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Mariia Vladimirova, Julyan Arbel, Stéphane Girard

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Mariia Vladimirova[1] & Julyan Arbel[1] & Stéphane Girard[1]

[1] *Univ. Grenoble Alpes, Inria, CNRS, LJK, 38000 Grenoble, France*
{mariia.valdimirova, julyan.arbel, stephane.girard}@inria.fr

Résumé. Des variables aléatoires à queue de type Weibull sont définies comme des variables aléatoires positives dont la queue droite se comporte de manière similaire à celle d'une variable aléatoire de Weibull. Nous introduisons des variables aléatoires de queue Weibull généralisées, des extensions qui ont une propriété de stabilité. Nous étudions la préservation des paramètres de queue de ces variables aléatoires sous des opérations telles que la multiplication, la puissance et la combinaison linéaire.

Mots-clés. Lois à queue de type Weibull, Lois généralisé à queue de type Weibull, Sous-Weibull, Stabilité

Abstract. Weibull-tail random variables are defined as non-negative random variables whose right tail behaves similarly to that of a Weibull random variable. We introduce generalized Weibull-tail random variables, extensions which have a property of stability. We study the preservation of tail parameters under operations like multiplication, power, and linear combination.

Keywords. Weibull-tail, Generalized Weibull-tail, Sub-Weibull, Stability

1 Introduction

The study of the distributional tail behavior arises in many applied probability models of different areas, such as hydrology (Strupczewski et al., 2011), finance (Rachev, 2003) and insurance risk theory (Ahmad et al., 2020). Since in most cases exact distributions are not available, deriving asymptotic relationships for their tail probabilities becomes essential. In this context, an important role is played by so-called Weibull-tail distributions (Gardes et al., 2011; Gardes and Girard, 2016).

A random variable X is called *Weibull-tail* with tail parameter $\beta > 0$ if its cumulative distribution function F satisfies

$$\overline{F}(x) = 1 - F(x) = \mathrm{e}^{-x^{\beta} l(x)}, \quad \text{for } x > 0, \tag{1}$$

where $l(x)$ is a slowly-varying function, i.e. it is a positive function such that

$$\lim_{x \to \infty} \frac{l(tx)}{l(x)} = 1, \quad \text{for all } t > 0.$$

We note $X \sim \mathrm{WT}(\beta)$. The family of Weibull-tail distributions includes a variety of fundamental distributions such as Gaussian ($\beta = 2$), exponential and gamma ($\beta = 1$), Weibull ($\beta > 0$), to name a few.

Some of the commonly used techniques to study the tail behavior is to consider probability tail bounds such as sub-Gaussian, sub-Exponential, or their generalization to sub-Weibull distributions (Vladimirova et al., 2020; Kuchibhotla and Chakrabortty, 2018). A non-negative random variable is called sub-Weibull with tail parameter $\theta > 0$ if its survival function is upper-bounded by that of a Weibull distribution:

$$\overline{F}(x) \leq a\mathrm{e}^{-bx^{1/\theta}}, \quad \text{for } x > 0 \text{ and some } a, b > 0. \tag{2}$$

This property ensures the existence of the moment generating function as well as bounds on moments. In contrast, the Weibull-tail properties characterize the survival or density functions without a hand on moments.

While tail parameters in Equation (1) and (2) of Weibull-tail and sub-Weibull properties are different, we can find some connections. Notice that for any constants $a, b > 0$ there exists a slowly-varying function $l(x) = b - \frac{\log a}{x^\beta}$ so that $a\mathrm{e}^{-bx^\beta} = \mathrm{e}^{-x^\beta l(x)}$. It means that if random variable X is sub-Weibull with parameter $\theta = 1/\beta > 0$, satisfying Equation (2), then survival function of X is upper-bounded by a Weibull-tail distribution with tail parameter β, satisfying Equation (1). If random variable X is Weibull-tail with tail parameter β, then from Potter's bounds, for $a_1, a_2 > 0$ we have

$$a_1\mathrm{e}^{-x^{\beta_1}} \leq \overline{F}(x) = \mathrm{e}^{-x^\beta l(x)} \leq a_2\mathrm{e}^{-x^{\beta_2}},$$

or $\mathrm{WT}(\beta) \subset \mathrm{SubW}(1/\beta_2)$ and $\mathrm{WT}(\beta) \not\subset \mathrm{SubW}(1/\beta_1)$ for x big enough and $\forall(\beta_1, \beta_2)$ such that $0 < \beta_2 < \beta < \beta_1$.

In this work, we study the properties of Weibull-tail random variables and introduce their stable extensions, i.e. generalized Weibull-tail random variables. Firstly, we show that multiplication by a constant doesn't change a random variable tail parameter (Lemma 2.1). A power of a Weibull-tail and generalized Weibull-tail random variable results into a distribution with a tail parameter divided by the power (Lemma 2.1). Further, Theorem 2.1 confirms that a sum of generalized Weibull-tail random variables (including the dependent ones) remains generalized Weibull-tail of tail parameter equal to the minimum among those of the terms. In addition, we consider a product of independent generalized Weibull-tail random variables in Theorem 2.2.

2 Weibull-tail properties

We begin by introducing the concept of a generalized random variable of the Weibull tail, which has an additional stability property:

Definition 2.1 (Generalized Weibull-tail)**.** A random variable X is called *generalized Weibull-tail* with tail parameter $\beta > 0$ if its survival function $\overline{F}$ is bounded by Weibull-tail functions of tail parameter β with possibly different slowly-varying functions l_1 and l_2:

$$\mathrm{e}^{-x^\beta l_1(x)} \leq \overline{F}(x) \leq \mathrm{e}^{-x^\beta l_2(x)}, \text{ for } x > 0. \tag{3}$$

We note $X \sim \mathrm{GWT}(\beta)$.

If $X \sim \mathrm{WT}(\beta)$ with slowly-varying function l, then by taking $l_1 = l_2 = l$, we have equality in Equation (3) and $X \sim \mathrm{GWT}(\beta)$. Therefore, Weibull-tail family of distributions is a subset of generalized Weibull-tail distributions of the same tail parameter: $\mathrm{WT}(\beta) \subset \mathrm{GWT}(\beta)$.

Lemma 2.1 (Power and multiplication by a constant)**.** *Let non-negative random variable X be Weibull-tail (generalized Weibull-tail) with tail parameter β, then aX^b is Weibull-tail (generalized Weibull-tail) with tail parameter $\frac{\beta}{b}$ for $a, b > 0$.*

Proof. For $a, b > 0$, the tail of $Y = aX^b$ is $\mathbb{P}\Big(aX^b \geq y\Big) = \mathbb{P}\Big(X \geq \big(\frac{y}{a}\big)^{1/b}\Big)$.

If X is Weibull-tail with tail parameter β, then $\mathbb{P}\left(X \geq x\right) = \mathrm{e}^{-x^\beta l(x)}$, where l is a slowly-varying function. It implies

$$\mathbb{P}\Big(aX^b \geq y\Big) = \mathrm{e}^{-y^{\beta/b}\tilde{l}(y)},$$

where $\tilde{l}(y) = \frac{l((y/a)^{1/b})}{a^{\beta/b}}$ is a slowly-varying function.

If X is generalized Weibull-tail with tail parameter β, then $\mathrm{e}^{-x^\beta l_1(x)} \leq \mathbb{P}\left(X \geq x\right) \leq \mathrm{e}^{-x^\beta l_2(x)}$, where l_1 and l_2 are slowly-varying functions. It implies

$$\mathrm{e}^{-y^{\beta/b}\tilde{l}_1(y)} \leq \mathbb{P}\Big(aX^b \geq y\Big) \leq \mathrm{e}^{-y^{\beta/b}\tilde{l}_2(y)},$$

where $\tilde{l}_i(y) = \frac{l_i((y/a)^{1/b})}{a^{\beta/b}}, i = 1, 2$ are slowly-varying functions. □

Theorem 2.1 (Sum of generalized Weibull-tail RVs)**.** *Let non-negative (possibly dependent) random variables X and Y be generalized Weibull-tail of parameters β_x and β_y. Then, $X + Y$ is generalized Weibull-tail of the parameter* $\min\{\beta_x, \beta_y\}$.

Proof. For any two distributions X and Y we have an upper bound

$$\mathbb{P}(X + Y \geq z) \leq \mathbb{P}(X \geq z/2 \cup Y \geq z/2) \leq \mathbb{P}(X \geq z/2) + \mathbb{P}(Y \geq z/2).$$

For non-negative random variables X and Y we have $\mathbb{P}(X + Y \geq z) \geq \mathbb{P}(X \geq z)$ and $\mathbb{P}(X + Y \geq z) \geq \mathbb{P}(Y \geq z)$. By combining these two inequalities we have a lower bound $\mathbb{P}(X + Y \geq z) \geq \max\{\mathbb{P}(X \geq z), \mathbb{P}(Y \geq z)\}$.

Thus, for survival function $\overline{F}_\Sigma$ of sum of random variables X and Y with survival functions $\overline{F}_X$ and $\overline{F}_Y$, the following inequality holds

$$\max\{\overline{F}_X(z), \overline{F}_Y(z)\} \leq \overline{F}_\Sigma(z) \leq 2\max\{\overline{F}_X(z/2), \overline{F}_Y(z/2)\}. \tag{4}$$

If those random variables are generalized Weibull-tail with tail parameters β_x and β_y, then there exist slowly-varying functions $l_1 = \max\{l_1^x, l_1^y\}$ and $l_2 = \min\{l_2^x, l_2^y\}$ with $l_1^x, l_2^x, l_1^y, l_2^y$ being slowly-varying function in the lower and upper bounds of generalized Weibull-tail X and Y respectively, such that

$$\mathrm{e}^{-z^\beta l_1(z)} \leq \max\{\overline{F}_X(z), \overline{F}_Y(z)\} \leq \mathrm{e}^{-z^\beta l_2(z)},$$

where $\beta = \min\{\beta_x, \beta_y\}$. Hence, Equation (4) is transformed into

$$\mathrm{e}^{-z^\beta l_1(z)} \leq \overline{F}_\Sigma(z) \leq \mathrm{e}^{-z^\beta \tilde{l}_2(z)},$$

where $\tilde{l}_2(z) = \frac{l_2(z/2)}{2^\beta}\left(1 - \frac{2^\beta \log 2}{z^\beta}\right)$ is slowly-varying. Thus, the survival function of the sum $X + Y$ is upper and lower-bounded by some Weibull-tail functions from family $\mathrm{WT}(\beta)$ where $\beta = \min\{\beta_x, \beta_y\}$. □

Theorem 2.2 (Product of generalized Weibull-tail RVs)**.** *The product of two independent non-negative generalized Weibull-tail random variables X and Y with tail parameters β_x and β_y is generalized Weibull-tail with tail parameter β such that $\frac{1}{\beta} = \frac{1}{\beta_x} + \frac{1}{\beta_y}$.*

Proof. Consider two independent non-negative generalized Weibull-tail random variables with tail parameters $\beta_x = \frac{1}{\theta_x}$ and $\beta_y = \frac{1}{\theta_y}$, $X \sim \mathrm{GWT}\left(\frac{1}{\theta_x}\right)$ and $Y \sim \mathrm{GWT}\left(\frac{1}{\theta_y}\right)$.

We want to prove that for some slowly-varying functions l_1 and l_2 the following holds for the product survival function :

$$\mathrm{e}^{-z^{\frac{1}{\theta_x+\theta_y}} l_1(z)} \leq \overline{F}_{XY}(z) = \mathbb{P}(XY \geq z) \leq \mathrm{e}^{-z^{\frac{1}{\theta_x+\theta_y}} l_2(z)}. \tag{5}$$

1. *Upper bound.* Firstly, notice that from the concavity of the logarithm for any $u, v > 0$ and $p \in (0,1)$ we have $\ln(pu + (1-p)v) \geq p\ln u + (1-p)\ln v$. Then $pu + (1-p)v \geq u^p v^{1-p}$. The change of variables $x = u^p$, $y = v^{1-p}$ implies

$$px^{1/p} + (1-p)y^{1/(1-p)} \geq xy. \tag{6}$$

From Equation (6), the upper bound of the product tail is

$$\mathbb{P}(XY \geq z) \leq \mathbb{P}\left(pX^{1/p} + (1-p)Y^{1/(1-p)} \geq z\right).$$

Lemma 2.1 implies that $pX^{1/p} \sim \mathrm{GWT}\left(\frac{p}{\theta_x}\right)$ and $(1-p)Y^{1/1-p} \sim \mathrm{GWT}\left(\frac{1-p}{\theta_y}\right)$. Taking $p = \frac{\theta_x}{\theta_x+\theta_y}$ and $1-p = \frac{\theta_y}{\theta_x+\theta_y}$, on the right-hand side we have a sum of two

independent generalized Weibull-tail random variables with tail parameter $\frac{1}{\theta_x+\theta_y}$. According to Theorem 2.1, this sum is generalized Weibull-tail with the same tail parameter $\frac{1}{\theta_x+\theta_y}$. It means that there exists slowly-varying function l_2 such that the tail of product XY is upper-bounded by

$$\mathbb{P}(XY \geq z) \leq \mathrm{e}^{-z^{\frac{1}{\theta_x+\theta_y}} l_2(z)}. \tag{7}$$

2. *Lower bound.* By dependence of X and Y, we have

$$\mathbb{P}(XY \geq z) \geq \mathbb{P}\left(X \geq z^{\frac{\theta_x}{\theta_x+\theta_y}}\right) \mathbb{P}\left(Y \geq z^{\frac{\theta_y}{\theta_x+\theta_y}}\right).$$

Since X and Y are generalized Weibull-tail, we can define function

$$l_1(z) = l_1^x(z^{\theta_x/(\theta_x+\theta_y)}) + l_1^y(z^{\theta_y/(\theta_x+\theta_y)})$$

with l_1^x and l_1^y being slowly-varying functions in the lower bounds of generalized Weibull-tail X and Y. Then, l_1 is slowly-varying and we have

$$\mathbb{P}(XY \geq z) \geq \mathrm{e}^{-z^{\frac{1}{\theta_x+\theta_y}} l_1(z)}. \tag{8}$$

Combining together Equations (7) and (8) with Definition 2.1 implies the statement of the theorem. □

Since a Weibull-tail random variable is a particular case of a generalized Weibull-tail random variable, a sum or a product of Weibull-tail random variables will also give a generalized Weibull-tail random variable.

Example 2.1 (Sum of Gaussian and exponential RVs)**.** A convolution of Gaussian $\mathcal{N}(0, \sigma^2)$ and exponential $\mathrm{Exp}(\lambda)$ distributions can be written in the following form: $f_\Sigma(z) = \mathrm{e}^{-\lambda z + \frac{\lambda^2\sigma^2}{2} + \log\frac{\lambda}{2} + \log\left(1+\mathrm{erf}\left(\frac{z-\lambda\sigma^2}{\sqrt{2}\sigma}\right)\right)}$, where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-y^2} dy$ is the error function. Since $\mathrm{erf}\left(\frac{z-\lambda\sigma^2}{\sqrt{2}\sigma}\right) \to 1$ when $z \to \infty$, for big enough z the convolution $f_\Sigma(z) \sim \mathrm{e}^{-\lambda z + K}$ with $K > 0$. Then, there exists a slowly-varying function l such that $f_\Sigma(z) \sim \mathrm{e}^{-zl(z)}$. It means that the convolution of Gaussian and exponential distributions is WT(1), like the exponential one which is heavier among exponential and Gaussian.

3 Conclusion and discussion

We introduce a notion of a generalized Weibull-tail distribution, an extended version of Weibull-tail distribution with a property of stability. We showed that a sum of generalized Weibull-tail distributions (including dependent) is generalized Weibull-tail with tail parameter equal to a tail parameter of the heaviest distribution among them.

Here we considered only non-negative distributions, i.e. the *right* tail. The theory might be generalized to distributions on $\mathbb{R}$ by taking into account the *left* tails. In that case one can include asymmetric distributions where left and right tails have different tail parameters.

In application to Bayesian neural networks, Vladimirova et al. (2019) proved that hidden units, or neurons, follow a sub-Weibull distribution (Vladimirova et al., 2020; Kuchibhotla and Chakrabortty, 2018) where the tail parameter depends on depth. Future work will study the applicability of Weibull-tail properties to Bayesian neural networks.

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