



# On the blow-up scenario for some modified Serre–Green–Naghdi equations

Billel Guelmame

## ► To cite this version:

Billel Guelmame. On the blow-up scenario for some modified Serre–Green–Naghdi equations. Non-linear Analysis: Theory, Methods and Applications, In press. hal-03264688v1

**HAL Id: hal-03264688**

**<https://hal.science/hal-03264688v1>**

Submitted on 18 Jun 2021 (v1), last revised 5 Jul 2022 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ON THE BLOW-UP SCENARIO FOR SOME MODIFIED SERRE–GREEN–NAGHDI EQUATIONS

BILLEL GUELMAME

ABSTRACT. The present paper deals with a modified Serre–Green–Naghdi (mSGN) system that has been introduced by Clamond et al. [4] to improve the dispersion relation. We present a precise blow-up scenario of the mSGN equations and we prove the existence of a class of solutions that develop singularities in finite time. All the presented results hold also for the Serre–Green–Naghdi system with weak surface tension.

**AMS Classification:** 35Q35; 35B65; 35B44; 76B15.

**Key words:** Serre–Green–Naghdi; dispersion; water waves; energy conservation; blow-up; shallow water.

## CONTENTS

1. Introduction	1
2. Main results	4
3. Preliminaries	5
4. Blow-up criteria	9
5. Blow-up results	12
References	14

## 1. INTRODUCTION

We consider in this paper the conservative modified Serre–Green–Naghdi system that was introduced by Clamond et al. [4]

$$h_t + [h u]_x = 0, \tag{1a}$$

$$[h u]_t + \left[ h u^2 + \frac{1}{2} g h^2 + \mathcal{R} \right]_x = 0, \tag{1b}$$

$$\mathcal{R} \stackrel{\text{def}}{=} \frac{1}{3} \left( 1 + \frac{3}{2} \beta \right) h^3 \left( -u_{tx} - u u_{xx} + u_x^2 \right) - \frac{1}{2} \beta g h^2 \left( h h_{xx} + \frac{1}{2} h_x^2 \right), \tag{1c}$$

where  $h$  denotes the depth of the fluid,  $u$  is the depth-averaged horizontal velocity,  $g$  is the gravitational acceleration and  $\beta$  is a free parameter. The classical Serre–Green–Naghdi system is recovered taking  $\beta = 0$ . The aim of this paper is to study the local (in time) well-posedness of (1), to obtain a precise blow-up criterion and to build smooth solutions that develop singularities in finite time.

Several modified Serre–Green–Naghdi equations have been derived and studied in the literature to optimise the linear dispersion [1, 2, 3, 7, 8, 18]. Some of those equations fail to conserve the energy and do not admit a variational principle, some other equations do not satisfy the Galilean invariance. In order to improve the dispersion of the classical Serre–Green–Naghdi (cSGN) system conserving its desirable properties, Clamond et al. [4] have modified the Lagrangian instead of modifying directly the cSGN equations to obtain the Lagrangian density

$$\frac{\mathcal{L}}{\rho} = \frac{1}{2} h u^2 + \frac{1}{6} (1 + \frac{3}{2} \beta) h^3 u_x^2 - \frac{1}{2} g h^2 - \frac{1}{4} \beta g h^2 h_x^2 + \phi \{h_t + [h u]_x\}, \quad (2)$$

where  $\phi$  is a Lagrange multiplier. The Euler–Lagrange equations of (2) lead to (1).

Smooth solutions of the modified Serre–Green–Naghdi (mSGN) equations (1) satisfy the energy equation

$$\mathcal{E}_t + \mathcal{Q}_x = 0, \quad (3)$$

with

$$\mathcal{E} \stackrel{\text{def}}{=} \frac{1}{2} h u^2 + \frac{1}{6} (1 + \frac{3}{2} \beta) h^3 u_x^2 + \frac{1}{2} g (h - \bar{h})^2 + \frac{1}{4} \beta g h^2 h_x^2, \quad (4)$$

$$\mathcal{Q} \stackrel{\text{def}}{=} u [\mathcal{E} + \mathcal{R} + \frac{1}{2} g h^2 - \frac{1}{2} g \bar{h}^2] + \frac{1}{2} \beta g h^3 h_x u_x, \quad (5)$$

where  $\bar{h}$  is the mean value of the depth  $h$  of the fluid. For  $\beta > 0$  and if  $h$  is far from zero ( $h \geq h_{\min} > 0$ ), the  $H^1$  norms of both  $u$  and  $h - \bar{h}$  are controlled by the energy. However, the energy of the cSGN equations ( $\beta = 0$ ) cannot control the  $L^2$  norm of  $h_x$ . This crucial property of mSGN (1) for  $\beta > 0$  is very important to build the small-energy solutions that develop singularities in finite time (Theorem 3 below). Several criteria have been proposed in [4] to chose the parameter  $\beta$ . The best value of  $\beta$  to improve the dispersion relation is  $\beta = 2/15 \approx 0.1333$ . To optimise the decay of a particular solitary wave of (1) studied in [4], one must take  $\beta = \frac{2}{3}(12\pi^{-2} - 1) \approx 0.1439$ . The value  $\beta \approx 0.34560$  approximates the inner angle of the crest of the solitary wave solution to the exact angle of the limiting solitary wave ( $120^\circ$ ). In the present paper, we consider only the case  $\beta > 0$ , and for the sake of simplicity we introduce

$$\alpha \stackrel{\text{def}}{=} 1 + \frac{3}{2} \beta. \quad (6)$$

The mSGN equations on the form (1) contains some terms with high-order derivatives and a term with a time derivative in the definition of  $\mathcal{R}$ . In order to obtain a simpler form of (1), we introduce the linear Sturm–Liouville operator

$$\mathcal{L}_h \stackrel{\text{def}}{=} h - \frac{1}{3} \alpha \partial_x h^3 \partial_x \quad (7)$$

and we apply  $\mathcal{L}_h^{-1}$  (the invertibility of  $\mathcal{L}_h$  is proved in Lemma 3 below) on the equation (1b) to obtain

$$u_t + u u_x + g h_x = -\mathcal{L}_h^{-1} \partial_x \left\{ \frac{2}{3} \alpha h^3 u_x^2 + \frac{1}{3} g h^3 h_{xx} - \frac{1}{4} \beta g h^2 h_x^2 \right\}. \quad (8)$$

In Lemma 3 below, we show that we gain one derivative with the operator  $\mathcal{L}_h^{-1} \partial_x$ , this is not enough to control the term  $\frac{1}{3} g h^3 h_{xx}$  in the right-hand side of (8). To get rid of this

term, we use (7) to rewrite (8) in the equivalent form

$$h_t + [hu]_x = 0, \quad (9a)$$

$$u_t + uu_x + \frac{\alpha-1}{\alpha} gh_x = -\mathcal{L}_h^{-1} \partial_x \left\{ \frac{2}{3} \alpha h^3 u_x^2 - \frac{1}{4} \beta g h^2 h_x^2 + \frac{g}{2\alpha} h^2 \right\}. \quad (9b)$$

The left-hand side of (9) is a symmetrisable  $2 \times 2$  hyperbolic system and the right-hand side is a zero-order non-local term. Then, the local well-posedness of (9) in  $H^s$  with  $s \geq 2$  can be obtained following the proof of symmetrisable hyperbolic systems. Beside the well-posedness result, we obtain in this paper a precise blow-up criteria. We also prove, using Riccati-type equations, the existence of a class of arbitrary small-energy initial data, such that the corresponding solutions develop singularities in finite time.

Other equations similar to (1) have been studied in the literature. For example, the Serre–Green–Naghdi equations with ‘weak’ surface tension [7, 9, 15, 19] that can be obtained replacing  $\mathcal{R}$  in (1) by

$$\mathcal{S} \stackrel{\text{def}}{=} \frac{1}{3} h^3 (-u_{tx} - u u_{xx} + u_x^2) - \gamma (h h_{xx} - \frac{1}{2} h_x^2),$$

where  $\gamma > 0$  is a constant (the surface tension coefficient divided by the density). The denomination ‘weak’ is due to the small-slope assumption ( $|h_x| \ll 1$ ). However, even with this assumption, large-amplitude waves can be approximated by those equations. Another interesting system is the dispersionless regularised Saint-Venant (rSV) system proposed by Clamond and Dutykh [5] that can be obtained replacing  $\mathcal{R}$  in (1) by

$$\mathcal{T} \stackrel{\text{def}}{=} \varepsilon h^3 (-u_{tx} - u u_{xx} + u_x^2) - \varepsilon g h^2 (h h_{xx} + \frac{1}{2} h_x^2),$$

where  $\varepsilon > 0$ . The classical Saint-Venant equations are recovered by taking  $\varepsilon = 0$ . The rSV equations have been studied in [5, 17, 20] and have been generalised recently to regularise the barotropic Euler equations [11]. In [17], Liu et al. have proved the local well-posedness of the rSV equations and they derived smooth solutions that cannot exist globally in time. The proofs presented in this paper for the mSGN equations can be generalised for the SGN equations with weak surface tension, for the rSV equations and also for the regularised barotropic Euler system [11]. The blow-up criterion proved here is more precise compared to the blow-up criterion in [17]. The key of the proof of the blow-up results in this paper is Lemma 5 below, a similar lemma have been proved in [17] for a short time (shorter than the existence time in general). In this paper, we show, with a shorter proof, that the same result holds true as long as the smooth solution exists.

This paper is organised as follows. The main results are introduced in Section 2. In Section 3, we prove some useful lemmas. Section 4 is devoted to obtain the precise blow-up scenario of strong solutions of the mSGN equations. In Section 5, we prove that some classical solutions cannot exist globally in time.

## 2. MAIN RESULTS

The first result of this paper is the local well-posedness of the system (9) in the Sobolev space

$$H^s(\mathbb{R}) \stackrel{\text{def}}{=} \left\{ f, \|f\|_{H^s(\mathbb{R})}^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi < +\infty \right\} \quad (10)$$

where  $s \geq 2$  is a real number.

**Theorem 1.** *Let  $\beta > 0$ ,  $\bar{h} > 0$  and  $s \geq 2$ , then, for any  $(h_0 - \bar{h}, u_0) \in H^s(\mathbb{R})$  satisfying  $\inf_{x \in \mathbb{R}} h_0(x) > 0$  there exists  $T > 0$  and  $(h - \bar{h}, u) \in C^0([0, T], H^s(\mathbb{R})) \cap C^1([0, T], H^{s-1}(\mathbb{R}))$  a unique solution of (9) such that*

$$\inf_{(t,x) \in [0,T] \times \mathbb{R}} h(t, x) > 0. \quad (11)$$

Moreover, the solution satisfies the conservation of the energy

$$\frac{d}{dt} \int_{\mathbb{R}} \left( \frac{1}{2} h u^2 + \frac{1}{6} \alpha h^3 u_x^2 + \frac{1}{2} g (h - \bar{h})^2 + \frac{1}{4} \beta g h^2 h_x^2 \right) dx = 0. \quad (12)$$

**Remark 1.** *The solution given in Theorem 1 depends continuously on the initial data, i.e., If  $(h_0 - \bar{h}, u_0), (\tilde{h}_0 - \bar{h}, \tilde{u}_0) \in H^s$ , such that  $h_0, \tilde{h}_0 \geq h^* > 0$ , and  $t \leq \min\{T, \tilde{T}\}$ , then there exists a constant  $C(\|(\tilde{h} - \bar{h}, \tilde{u})\|_{L^\infty([0,t], H^s)}, \|(h - \bar{h}, u)\|_{L^\infty([0,t], H^s)}) > 0$ , such that*

$$\|(h - \tilde{h}, u - \tilde{u})\|_{L^\infty([0,t], H^{s-1})} \leq C \|(h_0 - \tilde{h}_0, u_0 - \tilde{u}_0)\|_{H^s}. \quad (13)$$

The proof of Theorem 1 is classic and omitted in this paper. See [11, 13, 15, 17] and Theorem 3 of [12] for more details.

**Remark 2.** *If  $(h - \bar{h}, u) \in C^0([0, T], H^s(\mathbb{R}))$  and  $h$  satisfies (11) for some  $T > 0$ , then Theorem 1 ensures that the solution can be extended over  $[0, T]$ . In other words, if  $T_{\max}$  is the maximum time existence of the solution, then, we have the blow-up criterion*

$$T_{\max} < +\infty \implies \liminf_{t \rightarrow T_{\max}} \inf_{x \in \mathbb{R}} h(t, x) = 0 \quad \text{or} \quad \limsup_{t \rightarrow T_{\max}} \|(h - \bar{h}, u)\|_{H^s} = +\infty. \quad (14)$$

The blow-up criterion (14) can be improved as in [11, 12, 17] to

$$T_{\max} < +\infty \implies \liminf_{t \rightarrow T_{\max}} \inf_{x \in \mathbb{R}} h(t, x) = 0 \quad \text{or} \quad \limsup_{t \rightarrow T_{\max}} \|(h_x, u_x)\|_{L^\infty} = +\infty. \quad (15)$$

The last blow-up criterion ensures that if  $h > 0$  is far from zero, then, the blow-up will appear on the  $L^\infty$  norm of  $u_x$  or  $h_x$ . This result is improved in this paper, and we claim the two more precise criteria for blow-up mechanism.

**Theorem 2.** *Let  $\beta > 0$ , and let  $T_{\max}$  be the maximum time existence of the solution given by Theorem 1, then*

$$T_{\max} < +\infty \implies \liminf_{t \rightarrow T_{\max}} \inf_{x \in \mathbb{R}} h(t, x) = 0 \quad \text{or} \quad \begin{cases} \liminf_{t \rightarrow T_{\max}} \inf_{x \in \mathbb{R}} u_x(t, x) = -\infty, \\ \text{and} \\ \limsup_{t \rightarrow T_{\max}} \|h_x(t, x)\|_{L^\infty} = +\infty, \end{cases} \quad (16)$$

which is equivalent to second criterion

$$T_{max} < +\infty \implies \limsup_{t \rightarrow T_{max}} \|u_x(t, x)\|_{L^\infty} = +\infty \text{ and } \begin{cases} \liminf_{t \rightarrow T_{max}} \inf_{x \in \mathbb{R}} h(t, x) = 0, \\ \text{or} \\ \limsup_{t \rightarrow T_{max}} \|h_x(t, x)\|_{L^\infty} = +\infty. \end{cases} \quad (17)$$

Since  $H^1 \hookrightarrow L^\infty$  and the energy (12) is conserved, then,  $|h - \bar{h}|$  is controlled by the energy of the initial data (see Proposition 1 below). This ensures that if the initial energy is small enough then  $h$  is uniformly far from zero ( $\min_{t,x} h > 0$ ). In this case, the blow-up criterion (16) becomes

$$T_{max} < +\infty \implies \liminf_{t \rightarrow T_{max}} \inf_{x \in \mathbb{R}} u_x(t, x) = -\infty \quad \text{and} \quad \limsup_{t \rightarrow T_{max}} \|h_x\|_{L^\infty} = +\infty.$$

This blow-up criterion gives the precise blow-up of  $u_x$ , but it is not clear if  $h_x$  can blows-up on  $-\infty$  or  $+\infty$ . The following theorem shows that both cases are possible.

**Theorem 3.** *For any  $T > 0$  and  $K \in \left]0, \frac{g\sqrt{\beta}}{3\sqrt{2}}\bar{h}^3\right[$ , there exist*

- $(h_0 - \bar{h}, u_0) \in C_c^\infty(\mathbb{R})$  satisfying  $\int_{\mathbb{R}} \mathcal{E}_0 dx \leq K$  such that the corresponding solution of (9) blows-up at finite time  $T_{max} \leq T$  and

$$\inf_{[0, T_{max}] \times \mathbb{R}} u_x(t, x) = -\infty, \quad \sup_{[0, T_{max}] \times \mathbb{R}} h_x(t, x) = +\infty, \quad \inf_{[0, T_{max}] \times \mathbb{R}} h_x(t, x) > -\infty.$$

- $(\tilde{h}_0 - \bar{h}, \tilde{u}_0) \in C_c^\infty(\mathbb{R})$  satisfying  $\int_{\mathbb{R}} \tilde{\mathcal{E}}_0 dx \leq K$  such that the corresponding solution of (9) blows-up at finite time  $\tilde{T}_{max} \leq T$  and

$$\inf_{[0, \tilde{T}_{max}] \times \mathbb{R}} \tilde{u}_x(t, x) = -\infty, \quad \inf_{[0, \tilde{T}_{max}] \times \mathbb{R}} \tilde{h}_x(t, x) = -\infty, \quad \sup_{[0, \tilde{T}_{max}] \times \mathbb{R}} \tilde{h}_x(t, x) < +\infty.$$

**Remark 3.** *All the results presented above can also be proved for the SGN equations with weak surface tension, for the rSV equations [5] and also for the regularised barotropic Euler system [11].*

### 3. PRELIMINARIES

In this section, we recall some classical estimates and we prove some lemmas that are needed to prove Theorem 2 and Theorem 3.

**Lemma 1.** ([6]) *Let  $F \in C^{\tilde{m}+2}(\mathbb{R})$  with  $F(0) = 0$  and  $0 \leq s \leq \tilde{m}$ , then there exists a continuous function  $\tilde{F}$ , such that for all  $f \in H^s \cap W^{1,\infty}$  we have*

$$\|F(f)\|_{H^s} \leq \tilde{F}(\|f\|_{W^{1,\infty}}) \|f\|_{H^s}. \quad (18)$$

Let  $\Lambda$  be defined such that  $\widehat{\Lambda f} = (1 + \xi^2)^{\frac{1}{2}} \hat{f}$ , then we have the following estimate.

**Lemma 2.** ([14]) Let  $[A, B] \stackrel{\text{def}}{=} AB - BA$  be the commutator of the operators  $A$  and  $B$ . If  $r \geq 0$ , then  $\exists C > 0$  such that

$$\|fg\|_{H^r} \leq C (\|f\|_{L^\infty} \|g\|_{H^r} + \|f\|_{H^r} \|g\|_{L^\infty}), \quad (19)$$

$$\|[\Lambda^r, f]g\|_{L^2} \leq C (\|f_x\|_{L^\infty} \|g\|_{H^{r-1}} + \|f\|_{H^r} \|g\|_{L^\infty}). \quad (20)$$

Now, we recall the invertibility of the operator  $\mathcal{L}_h$  defined in (7) and that we gain two derivatives with  $\mathcal{L}_h^{-1}$ .

**Lemma 3.** Let  $\alpha > 0$  and  $0 < h \in W^{1,\infty}$  with  $h^{-1} \in L^\infty$ , then the operator  $\mathcal{L}_h$  is an isomorphism from  $H^2$  to  $L^2$  and  $\exists C_1 = C_1(\alpha, s, \|h^{-1}\|_{L^\infty}, \|h - \bar{h}\|_{W^{1,\infty}}) > 0$ ,  $C_2 = C_2(\alpha, \|h^{-1}\|_{L^\infty}, \|h\|_{L^\infty}) > 0$  such that

(1) If  $s \geq 0$ , then

$$\|\mathcal{L}_h^{-1} \partial_x \psi\|_{H^{s+1}} \leq C_1 (\|\psi\|_{H^s} + \|h - \bar{h}\|_{H^s} \|\mathcal{L}_h^{-1} \partial_x \psi\|_{W^{1,\infty}}), \quad (21a)$$

$$\|\mathcal{L}_h^{-1} \phi\|_{H^{s+1}} \leq C_1 (\|\phi\|_{H^s} + \|h - \bar{h}\|_{H^s} \|\mathcal{L}_h^{-1} \phi\|_{W^{1,\infty}}). \quad (21b)$$

(2) If  $s \geq 0$ , then

$$\|\mathcal{L}_h^{-1} \partial_x \psi\|_{H^{s+1}} \leq C_1 \|\psi\|_{H^s} (1 + \|h - \bar{h}\|_{H^s}), \quad (22a)$$

$$\|\mathcal{L}_h^{-1} \phi\|_{H^{s+1}} \leq C_1 \|\phi\|_{H^s} (1 + \|h - \bar{h}\|_{H^s}). \quad (22b)$$

(3) If  $\phi \in C_{\text{lim}} \stackrel{\text{def}}{=} \{f \in C(\mathbb{R}), f(+\infty), f(-\infty) \in \mathbb{R}\}$ , then  $\mathcal{L}_h^{-1} \phi$  is well defined and

$$\|\mathcal{L}_h^{-1} \phi\|_{W^{2,\infty}} \leq C_2 \|\phi\|_{L^\infty}. \quad (23)$$

(4) If  $\psi \in C_{\text{lim}} \cap L^1$ , then

$$\|\mathcal{L}_h^{-1} \partial_x \psi\|_{W^{1,\infty}} \leq C_2 (\|\psi\|_{L^\infty} + \|\psi\|_{L^1}). \quad (24)$$

The  $\mathcal{R}$  defined in (1c) contains some terms with two order derivatives, using (9), we can write  $\mathcal{R}$  without those high order derivatives involving the operator  $\mathcal{L}_h^{-1}$ . Then, using the previous lemma we show that the norm  $\|\mathcal{R}\|_{L^\infty}$  is controlled by  $\|(h, u, h^{-1})\|_{L^\infty}$ .

**Lemma 4.** Let  $(h - \bar{h}, u)$  be a smooth solution of (9), then for any  $T < T_{\max}$ , there exists  $C = C(\beta, \bar{h}, \|(h, u, h^{-1})\|_{L^\infty([0,T] \times \mathbb{R})}) > 0$ , such that

$$\|\mathcal{R}\|_{L^\infty([0,T] \times \mathbb{R})} \leq C. \quad (25)$$

*Proof.* From the definition of  $\mathcal{L}_h$ , we obtain that

$$(1 + \frac{1}{3} \alpha h^3 \partial_x \mathcal{L}_h^{-1} \partial_x) \Psi = h^3 \partial_x \mathcal{L}_h^{-1} \left( h \int_{-\infty}^x h^{-3} \Psi \right) \quad (26)$$

for any smooth function  $\Psi$ , such that  $\Psi(\pm\infty) = 0$ . Using (1c), (9b) and (26) we obtain

$$\begin{aligned} \mathcal{R} &= -\frac{1}{3} \alpha h^3 \partial_x [u_t + u u_x + \frac{\alpha-1}{\alpha} g h_x] + \frac{2}{3} \alpha h^3 u_x^2 - \frac{1}{4} \beta g h^2 h_x^2 \\ &= (1 + \frac{1}{3} \alpha h^3 \partial_x \mathcal{L}_h^{-1} \partial_x) \left\{ \frac{2}{3} \alpha h^3 u_x^2 - \frac{1}{4} \beta g h^2 h_x^2 + g \frac{h^2 - \bar{h}^2}{2\alpha} \right\} - g \frac{h^2 - \bar{h}^2}{2\alpha} \end{aligned} \quad (27)$$

$$= h^3 \partial_x \mathcal{L}_h^{-1} \left( h \int_{-\infty}^x \left( \frac{2}{3} \alpha u_x^2 - \frac{1}{4} \beta g h^{-1} h_x^2 + g h^{-3} \frac{h^2 - \bar{h}^2}{2\alpha} \right) \right) - g \frac{h^2 - \bar{h}^2}{2\alpha} \quad (28)$$

Using the conservation of the energy (12) we obtain

$$\begin{aligned} \left\| \int_{-\infty}^x \left( \frac{2}{3} \alpha u_x^2 - \frac{1}{4} \beta g h^{-1} h_x^2 + g h^{-3} \frac{h^2 - \bar{h}^2}{2\alpha} \right) \right\|_{L^\infty} &\leq \left\| \frac{2}{3} \alpha u_x^2 - \frac{1}{4} \beta g h^{-1} h_x^2 + g h^{-3} \frac{h^2 - \bar{h}^2}{2\alpha} \right\|_{L^1} \\ &\leq C_3(\|(h, h^{-1})\|_{L^\infty}) \mathcal{E}_0. \end{aligned}$$

Then, the inequality (25) follows directly from (23).  $\square$

Since we are considering the Serre–Green–Naghdi equations on the form (9) instead of (8), it is more convenient to use the following Riemann invariants<sup>1</sup>  $R$ ,  $S$  and their corresponding speeds of characteristics  $\lambda$ ,  $\mu$

$$R \stackrel{\text{def}}{=} u + 2 \sqrt{\frac{\alpha-1}{\alpha}} g h, \quad \lambda \stackrel{\text{def}}{=} u + \sqrt{\frac{\alpha-1}{\alpha}} g h, \quad (29)$$

$$S \stackrel{\text{def}}{=} u - 2 \sqrt{\frac{\alpha-1}{\alpha}} g h, \quad \mu \stackrel{\text{def}}{=} u - \sqrt{\frac{\alpha-1}{\alpha}} g h, \quad (30)$$

rather than the Riemann invariants of the classical Saint-Venant system. Then, the system (9) can be rewritten as

$$R_t + \lambda R_x = -\mathcal{L}_h^{-1} \partial_x \left\{ \frac{2}{3} \alpha h^3 u_x^2 - \frac{1}{4} \beta g h^2 h_x^2 + \frac{g}{2\alpha} h^2 \right\}, \quad (31a)$$

$$S_t + \mu S_x = -\mathcal{L}_h^{-1} \partial_x \left\{ \frac{2}{3} \alpha h^3 u_x^2 - \frac{1}{4} \beta g h^2 h_x^2 + \frac{g}{2\alpha} h^2 \right\}. \quad (31b)$$

Defining

$$\begin{aligned} P &\stackrel{\text{def}}{=} R_x = u_x + \sqrt{\frac{\alpha-1}{\alpha}} g h^{-\frac{1}{2}} h_x, \\ Q &\stackrel{\text{def}}{=} S_x = u_x - \sqrt{\frac{\alpha-1}{\alpha}} g h^{-\frac{1}{2}} h_x, \end{aligned}$$

we have

$$u_x = \frac{P + Q}{2}, \quad h_x = \frac{\sqrt{\alpha} h^{\frac{1}{2}}}{2 \sqrt{(\alpha-1)g}} (P - Q). \quad (32)$$

Let the characteristics  $X_a, Y_a$  starting from  $a$  defined as the solutions of the ordinary differential equations

$$\frac{d}{dt} X_a(t) = \lambda(t, X_a(t)), \quad X_a(0) = a \quad (33)$$

$$\frac{d}{dt} Y_a(t) = \mu(t, Y_a(t)), \quad Y_a(0) = a \quad (34)$$

Differentiation (31) with respect to  $x$ , and using (27) we obtain the Riccati-type equations

$$\frac{d^\lambda}{dt} P \stackrel{\text{def}}{=} P_t + \lambda P_x = -\frac{3}{8} P^2 + \frac{3}{8} Q^2 + P Q - 3 \alpha^{-1} h^{-3} \mathcal{R}, \quad (35a)$$

$$\frac{d^\mu}{dt} Q \stackrel{\text{def}}{=} Q_t + \mu Q_x = -\frac{3}{8} Q^2 + \frac{3}{8} P^2 + P Q - 3 \alpha^{-1} h^{-3} \mathcal{R}, \quad (35b)$$

---

<sup>1</sup>Those quantities are constants along the characteristics if the right-hand side of (9) is zero.



where  $\frac{d^\lambda}{dt}, \frac{d^\mu}{dt}$  denote the derivatives along the characteristics with the speeds  $\lambda$  and  $\mu$  respectively.

A key point to prove Theorem 2 and Theorem 3 is to control the term  $P^2$  in the Ricatti equation (35b) and the term  $Q^2$  in (35a). For that purpose, we prove in the following lemma that the integral of  $P^2$  along the  $X$  characteristics and the integral of  $Q^2$  along the  $Y$  characteristics are bounded.

**Lemma 5.** *Let  $\beta > 0$ ,  $\bar{h} > 0$  and  $(h_0 - \bar{h}, u_0) \in H^2$  initial data satisfying  $\inf_{x \in \mathbb{R}} h_0(x) > 0$  and let  $(h - \bar{h}, u)$  be the corresponding solution of (9) given by Theorem 1, let also  $t \in [0, T_{max}[$ , then, there exist*

$$A \left( \beta, \bar{h}, \|(h, u, h^{-1})\|_{L^\infty([0, t] \times \mathbb{R})}, \int \mathcal{E} dx \right) > 0, \quad B \left( \beta, \bar{h}, \|(h, u, h^{-1})\|_{L^\infty([0, t] \times \mathbb{R})}, \int \mathcal{E} dx \right) > 0,$$

such that for any  $x_2 \in \mathbb{R}$ , and for  $x_1 \in ]-\infty, x_2[$  the solution of  $X_{x_1}(t) = Y_{x_2}(t)$  (see Figure 1) we have

$$\int_0^t Q(s, X_{x_1}(s))^2 ds + \int_0^t P(s, Y_{x_2}(s))^2 ds \leq A t + B. \quad (36)$$

**Remark 4.** A similar result have been proved for the so-called variational wave equation with  $A = 0$  and  $B$  depends only on the energy of the initial data [10]. For the mSGN, additional terms appear, and a uniform (on time) bound cannot be obtained for large data.

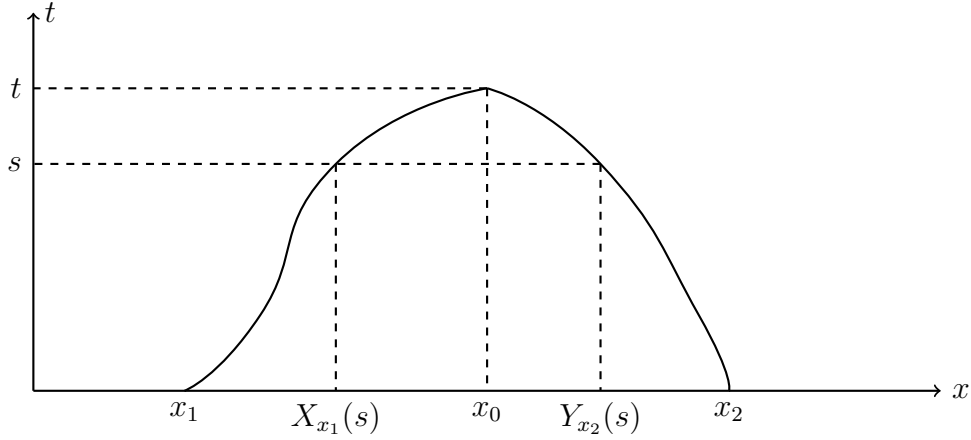


FIGURE 1. Characteristics.

*Proof.* Defining

$$\begin{aligned} \mathcal{B}_1 &\stackrel{\text{def}}{=} \sqrt{\frac{\alpha-1}{\alpha}} g h \left( \frac{1}{2} h u^2 + \frac{1}{2} g (h - \bar{h})^2 \right) - u \left( \mathcal{R} + \frac{1}{2} g h^2 - \frac{1}{2} g \bar{h}^2 \right), \\ \mathcal{B}_2 &\stackrel{\text{def}}{=} \sqrt{\frac{\alpha-1}{\alpha}} g h \left( \frac{1}{2} h u^2 + \frac{1}{2} g (h - \bar{h})^2 \right) + u \left( \mathcal{R} + \frac{1}{2} g h^2 - \frac{1}{2} g \bar{h}^2 \right), \end{aligned}$$

and using (25), one can prove that the quantity  $\|\mathcal{B}_1\|_{L^\infty} + \|\mathcal{B}_2\|_{L^\infty}$  is bounded. One can easily check that

$$\lambda \mathcal{E} - \mathcal{D} = \frac{\sqrt{6\alpha\beta g h^7}}{12} Q^2 + \mathcal{B}_1, \quad -\mu \mathcal{E} + \mathcal{D} = \frac{\sqrt{6\alpha\beta g h^7}}{12} P^2 + \mathcal{B}_2. \quad (37)$$

Since

$$\lambda - \mu = 2 \sqrt{\frac{\alpha-1}{\alpha}} g h \geq 2 \sqrt{\frac{\alpha-1}{\alpha}} g \|h^{-1}\|_{L^\infty}^{-\frac{1}{2}} > 0,$$

then  $x_1 < x_2$ . Integrating (3) on the set  $\{(s, x), s \in [0, t], X_{x_1}(s) \leq x \leq Y_{x_2}(s)\}$ , using the divergence theorem, the energy equation (12) and (37) one obtains (36).  $\square$

#### 4. BLOW-UP CRITERIA

The aim of this section is to prove Theorem 2, for that purpose, we consider  $s \geq 2$  and  $(h - \bar{h}, u)$  the solution of (9) given by Theorem 1 with the initial data  $(h_0 - \bar{h}, u_0)$  and we start by the following lemmas.

**Lemma 6.** *For  $T_{max} < +\infty$ , we consider the following properties*

$$\sup_{(t,x) \in [0, T_{max}] \times \mathbb{R}} u_x(t, x) < +\infty, \quad (38a)$$

$$\inf_{(t,x) \in [0, T_{max}] \times \mathbb{R}} h(t, x) > 0, \quad (38b)$$

$$\|(h, u)\|_{L^\infty([0, T_{max}] \times \mathbb{R})} < +\infty. \quad (38c)$$

Then, (38a)  $\iff$  (38b) and (38b)  $\implies$  (38c).

*Proof.* The proof of (38b)  $\implies$  (38c) follows directly from the conservation of the energy (12) and the embedding  $H^1 \hookrightarrow L^\infty$ .

Let the characteristic  $Z_a$  starting from  $a$  defined as the solutions of the ordinary differential equation

$$\frac{d}{dt} Z_a(t) = u(t, Z_a(t)), \quad Z_a(0) = a. \quad (39)$$

Denoting the derivatives along the characteristics with speed  $u$  by  $\frac{d^u}{dt}$  and using the conservation of the mass (9a) one obtains

$$\frac{d^u}{dt} h \stackrel{\text{def}}{=} h_t + u h_x = -u_x h, \quad \implies \quad h \geq \left( \inf_{x \in \mathbb{R}} h_0 \right) e^{-\sup_{t,x} u_x(t,x) T_{max}}. \quad (40)$$

The proof of (38a)  $\implies$  (38b) follows directly from the last inequality.

It only remains to prove the converse ((38b)  $\implies$  (38a)). Using the Young inequality

$$\pm P Q \leq \frac{3}{8} P^2 + \frac{2}{3} Q^2, \quad \pm P Q \leq \frac{3}{8} Q^2 + \frac{2}{3} P^2, \quad (41)$$

integrating (35a), (35b) along the characteristics, and using (25), (36) one obtains the existence of  $\tilde{A} > 0, \tilde{B} > 0$  which depend on  $\beta, \bar{h}, \|(h, u, h^{-1})\|_{L^\infty([0, T] \times \mathbb{R})}$  and  $\int \mathcal{E} dx$ , such

that

$$P(t, X_{x_1}(t)) \leq P_0(x_1) + \tilde{A}t + \tilde{B} \quad \forall (t, x_1) \in [0, T] \times \mathbb{R}, \quad (42a)$$

$$Q(t, Y_{x_2}(t)) \leq Q_0(x_2) + \tilde{A}t + \tilde{B} \quad \forall (t, x_2) \in [0, T] \times \mathbb{R}. \quad (42b)$$

The last inequalities imply that

$$\sup_{(t,x) \in [0, T_{max}] \times \mathbb{R}} P(t, x) < +\infty, \quad \text{and} \quad \sup_{(t,x) \in [0, T_{max}] \times \mathbb{R}} Q(t, x) < +\infty, \quad (43)$$

then, (38a) follows directly from (32).  $\square$

**Lemma 7.** *For  $T_{max} < +\infty$ , we consider the following properties*

$$\inf_{(t,x) \in [0, T_{max}] \times \mathbb{R}} u_x(t, x) > -\infty, \quad (44a)$$

$$\|h_x\|_{L^\infty([0, T_{max}] \times \mathbb{R})} < +\infty. \quad (44b)$$

Then, (38b)  $\implies$  ((44a)  $\iff$  (44b)).

*Proof.* We suppose that (38b) and (44a) are satisfied. Using Lemma 6 one obtains that  $\|u_x\|_{L^\infty}$  is bounded. Then (44b) follows directly from (43) and

$$h_x = \frac{\sqrt{\alpha} h^{\frac{1}{2}}}{\sqrt{(\alpha-1)g}} (u_x - Q) = \frac{\sqrt{\alpha} h^{\frac{1}{2}}}{\sqrt{(\alpha-1)g}} (P - u_x). \quad (45)$$

To prove the converse, we suppose that (38b) and (44b) are satisfied, then, using the Young inequality  $\pm ab \geq -\frac{1}{2}a^2 - \frac{1}{2}b^2$ , (35a) and

$$P^2 = Q^2 + 4\sqrt{\frac{\alpha-1}{\alpha}} g h^{-\frac{1}{2}} u_x h_x = Q^2 + 2\sqrt{\frac{\alpha-1}{\alpha}} g h^{-\frac{1}{2}} h_x (P + Q) \quad (46)$$

one obtains

$$\begin{aligned} \frac{d^\lambda}{dt} P &\geq -\frac{7}{8} P^2 - \frac{1}{8} Q^2 - 3\alpha^{-1} h^{-3} \mathcal{R} \\ &= -Q^2 - \frac{7}{4} \sqrt{\frac{\alpha-1}{\alpha}} g h^{-\frac{1}{2}} h_x P - \frac{7}{4} \sqrt{\frac{\alpha-1}{\alpha}} g h^{-\frac{1}{2}} h_x Q - 3\alpha^{-1} h^{-3} \mathcal{R} \\ &\geq -\frac{7}{4} \sqrt{\frac{\alpha-1}{\alpha}} g h^{-\frac{1}{2}} h_x P - \frac{3}{2} Q^2 - \frac{49}{32} \frac{\alpha-1}{\alpha} g h^{-1} h_x^2 - 3\alpha^{-1} h^{-3} \mathcal{R}. \end{aligned}$$

Using (44b), (25), (36) and Gronwall lemma, we obtain that  $\inf_{t,x} P > -\infty$ . Using again (44b) we obtain (44a).  $\square$

Now, we can prove Theorem 2. Note that Lemma 6 implies the equivalence between (16) and (17). Then, it only remains to prove (16). Step 1 is devoted to prove the blow-up criterion (15). The proof of (16) is given in Step 2.

*Proof of Theorem 2.*

**Step 1:** In order to prove (15), we suppose that  $\|(h_x, u_x)\|_{L^\infty([0, T_{max}] \times \mathbb{R})} < +\infty$ , (38b) and we prove that if  $T_{max} < +\infty$ , then  $\|(h - \bar{h}, u)\|_{L^\infty([0, T_{max}], H^s(\mathbb{R}))} < +\infty$  which contradicts with the definition of  $T_{max}$ . For that purpose, we define

$$\begin{aligned} W &\stackrel{\text{def}}{=} (h - \bar{h}, u)^\top \quad A(W) \stackrel{\text{def}}{=} \begin{pmatrix} \frac{\alpha-1}{\alpha} & 0 \\ 0 & h \end{pmatrix}, \quad B(W) \stackrel{\text{def}}{=} \begin{pmatrix} u & h \\ \frac{\alpha-1}{\alpha} g & u \end{pmatrix}, \\ \mathcal{F}(W) &\stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ -\mathcal{L}_h^{-1} \partial_x \left\{ \frac{2}{3} \alpha h^3 u_x^2 - \frac{1}{4} \beta g h^2 h_x^2 + \frac{g}{2\alpha} h^2 \right\} \end{pmatrix}, \end{aligned}$$

the system (9) becomes

$$W_t + B(W) W_x = \mathcal{F}(W). \quad (47)$$

Let  $(\cdot, \cdot)$  be the scalar product in  $L^2$  and  $E(W) \stackrel{\text{def}}{=} (\Lambda^s W, A \Lambda^s W)$ . Since  $AB$  is a symmetric matrix, straightforward calculations with (47) show that

$$\begin{aligned} E(W)_t &= -2 ([\Lambda^s, B] W_x, A \Lambda^s W) - 2 (B \Lambda^s W_x, A \Lambda^s W) \\ &\quad - 2 (\Lambda^s \mathcal{F}, A \Lambda^s W) + (\Lambda^s W, A_t \Lambda^s W) \\ &= -2 ([\Lambda^s, B] W_x, A \Lambda^s W) + (\Lambda^s W, (AB)_x \Lambda^s W) \\ &\quad - 2 (\Lambda^s \mathcal{F}, A \Lambda^s W) + (\Lambda^s W, A_t \Lambda^s W) \end{aligned} \quad (48)$$

Defining  $\bar{B} \stackrel{\text{def}}{=} B(\bar{h}, 0)$ , and using (20), (18) one obtains

$$\begin{aligned} |([\Lambda^s, B] W_x, A \Lambda^s W)| &\leq C \|A\|_{L^\infty} \|W\|_{H^s} (\|B_x - \bar{B}\|_{L^\infty} \|W_x\|_{H^{s-1}} + \|B - \bar{B}\|_{H^s} \|W_x\|_{L^\infty}) \\ &\leq \tilde{C}_1 \|W\|_{H^s}^2, \end{aligned}$$

where  $\tilde{C}_1$  is a positive constant that depends on  $\|W\|_{W^{1,\infty}}$  and  $\|h^{-1}\|_{L^\infty}$ . Using the conservation of the mass (9a) one obtains that

$$|(\Lambda^s W, (AB)_x \Lambda^s W)| + |(\Lambda^s W, A_t \Lambda^s W)| \leq \tilde{C}_2 \|W\|_{H^s}^2. \quad (49)$$

From the energy conservation (12), it is clear that

$$\left\| \frac{2}{3} \alpha h^3 u_x^2 - \frac{1}{4} \beta g h^2 h_x^2 + g \frac{h^2 - \bar{h}^2}{2\alpha} \right\|_{L^1} \leq \tilde{C}_3 \quad (50)$$

Then, using (21), (24), (19) and (18) we obtain

$$\|\mathcal{F}\|_{H^s} \leq \tilde{C}_4 \|W\|_{H^s}. \quad (51)$$

Since  $\|(h, h^{-1})\|_{L^\infty}$  is bounded, then,  $\|W\|_{H^s} \leq \tilde{C}_5 E(W)$ . Combining all the estimates above, one obtains

$$E(W)_t \leq \tilde{C} E(W), \quad (52)$$

where  $\tilde{C}$  does not depend on  $\|W\|_{H^s}$ . Then, Gronwall lemma implies that

$$\|(h - \bar{h}, u)\|_{L^\infty([0, T_{max}], H^s(\mathbb{R}))} \leq \tilde{C}_5 \|E(W)\|_{L^\infty([0, T_{max}])} < +\infty.$$

This ends the proof of (15).

**Step 2:** It remains to prove (16). We suppose that  $T_{max} < +\infty$  and (38b) is satisfied. The blow-up criterion (15) (previous step) insures that

$$\limsup_{t \rightarrow T_{max}} \|u_x\|_{L^\infty} = +\infty \quad \text{or} \quad \limsup_{t \rightarrow T_{max}} \|h_x\|_{L^\infty} = +\infty.$$

Lemma 6 implies that  $u_x$  is bounded from above, then

$$\liminf_{t \rightarrow T_{max}} \inf_{x \in \mathbb{R}} u_x(t, x) = -\infty \quad \text{or} \quad \limsup_{t \rightarrow T_{max}} \|h_x\|_{L^\infty} = +\infty.$$

Finally, Lemma 7, insures that if one of the quantities above blows-up, then the other one should also blow-up at the same time. Then

$$\liminf_{t \rightarrow T_{max}} \inf_{x \in \mathbb{R}} u_x(t, x) = -\infty \quad \text{and} \quad \limsup_{t \rightarrow T_{max}} \|h_x\|_{L^\infty} = +\infty. \quad \square$$

## 5. BLOW-UP RESULTS

The goal of this section is to prove Theorem 3, for that purpose, we consider smooth solutions with small energy. In the following proposition we prove that if the energy is small enough, then, the quantity  $\|(h, u, h^{-1})\|_{L^\infty}$  is uniformly bounded.

**Proposition 1.** *For  $\beta > 0$ ,  $\bar{h} > 0$ , let  $E$  be a positive number such that*

$$0 < E < \frac{g\sqrt{\beta}}{3\sqrt{2}} \bar{h}^3,$$

*Defining*

$$\begin{aligned} h_{min} &\stackrel{\text{def}}{=} \bar{h} - \sqrt{\frac{3\sqrt{2}E}{g\sqrt{\beta}\bar{h}}}, & h_{max} &\stackrel{\text{def}}{=} \bar{h} + \sqrt{\frac{3\sqrt{2}E}{g\sqrt{\beta}\bar{h}}}, \\ u_{max} &\stackrel{\text{def}}{=} -u_{min} \stackrel{\text{def}}{=} \sqrt{\max\{h_{min}^{-1}, 3\alpha^{-1}h_{min}^{-3}\}E}. \end{aligned}$$

*Then, for any  $(h - \bar{h}, u) \in H^1$ , if  $\int \mathcal{E} dx \leq E$ , we have*

$$0 < h_{min} \leq h \leq h_{max} < 2\bar{h}, \quad u_{min} \leq u \leq u_{max}, \quad (53)$$

**Remark 5.** *The conservation of the energy (12) and Proposition 1 insure that if the initial data satisfy  $\int \mathcal{E}_0 dx \leq E$ , then, as long the the solution exist, the quantity  $\|(h, u, h^{-1})\|_{L^\infty}$  is bounded by a constant that depends only on  $g, \gamma, \bar{h}$  and  $E$ . Thence, all the constants given in (23), (24), (25), (36) and (42) are universal and do not depend on the initial data and the solution.*

*Proof of Proposition 1.* The Young inequality  $\frac{1}{2}a^2 + \frac{1}{2}b^2 \geq \pm ab$  implies that

$$\begin{aligned}
E &\geq \int_{\mathbb{R}} \mathcal{E} \, dy \geq \int_{\mathbb{R}} \left( \frac{1}{2} g (h - \bar{h})^2 + \frac{1}{4} \beta g h^2 h_x^2 \right) dx \\
&\geq g \sqrt{\frac{\beta}{2}} \left( \int_{-\infty}^x (h - \bar{h}) h h_x \, dy - \int_x^{+\infty} (h - \bar{h}) h h_x \, dy \right) \\
&\geq \frac{g}{3} \sqrt{\frac{\beta}{2}} (h - \bar{h})^2 (2h + \bar{h}) \\
&\geq \frac{g}{3} \sqrt{\frac{\beta}{2}} \bar{h} (h - \bar{h})^2,
\end{aligned}$$

which implies that  $h_{\min} \leq h \leq h_{\max}$ . Doing the same estimates with  $u$  one obtains

$$\begin{aligned}
E &\geq \int_{\mathbb{R}} \mathcal{E} \, dy \geq \int_{\mathbb{R}} \left( \frac{1}{2} h u^2 + \frac{1}{6} \alpha h^3 u_x^2 \right) dy \\
&\geq \min \left\{ h_{\min}, \frac{1}{3} \alpha h_{\min}^3 \right\} \left( \int_{-\infty}^x u u_x \, dy - \int_x^{+\infty} u u_x \, dy \right) \\
&\geq \min \left\{ h_{\min}, \frac{1}{3} \alpha h_{\min}^3 \right\} |u|^2,
\end{aligned}$$

the last inequality ends the proof of  $u_{\min} \leq u \leq u_{\max}$ .  $\square$

*Proof of Theorem 3.* Since the proofs of the two parts of Theorem 3 are the same, we only prove the first part.

Let  $T > 0$ , and let  $\tilde{A}, \tilde{B}$  be the constants given in (42). From (25) we obtain that  $|\alpha^{-1} h^{-3} \mathcal{R}| \leq \tilde{C}^2$  for some  $\tilde{C} > 0$ . If the initial data satisfy  $\int \mathcal{E}_0 \, dx \leq E$ , then, the constants  $\tilde{A}, \tilde{B}$  and  $\tilde{C}$  are universal and depend only on  $g, \beta, \bar{h}$  and  $E$  (Remark 5). We choose  $\tilde{T}$  and  $\tilde{D}$  such that

$$0 < \tilde{T} \leq T, \quad 3\tilde{C}\tilde{T} \leq \frac{\pi}{4}, \quad \tilde{D} \stackrel{\text{def}}{=} 4 \max \left\{ \tilde{A}^2, \tilde{B}^2, \tilde{C}^2 \right\}, \quad (54)$$

and we choose the initial data  $(h_0 - \bar{h}, u_0) \in C_c^\infty(\mathbb{R})$  such that there exist  $x_1 \in \mathbb{R}$  satisfying

$$\int_{\mathbb{R}} \mathcal{E}_0 \, dx \leq E, \quad Q_0 \equiv 0, \quad P_0(x_1) < -2\sqrt{\tilde{D}}(\tilde{T} + 1), \quad P_0(x_1) < -\frac{8}{\tilde{T}}. \quad (55)$$

Let  $t < \min\{\tilde{T}, T_{\max}\}$ , then (35b) with the Young inequality  $PQ \geq -\frac{3}{8}P^2 - \frac{2}{3}Q^2$  imply that

$$\frac{d^\mu}{dt} Q \geq -\frac{25}{24}Q^2 - 3\alpha^{-1}h^{-3}\mathcal{R} \geq -3(Q^2 + \tilde{C}^2). \quad (56)$$

The last inequality with (42b) imply that for all  $x_1 \in \mathbb{R}$ , we have

$$-\tilde{C} \leq -\tilde{C} \tan(3\tilde{C}t) \leq Q(t, Y_{x_2}(t)) \leq \tilde{A}\tilde{T} + \tilde{B}, \quad (57)$$

which implies that  $Q$  cannot blow-up before  $t = \min \left\{ \tilde{T}, T_{max} \right\}$  and  $\|Q\|_{L^\infty} \leq \sqrt{\tilde{D}/4}(\tilde{T} + 1)$ . Using (35a) and the Young inequality  $PQ \leq \frac{1}{8}P^2 + 2Q^2$  one obtains

$$\begin{aligned} \frac{d^\lambda}{dt} P(t, X_{x_1}(t)) &\leq -\frac{1}{4}P(t, X_{x_1}(t))^2 + \frac{19}{8}Q(t, X_{x_1}(t))^2 + 3\tilde{C}^2 \\ &\leq -\frac{1}{4}P(t, X_{x_1}(t))^2 + \tilde{D}(\tilde{T} + 1)^2 \\ &\leq -\frac{1}{8}P(t, X_{x_1}(t))^2. \end{aligned}$$

The last inequality follows from  $P(t, X_{x_1}(t))^2 \geq 4\tilde{D}(\tilde{T} + 1)^2$ , which is true initially and holds because the map  $t \mapsto P(t, X_{x_1}(t))$  is decreasing and negative. Then  $T_{max} \leq \tilde{T} \leq T$  and from (42a) we obtain that

$$\inf_{[0, T_{max}] \times \mathbb{R}} P(t, x) = -\infty, \quad \sup_{[0, T_{max}] \times \mathbb{R}} P(t, x) < +\infty, \quad \sup_{[0, T_{max}] \times \mathbb{R}} |Q(t, x)| < +\infty. \quad \square$$

#### REFERENCES

- [1] J. S. Antunes Do Carmo. Boussinesq and Serre type models with improved linear dispersion characteristics: Applications. *Journal of hydraulic research.*, 51(6):719–727, 2013.
- [2] J. S. Antunes Do Carmo. Extended Serre equations for applications in intermediate water depths. *The Open Ocean Engineering Journal.*, 6(1):, 2013.
- [3] O. Castro-Orgaz and W. H. Hager. Boussinesq-and Serre-type models with improved linear dispersion characteristics: applications. *Journal of Hydraulic Research.*, 53(2): 282–284, 2015.
- [4] D. Clamond, D. Dutykh and D. Mitsotakis. Conservative modified Serre–Green–Naghdi equations with improved dispersion characteristics. *Comm. Nonlin. Sci. Numer. Simul.*, 45:245–257, 2017.
- [5] D. Clamond and D. Dutykh. Non-dispersive conservative regularisation of nonlinear shallow water (and isentropic Euler) equations. *Comm. Nonlin. Sci. Numer. Simul.*, 55:237–247, 2018.
- [6] A. Constantin and L. Molinet. The initial value problem for a generalized Boussinesq equation. *Differential and Integral equations.*, 15(9):1061–1072, 2002.
- [7] F. Dias and P. Milewski. On the fully-nonlinear shallow-water generalized Serre equations. *Physics Letters A.*, 374(8):1049–1053, 2010.
- [8] D. Dutykh, D. Clamond and D. Mitsotakis. Adaptive modeling of shallow fully nonlinear gravity waves. *RIMS Kokyuroku.*, 1947(4): 45–65, 2015.
- [9] D. Dutykh, M. Hoefer and D. Mitsotakis. Solitary wave solutions and their interactions for fully nonlinear water waves with surface tension in the generalized Serre equations. *Theoretical and Computational Fluid Dynamics.*, 32(3): 371–397, 2018.
- [10] R. T. Glassey, J. K. Hunter and Y. Zheng. Singularities of a variational wave equation. *Journal of differential equations.*, 129(1): 49–78, 1996.

- [11] B. Guelmame, D. Clamond, and S. Junca. Hamiltonian regularisation of the unidimensional barotropic Euler equations. *Preprint, hal-02907304*, 2020.
- [12] B. Guelmame, D. Clamond, and S. Junca. Local well-posedness of a Hamiltonian regularisation of the Saint-Venant system with uneven bottom. *Preprint, hal-02915262*, 2020.
- [13] S. Israwi. Large time existence for 1D Green–Naghdi equations. *Nonlinear Analysis*, 74(1):81–93, 2011.
- [14] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier–Stokes equations. *Communications on Pure and Applied Mathematics.*, 41(7):891–907, 1988.
- [15] D. Lannes. The Water Waves Problem: Mathematical Analysis and Asymptotics. *American Mathematical Society, AMS.*, 2013.
- [16] Y. A. Li. Linear stability of solitary waves of the Green–Naghdi equations. *Comm. Pure and Applied mathematics*, 54(5):501–536, 2001.
- [17] J.-L. Liu, R. L. Pego, and Y. Pu. Well-posedness and derivative blow-up for a dispersionless regularized shallow water system. *Nonlinearity.*, 32(11):4346–4376, 2019.
- [18] Z. B. Liu and Z. C. Sun. Two sets of higher-order Boussinesq-type equations for water waves. *Ocean engineering.*, 32(11-12):1296–1310, 2005.
- [19] D. Mitsotakis, D. Dutykh, A. Assylbekuly and D. Zhakebayev. On weakly singular and fully nonlinear travelling shallow capillary-gravity waves in the critical regime. *Physics Letters A.*, 381(20):1719–1726, 2017.
- [20] Y. Pu, R. L. Pego, D. Dutykh, and D. Clamond. Weakly singular shock profiles for a non-dispersive regularization of shallow-water equations. *Comm. Math. Sci.*, 16(5):1361–1378, 2018.

(**Billel Guelmame**) LJAD, INRIA, UNIVERSITÉ CÔTE D’AZUR, FRANCE.  
*Email address:* `billel.guelmame@univ-cotedazur.fr`