_1 + \gamma'_1 + r_1 + r_2 + r_3 + r_4, \gamma_2 + \gamma'_2 - r_1 - r_2 - r_3 - r_4). \end{aligned}$$

Since $r_1 + r_2 + r_3 + r_4 = 0$ implies $r_1 = r_2 = r_3 = r_4 = 0$, by the order we have put on $[\Lambda_P]$, we deduce

$$(\alpha \otimes \beta)(\alpha' \otimes \beta') \in \mathcal{F}_3^{[\mu] + [\mu'], [\lambda] + [\lambda'], [\gamma] + [\gamma']}.$$

Note that the filtration \mathcal{F}_3 , taking the action (3.9) into account, is crucial for this argument to work. Similar arguments work for any $n \geq 2$. This proves that $\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})$ is a graded algebra. We denote its product by \circ_n .

Next, we show that (3.4) implies the analogous property for the product \circ_n . For simplicity of notations let us again assume that $n = 2$. Recall that the product \circ_2 is defined on homogeneous elements $\overline{\alpha \otimes \beta} \in \text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu], [\lambda]}$ and $\overline{\alpha' \otimes \beta'} \in \text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})_{[\mu'], [\lambda']}$ by

$$\overline{\alpha \otimes \beta} \circ_n \overline{\alpha' \otimes \beta'} = (\alpha \otimes \beta)(\alpha' \otimes \beta') + \mathcal{F}_3^{<[\mu + \mu'], [\lambda + \lambda']}.$$

Clearly, (3.4) gives $(C(\mu_1) \circ C(\mu'_1)) \otimes (C(\mu_2) \circ C(\mu'_2)) = C([\mu + \mu'])$, and (3.12) gives

$$C([\mu]) \circ_n C([\mu']) \subset (C(\mu_1) \circ C(\mu'_1)) \otimes (C(\mu_2) \circ C(\mu'_2)).$$

The converse inclusion holds true as well, as one can see by expressing, reciprocally, the (componentwise) product of $\mathcal{L}_{0,1}^{\otimes n}$ in terms of the product of $\mathcal{L}_{0,n}$ via the formula (2.19). In conclusion,

$$C([\mu]) \circ_n C([\mu']) = C([\mu + \mu']).$$

We are left to show that (3.7) generalizes to $\mathcal{L}_{0,n}$. First, note that for every $1 \leq a \leq n$ the embedding $i_a: \mathcal{L}_{0,1} \rightarrow \mathcal{L}_{0,n}$ in (2.16) is a morphism of the filtered algebras $(\mathcal{L}_{0,1}, \mathcal{F}_2)$ and $(\mathcal{L}_{0,n}, \mathcal{F}_3)$, in the sense that

$$i_a(\mathcal{F}_2^{\mu, \lambda}) \subset \sum_{\gamma \in P} \mathcal{F}_3^{[\mu_a], [\lambda_a], [\gamma_a]},$$

where by definition $[\mu_a] = (0, \dots, 0, \mu, 0, \dots, 0)$ with μ on the a -th entry, and similarly $[\lambda_a] = (0, \dots, 0, \lambda, 0, \dots, 0)$ and $[\gamma_a] = (0, \dots, 0, \gamma, 0, \dots, 0)$. Therefore, the relation (3.7) yields in $\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})$ similar relations between elements of the form (matrix coefficient) $\otimes 1$, or $1 \otimes$ (matrix coefficient).

We now consider the case of “mixed” products. We give the details when $n = 2$, the general case being similar. Let us write the twist F in (2.18) as

$$F = \sum_{(F)} F_{(1)} \otimes F_{(2)} = \sum_{(F)} F_{(1)1} \otimes F_{(1)2} \otimes F_{(2)1} \otimes F_{(2)2},$$

that is, we set $F_{(1)1} := R_{(2)}^2 R_{(2)}^4$, $F_{(1)2} := R_{(2)}^1 R_{(2)}^3$, $F_{(2)1} := R_{(1)}^1 R_{(1)}^2$, $F_{(2)2} := R_{(1)}^3 R_{(1)}^4$. Put $d(\mu) := \dim(V_\mu)$, $\mu \in P_+$, and

$$\Delta^{(2)}(\mu_2 \phi_{l_2}^{k_2}) = \sum_{p,s=1}^{d(\mu_2)} \mu_2 \phi_p^{k_2} \otimes \mu_2 \phi_s^p \otimes \mu_2 \phi_{l_2}^s, \quad \Delta^{(2)}(\mu_1' \phi_{l_1'}^{k_1'}) = \sum_{p',s'=1}^{d(\mu_1')} \mu_1' \phi_{p'}^{k_1'} \otimes \mu_1' \phi_{s'}^{p'} \otimes \mu_1' \phi_{l_1'}^{s'}.$$

From (3.12), one obtains

$$\begin{aligned} (1 \otimes \mu_2 \phi_{l_2}^{k_2})(\mu_1' \phi_{l_1'}^{k_1'} \otimes 1) &= \sum_{(F)} \sum_{p,s=1}^{d(\mu_2)} \sum_{p',s'=1}^{d(\mu_1')} (\mu_1' \phi_{s'}^{p'}(\mu_1' \phi_{p'}^{k_1'}(F_{(2)1}) \mu_1' \phi_{l_1'}^{s'}(S(F_{(2)2})))) \\ &\quad \otimes (\mu_2 \phi_s^p(\mu_2 \phi_p^{k_2}(F_{(1)1}) \mu_2 \phi_{l_2}^s(S(F_{(1)2}))))). \end{aligned} \quad (3.13)$$

It is immediate that

$$\mu_1' \phi_{s'}^{p'} \otimes \mu_2 \phi_s^p \in C(\mu_1')_{w(e_{s'}^{\mu_1'}), w(e_{s'}^{\mu_1'}) - w(e_{p'}^{\mu_1'})} \otimes C(\mu_2)_{w(e_s^{\mu_2}), w(e_s^{\mu_2}) - w(e_p^{\mu_2})}.$$

As in (iv) above, for every $\mu \in P_+$ we order the weight vectors e_1^μ, \dots, e_m^μ so that $w(e_i^\mu) > w(e_j^\mu)$ implies $i < j$. With such an ordering the factorization $R = \Theta \hat{R}$ (see (2.4)) implies

$$\mu_2 \phi_p^{k_2}(F_{(1)1}) \mu_2 \phi_{l_2}^s(S(F_{(1)2})) = 0 \quad \text{unless } k_2 \geq p \text{ and } s \geq l_2,$$

and

$$\mu_1' \phi_{p'}^{k_1'}(F_{(2)1}) \mu_1' \phi_{l_1'}^{s'}(S(F_{(2)2})) = 0 \quad \text{unless } k_1' \leq p' \text{ and } s' \leq l_1'.$$

Since $s > l_2$, we have $w(e_s^{\mu_2}) \leq w(e_{l_2}^{\mu_2})$, and if $w(e_s^{\mu_2}) < w(e_{l_2}^{\mu_2})$, then $\mu_2 \phi_s^p \in \mathcal{F}_2^{< \mu_2, w(e_{l_2}^{\mu_2})}$. In this last situation, the summands $\mu_1' \phi_{s'}^{p'} \otimes \mu_2 \phi_s^p$ in the sum above vanish in $\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,2})$. In order to find all the non-zero summands, we have to consider also the weights with respect to the action (3.9).

Since $k_2 \geq p$ implies $w(e_{k_2}^{\mu_2}) \leq w(e_p^{\mu_2})$, we have $w(e_s^{\mu_2}) - w(e_p^{\mu_2}) \leq w(e_{l_2}^{\mu_2}) - w(e_{k_2}^{\mu_2})$. Therefore, the summands which are non-zero in $\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,2})$ have both weights $w(e_s^{\mu_2}) = w(e_{l_2}^{\mu_2})$ and $w(e_p^{\mu_2}) = w(e_{k_2}^{\mu_2})$. Doing similarly with the weights of $\mu'_1 \phi_{s'}^{p'}$, we find that also $w(e_{s'}^{\mu'_1}) = w(e_{l'_1}^{\mu'_1})$ and $w(e_{p'}^{\mu'_1}) = w(e_{k'_1}^{\mu'_1})$. When all these conditions on weights are satisfied, the corresponding components $F_{(1)1}, F_{(1)2}, F_{(2)1}, F_{(2)2}$ have zero weight. Therefore, the sum reduces to

$$\begin{aligned} & \sum_{(F)} \mu_2 \phi_{k_2}^{k_2}(F_{(1)1}) \mu_2 \phi_{l_2}^{l_2}(S(F_{(1)2})) \mu'_1 \phi_{k'_1}^{k'_1}(F_{(2)1}) \mu'_1 \phi_{l'_1}^{l'_1}(S(F_{(2)2})) \\ &= \langle \mu_2 \phi_{k_2}^{k_2} \otimes \mu_2 \phi_{l_2}^{l_2} \otimes \mu'_1 \phi_{k'_1}^{k'_1} \otimes \mu'_1 \phi_{l'_1}^{l'_1}, \Theta_{13} \Theta_{14}^{-1} \Theta_{24} \Theta_{23}^{-1} \rangle = q^{(w(e_{k_2}^{\mu_2}) - w(e_{l_2}^{\mu_2}), w(e_{k'_1}^{\mu'_1}) - w(e_{l'_1}^{\mu'_1}))}. \end{aligned}$$

Denoting by $q'_{k_2 l_2 k'_1 l'_1}$ this scalar, it follows

$$(1 \otimes \mu_2 \phi_{l_2}^{k_2}) \circ_2 (\mu'_1 \phi_{l'_1}^{k'_1} \otimes 1) = q'_{k_2 l_2 k'_1 l'_1} \mu'_1 \phi_{l'_1}^{k'_1} \otimes \mu_2 \phi_{l_2}^{k_2} = q'_{k_2 l_2 k'_1 l'_1} (\mu'_1 \phi_{l'_1}^{k'_1} \otimes 1) \circ_2 (1 \otimes \mu_2 \phi_{l_2}^{k_2}).$$

This is the relation analogous to (3.7) for mixed products in $\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,2})$.

Recall that in (3.8) we denoted by u_1, \dots, u_M the ordered list of matrix coefficients $\varpi_k \phi_i^j$. Let us order in a lexicographic way the elements $u_i \otimes u_j$, i.e., as a sequence $u_1^{(2)}, \dots, u_{M^2}^{(2)}$ such that the following condition holds: if $\varpi_{i'} \phi_{s'}^{t'} < \varpi_{k'} \phi_{i'}^{j'}$, or $\varpi_{i'} \phi_{s'}^{t'} = \varpi_{k'} \phi_{i'}^{j'}$ and $\varpi_{i'} \phi_s^t < \varpi_{k'} \phi_i^j$, then $u_a^{(2)} := \varpi_k \phi_i^j \otimes \varpi_{k'} \phi_{i'}^{j'}$ and $u_b^{(2)} := \varpi_{i'} \phi_s^t \otimes \varpi_{k'} \phi_{i'}^{j'}$ satisfy $u_b^{(2)} < u_a^{(2)}$. Then, for this ordering the polynomial relations (3.8) hold true for all $1 \leq u_i^{(2)} < u_j^{(2)} \leq M^2$. As described after (3.1), it follows that $\text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})$ is Noetherian. The filtration \mathcal{F}_3 being well founded, it implies that $\mathcal{L}_{0,n}$ is Noetherian too.

Finally, we consider the A -algebra $\mathcal{L}_{0,n}^A$, and prove it is Noetherian. We proceed in exactly the same way as for $\mathcal{L}_{0,n}$, changing the generators and replacing key arguments of the steps (i)–(iv) by the corresponding results over A . Let us describe these modifications step by step.

First, consider the case $n = 1$. Recall the A -lattices ${}_A \dot{C}(\lambda)$ (see (2.46)), and the decomposition (2.55) of \mathcal{O}_A into weight subspaces. In particular, have a decomposition into weight subspaces for the left coregular action,

$${}_A \dot{C}(\lambda) = \bigoplus_{\lambda' \in P} {}_A \dot{C}(\lambda)_{\lambda'}.$$

Define

$${}_A \mathcal{F}_2^{\mu, \lambda} := \bigoplus_{(\mu', \lambda') \preceq (\mu, \lambda)} {}_A \dot{C}(\mu')_{\lambda'}.$$

Recall that every A -module of matrix coefficients $(\text{End}({}_A V_\mu))^*$, $\mu \in P_+$, is contained in $\mathcal{O}_A(\leq \mu)$, and by inverting over $\mathbb{C}(q)$ the corresponding linear triangular system between basis elements, and using that the order relation \preceq is finer than \leq , we obtain an inclusion

$$\bigoplus_{\mu' \preceq \mu} {}_A \dot{C}(\mu') \subset \bigoplus_{\mu' \leq \mu} C(\mu')$$

(see (2.48)–(2.51)). It follows that ${}_A \mathcal{F}_2^{\mu, \lambda} = \mathcal{F}_2^{\mu, \lambda} \cap \mathcal{O}_A$, and therefore, like \mathcal{F}_2 the family ${}_A \mathcal{F}_2 := \{ {}_A \mathcal{F}_2^{\mu, \lambda} \}_{(\mu, \lambda) \in \Lambda}$ is a well-founded filtration of \mathcal{O}_A . Put ${}_A \mathcal{F}_2^{\prec \mu, \lambda} = \mathcal{F}_2^{\prec \mu, \lambda} \cap \mathcal{O}_A$, and consider the graded A -module $\text{Gr}_{{}_A \mathcal{F}_2}(\mathcal{L}_{0,1}^A)$ associated to ${}_A \mathcal{F}_2$. By (2.52)–(2.54) and the fact that $\mathcal{O}_A = \mathcal{L}_{0,1}^A$ as an A -module, we have the A -module decomposition

$$\text{Gr}_{{}_A \mathcal{F}_2}(\mathcal{L}_{0,1}^A) = \bigoplus_{(\mu, \lambda) \in \Lambda} {}_A C(\mu)_{\lambda},$$

where ${}_A C(\mu)_\lambda$ is the submodule of weight λ (for the left coregular action) of

$${}_A C(\mu) := (\text{End}({}_A V_\mu))^*.$$

Then, we can proceed as before. By step (i), we deduce that ${}_A \mathcal{F}_2$ is an algebra filtration of $\mathcal{L}_{0,1}^A$. By Proposition 2.12, the A -module ${}_A \dot{C}(\mu_1 + \mu_2)$ has multiplicity one in ${}_A \dot{C}(\mu_1) \otimes {}_A \dot{C}(\mu_2)$. In fact, by step (ii), ${}_A C(\mu_1 + \mu_2)$ has multiplicity one in ${}_A C(\mu_1) \otimes_A {}_A C(\mu_2)$, so exactly in the same way as (3.4), we obtain in $\text{Gr}_{{}_A \mathcal{F}_2}(\mathcal{L}_{0,1}^A)$ the equality

$${}_A C(\mu) \circ {}_A C(\nu) = {}_A C(\mu + \nu).$$

In step (iii), we fixed a basis of each space $C(\mu)$, consisting of a set of matrix coefficients $\{\mu \phi_j^i\}$ with respect to dual basis of weight vectors of the modules V_μ and V_μ^* . In step (iv), the basis elements of V_μ and V_μ^* were ordered by means of the weights, and we used the fact that the matrix coefficients in the spaces $C(\varpi_1), \dots, C(\varpi_m)$ form a generating set of the algebra $\text{Gr}_{{}_A \mathcal{F}_2}(\mathcal{L}_{0,1})$. The only property of the matrix coefficients used in the computations was that they are weight vectors for the left coregular action (and later, in the case $n > 1$, for the action (3.9)).

We can proceed exactly in the same manner by working with the A -modules of matrix coefficients ${}_A C(\mu)$. If one wishes to work at the level of \mathcal{O}_A , recall that any set of generators of \mathcal{O}_A generates $\mathcal{L}_{0,1}^A$ as well (see the proof of Proposition 2.18). Then, one can replace the basis $\{\mu \phi_j^i\}$ of each space $C(\mu)$ with the canonical basis $\dot{\mathbf{B}}[\mu]^*$ of ${}_A \dot{C}(\mu)$, and take the generating set of \mathcal{O}_A formed by the elements in $\dot{\mathbf{B}}[\varpi_i]^*$, $i = 1, \dots, m$ (see Proposition 2.10 and the comments thereafter). By the integrality properties satisfied by the R -matrix and the twists, all the computations in the proof of steps (iii) and (iv) can be done using such basis elements, and eventually take place over A (see [18, Propositions 4.10 and 6.9]). Therefore, we obtain a relation like (3.8) with coefficients $\alpha_{ij}^{st} \in A$. Since A is a Noetherian ring, again this proves $\text{Gr}_{{}_A \mathcal{F}_2}(\mathcal{L}_{0,1}^A)$, whence $\mathcal{L}_{0,1}^A$, are Noetherian rings.

This being done, the adaptation of the proof when $n > 1$ is immediate. The filtration \mathcal{F}_3 has to be replaced with ${}_A \mathcal{F}_3 := \left\{ {}_A \mathcal{F}_3^{[\mu],[\lambda],[\gamma]} \right\}_{([\mu],[\lambda],[\gamma])}$, where ${}_A \mathcal{F}_3^{[\mu],[\lambda],[\gamma]}$ is the A -module defined by

$${}_A \mathcal{F}_3^{[\mu],[\lambda],[\gamma]} = \bigoplus_{([\mu'],[\lambda'],[\gamma']) \preceq ([\mu],[\lambda],[\gamma])} {}_A \dot{C}([\mu'])_{[\lambda'],[\gamma]},$$

where

$${}_A \dot{C}([\mu])_{[\lambda],[\gamma]} = {}_A \dot{C}(\mu_1)_{\lambda_1, \gamma_1} \otimes_A \cdots \otimes_A {}_A \dot{C}(\mu_n)_{\lambda_n, \gamma_n},$$

and ${}_A \dot{C}(\mu)_{\lambda, \gamma}$ is the subspace of ${}_A \dot{C}(\mu)_\lambda$ of weight γ for the action (3.9). Then the proof proceeds in exactly the same way, replacing in (3.13) and all subsequent computations the matrix coefficients by the generators of \mathcal{O}_A provided by Proposition 2.10. This concludes the proof. \blacksquare

Theorem 3.4. *The algebra $\mathcal{M}_{0,n} = \mathcal{L}_{0,n}^{U_q}$ is Noetherian and generated over $\mathbb{C}(q)$ by a finite number of elements.*

Our method of proof follows closely that of the Hilbert–Nagata theorem (see [46]). Let us recall one version of this theorem. Let K be an arbitrary field, \mathfrak{A} a commutative algebra over K finitely generated by elements a_1, \dots, a_n , and G a group of algebra automorphisms of \mathfrak{A} .

Theorem 3.5. *If the action of G on \mathfrak{A} is completely reducible on finite-dimensional representations, then the ring \mathfrak{A}^G of invariants of \mathfrak{A} with respect to G is Noetherian and a finitely generated algebra over K .*

We recall here the main steps of the proof that we will adapt in order to prove Theorem 3.4:

- (a) From the complete reducibility of the action of G on \mathfrak{A} , one can define a linear map

$$R: \mathfrak{A} \rightarrow \mathfrak{A}^G$$

namely the projection onto the space of invariant elements along the sum of nontrivial isotypical components of \mathfrak{A} . This linear map is the Reynolds operator; we already discussed it in (2.23) in the case of U_q acting on $\mathcal{L}_{0,n}$. By the same arguments we developed there, it satisfies $R(hf) = hR(f)$ for every $f \in \mathfrak{A}$, $h \in \mathfrak{A}^G$.

- (b) Let I be an ideal of \mathfrak{A}^G . Then $I = R(\mathfrak{A}I) = \mathfrak{A}I \cap \mathfrak{A}^G$. Because $\mathfrak{A}I$ is an ideal of \mathfrak{A} , and \mathfrak{A} is Noetherian, there exist elements b_1, \dots, b_s , that can be chosen in $I \subset \mathfrak{A}^G$, such that $\mathfrak{A}I = \mathfrak{A}b_1 + \dots + \mathfrak{A}b_s$. Since $I = R(\mathfrak{A}I) = R(\mathfrak{A}b_1 + \dots + \mathfrak{A}b_s) = \mathfrak{A}^G b_1 + \dots + \mathfrak{A}^G b_s$, I is finitely generated over \mathfrak{A}^G . Therefore, \mathfrak{A}^G is Noetherian.
- (c) Let \mathfrak{B} be an algebra graded over \mathbb{N} (for simplicity of notations): $\mathfrak{B} = \bigoplus_{n=0}^{+\infty} \mathfrak{B}_n$, with $\mathfrak{B}_m \cdot \mathfrak{B}_n \subset \mathfrak{B}_{m+n}$. The augmentation ideal of \mathfrak{B} is $\mathfrak{B}^+ := \bigoplus_{n=1}^{+\infty} \mathfrak{B}_n$. If \mathfrak{B}^+ is a Noetherian ideal of \mathfrak{B} , then \mathfrak{B} is a finitely generated algebra over \mathfrak{B}_0 . This is [99, Lemma 2.4.5] (in that statement \mathfrak{B} is commutative, but this hypothesis is not necessary for the proof).
- (d) Assume that \mathfrak{A}^G is graded over \mathbb{N} (for simplicity of notations): $\mathfrak{A}^G = \bigoplus_{n=0}^{+\infty} \mathfrak{A}_n^G$ with $\mathfrak{A}_0^G = K$. Then $\mathfrak{A}^{G+} = \bigoplus_{n=1}^{+\infty} \mathfrak{A}_n^G$ is an ideal of \mathfrak{A}^G , which is Noetherian by (b) above. Applying (c), we deduce that \mathfrak{A}^G is a finitely generated algebra over K .

Proof of Theorem 3.4. Consider the filtration \mathcal{F} of $\mathcal{L}_{0,n}$ by the subspaces

$$\mathcal{F}^{[\mu]} = \bigoplus_{[\mu'] \preceq [\mu]} C([\mu']), \quad \mu \in P_+^n,$$

where P_+^n is given the lexicographic partial order induced from $[\Lambda]$. It is easily seen that \mathcal{F} is an algebra filtration: the coregular actions $\triangleright, \triangleleft$ fix globally each component $C(\mu)$ of $\mathcal{L}_{0,1}$, so the claim follows from (2.9), (2.17) and the fact that $C(\mu) \star C(\nu) \subset C(\mu + \nu)$ for all $\mu, \nu \in P_+$. Denote by $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})$ the corresponding graded algebra. As a vector space, we have

$$\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n}) = \bigoplus_{[\mu] \in P_+^n} C([\mu]). \quad (3.14)$$

Because each space $C([\mu])$ is stabilized by the coadjoint action of U_q , (3.14) has a key advantage on the refined decomposition (3.11). Indeed, since $\mathcal{L}_{0,n}$ is a U_q -module algebra, the action of U_q is well defined on $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})$ and gives it a structure of U_q -module algebra. As vector spaces, we have

$$\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q} = \bigoplus_{[\mu] \in P_+^n} C([\mu])^{U_q}.$$

Now we can adapt the different steps (a)–(d) recalled above:

- (a') The action of U_q on $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})$ is completely reducible. This follows from (3.14) and the fact that the spaces $C(\mu)$ are finite-dimensional and thus completely reducible U_q -modules. We can therefore define the Reynolds operator as in (a),

$$R: \text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n}) \rightarrow \text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}.$$

- (b') $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})$ is Noetherian, because (3.14) shows it is filtered by \mathcal{F}_3 , and the associated graded algebra $\text{Gr}_{\mathcal{F}_3}(\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})) = \text{Gr}_{\mathcal{F}_3}(\mathcal{L}_{0,n})$ is Noetherian by Theorem 3.1. As in (b), we deduce that $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$ is Noetherian. But $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q} = \text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n}^{U_q})$, which implies that $\mathcal{L}_{0,n}^{U_q}$ is Noetherian.

- (c') (and (d')) Then we can apply the steps (c)–(d). As a result $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$ is finitely generated, say by k non-zero elements $\bar{x}_1, \dots, \bar{x}_k$, which we may assume homogeneous.
- (e') We can now deduce that $\mathcal{L}_{0,n}^{U_q}$ is generated by elements x_i with leading terms the \bar{x}_i 's. Indeed, let $x \in \mathcal{L}_{0,n}^{U_q}$, and $[\mu] \in P_+^n$ such that $x \in \mathcal{F}^{[\mu]} \setminus \mathcal{F}^{<[\mu]}$, where $\mathcal{F}^{<[\mu]} := \bigoplus_{[\mu'] < [\mu]} C([\mu'])$. In $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})_{[\mu]}^{U_q} = \mathcal{F}^{[\mu]} / \mathcal{F}^{<[\mu]}$, we have

$$\bar{x} = \sum_{(i_1, \dots, i_k) \in I} \lambda_{(i_1, \dots, i_k)} \bar{x}_1^{i_1} \cdots \bar{x}_k^{i_k}$$

for some finite set $I \subset \mathbb{N}^k$, scalars $\lambda_{(i_1, \dots, i_k)} \in \mathbb{C}(q)$, and monomials $\bar{x}_1^{i_1} \cdots \bar{x}_k^{i_k}$ of degree $[\mu]$. By definition of the product in $\text{Gr}_{\mathcal{F}}(\mathcal{L}_{0,n})^{U_q}$,

$$\bar{x}_1^{i_1} \cdots \bar{x}_k^{i_k} = x_1^{i_1} \cdots x_k^{i_k} + \mathcal{F}^{<[\mu]},$$

so $x_1^{i_1} \cdots x_k^{i_k} \in \mathcal{F}^{[\mu]} \setminus \mathcal{F}^{<[\mu]}$, whence $\bar{x}_1^{i_1} \cdots \bar{x}_k^{i_k} = \overline{x_1^{i_1} \cdots x_k^{i_k}}$ and

$$x - \sum_{(i_1, \dots, i_k) \in I} \lambda_{(i_1, \dots, i_k)} x_1^{i_1} \cdots x_k^{i_k} \in \mathcal{F}^{<[\mu]}.$$

The conclusion follows by decreasing induction on $[\mu]$, since at last we terminate at $\mathcal{F}^{[0]} \cong \mathbb{C}(q)$.

By combining the steps (a') to (e'), we get that $\mathcal{M}_{0,n}$ is a Noetherian and finitely generated ring. ■

Remark 3.6.

- (1) Because $\mathcal{L}_{0,1}^{U_q}$ is the center of $\mathcal{L}_{0,1}$, (e') proves it is finitely generated. Of course this follows also from the isomorphism $\mathcal{L}_{0,1} \cong U_q^{\text{lf}}$ and the fact that the center of U_q^{lf} is the center of U_q (by Theorem 2.1), plus the well-known description of the latter.
- (2) In the \mathfrak{sl}_2 case the filtration \mathcal{F} on $\mathcal{L}_{0,n}^{U_q}$ should be related via the Wilson loop isomorphism (defined in [18, Section 8.2]) to the filtration of skein algebras of spheres with $n + 1$ punctures used in [93].

4 Proof of Theorem 1.2

As usual we let ϵ be a primitive l -th root of unity with l odd and $l > d_i$ for all $i \in \{1, \dots, m\}$. We now consider the specialization $\mathcal{L}_{0,n}^\epsilon$ of $\mathcal{L}_{0,n}$ at $q = \epsilon$, defined in Section 2.2.1. Recall the isomorphism of algebras $\eta^*: \mathcal{O}(G) \rightarrow \mathcal{Z}_0(\mathcal{O}_\epsilon)$ (see (2.71)), and that $\mathcal{L}_{0,n}^\epsilon = \mathcal{O}_\epsilon^{\otimes n}$ as a vector space. Consider the linear subspace of $\mathcal{L}_{0,n}^\epsilon$ defined by $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) := \mathcal{Z}_0(\mathcal{O}_\epsilon)^{\otimes n}$. This space is naturally a subalgebra of $\mathcal{O}_\epsilon^{\otimes n}$ (endowed with the componentwise product \star). In fact, we also have the following.

Proposition 4.1.

- (1) $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ is a central subalgebra of the algebra $\mathcal{L}_{0,n}^\epsilon$, and the $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -modules $\mathcal{L}_{0,n}^\epsilon$ and $\mathcal{O}_\epsilon^{\otimes n}$, with actions defined by the respective products of these algebras, do coincide.
- (2) $\mathcal{L}_{0,n}^\epsilon$ is a free $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module of rank $l^{n \cdot \dim \mathfrak{g}}$.
- (3) $(\eta^{*-1})^{\otimes n}: \mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) \rightarrow \mathcal{O}(G)^{\otimes n}$ is an isomorphism of algebras, and $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ is a Noetherian ring.
- (4) The $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module $\mathcal{L}_{0,n}^\epsilon$ is finite and Noetherian. Therefore, $\mathcal{L}_{0,n}^\epsilon$ is a Noetherian ring.

Note that the proof we give in (4) of the fact that $\mathcal{L}_{0,n}^\epsilon$ is Noetherian is independent from the proof of Theorem 3.1.

Proof. (1) Let us show that $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ is a central subalgebra of $\mathcal{L}_{0,n}^\epsilon$. In the case $n = 1$, the formula (2.9) implies that $\alpha\beta = \alpha \star \beta$ for all $\alpha \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ and $\beta \in \mathcal{L}_{0,1}^\epsilon$. Indeed, by (2.9) we have

$$\begin{aligned} \alpha\beta &= \sum_{(R),(R)} (R_{(2')}S(R_{(2)}) \triangleright \alpha) \star (R_{(1')} \triangleright \beta \triangleleft R_{(1)}) \\ &= \sum_{(R),(R),(\alpha),(\beta)} \alpha_{(1)} \star (\beta_{(1)}(R_{(1)}\alpha_{(3)}(S(R_{(2)})))\beta_{(3)}(R_{(1')}\alpha_{(2)}(R_{(2')}))\beta_{(2)}), \end{aligned}$$

where all components $\alpha_{(1)}, \alpha_{(2)}, \alpha_{(3)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$, since $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ is a Hopf subalgebra of \mathcal{O}_ϵ . But

$$\sum_{(R)} R_{(1)}\alpha_{(3)}(S(R_{(2)})) = S^{-1}(\Phi^-(S_{\mathcal{O}_\epsilon}(\alpha_{(3)}))) \in \mathcal{Z}_0(U_\epsilon),$$

since $\Phi^-(S_{\mathcal{O}_\epsilon}(\alpha_{(3)})) \in \mathcal{Z}_0(U_\epsilon)$ by Theorem 2.29 (2). Similarly, $\sum_{(R)} R_{(1')}\alpha_{(2)}(R_{(2')}) \in \mathcal{Z}_0(U_\epsilon)$. In general, these elements belong to $\mathcal{Z}_0(U_\epsilon)$ and not $\mathcal{Z}_0(U_\epsilon^{\text{ad}})$ because of the ‘‘diagonal’’ factor Θ of the R -matrix in (2.4). By Lemma 2.28, $\mathcal{Z}_0(U_A^{\text{ad}})$ acts by the trivial character ε (the counit) on specializations of Γ -modules. The action of $\mathcal{Z}_0(U_A)$ is the counit ε multiplied with some powers of $\epsilon^{1/D}$. However, [18, Propositions 4.1 and 4.10] show that such powers of $\epsilon^{1/D}$ eventually disappear in the sum above; this is because the sum can be rewritten in terms of copies of the quasi R -matrix \hat{R} in (2.4) and the pivotal element ℓ , instead of copies of R . Therefore,

$$\alpha\beta = \sum_{(\alpha),(\beta)} \alpha_{(1)} \star (\varepsilon(\beta_{(1)})\varepsilon(\alpha_{(3)})\varepsilon(\beta_{(3)})\varepsilon(\alpha_{(2)})\beta_{(2)}) = \alpha \star \beta. \quad (4.1)$$

This shows $\mathcal{L}_{0,1}^\epsilon$ and \mathcal{O}_ϵ coincide as modules over $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$. Next, we show that the subalgebras $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$ are central in $\mathcal{L}_{0,n}^\epsilon$ for all $a = 1, \dots, n$. This fact will conclude the proof that $\mathcal{L}_{0,n}^\epsilon$ and $\mathcal{O}_\epsilon^{\otimes n}$ coincide as $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -modules, because the subalgebras $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$ generate the space $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ in $(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$, and hence in $\mathcal{L}_{0,n}^\epsilon$ (this follows from the comment before (2.18)).

In order to show that $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$ is central in $\mathcal{L}_{0,n}^\epsilon$ for all $a = 1, \dots, n$, it is enough to show $\mathcal{Z}_0(\mathcal{O}_\epsilon)^{(a)}$ commutes with the elements of $\mathcal{L}_{0,n}^\epsilon$ supported by the tensor factors $(\mathcal{L}_{0,1}^\epsilon)^{(b)}$ with $b \neq a$. Since $(\alpha)^{(a)} \otimes (\beta)^{(b)} = ((\alpha)^{(a)} \otimes 1)(1 \otimes (\beta)^{(b)})$ by the definition, we have to show that $(1 \otimes (\beta)^{(b)})((\alpha)^{(a)} \otimes 1) = (\alpha)^{(a)} \otimes (\beta)^{(b)}$ whenever $\alpha \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$. We have (denoting $\sum_{(\alpha),(\alpha),(\alpha),(\alpha)}$ by $\sum_{(\alpha)^4}$, $\Delta(\alpha_{(1)}) = \sum_{(\alpha)} \alpha_{(1)(1)} \otimes \alpha_{(1)(2)}$ etc.):

$$\begin{aligned} (1 \otimes (\beta)^{(b)})((\alpha)^{(a)} \otimes 1) &= \sum_{(R^i)} (S(R_{(1)}^3 R_{(1)}^4) \triangleright \alpha \triangleleft R_{(1)}^1 R_{(1)}^2)^{(a)} \\ &\quad \otimes (S(R_{(2)}^1 R_{(2)}^3) \triangleright \beta \triangleleft R_{(2)}^2 R_{(2)}^4)^{(b)} \\ &= \sum_{(R^i),(\alpha)^4,(\beta)^2} (\alpha_{(2)})^{(a)} \otimes (\beta_{(2)})^{(b)} \\ &\quad \times \beta_{(1)}(\alpha_{(1)(2)}(R_{(1)}^2)R_{(2)}^2\alpha_{(3)(1)}(S(R_{(1)}^4))R_{(2)}^4) \\ &\quad \times \beta_{(3)}(\alpha_{(3)(2)}(R_{(1)}^3)R_{(2)}^3\alpha_{(1)(1)}(R_{(1)}^1)S(R_{(2)}^1)). \end{aligned}$$

By Theorem 2.29 (2), it follows that

$$\alpha_{(1)(2)}(R_{(1)}^2)R_{(2)}^2 = \Phi^+(\alpha_{(1)(2)}) \in \mathcal{Z}_0(U_\epsilon),$$

and similarly

$$\alpha_{(3)(1)}(S(R_{(1)}^4))R_{(2)}^4, \alpha_{(3)(2)}(R_{(1)}^3)R_{(2)}^3, \alpha_{(1)(1)}(R_{(1)}^1)S(R_{(2)}^1) \in \mathcal{Z}_0(U_\epsilon).$$

Denote by z any such element; $\mathcal{Z}_0(U_\epsilon^{\text{ad}})$ acts by the trivial character (the counit ε) on specializations of Γ -modules. By using [18, Proposition 6.2], arguing as above (4.1), we obtain that the expression of z in terms of the corresponding $\alpha_{(i)(j)}$ involves $\varepsilon(z) = \varepsilon(\alpha_{(i)(j)})$ only (no root $\epsilon^{1/D}$). It follows

$$\begin{aligned} & \beta_{(1)}(\alpha_{(1)(2)}(R_{(1)}^2)R_{(2)}^2\alpha_{(3)(1)}(S(R_{(1)}^4))R_{(2)}^4) \\ &= \varepsilon(\alpha_{(1)(2)}\alpha_{(3)(1)})\beta_{(1)}(1) = \varepsilon(\alpha_{(1)(2)})\varepsilon(\alpha_{(3)(1)})\varepsilon(\beta_{(1)}), \\ & \beta_{(3)}(\alpha_{(3)(2)}(R_{(1)}^3)R_{(2)}^3\alpha_{(1)(1)}(R_{(1)}^1)S(R_{(2)}^1)) = \varepsilon(\alpha_{(3)(2)})\varepsilon(\alpha_{(1)(1)})\varepsilon(\beta_{(3)}). \end{aligned}$$

Therefore, $(1 \otimes (\beta)^{(b)})((\alpha)^{(a)} \otimes 1) = (\alpha)^{(a)} \otimes (\beta)^{(b)}$. It follows that $\mathcal{L}_{0,n}^\epsilon = \mathcal{O}_\epsilon^{\otimes n}$ as modules over $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$; for instance when $n = 2$, given $\alpha', \beta' \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ we have $(\alpha' \otimes \beta')(\alpha \otimes \beta) = (\alpha' \otimes 1)(1 \otimes \beta')(\alpha \otimes 1)(1 \otimes \beta)$ immediately by (2.17), and $(1 \otimes \beta')(\alpha \otimes 1) = \alpha \otimes \beta' = (\alpha \otimes 1)(1 \otimes \beta')$ as above. Then $(\alpha' \otimes \beta')(\alpha \otimes \beta) = \alpha' \alpha \otimes \beta' \beta$. In particular, $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ is a central subalgebra of $\mathcal{L}_{0,n}^\epsilon$.

(2) Since $\mathcal{L}_{0,n}^\epsilon$ and $\mathcal{O}_\epsilon^{\otimes n}$ coincide as modules over $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon^{\otimes n})$, the claim follows from Theorem 2.29, that is, from [41, Theorem 7.2], which shows that \mathcal{O}_ϵ is a finitely generated projective module of rank $l^{\dim \mathfrak{g}}$ over $\mathcal{Z}_0(\mathcal{O}_\epsilon)$, and from the arguments of [28] (using that $K_0(\mathcal{O}(G)) = \mathbb{Z}$ by [87]), which imply that this module is free. Alternatively, it follows from the fact that \mathcal{O}_ϵ is a cleft extension of $\mathcal{O}(G)$ (see [6, Remark 2.18 (b)], and [25, Section 3.2]).

(3) The linear isomorphism $(\eta^{*-1})^{\otimes n}: \mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) \rightarrow \mathcal{O}(G)^{\otimes n}$ is an isomorphism of algebras because $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ is central in $\mathcal{L}_{0,n}^\epsilon$. It implies that $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ is a Noetherian ring, since $\mathcal{O}(G)^{\otimes n} = \mathcal{O}(G^n)$ and G^n is an affine algebraic variety.

(4) The fact that $\mathcal{L}_{0,n}^\epsilon$ is a finitely generated $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module follows from (2); an alternative proof of this fact will be provided at the end of the proof of Theorem 4.9. Since $\mathcal{L}_{0,n}^\epsilon$ is finite over $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ and $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ is Noetherian, $\mathcal{L}_{0,n}^\epsilon$ is a Noetherian $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module (e.g., by [7, Proposition 6.5]). It follows that $\mathcal{L}_{0,n}^\epsilon$ is a Noetherian ring (e.g., by [88, Chapter 1, Section 1.3]). \blacksquare

Recall that we denote $U_\epsilon^{\text{lf}} = U_A^{\text{lf}} \otimes_A \mathbb{C}_\epsilon$ (see (2.27)), and $\mathcal{Z}_0(U_\epsilon) \subset U_\epsilon$ is the central polynomial subalgebra generated by $E_{\beta_k}^l, F_{\beta_k}^l, L_i^{\pm l}$, for $k \in \{1, \dots, N\}$ and $i \in \{1, \dots, m\}$. Since $\Phi_1: \mathcal{L}_{0,1}^\epsilon \rightarrow U_\epsilon^{\text{lf}}$ is an embedding of algebras (see Corollary 2.25), it identifies $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ with a central subalgebra of U_ϵ^{lf} . Put $\mathcal{Z}_0(U_\epsilon^{\text{lf}}) := \Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))$. Recall Theorem 2.1, Proposition 2.24, and let $T^{(l)}, T_{2-}^{(l)}$ and $T_2^{(l)}$ be the subsets of T, T_{2-} and T_2 formed by the elements $K_{\lambda l}$ with $\lambda \in P, \lambda \in -2P_+$ and $\lambda \in 2P$, respectively.

Proposition 4.2. *We have $U_\epsilon = T_{2-}^{-1}U_\epsilon^{\text{lf}}[T/T_2] = \Phi_1(\mathcal{L}_{0,1}^\epsilon[d^{-1}])[T/T_2]$, and therefore the map $\Phi_1: \mathcal{L}_{0,1}^\epsilon[d^{-1}] \rightarrow T_{2-}^{-1}U_\epsilon^{\text{lf}}$ is an isomorphism.*

Moreover, $\mathcal{Z}(U_\epsilon^{\text{lf}}) = U_\epsilon^{\text{lf}} \cap \mathcal{Z}(U_\epsilon)$, and

$$\mathcal{Z}_0(U_\epsilon) = T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}})[T^{(l)}/T_2^{(l)}], \quad \mathcal{Z}(U_\epsilon) = T_{2-}^{(l)-1} \mathcal{Z}(U_\epsilon^{\text{lf}})[T^{(l)}/T_2^{(l)}].$$

Proof. The first claim follows immediately from Proposition 2.24 by specialization at $q = \epsilon$. For the second claim, the inclusion $U_\epsilon^{\text{lf}} \cap \mathcal{Z}(U_\epsilon) \subset \mathcal{Z}(U_\epsilon^{\text{lf}})$ is clear, and for the converse inclusion we only have to show that the elements of $\mathcal{Z}(U_\epsilon^{\text{lf}})$ commute with T . They commute with $T_2 \subset U_\epsilon^{\text{lf}}$, so the conjugation action by elements of T on $\mathcal{Z}(U_\epsilon^{\text{lf}})$ has order at most 2. Let $x \in \mathcal{Z}(U_\epsilon^{\text{lf}})$ with decomposition $x = \sum_i c_i x_i$ with all $c_i \in \mathbb{C}$ and x_i PBW basis vectors, and let $\lambda \in P$. We have $K_\lambda x K_{-\lambda} = \sum_i c_i q(x_i) x_i$, where $q(x_i) \in \epsilon^{\mathbb{Z}}$ satisfies $q(x_i)^2 = 1$ for all i . Because ϵ has odd order the only possibility is $q(x_i) = 1$, whence $K_\lambda x K_{-\lambda} = x$. The conclusion follows.

The inclusion $\mathcal{Z}_0(U_\epsilon^{\text{lf}}) \subset \mathcal{Z}_0(U_\epsilon)$ follows from the definition $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$, the formula $\Phi_1 = m \circ (\text{id} \otimes S^{-1}) \circ \Phi$, and the fact that Φ affords an embedding $\mathcal{Z}_0(\mathcal{O}_\epsilon) \rightarrow \mathcal{Z}_0(U_\epsilon(G^*))$

(see Theorem 2.29 (2)). Since $T^{(l)} \subset \mathcal{Z}_0(U_\epsilon)$, we obtain

$$T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}})[T^{(l)}/T_2^{(l)}] \subset \mathcal{Z}_0(U_\epsilon).$$

The proof of the converse inclusion is similar to that in Proposition 2.24. The isomorphism $\mathcal{Z}_0(\mathcal{O}_\epsilon)[\psi_{-l\rho}^{-1}] \rightarrow \mathcal{Z}_0(U_\epsilon(G^*))$ of Theorem 2.29 (2) implies

$$F_{\beta_k}^l K_{\beta_k}^l \otimes 1, 1 \otimes K_{\beta_k}^{-l} E_{\beta_k}^l \in \Phi(\mathcal{Z}_0(\mathcal{O}_\epsilon)[\psi_{-l\rho}^{-1}])$$

for every positive root β_k . Since $\psi_{-l\rho} = \Phi_1^{-1}(K_{-2l\rho}) = \psi_{-\rho}^l$ (the l -th power of $\psi_{-\rho}$ in $\mathcal{L}_{0,1}^\epsilon$), and

$$\Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[\psi_{-\rho}^{-l}]) = T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}}),$$

it follows that

$$F_{\beta_k}^l K_{\beta_k}^l, S^{-1}(E_{\beta_k}^l) K_{\beta_k}^l \in T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}}).$$

Hence $F_{\beta_k}^l, S^{-1}(E_{\beta_k}^l) \in T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}})[T^{(l)}/T_2^{(l)}]$. The sets $S^{-1}(E_{\beta_k}^l) \mathcal{Z}_0(U_\epsilon(\mathfrak{h}))$, $k = 1, \dots, N$, generate the subalgebra $\mathcal{Z}_0(U_\epsilon(\mathfrak{b}_+))$ of $\mathcal{Z}_0(U_\epsilon)$, so from the triangular decomposition $\mathcal{Z}_0(U_\epsilon) = \mathcal{Z}_0(U_\epsilon(\mathfrak{n}_-)) \mathcal{Z}_0(U_\epsilon(\mathfrak{h})) \mathcal{Z}_0(U_\epsilon(\mathfrak{n}_+))$ this proves the inclusion $\mathcal{Z}_0(U_\epsilon) \subset T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}})[T^{(l)}/T_2^{(l)}]$. From the isomorphism

$$\mathcal{Z}_0(U_\epsilon) \quad \bigotimes_{\mathcal{Z}_0(U_\epsilon) \cap \mathcal{Z}_1(U_\epsilon)} \quad \mathcal{Z}_1(U_\epsilon) \rightarrow \mathcal{Z}(U_\epsilon)$$

(see Theorem 2.27), and the fact that $\mathcal{Z}(U_q) \subset U_q^{\text{lf}}$ (whence $\mathcal{Z}_1(U_\epsilon) \subset \mathcal{Z}(U_\epsilon^{\text{lf}})$), the equality $\mathcal{Z}(U_\epsilon) = T_{2-}^{(l)-1} \mathcal{Z}(U_\epsilon^{\text{lf}})[T^{(l)}/T_2^{(l)}]$ follows at once. \blacksquare

Remark 4.3. Let us explain how this can be used to give an interpretation of the isomorphism $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \cong \mathcal{O}(G)$. Recall the notations introduced around Theorem 2.27. Since $G^* = U_+ T_G U_-$, we have $\mathcal{O}(G^*) = \mathcal{O}(U_+) \mathcal{O}(T_G) \mathcal{O}(U_-)$, and the map σ yields an identification

$$\mathcal{O}(G^0) = \mathcal{O}(U_+) \mathcal{O}(T_G/(2)) \mathcal{O}(U_-). \quad (4.2)$$

We can identify $\mathcal{O}(G^0)$ with the subalgebra $(\sigma|_{G^*})^*(\mathcal{O}(G^0)) \subset \mathcal{O}(G^*)$. Consider the exterior power $V = \wedge^N \mathfrak{g}$ endowed with the action $\wedge^N \text{Ad}$ of G . Put on \mathfrak{g} a basis consisting of one element e_α per root space \mathfrak{g}_α , along with a basis of \mathfrak{h} . Let $v \in V$ be the exterior power of the e_α 's for α negative, and v^* a dual vector such that $v^*(v) = 1$ and v^* vanishes on a T_G -invariant complement of v . It is classical that $G \setminus G^0$ has defining equation $\delta(g) = 0$, where δ is the matrix coefficient $\delta(g) = v^*(\pi_V(g)v)$ (see, e.g., [59, p. 174]). Hence $\mathcal{O}(G^0) = \mathcal{O}(G)[\delta^{-1}]$. On G^0 we have $\delta(u_+ t u_-) = \chi_{-2\rho}(t)$, where $\chi_{-2\rho}$ is the character of T_G associated to the root -2ρ . Now we can make the connection with U_ϵ . The isomorphism $\mathcal{Z}_0(U_\epsilon) \cong \mathcal{O}(G^*)$ of Theorem 2.27 (2) identifies $\mathcal{Z}_0(U_\epsilon(\mathfrak{h})) = \mathbb{C}[T^{(l)}]$ with $\mathcal{O}(T_G)$ by mapping K_λ to the character of T_G associated to λ . Therefore, it maps $\mathbb{C}[T^{(l)}]$ to $\mathcal{O}(T_G/(2))$, and $T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}})$ to $\mathcal{O}(G^0)$ by (4.2) and Proposition 4.2. Since $\mathcal{O}(G^0) = \mathcal{O}(G)[\delta^{-1}]$ and $T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}}) = \mathcal{Z}_0(U_\epsilon^{\text{lf}})[\ell^l]$, it follows that $\mathcal{Z}_0(U_\epsilon^{\text{lf}})$ and $\mathcal{O}(G)$ coincide after localization by ℓ^l and δ respectively. By using the Bruhat decomposition of G as in (4.6) in the proof of Theorem 4.9 below, one can deduce $\mathcal{Z}_0(U_\epsilon^{\text{lf}}) \cong \mathcal{O}(G)$, whence $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \cong \mathcal{O}(G)$ by injectivity of Φ_1 .

Let us make the following observation. Since $\mathcal{L}_{0,n}^\epsilon = \mathcal{L}_{0,n}^A \otimes_A \mathbb{C}_\epsilon$, with $\mathcal{L}_{0,n}^A = \mathcal{O}_A^{\otimes n}$ as an A -module, and a generating system of $\mathcal{O}_A^{\otimes n}$ is also a generating system of $\mathcal{L}_{0,n}^A$, it follows from Proposition 2.10 and the identities (2.56)–(2.57) that $\mathcal{L}_{0,n}^\epsilon$ is generated by elements of the form $\alpha_1 \otimes \dots \otimes \alpha_n$, where $\alpha_1, \dots, \alpha_n$ belong to the set C_{gen} of matrix coefficients lying on the first and last columns of the matrix representations of U_A^{res} in the canonical bases of the modules ${}_A V_{\varpi_i}$, $i = 1, \dots, m$. Denote by α^{*k} , $k \in \mathbb{N}$, the k -th power of an element $\alpha \in \mathcal{O}_A$.

Lemma 4.4. *For all $\alpha \in C_{\text{gen}}$, $\alpha^{*l} \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$.*

Proof. Recall that the Frobenius epimorphism $\eta: U_A^{\text{res}} \otimes_A \mathbb{C}_\epsilon \rightarrow U(\mathfrak{g})$ in (2.71) has kernel the ideal I generated by the elements $E_i, F_i, K_i - 1$, and $(K_i; p)_{q_i}$ where l does not divide p , $i = 1, \dots, m$. It follows that an element of \mathcal{O}_ϵ belongs to $\mathcal{Z}_0(\mathcal{O}_\epsilon) = \eta^*(\mathcal{O}(G))$ if and only if it vanishes on I . But this is immediate to check for the elements of the form α^{*l} with $\alpha \in C_{\text{gen}}$, using that K_i is grouplike and the pure summands of $\Delta(E_i)$ and $\Delta(F_i)$ have one component equal to 1 or $K_i^{\pm 1}$ and the other component equal to E_i or F_i . For instance,

$$\psi_{\varpi_i}^{*l}(K_i - 1) = \psi_{\varpi_i}(K_i)^l - 1 = \epsilon^{l(\alpha_i, \varpi_i)} - 1 = 0.$$

Similarly, for every $\alpha \in C_{\text{gen}}$, we find

$$\alpha^{*l}(E_i) = \alpha^{\otimes l}(\Delta^{(l)}(E_i)) = 0 \quad \text{and} \quad \alpha^{*l}(F_i) = \alpha^{*l}(K_i - 1) = 0. \quad \blacksquare$$

We need below explicit descriptions of the centers of $\mathcal{O}_\epsilon(\text{SL}_2)$ and $\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2)$ and their \mathcal{Z}_0 -subalgebras. Denote by a, b, c, d the standard generators of $\mathcal{O}_q(\text{SL}_2)$, i.e., the matrix coefficients in the basis of weight vectors $v_0, v_1 = F.v_0$ of the 2-dimensional irreducible representation V_1 of $U_q(\mathfrak{sl}_2)$. As above, denote by x^{*k} , $k \in \mathbb{N}$, the k -th power of an element $x \in \mathcal{O}_A(\text{SL}_2)$. The algebra $\mathcal{O}_A(\text{SL}_2)$ is generated by a, b, c, d ; the monomials $a^{*i} \star b^{*j} \star d^{*r}$ and $a^{*i} \star c^{*k} \star d^{*r}$, $i, j, k, r \in \mathbb{N}, k > 0$, form an A -basis of $\mathcal{O}_A(\text{SL}_2)$. The algebra $\mathcal{Z}_0(\mathcal{O}_\epsilon(\text{SL}_2))$ is generated by $a^{*l}, b^{*l}, c^{*l}, d^{*l}$; the monomials $a^{*il} \star b^{*jl} \star d^{*rl}$ and $a^{*il} \star c^{*kl} \star d^{*rl}$ form a basis of $\mathcal{Z}_0(\mathcal{O}_\epsilon(\text{SL}_2))$, and $\mathcal{Z}(\mathcal{O}_\epsilon(\text{SL}_2))$ is generated by $\mathcal{Z}_0(\mathcal{O}_\epsilon(\text{SL}_2))$ and the elements $b^{*(l-k)} \star c^{*k}$, $k = 0, \dots, l$ (see [41, Proposition 1.4 and the appendix]). We have the relation

$$a^{*l} \star d^{*l} - b^{*l} \star c^{*l} = 1 \tag{4.3}$$

and the Frobenius isomorphism of Parshall–Wang (see [92, Chapter 7]) coincides with the map

$$\text{FrpW}: \mathcal{O}(\text{SL}_2) \rightarrow \mathcal{Z}_0(\mathcal{O}_\epsilon(\text{SL}_2))$$

induced by η^* ; it sends the standard generators $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ of $\mathcal{O}(\text{SL}_2) = \mathcal{O}_1(\text{SL}_2)$ respectively to $a^{*l}, b^{*l}, c^{*l}, d^{*l}$. Finally, we have seen that $\mathcal{O}_\epsilon(\text{SL}_2)$ is a free $\mathcal{Z}_0(\mathcal{O}(\text{SL}_2))$ -module of rank l^3 (see Theorem 2.29 (3)). In [38], it is shown that a basis of this module is formed by the monomials $a^m b^n c^{s'}$ and $b^n c^{s''} d^r$, with the integers m, n, r, s', s'' in the range

$$1 \leq m \leq l-1, \quad 0 \leq n, r \leq l-1, \quad m \leq s' \leq l-1, \quad 0 \leq s'' \leq l-r-1. \tag{4.4}$$

Now consider $\mathcal{L}_{0,1}^A(\mathfrak{sl}_2)$. Recall that $\mathcal{L}_{0,1}^A = \mathcal{O}_A$ as U_A -modules. The algebra $\mathcal{L}_{0,1}^A(\mathfrak{sl}_2)$ is also generated by a, b, c, d ; a set of defining relations is (see [18, Section 5]):

$$\begin{aligned} ad &= da, & db &= q^2 bd, & cd &= q^2 dc, & ab - ba &= -(1 - q^{-2})bd, \\ cb - bc &= (1 - q^{-2})(da - d^2), & ac - ca &= (1 - q^{-2})dc, & ad - q^2 bc &= 1. \end{aligned} \tag{4.5}$$

The element $\omega := qa + q^{-1}d$ is central. Let T_k , $k \in \mathbb{N}$, be such that $T_k(x)/2$ is the k -th Chebyshev polynomial of the first type in the variable $x/2$. We have (see [18, Proposition 7.2], for the generalization to $\mathcal{L}_{0,n}^\epsilon(\mathfrak{sl}_2)$):

Lemma 4.5. *Let \mathcal{I} be the ideal of $\mathbb{C}[\omega, b^l, c^l, d^l]$ generated by $(T_l(\omega) - d^l)d^l - b^l c^l - 1$, we have*

$$\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2)) = \mathbb{C}[\omega, b^l, c^l, d^l]/\mathcal{I} \quad \text{and} \quad \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2)) = \mathbb{C}[T_l(\omega), b^l, c^l, d^l]/\mathcal{I}.$$

Here b^l, c^l, d^l are the l -th powers of b, c, d computed using the product of $\mathcal{L}_{0,1}^A(\mathfrak{sl}_2)$, not the product \star of $\mathcal{Z}_0(\mathcal{O}_\epsilon(\text{SL}_2))$. The above generator of \mathcal{I} can be interpreted as a determinant, and ω as a quantum trace on V_1 . The following has also been observed in [75].

Lemma 4.6. *Viewed as elements of $\mathcal{O}_A(\mathrm{SL}_2)$, $T_l(\omega) - d^l = a^{*l}$ and $x^l = x^{*l}$, $x \in \{b, c, d\}$.*

Proof. Let α and ϖ be the simple root and fundamental weight of \mathfrak{sl}_2 . In the notations of (2.70), we have $b = \psi_{-\varpi}^{-\alpha}$, $c = \psi_{-\varpi}^{\alpha}$, $d = \psi_{-\varpi}$; the formulas of Φ give

$$\Phi_1(b^{*l}) = (q - q^{-1})^l F^l, \quad \Phi_1(c^{*l}) = (q - q^{-1})^l E^l K^{-l}, \quad \Phi_1(d^{*l}) = K^{-l}.$$

These coincide respectively with $\Phi_1(b^l)$, $\Phi_1(c^l)$, $\Phi_1(d^l)$ (see [18, equation (5.3)]). By passing to the localization $\mathcal{O}_A(\mathrm{SL}_2)[d^{-1}]$, and using Parshall–Wang’s relation (4.3), one deduces easily

$$\Phi_1(a^{*l}) = K^l + (q - q^{-1})^{2l} F^l E^l = T_l(\Omega) - K^{-l},$$

where $\Omega = (\epsilon - \epsilon^{-1})^2 FE + \epsilon K + \epsilon^{-1} K^{-1}$ is the Casimir element, and $T_l(x)/2$ is the l -th Chebyshev polynomial of the first type in the variable $x/2$. We have $\Phi_1(\omega) = \Omega$, so $\Phi_1(a^{*l}) = T_l(\omega) - d^l$. The conclusion follows from the injectivity of Φ_1 . \blacksquare

This lemma proves that we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}(\mathrm{SL}_2) & \xrightarrow{\mathrm{Fr}_{\mathrm{PW}}} & \mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2)) & \hookrightarrow & \mathcal{O}_\epsilon(\mathrm{SL}_2) \\ & \searrow \mathrm{Fr} & \downarrow & & \downarrow \\ & & \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2)) & \hookrightarrow & \mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2), \end{array}$$

where $\mathrm{Fr}_{\mathrm{PW}}$ is Parshall–Wang’s Frobenius isomorphism recalled above, Fr is the Frobenius isomorphism introduced in [18, Definition 7.1], and the vertical arrows are the identifications as vector spaces (the middle one proved by Proposition 4.1).

Remark 4.7. By Lemma 4.5, the monomials $T_l(\omega)^i b^{jl} d^{rl}$ and $T_l(\omega)^i c^{kl} d^{rl}$, for $i, j, k, r \in \mathbb{N}$ and $k > 0$, form an A -basis of $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2))$. It is straightforward (though cumbersome) to express these basis elements in terms of the basis elements $a^{*il} \star b^{*jl} \star d^{*rl}$ and $a^{*il} \star c^{*kl} \star d^{*rl}$ of $\mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2))$, and conversely; this can be done by using Lemma 4.6, the formula (2.9) or the inverse one (expressing \star in terms of the product of $\mathcal{L}_{0,1}$, see [18, equation (4.6)]), and the formula of the coproduct $\Delta: \mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2) \rightarrow \mathcal{L}_{0,2}^\epsilon(\mathfrak{sl}_2)$ restricted to $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2))$ (given in [18, Lemma 7.5]).

Since $\mathcal{L}_{0,1}^A = \mathcal{O}_A$ as an A -module, the functionals t_i in Proposition 2.30 can be seen as maps $t_i: \mathcal{L}_{0,1}^A \rightarrow A$. Though the algebra structures of \mathcal{O}_ϵ and $\mathcal{L}_{0,1}^\epsilon$ are very different, $\mathcal{L}_{0,1}^\epsilon$ satisfies a result analogous to Proposition 2.30:

Proposition 4.8. *The maps $\triangleleft t_i$ preserve $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$, and they satisfy $(f \triangleleft t_i)(a) = f(n_i a)$ and $(f\alpha) \triangleleft t_i = (f \triangleleft t_i)(\alpha \triangleleft t_i)$ for every $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$, $a \in G$, $\alpha \in \mathcal{L}_{0,1}^\epsilon$.*

Proof. The first two claims follow from Proposition 2.30 and the definition $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$.

The last claim follows from the case $\mathfrak{g} = \mathfrak{sl}_2$, as in the proof of [41, Proposition 7.1]. In fact, it is enough to show that $t(fg) = t(f)t(g)$ for every $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2))$, $g \in \mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2)$; for completeness we explain this in Appendix C, see (C.3). A word of caution is needed: the proof of (C.3) uses that $\Delta: \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon \otimes \mathcal{O}_\epsilon$ is a morphism of algebras. The analogous property for $\mathcal{L}_{0,1}^\epsilon$ is that Δ yields a morphism of algebras $\Delta: \mathcal{L}_{0,1}^\epsilon \rightarrow \mathcal{L}_{0,2}^\epsilon$. Since the algebra structure of $\mathcal{L}_{0,2}^\epsilon$ is not the product one on $\mathcal{L}_{0,1}^\epsilon \otimes \mathcal{L}_{0,1}^\epsilon$, it is not true in general that

$$\sum_{(f),(g)} (f_{(1)} \otimes f_{(2)})(g_{(1)} \otimes g_{(2)}) = \sum_{(f),(g)} f_{(1)} g_{(1)} \otimes f_{(2)} g_{(2)}$$

for every $f, g \in \mathcal{L}_{0,1}^\epsilon$. However, it holds whenever $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$, since $\Delta(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)) \subset \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \otimes \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ and therefore $f_{(2)} \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$ commutes in $\mathcal{L}_{0,2}^\epsilon$ with any $g_{(1)} \in \mathcal{L}_{0,1}^\epsilon = \mathcal{O}_\epsilon$.

It is enough to prove the identity $t(fg) = t(f)t(g)$ when f ranges in a set of generators of the algebra $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}_2))$. So one can take f among, say, $T_l(\omega) - d^l = a^{*l}$ and $x^l = x^{*l}$, $x \in \{b, c, d\}$ (using Lemma 4.5). By (2.9) and Proposition C.1, we have

$$t(fg) = \sum_{(R),(R')} t(R_{(2')}S(R_{(2)}) \triangleright f) t(R_{(1')} \triangleright g \triangleleft R_{(1)}).$$

Expanding coproducts and using that $R^{-1} = (S \otimes \text{id})(R)$, we deduce

$$\begin{aligned} t(fg) &= \sum_{(f),(R),(R')} t(f_{(1)}) \langle f_{(2)}, R_{(2')}S(R_{(2)}) \rangle t(R_{(1')} \triangleright g \triangleleft R_{(1)}) \\ &= \sum_{(f),(R),(R')} t(f_{(1)}) t(\langle f_{(2)}, R_{(2')} \rangle R_{(1')} \triangleright g \triangleleft \langle f_{(3)}, S(R_{(2)}) \rangle R_{(1)}) \\ &= \sum_{(f)} t(f_{(1)}) t(S^{-1}(\Phi^-(f_{(2)})) \triangleright g \triangleleft S^{-2}(\Phi^-(f_{(3)}))) \\ &= \sum_{(f)} t(f_{(1)}) \langle g, S^{-2}(\Phi^-(f_{(3)})) \underline{w} S^{-1}(\Phi^-(f_{(2)})) \rangle \\ &= \sum_{(f)} t(f_{(1)}) \varepsilon(S^{-2}(\Phi^-(f_{(3)}))) \varepsilon(S^{-1}(\Phi^-(f_{(2)}))) t(g), \end{aligned}$$

where $\underline{w} \in \mathbb{U}_\Gamma$ is the quantum Weyl group element dual to t (see Appendix B), and in the last equality we used that Φ^- maps $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ into $\mathcal{Z}_0(U_\epsilon)$ (see Theorem 2.29 (2)), which acts on specializations of Γ -modules by the trivial character (the counit) $\varepsilon: U_\epsilon \rightarrow \mathbb{C}$. By (B.6)–(B.7), we have $t(a^{*l}) = t(d^{*l}) = 0$ and $t(b^{*l}) = 1$, $t(c^{*l}) = -1$. Now the computation of $t(fg)$ follows easily. For instance, taking $f = b^l = b^{*l}$, by using $\Delta(b^{*l}) = a^{*l} \otimes b^{*l} + b^{*l} \otimes d^{*l}$ and $\Delta(d^{*l}) = c^{*l} \otimes b^{*l} + d^{*l} \otimes d^{*l}$, we get

$$t(b^l g) = \varepsilon(S^{-2}(\Phi^-(b^{*l}))) \varepsilon(S^{-1}(\Phi^-(c^{*l}))) t(g) + \varepsilon(S^{-2}(\Phi^-(d^{*l}))) \varepsilon(S^{-1}(\Phi^-(d^{*l}))) t(g).$$

Since $b^{*l} \in \mathcal{O}_\epsilon(U_+)$, $\Phi^-(b^{*l}) = 0$. Also, it is immediate from the definition of Φ^- that $\Phi^-(d^{*l}) = \Phi^-(d)^l = L^l$; alternatively, one can bypass this computation by observing that Φ^- sets an isomorphism from $\mathcal{O}_\epsilon(T_G) = \mathcal{O}_\epsilon(B_+) \cap \mathcal{O}_\epsilon(B_-)$ to $\mathbb{C}[L^{\pm 1}] = U_\epsilon(\mathfrak{b}_+) \cap U_\epsilon(\mathfrak{b}_-)$, mapping a generator d to L or L^{-1} . We have $\varepsilon(L^l) = 1$, and therefore

$$t(b^l g) = t(g) = t(b^l) t(g).$$

The other cases $f = T_l(\omega) - d^l, c^l, d^l$ are similar. ■

Theorem 4.9. $\mathcal{L}_{0,n}^\epsilon$ is a free $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module of rank $l^{n \cdot \dim \mathfrak{g}}$, and $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ is a Noetherian ring and a finite, whence Noetherian, $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module.

Proof. We already proved the first claim in Proposition 4.1, and that $\mathcal{L}_{0,n}^\epsilon$ is a Noetherian $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module. For the second claim, it follows that the $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -submodule $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ is necessarily finitely generated. But $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ being Noetherian, $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ is then a Noetherian $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module and a Noetherian ring.

For the sake of clarity, let us provide a self-contained proof of the first claim, not invoking directly [28, 41] or [6, 25], but applying the same arguments directly to $\mathcal{L}_{0,n}^\epsilon$. Since $\mathcal{L}_{0,n}^\epsilon$ and $\mathcal{L}_{0,1}^{\otimes n}$ coincide as modules over $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) = \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$ by Proposition 4.1, the result will follow from

the case $n = 1$. Then we argue in four steps. First, using Theorem 2.1 we show that a certain localization of $\mathcal{L}_{0,1}^\epsilon$ is a free module of rank $l^{\dim \mathfrak{g}}$. Then, assuming that $\mathcal{L}_{0,1}^\epsilon$ is finitely generated and projective, we explain why it has constant rank $l^{\dim \mathfrak{g}}$ (this is very classical). Thirdly, we prove that $\mathcal{L}_{0,1}^\epsilon$ is finitely generated and projective as in [41, Theorem 7.2]. Finally, we obtain that it is a free module as in [28].

Recall Proposition 4.2: U_ϵ is a free $\Phi_1(\mathcal{L}_{0,1}^\epsilon[d^{-l}])$ -module of rank 2^m (note that $\mathcal{L}_{0,1}^\epsilon[d^{-l}] = \mathcal{L}_{0,1}^\epsilon[d^{-1}]$), $\mathcal{Z}_0(U_\epsilon)$ is free over

$$T_{2-}^{(l)-1} \mathcal{Z}_0(U_\epsilon^{\text{lf}}) = \Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}])$$

of rank 2^m . Since U_ϵ is also free of rank $l^{\dim \mathfrak{g}}$ over $\mathcal{Z}_0(U_\epsilon)$ (see Theorem 2.27 (1)), it is free over $\Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}])$ of rank $2^m l^{\dim \mathfrak{g}}$. The decomposition being unique, $\Phi_1(\mathcal{L}_{0,1}^\epsilon[d^{-l}])$ is free of rank $l^{\dim \mathfrak{g}}$ over $\Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}])$, and injectivity of Φ_1 implies that $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$ is free of rank $l^{\dim \mathfrak{g}}$ over $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$.

Assume now that $\mathcal{L}_{0,1}^\epsilon$ is finitely generated and projective. Let us show that its rank is $l^{\dim \mathfrak{g}}$. The localization $(\mathcal{L}_{0,1}^\epsilon)_P$ of $\mathcal{L}_{0,1}^\epsilon$ at any prime ideal P of $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ is a free module over $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)_P$ [96, Proposition 2.12.15]; the ranks of such modules are finite in number [96, Proposition 2.12.20]. If these ranks are all equal, then, by definition, it is the rank of $\mathcal{L}_{0,1}^\epsilon$ over $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$. This happens if $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ has no nontrivial (i.e., $\neq 1$) idempotent [96, Corollary 2.12.23], which is the case since it has no nontrivial zero divisors. To compute the rank, suppose P does not contain $d^l = \psi_{-\rho}^l$. Such ideals P are in 1-1 correspondence with the prime ideals of $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$ by the natural ring monomorphism $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \rightarrow \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$. The set $S = \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \setminus P$ is multiplicatively closed, and we have also a ring morphism $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}] \rightarrow S^{-1}\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$, which is also an injection (there are no zero divisors in $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$, whence in S). Then

$$(\mathcal{L}_{0,1}^\epsilon)_P = S^{-1}\mathcal{L}_{0,1}^\epsilon = \mathcal{L}_{0,1}^\epsilon[d^{-l}] \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]} S^{-1}\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$$

shows that $(\mathcal{L}_{0,1}^\epsilon)_P$ has over $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)_P = S^{-1}\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ the same rank $l^{\dim \mathfrak{g}}$ as $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$ over $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$. This proves our claim.

In order to show that $\mathcal{L}_{0,1}^\epsilon$ is finitely generated and projective over $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$, it is enough to show it is finite locally free, i.e., there are elements $d_i \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ such that the localizations $\mathcal{L}_{0,1}^\epsilon[d_i^{-1}]$ are finite free $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d_i^{-1}]$ -modules, and $\text{Maxspec}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))$ is covered by the open sets $U(d_i)$ made of the ideals not containing d_i (see [100, Lemma 77.2]).

We have seen above that $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$ is free of rank $l^{\dim \mathfrak{g}}$ over $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$. By Remark 4.3, $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}] \cong \mathcal{Z}_0(U_\epsilon^{\text{lf}})[\ell^l]$ is isomorphic to $\mathcal{O}(G^0)$, and $\mathcal{O}(G^0) = \mathcal{O}(G)[\delta^{-1}]$. Now, given $w \in W$ with a reduced expression $s_{i_1} \cdots s_{i_k}$, put $t_w = t_{i_1} \cdots t_{i_k}$. Let w be represented by $n_w = n_{i_1} \cdots n_{i_k}$ in $N(T_G)$. By Proposition 4.8, we have $(f \triangleleft t_w)(x) = f(n_w x)$ for every $f \in \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$, $x \in G$. Then

$$\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}] \triangleleft t_w \cong \mathcal{O}(n_w^{-1}G^0) \cong \mathcal{O}(G)[(\delta \triangleleft t_w)^{-1}]. \quad (4.6)$$

If b_1, \dots, b_r ($r := l^{\dim \mathfrak{g}}$) is a basis of $\mathcal{L}_{0,1}^\epsilon[d^{-l}]$ over $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d^{-l}]$, then $\mathcal{L}_{0,1}^\epsilon[d^{-l}] \triangleleft t_w$ is free over $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[(\delta \triangleleft t_w)^{-1}] \cong \mathcal{O}(n_w^{-1}G^0)$ with basis $b_1 \triangleleft t_w, \dots, b_r \triangleleft t_w$. Consider the Bruhat decomposition of G : any $g \in G$ can be written in the form $g = b_1 n b_2$, where $b_1, b_2 \in B_-, n \in W$. Hence $g = n n^{-1} b_1 n b_2 \in n B_+ B_- = n G^0$, and therefore

$$G = \bigcup_{w \in W} (B_- n_w B_-) = \bigcup_{w \in W} (n_w G^0).$$

For every $w \in W$, put $d_w^l := d^l \triangleleft t_w$. Under the isomorphism of $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ with $\mathcal{O}(G)$, we thus get that $\text{Maxspec}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))$ is covered by the open sets $U(d_w^l) \cong n_w G^0$, and $\mathcal{L}_{0,1}^\epsilon[d_w^{-l}]$ is finite free over $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)[d_w^{-l}]$. Therefore, $\mathcal{L}_{0,1}^\epsilon$ is finitely generated and projective over $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$.

Finally, let us explain why $\mathcal{L}_{0,1}^\epsilon$ is free over $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$, following the arguments of [28]. Let R be a commutative Noetherian ring, put $X = \text{Maxspec}(R)$, and let P be an R -module. Denote by R_I, P_I the localizations of R, P at a maximal ideal $I \in X$. Define the *f-rank* of P as $\text{f-rank}(P) = \inf_{I \in X} \{ \text{f-rank}_{R_I}(P_I) \}$, where $\text{f-rank}_{R_I}(P_I) = \sup \{ r \in \mathbb{N}, R_I^{\otimes r} \subset P_I \} \in \mathbb{N} \cup \{+\infty\}$ (i.e., the maximal dimension of a free summand of P_I). Bass' Cancellation theorem asserts that if P is projective and $\text{f-rank}(P) > \dim(X)$, and $P \oplus Q \cong M \oplus Q$ for some R -modules Q and M such that Q is finitely generated and projective, then $P \cong M$ (see [19, Section IV.3.5, p. 167 and p. 170], taking $A = R$, or [88, Section 11.7.13]). Let us apply this to $R = \mathcal{O}(G)$ and $P = \mathcal{L}_{0,1}^\epsilon$. We proved above that $\text{f-rank}_{R_I}(P_I) = l^{\dim \mathfrak{g}}$, a constant, and we have $l^{\dim \mathfrak{g}} > \dim \mathfrak{g} = \dim(G)$. By a result of Marlin [87], G being semisimple and simply connected the Grothendieck ring $K_0(\mathcal{O}(G))$ is isomorphic to \mathbb{Z} . Therefore, $\mathcal{L}_{0,1}^\epsilon \oplus Q \cong \mathcal{O}(G)^r$ for some free $\mathcal{O}(G)$ -module Q and $r \in \mathbb{N}$. Then Bass' cancellation implies $\mathcal{L}_{0,1}^\epsilon$ is free over $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) \cong \mathcal{O}(G)$. ■

5 Proof of Theorem 1.3

We begin with the following lemma, interesting by itself.

Lemma 5.1. *$\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ is a finite $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module and a Noetherian ring. Therefore, the ring $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ is integral over $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$.*

Proof. We know by Proposition 4.1 that $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ is a Noetherian ring, and $\mathcal{L}_{0,n}^\epsilon$ is a finite Noetherian $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module. Therefore, the submodule $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ is finitely generated. Being finite over $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$, it is necessarily a Noetherian ring (e.g., by [7, Proposition 7.2]).

Let $x \in \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$. The $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -submodule $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)[x]$ of $\mathcal{L}_{0,n}^\epsilon$ is finitely generated by the same argument. Using the fact that an element x is integral over $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ if and only if $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)[x]$ is a finitely generated $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ -module (e.g., by [7, Proposition 5.1]), this proves the last claim. ■

We will use the following notations. Let A be a ring with no nontrivial zero divisors. The center $Z = Z(A)$ is a commutative integral domain. We denote by $Q(Z)$ its field of fractions, and put

$$Q(A) := Q(Z) \otimes_Z A.$$

It is an algebra over its center $Q(Z)$. Since $\mathcal{L}_{0,n}^\epsilon$ has no nontrivial zero divisors [18, Proposition 6.30], we can take $A = \mathcal{L}_{0,n}^\epsilon$, or $A = (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$.

By the lemma, $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ is finite over $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$, so the ring $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon) \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)} Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ is a field. Necessarily it coincides with $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$, and therefore

$$Q(\mathcal{L}_{0,n}^\epsilon) = Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} \mathcal{L}_{0,n}^\epsilon = Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)} \mathcal{L}_{0,n}^\epsilon. \quad (5.1)$$

Recall that we denote by N the number of positive roots of \mathfrak{g} .

Theorem 5.2. *$Q(\mathcal{L}_{0,n}^\epsilon)$ is a division algebra and a central simple algebra of PI degree l^{Nn} .*

Proof. It follows from (5.1) and Theorem 4.9 that $Q(\mathcal{L}_{0,n}^\epsilon)$ is a vector space of dimension $l^n \cdot \dim \mathfrak{g}$ over $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$, and therefore has finite dimension over its center $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$. Because $\mathcal{L}_{0,n}^\epsilon$ has no nontrivial divisors [18, Proposition 6.30] and $Q(\mathcal{L}_{0,n}^\epsilon)$ is finite-dimensional over $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$, $Q(\mathcal{L}_{0,n}^\epsilon)$ is a division algebra, whence a central simple algebra. By classical theory (see, e.g., [88, Section 13.3.5], or [96, Corollary 2.3.25]), there is a finite extension \mathbb{F} of $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$, a *splitting field*, such that

$$\mathbb{F} \otimes_{Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))} Q(\mathcal{L}_{0,n}^\epsilon) = M_d(\mathbb{F}),$$

where $d \in \mathbb{N}$, the PI degree of $Q(\mathcal{L}_{0,n}^\epsilon)$, satisfies

$$d^2 = [Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))] = \frac{[Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))]}{[Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))]} \quad (5.2)$$

We have to show $d^2 = l^{2nN}$. We will obtain this equality by proving firstly that $d^2 \geq l^{2nN}$, and then $d^2 \leq l^{2nN}$.

In order to show that $d^2 \geq l^{2nN}$, it is enough to exhibit an irreducible representation V of $\mathcal{L}_{0,n}^\epsilon$ of dimension $k := l^{nN}$. Indeed, the representation map $\rho_V : \mathcal{L}_{0,n}^\epsilon \rightarrow \text{End}_{\mathbb{C}}(V)$ being surjective, given basis elements $v_1, \dots, v_{k^2} \in \text{End}(V)$, and elements $\alpha_1, \dots, \alpha_{k^2} \in \mathcal{L}_{0,n}^\epsilon$ such that $\rho(\alpha_i) = v_i$ for every $i \in \{1, \dots, k^2\}$, necessarily $\alpha_1, \dots, \alpha_{k^2}$ form a free family of $Q(\mathcal{L}_{0,n}^\epsilon)$. For, if there was a nontrivial relation $\sum_i z_i \alpha_i = 0$, with $z_i \in Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$, by clearing denominators and then applying the representation map ρ_V , we would get a nontrivial relation in $\text{End}_{\mathbb{C}}(V)$ between v_1, \dots, v_{k^2} .

Now, by Theorem 2.27 (1) (see [42, Section 20]), the dimension of a generic irreducible representation space of U_ϵ is l^N . Because $U_\epsilon = T_{2-}^{-1} U_\epsilon^{\text{lf}} [T/T_2]$ by Proposition 4.2, an irreducible representation of U_ϵ yields an irreducible representation of U_ϵ^{lf} . Moreover, the tensor product of n irreducible representation spaces of U_ϵ^{lf} of dimension l^N is an irreducible representation space of $(U_\epsilon^{\text{lf}})^{\otimes n}$ of dimension l^{nN} (see, e.g., [51, Theorem 3.10.2]). Applying the linear isomorphism $\psi_n = \Phi_n \circ (\Phi_1^{-1})^{\otimes n}$ in (2.21) thus provides an irreducible representation of $\mathcal{L}_{0,n}^\epsilon$ of dimension l^{nN} .

It remains to show $d^2 \leq l^{2nN}$, which by $[Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = l^{n(2N+m)}$ is equivalent to $[Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] \geq l^{mn}$. For this, it is enough to exhibit an extension of $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ contained in $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ and of degree l^{mn} . There is a very natural one, which we denote by $Q(\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon))$ and is constructed as follows. Consider for every $\lambda \in P_+$ the matrices

$$M_\lambda := ({}_{AV_\lambda} \phi_{e_k}^{e_l})_{k,l} \in \text{End}({}_A V_\lambda) \otimes \mathcal{L}_{0,n}^A, \quad M_\lambda^{(i)} := (({}_{AV_\lambda} \phi_{e_k}^{e_l})^{(i)})_{k,l} \in \text{End}({}_A V_\lambda) \otimes \mathcal{L}_{0,n}^A,$$

where $i = 1, \dots, n$, and as usual ${}_{AV_\lambda} \phi_{e_k}^{e_l}$ is a matrix coefficient of ${}_A V_\lambda$, $\{e_k\}$ the canonical basis of ${}_A V_\lambda$, and $({}_{V_\lambda} \phi_{e_k}^{e_l})^{(i)} := 1^{\otimes(i-1)} \otimes {}_{V_\lambda} \phi_{e_k}^{e_l} \otimes 1^{\otimes(n-i)}$. Set

$$\lambda\omega := \text{Tr}(\pi_{V_\lambda}(\ell)M_\lambda), \quad \lambda\omega^{(i)} := \text{Tr}(\pi_{V_\lambda}(\ell)M_\lambda^{(i)}),$$

where Tr is the standard trace on $\text{End}(V_\lambda)$. Clearly, $\lambda\omega \in \mathcal{L}_{0,1}^A$, $\lambda\omega^{(i)} \in \mathcal{L}_{0,n}^A$. By [18, Propositions 4.8 and 6.24], the family of elements $\prod_{i=1}^n \lambda_i \omega^{(i)}$, where $\lambda_1, \dots, \lambda_n \in P_+$, is a basis of $\mathcal{Z}(\mathcal{L}_{0,n})$; moreover the Alekseev map Φ_n affords an isomorphism from $\mathcal{Z}(\mathcal{L}_{0,n})$ to $\mathcal{Z}(U_q)^{\otimes n}$, and $\Phi_n(\lambda\omega^{(i)}) = (\Phi_1(\lambda\omega))^{(i)}$. For $n = 1$, specializing q to ϵ it follows

$$\mathcal{Z}_1(U_\epsilon) = \text{Vect}\{\Phi_1(\lambda\omega), \lambda \in P_+\}, \quad (5.3)$$

where $\mathcal{Z}_1(U_\epsilon)$ is defined before Theorem 2.27. Then, for every $i = 1, \dots, n$ define

$$\mathcal{Z}_{0,(i)}(\mathcal{L}_{0,n}^\epsilon) := \mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) \left[\{ \lambda\omega^{(i)}, \lambda \in P_+ \} \right]$$

and let $\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon) \subset \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ be the algebra generated by $\mathcal{Z}_{0,(1)}(\mathcal{L}_{0,n}^\epsilon), \dots, \mathcal{Z}_{0,(n)}(\mathcal{L}_{0,n}^\epsilon)$. The fields $Q(\mathcal{Z}_{0,(i)}(\mathcal{L}_{0,n}^\epsilon))$ are n linearly disjoint extensions of $Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$, so

$$[Q(\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = \prod_{i=1}^n [Q(\mathcal{Z}_{0,(i)}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))].$$

Now, by Proposition 4.2, we know that Φ_1 affords isomorphisms $Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)) \cong Q(\mathcal{Z}_0(U_\epsilon^{\text{lf}}))$ and $Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)) \cong Q(\mathcal{Z}(U_\epsilon^{\text{lf}}))$, and moreover

$$Q(\mathcal{Z}_0(U_\epsilon)) = Q(\mathcal{Z}_0(U_\epsilon^{\text{lf}}))(T^{(l)}/T_2^{(l)}), \quad Q(\mathcal{Z}(U_\epsilon)) = Q(\mathcal{Z}(U_\epsilon^{\text{lf}}))(T^{(l)}/T_2^{(l)}). \quad (5.4)$$

Computing via the field embedding $\Phi_1^{\otimes n}: Q(\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon)) \rightarrow Q(\mathcal{Z}(U_\epsilon^{\otimes n}))$, we deduce

$$\begin{aligned} & [Q(\mathcal{Z}_{0,(i)}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] \\ &= [\Phi_1^{\otimes n}(Q(\mathcal{Z}_{0,(i)}(\mathcal{L}_{0,n}^\epsilon))) : \Phi_1^{\otimes n}(Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)))] \\ &= [Q(\mathcal{Z}_0(U_\epsilon^{\text{lf}})^{\otimes n})[\{(\Phi_1(\lambda\omega))^{(i)}, \lambda \in P_+, i = 1, \dots, n\}] : Q(\mathcal{Z}_0(U_\epsilon^{\text{lf}})^{\otimes n})] \\ &= [Q(\mathcal{Z}_0(U_\epsilon)^{\otimes n})[\{(\Phi_1(\lambda\omega))^{(i)}, \lambda \in P_+, i = 1, \dots, n\}] : Q(\mathcal{Z}_0(U_\epsilon)^{\otimes n})] = l^m. \end{aligned}$$

The second and third equalities follow from (5.4) and the properties of Φ_1 recalled before it, and the last equality follows from Theorem 2.29 (2) and (5.3). As a result, we have

$$[Q(\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = l^{mn},$$

whence

$$[Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] \geq l^{mn}.$$

Since $[Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = l^{n(m+2N)}$, by (5.2) we obtain $d^2 \leq l^{2nN}$, which concludes the proof. \blacksquare

Remark 5.3. It follows $[Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))] = l^{mn}$ by the degree computation above, whence $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) = Q(\hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon))$. In [17], we prove that $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon) = \hat{\mathcal{Z}}_0(\mathcal{L}_{0,n}^\epsilon)$.

Theorem 5.4. $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$, $n \geq 2$, is a division algebra and a central simple algebra of PI degree $l^{N(n-1)-m}$.

Proof. The center of $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ contains $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$, so the finite-dimensionality of $Q(\mathcal{L}_{0,n}^\epsilon)$ over $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ implies the finite-dimensionality of $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ over its center. Since it has no non-zero divisors, this proves $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ is a division algebra.

Now denote by $\Delta^{(n)}: \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon^{\otimes n}$, $n \geq 2$, the n -fold coproduct, i.e., $\Delta^{(2)} := \Delta$, the standard coproduct of \mathcal{O}_ϵ , and $\Delta^{(n)} := (\text{id} \otimes \Delta) \circ \Delta^{(n-1)}$ for $n \geq 3$. Identifying $\mathcal{L}_{0,n}^\epsilon$ with $\mathcal{O}_\epsilon^{\otimes n}$ as a vector space, we consider $\Delta^{(n)}$ as a map $\Delta^{(n)}: \mathcal{L}_{0,1}^\epsilon \rightarrow \mathcal{L}_{0,n}^\epsilon$. It is an algebra morphism [18, Proposition 6.18], injective because $(\varepsilon^{\otimes(n-1)} \otimes \text{id})\Delta^{(n)} = \text{id}$. Then it extends uniquely to the fraction algebra $Q(\mathcal{L}_{0,1}^\epsilon)$. As noted above, $Q(\mathcal{L}_{0,1}^\epsilon) = Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)) \otimes_{\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)} \mathcal{L}_{0,1}^\epsilon$. Since $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon) = \mathcal{Z}_0(\mathcal{O}_\epsilon)$ is a Hopf subalgebra of \mathcal{O}_ϵ [41, Proposition 6.4], $\Delta^{(n)}$ maps $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ to $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$. Then, extending the scalars of $\Delta^{(n)}(Q(\mathcal{L}_{0,1}^\epsilon))$ by the field $Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$, consider the algebra

$$\begin{aligned} Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) &:= Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))} \Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon) \\ &= Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)))} \Delta^{(n)}(Q(\mathcal{L}_{0,1}^\epsilon)) \end{aligned}$$

$$\begin{aligned}
&= Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(Q(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)))} \Delta^{(n)}(Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))) \\
&\quad \otimes_{\Delta^{(n)}(Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)))} \Delta^{(n)}(Q(\mathcal{L}_{0,1}^\epsilon)).
\end{aligned}$$

By Proposition 5.2, $\Delta^{(n)}(Q(\mathcal{L}_{0,1}^\epsilon))$ is a $\Delta^{(n)}(Q(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)))$ -central simple algebra. The left factor is a field, so $Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$ is a central simple algebra over it (see, e.g., [96, Theorem 1.7.27], or [101, Lemma 4.9]). Note that the left factor can also be written as

$$\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) := Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\Delta^{(n)}(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))} \Delta^{(n)}(\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon))$$

for it contains $\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$, it is contained in its fraction field, and $\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ is a field because $\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)$ is finite over $\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)$ and has no nontrivial zero divisors. Note that

$$[\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))] = l^m.$$

We proved in [18, Proposition 6.19] that the ring $(\mathcal{L}_{0,n}^A)^{U_A}$ is the centralizer of $\Delta^{(n)}(\mathcal{L}_{0,1}^A)$ in $\mathcal{L}_{0,n}^A$; the same arguments show that $(\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$ is the centralizer of $\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)$ in $\mathcal{L}_{0,n}^\epsilon$. So the algebra

$$Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) := Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) \otimes_{\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)} (\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}$$

is the centralizer of $Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$ in $Q(\mathcal{L}_{0,n}^\epsilon)$. Since the latter is simple, we can apply the double centralizer theorem (see, e.g., [96, Theorem 7.1.9], or [101, Theorem 7.1]): $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ is a simple algebra, we have

$$[Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))] = \frac{[Q(\mathcal{L}_{0,n}^\epsilon) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]}{[Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon)) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]} = l^{2nN-(2N+m)},$$

and the centralizer of $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ is $Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$. In particular, $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ has center $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) \cap Q_{\mathcal{Z}}(\Delta^{(n)}(\mathcal{L}_{0,1}^\epsilon))$, which is easily shown to be $\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$. It then follows

$$\begin{aligned}
[Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) : \tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))] &= \frac{[Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon}) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]}{[\tilde{Q}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)) : Q(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))]} \\
&= l^{2nN-(2N+m)} \cdot l^{-m} = l^{2(N(n-1)-m)}.
\end{aligned}$$

Therefore, $Q((\mathcal{L}_{0,n}^\epsilon)^{U_\epsilon})$ is a central simple algebra of PI degree $l^{N(n-1)-m}$. ■

A Low and up crystal structures in the \mathfrak{sl}_2 case

Let $k \in \mathbb{N}$, and denote by V_k the simple $U_q^{\text{ad}}(\mathfrak{sl}_2)$ module of dimension $k+1$. It has a basis v_0, \dots, v_k such that

$$\begin{aligned}
K.v_j &= q^{k-2j}v_j, & F.v_j &= [j+1]_q v_{j+1} \quad \text{if } j < k, & F.v_k &= 0, \\
E.v_j &= [k-j+1]_q v_{j-1} \quad \text{if } j > 0, & E.v_0 &= 0.
\end{aligned}$$

This basis defines the full A -sublattice ${}_A V_k$, which is left invariant by U_A^{res} , and we have

$$F^{(a)}.v_j = \begin{bmatrix} j+a \\ a \end{bmatrix}_q v_{j+a}, \quad E^{(a)}.v_j = \begin{bmatrix} k-j+a \\ a \end{bmatrix}_q v_{j-a}, \quad a \geq 0.$$

The action of the Kashiwara operator \tilde{e}, \tilde{f} on V_k are given by $\tilde{f}(v_j) = v_{j+1}$, $\tilde{e}(v_j) = v_{j-1}$.

The crystal basis $(\mathcal{L}^{\text{low}}, \mathcal{B}^{\text{low}})$ at $q = 0$ is formed by the \mathcal{A}_0 -sublattice \mathcal{L}^{low} generated by v_0, \dots, v_k , and \mathcal{B}^{low} by the images $\bar{v}_0, \dots, \bar{v}_k$ of these vectors in $\mathcal{L}^{\text{low}}/q\mathcal{L}^{\text{low}}$.

The bilinear form $\langle \cdot \rangle_k$ defined by (2.39) is easily computed

$$\langle v_i, v_j \rangle_k = \langle F^{(i)}.v_0, F^{(j)}.v_0 \rangle_k = \langle v_0, E^{(i)}F^{(j)}.v_0 \rangle_k = \begin{bmatrix} k \\ i \end{bmatrix}_q \delta_{i,j}.$$

By definition,

$${}_A V_k^{\text{up}} = \{v \in V_k, \langle v, {}_A V_k \rangle_k \subset A\} = \bigoplus_{j=0}^k A v_j^{\text{up}},$$

where

$$v_j^{\text{up}} = \begin{bmatrix} k \\ j \end{bmatrix}_q^{-1} v_j.$$

The upper crystal basis $(\mathcal{L}^{\text{up}}, \mathcal{B}^{\text{up}})$ at $q = 0$ is formed by the \mathcal{A}_0 -sublattice \mathcal{L}^{up} generated by $v_0^{\text{up}}, \dots, v_k^{\text{up}}$, and \mathcal{B}^{up} by the images $\bar{v}_0^{\text{up}}, \dots, \bar{v}_k^{\text{up}}$ of these vectors in $\mathcal{L}^{\text{up}}/q\mathcal{L}^{\text{up}}$.

Using that $[n]_q \in q^{1-n}(1 + q\mathcal{A}_0)$, we obtain

$$\begin{bmatrix} k \\ j \end{bmatrix}_q \in q^{j^2 - kj}(1 + q\mathcal{A}_0).$$

As a result, we get $\bar{v}_j^{\text{up}} = q^{kj - j^2} \bar{v}_j$, which is exactly the relation (2.41) relating the low and up crystal bases, with $\lambda = k\varpi_1$, $\mu = (k - 2j)\varpi_1$.

B Quantum Weyl group

We recall some of the formulas of [31]. Let $e_q(z)$ be the formal power series in z with coefficients in $\mathbb{C}(q)$ defined by

$$e_q(z) = \sum_{n=0}^{+\infty} \frac{z^n}{(n)_{q^2}!}.$$

We first consider the case of $\mathfrak{g} = \mathfrak{sl}_2$. As explained in [18, Section 3], the Cartan element $H \in \mathfrak{g}$ defines an element of $\mathbb{U}_q(\mathfrak{sl}_2)$. Viewed as elements of $\mathbb{U}_q(\mathfrak{sl}_2)$ we have $L = q^{H/2}$. The series $\Theta = q^{H \otimes H/2}$ defines an element of $\mathbb{U}_q(\mathfrak{sl}_2)^{\hat{\otimes} 2}$, its image under multiplication being $q^{H^2/2}$. The R -matrix can be expressed as $R = \Theta \hat{R}$ where $\hat{R} = e_{q^{-1}}((q - q^{-1})E \otimes F)$ is a well defined element of $\mathbb{U}_q(\mathfrak{sl}_2)^{\hat{\otimes} 2}$. Consider the Lusztig [82] braid group automorphism of $U_q(\mathfrak{sl}_2)$, defined by

$$T(L) = L^{-1}, \quad T(E) = -FK^{-1}, \quad T(F) = -KE. \quad (\text{B.1})$$

For every $x \in U_q(\mathfrak{sl}_2)$ it satisfies: $\Delta(T(x)) = \hat{R}^{-1}(T \otimes T)(\Delta(x))\hat{R}$. Define the quantum Weyl group element $\hat{w} \in \mathbb{U}_q(\mathfrak{sl}_2)$ by Saito's formula [97]:

$$\hat{w} = e_{q^{-1}}(F)q^{-H^2/4}e_{q^{-1}}(-E)q^{-H^2/4}e_{q^{-1}}(F)q^{-H/2}. \quad (\text{B.2})$$

For every $x \in U_q(\mathfrak{sl}_2)$, it satisfies

$$T(x) = \hat{w}x\hat{w}^{-1}, \quad \Delta(\hat{w}) = \hat{R}^{-1}(\hat{w} \otimes \hat{w}), \quad (\text{B.3})$$

$$\hat{w}^2 = q^{H^2/2} \xi \theta, \quad (\text{B.4})$$

where $\theta \in \mathbb{U}_q(\mathfrak{sl}_2)$ is the ribbon element, and $\xi \in \mathbb{U}_q(\mathfrak{sl}_2)$ is the central group element whose value on the type 1 simple module V_k of $U_q^{\text{ad}}(\mathfrak{sl}_2)$ of dimension $k + 1$ is the scalar endomorphism $(-1)^k \text{id}_{V_k}$.

In order to compare our setting to the one of [41], we need an explicit formula of \hat{w} . Using the basis v_j of V_k of Appendix A, (B.1), (B.3) and (B.4), we obtain

$$\hat{w}v_j = (-1)^j q^{-j(k-j-1)-k} v_{k-j}. \quad (\text{B.5})$$

In [41], another quantum Weyl group element \underline{w} is defined. It is dual to the Vaksman–Soibelman functional $t: \mathcal{O}_q(\text{SL}_2) \rightarrow \mathbb{C}(q)$ of [98, 102], that is, $t(\alpha) = \langle \alpha, \underline{w} \rangle$ for all $\alpha \in \mathcal{O}_q(\text{SL}_2)$. By comparing (B.5) with the formulas defining the action of t in [41, Section 1.7], we find $\underline{w} = \xi \hat{w} K$ and the basis vectors w_r^p of [41], where $p \in (1/2)\mathbb{N}$ and $r \in \{-p, -p+1, \dots, p-1, p\}$, are related to the vectors v_j above as follows: $v_j = \lambda_j w_r^p$, where $k = 2p$, $j = p - r$, $\lambda_0 = 1$, $\lambda_1 = [k]q^{-k}$, and

$$\lambda_j = \frac{[k]!}{[j]![k-(j-2)]!} q^{j(j+1)-j(k+2)}, \quad j \geq 2.$$

Explicit formulas of the evaluation of t on basis vectors of $\mathcal{O}_q(\text{SL}_2)$ can be computed. We get

$$t(\tilde{a}^{\star m} \star \tilde{b}^{\star n} \star \tilde{d}^{\star p}) = \delta_{m,p} q^{-np} \prod_{i=1}^p (1 - q^{-2i}), \quad (\text{B.6})$$

$$t(\tilde{a}^{\star m} \star \tilde{c}^{\star n} \star \tilde{d}^{\star p}) = (-1)^n \delta_{m,p} q^{-n(p+1)} \prod_{i=1}^p (1 - q^{-2i}), \quad (\text{B.7})$$

where $\tilde{a} = a$, $\tilde{b} = qb$, $\tilde{c} = q^{-1}c$, $\tilde{d} = d$ and as usual a, b, c, d are the standard generators of $\mathcal{O}_q(\text{SL}_2)$, i.e., the matrix coefficients in the basis of weight vectors v_0, v_1 of the 2-dimensional irreducible representation V_1 of $U_q(\mathfrak{sl}_2)$ such that $K.v_0 = qv_0$ and $v_1 = F.v_0$. Here we have introduced the generators $\tilde{a}, \dots, \tilde{d}$ to facilitate the comparison with the formulas in [41]; these generators come naturally in their setup because they use different generators E_i and F_i of $U_q(\mathfrak{g})$, which in our notations can be written respectively as $K_i^{-1}E_i$ and F_iK_i .

The formulas (B.6)–(B.7) can be shown by two independent methods. The first uses a definition of t as a *GNS* state associated to an infinite-dimensional representation of $\mathcal{O}_q(\text{SL}_2)$, as recalled in [41, Section 1.6]. The second is to write, e.g.,

$$t(\tilde{a}^{\star m} \star \tilde{b}^{\star n} \star \tilde{d}^{\star p}) = \langle \tilde{a}^{\otimes m} \otimes \tilde{b}^{\otimes n} \otimes \tilde{d}^{\otimes p}, \Delta^{(m+n+p)}(\underline{w}) \rangle \quad (\text{B.8})$$

and to use explicit expressions of $\Delta^{(m+n+p)}(\underline{w})$ when represented on $V_1^{\otimes(m+n+p)}$. In general, one can check that

$$\begin{aligned} \Delta^{(n)}(\hat{\omega}) &= (\Delta^{(n-1)} \otimes \text{id})(\hat{R}^{-1})((\Delta^{(n-2)} \otimes \text{id})(\hat{R}^{-1}) \otimes \text{id}) \cdots ((\Delta \otimes \text{id})(\hat{R}^{-1}) \otimes \text{id}^{\otimes(n-3)}) \\ &\quad \times (\hat{R}^{-1} \otimes \text{id}^{\otimes(n-2)}) \hat{\omega}^{\otimes n}. \end{aligned}$$

By (B.5) or (B.6)–(B.7), we see that \hat{w} (or \underline{w}) and t are well defined on the integral forms,

$$\hat{w} \in \mathbb{U}_\Gamma, \quad t: \mathcal{O}_A(\text{SL}_2) \rightarrow A.$$

We now consider the case where \mathfrak{g} is of rank $m \geq 2$. To each simple root α_i , $1 \leq i \leq m$, is associated the subalgebra of U_q generated by E_i, F_i, L_i, L_i^{-1} . It is a copy of $U_{q_i}(\mathfrak{sl}_2)$, where $q_i = q^{d_i}$. Let \hat{w}_i be the corresponding quantum Weyl group element in $\mathbb{U}_q = \mathbb{U}_q(\mathfrak{g})$, defined by Saito's

formula (B.2), replacing H, E, F by H_i, E_i and F_i . Also, denote by $\nu_i: \mathcal{O}_q \rightarrow \mathcal{O}_{q_i}(\mathrm{SL}_2)$ the projection map dual to the inclusion $U_{q_i}(\mathfrak{sl}_2) \otimes_{\mathbb{C}(q_i)} \mathbb{C}(q) \hookrightarrow U_q$, and put $t_i = t \circ \nu_i$. Let \underline{w}_i be the corresponding quantum Weyl group element in \mathbb{U}_q , i.e., $t_i(\alpha) = \langle \alpha, \underline{w}_i \rangle$ for all $\alpha \in \mathcal{O}_q$. On integral forms they yield well-defined elements $\hat{w}_i, \underline{w}_i \in \mathbb{U}_\Gamma$ and $t_i: \mathcal{O}_A \rightarrow A$ (see [41, Proposition 5.1], and [84] for a different construction). They satisfy the defining relations of the braid group $\mathcal{B}(\mathfrak{g})$ of \mathfrak{g} [70]:

$$\begin{aligned} \hat{w}_i \hat{w}_j \hat{w}_i &= \hat{w}_j \hat{w}_i \hat{w}_j & \text{if } a_{ij} a_{ji} &= 1, \\ (\hat{w}_i \hat{w}_j)^k &= (\hat{w}_j \hat{w}_i)^k & \text{for } k = 1, 2, 3 & \text{ if } a_{ij} a_{ji} = 0, 2, 3, \end{aligned}$$

and similarly by replacing \hat{w}_i with \underline{w}_i , or with t_i (see [98] for the latter). The Weyl group $W = W(\mathfrak{g}) = N(T_G)/T_G$ is generated by the reflections s_i associated to the simple roots α_i . Denote by $n_i \in N(T_G)$ a representative of s_i . Let $w \in W$ and denote by $w = s_{i_1} \dots s_{i_k}$ a reduced expression. Because of the braid group relations the elements $\hat{w} = \hat{w}_{i_1} \dots \hat{w}_{i_k}$, $\underline{w} = \underline{w}_{i_1} \dots \underline{w}_{i_k}$ and the functional $t_w = t_{i_1} \dots t_{i_k}$ do not depend on the choice of reduced expression. The Lusztig [82] braid group automorphism $T_w: \Gamma \rightarrow \Gamma$ associated to w satisfies (see [41])

$$T_w(x) = \hat{w}x\hat{w}^{-1}, \quad x \in \Gamma.$$

Let w_0 be the longest element in W . We have

$$\Delta(\hat{w}_0) = \hat{R}^{-1}(\hat{w}_0 \otimes \hat{w}_0), \tag{B.9}$$

where as usual $R = \Theta \hat{R}$.

C Regular action on \mathcal{O}_ϵ

The following result is proved in [41, Section 1.10]. For completeness, let us give a (different) proof. Recall from (2.72) that we may identify $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ with $\mathcal{O}(G)$.

Proposition C.1. *For every $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$, $g \in \mathcal{O}_\epsilon$, we have*

$$t_i(f) = f(n_i), \tag{C.1}$$

$$t_i(f \star g) = t_i(f)t_i(g). \tag{C.2}$$

Proof. It is sufficient to prove the results for SL_2 because $\nu_i: \mathcal{O}_\epsilon \rightarrow \mathcal{O}_{\epsilon_i}(\mathrm{SL}_2)$ is a morphism of Hopf algebras and $\nu_i(\mathcal{Z}_0(\mathcal{O}_\epsilon)) \subset \mathcal{Z}_0(\mathcal{O}_{\epsilon_i}(\mathrm{SL}_2))$. In this case, (C.1) can be proved by using (B.6)–(B.7), evaluating t on basis elements of $\mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2))$ as is done in [41, Lemma 1.5 (a)]. Such a basis is formed by monomials like in (B.6)–(B.7), with all exponents divisible by l ; then for instance

$$t(\tilde{a}^{\star ml} \star \tilde{b}^{\star nl} \star \tilde{d}^{\star pl}) = \delta_{p,0} \delta_{m,0} = \underline{a}^m \underline{b}^n \underline{d}^p(n),$$

where $\underline{a}, \dots, \underline{d}$ are the generators of $\mathcal{O}(G) = \mathcal{O}_1(G)$ corresponding to a, \dots, d , and we take

$$n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

as representative of the reflection s generating the Weyl group $W(\mathfrak{sl}_2)$. Here is an alternative proof of (C.1): (C.2) shows that t is a homomorphism on $\mathcal{Z}_0(\mathcal{O}_\epsilon(\mathrm{SL}_2))$, so by proving (C.2) at first one is reduced to check (C.1) on the generators $a^{\star l}, \dots, d^{\star l}$, which is easy by means of (B.8) and (B.9).

We provide a proof of (C.2) that we find more conceptual than the one in [41, Lemma 1.5 (b)] (which uses again (B.6)–(B.7)). As above, let us denote $\underline{w} = \xi \hat{w} K$. For any $f, g \in \mathcal{O}_\epsilon$, we have

$$\begin{aligned} t(f \star g) &= (f \otimes g)(\Delta(\underline{w})) = (f \otimes g)(\widehat{R}^{-1}(\underline{w} \otimes \underline{w})) = \sum_{(\widehat{R}^{-1})} f((\widehat{R}^{-1})_{(1)} \underline{w}) g((\widehat{R}^{-1})_{(2)} \underline{w}) \\ &= \sum_{(\widehat{R}^{-1}), (f)} f_{(1)}((\widehat{R}^{-1})_{(1)}) f_{(2)}(\underline{w}) g((\widehat{R}^{-1})_{(2)} \underline{w}) = \sum_{(f)} f_{(2)}(\underline{w}) g((f_{(1)} \otimes \text{id})(\widehat{R}^{-1}) \underline{w}). \end{aligned}$$

Assume now $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon(\text{SL}_2))$. Since $\mathcal{Z}_0(\mathcal{O}_\epsilon(\text{SL}_2))$ is a Hopf subalgebra of $\mathcal{O}_\epsilon(\text{SL}_2)$, we have $f_{(1)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon(\text{SL}_2))$. From Theorem 2.29 (2), we deduce

$$(f_{(1)} \otimes \text{id})(\widehat{R}^{-1}) \in U_\epsilon(\mathfrak{n}_-) \cap \mathcal{Z}_0(U_\epsilon^{\text{ad}}).$$

Denote by z this element. Note that from its expression we have $\epsilon(z) = \epsilon(f_{(1)})$. Now $g(z\underline{w}) = \sum_{(g)} g_{(1)}(z) g_{(2)}(\underline{w})$, but $g_{(1)}$ is a linear combination of matrix elements of Γ -modules, on which $\mathcal{Z}_0(U_\epsilon^{\text{ad}})$ acts by the trivial character. Therefore,

$$g(z\underline{w}) = \sum_{(g)} \epsilon(z) g_{(1)}(1) g_{(2)}(\underline{w}) = \epsilon(z) g(\underline{w}) = \epsilon(f_{(1)}) g(\underline{w}),$$

and eventually

$$t(f \star g) = \sum_{(f)} f_{(2)}(\underline{w}) \epsilon(f_{(1)}) g(\underline{w}) = t(f) t(g).$$

This concludes the proof. ■

For the sake of completeness, let us show how this result implies:

Proof of Proposition 2.30 (i.e., [41, Proposition 7.1]). We have $f \triangleleft t_i = \sum_{(f)} t_i(f_{(1)}) f_{(2)}$, $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$. Since $\mathcal{Z}_0(\mathcal{O}_\epsilon)$ is a Hopf subalgebra of \mathcal{O}_ϵ , $f_{(2)} \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$ and therefore the maps $\triangleleft t_i: \mathcal{O}_\epsilon \rightarrow \mathcal{O}_\epsilon$ preserve $\mathcal{Z}_0(\mathcal{O}_\epsilon)$. Moreover, $(f \triangleleft t_i)(a) = \sum_{(f)} f_{(1)}(n_i) f_{(2)}(a) = f(n_i a)$, $a \in G$, by (C.1).

It remains to show that $(f \star \alpha) \triangleleft t_i = (f \triangleleft t_i)(\alpha \triangleleft t_i)$ for every $f \in \mathcal{Z}_0(\mathcal{O}_\epsilon)$, $\alpha \in \mathcal{O}_\epsilon$. We have

$$\begin{aligned} (f \star g) \triangleleft t_i &= \sum_{(f \star g)} t_i((f \star g)_{(1)}) (f \star g)_{(2)} = \sum_{(f), (g)} t_i(f_{(1)} \star g_{(1)}) f_{(2)} \star g_{(2)} \\ &= \sum_{(f), (g)} t(\nu_i(f_{(1)}) \nu_i(g_{(1)})) f_{(2)} \star g_{(2)} \\ &= \sum_{(f), (g)} t(\nu_i(f_{(1)})) t(\nu_i(g_{(1)})) f_{(2)} \star g_{(2)}, \end{aligned} \tag{C.3}$$

using that ν_i is a homomorphism in the third equality, and (C.2) in the last one. The result is just $(f \triangleleft t_i)(g \triangleleft t_i)$. ■

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