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# Asymptotic analysis of a Bingham fluid in a thin T-like shaped structure.

Renata Bunoiu\*, Antonio Gaudiello<sup>†</sup> and Angelo Leopardi<sup>‡</sup>

## Abstract

We study the steady incompressible flow of a Bingham fluid in a thin T-like shaped domain, under the action of given external forces and with no-slip boundary condition on the whole boundary of the domain. This phenomenon is described by non linear variational inequalities. By letting the parameter describing the thickness of the thin domain tend to zero, we derive two uncoupled problems corresponding to the two branches of the T-like shaped structure. We then analyze and give a physical justification of the limit problem.

Keywords: Non-Newtonian fluids, junctions, thin structures

2010 AMS subject classifications: 76A05, 76A99, 74K30

## 1 Introduction

In this paper we study the asymptotic behavior of a Bingham fluid in a T-like shaped domain. More precisely, for every  $n \in \mathbb{N}$  let  $h_n \in ]0, 1[$  be a small parameter and let  $\Omega_n$  be a thin two-dimensional T-like shaped domain defined by (see Fig. 1)

$$\Omega_n = \Omega_n^a \cup \Omega_n^b, \quad \Omega_n^a = \left] -\frac{h_n}{2}, \frac{h_n}{2} \left[ \times [0, 1[, \quad \Omega_n^b = \left] -\frac{1}{2}, \frac{1}{2} \left[ \times ]-h_n, 0[. \quad (1.1)$$

In  $\Omega_n$  we consider the steady incompressible flow of a Bingham fluid (see [1]) having viscosity  $\mu h_n^2$  and yield stress  $g h_n$ , with  $\mu$  and  $g$  two strictly positive constants. The fluid moves under the action of given external forces  $f_n$  and obeys no-slip condition on the whole boundary of the domain (see [22]). The goal of our paper is to study the asymptotic behavior of this fluid when

$$\lim_n h_n = 0. \quad (1.2)$$

The Bingham fluid is a fluid which has a non linear rheological behavior (see Section 2). It

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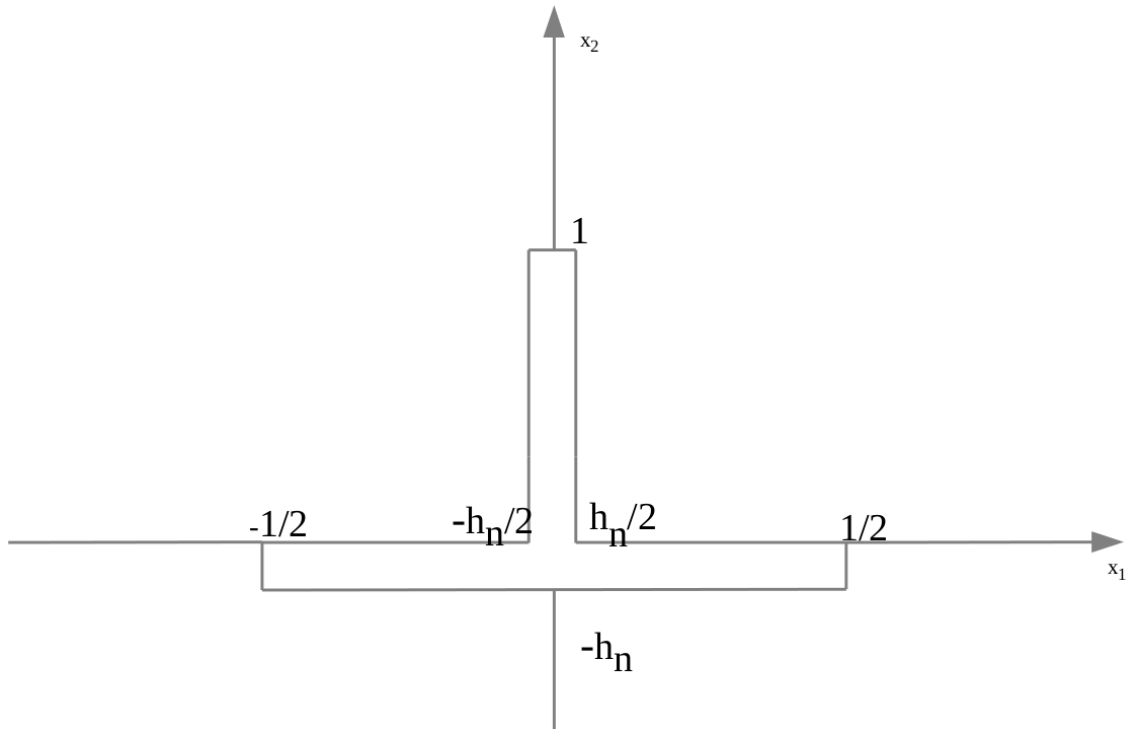


Figure 1: the thin two-dimensional T-like shaped domain  $\Omega_n$

is a non-Newtonian fluid which moves like a rigid body when a certain function of the stress tensor is below a given threshold. Beyond this threshold, it obeys a non linear constitutive law (see (2.2)). As examples of such a fluid we can mention some paints, the mud which can be used for the oil extraction, the volcanic lava, etc. Also the blood can be assumed in first approximation as a Bingham fluid; more precisely, it has a yield stress as the Bingham fluid, but at high shear stress it behaves as a pseudo-plastic fluid.

As proposed in [12], from a mathematical point of view, the velocity  $\mathbf{u}_n$  and the pressure  $r_n$  of a Bingham fluid in  $\Omega_n$  solve the following non linear variational inequality

$$\left\{ \begin{array}{l} (\mathbf{u}_n, r_n) \in (H_0^1(\Omega_n))^2 \times (L^2(\Omega_n)/\mathbb{R}), \quad \text{div}(\mathbf{u}_n) = 0 \text{ in } \Omega_n, \\ \int_{\Omega_n} (h_n^2 \mu D\mathbf{u}_n D(\mathbf{v} - \mathbf{u}_n) + h_n g |D\mathbf{v}| - h_n g |D\mathbf{u}_n|) dx_1 dx_2 \\ \geq \int_{\Omega_n} \mathbf{f}_n (\mathbf{v} - \mathbf{u}_n) dx_1 dx_2 - \langle \nabla r_n, \mathbf{v} - \mathbf{u}_n \rangle_{(H^{-1}(\Omega_n))^2, (H_0^1(\Omega_n))^2} \quad \forall \mathbf{v} \in (H_0^1(\Omega_n))^2. \end{array} \right. \quad (1.3)$$

It is not an easy task to solve this variational inequality numerically due to the presence of the small parameter  $h_n$  which impedes the construction of suitable meshes. For this reason an asymptotic analysis is required when  $h_n$  vanishes.

First we reformulate problem (1.3) on a fixed domain through two appropriate rescalings according to [8] (see (3.1)-(3.9)), and in (3.10) we impose appropriate convergence assumptions on the rescaled external forces. A distinctive feature of the result proved in this paper

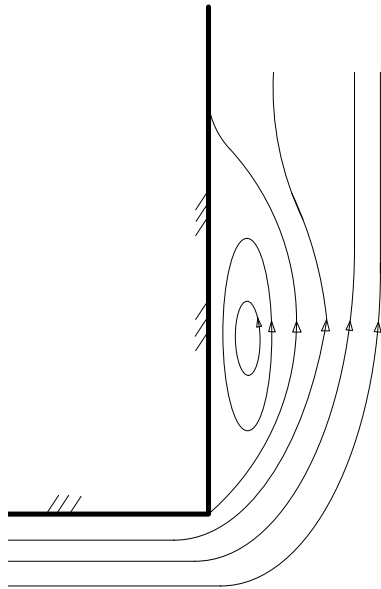


Figure 2: Flow in a  $90^\circ$  bend

is that the limit problem obtained letting  $h_n$  tend to zero decouples into two independent problems associated to the two different branches of the T-domain, as stated in Theorem 4.1. This behaviour can be explained with the separation of the flow in a  $90^\circ$  bend. In fact (for instance, see [22]) in a  $90^\circ$  bend the streamlines have a gradual change of direction, and a recirculation zone appears near the junction (see Fig. 2) This recirculation zone contracts the flow, i.e. it reduces the passage area between the two branches of the T-shaped domain. In the limit problem considered in the present paper such a recirculation near the junction is emphasized and its effect produces the decoupling of the flow in the two branches. Now we limit ourself to describe what happens in the horizontal branch, since a similar situation occurs in the vertical branch, up to a permutation of the axes. In the horizontal branch, the second component of the limit velocity vanishes. Indeed, following the separation of the two branches, now the flow in each branch reduces in a recirculation (see Fig. 3), then, the mass transfer through a transversal section is zero. Clearly when the thickness of the branch tends to zero this implies the limit zero velocity in the transversal direction to the branch. As the first component of the velocity is concerned, it solves variational inequality (4.4), stated in an anisotropic functional space. This inequality involves the first component of the limit force. If this component is independent of the transversal direction, also the first component of the velocity is zero (see Remark 8.2). This behavior can be explained because in this case the velocity is constant along transversal directions. Consequently, due to the presence of the wall, the conservation of the mass implies the zero velocity. On the other hand, if the first component of the force depends also on the transversal direction and the first component of the velocity is non-identically zero, then the variational inequality (4.4) modelizes a flow

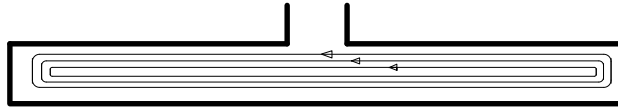


Figure 3: Recirculation in the horizontal branch

obeying to a lower-dimensional "Bingham-like" law (see Remark 8.4). This law is used in engineering literature for describing Bingham flow in thin domains (for instance, see [25]).

If in the original problem one assumes  $g = 0$  in the yield stress, the corresponding fluid is Newtonian, the flow is governed by the Stokes equations and a similar analysis can be performed (see Section 8).

The paper is organized as follows. In Section 2 we introduce the physical properties of the fluid and the variational inequality modelizing the flow. In Section 3 we rescale the problem on a fixed domain. The main converge result, Theorem 4.1, is stated in Section 4. In order to prove this theorem, in Section 5 and in Section 6 we obtain sharp *a priori* estimates on the velocity and on the pressure, respectively. Section 7 is devoted to the identification of the variational inequality satisfied by the limit velocity and pressure. An analysis of the limit problem is performed in Section 8.

Thin domains with T-junctions are of wide interest in fluid dynamics. For instance, in [9] the effect of a T-junction in a micro-fluidic device for the controlled formation of water-in-oil dispersion is analyzed. For other numerical and experimental studies of flows in thin junctions we refer to [18], [30], and review [11]. For the asymptotic analysis of Navier-Stokes flow in thin multi-domain we refer to [27] and [28]. About the study of this type of junction in other contexts, we refer to [13], [14], [15], [16], [17], [20], [19], [21], [23], [26].

The physical description of the Bingham fluid was introduced in [1] and its mathematical model for a bounded domain was given in [12] as a non linear variational inequality. The existence of the velocity and of the pressure for such a flow was there proved in the case of a two-dimensional domain. For further mathematical considerations on the Bingham fluid and other non-Newtonian fluids we refer to [10]. The first asymptotic mathematical analysis of the Bingham flow in a thin domain was performed in [6] and [7] in the two-dimensional and the three-dimensional case, respectively. For other studies of the Bingham flow in domains depending on a small parameter, in particular in the frame of the homogenization theory,

we refer the reader to [24], [3], [4], and [5].

## 2 The setting of the problem

In the domain  $\Omega_n$  defined by (1.1) we consider the non linear flow of a Bingham fluid. If we denote by  $u_n$  and  $r_n$  the velocity and the pressure of this fluid, respectively, the corresponding stress tensor is defined by

$$\sigma_{ij}(\mathbf{u}_n, r_n) = -r_n \delta_{ij} + gh_n \frac{e_{ij}(\mathbf{u}_n)}{\sqrt{e_{II}(\mathbf{u}_n)}} + 2\mu h_n^2 e_{ij}(\mathbf{u}_n), \quad (2.1)$$

where  $\delta_{ij}$  is the Kronecker symbol,  $g$  is a strictly positive constant related to the yield stress of the fluid,  $\mu$  is a strictly positive constant related to the viscosity of the fluid,  $e(\mathbf{u}_n)$  is the strain tensor with entries given by

$$e_{ij}(\mathbf{u}_n) = \frac{1}{2} \left( \frac{\partial u_{n,i}}{\partial x_j} + \frac{\partial u_{n,j}}{\partial x_i} \right), \quad 1 \leq i, j \leq 2,$$

and  $e_{II}(\mathbf{u}_n)$  is defined by

$$e_{II}(\mathbf{u}_n) = \frac{1}{2} \sum_{i,j=1}^2 e_{ij}(\mathbf{u}_n) e_{ij}(\mathbf{u}_n).$$

Moreover, we define

$$\begin{aligned} \sigma_{ij}^e(\mathbf{u}_n) &= gh_n \frac{e_{ij}(\mathbf{u}_n)}{\sqrt{e_{II}(\mathbf{u}_n)}} + 2\mu h_n^2 e_{ij}(\mathbf{u}_n), \\ \sigma_{II}(\mathbf{u}_n) &= \frac{1}{2} \sum_{i,j=1}^2 \sigma_{ij}^e(\mathbf{u}_n) \sigma_{ij}^e(\mathbf{u}_n). \end{aligned}$$

We remark that relation (2.1), which represents the constitutive law of the Bingham fluid, is valid only if  $e_{II}(\mathbf{u}_n) \neq 0$ . In [12] it is shown that this constitutive law is equivalent with the following one:

$$\begin{cases} \sqrt{\sigma_{II}(\mathbf{u}_n)} \leq gh_n \Leftrightarrow e_{ij}(\mathbf{u}_n) = 0, \\ \sqrt{\sigma_{II}(\mathbf{u}_n)} > gh_n \Leftrightarrow e_{ij}(\mathbf{u}_n) = \frac{1}{2\mu h_n^2} \left( 1 - \frac{gh_n}{\sqrt{\sigma_{II}(\mathbf{u}_n)}} \right) \sigma_{ij}^e(\mathbf{u}_n). \end{cases} \quad (2.2)$$

We point out that this is a threshold law: as long as the shear stress  $\sqrt{\sigma_{II}(\mathbf{u}_n)}$  is below  $gh_n$ , the fluid behaves as a rigid solid. When the value of the shear stress  $\sqrt{\sigma_{II}(\mathbf{u}_n)}$  exceeds  $gh_n$ , then the fluid flows obeying a non linear law.

We also suppose that the fluid is incompressible, which means that its velocity is divergence free

$$\operatorname{div} \mathbf{u}_n = \frac{\partial u_{n,1}}{\partial x_1} + \frac{\partial u_{n,2}}{\partial x_2} = 0 \quad \text{in } \Omega_n. \quad (2.3)$$

Moreover, we apply to the fluid a given external force  $f_n$  belonging to  $(L^2(\Omega_n))^2$ , and we then have the following relations

$$\frac{\partial \sigma_{i1}}{\partial x_1} + \frac{\partial \sigma_{i2}}{\partial x_2} = u_{n,i} \quad \text{in } \Omega_n, \quad i = 1, 2. \quad (2.4)$$

Furthermore, we assume the no-slip condition to the boundary of the domain, which reads

$$u_n = 0 \quad \text{on } \partial\Omega_n. \quad (2.5)$$

In [12] it is shown that the velocity  $u_n$  satisfying (2.1), (2.3), (2.4) and (2.5) solves the following variational inequality

$$\left\{ \begin{array}{l} u_n \in (H_0^1(\Omega_n))^2, \quad \text{div}(u_n) = 0 \text{ in } \Omega_n, \\ \int_{\Omega_n} (h_n^2 \mu D u_n D(v - u_n) + h_n g |Dv| - h_n g |D u_n|) dx_1 dx_2 \\ \geq \int_{\Omega_n} f_n (v - u_n) dx_1 dx_2 \quad \forall v \in (H_0^1(\Omega_n))^2 : \text{div}(v) = 0 \text{ in } \Omega_n. \end{array} \right. \quad (2.6)$$

where  $Dv$  is the matrix denoting the gradient of  $v$ , for a function  $v$  in  $(H_0^1(\Omega_n))^2$ . For each  $n$ , this inequality admits a unique solution  $u_n$ .

According to [12], problem (2.6) is equivalent to the following one

$$\left\{ \begin{array}{l} (u_n, r_n) \in (H_0^1(\Omega_n))^2 \times (L^2(\Omega_n)/\mathbb{R}), \quad \text{div}(u_n) = 0 \text{ in } \Omega_n, \\ \int_{\Omega_n} (h_n^2 \mu D u_n D(v - u_n) + h_n g |Dv| - h_n g |D u_n|) dx_1 dx_2 \\ \geq \int_{\Omega_n} f_n (v - u_n) dx_1 dx_2 - \langle \nabla r_n, v - u_n \rangle_{(H^{-1}(\Omega_n))^2, (H_0^1(\Omega_n))^2} \quad \forall v \in (H_0^1(\Omega_n))^2 \end{array} \right. \quad (2.7)$$

which admits a solution  $(u_n, r_n)$ , such that  $u_n$  is unique, but  $r_n$  is not unique.

The aim of this paper is to study the asymptotic behavior, as  $n$  diverges, of problem (2.7), under assumption (1.2) and suitable assumption on the given data  $\{f_n\}_{n \in \mathbb{N}}$ .

For  $n \in \mathbb{N}$ , let  $u_n \in (H_0^1(\Omega_n))^2$  be the solution of problem (2.7) and let  $r_n \in L^2(\Omega_n)/\mathbb{R}$  be a solution of problem (2.7). Obviously we have

$$\langle \nabla r_n, v - u_n \rangle_{(H^{-1}(\Omega_n))^2, (H_0^1(\Omega_n))^2} = \left\langle \nabla \left( r_n - \frac{1}{|\Omega_n^a|} \int_{\Omega_n^a} r_n dx_1 dx_2 \right), v - u_n \right\rangle_{(H^{-1}(\Omega_n))^2, (H_0^1(\Omega_n))^2}.$$

By setting

$$p_n = r_n - \frac{1}{|\Omega_n^a|} \int_{\Omega_n^a} r_n dx_1 dx_2,$$

the function  $p_n \in L^2(\Omega_n)$  is such that

$$\int_{\Omega_n^a} p_n dx_1 dx_2 = 0 \quad (2.8)$$

and

$$\begin{aligned} \langle \nabla r_n, v - u_n \rangle_{(H^{-1}(\Omega_n))^2, (H_0^1(\Omega_n))^2} &= \langle \nabla p_n, v - u_n \rangle_{(H^{-1}(\Omega_n))^2, (H_0^1(\Omega_n))^2} \\ &= \int_{\Omega_n} p_n \operatorname{div} (v - u_n) dx_1 dx_2. \end{aligned}$$

So, if  $r_n$  is a solution for problem (2.7), then  $p_n$  too is a solution for problem (2.7). Consequently, problem (2.6) is equivalent with the following one

$$\left\{ \begin{array}{l} (u_n, p_n) \in (H_0^1(\Omega_n))^2 \times (L^2(\Omega_n)/\mathbb{R}), \quad \operatorname{div}(u_n) = 0 \text{ in } \Omega_n, \\ \int_{\Omega_n} (h_n^2 \mu D u_n D (v - u_n) + h_n g |Dv| - h_n g |D u_n|) dx_1 dx_2 \\ \geq \int_{\Omega_n} (f_n (v - u_n) + p_n \operatorname{div} (v - u_n)) dx_1 dx_2 \quad \forall v \in (H_0^1(\Omega_n))^2 \end{array} \right. \quad (2.9)$$

In a similar way, setting

$$q_n = r_n - \frac{1}{|\Omega_n^b|} \int_{\Omega_n^b} r_n dx_1 dx_2, \quad (2.10)$$

the function  $q_n \in L^2(\Omega_n)$  satisfies

$$\int_{\Omega_n^b} q_n dx_1 dx_2 = 0$$

and

$$\begin{aligned} \langle \nabla r_n, v - u_n \rangle_{(H^{-1}(\Omega_n))^2, (H_0^1(\Omega_n))^2} &= \langle \nabla q_n, v - u_n \rangle_{(H^{-1}(\Omega_n))^2, (H_0^1(\Omega_n))^2} \\ &= \int_{\Omega_n} q_n \operatorname{div} (v - u_n) dx_1 dx_2. \end{aligned}$$

Moreover, the couple  $(u_n, q_n) \in (H_0^1(\Omega_n))^2 \times (L^2(\Omega_n)/\mathbb{R})$  is a solution of problem (2.9) with  $q_n$  instead of  $p_n$ .

### 3 The rescaled problem

Set now

$$\left\{ \begin{array}{l} \Omega^a = \left] -\frac{1}{2}, \frac{1}{2} \left[ \times \right] 0, 1 \left[ , \quad \Omega^b = \left] -\frac{1}{2}, \frac{1}{2} \left[ \times \right] -1, 0 \left[ , \\ \Gamma^a = \partial\Omega^a \setminus \left( \left] -\frac{1}{2}, \frac{1}{2} \left[ \times \{0\} \right) \right), \quad \Gamma_n^b = \partial\Omega^b \setminus \left( \left] -\frac{h_n}{2}, \frac{h_n}{2} \left[ \times \{0\} \right) \right) \quad n \in \mathbb{N}. \end{array} \right. \quad (3.1)$$



In order to pass to the limit, the first step consists in the reformulation of problems (2.6) and (2.9) in the domain  $\Omega^a \cup \Omega^b$ , independent of  $n$ . This is done as usual, by a domain dilatation technique (see [8]), through the maps

$$x = (x_1, x_2) \in \Omega^a \longrightarrow (h_n x_1, x_2) \in \text{Int}(\Omega_n^a), \quad x = (x_1, x_2) \in \Omega^b \longrightarrow (x_1, h_n x_2) \in \Omega_n^b, \quad (3.2)$$

where  $\text{Int}(\Omega_n^a)$  denotes the interior of  $\Omega_n^a$  (and the analogous for  $\Omega_n^b$ ). More precisely, for every  $n \in \mathbb{N}$  we set

$$\left\{ \begin{array}{l} D_n^a : v^a \in (H^1(\Omega^a))^2 \longrightarrow \left( \frac{1}{h_n} \partial_{x_1} v^a, \partial_{x_2} v^a \right) \in (L^2(\Omega^a))^2 \times (L^2(\Omega^a))^2, \\ D_n^b : v^b \in (H^1(\Omega^b))^2 \longrightarrow \left( \partial_{x_1} v^b, \frac{1}{h_n} \partial_{x_2} v^b \right) \in (L^2(\Omega^b))^2 \times (L^2(\Omega^b))^2, \\ \text{div}_n^a : v^a = (v_1^a, v_2^a) \in (H^1(\Omega^a))^2 \longrightarrow \frac{1}{h_n} \partial_{x_1} v_1^a + \partial_{x_2} v_2^a \in L^2(\Omega^a), \\ \text{div}_n^b : v^b = (v_1^b, v_2^b) \in (H^1(\Omega^b))^2 \longrightarrow \partial_{x_1} v_1^b + \frac{1}{h_n} \partial_{x_2} v_2^b \in L^2(\Omega^b), \end{array} \right. \quad (3.3)$$

$$\left\{ \begin{array}{l} f_n^a : x = (x_1, x_2) \in \Omega^a \longrightarrow f_n(h_n x_1, x_2), \\ f_n^b : x = (x_1, x_2) \in \Omega^b \longrightarrow f_n(x_1, h_n x_2), \end{array} \right. \quad (3.4)$$

$$\left\{ \begin{array}{l} \mathcal{V}_n = \left\{ (v^a, v^b) \in (H^1(\Omega^a))^2 \times (H^1(\Omega^b))^2 : v^a = 0 \text{ on } \Gamma^a, \quad v^b = 0 \text{ on } \Gamma_n^b, \right. \\ \left. v^a(x_1, 0) = v^b(h_n x_1, 0) \text{ on } ]-\frac{1}{2}, \frac{1}{2}[ \right\}, \end{array} \right. \quad (3.5)$$

and

$$\tilde{\mathcal{V}}_n = \left\{ (v^a, v^b) \in \mathcal{V}_n : \text{div}_n^a(v^a) = 0 \text{ in } \Omega^a, \quad \text{div}_n^b(v^b) = 0 \text{ in } \Omega^b \right\}. \quad (3.6)$$

Now, if  $u_n$  solves (2.6) (or equivalently  $(u_n, p_n)$  solves problem (2.9)), then  $((u_n^a, u_n^b), (p_n^a, p_n^b))$  defined by

$$\left\{ \begin{array}{l} u_n^a(x_1, x_2) = u_n(h_n x_1, x_2) \text{ in } \Omega^a, \quad u_n^b(x_1, x_2) = u_n(x_1, h_n x_2) \text{ in } \Omega^b, \\ p_n^a(x_1, x_2) = p_n(h_n x_1, x_2) \text{ in } \Omega^a, \quad p_n^b(x_1, x_2) = p_n(x_1, h_n x_2) \text{ in } \Omega^b, \end{array} \right. \quad (3.7)$$

solves

$$\left\{ \begin{array}{l} u_n = (u_n^a, u_n^b) \in \tilde{\mathcal{V}}_n, \\ \int_{\Omega^a} (h_n^2 \mu D_n^a u_n^a D_n^a (v^a - u_n^a) + h_n g |D_n^a v^a| - h_n g |D_n^a u_n^a|) dx_1 dx_2 \\ + \int_{\Omega^b} (h_n^2 \mu D_n^b u_n^b D_n^b (v^b - u_n^b) + h_n g |D_n^b v^b| - h_n g |D_n^b u_n^b|) dx_1 dx_2 \\ \geq \int_{\Omega^a} f_n^a (v^a - u_n^a) dx_1 dx_2 + \int_{\Omega^b} f_n^b (v^b - u_n^b) dx_1 dx_2 \quad \forall v = (v^a, v^b) \in \tilde{\mathcal{V}}_n. \end{array} \right. \quad (3.8)$$

or equivalently

$$\left\{ \begin{array}{l} u_n = (u_n^a, u_n^b) \in \tilde{\mathcal{V}}_n, \quad p_n = (p_n^a, p_n^b) \in (L^2(\Omega^a) \times L^2(\Omega^b)) / \mathbb{R}, \\ \int_{\Omega^a} (h_n^2 \mu D_n^a u_n^a D_n^a (v^a - u_n^a) + h_n g |D_n^a v^a| - h_n g |D_n^a u_n^a|) dx_1 dx_2 \\ + \int_{\Omega^b} (h_n^2 \mu D_n^b u_n^b D_n^b (v^b - u_n^b) + h_n g |D_n^b v^b| - h_n g |D_n^b u_n^b|) dx_1 dx_2 \\ \geq \int_{\Omega^a} (f_n^a (v^a - u_n^a) + p_n^a \operatorname{div}_n^a (v^a - u_n^a)) dx_1 dx_2 \\ + \int_{\Omega^b} (f_n^b (v^b - u_n^b) + p_n^b \operatorname{div}_n^b (v^b - u_n^b)) dx_1 dx_2 \quad \forall v = (v^a, v^b) \in \mathcal{V}_n, \end{array} \right. \quad (3.9)$$

**Remark 3.1.** Due to relation (2.8), a change of variables shows that the solution  $(p_n^a, p_n^b)$  of problem (3.9) is such that  $\int_{\Omega^a} p_n^a dx_1 dx_2 = 0$ . In a similar way, according to (2.10), there exists a solution  $(q_n^a, q_n^b)$  of problem (3.9) such that the function  $q_n^b$  satisfies  $\int_{\Omega^b} q_n^b dx_1 dx_2 = 0$ .

Conversely, if  $((u_n^a, u_n^b), (p_n^a, p_n^b))$  solves (3.8) (or equivalently (3.9)), then  $(u_n, p_n)$  defined by

$$\left\{ \begin{array}{l} u_n(x_1, x_2) = u_n^a \left( \frac{x_1}{h_n}, x_2 \right) \text{ in } \Omega_n^a, \quad u_n(x_1, x_2) = u_n^b \left( x_1, \frac{x_2}{h_n} \right) \text{ in } \Omega_n^b, \\ p_n(x_1, x_2) = p_n^a \left( \frac{x_1}{h_n}, x_2 \right) \text{ in } \Omega_n^a, \quad p_n(x_1, x_2) = p_n^b \left( x_1, \frac{x_2}{h_n} \right) \text{ in } \Omega_n^b, \end{array} \right.$$

solves (2.6) (or equivalently (2.9)). Therefore, the goal of this paper becomes to study the asymptotic behavior, as  $n$  diverges, of problem (3.9). To this aim, we assume

$$\left\{ \begin{array}{l} f_n^a \rightarrow f^a = (f_1^a, f_2^a) \text{ strongly in } (L^2(\Omega^a))^2, \\ f_n^b \rightarrow f^b = (f_1^b, f_2^b) \text{ strongly in } (L^2(\Omega^b))^2. \end{array} \right. \quad (3.10)$$

## 4 The main results

In order to give the main result of our paper, namely the convergence of the rescaled initial problem to a limit problem stated in some anisotropic spaces, according to [6] we first introduce the two applications

$$T^a : w \in L^2(\Omega^a) \rightarrow T^a(w) \in L^2(]0, 1[) \quad \text{and} \quad T^b : w \in L^2(\Omega^b) \rightarrow T^b(w) \in L^2(]-\frac{1}{2}, \frac{1}{2}[)$$

defined by

$$T^a(w) : x_2 \in ]0, 1[ \rightarrow \int_{-\frac{1}{2}}^{\frac{1}{2}} w(x_1, x_2) dx_1 \quad \text{and} \quad T^b(w) : x_1 \in ]-\frac{1}{2}, \frac{1}{2}[ \rightarrow \int_{-1}^0 w(x_1, x_2) dx_2, \quad (4.1)$$

respectively.

Let

$$W^a = \left\{ w^a \in L^2(\Omega^a) : \partial_{x_1} w^a \in L^2(\Omega^a), \quad w^a = 0 \text{ on } \left\{ \pm \frac{1}{2} \right\} \times ]0, 1[, \right.$$

$$\left. T^a(w^a) \in H_0^1(]0, 1[) \right\},$$

$$W^b = \left\{ w^b \in L^2(\Omega^b) : \partial_{x_2} w^b \in L^2(\Omega^b), \quad w^b = 0 \text{ on } ]-\frac{1}{2}, \frac{1}{2}[ \times \{-1, 0\}, \right.$$

$$\left. T^b(w^b) \in H_0^1\left(]-\frac{1}{2}, \frac{1}{2}[ \right) \right\}.$$

$W^a$  and  $W^b$  are two anisotropic Hilbert spaces equipped with the inner products

$$(w^a, z^a) = \int_{\Omega^a} (w^a z^a + \partial_{x_1} w^a \partial_{x_1} z^a) dx_1 dx_2 + \int_0^1 (T^a(w^a))' (T^a(z^a))' dx_2$$

and

$$(w^b, z^b) = \int_{\Omega^b} (w^b z^b + \partial_{x_2} w^b \partial_{x_2} z^b) dx_1 dx_2 + \int_{-\frac{1}{2}}^{\frac{1}{2}} (T^b(w^b))' (T^b(z^b))' dx_1,$$

respectively. Let

$$\begin{cases} W_0^a = \{w^a \in W^a(\Omega^a) : T^a(w^a) = 0 \text{ in } ]0, 1[\}, \\ W_0^b = \{w^b \in W^b(\Omega^b) : T^b(w^b) = 0 \text{ in } ]-\frac{1}{2}, \frac{1}{2}[\}. \end{cases} \quad (4.2)$$

**Theorem 4.1.** *Assume (1.2) and (3.10). For every  $n \in \mathbb{N}$  let  $((u_n^a, u_n^b), (p_n^a, p_n^b))$  and  $((u_n^a, u_n^b), (q_n^a, q_n^b))$  be two solutions of problem (3.9) such that*

$$\int_{\Omega^a} p_n^a dx_1 dx_2 = 0, \quad \int_{\Omega^b} q_n^b dx_1 dx_2 = 0 \quad \forall n \in \mathbb{N}.$$

Let  $W_0^a(\Omega^a)$  and  $W_0^b(\Omega^b)$  be defined in (4.2).

Then, there exist  $u_2^a \in W_0^a(\Omega^a)$  and  $u_1^b \in W_0^b(\Omega^b)$  such that

$$\begin{cases} u_n^a \rightharpoonup (0, u_2^a) \text{ weakly in } (L^2(\Omega^a))^2, \\ \partial_{x_1} u_n^a \rightharpoonup (0, \partial_{x_1} u_2^a) \text{ weakly in } (L^2(\Omega^a))^2, \\ h_n u_n^a \rightharpoonup 0 \text{ weakly in } (H^1(\Omega^a))^2, \end{cases} \quad \begin{cases} u_n^b \rightharpoonup (u_1^b, 0) \text{ weakly in } (L^2(\Omega^b))^2, \\ \partial_{x_2} u_n^b \rightharpoonup (\partial_{x_2} u_1^b, 0) \text{ weakly in } (L^2(\Omega^b))^2, \\ h_n u_n^b \rightharpoonup 0 \text{ weakly in } (H^1(\Omega^b))^2. \end{cases}$$

There exist an increasing sequence of positive numbers, still denoted by  $\{n\}$ ,  $p^a \in L^2(\Omega^a)$  independent of  $x_1$ , and  $q^b \in L^2(\Omega^b)$  independent of  $x_2$  (in possible dependence on the subsequence), such that

$$\begin{aligned} \int_0^1 p^a dx_2 = 0, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} q^b dx_1 = 0, \\ p_n^a \rightharpoonup p^a \text{ weakly in } L^2(\Omega^a), \quad q_n^b \rightharpoonup q^b \text{ weakly in } L^2(\Omega^b). \end{aligned}$$

Moreover,  $(u_2^a, p^a)$  solves

$$\begin{cases} u_2^a \in W_0^a(\Omega^a), & p^a \in L^2(\Omega^a), \\ \int_{\Omega^a} (\mu \partial_{x_1} u_2^a \partial_{x_1} (w^a - u_2^a) + g (|\partial_{x_1} w^a| - |\partial_{x_1} u_2^a|)) dx_1 dx_2 \\ \geq \int_{\Omega^a} (f_2^a (w^a - u_2^a) + p^a \partial_{x_2} (w^a - u_2^a)) dx_1 dx_2 \quad \forall w^a \in W^a(\Omega^a), \end{cases} \quad (4.3)$$

and  $(u_1^b, q^b)$  solves

$$\begin{cases} u_1^b \in W_0^b(\Omega^b), & q^b \in L^2(\Omega^b), \\ \int_{\Omega^b} (\mu \partial_{x_2} u_1^b \partial_{x_2} (w^b - u_1^b) + g (|\partial_{x_2} w^b| - |\partial_{x_2} u_1^b|)) dx_1 dx_2 \\ \geq \int_{\Omega^b} (f_1^b (w^b - u_1^b) + q^b \partial_{x_1} (w^b - u_1^b)) dx_1 dx_2 \quad \forall w^b \in W^b(\Omega^b), \end{cases} \quad (4.4)$$

where  $f_2^a$  and  $f_1^b$  are given by (3.10). Furthermore,  $u_2^a$  and  $u_1^b$  are unique.

## 5 *A priori* estimates and a convergence result for the velocity

In order to use compactness results for passing to the limit in the terms involving the velocity, we first derive *a priori* estimates for this function.

**Proposition 5.1.** *Assume (1.2) and (3.10). For every  $n \in \mathbb{N}$  let  $(u_n^a, u_n^b)$  be the solution of (3.8). Then there exists a positive constant  $c$  independent on  $n$  such that*

$$h_n \|D_n^a u_n^a\|_{(L^2(\Omega^a))^4} \leq c, \quad h_n \|D_n^b u_n^b\|_{(L^2(\Omega^b))^4} \leq c \quad \forall n \in \mathbb{N}, \quad (5.1)$$

$$\|u_n^a\|_{(L^2(\Omega^a))^2} \leq c, \quad \|u_n^b\|_{(L^2(\Omega^b))^2} \leq c \quad \forall n \in \mathbb{N}. \quad (5.2)$$

*Proof.* By choosing  $(v^a, v^b) = (0, 0)$  and  $(v^a, v^b) = (2u_n^a, 2u_n^b)$  in (3.8) we obtain

$$\begin{cases} \int_{\Omega^a} (-h_n^2 \mu |D_n^a u_n^a|^2 - h_n g |D_n^a u_n^a|) dx_1 dx_2 \\ + \int_{\Omega^b} (-h_n^2 \mu |D_n^b u_n^b|^2 - h_n g |D_n^b u_n^b|) dx_1 dx_2 \\ \geq - \int_{\Omega^a} f_n^a u_n^a dx_1 dx_2 - \int_{\Omega^b} f_n^b u_n^b dx_1 dx_2 \quad \forall n \in \mathbb{N}, \end{cases} \quad (5.3)$$

and

$$\left\{ \begin{array}{l} \int_{\Omega^a} (h_n^2 \mu |D_n^a u_n^a|^2 + h_n g |D_n^a u_n^a|) dx_1 dx_2 + \int_{\Omega^b} (h_n^2 \mu |D_n^b u_n^b|^2 + h_n g |D_n^b u_n^b|) dx_1 dx_2 \\ \geq \int_{\Omega^a} f_n^a u_n^a dx_1 dx_2 + \int_{\Omega^b} f_n^b u_n^b dx_1 dx_2 \quad \forall n \in \mathbb{N}, \end{array} \right. \quad (5.4)$$

respectively. The comparison of (5.3) and (5.4) implies

$$\left\{ \begin{array}{l} \int_{\Omega^a} (h_n^2 \mu |D_n^a u_n^a|^2 + h_n g |D_n^a u_n^a|) dx_1 dx_2 + \int_{\Omega^b} (h_n^2 \mu |D_n^b u_n^b|^2 + h_n g |D_n^b u_n^b|) dx_1 dx_2 \\ = \int_{\Omega^a} f_n^a u_n^a dx_1 dx_2 + \int_{\Omega^b} f_n^b u_n^b dx_1 dx_2 \quad \forall n \in \mathbb{N}. \end{array} \right. \quad (5.5)$$

On the other hand, the homogeneous boundary conditions of  $u_n^a$  on  $\Gamma^a$  and of  $u_n^b$  on  $\Gamma_n^b$  give

$$\left\{ \begin{array}{l} \|u_n^a\|_{(L^2(\Omega^a))^2} \leq \|\partial_{x_1} u_n^a\|_{(L^2(\Omega^a))^2} \leq \|h_n D_n^a u_n^a\|_{(L^2(\Omega^a))^4}, \\ \|u_n^b\|_{(L^2(\Omega^b))^2} \leq \|\partial_{x_2} u_n^b\|_{(L^2(\Omega^b))^2} \leq \|h_n D_n^b u_n^b\|_{(L^2(\Omega^b))^4} \quad \forall n \in \mathbb{N}. \end{array} \right. \quad (5.6)$$

The use of the Hölder inequality in the right-hand side of (5.5) and estimates (5.6) imply

$$\left\{ \begin{array}{l} \mu \int_{\Omega^a} |h_n D_n^a u_n^a|^2 dx_1 dx_2 + \mu \int_{\Omega^b} |h_n D_n^b u_n^b|^2 dx_1 dx_2 \\ \leq \|f_n^a\|_{(L^2(\Omega^a))^2} \|u_n^a\|_{(L^2(\Omega^a))^2} + \|f_n^b\|_{(L^2(\Omega^b))^2} \|u_n^b\|_{(L^2(\Omega^b))^2} \\ \leq \|f_n^a\|_{(L^2(\Omega^a))^2} \|h_n D_n^a u_n^a\|_{(L^2(\Omega^a))^4} + \|f_n^b\|_{(L^2(\Omega^b))^2} \|h_n D_n^b u_n^b\|_{(L^2(\Omega^b))^4} \quad \forall n \in \mathbb{N}. \end{array} \right. \quad (5.7)$$

Finally, (5.1) follows from (5.7), (3.10), and the following inequality

$$(\alpha + \beta)^2 \leq 2(\alpha^2 + \beta^2) \quad \forall \alpha, \beta \in \mathbb{R},$$

while (5.2) follows from (5.6) and (5.1).  $\square$

Proposition 5.1 implies the following result.

**Proposition 5.2.** *Let  $T^a$  and  $T^b$  be defined in (4.1). Assume (1.2) and (3.10). For every  $n \in \mathbb{N}$  let  $(u_n^a, u_n^b)$  be the solution of (3.8). Then there exist a subsequence of  $\mathbb{N}$ , still denoted by  $\{n\}$ ,  $u^a \in (L^2(\Omega^a))^2$  with  $\partial_{x_1} u^a \in (L^2(\Omega^a))^2$ , and  $u^b \in (L^2(\Omega^b))^2$  with  $\partial_{x_2} u^b \in (L^2(\Omega^b))^2$  (in possible dependence on the subsequence), such that*

$$\left\{ \begin{array}{l} u_n^a \rightharpoonup u^a \text{ weakly in } (L^2(\Omega^a))^2, \\ \partial_{x_1} u_n^a \rightharpoonup \partial_{x_1} u^a \text{ weakly in } (L^2(\Omega^a))^2, \\ h_n u_n^a \rightharpoonup 0 \text{ weakly in } (H^1(\Omega^a))^2, \end{array} \right. \quad \left\{ \begin{array}{l} u_n^b \rightharpoonup u^b \text{ weakly in } (L^2(\Omega^b))^2, \\ \partial_{x_2} u_n^b \rightharpoonup \partial_{x_2} u^b \text{ weakly in } (L^2(\Omega^b))^2, \\ h_n u_n^b \rightharpoonup 0 \text{ weakly in } (H^1(\Omega^b))^2. \end{array} \right. \quad (5.8)$$

Moreover,

$$u^a = 0 \text{ on } \{\pm \frac{1}{2}\} \times ]0, 1[, \quad u^b = 0 \text{ on } ]-\frac{1}{2}, \frac{1}{2}[ \times \{-1\}, \quad (5.9)$$

$$u^b = 0 \text{ on } ]-\frac{1}{2}, \frac{1}{2}[ \times \{0\}, \quad (5.10)$$

$$u_1^a = 0 \text{ in } \Omega^a, \quad u_2^b = 0 \text{ in } \Omega^b, \quad (5.11)$$

$$T^a(u_2^a) = 0 \text{ in } ]0, 1[, \quad T^b(u_1^b) = 0 \text{ in } ]-\frac{1}{2}, \frac{1}{2}[. \quad (5.12)$$

*Proof.* Statements in (5.8) and in (5.9) follow from Proposition 5.1.

Statement in (5.10) is obtained taking into account that up to a subsequence we have the convergences

$$u_n^b(\cdot, 0) \rightharpoonup u^b(\cdot, 0) \text{ weakly in } (L^2(]-\frac{1}{2}, \frac{1}{2}[)) ^2$$

and

$$\chi_{]-\frac{h_n}{2}, \frac{h_n}{2}[} \rightarrow 0 \text{ strongly in } L^2(]-\frac{1}{2}, \frac{1}{2}[),$$

and then passing to the limit, as  $n$  diverges, in

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} u_n^b(x_1, 0) \varphi(x_1) dx_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \chi_{]-\frac{h_n}{2}, \frac{h_n}{2}[}(x_1) u_n^b(x_1, 0) \varphi(x_1) dx_1 \quad \forall \varphi \in (C_0^\infty(]-\frac{1}{2}, \frac{1}{2}[))^2, \quad \forall n \in \mathbb{N},$$

which holds true since  $u_n^b = 0$  on  $(]-\frac{1}{2}, \frac{1}{2}[ \setminus ]-\frac{h_n}{2}, \frac{h_n}{2}[) \times \{0\}$ .

The first equality in (5.11) is obtained passing to the limit in

$$\partial_{x_1} (u_n^a)_1 + h_n \partial_{x_2} (u_n^a)_2 = h_n \operatorname{div}_n^a (u_n^a) = 0 \text{ in } \Omega^a, \quad \forall n \in \mathbb{N},$$

and taking into account the last two convergences in the left-hand side of (5.8), and the left-hand side in (5.9). Similarly, one proves the right-hand side in (5.11).

As far as the proof of the left-hand side in (5.12) is concerned, equality

$$\operatorname{div}_n^a (u_n^a) = 0 \text{ in } \Omega^a, \quad \forall n \in \mathbb{N},$$

gives

$$\int_{\Omega^a} \left( \frac{1}{h_n} \partial_{x_1} (u_n^a)_1 + \partial_{x_2} (u_n^a)_2 \right) \varphi(x_2) dx_1 dx_2 = 0 \quad \forall \varphi \in C_0^\infty(]0, 1[), \quad \forall n \in \mathbb{N}. \quad (5.13)$$

On the other side, the boundary conditions on  $u_n^a$  give

$$\begin{cases} \int_{\Omega^a} \partial_{x_1} (u_n^a)_1 \varphi(x_2) dx_1 dx_2 = \int_{\partial\Omega^a} (u_n^a)_1 \varphi(x_2) n_1 dx_1 dx_2 = 0 \\ \forall \varphi \in C_0^\infty(]0, 1[), \quad \forall n \in \mathbb{N}, \end{cases} \quad (5.14)$$

where  $n_1$  is the first component of the exterior unit normal on  $\partial\Omega^a$ . Combining (5.13) and (5.14) provides

$$\int_0^1 \partial_{x_2} (T^a((u_n^a)_2)) \varphi(x_2) dx_2 = \int_{\Omega^a} \partial_{x_2} (u_n^a)_2 \varphi(x_2) dx_1 dx_2 = 0 \quad \forall \varphi \in C_0^\infty(]0, 1[), \quad \forall n \in \mathbb{N}.$$

Consequently, one has

$$\partial_{x_2} (T^a ((u_n^a)_2)) = 0 \text{ in } ]0, 1[, \quad \forall n \in \mathbb{N},$$

which implies

$$T^a ((u_n^a)_2) = 0 \text{ in } ]0, 1[, \quad \forall n \in \mathbb{N}, \quad (5.15)$$

since  $T^a ((u_n^a)_2)(1) = 0$ . Finally, combining the first convergence in the left-hand side in (5.8) and (5.15) proves the left-hand side in (5.12), since  $T^a$  is weakly continuous, being strongly continuous.

The proof of the right-hand side in (5.12) is more sophisticated, since one does not know if  $u_n^b = 0$  on  $] - \frac{h_n}{2}, \frac{h_n}{2}[\times\{0\}$ . As above,

$$\operatorname{div}_n^b(u_n^b) = 0 \text{ in } \Omega^b, \quad \forall n \in \mathbb{N},$$

gives

$$\int_{\Omega^b} \left( \partial_{x_1} (u_n^b)_1 + \frac{1}{h_n} \partial_{x_2} (u_n^b)_2 \right) \varphi(x_1) dx_1 dx_2 = 0 \quad \forall \varphi \in C_0^\infty(]-1, 1[), \quad \forall n \in \mathbb{N}. \quad (5.16)$$

Now, fix  $\delta \in ]0, \frac{1}{2}[$ . The boundary conditions on  $u_n^b$  give

$$\begin{cases} \int_{\Omega^b} \partial_{x_2} (u_n^b)_2 \varphi(x_1) dx_1 dx_2 = \int_{\partial\Omega^b} (u_n^b)_2 \varphi(x_1) n_2 dx_1 dx_2 = 0 \\ \forall \varphi \in C_0^\infty(]-\frac{1}{2}, -\delta[ \cup ]\delta, \frac{1}{2}[), \quad \forall n \in \mathbb{N} : \frac{h_n}{2} < \delta, \end{cases} \quad (5.17)$$

where  $n_2$  is the second component of the exterior unit normal on  $\partial\Omega^b$ . Combining (5.16) and (5.17) provides

$$\begin{cases} \int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_{x_1} (T^b ((u_n^b)_1)) \varphi(x_1) dx_1 = \int_{\Omega^b} \partial_{x_1} (u_n^b)_1 \varphi(x_1) dx_1 dx_2 = 0 \\ \forall \varphi \in C_0^\infty(]-\frac{1}{2}, -\delta[ \cup ]\delta, \frac{1}{2}[), \quad \forall n \in \mathbb{N} : \frac{h_n}{2} < \delta. \end{cases}$$

Consequently, one has

$$\partial_{x_1} (T^b ((u_n^b)_1)) = 0 \text{ in } ] - \frac{1}{2}, -\delta[ \cup ]\delta, \frac{1}{2}[, \quad \forall n \in \mathbb{N} : \frac{h_n}{2} < \delta,$$

which implies

$$T^b ((u_n^b)_1) = 0 \text{ in } ] - \frac{1}{2}, -\delta[ \cup ]\delta, \frac{1}{2}[, \quad \forall n \in \mathbb{N} : \frac{h_n}{2} < \delta, \quad (5.18)$$

since  $T^b ((u_n^b)_1)(\pm\frac{1}{2}) = 0$ . Combining the first convergence in the right-hand side of (5.8) and (5.18) provides

$$T^b ((u^b)_1) = 0 \text{ in } ] - \frac{1}{2}, -\delta[ \cup ]\delta, \frac{1}{2}[, \quad (5.19)$$

since  $T^b$  is weakly continuous, being strongly continuous. Finally, the right-hand side in (5.12) follows from (5.19) by the arbitrary choice of  $\delta \in ]0, \frac{1}{2}[$ .  $\square$

## 6 *A priori* estimates and a convergence result for the pressure

**Proposition 6.1.** *For every  $n \in \mathbb{N}$  let  $((u_n^a, u_n^b), (p_n^a, p_n^b))$  be a solution of (3.9). Then there exists a positive constant  $c$  independent on  $n$  such that*

$$\|\partial_{x_1} p_n^a\|_{H^{-1}(\Omega^a)} \leq h_n c, \quad \|\partial_{x_2} p_n^a\|_{H^{-1}(\Omega^a)} \leq c, \quad \|p_n^a\|_{L^2(\Omega^a)} \leq c \left(1 + \left| \int_{\Omega^a} p_n^a dx_1 dx_2 \right| \right), \quad (6.1)$$

$$\|\partial_{x_1} p_n^b\|_{H^{-1}(\Omega^b)} \leq c, \quad \|\partial_{x_2} p_n^b\|_{H^{-1}(\Omega^b)} \leq c h_n, \quad \|p_n^b\|_{L^2(\Omega^b)} \leq c \left(1 + \left| \int_{\Omega^b} p_n^b dx_1 dx_2 \right| \right), \quad (6.2)$$

for every  $n \in \mathbb{N}$ .

*Proof.* Choosing  $(v^a, v^b) = (w^a + u_n^a, u_n^b)$  as test functions in (3.9), with  $w^a \in (H_0^1(\Omega^a))^2$ , gives

$$\begin{cases} \int_{\Omega^a} (h_n^2 \mu D_n^a u_n^a D_n^a w^a + h_n g |D_n^a(w^a + u_n^a)| - h_n g |D_n^a u_n^a|) dx_1 dx_2 \\ \geq \int_{\Omega^a} (f_n^a w^a + p_n^a \operatorname{div}_n^a w^a) dx_1 dx_2 \quad \forall w^a \in (H_0^1(\Omega^a))^2, \quad \forall n \in \mathbb{N}. \end{cases} \quad (6.3)$$

Applying the Hölder inequality in (6.3) provides

$$\begin{cases} \int_{\Omega^a} p_n^a \operatorname{div}_n^a w^a dx_1 dx_2 \leq \\ h_n^2 \mu \|D_n^a u_n^a\|_{(L^2(\Omega^a))^4} \|D_n^a w^a\|_{(L^2(\Omega^a))^4} + h_n g \|D_n^a w^a\|_{(L^1(\Omega^a))^4} \\ + \|f_n^a\|_{(L^2(\Omega^a))^2} \|w^a\|_{(L^2(\Omega^a))^2} \quad \forall w^a \in (H_0^1(\Omega^a))^2, \quad \forall n \in \mathbb{N}. \end{cases}$$

Consequently, using (3.10), (5.1), the Hölder inequality, and the Poincaré inequality ensure the existence of a positive constant  $c$  such that

$$\int_{\Omega^a} p_n^a \operatorname{div}_n^a w^a dx_1 dx_2 \leq c h_n \|D_n^a w^a\|_{(L^2(\Omega^a))^4} \quad \forall w^a \in (H_0^1(\Omega^a))^2, \quad \forall n \in \mathbb{N},$$

that is

$$\left| \int_{\Omega^a} p_n^a \operatorname{div}_n^a w^a dx_1 dx_2 \right| \leq c h_n \|D_n^a w^a\|_{(L^2(\Omega^a))^4} \quad \forall w^a \in (H_0^1(\Omega^a))^2, \quad \forall n \in \mathbb{N}. \quad (6.4)$$

Now, in (6.4) choosing  $w^a = (w, 0)$  and then  $w^a = (0, w)$ , with  $w \in H_0^1(\Omega^a)$ , get

$$\left| \int_{\Omega^a} p_n^a \partial_{x_1} w dx_1 dx_2 \right| \leq c h_n \|Dw\|_{(L^2(\Omega^a))^2} \quad \forall w \in H_0^1(\Omega^a), \quad \forall n \in \mathbb{N}, \quad (6.5)$$

and

$$\left| \int_{\Omega^a} p_n^a \partial_{x_2} w dx_1 dx_2 \right| \leq c \|Dw\|_{(L^2(\Omega^a))^2} \quad \forall w \in H_0^1(\Omega^a), \quad \forall n \in \mathbb{N}. \quad (6.6)$$

The first two estimates in (6.1) follow from (6.5) and (6.6), respectively. The third one follows from Lemma 6.1 in [29] and from the first two estimates in (6.1).

Similarly, one can prove (6.2).  $\square$



Proposition 5.2 and Proposition 6.1 imply the following result.

**Corollary 6.2.** *Assume (1.2) and (3.10). For every  $n \in \mathbb{N}$  let  $((u_n^a, u_n^b), (p_n^a, p_n^b))$  and  $((u_n^a, u_n^b), (q_n^a, q_n^b))$  be two solutions of (3.9) such that*

$$\int_{\Omega^a} p_n^a dx_1 dx_2 = 0, \quad \int_{\Omega^b} q_n^b dx_1 dx_2 = 0 \quad \forall n \in \mathbb{N}.$$

*Then there exist an increasing sequence of positive numbers, still denoted by  $\{n\}$ ,  $p^a \in L^2(\Omega^a)$  independent of  $x_1$ , and  $q^b \in L^2(\Omega^b)$  independent of  $x_2$  (in possible dependence on the subsequence), such that*

$$\int_0^1 p^a dx_2 = 0, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} q^b dx_1 = 0, \quad (6.7)$$

$$p_n^a \rightharpoonup p^a \text{ weakly in } L^2(\Omega^a), \quad q_n^b \rightharpoonup q^b \text{ weakly in } L^2(\Omega^b). \quad (6.8)$$

## 7 Proof of Theorem 4.1

*Proof.* By virtue of Proposition 5.2 and Corollary 6.2, there exist an increasing sequence of positive numbers, still denoted by  $\{n\}$ , and  $u^a \in (L^2(\Omega^a))^2$  with  $\partial_{x_1} u^a \in (L^2(\Omega^a))^2$ ,  $u^b \in (L^2(\Omega^b))^2$  with  $\partial_{x_2} u^b \in (L^2(\Omega^b))^2$ ,  $p^a \in L^2(\Omega^a)$  independent of  $x_1$ , and  $q^b \in L^2(\Omega^b)$  independent of  $x_2$  (in possible dependence on the subsequence), such that (5.8)-(5.12), (6.7) and (6.8) hold true. It remains to prove that  $(u_2^a, p^a)$  solves the variational inequality in (4.3) and  $(u_1^b, q^b)$  solves the variational inequality in (4.4). The uniqueness of  $u_2^a$  and  $u_1^b$  respectively can be proved as in [7].

In (3.8) choosing  $(v^a, v^b) = ((0, 0), (0, 0))$  first, then  $(v^a, v^b) = (2u_n^a, 2u_n^b)$ , and comparing provide

$$\begin{aligned} & \int_{\Omega^a} (h_n^2 \mu |D_n^a u_n^a|^2 + h_n g |D_n^a u_n^a|) dx_1 dx_2 + \int_{\Omega^b} (h_n^2 \mu |D_n^b u_n^b|^2 + h_n g |D_n^b u_n^b|) dx_1 dx_2 \\ &= \int_{\Omega^a} f_n^a u_n^a dx_1 dx_2 + \int_{\Omega^b} f_n^b u_n^b dx_1 dx_2, \quad \forall n \in \mathbb{N}. \end{aligned} \quad (7.1)$$

Combining (3.9) and (7.1) gives

$$\begin{aligned} & \int_{\Omega^a} (h_n^2 \mu D_n^a u_n^a D_n^a v^a + h_n g |D_n^a v^a|) dx_1 dx_2 + \int_{\Omega^b} (h_n^2 \mu D_n^b u_n^b D_n^b v^b + h_n g |D_n^b v^b|) dx_1 dx_2 \\ & \geq \int_{\Omega^a} (f_n^a v^a + p_n^a \operatorname{div}_n^a v^a) dx_1 dx_2 + \int_{\Omega^b} (f_n^b v^b + p_n^b \operatorname{div}_n^b v^b) dx_1 dx_2, \quad \forall (v^a, v^b) \in \mathcal{V}_n. \end{aligned} \quad (7.2)$$

Passing to the limit in a suitable subsequence of (7.2), as  $n \rightarrow +\infty$ , with test functions  $(v^a, v^b) = ((0, w^a), (0, 0))$  such that  $w^a \in H_0^1(\Omega^a)$ , and using (5.8), (6.8), and assumption (3.10) provide

$$\int_{\Omega^a} (\mu \partial_{x_1} u_2^a \partial_{x_1} w^a + g |\partial_{x_1} w^a|) dx_1 dx_2 \geq \int_{\Omega^a} (f_2^a w^a + p^a \partial_{x_2} w^a) dx_1 dx_2, \quad \forall w^a \in H_0^1(\Omega^a). \quad (7.3)$$

By a density argument (7.3) holds true also with  $w^a$  belonging to space  $W^a$  defined in Section 4. In particular, choosing  $w^a = u_2^a \in W_0^a \subset W^a$  in (7.3) gives

$$\int_{\Omega^a} (\mu |\partial_{x_1} u_2^a|^2 + g |\partial_{x_1} u_2^a|) dx_1 dx_2 \geq \int_{\Omega^a} f_2^a u_2^a dx_1 dx_2, \quad (7.4)$$

since  $p^a$  is independent of  $x_1$  and  $T^a(u_2^a) = 0$  in  $]0, 1[$ .

Now replacing  $q_n^a$  with  $q_n^b$  in (3.9) gives formula (7.2) with  $q_n^a$  replaced by  $q_n^b$ . Then, passing to the limit in a suitable subsequence of this new formula, as  $n \rightarrow +\infty$ , with test functions  $(v^a, v^b) = ((0, 0), (w^b, 0))$  such that  $w^b \in H_0^1(\Omega^b)$ , and arguing as above provide

$$\int_{\Omega^b} (\mu \partial_{x_2} u_1^b \partial_{x_2} w^b + g |\partial_{x_2} w^b|) dx_1 dx_2 \geq \int_{\Omega^b} (f_1^b w^b + q^b \partial_{x_2} w^b) dx_1 dx_2, \quad \forall w^b \in H_0^1(\Omega^b), \quad (7.5)$$

and consequently

$$\int_{\Omega^b} (\mu |\partial_{x_2} u_1^b|^2 + g |\partial_{x_2} u_1^b|) dx_1 dx_2 \geq \int_{\Omega^b} f_1^b u_1^b dx_1 dx_2. \quad (7.6)$$

Adding (7.6) to (7.4) gives

$$\begin{aligned} & \int_{\Omega^a} (\mu |\partial_{x_1} u_2^a|^2 + g |\partial_{x_1} u_2^a|) dx_1 dx_2 + \int_{\Omega^b} (\mu |\partial_{x_2} u_1^b|^2 + g |\partial_{x_2} u_1^b|) dx_1 dx_2 \\ & \geq \int_{\Omega^a} f_2^a u_2^a dx_1 dx_2 + \int_{\Omega^b} f_1^b u_1^b dx_1 dx_2. \end{aligned} \quad (7.7)$$

Now passing to the limit in (7.1) and using a l.s.c. argument, (5.8), and (3.10) provide

$$\begin{aligned} & \int_{\Omega^a} f_2^a u_2^a dx_1 dx_2 + \int_{\Omega^b} f_1^b u_1^b dx_1 dx_2 \\ & \geq \int_{\Omega^a} (\mu |\partial_{x_1} u_2^a|^2 + g |\partial_{x_1} u_2^a|) dx_1 dx_2 + \int_{\Omega^b} (\mu |\partial_{x_2} u_1^b|^2 + g |\partial_{x_2} u_1^b|) dx_1 dx_2. \end{aligned} \quad (7.8)$$

Comparing (7.7) and (7.8) implies

$$\begin{aligned} & \int_{\Omega^a} (\mu |\partial_{x_1} u_2^a|^2 + g |\partial_{x_1} u_2^a|) dx_1 dx_2 + \int_{\Omega^b} (\mu |\partial_{x_2} u_1^b|^2 + g |\partial_{x_2} u_1^b|) dx_1 dx_2 \\ & = \int_{\Omega^a} f_2^a u_2^a dx_1 dx_2 + \int_{\Omega^b} f_1^b u_1^b dx_1 dx_2. \end{aligned} \quad (7.9)$$

Now comparing (7.4), (7.6), and (7.9) implies

$$\begin{cases} \int_{\Omega^a} (\mu |\partial_{x_1} u_2^a|^2 + g |\partial_{x_1} u_2^a|) dx_1 dx_2 = \int_{\Omega^a} f_2^a u_2^a dx_1 dx_2, \\ \int_{\Omega^b} (\mu |\partial_{x_2} u_1^b|^2 + g |\partial_{x_2} u_1^b|) dx_1 dx_2 = \int_{\Omega^b} f_1^b u_1^b dx_1 dx_2. \end{cases} \quad (7.10)$$

Eventually, (7.3), (7.5), (7.10), and a density argument (see [7]) ensure that  $(u_2^a, p^a)$  solves the variational inequality in (4.3) and  $(u_1^b, q^b)$  solves the variational inequality in (4.4).  $\square$

## 8 Analysis of the limit problem

This section is devoted to the analysis of limit problems (4.3) and (4.4) obtained in Theorem 4.1.

**Remark 8.1.** *We first notice that these two problems are not coupled and consequently they can be treated independently. From a physical point of view, this behavior can be explained with the presence of a recirculation zone close to the junction that reduces the passage area through the two branches. The formation of this zone is due to the separation of the flow in a 90° bend (see [22]). In our limit problem the effect of such a recirculation produces the decoupling of the two branches.*

*We next remark that, up to a permutation of the axes, both limit problems (4.3) and (4.4) are of the same type and for this reason we focus our attention on problem (4.4), that we recall below.*

Let  $f_1^b \in L^2(\Omega^b)$  be given in (3.10). Find  $(u_1^b, q^b)$  such that

$$\left\{ \begin{array}{l} u_1^b \in W_0^b(\Omega^b), \quad q^b \in L^2(\Omega^b), \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} q^b dx_1 = 0, \\ \int_{\Omega^b} (\mu \partial_{x_2} u_1^b \partial_{x_2} (w^b - u_1^b) + g (|\partial_{x_2} w^b| - |\partial_{x_2} u_1^b|)) dx_1 dx_2 \\ \geq \int_{\Omega^b} (f_1^b (w^b - u_1^b) + q^b \partial_{x_1} (w^b - u_1^b)) dx_1 dx_2 \quad \forall w^b \in W^b(\Omega^b), \end{array} \right. \quad (8.1)$$

where  $\Omega^b = ]-\frac{1}{2}, \frac{1}{2}[ \times ]-1, 0[$ . According to [6], this problem is equivalent to the following problem

$$\left\{ \begin{array}{l} u_1^b \in W_0^b(\Omega^b), \\ \int_{\Omega^b} (\mu \partial_{x_2} u_1^b \partial_{x_2} (w^b - u_1^b) + g (|\partial_{x_2} w^b| - |\partial_{x_2} u_1^b|)) dx_1 dx_2 \\ \geq \int_{\Omega^b} f_1^b (w^b - u_1^b) dx_1 dx_2 \quad \forall w^b \in W_0^b(\Omega^b). \end{array} \right. \quad (8.2)$$

**Remark 8.2.** *If  $f_1^b$  is independent of  $x_2$ , then  $u_1^b = 0$  is the solution of limit problem (8.2). Indeed, suppose that  $f_1^b$  is independent of  $x_2$ . By choosing in the variational formulation (8.2) a test function  $w^b \in W_0^b(\Omega^b)$ , the right hand side vanishes, as follows:*

$$\int_{\Omega^b} f_1^b (w^b - u_1^b) dx_1 dx_2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( f_1^b(x_1) \int_{-1}^0 (w^b - u_1^b) dx_2 \right) dx_1 = 0. \quad (8.3)$$

*Then the problem becomes to find  $u_1^b \in W_0^b(\Omega^b)$  such that*

$$\int_{\Omega^b} (\mu \partial_{x_2} u_1^b \partial_{x_2} (w^b - u_1^b) + g (|\partial_{x_2} w^b| - |\partial_{x_2} u_1^b|)) dx_1 dx_2 \geq 0 \quad \forall w^b \in W_0^b(\Omega^b).$$

We remark that  $u_1^b = 0$  satisfies this last inequality for any  $w^b \in W_0^b(\Omega^b)$ . Indeed,  $u_1^b = 0$  implies

$$g \int_{\Omega^b} |\partial_{x_2} w^b| dx_1 dx_2 \geq 0 \quad \forall w^b \in W_0^b(\Omega^b).$$

This last inequality being always satisfied and the solution  $u_1^b$  of problem (8.2) being unique, we obtain the result.

**Remark 8.3.** Let  $f_1^b$  be dependent on  $x_2$ . Then, if  $u_1^b = 0$  is solution of the limit problem (8.2), one has

$$g \int_{\Omega^b} |\partial_{x_2} w^b| dx_1 dx_2 \geq \int_{\Omega^b} f_1^b w^b dx_1 dx_2 \quad \forall w^b \in W_0^b(\Omega^b). \quad (8.4)$$

Indeed, by taking  $w^b \in W_0^b(\Omega^b)$  and  $u_1^b = 0$  in the variational inequality (8.2), relation (8.4) directly follows.

**Remark 8.4.** In the sequel we are interested on the solution of problem (8.1) in the case when the function  $f_1^b$  depends on  $x_2$  and is such that

$$g \int_{\Omega^b} |\partial_{x_2} w^b| dx_1 dx_2 < \int_{\Omega^b} f_1^b w^b dx_1 dx_2 \quad \forall w^b \in W_0^b(\Omega^b).$$

According to [7], one can derive from problem (8.1) the following differential equation stated in the domain  $\Omega^b$

$$-\partial_{x_2} [\mu \partial_{x_2} u_1^b + g \operatorname{sgn}(\partial_{x_2} u_1^b)] = f_1^b - q^b,$$

which is valid if  $\partial_{x_2} u_1^b \neq 0$  and where  $\operatorname{sgn}$  states for the signum function.

As in [7] and [6], we define, up to an additive function of  $x_1$ , the limiting stress tensor  $\sigma_{12}$  by

$$\sigma_{12} = -\partial_{x_1} q^b + \sigma_{12}^*, \quad (8.5)$$

with  $\sigma_{12}^* = \mu \partial_{x_2} u_1^b + g \operatorname{sgn}(\partial_{x_2} u_1^b)$ . This is to be compared with the following lower-dimensional ‘‘Bingham-like’’ law, used in the engineering literature for describing Bingham flow in thin domains, as for instance in [25]:

$$\begin{cases} \mu \partial_{x_2} u_1^b = 0 & \Leftrightarrow |\tau| \leq 0, \\ \mu \partial_{x_2} u_1^b = \tau - g \operatorname{sgn}(\partial_{x_2} u_1^b) & \Leftrightarrow |\tau| > 0, \end{cases} \quad (8.6)$$

where  $\tau$  represents the constraint. We notice that, if  $\partial_{x_2} u_1^b \neq 0$ , then relation (8.5) implies  $|\sigma_{12}^*| > g$ . Consequently, if  $|\sigma_{12}^*| \leq g$ , we have  $\partial_{x_2} u_1^b = 0$ , which is exactly the lower-dimensional ‘‘Bingham-like’’ law above.

We end our analysis by recalling that the newtonian fluid can be seen as a particular case of the Bingham fluid, corresponding to  $g = 0$ . Indeed, taking  $g = 0$  in (2.1), the stress

tensor becomes  $\sigma_{ij}(u_n, r_n) = -r_n \delta_{ij} + 2\mu h_n^2 e_{ij}(u_n)$ , which corresponds to a newtonian fluid. The initial problem corresponds in this case to the following Stokes system

$$\left\{ \begin{array}{l} (u_n, r_n) \in (H_0^1(\Omega_n))^2 \times (L^2(\Omega_n)/\mathbb{R}), \quad \operatorname{div}(u_n) = 0 \text{ in } \Omega_n, \\ \mu \int_{\Omega_n} h_n^2 D u_n D v dx_1 dx_2 = \int_{\Omega_n} f_n v dx_1 dx_2 - \langle \nabla r_n, v \rangle_{(H^{-1}(\Omega_n))^2, (H_0^1(\Omega_n))^2} \quad \forall v \in (H_0^1(\Omega_n))^2 \end{array} \right.$$

which admits a unique solution  $(u_n, r_n)$ .

By performing a similar analysis, we obtain to the limit the following partial differential equation stated in the domain  $\Omega^b$

$$-\mu \frac{\partial^2 u_b^1(x_1, x_2)}{\partial x_2^2} = f_1^b - q^b$$

and the corresponding limiting stress tensor

$$\sigma_{12} = -\partial_{x_1} q^b + \mu \partial_{x_2} u_b^1.$$

For an asymptotic analysis of the Stokes system with the method of "partial asymptotic decomposition of domain" we refer to [2].

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