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MULTIVARIATE HAWKES PROCESSES ON INHOMOGENEOUS RANDOM GRAPHS

ZOÉ AGATHE-NERINE

ABSTRACT. We consider a population of N interacting neurons, represented by a multivariate Hawkes process : the firing rate of each neuron depends on the history of the connected neurons. Contrary to the mean-field framework where the interaction occurs on the complete graph, the connectivity between particles is given by a random possibly diluted and inhomogeneous graphs where the probability of presence of each edge depends on the spatial position of its vertices. We address the well-posedness of this system and the behaviour as $N \rightarrow \infty$. A crucial issue will be to understand how spatial inhomogeneity influences the large time behaviour of the system.

Keywords. Multivariate nonlinear Hawkes processes, Mean-field systems, Neural networks, Spatially extended system, Random graph, Graph convergence.

AMS Classification. 60F15, 60G55, 44A35,92B20.

1. INTRODUCTION

1.1. **Biological and mathematical context.** Neurons are cells specialised in the reception, integration and transfer of information in the brain. A propagating electrical signal is transmitted from a neuron to the others in terms of all-or-none emission of action potential also called *spike* which is a stereotyped phenomenon. More precisely, neurons possess a permeable membrane which allows ion exchanges. Without stimulus, the difference of respective ion concentrations induces a voltage gradient called resting potential. This potential evolves depending on the information received from other neurons : a presynaptic neuron emitting a spike leads to the release of neurotransmitters, and induces a change in the ions distribution around the membrane of post-synaptic neurons. If the stimulus reaches a sufficient threshold, the neuron generates an action potential, the *synaptic integration*.

The progress of monitoring methods as MRI (Magnetic Resonance Imaging) and ECG (Electrocardiography) since the 50's led to a better understanding of the physiology of a neuron. As a result, the implementation of mathematical models started with the Hodgkin-Huxley model [39] (in 1952) describing the evolution of the membrane potential in terms of a system of four ODEs, further simplified in two equations by FitzHugh [33] and Nagumo [52] (in 1962).

Stochasticity is intrinsic to the neuronal activity : noise in neuronal systems may come from different sources. To name a few, randomness accounts for the emergence of spontaneous spikes [31], failed propagation [62], and the stochastic opening and closing of the ion channels (the probability of the channel being open or closed depends on the membrane potential). Stochasticity is also present at the scale of a whole population in the large variability of synaptic connections between neurons. From a mathematical perspective, this naturally led to diffusion models : mean-field Hodgkin-Huxley and FitzHugh-Nagumo's models in [3], mean-field Piecewise Deterministic Markov Processes (PDMP) in [48, 19].

Another popular model is the integrate-and-fire dynamic, first introduced in the seminal work of Lapicque [43], and still studied mathematically, as e.g. in [22].

The previous approach on the modeling of the membrane potential, and typically leads to non-linear Fokker–Planck equations whose large time behaviour is often hard to determine analytically. A usual approach in this context (that we follow here) leads to more tractable and explicitly solvable models : as spikes are stereotyped, all the information is coded in the duration of time between the spikes. Hence we model the activity of a neuron by a point process where each point represents the time of a spike. In this context, the framework of *Hawkes processes* is particularly relevant since it can account for the dependence of the activity of a neuron on the past of the whole population : the spike of one neuron can trigger others spikes. Hawkes processes have been first introduced in [34] in 1971 to model earthquakes, and have been thoroughly studied since (with applications to seismology [54] and finance [37]). It is not possible to account for the vast mathematical literature on Hawkes processes since the seminal works of [34, 35, 11], we refer nonetheless to [23, 38, 14] and references therein.

In this paper, the main issue we concentrate on is the structure of interaction between neurons. There is indeed experimental evidences that neurons are spatially organized [10, 51]. The first approach which does not take into account this spatial structure assumes a complete graph of interaction (mean-field framework). Mean-field analysis goes back to [49, 63], originally for diffusion models as in [3]. The litterature on mean-field analysis is huge and does not restrict to neuroscience applications (see the following references as far as neurosciences are concerned: integrate and fire models [22], PDMP [48, 19]). As for mean-field Hawkes processes, similar models have been considered in [23, 36, 38] and expanded with additional features (age dependence in [14, 59], inhibition in [20, 30, 60]). What makes the mean-field analysis for Hawkes processes particularly tractable is that the large population limit is given in terms of an inhomogeneous Poisson process whose intensity solves a convolution equation [23].

The spatial organization in the brain has been originally analysed mathematically from a phenomenological perspective : we may refer to the celebrated neural field equation [66, 1, 12], which has given a macroscopic description of excitable units with non-local interaction. Several works have extended the mean-field framework to take into account the presence of a macroscopic spatial structure in the interaction (originally for diffusion models [64, 47, 13], as well as for Hawkes processes [28, 15]). More specifically, [15] has given a mesoscopic interpretation of the neural field equation in terms of the limit of spatially extended Hawkes processes interacting through a mesoscopic spatial kernel.

The main contribution of this paper is to go further and provide a microscopic interpretation of this spatial structure in terms of random graphs. We assume that the interaction between neurons is given by a possibly inhomogeneous and diluted graph, where the probability of presence of an edge depends on the positions of its vertices. The main example that we have in mind concerns the class of W -random graph (see [26, 44, 41, 8, 9] for the theory behind), that includes homogeneous Erdős Rényi graphs. The only previous works so far on particle systems with similar interaction address the case of diffusions. Laws of Large Numbers (LLN) and large deviations results on homogeneous Erdős Rényi graphs have been considered in [24, 18, 55] and further extended to the inhomogenous case in [5, 46, 4, 50] on a bounded time interval. The behaviour of such systems on time scale that is no longer bounded (but may depend on the size of the population) is more difficult, and remains largely open so far (in this direction, see [17]). The present work is, to the best of our knowledge, the first paper to address similar issues to Hawkes processes.

We prove here quenched LLN results on a bounded time interval. The behaviour of the system on unbounded time scale will be the object of future works. Note also that all the present works address the case where the graph of interaction has diverging degrees. The case with sparse interaction (see [56, 42] in the diffusive case) remains open for Hawkes processes and will be the object of future works.

1.2. Our model. The aim of this paper is to describe the behavior in large population and large time of a network of particles interacting on a spatially structured random graph. Let N be the size of the population, consider the multivariate Hawkes process $\left(Z_1^{(N)}(t), \dots, Z_N^{(N)}(t)\right)_{t>0}$: for $i = 1 \dots N$, the i th neuron is located on $x_i \in I$ where $I \subset \mathbb{R}^d$ is a spatial domain, $Z_i^{(N)}(t)$ counts the number of spikes during the time interval $[0, t]$. Its conditional intensity at time t is given by

$$\lambda_i^{(N)}(t) = f \left(u_0(t, x_i) + \frac{1}{N} \sum_{j=1}^N w_{ij}^{(N)} \int_0^{t-} h(t-s) dZ_j^{(N)}(s) \right). \quad (1.1)$$

Here, $f : \mathbb{R} \rightarrow \mathbb{R}_+$ represents the synaptic integration, $u_0 : \mathbb{R}_+ \times I \rightarrow \mathbb{R}$ a spontaneous activity of the neuron, $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ a memory function which models how a past jump of the system affects the present intensity. The novelty here is $w_{ij}^{(N)}$, representing the random inhomogeneous interaction between the neurons i and j that depends on their positions x_i and x_j . I represents the spatial domain of the neuron (suppose e.g. that $I = [0, 1]$ or $I = \mathbb{R}^d$). We refer to Section 2 for precise definitions.

We study the behavior of the process $\left(Z_1^{(N)}(t), \dots, Z_N^{(N)}(t)\right)_{t>0}$ as $N \rightarrow \infty$ and $t \rightarrow \infty$. The large population limit is described in terms of an inhomogeneous Poisson process whose intensity involves the macroscopic spatial structure of the graph. Note that the large population convergence is considered for a fixed realization of the graph (quenched model). A second aspect of the present work of independent interest will be to analyse the long time dynamics of the macroscopic process. We generalise the phase transition already observed for mean-field linear Hawkes processes [23]. An important issue will be to understand how the inhomogeneity of the graph influences the long time dynamics. This will be illustrated by different examples and simulations.

1.3. Organisation of the paper. After introducing some general notations, we start in Section 2 by defining formally the process of interest (2.2). The well-posedness of such process is treated by Proposition 2.5. We study the large population behavior of the process $\left(Z_1^{(N)}(t), \dots, Z_N^{(N)}(t)\right)_{t>0}$ in Section 2.3. We show, under suitable hypotheses on the parameters, that the behavior of a neuron located in $x \in I$ in a infinite system is described by the intensity $\lambda(\cdot, x)$ solving

$$\lambda(t, x) = f \left(u_0(t, x) + \int_I W(x, y) \int_0^t h(t-s) \lambda(s, y) ds \nu(dy) \right). \quad (1.2)$$

Here, $W : I \times I \rightarrow \mathbb{R}_+$ is seen as the limit interaction kernel, and ν describes the macroscopic law of the distribution of the positions. Well-posedness and regularity of (1.2) is considered in Proposition 2.7. In Section 3, we study the behaviour of the process (2.2) in large population in Theorems 3.10 and 3.12. The behaviour of the empirical measure and respectively the spatial profile (Definition 3.16) is analysed in Section 3.4 (resp. Section 3.5). In Section 4, we study the behavior of (1.2) in the linear case, that

is when $f = Id$. We extend the phase transition observed without spatial structure seen in [23] to a general interaction kernel W . Finally in Section 5, we apply our results to concrete cases and present some simulations. The proofs are gathered in the last Sections.

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2. A SYSTEM OF N INTERACTING PARTICLES ON A GRAPH AND ITS LIMIT

2.1. Notation. We write $\|\cdot\|$ for the usual Euclidian norm in \mathbb{R}^n , $\|(x_1, \dots, x_n)\| = (|x_1|^2 + \dots + |x_n|^2)^{\frac{1}{2}}$. For (E, \mathcal{A}, μ) a measured space, for a function h in $L^p(E, \mu)$ with $p \geq 1$, we write $\|h\|_{E, \mu, p} := (\int_E |h|^p d\mu)^{\frac{1}{p}}$. When $p = 2$, we write as $\langle f, g \rangle_{E, \mu} = \int_E fgd\mu$ the scalar product. Without ambiguity, we may omit the notation (E, μ) or μ . For instance, for $T > 0$ and h in $L^p([0, T])$, we write $\|h\|_{[0, T], p} := (\int_0^T |h(t)|^p dt)^{\frac{1}{p}}$. When we omit the notation $[0, T]$, the integration is on \mathbb{R}_+ . For a real-valued bounded function g on a space E , we write $\|g\|_\infty := \|g\|_{E, \infty} = \sup_{x \in E} |g(x)|$. If d is a distance on E , we denote by $\|f\|_L = \sup_{x \neq y} |f(x) - f(y)|/d(x, y)$ the Lipchitz seminorm of a real-valued function f on E . We also denote by $\|f\|_{BL} := \|f\|_L + \|f\|_{E, \infty}$ the bounded Lipschitz norm of f . For μ and ν measures on E , we define

$$d_{BL}(\mu, \nu) := \sup_{g, \|g\|_{BL} \leq 1} \left| \int_E g(d\mu - d\nu) \right|. \quad (2.1)$$

We denote by $\mathbb{D}([0, T], \mathbb{N})$ the space of càdlàg (right continuous with left limits) functions defined on $[0, T]$ and taking values in \mathbb{N} .

2.2. The model.

2.2.1. Definitions. The graph of interaction of (1.1) is constructed as follows :

Definition 2.1. On a common probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \mathbb{P})$, we consider a sequence $(x_i^{(N)})_{N \geq 1, i \in \llbracket 1, N \rrbracket}$ of positions (possibly random) and a family of random variables $\xi^{(N)} = (\xi_{ij}^{(N)})_{N \geq 1, i, j \in \llbracket 1, N \rrbracket}$ on $\tilde{\Omega}$ such that under \mathbb{P} , for any $N \geq 1$ and $i, j \in \llbracket 1, N \rrbracket$, conditionned on the positions (x_1, \dots, x_N) , $\xi^{(N)}$ is a collection of mutually independent Bernoulli random variables such that for $1 \leq i, j \leq N$, $\xi_{ij}^{(N)}$ has parameter $W_N(x_i, x_j)$. We assume that the particles in (1.1) are connected according to the oriented graph $\mathcal{G}^{(N)} = (\{1, \dots, N\}, \xi^{(N)})$. For any i and j , $\xi_{ij}^{(N)} = 1$ encodes for the presence of the edge $j \rightarrow i$ and $\xi_{ij}^{(N)} = 0$ for its absence.

It is possible to construct by coupling this graph for all N simultaneously : consider a sequence of fixed positions in $I(x_1, \dots, x_N, \dots)$ (for each $N \geq 1$, $x_i^{(N)} = x_i$) and a sequence $(U_k)_{k \in \mathbb{N}}$ of i.i.d. variables of distribution $\mathcal{U}(0, 1)$. Fix an arbitrary bijection $\phi : \mathbb{N}^2 \mapsto \mathbb{N}$, and define $\xi_{ij}^{(N)} = \mathbf{1}_{\{U_{\phi(i, j)} \leq W_N(x_i, x_j)\}}$: conditioned on the positions (x_1, \dots, x_N) , $\xi^{(N)}$ is indeed a collection of independent Bernoulli random variables and $\xi_{ij}^{(N)} \sim \mathcal{B}(W_N(x_i, x_j))$.

We now fix these sequences, and work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ rich enough for all the following processes can be defined. We denote by \mathbf{E} the expectation under \mathbf{P} .

Definition 2.2. Let $(\pi_i(ds, dz))_{1 \leq i \leq N}$ be a sequence of i.i.d. Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure $dsdz$. A (\mathcal{F}_t) -adapted multivariate counting process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$, denoted by $(Z_1^{(N)}(t), \dots, Z_N^{(N)}(t))_{t \geq 0}$ is called a multivariate Hawkes process with parameters $(N, f, \xi^{(N)}, W_N, u_0, h)$ associated with the positions $(x_1^{(N)}, \dots, x_N^{(N)})$ if \mathbf{P} -almost surely, for all $t \geq 0$ and $i \in \llbracket 1, N \rrbracket$:

$$Z_i^{(N)}(t) = \int_0^t \int_0^\infty \mathbf{1}_{\{z \leq \lambda_i^{(N)}(s)\}} \pi_i(ds, dz) \quad (2.2)$$

with $\lambda_i^{(N)}(t)$ defined by

$$\lambda_i^{(N)}(t) = f \left(u_0(t, x_i) + \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \xi_{ij}^{(N)} \int_0^{t-} h(t-s) dZ_j^{(N)}(s) \right). \quad (2.3)$$

We denote by $\kappa_i^{(N)} \geq 0$ a non-negative dilution parameter which can depend on $\underline{x}^{(N)}$, and $\xi^{(N)}$. The idea behind this dilution parameter is that $\kappa_i^{(N)} \simeq \frac{N}{\mathbb{E}[deg_N(i)]}$ (so that the interaction term remains of order 1 as $N \rightarrow \infty$). The interaction in (1.1) is fixed as $w_{ij}^{(N)} = \kappa_i^{(N)} \xi_{ij}^{(N)}$.

Remark 2.3. Note that (see Proposition 3 of [23]) the process $(Z_1^{(N)}(t), \dots, Z_N^{(N)}(t))_{t \geq 0}$ defined by (2.2) is such that \mathbf{P} -almost surely, $Z_i^{(N)}$ and $Z_j^{(N)}$ do not jump imultaneously for all $i \neq j$, and for all $i \in \llbracket 1, N \rrbracket$, the compensator of $Z_i^{(N)}(t)$ is $\int_0^t \lambda_i^{(N)}(s) ds$.

2.2.2. *Existence.* We first provide well-posedness results of $(Z_1^{(N)}, \dots, Z_N^{(N)})$ given by (2.2). We require the following assumptions :

Hypothesis 2.4. We suppose that f is Lipschitz continuous with Lipschitz constant $L_f \geq 0$, that h is locally square integrable on $[0, +\infty)$. We also suppose that $(t, x) \mapsto u_0(t, x)$ is continuous in t and Lipschitz continuous in x (uniformly in t) with Lipschitz constant $L_{u_0} \geq 0$. Moreover u_0 is supposed bounded uniformly in (t, x) : there exists $\|u_0\|_\infty \geq 0$ such that for all $t \geq 0$ and $x \in \mathbb{R}^d$, $|u_0(t, x)| \leq \|u_0\|_\infty$.

Proposition 2.5. *Under Hypothesis 2.4, for a fixed realisation of $(\pi_i)_{1 \leq i \leq N}$, there exists a pathwise unique multivariate Hawkes process (in the sense of Definition 2.2) with parameters $(N, f, \xi^{(N)}, W_N, u_0, h, \underline{x}^{(N)})$ such that $(\sup_{1 \leq i \leq N} \mathbf{E}[Z_i^{(N)}(t)])_{t \geq 0}$ is locally bounded.*

The proof of Proposition 2.5 will be given in Section 6.1.

2.3. **Large population limit process.** We want to study the behavior of the process defined in Definition 2.2 when $N \rightarrow \infty$. After some heuristics, we show the well-posedness of the limit of the system 2.2.

2.3.1. *Heuristics.* The point of the paragraph is to motivate the proper limit for the particle system (2.2) as $N \rightarrow \infty$.

A minimal requirement for the macroscopic limit of (2.2) as $N \rightarrow \infty$ is that the empirical distribution of the positions $\nu^{(N)} := \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$ has itself a macroscopic limit ν . We will consider below different scenarios under which such LLN effectively holds. Concerning the macroscopic behavior of the graph, another minimal requirement needed is that in a certain way to define later, the graph $\mathcal{G}^{(N)}$ given in Definition 2.2 converges towards a macroscopic interaction kernel $W : I \times I \rightarrow \mathbb{R}_+$. We refer to Section 3.2 for more precise statements.

Then, as $N \rightarrow \infty$, an informal LLN argument shows that the empirical mean in (2.3) will become an expectation w.r.t both the candidate limit for $Z_i^{(N)}$ and w.r.t the macroscopic law ν of the positions. Hence, the macroscopic description of a neuron at position $x \in I$ should be described in terms of its intensity $\lambda(t, x)$ solving (1.2).

This heuristics provides with a limit process for a particle located in x in terms of an inhomogeneous Poisson point process with deterministic intensity $\lambda(\cdot, x)$ satisfying (1.2).

2.3.2. *Well-posedness of the macroscopic limit.* We propose a framework under which (1.2) is well-posed, with more hypotheses on the regularity of the parameter functions we consider.

Hypothesis 2.6. Define the macroscopic indegree at position x by

$$D(x) = \int_I W(x, y) \nu(dy). \quad (2.4)$$

We assume that it has a Hölder regularity : there exist $C_w > 0$ and $\iota \in]0, 1]$ such that for all $(x, x') \in I \times I$, W verifies

$$\int_I |W(x, y) - W(x', y)| \nu(dy) \leq C_w \|x - x'\|^\iota. \quad (2.5)$$

Moreover, it is uniformly bounded on I :

$$\sup_{x \in I} D(x) =: C_{W_1} < \infty. \quad (2.6)$$

Proposition 2.7. *Let $T > 0$. Under Hypotheses 2.4 and 2.6, there exists a unique solution λ solution of (1.2) continuous and bounded on $[0, T] \times I$. Moreover, there exists $C_\lambda > 0$ depending on (f, u_0, W, h, ν, T) such that for all $(t, x, z) \in [0, T] \times I \times I$,*

$$|\lambda(t, x) - \lambda(t, z)| \leq C_\lambda (\|x - z\| + \|x - z\|^\iota) =: C_\lambda \phi(\|x - z\|). \quad (2.7)$$

Moreover, if u_0 is continuously differentiable in time and $\frac{\partial u_0}{\partial t}$ is bounded on $[0, T] \times I$, h is continuous and piecewise continuously differentiable, and $f = Id$, then λ is differentiable in time and

$$\frac{\partial \lambda}{\partial t}(t, x) = \frac{\partial u_0}{\partial t}(t, x) + h(t) \int_I W(x, y) \lambda(0, y) \nu(dy) + \int_I \int_0^t h(t-s) W(x, y) \frac{\partial \lambda}{\partial t}(s, y) \nu(dy) ds, \quad (2.8)$$

and $\left| \frac{\partial \lambda}{\partial t} \right|$ is bounded on $[0, T] \times I$.

The proof of Proposition 2.7 will be given in Section 6.2. Note that Proposition 2.7 provides the existence of a unique solution λ of (1.2) that is continuous on $\mathbb{R}_+ \times I$ and locally bounded.

3. CONVERGENCE OF THE MODEL IN LARGE POPULATION

3.1. Coupling. To verify that our heuristics about the large population behavior is correct, we introduce a suitable coupling between the process defined in (2.2) and a multivariate Poisson process with intensity $\lambda(\cdot, x)$ solving (1.2) for $x \in \underline{x}^{(N)}$.

Definition 3.1. Let $(\pi_i(ds, dz))_{1 \leq i \leq N}$ be the same family of i.i.d. Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ as in Definition 2.2. For this family, we construct for all i in $\llbracket 1, N \rrbracket$:

$$\bar{Z}_i(t) = \int_0^t \int_0^\infty \mathbf{1}_{\{z \leq \lambda(s, x_i)\}} \pi_i(ds, dz) \quad (3.1)$$

with λ satisfying (1.2). Each process \bar{Z}_i is an inhomogenous Poisson process with (deterministic) intensity $\lambda(\cdot, x_i)$, and as the family (π_i) is independent, the processes $(\bar{Z}_i)_{i=1, \dots, N}$ are also independent.

3.2. Hypotheses. Regarding the behavior of the graph when $N \rightarrow \infty$, we use here the formalism of graph convergence developed in [44] and introduce different norms on I^2 . The key notion is to represent graphs in term of graphons, that are positive kernels defined on I^2 . Note that we will not necessarily restrict ourselves to the case usually considered in the literature, that is symmetric and bounded graphons.

Definition 3.2. Let W be a \mathbb{R} -valued function defined on $I \times I$, where I is a measured space endowed with some probability measure ν . When the following terms are correctly defined, we write :

$$\|W\|_{\square, \nu} := \sup_{S, T \subset I} \left| \int_{S \times T} W(x, y) \nu(dx) \nu(dy) \right|, \quad (3.2)$$

$$\|W\|_{\infty \rightarrow 1, \nu} := \sup_{\|g\|_\infty \leq 1} \int_I \left| \int_I W(x, y) g(y) \nu(dy) \right| \nu(dx), \quad (3.3)$$

$$\|W\|_{\infty \rightarrow \infty, \nu} := \sup_{\|g\|_\infty \leq 1} \sup_{x \in I} \left| \int_I W(x, y) g(y) \nu(dy) \right|. \quad (3.4)$$

These norms go back to the formalism of graph convergence introduced in [44, 26] and further developed in [7, 8, 9] (and references therein). The last two norms can be seen as the norms of the linear operator $T_W : g \mapsto (x \mapsto \int_I W(x, y) g(y) \nu(dy))$ when considering respectively $T_W : L^\infty(I, \nu) \rightarrow L^1(I, \nu)$ and $T_W : L^\infty(I, \nu) \rightarrow L^\infty(I, \nu)$. We also define the cut-distance between two functions by

$$d_{\square, \nu}(W_1, W_2) = \|W_1 - W_2\|_{\square, \nu}.$$

Remark 3.3. Lemma 8.11 of [44] gives that $\|\cdot\|_{\square, \nu}$ and $\|\cdot\|_{\infty \rightarrow 1, \nu}$ are equivalent: if W is a function defined on I^2 with values in \mathbb{R} , then

$$\|W\|_{\square, \nu} \leq \|W\|_{\infty \rightarrow 1, \nu} \leq 4\|W\|_{\square, \nu}. \quad (3.5)$$

Usual representations of graphon consists in taking $I = [0, 1]$ endowed with Lebesgue measure. We extend this definition to the general case where ν is a general probability on I . To do this, we require the following assumption.

Hypothesis 3.4. The measure ν is absolutely continuous w.r.t Lebesgue measure on \mathbb{R}^d .

It implies that, for every $N \geq 1$, we can find $(B_1^{(N)}, \dots, B_N^{(N)}) \subset I^N$ such that $\nu(B_i^{(N)}) = \frac{1}{N}$ for all $i = 1, \dots, N$, and $I = \sqcup_{i=1}^N B_i^{(N)}$. We denote this partition as \mathcal{P}_N . Without ambiguity, we will forget the index (N) and only write (B_1, \dots, B_N) .

Example 3.5. If $I = \mathbb{R}^d$, we denote by $\nu^{(1)}, \dots, \nu^{(d)}$ the marginals densities ν . The probability measure $\nu^{(1)}$ is then absolutely continuous w.r.t Lebesgue measure on \mathbb{R} . For $i = 1 \dots N$, define y_i as the quantile of order $\frac{i}{N}$ of the distribution $\nu^{(1)}$ and set $y_0 = -\infty$ and $y_N = +\infty$: $\nu^{(1)}((y_i, y_{i+1})) = \frac{1}{N}$ for all $i = 0, \dots, N - 1$. We take then $B_i := (y_{i-1}, y_i] \times \mathbb{R}^{d-1}$.

With this partition, we define the following step-function on I^2 associated with a directed graph A_N with weight a_{ij} on the edge $j \rightarrow i$ as :

$$W^{A_N}(u, v) = \sum_{i=1}^N \sum_{j=1}^N a_{ij} \mathbf{1}_{B_i}(u) \mathbf{1}_{B_j}(v), \quad (u, v) \in I^2. \quad (3.6)$$

Definition 3.6. We denote by $\mathcal{G}_N^{(1)}$ the directed weighted graph with vertices $\{1, \dots, N\}$ such that every edge $j \rightarrow i$ is present, and with weight $\kappa_i^{(N)} W_N(x_i, x_j)$.

Here $\mathcal{G}_N^{(1)}$ represents the average version of the graph $\mathcal{G}^{(N)}$ (where $\xi_{ij} \sim \mathcal{B}(W_N(x_i, x_j))$ has been replaced by $\mathbb{E}(\xi_{ij})$), renormalized by the dilution coefficient $\kappa_i^{(N)}$. A key argument of Theorems 3.10 and 3.12 will be to show that $\mathcal{G}^{(N)}$ and $\mathcal{G}_N^{(1)}$ are close as $N \rightarrow \infty$ through concentration arguments that require the following assumptions on W_N .

Hypothesis 3.7. We suppose some uniformity in the parameters: there exist $\kappa_N \geq 1$ and $w_N \in]0, 1]$ such that :

$$\max_{i \in [1, N]} (\kappa_i^{(N)}) \leq \kappa_N, \quad (3.7)$$

$$\max_{i, j \in [1, N]} (W_N(x_i, x_j)) \leq w_N, \quad (3.8)$$

$$\frac{1}{\kappa_N} \leq w_N \leq 1, \quad (3.9)$$

and asymptotically:

$$\kappa_N^2 w_N \underset{N \rightarrow \infty}{=} o\left(\frac{N}{\log(N)}\right) \quad \text{and} \quad \frac{\kappa_N}{N} \underset{N \rightarrow \infty}{\rightarrow} 0. \quad (3.10)$$

We also suppose that there exists $C_W > 0$ independent of N such that

$$\sup_{i \in [1, N]} \frac{1}{N} \sum_{j=1}^N \kappa_i^{(N)} W_N(x_i, x_j) \leq C_W. \quad (3.11)$$

To illustrate the above conditions, think of the case where W_N is a constant equal to ρ_N with $\rho_N \xrightarrow{N \rightarrow \infty} 0$. This corresponds to a diluted Erdős-Rényi graph random graph.

In this case, we can take $w_N = \rho_N$ and $\kappa_i^{(N)} = \kappa_N = \frac{1}{\rho_N}$. Then (3.10) boils down to $\rho_N \gg \frac{\log(N)}{N}$. Inequality (3.11) is the microscopic counterpart of (2.6) : we require that the indegrees of vertices in $\mathcal{G}_N^{(1)}$ are uniformly bounded.

3.3. Convergence. We study the proximity between the particle systems (2.2) and its macroscopic limit (3.1). We show two theorems that require different sets of hypotheses on the parameter functions, under two main scenarios.

Definition 3.8. We consider different frameworks for the choices of the positions:

- (1) **Random spatial distribution :** We consider a random sequence $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_N, \dots)$ of i.i.d variables distributed according to ν on I , and we set for all $N \geq 1$ $\underline{x}_N = (x_1, \dots, x_N)$ as the lexical re-ordering of the N first positions $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_N)$. We also assume that there exists some $\chi > 5$ such that $\|W\|_{L^\chi(I^2, \nu \otimes \nu)} < \infty$.
- (2) **Deterministic regular distribution of the positions :** For every $N \geq 1$ and $1 \leq i \leq N$, $x_i^{(N)} = \frac{i}{N}$ and $I = [0, 1]$ endowed with $\nu(dx) = dx$. We also assume that W is piecewise continuous on $[0, 1]^2$.

3.3.1. *First case : convergence in average.*

Hypothesis 3.9. We suppose that the annealed graph $\mathcal{G}_N^{(1)}$ converges to W for the cut-distance :

$$d_{\square, \nu} \left(W^{\mathcal{G}_N^{(1)}}, W \right) \xrightarrow{N \rightarrow \infty} 0, \quad (3.12)$$

as well as

$$\sup_{j \in [1, N]} \frac{1}{N} \sum_{i=1}^N \kappa_i^{(N)} W_N(x_i, x_j) \leq C_W. \quad (3.13)$$

Note that (3.12) implies that $\|W^{\mathcal{G}_N^{(1)}} - W\|_{\infty \rightarrow 1, \nu} \xrightarrow{N \rightarrow \infty} 0$ (see Remark 3.3). Note that the hypothesis (3.13) differs from (3.11) in the sense that (3.13) asks for a uniform bound on the outdegree (that is, the number of tail ends adjacent to a vertex) whereas (3.11) relates to a uniform bound on the indegree.

Theorem 3.10. *Let $T > 0$. Suppose that the sequence of positions $(\underline{x}_N)_N$ satisfies one of the scenarios of Definition 3.8. Then, under the set of Hypotheses 2.4 , 2.6 , 3.7 , 3.4 and 3.9, we have*

$$\frac{1}{N} \sum_{i=1}^N \mathbf{E} \left[\sup_{t \in [0, T]} \left| Z_i^{(N)}(t) - \bar{Z}_i(t) \right| \right] \xrightarrow{N \rightarrow \infty} 0 \quad (3.14)$$

for \mathbb{P} -almost realisations of the connectivity sequence $(\xi^{(N)})_{N \geq 1}$ and positions $(\underline{x}_N)_{N \geq 1}$.

The proof of Theorem 3.10 will be given in Section 7.2.

3.3.2. *Second case : convergence of the supremum.*

Some graphs do not satisfy (3.13), see the examples of Section 5.1.2. We propose here another result of convergence that does not require the control (3.13), but ask in return for a stronger convergence of the graphons.

Hypothesis 3.11. We suppose that

$$\|W^{\mathcal{G}_N^{(1)}} - W\|_{\infty \rightarrow \infty, \nu} \xrightarrow{N \rightarrow \infty} 0. \quad (3.15)$$

Theorem 3.12. *Let $T > 0$. Suppose that the sequence of positions $(\underline{x}_N)_N$ satisfies one of the scenarios of Definition 3.8. Consider the coupling introduced in Definition 3.1. Then, under the set of Hypotheses 2.4 , 2.6 , 3.7 and 3.11, we have*

$$\max_{1 \leq i \leq N} \mathbf{E} \left[\sup_{t \in [0, T]} \left| Z_i^{(N)}(t) - \bar{Z}_i(t) \right| \right] \xrightarrow{N \rightarrow \infty} 0. \quad (3.16)$$

\mathbb{P} -almost surely.

The proof of Theorem 3.12 will be given in Section 7.3.

Remark 3.13. Theorems 3.10 and 3.12 are quenched results, valid for fixed realisations of the graph. In this case, the speed convergence is unknown. Nevertheless, if we integrate also with respect to the graph (annealed case), one can obtain explicit speed of convergence as follows :

$$\begin{aligned} \max_{1 \leq i \leq N} \mathbb{E} \mathbb{E} \left[\sup_{t \in [0, T]} \left| Z_i^{(N)}(t) - \bar{Z}_i(t) \right| \right] &\leq C_T \frac{\kappa_N \sqrt{w_N}}{\sqrt{N}}, \\ \frac{1}{N} \sum_{i=1}^N \mathbb{E} \mathbb{E} \left[\sup_{t \in [0, T]} \left| Z_i^{(N)}(t) - \bar{Z}_i(t) \right| \right] &\leq C_T \frac{\kappa_N \sqrt{w_N}}{\sqrt{N}}. \end{aligned}$$

Working in the annealed case simplifies considerably the proof (left to the reader), the previous estimate can be easily derived from the calculation done in the proofs of the previous theorems.

3.4. Consequence on the empirical measure. A direct consequence of Theorems 3.10 and 3.12 concerns the behavior as $N \rightarrow \infty$ of the empirical distribution on the space of trajectories associated with a position $S := \mathbb{D}([0, T], \mathbb{N}) \times I$.

Definition 3.14. We define the following measures on S :

$$\mu_N(d\eta, dx) := \frac{1}{N} \sum_{i=1}^N \delta_{(Z_i^{(N)}([0, T]), x_i^{(N)})}(d\eta, dx), \text{ and} \quad (3.17)$$

$$\mu_\infty(d\eta, dx) := P_{[0, T], \infty}(d\eta | x) \nu(dx), \quad (3.18)$$

where $P_{[0, T], \infty}(\cdot | x)$ is the law of an inhomogeneous Poisson point process with intensity $(\lambda(t, x))_{0 \leq t \leq T}$ (solution of (1.2)). Note that μ_∞ (resp. μ_N) is a probability measure (resp. random probability measure) on $\mathbb{D}([0, T], \mathbb{N}) \times I$.

Theorem 3.15. *Under the assumptions of Theorem 3.10 or Theorem 3.12, we have*

$$\mathbf{E} [d_{BL}(\mu_N, \mu_\infty)] \xrightarrow{N \rightarrow \infty} 0. \quad (3.19)$$

for \mathbb{P} -almost realisations of the connectivity sequence $(\xi^{(N)})_{N \geq 1}$ and positions $(\underline{x}_N)_{N \geq 1}$ under Scenarios of Definition 3.8, where d_{BL} is the bounded Lipschitz distance introduced in (2.1).

The proof of Theorem 3.15 will be given in Section 8.1. We can see this result as an extension of Theorems 1 and 2 of [15], where the memory function is an exponential kernel and the interaction comes from a fixed interaction kernel that depends on the positions.

3.5. Spatial profile. We consider here that we are under the conditions of Scenario (2) of Definition 3.8.

Definition 3.16. Define the following random profile

$$U_N(t, x) := \sum_{i=1}^N U_{i, N}(t) \mathbf{1}_{x \in (\frac{i-1}{N}, \frac{i}{N}]}, \text{ where} \quad (3.20)$$

$$U_{i,N}(t) := u_0(t, x_i) + \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \xi_{ij}^{(N)} \int_0^{t-} h(t-s) dZ_j^{(N)}(s), \quad (3.21)$$

and the deterministic profile

$$u(t, x) := u_0(t, x) + \int_I W(x, y) \int_0^t h(t-s) \lambda(s, y) ds \nu(dy). \quad (3.22)$$

We see from Proposition 2.7 that u is continuous and bounded.

Note that $\lambda_i^{(N)}(t) = f(U_{i,N}(t))$ and that $U_{i,N}(t)$ describes the accumulated activity of neuron i up to time t . Similar quantity has already been considered in [15] in the case $h(t) = e^{-\alpha t}$ with a deterministic graph of interaction. Note also that in this case, (3.22) is the solution of the scalar neural field equation

$$\partial_t u(t, x) = -\alpha u(t, x) + \int_I W(x, y) f(u(t, y)) \nu(dy).$$

It has been extensively studied in the literature as an important example of macroscopic structured model with non local interaction (see [1, 66, 12]).

We study the closeness between U_N and u .

Proposition 3.17. *Under the Hypotheses of Theorem 2.18,*

$$\mathbf{E} \left[\int_0^T \int_0^1 |U_N(t, x) - u(t, x)| dx dt \right] \xrightarrow{N \rightarrow \infty} 0, \quad (3.23)$$

for \mathbb{P} -almost realisations of the connectivity sequence $(\xi^{(N)})_{N \geq 1}$ and positions $(\underline{x}_N)_{N \geq 1}$.

The proof of Proposition 3.17 will be given in Section 8.2.

4. LARGE TIME BEHAVIOR OF THE LIMIT PROCESS IN THE LINEAR CASE

We want to see how the limiting intensity (1.2) behaves as $t \rightarrow \infty$. We restrict here to the following *linear case*, that is when $f = Id$:

$$\lambda(t, x) = u_0(t, x) + \int_I W(x, y) \int_0^t h(t-s) \lambda(s, y) ds \nu(dy) \quad (4.1)$$

on $\mathbb{R}_+ \times I$. The case without spatial interaction, that is $\lambda(t) = u_0 + \int_0^t h(t-s) \lambda(s) ds$ is standard and has been studied in details in the literature ([23], Theorems 10 and 11). Depending on the value of $\|h\|_1$, there is a phase transition in the behavior of such λ when $t \rightarrow \infty$: in the subcritical case ($\|h\|_1 < 1$), $\lambda(t) \xrightarrow{t \rightarrow \infty} \frac{u_0}{1 - \|h\|_1}$ and in the supercritical case ($\|h\|_1 > 1$), $\lambda(t) \xrightarrow{t \rightarrow \infty} \infty$. The point of the present paragraph is to extend this result to the spatial case. We require the following assumptions :

Hypothesis 4.1. In addition to the Hypotheses 2.4 and 2.6, we suppose that h is non-negative, in $L^1(\mathbb{R}^+)$ and piecewise continuously differentiable. We also suppose that u_0 is positive, continuously differentiable in time with $\frac{\partial u_0}{\partial t}$ bounded on any $[0, T] \times I$, and that there exists u Lipschitz continuous on I such that $\sup_{x \in I} |u_0(t, x) - u(x)| \xrightarrow{t \rightarrow \infty} 0$.

To describe the phase transition, we introduce the following linear operator

$$\begin{aligned} T_W : L^\infty(I) &\longrightarrow L^\infty(I) \\ g &\longmapsto (T_W g : x \longmapsto \int_I W(x, y)g(y)\nu(dy)). \end{aligned} \quad (4.2)$$

The continuity of T_W follows directly from (3.11), and we have $\|T_W\| \leq C_{W_1}$. We denote by $r_\infty(T_W)$ the spectral radius of T_W :

$$r_\infty := r_\infty(T_W) = \sup_{\sigma \in Sp(T_W)} |\sigma| = \lim_{n \rightarrow \infty} \|T_W^n\|^{\frac{1}{n}}.$$

The phase transition is given in terms of $\|h\|_{1r_\infty} < 1$ (subcritical) and $\|h\|_{1r_\infty} > 1$ (supercritical). The two cases are described separately below,

4.1. The exponential case. Previous works [15] have considered the case $h(t) = e^{-\alpha t}$ with $\alpha > 0$. The term α is then called the leakage rate. Note that in this case, the dynamics becomes markovian [27]. At the large population limit, the spatial profile seen in Section 3.5 is in this case linked to the scalar neural field equation [15]. In the exponential case, with the introduction of the operator T_W we can give an explicit solution of (4.1).

Proposition 4.2. *In the exponential case $h(t) = e^{-\alpha t}$, the solution of (4.1) when u_0 does not depend on time is explicitly given by*

$$\lambda(t, x) = e^{-\alpha t} e^{tT_W} u_0(x) + \alpha \int_0^t e^{-\alpha(t-s)} e^{(t-s)T_W} u_0(x) ds, \quad (4.3)$$

where e^{tT_W} , $t \geq 0$ is the semigroup of the bounded operator T_W defined as $e^{tT_W} u := \sum_{k=0}^{\infty} \frac{t^k}{k!} T_W^k u$, $u \in L^\infty(I)$.

Proof. Define for $t \geq 0$ $A(t) := x \mapsto e^{\alpha t} \lambda(t, x)$. It solves in $L^\infty(I)$ $\frac{d}{dt} A(t) = \alpha e^{\alpha t} u_0 + T_W A(t)$ with $A(0) = \lambda(0, \cdot) = u_0$. A variation of constants formula gives $A(t) = e^{tT_W} u_0 + \alpha \int_0^t e^{(t-s)T_W} e^{\alpha s} u_0 ds$, and (4.3) follows by definition of A . \square

Example 4.3. Consider the particular case of Expected Degree Distribution (EED) (see [16, 58]): where $W(x, y) = f(x)g(y)$ with f and g two positive functions on I such that $f, g \in L^2(I, \nu)$. Without any loss of generality, we assume $\int_I g d\nu = 1$ and then $D(x) = f(x)$. In this case, for $u_0(t, x) = u_0(x)$, we have explicitly that for $k \geq 1$ $T_W^k u = f \langle g, u \rangle \langle f, g \rangle^{k-1}$ so that the solution of (4.3) is given by

$$\lambda(t, x) = u_0(x) + \frac{\langle g, u \rangle}{\alpha - \langle f, g \rangle} \left(1 - e^{t(\langle f, g \rangle - \alpha)} \right) f(x).$$

The large time behavior depends then explicitly on the sign of $\langle f, g \rangle - \alpha$:

$$\begin{aligned} \langle f, g \rangle > \alpha &\Rightarrow \forall x \in I, \quad \lambda(t, x) \xrightarrow[t \rightarrow \infty]{} +\infty \text{ and} \\ \langle f, g \rangle < \alpha &\Rightarrow \forall x \in I, \quad \lambda(t, x) \xrightarrow[t \rightarrow \infty]{} u_0(x) + \frac{\langle g, u \rangle}{\alpha - \langle f, g \rangle} f(x). \end{aligned}$$

We now consider the general case.

4.2. **Subcritical case.** We assume that we are in the subcritical case:

$$\|h\|_1 r_\infty < 1. \quad (4.4)$$

The main result is the following

Theorem 4.4. *Suppose that we are in the subcritical case (4.4). Under Hypotheses 2.6 and 4.1*

- *there exists a unique function $\ell : I \mapsto \mathbb{R}^+$ solution of*

$$\ell(x) = u(x) + \|h\|_1 \int_I W(x, y) \ell(y) \nu(dy), \quad (4.5)$$

continuous and bounded on I . Moreover, there exists $C_\ell > 0$ such that for all $(x, y) \in I^2$,

$$|\ell(x) - \ell(y)| \leq C_\ell \phi(\|x - y\|), \quad (4.6)$$

where ϕ is given in (2.7).

- *for any $x \in I$, we have the convergence*

$$\lambda(t, x) \xrightarrow[t \rightarrow \infty]{} \ell(x) \quad (4.7)$$

The proof of Theorem 4.4 will be given in Section 9.1. We are now in position to address the question that motivates our paper : to what extent the inhomogeneity of the underlying graph influences the microscopic dynamics ?

Proposition 4.5. *In the subcritical case (4.4), ℓ solution of (4.5) is explicitly defined by*

$$\ell = \sum_{k=0}^{\infty} \|h\|_1^k T_W^k u. \quad (4.8)$$

In particular, if u_0 is constant, ℓ is uniform (i.e. $\ell(x) = \ell$ for every $x \in I$) if and only if the indegree is uniform ($D(x) = \int_I W(x, y) \nu(dy) = D$ for every $x \in I$). In such case, $r_\infty = D$.

Note that (4.8) informs us about the influence of the macroscopic graph W on the dynamics : with u_0 constant, we have

$$\ell(x) = u_0 \left(\sum_{k=0}^{\infty} \|h\|_1^k D^{(k)}(x) \right), \quad (4.9)$$

where $D^{(0)} = 1$, $D^{(1)} = D(x)$ and $D^{(k+1)} = T_W D^{(k)}$. We see from (4.9) that in order to understand $\ell(x)$, one need to explore the structure of the macroscopic graph around x .

Proof. (4.5) can be written $\ell = u + \|h\|_1 T_W \ell$ which leads to $\|h\|_1 \left(\frac{Id}{\|h\|_1} - T_W \right) \ell = u$. As $r_\infty < \frac{1}{\|h\|_1}$ in the subcritical case, $\left(\frac{Id}{\|h\|_1} - T_W \right)$ is invertible (recall that $r_\infty = \sup_{\sigma \in Sp(T_W)} |\sigma|$) and then $\ell = (Id - \|h\|_1 T_W)^{-1} u = \sum_{k=0}^{\infty} \|h\|_1^k T_W^k u$.

We take now u_0 constant. Theorem 4.4 gives the existence of a unique ℓ satisfying (4.5). Assume that this solution is a constant function ℓ_0 , then for all $x \in I$ we have from (4.5) $\ell_0 = u_0 + \|h\|_1 \ell_0 \int_I W(x, y) \nu(dy)$ thus $\int_I W(x, y) \nu(dy)$ is constant and is equal to $\frac{\ell_0 - u_0}{\|h\|_1}$. Respectively, assume $\int_I W(x, y) \nu(dy)$ constant and equal to D . Then, a direct computation gives $\|T_W f\|_\infty \leq \|f\|_\infty D$ hence (as $r_\infty = \lim_{n \rightarrow \infty} \|T_W^n\|^{1/n}$) $r_\infty \leq D$. As $T_W \mathbf{1} = D \mathbf{1}$ (where $\mathbf{1}(x) \equiv 1$), we have $D \leq r_\infty$ thus $D = r_\infty$. The subcritical case can

then be written as $\|h\|_1 D < 1$ and we can define $\ell_0 := \frac{u_0}{1 - \|h\|_1 D} > 0$. The constant function ℓ_0 is continuous, bounded and solution of (4.5) which is unique : thus the solution of (4.5) is indeed constant. \square

4.3. Supercritical case. We assume that we are in the supercritical case :

$$\|h\|_1 r_\infty > 1. \quad (4.10)$$

Note again that, in the case without space interaction ($W = 1$), (4.10) reduces to $\|h\|_1 > 1$ and it can be shown (see [23], Theorem 11) that $\lambda(t) \sim \alpha e^{\beta t} \rightarrow \infty$ for some $\alpha, \beta > 0$. In our context with nontrivial W , note that one does not expect to have $\lambda(t, x) \xrightarrow[t \rightarrow \infty]{} \infty$ uniformly on x as one can see from the obvious following example : take $W(x, y) = \alpha \mathbf{1}_{[0, \frac{1}{2})^2}(x, y) + \beta \mathbf{1}_{[\frac{1}{2}, 1]^2}(x, y)$ for $\alpha > \beta$, then $r_\infty = \frac{\alpha}{2}$. This corresponds to the case of two disconnected mean-field components A (for neurons with positions in $I_A = [0, \frac{1}{2})$) and B (for neurons with positions in $I_B = [\frac{1}{2}, 1]$). The critical parameter for population A (resp. B) is hence $\alpha_c = \frac{2}{\|h\|_1}$ (resp. $\beta_c = \frac{2}{\|h\|_1}$). Taking now $\alpha > \alpha_c$ and $\beta < \beta_c$, (4.10) is satisfied but one does not have $\lambda(t, x) \xrightarrow[t \rightarrow \infty]{} \infty$ uniformly on x as the population B is subcritical, we only have $\lambda(t, x) \xrightarrow[t \rightarrow \infty]{} \infty$ for $x \in I_A$.

In order to avoid such trivial examples, we assume that the graphon W is sufficiently connected in the following way. Defining for $k \geq 1$:

$$W^{(k)}(x, y) := \int_{I \times \dots \times I} W(x, x_1) \cdots W(x_{k-1}, y) dx_1 \cdots dx_{k-1},$$

we assume primitivity of W i.e. that there exists k such that

$$W^{(k)} > 0. \quad (4.11)$$

Note that $W^{(k)}$ is the kernel of the operator T_W^k . To understand (4.11), think of the finite dimensional case with N particles interacting through a connectivity matrix A . In this context, A being primitive means the existence of some $k \geq 1$ such that $A^k(i, j) > 0$ for all i, j . Hypothesis (4.11) is the exact counterpart in infinite dimension. We also assume the more technical assumptions :

Hypothesis 4.6.

$$\sup_x \int_I W(x, y)^2 \nu(dy) =: C_{W_2} < \infty, \quad (4.12)$$

and

$$\forall (x, y) \in I^2, \quad W(x, y) = W(y, x). \quad (4.13)$$

We also assume that we can define the Laplace transform of h for any $z \geq 0$: $\mathcal{L}(h)(z) := \int_0^\infty e^{-tz} h(t) dt$. Having h of polynomial growth works for instance.

Proposition 4.7. *Suppose that we are in the subcritical case (4.10). Under Hypotheses 4.1 and (4.6), $\int_I \lambda(t, x)^2 \nu(dx) \xrightarrow[t \rightarrow \infty]{} \infty$.*

The proof of Proposition 4.7 will be given in Section 9.2.

Remark 4.8. Proposition 4.7 provides a divergence result in L^2 norm, that is not uniform in x . But with more restrictive hypotheses on the connectivity of W (without supposing W symmetric), one can easily derive uniform divergence result. Assume $0 < \inf_{x \in I} u(x) =: \underline{u} < \infty$ and $\|h\|_1 \inf_{x \in I} \|W(x, \cdot)\|_{1, \nu} > 1$, we have then that $\inf_{x \in I} \lim_{t \rightarrow \infty} \lambda(t, x) = +\infty$.

Note that by Fatou Lemma, $\inf_{x \in I} \|W(x, \cdot)\|_{1, \nu} \leq r_\infty$, hence it also implies the result of Proposition 4.7.

Proof. Let $v(t, x) := \inf_{s \geq t} \lambda(s, x)$. For all $x \in I$, set $\underline{\ell}(x) = \liminf_{t \rightarrow \infty} \lambda(t, x)$. We have for all $t > 0$

$$\begin{aligned} \lambda(t, x) &= u_0(t, x) + \underbrace{\int_0^{\frac{t}{2}} \int_I W(x, y) h(t-s) \lambda(s, y) \nu(dy) ds}_{\geq 0} + \int_{\frac{t}{2}}^t \int_I W(x, y) h(t-s) \underbrace{\lambda(s, y)}_{\geq v(\frac{t}{2}, y)} \nu(dy) ds \\ &\geq u_0(t, x) + \int_0^{\frac{t}{2}} h(s) ds \int_I W(x, y) v\left(\frac{t}{2}, y\right) \nu(dy), \end{aligned}$$

then taking $\liminf_{t \rightarrow \infty}$, we obtain as $v(\cdot, y)$ is non decreasing by monotone convergence

$$\inf_{x \in I} \underline{\ell}(x) \geq \underline{u} + \inf_{y \in I} \underline{\ell}(y) \|h\|_1 \inf_{x \in I} \int_I W(x, y) \nu(dy).$$

As u is positive and $\|h\|_1 \inf_{x \in I} \|W(x, \cdot)\|_{1, \nu} > 1$ (subcritical case), it implies that $\inf_{x \in I} \underline{\ell}(x) = \inf_{x \in I} \lim_{t \rightarrow \infty} \inf_{s \geq t} \lambda(s, x) = +\infty$ hence the result. \square

5. APPLICATIONS

The point of the present section is to give examples of graphs ($\mathcal{G}^{(N)}$) and corresponding graphons that satisfy the hypothesis of the paper. The main class of examples we have in mind fall into the framework of W -random graphs, see [45].

5.1. A general class of examples. We give a general class of graph that fulfills the hypotheses of our model, it can be found in [46]. Given a positive measurable kernel $(x, y) \mapsto \mathcal{P}(x, y)$ on I^2 , for any $N \geq 1$ we consider the interaction kernel

$$W_N(x, y) := \rho_N \min\left(\frac{1}{\rho_N}, \mathcal{P}(x, y)\right) \quad (5.1)$$

with $\rho_N \in [0, 1]$ fixed. If \mathcal{P} is bounded, by modifying ρ_N we can suppose $\|\mathcal{P}\|_\infty = 1$ and then $W_N(x, y) = \rho_N \mathcal{P}(x, y)$. Then, one needs to distinguish two cases here: the dense case when $\lim_{N \rightarrow \infty} \rho_N = \rho > 0$ and the diluted case when $\rho_N \rightarrow 0$.

5.1.1. Uniformly bounded degrees. Suppose $\sup_x \int_I \mathcal{P}(x, y) \nu(dy) < \infty$. On the choice of $\kappa_i^{(N)}$, recall that this prefactor in (2.3) was here to ensure that the interaction remains of order 1 as $N \rightarrow \infty$. In the dense case renormalization is not necessary, one can take $\kappa_i^{(N)} = 1$; and in the diluted case we can take $\kappa_i^{(N)} = \frac{1}{\rho_N}$. In either case, we take $w_N = \rho_N$.

To satisfy Hypothesis 3.7, we require $N\rho_N \xrightarrow{N \rightarrow \infty} \infty$ and $\frac{1}{\rho_N} \xrightarrow{N \rightarrow \infty} o\left(\frac{N}{\log(N)}\right)$. Note that Hypothesis 3.9 or Hypothesis 3.11 with $W = \mathcal{P}$ are satisfied under regularity assumption on \mathcal{P} , see Prop. 3.2, 3.4, 3.6 and 3.9 of [46]. Note that if $\rho_N = 1$, it is a direct consequence of Proposition 7.4 as $W^{\mathcal{G}_N^{(1)}} = W^{\mathcal{G}_N^{(2)}}$ then.

Typical examples include the classic Erdős-Rényi graph with $\mathcal{P} = 1$ (hence $W_N = \rho_N$ is uniform), interaction with the P -nearest neighbors (see [57]), or the EDD model previously defined in Example 4.3. These examples are thoroughly detailed in the next part.

5.1.2. *Unbounded degrees.* Suppose that \mathcal{P} satisfies for all $x \in I$: $\int_I \mathcal{P}(x, y)^2 \nu(dy) < \infty$ and $\mathcal{P}_* := \inf_{z \in I} \int \mathcal{P}(z, y) \nu(dy) > 0$, but $\sup_x \int_I \mathcal{P}(x, y) \nu(dy) = \infty$. Then we take $\kappa_i^{(N)} = \frac{\mathcal{P}(x, y)}{\int_I \mathcal{P}(x, z) \nu(dz)}$.

$$\frac{1}{\rho_N \sum_{j=1}^N \min\left(\frac{1}{\rho_N}, \mathcal{P}(x_i, x_j)\right)},$$

Some examples in this situation can be found in [46], Section 3.4. For instance, consider $\mathcal{P}(x, y) = \frac{1}{x^\alpha} g(y)$ with g a probability measure on $[0, 1]$ and $\alpha < \frac{1}{2}$.

We present in the following different concrete examples of application of our results. We focus on the framework $I = [0, 1]$ with the regular distribution of the positions $x_i^{(N)} = \frac{i}{N}$, $1 \leq i \leq N$ and ν the Lebesgue measure. We take $f = Id$ to apply the results of Section 4.

5.2. **Example : Erdős-Rényi graph.** We can consider two different frameworks for the Erdős-Rényi graph. First, we can consider (5.1) with $W_N(x, y) = \rho_N$ (and $\mathcal{P} = 1$) for any $(x, y) \in I^2$ for some $\rho_N \in (0, 1)$. $\mathcal{G}^{(N)}$ is then a Erdős-Rényi random graph. Keeping the previous normalization $\kappa_i^{(N)} = \kappa_N := \frac{1}{\rho_N}$, as seen previously the sequence of graphs

converges toward the graphon $W = 1$ when $N\rho_N \xrightarrow{N \rightarrow \infty} \infty$ and $\frac{1}{\rho_N} \underset{N \rightarrow \infty}{=} o\left(\frac{N}{\log(N)}\right)$, as the threshold obtained in [18].

Following the previous framework we suppose here that we have either $(\rho_N \rightarrow \rho > 0, \kappa_i \equiv 1)$ or $(\rho_N \rightarrow 0, \kappa_i \equiv \frac{1}{\rho_N})$. Note that in both case the natural limiting graphon W is given by $W \equiv \rho$ (with $\rho = 1$ in the diluted case). Note that (2.5) is satisfied, and we can apply Theorems 3.10 and 3.12 (the convergence of graphs is seen in Proposition 7.4). As the degree is constant, Proposition 4.5 gives $r_\infty = \rho$. As Hypothesis 4.6 is satisfied, there is a transition phase around $\rho_c = \frac{1}{\|h\|_1}$.

5.2.1. *Subcritical case* : $\|h\|_1 \rho < 1$. Theorem 4.4 gives that for any $x \in I$,

$$\lambda(t, x) \xrightarrow{t \rightarrow \infty} \ell(x) = \frac{u(x)(1 - \|h\|_1 \rho) + \|u\|_{I, \nu, 1} \|h\|_1 \rho}{1 - \|h\|_1 \rho},$$

and we give an example of simulation in Figure 1. Note that if u_0 is constant, then $\ell = \frac{u}{1 - \|h\|_1 \rho}$. We give an example of simulation in this case in Figure 2.

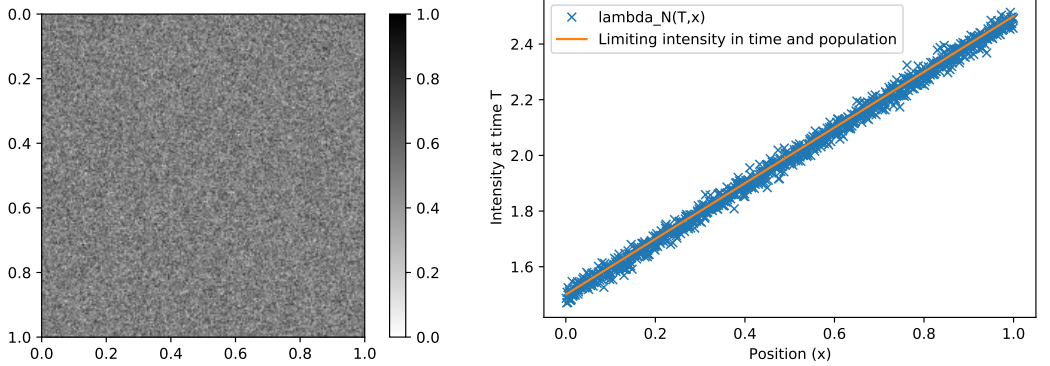
5.2.2. *Supercritical case* : $\|h\|_1 \rho > 1$. As W is constant, we can directly apply Remark 4.8 and obtain $\inf_{x \in I} \lim_{t \rightarrow \infty} \lambda(t, x) = +\infty$.

5.3. **Example : P-nearest neighbor model** [57]. Consider the kernel $W(x, y) = \mathbf{1}_{d_{\mathcal{S}_1}(x, y) < r}$ for any $(x, y) \in I^2$ for some fixed $r \in (0, \frac{1}{2})$ and with

$$d_{\mathcal{S}_1}(x, y) = \min(|x - y|, 1 - |x - y|). \quad (5.2)$$

It means that the particles at positions x and y interacts if and only if they are at distance less than r on the circle $\mathcal{S}_1 := \mathbb{R}/[0, 1]$. Note that this corresponds to a deterministic graph. As (2.5) is satisfied - for any $(x, x') \in I^2$, $\int_I |W(x, y) - W(x', y)| \nu(dy) = \int_0^1 |\mathbf{1}_{|x-y|<r} - \mathbf{1}_{|x'-y|<r}| dy \leq 4|x - x'|$, we can apply Theorems 3.10 and 3.12.

As for any $x \in I$, $\int_I W(x, y) dy = 2r$, Proposition 4.5 gives that $r_\infty = 2r$. The assumptions (4.12) and (4.13) are trivially verified, and as $W^{(k)}$ is positive for $k :=$


 (a) Matrix of $\mathcal{G}^{(N)}$

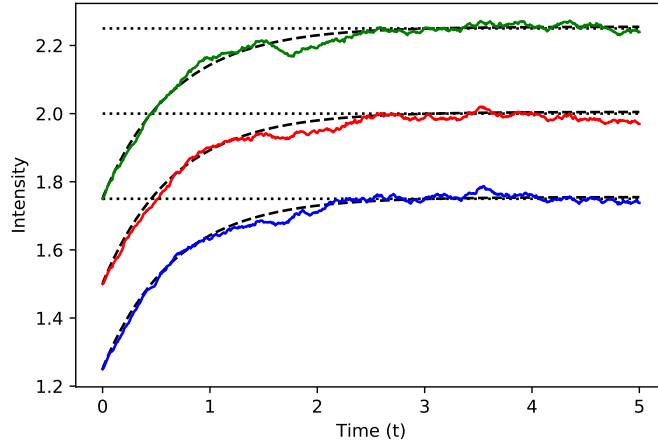
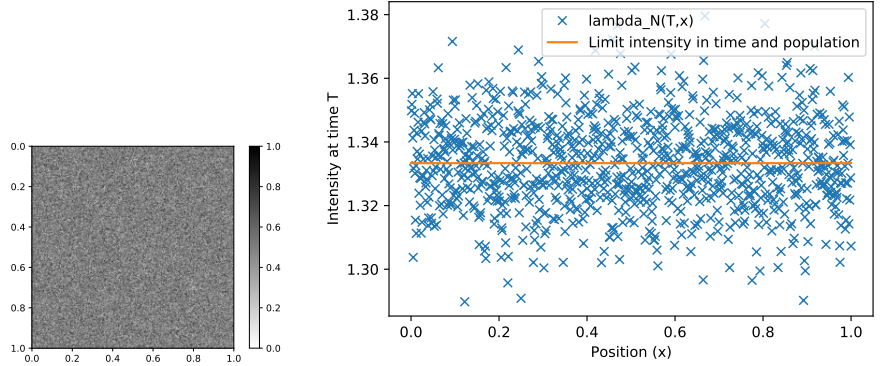
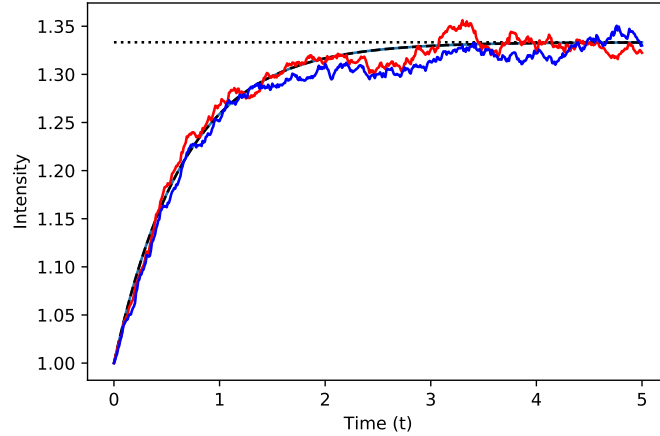
 (b) Each dot represents $\lambda_N(T, x)$ for $x \in \underline{x}^{(N)}$, and the plain line corresponds to the macroscopic limit $\ell(x)$.

 (c) Evolution of microscopic and macroscopic intensities of three particles at positions $x=0.25$ (blue - the lowest), 0.5 (red) and 0.75 (green - the highest). In each case, the colored line represents $\lambda_N(t, x)$, the dashed line represents $\lambda(t, x)$ and the dotted line represents the limit $\ell(x)$.

 FIGURE 1. Simulation of Example 5.2 with inhomogeneous u_0

We chose $h(t) = e^{-\alpha t}$ with $\alpha = 2$, $p = 0.5$ for the Erdős Rényi graph and $u_0(t, x) = x + 1$. We are then in the subcritical case as $\|h\|_1 p < 1$ and the limiting intensity is given by $\ell(x) = x + \frac{1}{2}$. We run a simulation for $N = 1000$ particles and a final time $T = 5$: in 1a, we show the matrix of the Erdős-Rényi graph $\mathcal{G}^{(N)}$. In 1b, we represent the spatial distribution of intensities at fixed time T . In 1c, we show the time evolution of the intensities for different positions. Note here that the inhomogeneity of $\ell(x)$ is due to the inhomogeneity of the u_0 , not of the graph.

$\inf \{n \geq 0, nr \geq \frac{1}{2}\}$, Hypothesis 4.6 is satisfied and there is a transition phase around $r_c = \frac{1}{2\|h\|_1}$. In the subcritical case ($r < r_c$), Proposition 4.5 gives that when u_0 is constant the limiting intensity is explicit and $\ell = \frac{u_0}{1 - 2r\|h\|_1}$. We give an example of simulation in

(a) Matrix of $\mathcal{G}^{(N)}$ (b) Each dot represents $\lambda_N(T, x)$ for $x \in \underline{x}^{(N)}$, and the plain line corresponds to the macroscopic limit $\ell(x)$.(c) Evolution of microscopic and macroscopic intensities of two particles at positions $x=0.5$ (red) and 0.75 (blue). The colored lines represent $\lambda_N(t, x)$, the dashed line represents $\lambda(t)$ and the dotted line represents the limit ℓ .FIGURE 2. Simulation of Example 5.2 with homogeneous u_0

We chose $h(t) = e^{-\alpha t}$ with $\alpha = 2$, $p = 0.5$ for the Erdős Rényi graph and $u_0(t, x) = 1$, we are then in the subcritical case ($\|h\|_{1p} = \frac{1}{4} < 1$). As the graph is homogeneous in space and the self-activity is constant, the limit solution of (1.2) does not depend of the position: $\lambda(t) = \frac{4}{3} - \frac{1}{3}e^{-\frac{2}{3}t}$. The limiting intensity is then constant $\ell = \frac{4}{3}$. We run a simulation for $N = 1000$ particles and a final time $T = 5$. In 2a, we show the matrix of the Erdős-Rényi graph $\mathcal{G}^{(N)}$. In 2c, we show the time evolution of the intensities for different positions. In 2b, we represent the spatial distribution of intensities at fixed time T .

this case in Figure 3. In the supercritical case ($r > r_c$), as the degree is constant, we can directly apply Remark 4.8 and obtain $\inf_{x \in I} \lim_{t \rightarrow \infty} \lambda(t, x) = +\infty$.

5.4. Example : Inhomogeneous graph with EDD [16]. Recall Example 4.3: $W(x, y) = f(x)g(y)$ with f and g two positive bounded functions on I such that $f, g \in L^2(I, \nu)$ and

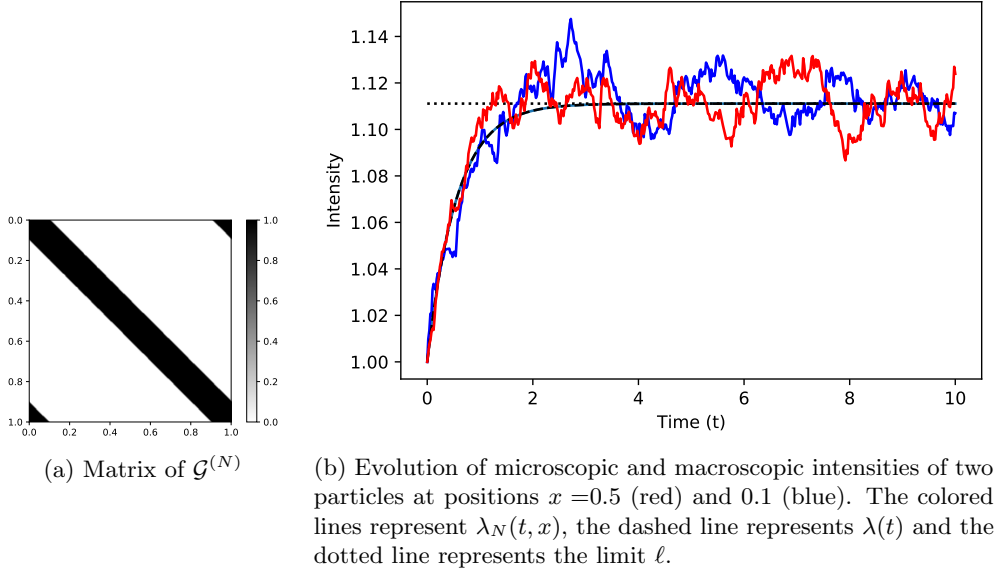


FIGURE 3. Simulation of Example 5.3 in the subcritical case

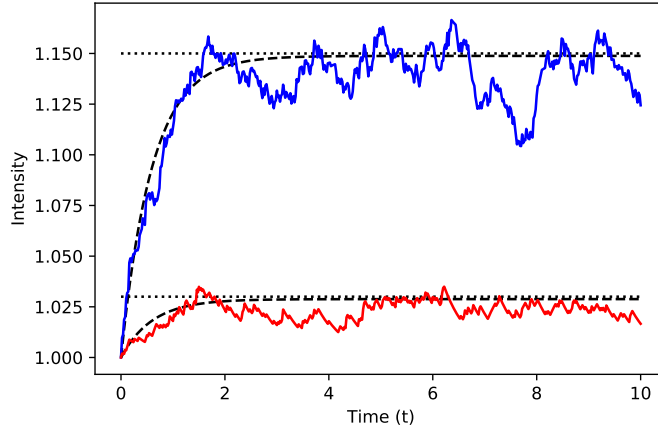
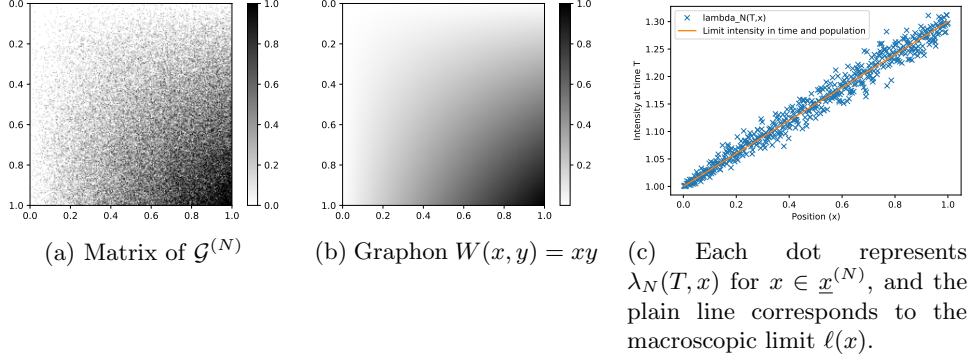
We chose $h(t) = e^{-2t}$, $r = 0.1$ and $u_0(t, x) = 1$, hence we are in the subcritical case as $2r\|h\|_1 < 1$. The graph is not homogeneous in space but has a symmetry and the self-activity u_0 is constant, hence the solution of (1.2) does not depend of the position : $\lambda(t) = \frac{10}{9} - \frac{1}{9}e^{-\frac{9}{5}t}$. The limiting intensity is constant $\ell = \frac{10}{9}$. We run a simulation for $N = 500$ particles and a final time $T = 10$: in 3a, we show the matrix of the graph $\mathcal{G}^{(N)}$ obtained. In 3b, we show the matrix of the Erdős-Rényi graph $\mathcal{G}^{(N)}$. In 2c, we show the time evolution of the intensities for different positions. We see that the simulated intensities follow indeed the behavior expected, as they are close to $\lambda(t, x)$ and converge toward a constant limit ℓ .

$\int_I g d\nu = 1$. We also suppose that f satisfies a Hölder condition for $\iota \in (0, 1]$ and is bounded. Note that the indegree is $D(x) = f(x)$. Hypothesis 2.6 is satisfied and we can apply Theorems 3.10 and 3.12. The operator T_W is then defined as $T_W k(x) = f(x)\langle g, k \rangle$ for $k \in L^\infty$. An iteration gives $T_W^n = \langle f, g \rangle^{n-1} T_W$ for all $n \geq 1$, and then $r_\infty = \langle f, g \rangle$. The following results generalize Example 4.3 where h is fixed as a decreasing exponential ($h(t) = e^{-\alpha t}$, $\|h\|_1 = \frac{1}{\alpha}$) : the transition phase described with the sign of $\langle f, g \rangle - \alpha$ can be written with the quantity $r_\infty \|h\|_1$.

5.4.1. *Subcritical case* : $\|h\|_1 \langle f, g \rangle < 1$. Theorem 4.4 gives that for any $x \in I$, $\lambda(t, x) \xrightarrow[t \rightarrow \infty]{} \ell(x)$ where ℓ is the solution of (4.5), that is

$$\ell(x) = u(x) + \|h\|_1 \frac{f(x)\langle u, g \rangle}{1 - \|h\|_1 \langle f, g \rangle}.$$

5.4.2. *Particular case* : $f=g$. Suppose that f is positive on I , then Hypothesis 4.6 is satisfied and there is a transition phase at $\|h\|_1 \langle f, f \rangle = 1$. We give an example of simulation in this case in Figure 3.



(d) Evolution of microscopic and macroscopic intensities of two particles at positions $x=0.5$ (blue - the highest) and 0.3 (red - the lowest). In each case, the colored line represents $\lambda_N(t, x)$, the dashed line represents $\lambda(t, x)$ and the dotted line represents the limit $\ell(x)$.

FIGURE 4. Simulation of Example 5.4

We chose $h(t) = e^{-\alpha t}$ with $\alpha = 2$, $u_0(t, x) = 1$ and $f = g = Id$, that is $W(x, y) = xy$: we are in the subcritical case ($\|h\|_1 \langle f, f \rangle < 1$) and the limiting intensity is $\ell(x) = 1 + \frac{3}{10}x$. We run a simulation for $N = 500$ particles and a final time $T = 10$: in 4b, we represent the graphon W , and in 4a we show the matrix random graph $\mathcal{G}^{(N)}$ obtained. In 4c, we represent the spatial distribution of intensities at fixed time T . In 4d, we show the time evolution of the intensities for different positions. Note here that the inhomogeneity of $\ell(x)$ is only due to the inhomogeneity of the kernel W .

5.5. Example : Multi-class interaction populations. Another interesting case concerns the case of deterministic and inhomogeneous graphs modeling the macroscopic organization of neurons into vertical columns. A generic construction is the following : divide $I = (0, 1]$ into P consecutive subintervals I_j with respective length $\alpha_j > 0$, that is $I_j = (\alpha_1 + \dots + \alpha_{j-1}, \alpha_1 + \dots + \alpha_j]$ and $\alpha_1 + \dots + \alpha_P = 1$. Take any connectivity matrix M between these P populations, $M = (m_{ij})_{1 \leq i, j \leq P}$ with $m_{ij} \in \{0, 1\}$ modeling the deterministic connection between subpopulations i and j . Take the self-activity fixed for each population, described by $u_0(t) = (u_{0,i}(t))_{1 \leq i \leq P}$ and converging towards $u = (u_i)_{1 \leq i \leq P}$ as

$t \rightarrow \infty$. Define finally $W(x, y) = \sum_{i,j=1}^P m_{ij} \mathbf{1}_{x \in I_i} \mathbf{1}_{y \in I_j}$, as well as $\tilde{M} = (\alpha_j m_{ij})_{1 \leq i, j \leq P}$. Then $r_\infty = \rho(\tilde{M})$ where $\rho(\tilde{M})$ is the spectral radius of \tilde{M} so that the phase transition described above is given here in terms of $\rho(\tilde{M}) \|h\|_1 < 1$ or $\rho(\tilde{M}) \|h\|_1 > 1$.

The limiting intensity $\lambda(t, x)$ is constant on each population, and can then be described by the vector $\tilde{\lambda}(t) = (\lambda_i(t))_{1 \leq i \leq P}$ which solves $\tilde{\lambda}(t) = u_0(t) + \int_0^t h(t-s) \tilde{M} \tilde{\lambda}(s) ds$. In the subcritical case, the limit ℓ is piecewise constant (on each population) and is described by the vector $\ell = (\ell_i)_{1 \leq i \leq P}$ solving $\ell = u + \|h\|_1 \tilde{M} \ell$. In the supercritical case, $\sum_{i=1}^P \alpha_i \lambda_i(t)^2 \xrightarrow{t \rightarrow \infty} \infty$ when \tilde{M} is symmetric and primitive.

Remark. Note that a closer look to the proof of Theorem 2.3 and (2.5) of [2] shows that $\lambda_i(t) \rightarrow \infty$ for all $i = 1 \dots p$ in the simpler case when M is only irreducible but not necessarily symmetric nor primitive (e.g. the case considered in [28]).

6. PROOFS : EXISTENCE AND UNIQUENESS OF THE MODEL AND ITS LIMIT

6.1. Proof of Proposition 2.5. We study the pathwise uniqueness by considering the total variation distance between two such processes. We show the existence by constructing a Cauchy sequence adapted and using a Picard iteration argument.

We follow the structure of the proof proposed in [23] (Theorem 6). We consider a family of independent Poisson measures $(\pi_i(ds, dz))_{1 \leq i \leq N}$ with intensity $dsdz$. In this proof, we denote $\kappa_i^{(N)} \xi_{ij}^{(N)}$ by w_{ij} . We start by showing uniqueness and we omit the notation (N) for simplicity. We set $(Z_i(t))_{i \in \llbracket 1, N \rrbracket, t \geq 0}$ and $(\underline{Z}_i(t))_{i \in \llbracket 1, N \rrbracket, t \geq 0}$ two solutions of the system (2.2) such that $\mathbf{E}[Z_i(t)] < +\infty$ and $\mathbf{E}[\underline{Z}_i(t)] < +\infty$ for any $i \in \llbracket 1, N \rrbracket$ and $t \geq 0$.

For any $i \in \llbracket 1, N \rrbracket$, we consider the total variation distance between Z_i and \underline{Z}_i on $[0, t]$:

$$\Delta_i(t) := \int_0^t |d(Z_i(s) - \underline{Z}_i(s))|.$$

$\Delta_i(t)$ counts the number of unshared jumps between Z_i and \underline{Z}_i on $[0, t]$. We denote respectively by λ_i and $\underline{\lambda}_i$ the stochastic intensities of Z_i and \underline{Z}_i . As they are constructed on the same Poisson measure π_i , the unshared jumps are the points of π_i located between the two intensities, thus we have

$$\Delta_i(t) = \int_0^t \int_0^{+\infty} |\mathbf{1}_{\{z \leq \lambda_i(s)\}} - \mathbf{1}_{\{z \leq \underline{\lambda}_i(s)\}}| \pi_i(ds, dz).$$

Setting $\delta_i(t) := \mathbf{E}[\Delta_i(t)]$, we obtain with Fubini's Theorem

$$\delta_i(t) = \mathbf{E} \left[\int_0^t \int_0^{+\infty} |\mathbf{1}_{\{z \leq \lambda_i(s)\}} - \mathbf{1}_{\{z \leq \underline{\lambda}_i(s)\}}| dz ds \right] = \int_0^t \mathbf{E} [|\lambda_i(s) - \underline{\lambda}_i(s)|] ds.$$

Using (2.3) and as f is Lipschitz continuous (Hypothesis 2.4), we have

$$\begin{aligned} \delta_i(t) &= \int_0^t \mathbf{E} \left[\left| f \left(u_0(s, x_i) + \frac{1}{N} \sum_{j=1}^N w_{ij} \int_{]0, s[} h(s-u) dZ_j^{(N)}(u) \right) \right. \right. \\ &\quad \left. \left. - f \left(u_0(s, x_i) + \frac{1}{N} \sum_{j=1}^N w_{ij} \int_{]0, s[} h(s-u) d\underline{Z}_j^{(N)}(u) \right) \right| \right] ds \\ &\leq L_f \frac{1}{N} \sum_{j=1}^N w_{ij} \mathbf{E} \left[\int_0^t \int_{]0, s[} |h(s-u)| d\Delta_j(u) ds \right]. \end{aligned}$$

We apply Lemma A.1 (Δ_i is with finite variations, $\Delta_i(0) = 0$ and h is locally integrable) and obtain

$$\delta_i(t) \leq L_f \frac{1}{N} \sum_{j=1}^N w_{ij} \int_0^t |h(t-s)| \delta_j(s) ds.$$

We set $\delta(t) = \sum_{i=1}^N \delta_i(t)$ and $W_N = \max_{(i,j) \in \llbracket 1, N \rrbracket^2} w_{ij}$. Then, summing on i , we have

$$\delta(t) \leq L_f W_N \int_0^t |h(t-s)| \delta(s) ds.$$

Since h is locally integrable, δ is non-negative and locally bounded, we can apply Lemma A.3 (i) and obtain that $\delta(t) = 0$ for all $t \geq 0$. As each Δ_i is non-negative, we obtain that for all $i \in \llbracket 1, N \rrbracket$ and $t \geq 0$, $\Delta_i(t) = 0$ almost surely. Hence $Z_i(t) = \underline{Z}_i(t)$ almost surely for all $i \in \llbracket 1, N \rrbracket$ and $t \geq 0$, which gives the uniqueness.

We show now the existence of a process satisfying (2.2). To do it, we procede by iteration : for all $i \in \llbracket 1, N \rrbracket$ and $t \geq 0$: let $Z_{i,0}(t) = 0$. Then, for all $n \geq 0$ we set:

$$Z_{i,n+1}(t) = \int_0^t \int_0^{+\infty} \mathbf{1}_{\{z \leq f(u_0(t, x_i) + \frac{1}{N} \sum_{j=1}^N w_{ij} \int_0^{s-} h(s-u) dZ_{j,n}(u))\}} \pi_i(ds, dz).$$

With i and n fixed, such a process $(Z_{i,n+1})$ exists : it is a counting process with stochastic intensity $\lambda_{i,n+1}(t) = f\left(u_0(t, x_i) + \frac{1}{N} \sum_{j=1}^N w_{ij} \int_0^{t-} h(t-u) dZ_{j,n}(u)\right)$. As for the uniqueness, we set for all $i \in \llbracket 1, N \rrbracket, n \geq 0$ and $t \geq 0$, $\delta_{i,n}(t) = \mathbf{E} \left[\int_0^t |dZ_{i,n+1}(s) - dZ_{i,n}(s)| \right]$ and $\delta_n(t) = \sum_{i=1}^N \delta_{i,n}(t)$.

As it was done previously, we find :

$$\begin{aligned} \delta_{i,n+1}(t) &= \mathbf{E} \left[\int_0^t |dZ_{i,n+2}(s) - dZ_{i,n+1}(s)| \right] = \mathbf{E} \left[\int_0^t \int_0^{+\infty} \left| \mathbf{1}_{\{z \leq \lambda_{i,n+2}(s)\}} - \mathbf{1}_{\{z \leq \lambda_{i,n+1}(s)\}} \right| dz ds \right] \\ &\leq \int_0^t \mathbf{E} \left[L_f \left| \frac{1}{N} \sum_{j=1}^N w_{ij} \int_{]0, s[} h(s-u) (dZ_{j,n+1}(u) - dZ_{j,n}(u)) \right| \right] ds. \end{aligned}$$

Summing on i and using Lemma A.1 we obtain

$$\delta_{n+1}(t) \leq L_f W_N \int_0^t |h(t-s)| \delta_n(s) ds. \quad (6.1)$$

We want to apply Lemma A.3(ii), but for this we have to show that δ_n is locally bounded. We note $m_{i,n}(t) = \mathbf{E} [Z_{i,n}(t)]$ and $v_n(t) = \sum_{i=1}^N m_{i,n}(t)$. By construction,

$$m_{i,n+1}(t) = \mathbf{E} \left[\int_0^t \int_0^{+\infty} \mathbf{1}_{\{z \leq f(u_0(s, x_i) + \frac{1}{N} \sum_{j=1}^N w_{ij} \int_0^{s-} h(s-u) dZ_{j,n}(u))\}} \pi_i(ds, dz) \right].$$

As π_i is a random Poisson measure with intensity $dsdz$, we have

$$m_{i,n+1}(t) = \mathbf{E} \left[\int_0^t f \left(u_0(s, x_i) + \frac{1}{N} \sum_{j=1}^N w_{ij} \int_0^{s-} h(s-u) dZ_{j,n}(u) \right) ds \right].$$

By Hypothesis 2.4, we have that $f(y) \leq f(0) + L_f|y|$ for all y so that :

$$m_{i,n+1}(t) \leq f(0)t + L_f\|u_0\|_\infty t + \frac{1}{N} \sum_{j=1}^N w_{ij} \int_0^t \int_0^{s-} |h(s-u)| dm_{j,n}(u) ds.$$

Applying Lemma A.1 and summing on i we obtain

$$v_{n+1}(t) \leq Nt(f(0) + L_f\|u_0\|_\infty) + W_N \int_0^t |h(t-s)| v_n(s) ds. \quad (6.2)$$

As $v_0 = 0$ and h is locally integrable, by induction v_n is locally bounded for all $n \geq 0$. Yet $\delta_n(t) = \sum_{i=1}^N \mathbf{E} \left[\int_0^t |dZ_{i,n+1}(s) - dZ_{i,n}(s)| \right] \leq v_{n+1}(t) + v_n(t)$ hence δ_n is indeed locally bounded for all n . Lemma A.3(ii) and (6.1) give then that for all $T > 0$, there exists C_T such that $\sup_{t \in [0, T]} \sum_{n \geq 0} \delta_n(t) \leq C_T < \infty$. Thus we have

$$\sup_{t \in [0, T]} \sum_{n \geq 0} \sum_{i=1}^N \mathbf{E} \left[\int_0^t |dZ_{i,n+1}(s) - dZ_{i,n}(s)| \right] \leq C_T < \infty.$$

Thus for i fixed, the sequence of random variables $(Z_{i,n})_n$ is Cauchy in L^1 on the space $D([0, t], \mathbb{R})$ with the expectation of the total variation distance. Then there exists Z_i such that $\mathbf{E} \left[\int_0^T |dZ_{i,n}(s) - dZ_i(s)| \right] \xrightarrow{n \rightarrow \infty} 0$. From this convergence and a diagonal argument, there exists an extraction φ such that for all i ,

$$\int_0^T |dZ_{i,\varphi(n)}(s) - dZ_i(s)| \xrightarrow{n \rightarrow \infty} 0.$$

Since $\int_0^T |dZ_{i,\varphi(n)}(s) - dZ_i(s)|$ is an integer, $Z_{i,\varphi(n)}$ is a.s. stationary and one obtain from this that the righthand of

$$Z_{i,\varphi(n)+1}(t) = \int_0^t \int_0^{+\infty} \mathbf{1}_{\{z \leq f(u_0(t, x_i) + \frac{1}{N} \sum_{j=1}^N w_{ij} \int_0^{s-} h(s-u) dZ_{j,\varphi(n)}(u))\}} \pi_i(ds, dz) \quad (6.3)$$

is equal to $\int_0^t \int_0^{+\infty} \mathbf{1}_{\{z \leq f(u_0(t, x_i) + \frac{1}{N} \sum_{j=1}^N w_{ij} \int_0^{s-} h(s-u) dZ_j(u))\}} \pi_i(ds, dz)$. Hence the left-hand of (6.3) converges too, towards some $\tilde{Z}_i(t)$. It remains to show that $\tilde{Z} = Z$. Fatou's Lemma gives

$$\mathbf{E} \left[\int_0^T |dZ_i(s) - d\tilde{Z}_i(s)| \right] \leq \liminf_{n \rightarrow \infty} \mathbf{E} \left[\int_0^T |dZ_{i,\varphi(n)}(s) - dZ_{i,\varphi(n)+1}(s)| \right] = 0$$

as $(Z_{i,n})_n$ is a Cauchy sequence. We have then that the limit process verifies a.s.

$$Z_i(t) = \int_0^t \int_0^{+\infty} \mathbf{1}_{\{z \leq f(u_0(t, x_i) + \frac{1}{N} \sum_{j=1}^N w_{ij} \int_0^{s-} h(s-u) dZ_j(u))\}} \pi_i(ds, dz).$$

This gives the existence of multivariate Hawkes process $(Z_1(t), \dots, Z_N(t))_{t \geq 0}$ satisfying (2.2). Now let us verify that $t \mapsto \sup_{1 \leq i \leq N} \mathbf{E}[Z_i(t)]$ is locally bounded. Recall (6.2) : as v_n is locally bounded, by Lemma A.3(iii) for all $T > 0$, there exists C_T such that $\sup_{t \in [0, T]} \sup_{n \geq 0} v_n(t) \leq C_T < +\infty$ hence

$$\sup_{t \in [0, T]} \sup_{n \geq 0} \sum_{i=1}^N \mathbf{E}[Z_{i,n}(t)] \leq C_T < +\infty$$

and by dominated convergence, for all $T > 0$, $\sup_{t \in [0, T]} \sum_{i=1}^N \mathbf{E}[Z_i(t)] < +\infty$ and the proof is concluded.

□

6.2. Proof of Proposition 2.7. We show existence and uniqueness of a continuous and bounded solution to equation (1.2). We follow the proof proposed in [15] (Proposition 5), with major changes to accomodate our hypotheses.

We show existence and uniqueness of λ by applying Banach fixed-point Theorem. We consider the map F defined on $\mathcal{C}_b([0, T] \times I, \mathbb{R}_+)$ (the set of bounded continuous functions defined on $[0, T] \times I$ taking non-negative values) by, for any $g \in \mathcal{C}_b([0, T] \times I, \mathbb{R}_+)$:

$$F(g)(t, x) = f \left(u_0(t, x) + \int_I W(x, y) \int_0^t h(t-s)g(s, y)ds \nu(dy) \right) \text{ for all } (t, x) \in [0, T] \times I.$$

First, we check that F takes values in $\mathcal{C}_b([0, T] \times I, \mathbb{R}_+)$: consider $g \in \mathcal{C}_b([0, T] \times I, \mathbb{R}_+)$. Let us show that $F(g)$ is **bounded**. Fix $(t, x) \in [0, T] \times I$. As f is Lipschitz continuous, we have :

$$F(g)(t, x) \leq f(0) + L_f \|u_0\|_\infty + L_f \int_{\mathbb{R}^d} |W(x, y)| \int_0^t |h(t-s)|g(s, y)ds \nu(dy).$$

As g is bounded and h is locally integrable (Hypothesis 2.4), we have

$$\sup_{t \in [0, T], x \in I} F(g)(t, x) \leq f(0) + L_f \|u_0\|_\infty + L_f \|h\|_{[0, T], 1} \|g\|_\infty \sup_{x \in I} \int_{\mathbb{R}^d} W(x, y) \nu(dy) < \infty,$$

where we used Hypothesis (2.6).

We check now that $F(g)$ is **continuous**. We show the sequential continuity : we fix $(t, x) \in [0, T] \times I$ and a sequence (t_n, x_n) converging to (t, x) . As f is Lipschitz continuous, we have :

$$\begin{aligned} & |F(g)(t_n, x_n) - F(g)(t, x)| \leq L_f |u_0(t_n, x_n) - u_0(t, x)| \\ & + L_f \left| \int_I W(x_n, y) \int_0^{t_n} h(t_n - s)g(s, y)ds \nu(dy) - \int_I W(x, y) \int_0^t h(t - s)g(s, y)ds \nu(dy) \right|. \end{aligned} \tag{6.4}$$

The first term $L_f |u_0(t_n, x_n) - u_0(t, x)|$ tend to 0 when n tends to infinity as u_0 is continuous in time and space by Hypothesis 2.4. To show the convergence of the second term, we use the following bound :

$$\begin{aligned} & \left| \int_I W(x_n, y) \int_0^{t_n} h(t_n - s)g(s, y)ds \nu(dy) - \int_{\mathbb{R}^d} W(x, y) \int_0^t h(t - s)g(s, y)ds \nu(dy) \right| \\ & \leq \left| \int_I (W(x_n, y) - W(x, y)) \int_0^{t_n} h(t_n - s)g(s, y)ds \nu(dy) \right| \\ & + \left| \int_I W(x, y) \left(\int_0^{t_n} h(t_n - s)g(s, y)ds - \int_0^t h(t - s)g(s, y)ds \right) \nu(dy) \right| =: A_n + B_n \end{aligned}$$

As h is locally integrable, g bounded, we can upper bound A_n immediately :

$$A_n \leq \|h\|_{[0, T], 1} \|g\|_\infty \int_I |W(x_n, y) - W(x, y)| \nu(dy) \leq \|h\|_{[0, T], 1} \|g\|_\infty C_w \|x - x_n\|^t \xrightarrow{n \rightarrow \infty} 0,$$

using (2.5). To study the convergence of B_n , we do a substitution and split the integral in two :

$$\begin{aligned} B_n &\leq \int |W(x, y)| \left| \int_0^{t_n} h(u)g(t_n - u, y)du - \int_0^t h(u)g(t - u, y)ds \right| \nu(dy) \\ &\leq \int |W(x, y)| \left(\int_0^t |h(u)| |g(t_n - u, y) - g(t - u, y)| du \right) \nu(dy) \\ &\quad + \int |W(x, y)| \left(\int_{\min(t, t_n)}^{\max(t, t_n)} |h(u)|g(t_n - u, y)du \right) \nu(dy) =: a_n + b_n. \end{aligned}$$

Since g is continuous, for all $y \in I$, $\int_0^t |h(u)| |g(t_n - u, y) - g(t - u, y)| du \xrightarrow[n \rightarrow \infty]{} 0$, and since $\int_0^t |h(u)| |g(t_n - u, y) - g(t - u, y)| du \leq 2\|h\|_{[0, T], 1}\|g\|_\infty$, we see from dominated convergence theorem that $a_n \xrightarrow[n \rightarrow \infty]{} 0$. We focus on the term b_n

$$\int_{\min(t, t_n)}^{\max(t, t_n)} |h(u)|g(t_n - u, y)du \leq \|g\|_\infty \int_0^T |h(u)|\mathbf{1}_{[\min(t, t_n), \max(t, t_n)]}(u)du.$$

Yet $|h(u)|\mathbf{1}_{[\min(t, t_n), \max(t, t_n)]}(u) \xrightarrow[n \rightarrow \infty]{} 0$, and we obtain $b_n \xrightarrow[n \rightarrow \infty]{} 0$ by dominated convergence as h is locally integrable. We have shown that for all $(t, x) \in [0, T] \times I$, $\lim_{n \rightarrow \infty} |F(g)(t_n, x_n) - F(g)(t, x)| = 0$ for any sequence (t_n, x_n) tending to (t, x) : $F(g)$ is continuous.

We show now that there exists a constant $C > 0$ such that for all $(t, x, z) \in [0, T] \times I^2$:

$$|F(g)(t, x) - F(g)(t, z)| \leq C (\|x - z\| + \|x - z\|^\ell). \quad (6.5)$$

Let $(t, x, z) \in [0, T] \times I \times I$. As done previously (f and u_0 are Lipschitz continuous), we have :

$$|F(g)(t, x) - F(g)(t, z)| \leq L_f L_{u_0} \|x - z\| + L_f \int_I \int_0^t |h(t - s)|g(s, y)ds |W(x, y) - W(z, y)|\nu(dy).$$

Since g is bounded, h is locally integrable, using (2.5)

$$|F(g)(t, x) - F(g)(t, z)| \leq L_f L_{u_0} \|x - z\| + L_f \|g\|_\infty \|h\|_{[0, T], 1} C_w \|x - z\|^\ell,$$

which gives (6.5) .

Hence, $\mathcal{C}_b([0, T] \times I, \mathbb{R}_+)$ is stable by F . We are going to prove that F admits a unique fixed point, which is λ satisfying (1.2). To do it, we show that some iteration of F is contractive, and then the Banach fixed-point Theorem gives the result.

Let $t \in [0, T]$, g and \tilde{g} be two functions in $\mathcal{C}_b([0, t] \times I, \mathbb{R}_+)$. We use the distance $D_t(g, \tilde{g}) := \sup_{s \in [0, t]} \sup_{x \in I} |g(s, x) - \tilde{g}(s, x)|$ which makes the space $\mathcal{C}_b([0, t] \times I, \mathbb{R}_+)$ complete. Naturally, for all $s \leq t$, $D_s(g, \tilde{g}) \leq D_t(g, \tilde{g})$. Let $x \in \mathbb{R}^d$. As previously,

$$|F(g)(t, x) - F(\tilde{g})(t, x)| \leq L_f \left(\sup_{z \in I} \int_I |W(z, y)|\nu(dy) \right) \int_0^t |h(t - s)|D_s(g, \tilde{g})ds.$$

Using Cauchy-Schwarz inequality, as h is in L_{loc}^2 under Hypothesis 2.4

$$|F(g)(t, x) - F(\tilde{g})(t, x)| \leq L_f \left(\sup_{z \in I} \int_I |W(z, y)|\nu(dy) \right) \|h\|_{[0, T], 2} \left(\int_0^t (D_s(g, \tilde{g}))^2 ds \right)^{\frac{1}{2}}.$$

Using (2.6), we have then shown the existence of a constant $C(f, w, \nu, h, T, p)$ such that for all mappings g and \tilde{g} , for all $t \in [0, T]$:

$$D_t(F(g), F(\tilde{g})) \leq C \left(\int_0^t (D_s(g, \tilde{g}))^2 ds \right)^{\frac{1}{2}}. \quad (6.6)$$

By induction on $k \in \mathbb{N}$, with (6.6), we show that for all $t \in [0, T]$ and for any mappings g and \tilde{g} : $D_t(F^k(g), F^k(\tilde{g})) \leq C^k \left(\frac{t^k}{k!} \right)^{\frac{1}{2}} D_t(g, \tilde{g})$. The initialisation is immediate, and then for $k \geq 0$:

$$\begin{aligned} D_t(F^{k+1}(g), F^{k+1}(\tilde{g})) &\leq C \left(\int_0^t \left(D_s(F^k(g), F^k(\tilde{g})) \right)^2 ds \right)^{\frac{1}{2}} && \text{by (6.6)} \\ &\leq C \left(\int_0^t C^{2k} \frac{s^k}{k!} D_s(g, \tilde{g})^2 ds \right)^{\frac{1}{2}} && \text{by induction hypothesis} \\ &\leq C^{k+1} \left(\frac{t^{k+1}}{(k+1)!} \right)^{\frac{1}{2}} D_t(g, \tilde{g}), \end{aligned}$$

which concludes the induction. We have then for all k and any functions g and \tilde{g} of $\mathcal{C}_b([0, T] \times I, \mathbb{R}_+)$, the k -th iteration of F verifies $D_T(F^k(g), F^k(\tilde{g})) \leq C^k \left(\frac{T^k}{k!} \right)^{\frac{1}{2}} D_T(g, \tilde{g})$.

Hence there exists a rank k such that F^k is contractive, thus has a unique fixed point which is also then the unique fixed point of F in $\mathcal{C}_b([0, T] \times I, \mathbb{R}_+)$ that we call λ . Furthermore, we have shown that any image by F verifies the property (6.5), so in particular λ verifies it too and (2.7) is then true (with C_λ the constant of equation (6.5) for $g = \lambda$).

We focus now on the second part of Proposition 2.7 under restricted hypotheses on the parameter functions: we consider u_0 continuously differentiable in time and $\frac{\partial u_0}{\partial t}$ bounded on $[0, T] \times I$, h continuous and piecewise continuously differentiable, and $f = Id$. First, we ensure that (2.8) admits a unique continuous bounded solution. Then, by studying a sequence of functions that converges towards λ , we show that λ is differentiable in time and $\frac{\partial \lambda}{\partial t}$ satisfies (2.8).

Using the same method as done above, one can show that the map G defined on $\mathcal{C}_b([0, T] \times I, \mathbb{R}_+)$ by

$$G(g)(t, x) = \frac{\partial u_0}{\partial t}(t, x) + h(t) \int_I W(x, y) \lambda(0, y) \nu(dy) + \int_I \int_0^t h(t-s) W(x, y) g(s, y) \nu(dy) ds \quad \forall (t, x) \in [0, T] \times I$$

admits a unique fixed point called μ . Moreover, we can introduce a sequence of function $(\mu_n)_n$ that converges uniformly towards μ defined by iteration, $\mu_0 = 0$ and

$$\mu_{n+1}(t, x) := \frac{\partial u_0}{\partial t}(t, x) + h(t) \int_I W(x, y) \lambda(0, y) \nu(dy) + \int_I \int_0^t h(t-s) W(x, y) \mu_n(s, y) \nu(dy) ds.$$

Similarly, we can introduce a sequence of function $(\lambda_n)_n$ that converges uniformly towards λ defined by iteration, $\lambda_0 = 0$ and

$$\lambda_{n+1}(t, x) := u_0(t, x) + \int_I \int_0^t h(t-s) W(x, y) \lambda_n(s, y) \nu(dy) ds.$$

By induction, for every n , λ_n is differentiable in time and bounded and then, by integration by part we obtain

$$\frac{\partial \lambda_{n+1}}{\partial t}(t, x) = \frac{\partial u_0}{\partial t}(t, x) + h(t) \int_I W(x, y) \lambda_n(0, y) \nu(dy) + \int_0^t \int_I W(x, y) h(t-s) \frac{\partial \lambda_n}{\partial s}(s, y) \nu(dy) ds. \quad (6.7)$$

Now, we can compare μ_n and $\frac{\partial \lambda_n}{\partial t}$: for any $(t, x) \in [0, T] \times I$,

$$\begin{aligned} \left| \mu_{n+1}(t, x) - \frac{\partial \lambda_{n+1}}{\partial t}(t, x) \right| &= \left| h(t) \int_I W(x, y) (\lambda(0, y) - \lambda_n(0, y)) \nu(dy) \right. \\ &\quad \left. + \int_0^t \int_I W(x, y) h(t-s) \left(\mu_n(s, y) - \frac{\partial \lambda_n}{\partial s}(s, y) \right) \nu(dy) ds \right| \\ \left\| \mu_{n+1}(t, \cdot) - \frac{\partial \lambda_{n+1}}{\partial t}(t, \cdot) \right\|_\infty &\leq h(t) C_{W_1} \|\lambda(0, \cdot) - \lambda_n(0, \cdot)\|_\infty + C_{W_1} \int_0^t h(t-s) \left\| \mu_n(s, \cdot) - \frac{\partial \lambda_n}{\partial s}(s, \cdot) \right\|_\infty ds. \end{aligned}$$

We obtain, as (λ_n) converges uniformly to λ ,

$$\limsup_{n \rightarrow \infty} \left\| \mu_{n+1}(t, \cdot) - \frac{\partial \lambda_{n+1}}{\partial t}(t, \cdot) \right\|_\infty \leq C_{W_1} \int_0^t h(t-s) \limsup_{n \rightarrow \infty} \left\| \mu_n(s, \cdot) - \frac{\partial \lambda_n}{\partial s}(s, \cdot) \right\|_\infty ds.$$

This gives from Lemma A.3 (i), provided that $\limsup_{n \rightarrow \infty} \left\| \mu_n(s, \cdot) - \frac{\partial \lambda_n}{\partial s}(s, \cdot) \right\|_\infty$ is finite, that $\sup_{t \in [0, T]} \limsup_{n \rightarrow \infty} \left\| \mu_n(t, \cdot) - \frac{\partial \lambda_n}{\partial t}(t, \cdot) \right\|_\infty = 0$. It implies that, as (μ_n) converges uniformly to μ , so does $\left(\frac{\partial \lambda_n}{\partial t} \right)_n$, and then as λ is differentiable, $\lambda_n \xrightarrow[n \rightarrow \infty]{\text{uniformly}} \lambda$ and $\frac{\partial \lambda_n}{\partial t} \xrightarrow[n \rightarrow \infty]{\text{uniformly}} \frac{\partial \lambda}{\partial t}$, we obtain $\frac{\partial \lambda}{\partial t} = \mu$. It remains to check that $\limsup_{n \rightarrow \infty} \left\| \mu_n(s, \cdot) - \frac{\partial \lambda_n}{\partial s}(s, \cdot) \right\|_\infty$ is finite. As (λ_n) converges to λ , it is uniformly bounded and as $\frac{\partial u_0}{\partial t}$ is bounded, we can find g locally bounded such that, from (6.7), $\left\| \frac{\partial \lambda_{n+1}}{\partial t}(t, \cdot) \right\|_\infty \leq g(t) + C_{W_1} \int_0^t h(t-s) \left\| \frac{\partial \lambda_n}{\partial s}(s, \cdot) \right\|_\infty ds$. Lemma A.3 (iii) gives then that $\sup_{s \in [0, T]} \sup_{n \geq 0} \left\| \frac{\partial \lambda_n}{\partial s}(s, \cdot) \right\|_\infty < \infty$. We can do the same for (μ_n) and obtain $\sup_{s \in [0, T]} \sup_{n \geq 0} \|\mu_n(s, \cdot)\|_\infty < \infty$ which concludes to $\limsup_{n \rightarrow \infty} \left\| \mu_n(s, \cdot) - \frac{\partial \lambda_n}{\partial s}(s, \cdot) \right\|_\infty < \infty$ for any $s \in [0, T]$. \square

7. PROOFS : CONVERGENCE OF THE PROCESS (2.2)

7.1. Toolbox. We present useful results that come up in the main proofs.

Proposition 7.1. *Recall the definitions of κ_N and w_N in Hypothesis 3.7. Let (α_{ij}) and (α_{ijk}) such that for every $(i, j, l) \in \llbracket 1, N \rrbracket^3$, $|\alpha_{lj}| \leq 1$ and $|\alpha_{ijl}| \leq 1$. Define*

$$X_j := \frac{\kappa_N}{N} \sum_{l=1}^N \alpha_{lj} \bar{\xi}_{lj}, \quad \tilde{X}_i := \frac{\kappa_N}{N} \sum_{l=1}^N \alpha_{il} \bar{\xi}_{il}, \quad X_{ij} := \frac{\kappa_N}{N} \sum_{l=1}^N \alpha_{ijl} \bar{\xi}_{il},$$

with $\bar{\xi}_{lk} := \xi_{lk}^{(N)} - W_N(x_l, x_k)$. Then, under Hypothesis 3.7, \mathbb{P} -almost surely if N is large enough :

$$\sup_{1 \leq j \leq N} |X_j| \leq \varepsilon_N, \quad \sup_{1 \leq i \leq N} |\tilde{X}_i| \leq \varepsilon_N \quad \text{and} \quad \sup_{1 \leq i, j \leq N} |X_{ij}| \leq \varepsilon_N,$$

for $\varepsilon_N := 32 \frac{\kappa_N^2 w_N}{N} \log(N)$.

Note that under Hypothesis 3.7, $\varepsilon_N \xrightarrow{N \rightarrow \infty} 0$.

Proof. The proof relies on the use of Lemma A.7.

We derive a uniform bound on $(X_j)_{j \in [1, N]}$: fixing j , we apply Lemma A.7 for the choice $U_l = \xi_{lj}^{(N)}$, $p_l = W_N(x_l, x_j)$ (note that (3.8) yields that $p_l \leq w_N$), $v_l = \alpha_{lj}$ and the constant $\kappa_N > 0$. We obtain, taking the supremum on j and a union bound :

$$\mathbb{P} \left(\sup_{j \in [1, N]} |X_j| > \varepsilon_N \right) \leq 2N \exp \left(-16 \log(N) B \left(4\sqrt{2} \left(\frac{\log(N)}{N w_N} \right)^{\frac{1}{2}} \right) \right).$$

As $B(u) = u^{-2} ((1+u) \log(1+u) - u) \rightarrow \frac{1}{2}$ when $u \rightarrow 0$ and $\frac{\log(N)}{N w_N} \leq \frac{\log(N)}{N} \kappa_N^2 w_N \rightarrow 0$ when $N \rightarrow \infty$ using (3.9) and (3.10), we can choose a deterministic p such that for all $N \geq p$, $B \left(4\sqrt{2} \left(\frac{\log(N)}{N w_N} \right)^{\frac{1}{2}} \right) \geq \frac{3}{16}$. We then have if $N \geq p$:

$$\mathbb{P} \left(\sup_{j \in [1, N]} |X_j| > \varepsilon_N \right) \leq 2N \exp(-3 \log(N)) = \frac{2}{N^2}.$$

Hence, by Borel-Cantelli Lemma, there exists $\tilde{\mathcal{O}} \in \mathcal{F}$ such that $\mathbb{P}(\tilde{\mathcal{O}}) = 1$ and on $\tilde{\mathcal{O}}$, there exists $\tilde{N} < \infty$ such that if $N \geq \tilde{N}$, $\sup_{j \in [1, N]} |X_j| \leq \varepsilon_N$. We can show similarly that $\sup_{1 \leq i \leq N} |\tilde{X}_i| \leq \varepsilon_N$. To show the result on (X_{ij}) , we use the same Lemma A.7 but we need to lower-bound $B \left(4\sqrt{2} \left(\frac{\log(n)}{n w_n} \right)^{\frac{1}{2}} \right)$ differently : we can choose a deterministic \tilde{p} for all $n \geq \tilde{p}$, $B \left(4\sqrt{2} \left(\frac{\log(n)}{n w_n} \right)^{\frac{1}{2}} \right) \geq \frac{1}{4}$ and then the same argument as before works to obtain $\mathbb{P} \left(\sup_{i, j \in [1, N]} |X_{ij}(T)| > \varepsilon_N \right) \leq 2N^2 \exp \left(-16 \log(N) \frac{1}{4} \right) \leq \frac{2}{N^2}$, and we conclude by Borel-Cantelli Lemma. \square

Corollary 7.2. *Under the set of Hypothesis 3.7, we have \mathbb{P} -almost surely if N is large enough :*

$$\sup_{1 \leq j \leq N} \left(\sum_{i=1}^N \frac{\kappa_i^{(N)}}{N} \xi_{ij}^{(N)} \right) \leq 1 + \sup_{1 \leq j \leq N} \left(\sum_{i=1}^N \frac{\kappa_i^{(N)}}{N} W_N(x_i, x_j) \right) \quad (7.1)$$

$$\sup_{1 \leq i \leq N} \left(\sum_{j=1}^N \frac{\kappa_j^{(N)}}{N} \xi_{ij}^{(N)} \right) \leq 1 + \sup_{1 \leq i \leq N} \left(\sum_{j=1}^N \frac{\kappa_j^{(N)}}{N} W_N(x_i, x_j) \right) \quad (7.2)$$

$$\frac{1}{N^3} \sum_{i, j=1}^N \left(\kappa_i^{(N)} \right)^2 \xi_{ij}^{(N)} \leq \frac{\kappa_N}{N} \left(1 + \sum_{i, j=1}^N \frac{\kappa_i^{(N)}}{N^2} W_N(x_i, x_j) \right). \quad (7.3)$$

Proof. It is a direct application of Proposition 7.1 (as for any i , $\left| \frac{\kappa_i^{(N)}}{\kappa_N} \right| \leq 1$ with (3.7)), as ε_N defined in Proposition 7.1 tends to 0 as $N \rightarrow \infty$ under Hypothesis 3.7, hence for N large enough $\varepsilon_N \leq 1$. \square

We introduce an auxilliary graph that intervenes in the proofs.

Definition 7.3. We denote by $\mathcal{G}_N^{(2)}$ the directed weighted graph with vertices $\{1, \dots, N\}$ such that every edge $j \rightarrow i$ is present, and with weight $W(x_i, x_j)$.

The following Proposition is technical, and its proof is postponed in Section 7.4.

Proposition 7.4. *Under the Scenarios of Definition 3.8,*

$$d_{\square, \nu} \left(W^{\mathcal{G}_N^{(2)}}, W \right) \xrightarrow{N \rightarrow \infty} 0, \quad (7.4)$$

$$\|W^{\mathcal{G}_N^{(2)}} - W\|_{\infty \rightarrow \infty, \nu} \xrightarrow{N \rightarrow \infty} 0, \quad (7.5)$$

$$\max_{1 \leq i \leq N} \int_0^T \left| \int_I W(x_i, x) \gamma(s, x) \left(\nu^{(N)}(dx) - \nu(dx) \right) \right| ds \xrightarrow{N \rightarrow \infty} 0 \quad (7.6)$$

and

$$\frac{1}{N} \sum_{i=1}^N \int_0^T \left| \int_I W(x_i, x) \gamma(s, x) \left(\nu^{(N)}(dx) - \nu(dx) \right) \right| ds \xrightarrow{N \rightarrow \infty} 0, \quad (7.7)$$

where $\gamma(s, x) := \int_0^s h(s-u) \lambda(u, x) du$.

Note that if (7.4) and (7.7) are satisfied in another configuration of positions that in the Scenarios of Definition 3.8, Theorem 3.10 still applies. Likewise, if (7.5) and (7.6) are satisfied, Theorem 3.12 still applies.

7.2. Proof of Theorem 3.10. Recall the definitions of $Z_i^{(N)}$ and \bar{Z}_i in (2.2) and (3.1). We remind that we consider the sequences $(\underline{x}_N)_{N \geq 1}$ and $(\xi_{ij}^{(N)})_{\substack{N \geq 1 \\ i, j \in \llbracket 1, N \rrbracket}}$ fixed (our result is quenched). Let $t \in [0, T]$. For each $i \in \llbracket 1, N \rrbracket$, let $\Delta_i^{(N)}(t)$ be the total variation distance between $Z_i^{(N)}$ and \bar{Z}_i on $[0, t]$:

$$\Delta_i^{(N)}(t) = \int_0^t \left| d \left(Z_i^{(N)}(s) - \bar{Z}_i(s) \right) \right|. \quad (7.8)$$

Remark that we always have $\sup_{t \in [0, T]} \left| Z_i^{(N)}(t) - \bar{Z}_i(t) \right| \leq \Delta_i^{(N)}(t)$.

We then have, using the Lipschitz continuity of f :

$$\begin{aligned} \mathbf{E} \left[\Delta_i^{(N)}(t) \right] &= \mathbf{E} \left[\int_0^t \int_0^\infty \left| \mathbf{1}_{\{z \leq \lambda_i^{(N)}(s)\}} - \mathbf{1}_{\{z \leq \lambda(s, x_i)\}} \right| \pi_i(ds, dz) \right] = \int_0^t \mathbf{E} \left[\left| \lambda_i^{(N)}(s) - \lambda(s, x_i) \right| \right] ds \\ &\leq L_f \int_0^t \mathbf{E} \left[\left| \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \xi_{ij}^{(N)} \int_0^{s-} h(s-u) dZ_j^{(N)}(u) - \int_I W(x_i, y) \int_0^{s-} h(s-u) \lambda(u, y) du \nu(dy) \right| \right] ds \\ &\leq L_f \left(\sum_{k=1}^5 A_{i,t,k}^{(N)} \right) \end{aligned} \quad (7.9)$$

where

$$A_{i,t,1}^{(N)} := \int_0^t \mathbf{E} \left[\left| \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \xi_{ij}^{(N)} \int_0^{s-} h(s-u) \left(dZ_j^{(N)}(u) - d\bar{Z}_j(u) \right) \right| \right] ds, \quad (7.10)$$

$$A_{i,t,2}^{(N)} := \int_0^t \mathbf{E} \left[\left| \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \xi_{ij}^{(N)} \int_0^{s-} h(s-u) \left(d\bar{Z}_j(u) - \lambda(u, x_j) du \right) \right| \right] ds, \quad (7.11)$$

$$A_{i,t,3}^{(N)} := \int_0^t \left| \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \left(\xi_{ij}^{(N)} - W_N(x_i, x_j) \right) \int_0^s h(s-u) \lambda(u, x_j) du \right| ds, \quad (7.12)$$

$$A_{i,t,4}^{(N)} := \int_0^t \left| \frac{1}{N} \sum_{j=1}^N \left(\kappa_i^{(N)} W_N(x_i, x_j) - W(x_i, x_j) \right) \int_0^s h(s-u) \lambda(u, x_j) du \right| ds \text{ and } (7.13)$$

$$A_{i,t,5}^{(N)} := \int_0^t \left| \frac{1}{N} \sum_{j=1}^N W(x_i, x_j) \int_0^s h(s-u) \lambda(u, x_j) du - \int_{\mathbb{R}^d} W(x_i, y) \int_0^s h(s-u) \lambda(u, y) du \nu(dy) \right| ds. \quad (7.14)$$

We are going to control each term $\frac{1}{N} \sum_{i=1}^N A_{i,t,k}^{(N)}$.

$A_{i,t,1}^{(N)}$ captures the proximity between the particle system $Z_i^{(N)}$ with its meanfield counterpart \bar{Z}_i at the same position. We have, as the graph $(\xi^{(N)})$ is fixed,

$$\begin{aligned} A_{i,t,1}^{(N)} &= \int_0^t \mathbf{E} \left[\left| \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \xi_{ij}^{(N)} \int_0^{s-} h(s-u) \left(dZ_j^{(N)}(u) - d\bar{Z}_j(u) \right) \right| \right] ds \\ &\leq \frac{1}{N} \sum_{j=1}^N \kappa_i^{(N)} \xi_{ij}^{(N)} \mathbf{E} \left[\int_0^t \int_0^{s-} |h(s-u)| \left| d \left(\Delta_j^{(N)}(u) \right) \right| ds \right]. \end{aligned}$$

We use the Lemma A.1 so that

$$A_{i,t,1}^{(N)} \leq \frac{1}{N} \sum_{j=1}^N \kappa_i^{(N)} \xi_{ij}^{(N)} \mathbf{E} \left[\int_0^t |h(t-s)| \Delta_j^{(N)}(s) ds \right],$$

then we have, summing :

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N A_{i,t,1}^{(N)} &\leq \frac{1}{N} \sum_{j=1}^N \left(\sum_{i=1}^N \frac{\kappa_i^{(N)}}{N} \xi_{ij}^{(N)} \right) \int_0^t |h(t-s)| \mathbf{E} \left[\Delta_j^{(N)}(s) \right] ds \\ &\leq \sup_{1 \leq j \leq N} \left(\sum_{i=1}^N \frac{\kappa_i^{(N)}}{N} \xi_{ij}^{(N)} \right) \int_0^t |h(t-s)| \mathbf{E} \left[\frac{1}{N} \sum_{j=1}^N \Delta_j^{(N)}(s) \right] ds. \end{aligned}$$

We use (7.1) and (3.13) to obtain \mathbb{P} -almost surely if N is large enough the bound

$$\frac{1}{N} \sum_{i=1}^N A_{i,t,1}^{(N)} \leq (1 + C_W) \int_0^t |h(t-s)| \mathbf{E} \left[\frac{1}{N} \sum_{j=1}^N \Delta_j^{(N)}(s) \right] ds. \quad (7.15)$$

$A_{i,t,2}^{(N)}$ captures the proximity between the limit process and its expectation. We have that

$$A_{i,t,2}^{(N)} = \int_0^t \mathbf{E} \left[\left| \frac{1}{N} \sum_{j=1}^N (V_j^i(s) - \mathbf{E} [V_j^i(s)]) \right|^2 \right] ds,$$

where $V_j^i(s) = \kappa_i^{(N)} \xi_{ij}^{(N)} \int_0^{s-} h(s-u) d\bar{Z}_j(u)$ is a family of independent random variables (by independence of the π_i). Note that $\mathbf{E} [V_j^i(s)] = \kappa_i^{(N)} \xi_{ij}^{(N)} \int_0^s h(s-u) \lambda(u, x_j) du$. Define $M_j^i(s) := V_j^i(s) - \mathbf{E} [V_j^i(s)]$, which can also be written as

$$M_j^i(s) = \int_0^s \int_0^\infty \mathbf{1}_{\{z \leq \lambda(u, x_j)\}} \kappa_i^{(N)} \xi_{ij}^{(N)} h(s-u) \pi_i(du, dz) - \int_0^s \int_0^\infty \mathbf{1}_{\{z \leq \lambda(u, x_j)\}} \kappa_i^{(N)} \xi_{ij}^{(N)} h(s-u) dudz.$$

It is a square integrable martingale and

$$\begin{aligned} \text{Var} (V_j^i(s)) &= \mathbf{E} [M_j^i(s)^2] = \mathbf{E} \left[\int_0^s \int_0^\infty \left(\mathbf{1}_{\{z \leq \lambda(u, x_j)\}} \kappa_i^{(N)} \xi_{ij}^{(N)} h(s-u) \right)^2 dudz \right] \\ &= \int_0^s \left(\kappa_i^{(N)} \right)^2 \xi_{ij}^{(N)} h(s-u)^2 \lambda(u, x_j) du. \end{aligned}$$

Thus summing on i and using Lemma A.2,

$$\frac{1}{N} \sum_{i=1}^N A_{i,t,2}^{(N)} \leq \frac{1}{N} \sum_{i=1}^N \int_0^t \frac{1}{N} \sqrt{\sum_{j=1}^N \int_0^s \left(\kappa_i^{(N)} \right)^2 \xi_{ij}^{(N)} h(s-u)^2 \lambda(u, x_j) du} ds.$$

We use then Jensen inequality to both uniform measures on $\{1, \dots, N\}$ and $[0, t]$ to obtain :

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N A_{i,t,2}^{(N)} &\leq \frac{t}{N} \int_0^t \sqrt{\frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N \int_0^s \left(\kappa_i^{(N)} \right)^2 \xi_{ij}^{(N)} h(s-u)^2 \lambda(u, x_j) du} \frac{ds}{t} \\ &\leq \frac{t}{N} \sqrt{\frac{1}{Nt} \sum_{i,j=1}^N \left(\kappa_i^{(N)} \right)^2 \xi_{ij}^{(N)} \int_0^t \int_0^s h(s-u)^2 \lambda(u, x_j) duds} \end{aligned}$$

By Hypothesis 2.4 on h , we have

$$\frac{1}{N} \sum_{i=1}^N A_{i,t,2}^{(N)} \leq t \|h\|_{t,2} \sqrt{\|\lambda\|_{[0,t] \times I, \infty}} \sqrt{\frac{1}{N^3} \sum_{i,j=1}^N \left(\kappa_i^{(N)} \right)^2 \xi_{ij}^{(N)}}.$$

We use (7.3) and (3.11) to obtain \mathbb{P} -almost surely if N is large enough the bound :

$$\frac{1}{N} \sum_{i=1}^N A_{i,t,2}^{(N)} \leq t \|h\|_{t,2} \sqrt{\|\lambda\|_{[0,t] \times I, \infty}} \sqrt{\frac{\kappa_N}{N}} (1 + C_W). \quad (7.16)$$

$A_{i,t,3}^{(N)}$ captures the proximity between the realization of the graph $(\xi^{(N)})$ and its expectation. We define for $(s, x, y) \in [0, T] \times I \times I$:

$$\gamma(s, x) := \int_0^s h(s-u)\lambda(u, x)du, \quad (7.17)$$

$$\Gamma_T(x, y) := \int_0^T \gamma(s, x)\gamma(s, y)ds. \quad (7.18)$$

Note that we always have $|\gamma(s, x)| \leq \|h\|_{s,1}\|\lambda\|_{[0,s]\times I,\infty} =: \gamma_{s,\infty}$ and $0 \leq \Gamma_T(x, y) \leq T\gamma_{T,\infty}^2$. Recall that $\bar{\xi}_{ij} := \xi_{ij}^{(N)} - W_N(x_i, x_j)$. Then $A_{i,t,3}^{(N)} = \int_0^t \left| \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \bar{\xi}_{ij} \gamma(s, x_j) \right| ds \leq \int_0^T \left| \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \bar{\xi}_{ij} \gamma(s, x_j) \right| ds$. Note that one cannot apply Proposition 7.1 directly in the integrand since the a.s. would depend on $s \in [0, T]$. Therefore, we control its square, by Jensen inequality :

$$A_{i,t,3}^{(N)2} \leq T \int_0^T \left(\frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \bar{\xi}_{ij} \gamma(s, x_j) \right)^2 ds = T^2 \gamma_{T,\infty}^2 \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \bar{\xi}_{ij} X_{ij},$$

with $X_{ij} := \frac{\kappa_i^{(N)}}{N} \sum_{l=1}^N \bar{\xi}_{il} \frac{\Gamma_T(x_j, x_l)}{T\gamma_{T,\infty}^2}$. By Proposition 7.1, \mathbb{P} -almost surely for N large enough, $\sup_{1 \leq i, j \leq N} |X_{ij}| \leq \varepsilon_N$, thus

$$\begin{aligned} A_{i,t,3}^{(N)2} &\leq T^2 \gamma_{T,\infty}^2 \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \left(\xi_{ij}^{(N)} + W_N(x_i, x_j) \right) \sup_{i,j} |X_{ij}| \\ &\leq T^2 \gamma_{T,\infty}^2 \varepsilon_N \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \left(\xi_{ij}^{(N)} + W_N(x_i, x_j) \right). \end{aligned}$$

Taking the square root then summing on i , we use the discrete Jensen inequality to obtain

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N A_{i,t,3}^{(N)} &\leq \sqrt{\varepsilon_N} T \gamma_{T,\infty} \frac{1}{N} \sum_{i=1}^N \sqrt{\frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \left(\xi_{ij}^{(N)} + W_N(x_i, x_j) \right)} \\ &\leq \sqrt{\varepsilon_N} T \gamma_{T,\infty} \sqrt{\sum_{i=1}^N \frac{\kappa_i^{(N)}}{N^2} \sum_{j=1}^N \left(\xi_{ij}^{(N)} + W_N(x_i, x_j) \right)}, \end{aligned}$$

if N is large enough \mathbb{P} -almost surely. Using (7.2) and (3.11), we have

$$\begin{aligned}
 \frac{1}{N} \sum_{i=1}^N A_{i,t,3}^{(N)} &\leq \sqrt{\varepsilon_N} T \gamma_{T,\infty} \sqrt{\frac{1}{N} \sum_{i=1}^N \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \xi_{ij}^{(N)} + \frac{1}{N} \sum_{i=1}^N \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N W_N(x_i, x_j)} \\
 &\leq \sqrt{\varepsilon_N} T \gamma_{T,\infty} \sqrt{1 + C_W + \sup_{i \in \llbracket 1, N \rrbracket} \left(\frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N W_N(x_i, x_j) \right)} = \sqrt{\varepsilon_N} T \gamma_{T,\infty} \sqrt{1 + 2C_W}.
 \end{aligned} \tag{7.19}$$

$A_{i,t,4}^{(N)}$ captures the proximity between the law of the graph on N particles and the limit graphon W . Recall the definition of γ in (7.17) the graphs introduced in Definitions 3.6 and 7.3. Denoting by $c(s) = (c_j(s))_{1 \leq j \leq N} = \left(\frac{\gamma(s, x_j)}{\gamma_{t,\infty}} \right)_{1 \leq j \leq N} \in [-1, 1]^N$, we obtain using the notations (3.6) and introducing for any $c = (c_1, \dots, c_N) \in [-1, 1]^N$ the step function $g^c(v) = \sum_{l=1}^N c_l \mathbf{1}_{B_l}(v)$ for $v \in I$, after summation :

$$\begin{aligned}
 \frac{1}{N} \sum_{i=1}^N A_{i,t,4}^{(N)} &= \gamma_{t,\infty} \int_0^t \int \left| \int \left(W^{\mathcal{G}_N^{(1)}}(u, v) - W^{\mathcal{G}_N^{(2)}}(u, v) \right) g^{c(s)}(v) \nu(dv) \right| \nu(du) ds \\
 &= T \gamma_{T,\infty} \|W^{\mathcal{G}_N^{(1)}} - W^{\mathcal{G}_N^{(2)}}\|_{\infty \rightarrow 1, \nu},
 \end{aligned}$$

where $\|\cdot\|_{\infty \rightarrow 1, \nu}$ is defined in (3.3). Hence, with Remark 3.3 we obtain :

$$\frac{1}{N} \sum_{i=1}^N A_{i,t,4}^{(N)} \leq 4T \gamma_{T,\infty} \left(d_{\square, \nu} \left(W^{\mathcal{G}_N^{(1)}}, W \right) + d_{\square, \nu} \left(W^{\mathcal{G}_N^{(2)}}, W \right) \right) \tag{7.20}$$

We use (7.4) to deal with $d_{\square, \nu} \left(W^{\mathcal{G}_N^{(2)}}, W \right)$.

$\frac{1}{N} \sum_{i=1}^N A_{i,t,5}^{(N)}$ captures the proximity between the empirical measure of the positions of N particles $\mu^{(N)}$ and its limit ν . We denote by $A_{t,5}^{(N)} = \frac{1}{N} \sum_{i=1}^N A_{i,t,5}^{(N)}$, it is controlled with (7.7).

Combining (7.15), (7.16), (7.19) and (7.20), we obtain if N is large enough \mathbb{P} -almost surely for every $t \in [0, T]$:

$$\begin{aligned}
 \mathbf{E} \left[\frac{1}{N} \sum_{i=1}^N \Delta_i^{(N)}(t) \right] &\leq C_1 \int_0^t |h(t-s)| \mathbf{E} \left[\frac{1}{N} \sum_{j=1}^N \Delta_j^{(N)}(s) \right] ds + C_2 \sqrt{\frac{\kappa_N}{N}} + C_3 \sqrt{\varepsilon_N} \\
 &\quad + C_4 d_{\square, \nu} \left(W^{\mathcal{G}_N^{(1)}}, W \right) + C_4 \gamma_{T,\infty} d_{\square, \nu} \left(W^{\mathcal{G}_N^{(2)}}, W \right) + L_f A_{t,5}^{(N)} \tag{7.21}
 \end{aligned}$$

with C_1, C_2, C_3, C_4 constants depending on L_f, C_W, h and T .

We apply Lemma A.5 with $u(t) = \mathbf{E} \left[\frac{1}{N} \sum_{i=1}^N \Delta_i^{(N)}(t) \right]$ on $[0, T]$, and remind that $\sup_{t \in [0, T]} \left| Z_i^{(N)}(t) - \bar{Z}_i(t) \right| \leq \Delta_i^{(N)}(t)$ to obtain \mathbb{P} -almost surely on the realisation of $(\xi^{(N)})$ if N is large enough :

$$\frac{1}{N} \sum_{i=1}^N \mathbf{E} \left[\sup_{t \in [0, T]} \left| Z_i^{(N)}(t) - \bar{Z}_i(t) \right| \right] \leq C \left(\sqrt{\frac{\kappa_N}{N}} + \sqrt{\varepsilon_N} + d_{\square, \nu} \left(W^{\mathcal{G}_N^{(1)}}, W \right) + d_{\square, \nu} \left(W^{\mathcal{G}_N^{(2)}}, W \right) + A_{T,5}^{(N)} \right) \quad (7.22)$$

with $C = \sqrt{2} \max(C_2, C_3, C_4, L_f) \exp \left(C_1^2 \|h\|_{T,2}^2 \right)$.

By (3.10), $\varepsilon_N \xrightarrow{N \rightarrow \infty} 0$, by (3.10) $\frac{\kappa_N}{N} \xrightarrow{N \rightarrow \infty} 0$ and by (3.12) $d_{\square, \nu} \left(W^{\mathcal{G}_N^{(1)}}, W \right) \xrightarrow{N \rightarrow \infty} 0$. Combining with Proposition 7.4, it concludes the proof of (3.14). \square

7.3. Proof of Theorem 3.12. The following proof is almost the same as the proof of Theorem 3.10 with changes due to the fact that we take now the maximum on i . Let us go back to the inequality (7.9). We are going to control each term $\max_{1 \leq i \leq N} A_{i,t,k}^{(N)}$.

Concerning $A_{i,t,1}^{(N)}$, the same estimate as in the proof of Theorem 3.10 leads now to

$$\max_{1 \leq i \leq N} A_{i,t,1}^{(N)} \leq \max_{1 \leq i \leq N} \left(\frac{1}{N} \sum_{j=1}^N \kappa_i^{(N)} \xi_{ij}^{(N)} \right) \int_0^t |h(t-s)| \max_{1 \leq i \leq N} \mathbf{E} \left[\Delta_i^{(N)}(s) \right] ds.$$

We use (7.2) and (3.11) to obtain \mathbb{P} -almost surely if N is large enough : (7.2)

$$\max_{1 \leq i \leq N} A_{i,t,1}^{(N)} \leq (1 + C_W) \int_0^t |h(t-s)| \max_{1 \leq i \leq N} \mathbf{E} \left[\Delta_i^{(N)}(s) \right] ds. \quad (7.23)$$

Note that here, we do not use the same control as is the proof of Theorem 3.10, we only need the uniformly bounded indegree.

Concerning $A_{i,t,2}^{(N)}$, we obtain as in the proof of Theorem 3.10

$$A_{i,t,2}^{(N)} \leq \int_0^t \frac{1}{N} \sqrt{\sum_{j=1}^N \int_0^s \left(\kappa_i^{(N)} \right)^2 \xi_{ij}^{(N)} h(s-u)^2 \lambda(u, x_j) du} ds.$$

We use Jensen inequality on the probability measure $\frac{1}{t} dt$ on $[0, t]$ and then the boundedness of h and λ to obtain

$$\begin{aligned} A_{i,t,2}^{(N)} &\leq \frac{t}{N \sqrt{t}} \kappa_i^{(N)} \sqrt{\sum_{j=1}^N \xi_{ij}^{(N)} \int_0^t \int_0^s h(s-u)^2 \lambda(u, x_j) du ds} \\ &\leq \|h\|_{t,2} \sqrt{\|\lambda\|_{[0,t] \times \mathbb{R}^d, \infty}} \frac{\sqrt{t}}{N} \kappa_i^{(N)} \sqrt{\sum_{j=1}^N \xi_{ij}^{(N)}}, \end{aligned}$$

and taking the maximum leads to

$$\max_{1 \leq i \leq N} A_{i,t,2}^{(N)} \leq \sqrt{\frac{\kappa_N}{N}} \sqrt{\max_{1 \leq i \leq N} \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \xi_{ij}^{(N)}} \|h\|_{t,2} \sqrt{t \|\lambda\|_{[0,t] \times \mathbb{R}^d, \infty}}.$$

Using as before (7.2) and (3.11), we obtain \mathbb{P} -almost surely if N is large enough

$$\max_{1 \leq i \leq N} A_{i,t,2}^{(N)} \leq \sqrt{\frac{\kappa_N}{N}} \sqrt{1 + C_W} \|h\|_{t,2} \sqrt{t \|\lambda\|_{[0,t] \times \mathbb{R}^d, \infty}}. \quad (7.24)$$

Concerning $A_{i,t,3}^{(N)}$, we obtain as in the proof of Theorem 3.10 that \mathbb{P} -almost surely

$$A_{i,t,3}^{(N)} \leq T\gamma_{T,\infty} \sqrt{\varepsilon_N \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \left(\xi_{ij}^{(N)} + W_N(x_i, x_j) \right)},$$

hence taking the maximum and using (7.2) and (3.11), we obtain \mathbb{P} -almost surely if N is large enough

$$\max_{1 \leq i \leq N} A_{i,t,3}^{(N)} \leq T\gamma_{T,\infty} \sqrt{\varepsilon_N} \sqrt{1 + 2C_W}. \quad (7.25)$$

Concerning $A_{i,t,4}^{(N)}$, we recognise

$$A_{i,t,4}^{(N)} = \gamma_{t,\infty} \int_0^t \left| \frac{1}{N} \sum_{j=1}^N \left(\kappa_i^{(N)} W_N(x_i, x_j) - W(x_i, x_j) \right) c_j(s) \right| ds.$$

We obtain, using Definitions 3.6 and 7.3 with Example 3.5 that as

$$\begin{aligned} & \sup_{1 \leq i \leq N} \left| \frac{1}{N} \sum_{j=1}^N \left(\kappa_i^{(N)} W_N(x_i, x_j) - W(x_i, x_j) \right) c_j(s) \right| \\ &= \sup_{u \in I} \left| \int \left(W^{\mathcal{G}_N^{(1)}}(u, v) - W^{\mathcal{G}_N^{(2)}}(u, v) \right) g^{c(s)}(v) \nu(dv) \right|, \end{aligned}$$

we have

$$\begin{aligned} \sup_{1 \leq i \leq N} A_{i,t,4}^{(N)} &= \gamma_{t,\infty} \sup_{1 \leq i \leq N} \int_0^t \left| \frac{1}{N} \sum_{j=1}^N \left(\kappa_i^{(N)} W_N(x_i, x_j) - W(x_i, x_j) \right) c_j(s) \right| ds \\ &\leq \gamma_{t,\infty} \int_0^t \sup_{g, \|g\|_\infty \leq 1} \sup_{u \in I} \left| \int \left(W^{\mathcal{G}_N^{(1)}}(u, v) - W^{\mathcal{G}_N^{(2)}}(u, v) \right) g(v) \nu(dv) \right| ds \\ &\leq T\gamma_{T,\infty} \|W^{\mathcal{G}_N^{(1)}} - W^{\mathcal{G}_N^{(2)}}\|_{\infty \rightarrow \infty, \nu} \leq T\gamma_{T,\infty} \left(\|W^{\mathcal{G}_N^{(1)}} - W\|_{\infty \rightarrow \infty, \nu} + \|W - W^{\mathcal{G}_N^{(2)}}\|_{\infty \rightarrow \infty, \nu} \right). \end{aligned} \quad (7.26)$$

Concerning $A_{i,t,5}^{(N)}$, we denote by $\tilde{A}_{t,5}^{(N)} = \max_{1 \leq i \leq N} A_{i,t,5}^{(N)}$. It is controlled with (7.6).

Combining (7.23), (7.24), (7.25) and (7.26), we obtain if N is large enough \mathbb{P} -almost surely for every $t \in [0, T]$:

$$\begin{aligned} \mathbf{E} \left[\max_{1 \leq i \leq N} \Delta_i^{(N)}(t) \right] &\leq C_1 \int_0^t |h(t-s)| \mathbf{E} \left[\max_{1 \leq i \leq N} \Delta_i^{(N)}(s) \right] ds + C_2 \sqrt{\frac{\kappa_N}{N}} \\ &\quad + C_3 \sqrt{\varepsilon_N} + C_4 \|W^{\mathcal{G}_N^{(1)}} - W\|_{\infty \rightarrow \infty, \nu} + C_4 \|W^{\mathcal{G}_N^{(2)}} - W\|_{\infty \rightarrow \infty, \nu} + L_f \tilde{A}_{t,5}^{(N)} \end{aligned} \quad (7.27)$$

with C_1, C_2, C_3 and C_4 constants depending on h, f, C_W and T .

We apply the Lemma A.5 with $u(t) = \mathbf{E} \left[\max_{1 \leq i \leq N} \Delta_i^{(N)}(t) \right]$ on $[0, T]$, and remind $\sup_{t \in [0, T]} \left| Z_i^{(N)}(t) - \bar{Z}_i(t) \right| \leq \Delta_i^{(N)}(t)$ to obtain \mathbb{P} -almost surely on the realisation of $(\xi^{(N)})$ if N is large enough :

$$\begin{aligned} \max_{1 \leq i \leq N} \mathbf{E} \left[\sup_{t \in [0, T]} \left| Z_i^{(N)}(t) - \bar{Z}_i(t) \right| \right] &\leq C \left(\sqrt{\frac{\kappa_N}{N}} + \sqrt{\varepsilon_N} + \|W^{\mathcal{G}_N^{(1)}} - W\|_{\infty \rightarrow \infty, \nu} \right. \\ &\quad \left. + \|W^{\mathcal{G}_N^{(2)}} - W\|_{\infty \rightarrow \infty, \nu} + \tilde{A}_{t, 5}^{(N)} \right) \end{aligned} \quad (7.28)$$

with $C = \sqrt{2} \max(C_2, C_3, C_4, L_f) \exp\left(C_1^2 \|h\|_{T, 2}^2 T\right)$.

By (3.10), $\varepsilon_N \xrightarrow{N \rightarrow \infty} 0$, by (3.10) $\frac{\kappa_N}{N} \xrightarrow{N \rightarrow \infty} 0$ and by (3.15) $\|W^{\mathcal{G}_N^{(1)}} - W\|_{\infty \rightarrow \infty, \nu} \xrightarrow{N \rightarrow \infty}$

0. Combining with Proposition 7.4, it concludes the proof of (3.16). \square

7.4. Proofs : Application to the Scenarios of Definition 3.8. In this section, we prove Proposition 7.4. We start by presenting a toolbox of results that come up in the main proof.

7.4.1. *Toolbox.*

Lemma 7.5. *Let $(\tilde{x}_i)_{i \geq 1}$ be a sequence of i.i.d positions on $[0, 1]$ with law $\mathcal{U}[0, 1]$. For all $N \geq 1$ and for $i = 1, \dots, N$, define $x_i = \tilde{x}_{(i)}$ as the order statistics of $(\tilde{x}_1, \dots, \tilde{x}_N)$ (i.e. $\{\tilde{x}_1, \dots, \tilde{x}_N\} = \{x_1, \dots, x_N\}$ and $x_1 < \dots < x_N$). Then, for any borelian sets A and B of $]0, 1]$,*

$$\frac{1}{N} \sum_{i=1}^N \mathbf{1}_{x_i \in A, \frac{i}{N} \in B} \xrightarrow{N \rightarrow \infty} \lambda(A \cap B) \quad a.s. \quad (7.29)$$

where λ denotes the Lebesgue measure on $[0, 1]$.

Proof. It is sufficient to show that for all $(t, t') \in]0, 1]^2$,

$$\frac{1}{N} \sum_{i=1}^N \mathbf{1}_{x_i \leq t, \frac{i}{N} \leq t'} \xrightarrow{N \rightarrow \infty} \min(t, t') \quad a.s.$$

We introduce the uniform sample quantile function as in [21] : define for any $y \in [0, 1]$

$$U_N(y) = \begin{cases} 0 & \text{if } y = 0 \\ x_k & \text{if } \frac{k-1}{N} < y \leq \frac{k}{N}, \quad k \in \llbracket 1, N \rrbracket. \end{cases} \quad (7.30)$$

First, we show that $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \mathbf{1}_{x_i \leq t, \frac{i}{N} \leq t'} = \lim_{N \rightarrow \infty} \int_0^{t'} \mathbf{1}_{U_N(y) \leq t} dy$. We note k the integer such that $x_k \leq t < x_{k+1}$ (and $k = 0$ if $x_1 > t$).

If $t' \geq \frac{k}{N}$, then

$$\frac{1}{N} \sum_{i=1}^N \mathbf{1}_{x_i \leq t, \frac{i}{N} \leq t'} = \frac{1}{N} \sum_{i=1}^N \mathbf{1}_{x_i \leq t, i \leq Nt'} = \frac{1}{N} \sum_{i=1}^{Nt'} \mathbf{1}_{x_i \leq t} = \frac{k}{N},$$

and $\int_0^{t'} \mathbf{1}_{U_N(y) \leq t} dy = \int_0^{t'} \mathbf{1}_{y \leq \frac{k}{N}} dy = \frac{k}{N}$. If $t' < \frac{k}{N}$, $\int_0^{t'} \mathbf{1}_{U_N(y) \leq t} dy = \int_0^{t'} \mathbf{1}_{y \leq \frac{k}{N}} dy = t'$, and

$$\frac{1}{N} \sum_{i=1}^N \mathbf{1}_{x_i \leq t, \frac{i}{N} \leq t'} = \frac{1}{N} \sum_{i=1}^k \mathbf{1}_{i \leq Nt'} = \frac{\lfloor Nt' \rfloor}{N} \xrightarrow{N \rightarrow \infty} t'.$$

Then, we know from [21] that $\sup_{0 \leq y \leq 1} |U_N(y) - y| \xrightarrow[N \rightarrow \infty]{a.s.} 0$, and hence almost surely, for any fixed $y \in [0, 1]$, $U_N(y) \xrightarrow[N \rightarrow \infty]{p.s.} y$ and by dominated convergence

$$\int_0^{t'} \mathbf{1}_{U_N(y) \leq t} dy \xrightarrow[N \rightarrow \infty]{} \int_0^{t'} \mathbf{1}_{y \leq t} dy = \min(t, t'),$$

which concludes the proof. \square

Proposition 7.6. *Under the Scenario (1) of Definition 3.8, for any function g such that $\|g\|_{L^\chi(I \times I), \nu \times \nu} < \infty$ with $\chi > 5$,*

$$\sup_{1 \leq i \leq N} \int_I g(x_i, y) \left(\nu^{(N)}(dy) - \nu(dy) \right) \xrightarrow[N \rightarrow \infty]{} 0 \quad (7.31)$$

\mathbb{P} -almost surely on the realisation of the sequence $(\underline{x}^{(N)})_N$.

Proof. Fix $M > 0$, and define the function $p_M(u) = u \mathbf{1}_{|u| \leq M} + M \mathbf{1}_{u > M} - M \mathbf{1}_{u < -M}$ on \mathbb{R} . Set $g_M = p_M \circ g$. The following arguments come from [46] in the proof of Proposition 3.4. We have

$$\begin{aligned} \sup_{1 \leq i \leq N} \int_I g(x_i, y) \left(\nu^{(N)}(dy) - \nu(dy) \right) &\leq \sup_{1 \leq i \leq N} \frac{1}{N} \sum_{j=1}^N |g(x_i, x_j) - g_M(x_i, x_j)| \\ &+ \sup_{1 \leq i \leq N} \int_I |g(x_i, y) - g_M(x_i, y)| \nu(dy) + \sup_{1 \leq i \leq N} \int_I g_M(x_i, y) \left(\nu^{(N)}(dy) - \nu(dy) \right) =: (I) + (II) + (III). \end{aligned}$$

To study (I), note that $|g(x, y) - g_M(x, y)| = |g(x, y) - g_M(x, y)| \mathbf{1}_{|g(x, y)| > M} \leq 2|g(x, y)| \mathbf{1}_{|g(x, y)| > M}$, and that for any independent X, Y with law ν

$$\begin{aligned} \mathbb{E} [|g(X, Y)| \mathbf{1}_{|g(X, Y)| > M}] &= \sum_{l=0}^{+\infty} \mathbb{E} \left[|g(X, Y)| \mathbf{1}_{2^l M < |g(X, Y)| \leq 2^{l+1} M} \right] \\ &\leq \sum_{l=0}^{+\infty} 2^{l+1} M \left(\mathbb{P} \left(|g(X, Y)| > 2^l M \right) - \mathbb{P} \left(|g(X, Y)| > 2^{l+1} M \right) \right) \\ &= 2M \mathbb{P} (|h(X, Y)| > M) + \sum_{l=1}^{+\infty} 2^l M \mathbb{P} \left(|g(X, Y)| > 2^l M \right) \\ &\leq \mathbb{E} [|g(X, Y)|^\chi] \left(\frac{2}{M^{\chi-1}} + \sum_{l=1}^{+\infty} \frac{2^l M}{(2^l M)^\chi} \right) \leq \frac{3 \mathbb{E} [|g(X, Y)|^\chi]}{M^{\chi-1}}, \end{aligned}$$

where we used Markov inequality. As $\mathbf{E} \left[\frac{1}{N} \sum_{l=1}^N |g(x_i, x_l) - g_M(x_i, x_l)| \right] \leq \frac{2}{N} \sum_{l=1}^N \mathbb{E} [|g(x_i, x_l)| \mathbf{1}_{|g(x_i, x_l)| > M}]$,

it implies, for the choice of $M = N^{\delta_1}$ with $\delta_1 > 0$ to define later, using Markov inequality and a union bound that

$$\mathbb{P} \left((I) > \frac{1}{N^{\delta_2}} \right) \leq \frac{6 \mathbb{E} [|g(X, Y)|^\chi]}{N^{\delta_1(\chi-1) - \delta_2 - 1}}.$$

Similarly, we can show that

$$\mathbb{P} \left((II) > \frac{1}{N^{\delta_2}} \right) \leq \frac{6 \mathbb{E} [|g(X, Y)|^\chi]}{N^{\delta_1(\chi-1) - \delta_2 - 1}}.$$

We will use the two precedent bounds with Borel-Cantelli Lemma to deduce that \mathbb{P} -almost surely, $(I) + (II) \xrightarrow[N \rightarrow \infty]{} 0$ by asking $\delta_1(\chi - 1) - \delta_2 - 1 > 1$. To deal with (III)

we use the boundedness of g_M . Note that (III) can be re-written $\sup_{1 \leq i \leq N} \frac{1}{N} \sum_{l=1}^N Y_l^{(i),M}$ with $Y_l^{(i),M} := g_M(x_i, x_l) - \int_I g_M(x_i, y) \nu(dy) = g_M(x_i, x_l) - \mathbb{E}[g_M(x_i, Y)|x_i]$. We set $\mathcal{F}_l^{(i)} = \sigma(x_i, x_1, \dots, x_l)$. We have for $l \neq i$

$$\mathbb{E} \left[Y_l^{(i),M} \middle| \mathcal{F}_{l-1}^{(i)} \right] = \mathbb{E} \left[U_M(x_i, x_l) - \mathbb{E}_Y [U_M(x_i, Y)|x_i] \middle| \mathcal{F}_{l-1}^{(i)} \right] = 0.$$

As $\left| Y_l^{(i),M} \right| \leq 2M$, we can then apply Lemma A.6 : for any $x > 0$,

$$\mathbb{P} \left(\frac{1}{N-1} \sum_{\substack{l=1 \\ l \neq i}}^N \frac{Y_l^{(i),M}}{2M} \geq x \right) \leq \exp \left(-(N-1) \frac{x^2}{2} B(x) \right)$$

with the function B defined in (A.2). We consider a sequence ε_N such that $\varepsilon_N \xrightarrow[N \rightarrow \infty]{} 0$

(we precise later which one), and apply the precedent result with $x = \frac{\varepsilon_N N}{2M(N-1)}$. As

$B(u) = u^{-2} ((1+u) \log(1+u) - u) \rightarrow \frac{1}{2}$ when $u \rightarrow 0$, we can choose a deterministic p such that for all $N \geq p$, $B \left(\frac{\varepsilon_N N}{2M(N-1)} \right) \geq \frac{1}{4}$. We then have if $N \geq p$:

$$\mathbb{P} \left(\frac{1}{N} \sum_{\substack{l=1 \\ l \neq i}}^N Y_l^{(i),M} \geq \varepsilon_N \right) \leq \exp \left(-\frac{1}{32M^2} \frac{\varepsilon_N^2 N^2}{N-1} \right),$$

doing the same for $-Y_l^{(i)}$ and with a union bound we obtain

$$\mathbb{P} \left(\sup_{1 \leq i \leq N} \left| \frac{1}{N} \sum_{\substack{l=1 \\ l \neq i}}^N Y_l^{(i),M} \right| \geq \varepsilon_N \right) \leq 2N \exp \left(-\frac{1}{32M^2} \frac{\varepsilon_N^2 N^2}{N-1} \right).$$

It is sufficient to find ε_N such that $\varepsilon_N \xrightarrow[N \rightarrow \infty]{} 0$ and $\sum 2N \exp \left(-\frac{1}{32M^2} \frac{\varepsilon_N^2 N^2}{N-1} \right) < \infty$ to

conclude by Borel-Cantelli's Lemma, \mathbb{P} -almost surely if N is large enough $\sup_{1 \leq i \leq N} \left| \frac{1}{N} \sum_{\substack{l=1 \\ l \neq i}}^N Y_l^{(i),M} \right| \leq$

ε_N . We set then $\varepsilon_N^2 := 32M^2(N-1)N^\gamma$, and require $\gamma > -2$ and $\gamma < -1 - 2\delta_1$. As $Y_i^{(i),M}$ is bounded (by $2M$), adding the term $\frac{1}{N} Y_i^{(i),M}$ does not change the convergence if $\delta_1 < 1$ which is already asked for the conditions on ε_N (recall $M = N^{\delta_1}$).

All we remain to find is parameters $(\delta_1, \delta_2, \gamma)$ such that $\delta_1 > 0$, $\delta_2 > 0$, $\delta_1(\chi - 1) - \delta_2 - 1 > 1$, $\gamma > -2$ and $\gamma < -1 - 2\delta_1$ (to ensure that the probabilities obtained with (I) , (II) and (III) are summable and the sufficient conditions on ε). As $\chi > 5$, any choices such that

$\delta_1 \in (0, \frac{1}{2})$ and $\delta_2 \in (0, 1)$ work (as $\delta_1(\chi - 1) - 1 < 1$) with $\gamma \in (-2, -1 - 2\delta_1)$, and we obtain (7.31) \mathbb{P} -almost surely. \square

Corollary 7.7. *Under Scenario (1) of Definition 3.8, we define*

$$\epsilon_{i,1} := \int_{I \times I} W(x_i, y)W(x_i, z)\Gamma(y, z) \left(\nu^{(N)}(dy)\nu^{(N)}(dz) - \nu(dy)\nu^{(N)}(dz) \right) \quad (7.32)$$

$$\epsilon_{i,2} := \int_{I \times I} W(x_i, y)W(x_i, z)\Gamma(y, z) \left(\nu(dy)\nu^{(N)}(dz) - \nu(dy)\nu(dz) \right), \quad (7.33)$$

where Γ is defined in (7.18). Then under Hypothesis 3.7, \mathbb{P} -almost surely,

$$\sup_{1 \leq i \leq N} \epsilon_{i,1} \xrightarrow{N \rightarrow \infty} 0 \quad \text{and} \quad \sup_{1 \leq i \leq N} \epsilon_{i,2} \xrightarrow{N \rightarrow \infty} 0.$$

Proof. Note that $\epsilon_{i,2} = \int_I \phi(x_i, z) (\nu^{(N)}(dz) - \nu(dz))$, with $\phi(x, z) := W(x, z) \int_I W(x, y)\Gamma(y, z)\nu(dy)$. As Γ is bounded, $|\phi(x, z)| \leq |W(x, z)| \|\Gamma\|_\infty C_{W_1}$ and since $W \in L^X(I^2, \nu \times \nu)$, $\|\phi\|_{L^X(I \times I), \nu \times \nu} < \infty$, (7.33) is an immediate application of Proposition 7.6. Similarly, note that $\epsilon_{i,1} = \int_I g_N(x_i, y) (\nu^{(N)}(dy) - \nu(dy))$, with $g_N(x, y) := W(x, y) \int_I W(x, z)\Gamma(y, z)\nu^{(N)}(dz)$. Define also $g(x, y) := W(x, y) \int_I W(x, z)\Gamma(y, z)\nu(dz)$, and then

$$\epsilon_{i,1} = \int_I (g_N(x_i, y) - g(x_i, y)) (\nu^{(N)}(dy) - \nu(dy)) + \int_I g(x_i, y) (\nu^{(N)}(dy) - \nu(dy)).$$

We have immediatly (as done with (7.33)) that $\sup_{1 \leq i \leq N} \int_I g(x_i, y) (\nu^{(N)}(dy) - \nu(dy)) \xrightarrow{N \rightarrow \infty} 0$.

For the other term, that we denote by $\epsilon_{i,3}$, we have $\epsilon_{i,3} = \int_I W(x_i, y)\alpha_N(x_i, y) (\nu^{(N)}(dy) - \nu(dy))$ where $\alpha_N(x_i, y) := \int_I W(x_i, z)\Gamma(y, z) (\nu^{(N)}(dz) - \nu(dz))$. As Γ is bounded, Proposition 7.6 (and its proof) gives that $\alpha_N(x_i, y) \xrightarrow{N \rightarrow \infty} 0$ uniformly in i and y . Another application of Proposition 7.6 gives then that $\sup_{1 \leq i \leq N} \epsilon_{i,3} \xrightarrow{N \rightarrow \infty} 0$ which concludes the proof. \square

7.4.2. Proof of Proposition 7.4 for Scenario (1). We treat the estimates (7.4), (7.5), (7.6) and (7.7) separately.

Proof of (7.4). We remind that we want to prove $d_{\square, \nu} \left(W_N^{\mathcal{G}_N^{(2)}}, W \right) \xrightarrow{N \rightarrow \infty} 0$, when the positions are i.i.d according to ν on I . Recall the definition of (x_1, \dots, x_N) as the lexicographic reordering of the i.i.d. sample $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_N)$. The proof is organised as follow : we start by looking at the case $d = 1$, $I = [0, 1]$ and ν is the Lebesgue measure on I , and then extend to the general case. We start by approximating W by \tilde{W} continuous in $L^1(I^2, \nu)$.

Step 1 - Approximation of W in norm L^1 . We first prove that for $\varepsilon > 0$, there exists $m \geq 1$ sufficiently large such that $\|W - W_{\mathcal{P}_m}\|_{L^1(I^2)} \leq \varepsilon$.

We fix $\varepsilon > 0$. As $W \in L^1(I^2, \nu)$, there exists \tilde{W} continuous such that $\|W - \tilde{W}\|_{1, \nu} \leq \frac{\varepsilon}{3}$. As \tilde{W} is also uniformly continuous, there exists $\eta > 0$ such that if $\|u - u'\| + \|v - v'\| \leq \eta$, $|\tilde{W}(u, v) - \tilde{W}(u', v')| \leq \frac{\varepsilon}{3}$. We fix m large enough such that $\frac{1}{m} \leq \eta$, and denote by $\mathcal{P}_m = \sqcup_{i=1}^m J_i$ the partition $J_i = (\frac{i-1}{m}, \frac{i}{m}]$. It verifies then, for each $i \in (1, \dots, m)$ $\text{Diam}(J_i) \leq \eta$. We define the step function (which average the values of W over cells obtained with the partition)

$$W_{\mathcal{P}_m}(u, v) := m^2 \sum_{i,j=1}^m \int_{J_i \times J_j} W(x, y) \nu(dx) \nu(dy) \mathbf{1}_{J_i}(u) \mathbf{1}_{J_j}(v). \quad (7.34)$$

We note $\mathcal{G}_N^{(3)}$ the directed weighted graph with vertices $\{1, \dots, N\}$ such that every edge $j \rightarrow i$ is present, with weight $W_{\mathcal{P}_m}(x_i, x_j)$. We use it to upper-bound the cut-distance between W and $W^{\mathcal{G}_N^{(2)}}$:

$$\begin{aligned} d_{\square, \nu} \left(W^{\mathcal{G}_N^{(2)}}, W \right) &\leq \|W^{\mathcal{G}_N^{(2)}} - W\|_{1, \nu} \\ &\leq \|W^{\mathcal{G}_N^{(2)}} - W^{\mathcal{G}_N^{(3)}}\|_{1, \nu} + \|W^{\mathcal{G}_N^{(3)}} - W_{\mathcal{P}_m}\|_{1, \nu} + \|W_{\mathcal{P}_m} - W\|_{1, \nu}. \end{aligned} \quad (7.35)$$

We are going to control each of these terms in (7.35) in the following steps.

Step 2 - Control of $\|W_{\mathcal{P}_m} - W\|_{1, \nu}$. We have

$$\|W_{\mathcal{P}_m} - W\|_{1, \nu} \leq \|W_{\mathcal{P}_m} - \tilde{W}_{\mathcal{P}_m}\|_{1, \nu} + \|\tilde{W}_{\mathcal{P}_m} - \tilde{W}\|_{1, \nu} + \|\tilde{W} - W\|_{1, \nu}.$$

As $\|W - \tilde{W}\|_{1, \nu} \leq \frac{\varepsilon}{3}$, and as for any partition \mathcal{P} , $\|W_{\mathcal{P}}\|_{1, \nu} \leq \|W\|_{1, \nu}$, we have $\|W_{\mathcal{P}_m} - \tilde{W}_{\mathcal{P}_m}\|_{1, \nu} \leq \frac{\varepsilon}{3}$ and

$$\begin{aligned} \|\tilde{W}_{\mathcal{P}_m} - \tilde{W}\|_{1, \nu} &= \sum_{i,j=1}^m \int_{J_i} \int_{J_j} \left| \tilde{W}(u, v) - m^2 \int_{J_i} \int_{J_j} \tilde{W}(x, y) \nu(dx) \nu(dy) \right| \nu(du) \nu(dv) \\ &\leq \sum_{i,j=1}^m \int_{J_i} \int_{J_j} m^2 \int_{J_i} \int_{J_j} \left| \tilde{W}(u, v) - \tilde{W}(x, y) \right| \nu(dx) \nu(dy) \nu(du) \nu(dv) \leq \frac{\varepsilon}{3}, \end{aligned}$$

hence $\|W_{\mathcal{P}_m} - W\|_{1, \nu} \leq \varepsilon$.

Step 3 - Control of $\|W^{\mathcal{G}_N^{(2)}} - W^{\mathcal{G}_N^{(3)}}\|_{1, \nu}$. For all $N \geq 1$, we recall $(B_1^{(N)}, \dots, B_N^{(N)})$ the partition of I with $B_i = \left(\frac{i-1}{N}, \frac{i}{N} \right]$ (we omit by simplicity the (N)). Using the notation introduced in (3.6) we have

$$\begin{aligned} \|W^{\mathcal{G}_N^{(2)}} - W^{\mathcal{G}_N^{(3)}}\|_{1, \nu} &= \sum_{i,j=1}^N \int_{B_i} \int_{B_j} |W(x_i, x_j) - W_{\mathcal{P}_m}(x_i, x_j)| \nu(du) \nu(dv) \\ &= \frac{1}{N^2} \sum_{i,j=1}^N |W(x_i, x_j) - W_{\mathcal{P}_m}(x_i, x_j)| =: \frac{1}{N^2} \sum_{i,j=1}^N F(x_i, x_j). \end{aligned} \quad (7.36)$$

We use the following proposition to show that it converges almost surely to $\|W_{\mathcal{P}_m} - W\|_{1, \nu}$.

Proposition 7.8 (Hoeffding [40]). *Let X_1, X_2, \dots be a sequence of i.i.d random variables with distribution ν , and f a real-valued measurable function. Then if $\mathbb{E}[|f(X_1, X_2)|] < +\infty$,*

$$\frac{1}{N(N-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^N f(X_i, X_j) \xrightarrow[N \rightarrow \infty]{a.s.} \mathbb{E}[f(X_1, X_2)] = \int \int f(x, y) \nu(dx) \nu(dy). \quad (7.37)$$

We have indeed

$$\frac{1}{N^2} \sum_{i,j=1}^N F(x_i, x_j) = \frac{1}{N^2} \sum_{i=1}^N F(x_i, x_i) + \underbrace{\frac{N(N-1)}{N^2}}_{\rightarrow 1} \underbrace{\frac{1}{N(N-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^N F(x_i, x_j)}_{\rightarrow \int \int F(x,y) \nu(dx) \nu(dy) \text{ a.s.}}$$

and

$$\frac{1}{N^2} \sum_{i=1}^N F(x_i, x_i) \leq \frac{1}{N} \left(\frac{1}{N} \sum_{i=1}^N |W(x_i, x_i)| + \frac{1}{N} \sum_{i=1}^N |W_{\mathcal{P}_m}(x_i, x_i)| \right) \xrightarrow{N \rightarrow \infty} 0$$

as the sums are controlled by Hypothesis 2.6.

Step 4 - Control of $\|W_{\mathcal{G}_N^{(3)}} - W_{\mathcal{P}_m}\|_{1,\nu}$. We have

$$\|W_{\mathcal{G}_N^{(3)}} - W_{\mathcal{P}_m}\|_{1,\nu} = \sum_{i,j=1}^N \int_{B_i} \int_{B_j} |W_{\mathcal{P}_m}(x_i, x_j) - W_{\mathcal{P}_m}(x, y)| \nu(dx) \nu(dy). \quad (7.38)$$

Recalling (7.34) and setting $\alpha_{kl} = m^2 \int_{J_k \times J_l} W(u, v) \nu(du) \nu(dv)$ we have

$$\|W_{\mathcal{G}_N^{(3)}} - W_{\mathcal{P}_m}\|_{1,\nu} = \sum_{k,l} \sum_{k',l'} |\alpha_{kl} - \alpha_{k'l'}| \sum_{i,j=1}^N \mathbf{1}_{J_k \times J_l}(x_i, x_j) \nu(J_{k'} \cap B_i) \nu(J_{l'} \cap B_j).$$

We consider N large enough ($N > m$) such that every box $B_i =]\frac{i-1}{N}, \frac{i}{N}]$ (of size $\frac{1}{N}$) is inside a larger box $J_{k'} =]\frac{k'-1}{m}, \frac{k'}{m}]$ (of size $\frac{1}{m}$) (note that there might be some B_i that are on two different parts of the partition \mathcal{P}_m , but we can neglect this contribution - at most of order $\frac{m}{N} \xrightarrow{N \rightarrow \infty} 0$). Then $\nu(J_{k'} \cap B_i) = \mathbf{1}_{\{B_i \subset J_{k'}\}} \nu(B_i) = \frac{\mathbf{1}_{\{\frac{i}{N} \in J_{k'}\}}}{N}$, and

$$\begin{aligned} \|W_{\mathcal{G}_N^{(3)}} - W_{\mathcal{P}_m}\|_{1,\nu} &\leq \sum_{k,l=1}^m \sum_{k',l'=1}^m |\alpha_{kl} - \alpha_{k'l'}| \sum_{i,j=1}^N \frac{1}{N^2} \mathbf{1}_{\{(x_i, x_j) \in J_k \times J_l, B_i \subset J_{k'}, B_j \subset J_{l'}\}} \\ &\leq \sum_{k,l=1}^m \sum_{k',l'=1}^m |\alpha_{kl} - \alpha_{k'l'}| \left(\frac{1}{N} \sum_{i=1}^N \mathbf{1}_{\{x_i \in J_k, \frac{i}{N} \in J_{k'}\}} \right) \left(\frac{1}{N} \sum_{j=1}^N \mathbf{1}_{\{x_j \in J_l, \frac{j}{N} \in J_{l'}\}} \right). \end{aligned}$$

Then, from Lemma 7.5, $\frac{1}{N} \sum_{i=1}^N \mathbf{1}_{\{x_i \in J_k, \frac{i}{N} \in J_{k'}\}} \xrightarrow{N \rightarrow \infty, \text{ a.s.}} \lambda(J_k \cap J_{k'}) = \frac{1}{m} \mathbf{1}_{k=k'}$, thus almost-surely (on the realisation of the sequence of positions) we have $\|W_{\mathcal{G}_N^{(3)}} - W_{\mathcal{P}_m}\|_{1,\nu} \xrightarrow{N \rightarrow \infty} 0$.

Conclusion when the positions are uniformly drawn - From (7.35), we have shown then that $\limsup_{N \rightarrow \infty} d_{\square, \nu} \left(W_{\mathcal{G}_N^{(2)}}, W \right) \leq \varepsilon$ which concludes the proof for the case $x_i \sim \mathcal{U}(0, 1)$.

Generalisation : from $[0, 1]$ to $[0, 1]^d$ - Consider the case $x_i = \left(u_i^{(1)}, \dots, u_i^{(d)} \right)$ where $\left(u_i^{(j)} \right)_{1 \leq j \leq d}$ are drawn uniformly on $(0, 1]$ (but not necessarily independent), and the

partition $I = (0, 1]^d = \bigsqcup_{i=1}^N B_i^{(N)} = \bigsqcup_{i=1}^N \left(\left(\frac{i-1}{N}, \frac{i}{N} \right] \times (0, 1]^{d-1} \right)$. Proposition 7.8 still apply, and the treatment of the terms $\|W^{\mathcal{G}_N^{(2)}} - W\|_{1,\nu}$ and $\|W_{\mathcal{P}_m} - W\|_{1,\nu}$ in (7.35) remains the same. For the term $\|W^{\mathcal{G}_N^{(3)}} - W_{\mathcal{P}_m}\|_{1,\nu}$, it suffices to note that the chosen partition $\bigsqcup B_i^{(N)}$ only affects the first coordinates to conclude by the same arguments.

Generalisation : from $[0, 1]^d$ to \mathbb{R}^d - Consider the case $I \subset \mathbb{R}^d$. From Sklar's theorem (originally in [61], see Theorem 2.3.3 of [53]) we have :

$$f_\nu(x^{(1)}, \dots, x^{(d)}) = c(F_1(x^{(1)}), \dots, F_d(x^{(d)})) f_1(x^{(1)}) \dots f_d(x^{(d)}),$$

where c is the copula density function of ν , f_i the i -th marginal probability density functions, F_i the i -th marginal cumulative distribution functions and f_ν the joint probability function of the law ν (that satisfies $\nu(dx) = f_\nu(x^{(1)}, \dots, x^{(d)}) dx^{(1)} \dots dx^{(d)}$). It implies that, for the change of variables $u = (F_1(x^{(1)}), \dots, F_d(x^{(d)}))$, we have $c(u) du = f_\nu(x) dx$. Define also $u_i = (F_1(x_i^{(1)}), \dots, F_d(x_i^{(d)}))$ and

$$W_F(u, v) := W \left((F_1^{-1}(u^{(1)}), \dots, F_d^{-1}(u^{(d)})), (F_1^{-1}(v^{(1)}), \dots, F_d^{-1}(v^{(d)})) \right),$$

the previous change of variable gives then

$$\begin{aligned} \|W^{\mathcal{G}_N^{(2)}} - W\|_{1,\nu} &= \sum_{i,j=1}^N \int_{B_i} \int_{B_j} |W(x_i, x_j) - W(x, y)| \nu(dx) \nu(dy) \\ &= \sum_{i,j=1}^N \int_{\left(\frac{i-1}{N}, \frac{i}{N}\right] \times \mathbb{R}^{d-1}} \int_{\left(\frac{j-1}{N}, \frac{j}{N}\right] \times \mathbb{R}^{d-1}} |W_F(u_i, u_j) - W_F(u, v)| c(u) c(v) du dv. \end{aligned}$$

The precedent case gives immediately the result.

Proof of (7.5). We remind that we want to prove $\|W^{\mathcal{G}_N^{(2)}} - W\|_{\infty \rightarrow \infty, \nu} \xrightarrow{N \rightarrow \infty} 0$. As in the proof of (7.4), we start with the case $I = [0, 1]$, $\tilde{x}_i \sim \mathcal{U}(0, 1)$ i.i.d. (then ν is the Lebesgue measure). What changes is that we no longer integrate with respect to the first variable, but we take the supremum. The approximation in $L^1(I^2)$ is not adapted anymore, thus we approximate W differently. Recall that

$$\|W^{\mathcal{G}_N^{(2)}} - W\|_{\infty \rightarrow \infty, \nu} = \sup_{g, \|g\|_\infty \leq 1} \sup_{u \in I} \left| \int_I (W^{\mathcal{G}_N^{(2)}}(u, v) - W(u, v)) g(v) \nu(dv) \right|.$$

Step 1 - A first bound. Fixing g such that $\|g\|_\infty \leq 1$ and $u \in I$, for any N there exists a unique i such that $u \in B_i^{(N)} = \left(\frac{i-1}{N}, \frac{i}{N} \right]$. Then

$$\begin{aligned} \left| \int_I (W^{\mathcal{G}_N^{(2)}}(u, v) - W(u, v)) g(v) \nu(dv) \right| &= \left| \sum_{j=1}^N \int_{B_j} (W(x_i, x_j) - W(u, v)) g(v) \nu(dv) \right| \\ &\leq \left| \sum_{j=1}^N \int_{B_j} (W(x_i, x_j) - W(x_i, v)) g(v) \nu(dv) \right| \\ &\quad + \left| \int_I (W(x_i, v) - W(u, v)) g(v) \nu(dv) \right| =: A(g, u) + B(g, u). \end{aligned}$$

Step 2 - Upper-bound of $A(g, u)$ by approximated functions independent of g . As $\|g\|_\infty \leq 1$, we have $A(g, u) \leq \sum_{j=1}^N \int_{B_j} |W(x_i, x_j) - W(x_i, v)| dv$. Note that it does not depend anymore on g and it depends on u only by the index i . To control this term, we first approximate W by a stepfunction in $L^1(I)$, $\tilde{W}_{\mathcal{P}_m}$. Introduce $(\varphi_\eta)_{\eta>0}$ as $\varphi_\eta(x) = \eta^{-1} \phi(\frac{x}{\eta})$ where ϕ is a non-negative continuous function of I with $\int \phi = 1$. Define for all $x \in I$ $\tilde{W}_\eta(x, \cdot) := W(x, \cdot) * \varphi_\eta$. Note that $y \mapsto \tilde{W}_\eta(x, y) \in \mathbb{R}$ is a continuous function for all $x \in [0, 1]$. As for any $(x, x') \in I^2$, $\|W(x, \cdot) - W(x', \cdot)\|_1 \leq C_w \|x - x'\|^t$ using (2.5), $x \mapsto W(x, \cdot)$ is continuous from $[0, 1]$ to $L^1(I)$, so that the set of functions $F := \{W(x, \cdot), x \in [0, 1]\}$ is compact. Hence, for $\varepsilon > 0$, we can then find $p \geq 1$ and p positions y_1, \dots, y_p such that $F \subset \cup_{k=1}^p B_{L^1}(W(y_k, \cdot), \varepsilon)$. Then, there exists $\eta > 0$ such that for all $k \leq p$, $\|\tilde{W}_\eta(y_k, \cdot) - W(y_k, \cdot)\|_{I,1} \leq \varepsilon$. From now, we may omit the notation η for \tilde{W} .

Let $m \geq 1$ and $\mathcal{P}_m = \sqcup_{i=1}^m J_i$ for $J_r = \left(\frac{r-1}{m}, \frac{r}{m}\right]$ the regular partition of I of order m . For any kernel H on I^2 , define

$$H_{\mathcal{P}_m}(x, v) := m \sum_{r=1}^m \left(\int_{J_r} H(x, y) dy \right) \mathbf{1}_{J_r}(v). \quad (7.39)$$

Note that $H \mapsto H_{\mathcal{P}_m}$ is continuous : $\|H_{\mathcal{P}_m}\|_{L^1(I^2)} \leq \|H\|_{L^1(I^2)}$. Note that this definition is different from the one used in the proof of (7.4) where we integrated on both variables.

By continuity of $y \mapsto \tilde{W}(y_k, y)$ for all $k = 1 \dots p$, there exists $m \geq 1$ such that $\sup_{1 \leq l \leq p} \|\tilde{W}(y_l, \cdot) - \tilde{W}_{\mathcal{P}_m}(y_l, \cdot)\|_\infty \leq \varepsilon$, and thus $\sup_{1 \leq l \leq p} \int \left| \tilde{W}(y_l, y) - \tilde{W}_{\mathcal{P}_m}(y_l, y) \right| dy \leq \varepsilon$.

Then, for any $x \in I$,

$$\begin{aligned} \|W(x, \cdot) - \tilde{W}_{\mathcal{P}_m}(x, \cdot)\|_{I,1} &\leq \underbrace{\|W(x, \cdot) - W(y_l, \cdot)\|_{I,1}}_{\leq \varepsilon \text{ by the cover of } F} + \underbrace{\|W(y_l, \cdot) - \tilde{W}(y_l, \cdot)\|_{I,1}}_{\leq \varepsilon \text{ by the choice of } \eta} \\ &+ \underbrace{\|\tilde{W}(y_l, \cdot) - \tilde{W}_{\mathcal{P}_m}(y_l, \cdot)\|_{I,1}}_{\leq \varepsilon \text{ by the choice of } m} + \|\tilde{W}_{\mathcal{P}_m}(y_l, \cdot) - \tilde{W}_{\mathcal{P}_m}(x, \cdot)\|_{I,1} \\ &\leq 3\varepsilon + \|\tilde{W}(y_l, \cdot) - \tilde{W}(x, \cdot)\|_{I,1} \leq 4\varepsilon, \end{aligned}$$

where we used the fact that for any partition \mathcal{P} , $\|\tilde{W}_{\mathcal{P}}\|_{I,1} \leq \|\tilde{W}\|_{I,1}$ and $\|(W(x, \cdot) - W(y_l, \cdot)) * \varphi_\eta\|_{I,1} \leq \|W(x, \cdot) - W(y_l, \cdot)\|_{I,1} \cdot \|\varphi_\eta\|_{I,1} \leq \varepsilon$.

Using this approximation, we can now upper bound $A(g, u)$ independently of the choice of g and relying on the choice of u only by the index i such that $u \in B_i^{(N)}$: we have

$$\begin{aligned} A(g, u) &\leq \frac{1}{N} \sum_{j=1}^N \left| W(x_i, x_j) - \tilde{W}_{\mathcal{P}_m}(x_i, x_j) \right| + \sum_{j=1}^N \int_{B_j} \left| \tilde{W}_{\mathcal{P}_m}(x_i, x_j) - \tilde{W}_{\mathcal{P}_m}(x_i, v) \right| dv \\ &+ \sum_{j=1}^N \int_{B_j} \left| \tilde{W}_{\mathcal{P}_m}(x_i, v) - W(x_i, v) \right| dv =: A_1^{(i)} + A_2^{(i)} + A_3^{(i)}. \end{aligned}$$

Step 3 - Uniform control of the $A_k^{(i)}$. As $A_3^{(i)} = \|\tilde{W}_{\mathcal{P}_m}(x_i, \cdot) - W(x_i, \cdot)\|_{I,1}$, we control it by the work done previously independently of the index i (see Step 2): $\sup_i A_3^{(i)} \xrightarrow{N \rightarrow \infty} 0$. Set $g(x, y) := W(x, y) - \tilde{W}_{\mathcal{P}_m}(x, y)$, and as $W \in L^X(I^2)$, so does g . We can then apply

Proposition 7.6 and we obtain $\sup_{1 \leq i \leq N} |A_3^{(i)} - A_1^{(i)}| = \sup_{1 \leq i \leq N} \int_I g(x_i, y) (\nu^{(N)}(dy) - \nu(dy)) \xrightarrow{N \rightarrow \infty} 0$.

We focus now on $A_2^{(i)}$ and show that $\sup_x \sum_{j=1}^N \int_{B_j} |\tilde{W}_{\mathcal{P}_m}(x, x_j) - \tilde{W}_{\mathcal{P}_m}(x, v)| dv$ tends to 0 : denoting by $\alpha_k(x) = m \int_{J_k} \tilde{W}(x, y) dy$, we have

$$\begin{aligned} \sum_{j=1}^N \int_{B_j} |\tilde{W}_{\mathcal{P}_m}(x, x_j) - \tilde{W}_{\mathcal{P}_m}(x, v)| dv &= \sum_{j=1}^N \int_{B_j} \left| \sum_{k=1}^m \alpha_k(x) \mathbf{1}_{J_k}(x_j) - \sum_{k'=1}^m \alpha_{k'}(x) \mathbf{1}_{J_{k'}}(v) \right| dv \\ &\leq \sum_{k, k'=1}^m |\alpha_k(x) - \alpha_{k'}(x)| \sum_{j=1}^N \mathbf{1}_{J_k}(x_j) |J_{k'} \cap B_j|. \end{aligned}$$

Similarly at what has been done in Step 4 for the proof of (7.4), we consider N large enough ($N > m$) such that every box $B_i =]\frac{i-1}{N}, \frac{i}{N}]$ is inside a larger box $J_{k'} =]\frac{k'-1}{m}, \frac{k'}{m}]$, then $\nu(J_{k'} \cap B_j) = \mathbf{1}_{\{B_j \subset J_{k'}\}} \nu(B_j) = \frac{\mathbf{1}_{\{\frac{j}{N} \in J_{k'}\}}}{N}$ and

$$\sum_{j=1}^N \int_{B_j} |\tilde{W}_{\mathcal{P}_m}(x, x_j) - \tilde{W}_{\mathcal{P}_m}(x, v)| dv \leq \sum_{\substack{k, k'=1 \\ k \neq k'}}^m |\alpha_k(x) - \alpha_{k'}(x)| \sum_{j=1}^N \frac{1}{N} \mathbf{1}_{\{x_j \in J_k, \frac{j}{N} \in J_{k'}\}}.$$

As $\alpha_k(x) \leq m \int_I \tilde{W}(x, y) dy \leq m C_{W_1}$ which is independent of x and k ,

$$\sum_{j=1}^N \int_{B_j} |\tilde{W}_{\mathcal{P}_m}(x, x_j) - \tilde{W}_{\mathcal{P}_m}(x, v)| dv \leq 2m C_{W_1} \sum_{\substack{k, k'=1 \\ k \neq k'}}^m \sum_{j=1}^N \frac{1}{N} \mathbf{1}_{\{x_j \in J_k, \frac{j}{N} \in J_{k'}\}}.$$

From Lemma 7.5, $\frac{1}{N} \sum_{j=1}^N \mathbf{1}_{\{x_j \in J_k, \frac{j}{N} \in J_{k'}\}} \xrightarrow[N \rightarrow \infty]{a.s.} \lambda(J_k \cap J_{k'}) = \frac{1}{m} \mathbf{1}_{k=k'}$, thus almost-surely (on the realisation of the sequence of positions) $\sum_{j=1}^N \int_{B_j} |\tilde{W}_{\mathcal{P}_m}(x, x_j) - \tilde{W}_{\mathcal{P}_m}(x, v)| dv$ tends to 0 independently on the choice of x . We have shown that $\sup_{g, \|g\|_\infty \leq 1} \sup_{u \in I} A(g, u) \xrightarrow{N \rightarrow \infty} 0$ \mathbb{P} -almost surely.

Step 4 - Control of $B(g, u)$ and conclusion. Using(2.5) from Hypothesis 2.6, we have

$$B(g, u) \leq \int_I |W(x_i, v) - W(u, v)| \|g\|_\infty \nu(dv) \leq C_w \|x_i - u\|^t.$$

Let us show that $\sup_{x \in I} \sum_{i=1}^N \mathbf{1}_{B_i}(x) \|x_i - x\|^t \xrightarrow{N \rightarrow \infty} 0$. Recall (7.30) : we have $\sum_{i=1}^N \mathbf{1}_{B_i}(x) \|x_i - x\|^t = \|U_N(x) - x\|^t$ by definition of U_N , the uniform sample quantile function. As we know from [21] that $\sup_{0 \leq y \leq 1} |U_N(y) - y| \xrightarrow{N \rightarrow \infty}{a.s.} 0$, almost surely $\sup_{g, \|g\|_\infty \leq 1} \sup_{u \in I} B(g, u) \xrightarrow{N \rightarrow \infty} 0$.

It concludes the proof for (7.5).

Proof of (7.7) and (7.6). The term of interest is $A_{i,T,5}^{(N)}$, defined in (7.14), we have by Jensen's inequality

$$\begin{aligned} A_{i,T,5}^{(N)2} &\leq \left(\int_0^T \left| \int_I F(x_i, y, s) \left(\nu^{(N)}(dy) - \nu(dy) \right) \right| ds \right)^2 \\ &\leq T \int_0^T \left(\frac{1}{N} \sum_{j=1}^N \left(F(x_i, x_j, s) - \int_I F(x_i, y, s) \nu(dy) \right) \right)^2 ds. \\ &\leq \frac{T}{N^2} \sum_{j,l=1}^N \int_0^T \left(F(x_i, x_j, s) F(x_i, x_l, s) + \left(\int_I F(x_i, y, s) \nu(dy) \right)^2 \right. \\ &\quad \left. - 2F(x_i, x_j, s) \int_I F(x_i, y, s) \nu(dy) \right) ds. \end{aligned}$$

As for any y and z , $F(x_i, y, s) F(x_i, z, s) = W(x_i, y)W(x_i, z)\gamma(s, y)\gamma(s, z)$, setting $\Gamma(y, z) = \int_0^T \gamma(s, y)\gamma(s, z)ds$ we obtain

$$\begin{aligned} A_{i,T,5}^{(N)2} &\leq \frac{T}{N^2} \sum_{j,l=1}^N W(x_i, x_j)W(x_i, x_l)\Gamma(x_j, x_l) - \frac{2T}{N} \sum_{j=1}^N \int_I W(x_i, x_j)W(x_i, y)\Gamma(x_j, y)\nu(dy) \\ &\quad + T \int_{I^2} W(x_i, y)W(x_i, z)\Gamma(y, z)\nu(dy)\nu(dz) \\ &= T \int_{I \times I} W(x_i, y)W(x_i, z)\Gamma(y, z) \left(\nu^{(N)}(dy)\nu^{(N)}(dz) - 2\nu^{(N)}(dy)\nu(dz) + \nu(dy)\nu(dz) \right) \\ &= T(\epsilon_{i,1} + \epsilon_{i,2}) \end{aligned} \tag{7.40}$$

where $\epsilon_{i,1}$ and $\epsilon_{i,2}$ are defined and studied in Corollary 7.7. Taking the square root and then summing on i or taking the supremum, (7.7) and (7.6) follow.

Remark 7.9. If we ask for more regularity of W , we can have a more direct proof of (7.7). Assume that there exist $L_W > 0$ and $M_W > 0$ such that

$$\sup_{x \in I} \sup_{y \neq y'} \frac{|W(x, y) - W(x, y')|}{\|y - y'\|} \leq L_W \text{ and } \sup_{x, y \in I} |W(x, y)| \leq M_W.$$

Remark then that Hypothesis 2.6 is trivially satisfied with $\iota = 1$ and $C_w = L_W$, which implies that λ is uniformly Lipschitz continuous in the second variable (in (2.7), $\phi(x) = 2\|x\|$). We show first that F defined above in (7.36) is also uniformly Lipschitz continuous in the second variable : for any $(x, y, y', s) \in I^3 \times [0, T]$,

$$F(x, y, s) - F(x, y', s) = (W(x, y) - W(x, y')) \gamma(s, y) + W(x, y') \int_0^s h(s-u) (\lambda(u, y) - \lambda(u, y')) du,$$

then

$$\begin{aligned} |F(x, y, s) - F(x, y', s)| &\leq |W(x, y) - W(x, y')| |\gamma(s, y)| + |W(x, y')| \int_0^s |h(s-u)| |\lambda(u, y) - \lambda(u, y')| du \\ &\leq \|h\|_{T,1} \|y - y'\| (\|\lambda\|_{[0,T] \times I, \infty} L_W + 2M_W C_\lambda) =: L_F \|y - y'\|, \end{aligned}$$

with $L_F > 0$ independent of the choice of s and x .

As $A_{i,t,5}^{(N)} = \int_0^t \left| \int_I F(x_i, y, s) (\nu^{(N)}(dy) - \nu(dy)) \right| ds$ and F is uniformly Lipschitz continuous in the second variable with constant L_F , we have

$$\frac{1}{N} \sum_{i=1}^N A_{i,T,5}^{(N)} \leq L_F \sup_{g \in BL} \int_0^T \left| \int_I g(y) (\nu^{(N)}(dy) - \nu(dy)) \right| ds \leq T L_F d_{BL}(\nu^{(N)}, \nu) \xrightarrow{N \rightarrow \infty} 0$$

by Varadarajan Theorem (see [29] Theorem 11.4.1 and [65]).

7.4.3. Proof of Proposition 7.4 for Scenario (2). As the positions are regularly distributed, we use regularity on the interaction kernel W . Recall that $I = [0, 1]$, $x_i^{(N)} = \frac{i}{N}$, and $\nu(dx) = dx$. We focus on the case W continuous. When W is piecewise continuous, the same results follow as we can work on each rectangles where W can be extended in a continuous function, and these rectangles are in finite number.

Proof of (7.4). Using Remark 3.3 and (3.3), we have

$$\begin{aligned} d_{\square} \left(W^{\mathcal{G}_N^{(2)}}, W \right) &\leq \|W^{\mathcal{G}_N^{(2)}} - W\|_{\infty \rightarrow 1} = \sup_{\|f\|_{\infty}, \|g\|_{\infty} \leq 1} \left| \int \int \left(W^{\mathcal{G}_N^{(2)}} - W \right) (x, y) f(x) g(y) dx dy \right| \\ &\leq \int \int \left| \left(W^{\mathcal{G}_N^{(2)}} - W \right) (x, y) \right| dx dy = \|W^{\mathcal{G}_N^{(2)}} - W\|_{L^1, [0,1]^2} \\ &= \sum_{i,j=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{j-1}{N}}^{\frac{j}{N}} \left| W \left(\frac{i}{N}, \frac{j}{N} \right) - W(x, y) \right| dx dy. \end{aligned}$$

As W is continuous on the compact $[0, 1]^2$ in this scenario, it is uniformly continuous due to Heine-Cantor theorem thus for any $\varepsilon > 0$, there exists $\eta > 0$ such that $|x - x'| + |y - y'| \leq \eta \Rightarrow |W(x, y) - W(x', y')| < \varepsilon$. For N large enough, $\frac{1}{N} < \eta$ and then (7.4) holds as $d_{\square} \left(W^{\mathcal{G}_N^{(2)}}, W \right) \leq \sum_{i,j=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{j-1}{N}}^{\frac{j}{N}} \varepsilon dx dy = \varepsilon$.

Proof of (7.5). Remind that

$$\|W^{\mathcal{G}_N^{(2)}} - W\|_{\infty \rightarrow \infty, \nu} = \sup_{g, \|g\|_{\infty} \leq 1} \sup_{u \in [0,1]} \left| \int_0^1 \left(W^{\mathcal{G}_N^{(2)}}(u, v) - W(u, v) \right) g(v) dv \right|.$$

As done for (7.4), we use the uniform continuity of W : for any $\varepsilon > 0$, we take $\eta > 0$ such that $|x - x'| + |y - y'| \leq \eta \Rightarrow |W(x, y) - W(x', y')| < \varepsilon$. Fix g such that $\|g\|_{\infty} \leq 1$ and $u \in]0, 1]$, for any N there exists a unique i such that $u \in B_i^{(N)} = \left(\frac{i-1}{N}, \frac{i}{N} \right]$. For N large enough, $\frac{2}{N} < \eta$ and we have then

$$\begin{aligned} \left| \int_0^1 \left(W^{\mathcal{G}_N^{(2)}}(u, v) - W(u, v) \right) g(v) dv \right| &= \left| \sum_{j=1}^N \int_{B_j} \left(W \left(\frac{i}{N}, \frac{j}{N} \right) - W(u, v) \right) g(v) dv \right| \\ &\leq \sum_{j=1}^N \int_{\frac{j-1}{N}}^{\frac{j}{N}} \left| W \left(\frac{i}{N}, \frac{j}{N} \right) - W(u, v) \right| |g(v)| dv \leq \varepsilon. \end{aligned}$$

independently from the choices of g and u : we have shown for this Scenario that $\|W^{\mathcal{G}_N^{(2)}} - W\|_{\infty \rightarrow \infty, \nu} \xrightarrow{N \rightarrow \infty} 0$.

Proof of (7.7) and (7.6). As W is continuous on $[0, 1]^2$ and $(s, y) \mapsto \gamma(s, y) = \int_0^s h(s-u)\lambda(u, y)du$ is also continuous on $[0, T] \times [0, 1]$ as a convolution between h locally integrable and λ continuous, the application $(x, y, s) \mapsto F(x, y, s) = W(x, y)\gamma(s, y)$ is continuous on the compact $K = [0, 1] \times [0, 1] \times [0, T]$, it is uniformly continuous due to Heine-Cantor theorem. Then, for $\varepsilon > 0$, there exists $\eta > 0$ such that for any (x, y, s) and (x', y', s') in K , $|x - x'| + |y - y'| + |s - s'| \leq \eta \Rightarrow |F(x, y, s) - F(x', y', s')| < \varepsilon$. For N large enough, $\frac{1}{N} < \eta$ and we have then

$$\begin{aligned} A_{i,T,5}^{(N)} &= \int_0^T \left| \sum_{j=1}^N \int_{\frac{j-1}{N}}^{\frac{j}{N}} F(x_i, x_j, s) dy - \sum_{j=1}^N \int_{\frac{j-1}{N}}^{\frac{j}{N}} F(x_i, y, s) dy \right| ds \\ &\leq \int_0^T \sum_{j=1}^N \int_{\frac{j-1}{N}}^{\frac{j}{N}} |F(x_i, x_j, s) - F(x_i, y, s)| dy ds \leq T\varepsilon. \end{aligned} \quad (7.41)$$

Summing on i or taking the supremum, (7.7) and (7.6) follow.

8. PROOFS : THE EMPIRICAL MEASURE AND THE SPATIALE PROFILE

8.1. Proof of Theorem 3.15. We prove the convergence of $\mathbf{E} [d_{BL}(\mu_N, \mu_\infty)] \xrightarrow[N \rightarrow \infty]{} 0$.

Proof. Some of the following arguments come from [15]. We consider $\mathbb{D}([0, T], \mathbb{N})$ with the distance d_0 introduced in [6] (§14) which makes it complete, and we have for any η, ζ in $\mathbb{D}([0, T], \mathbb{N})$, $d_0(\eta, \zeta) \leq \sup_{t \leq T} |\eta(t) - \zeta(t)|$. Recall

$$d_{BL}(\mu_N, \mu_\infty) = \sup_{\phi, \|\phi\|_{BL} \leq 1} \left| \int \phi d\mu_N - \int \phi d\mu_\infty \right|.$$

We start by proving that for any ϕ fixed, $\mathbf{E} \left| \int \phi (d\mu_N - d\mu_\infty) \right| \xrightarrow[N \rightarrow \infty]{} 0$. By an argument of compactness, we show how it implies (3.19).

Step 1 - Convergence when ϕ is fixed. We fix ϕ a real-valued function on S such that $\|\phi\|_{BL} \leq 1$. Then with the coupling introduced in Definition 3.1 :

$$\begin{aligned} \mathbf{E} \left| \int \phi (d\mu_N - d\mu_\infty) \right| &= \mathbf{E} \left| \frac{1}{N} \sum_{i=1}^N \phi \left(Z_i^{(N)}, x_i \right) - \int \phi(\eta, x) P_{[0, T], \infty}(d\eta|x) \nu(dx) \right| \\ &\leq \mathbf{E} \left| \frac{1}{N} \sum_{i=1}^N \left(\phi \left(Z_i^{(N)}, x_i \right) - \phi \left(\bar{Z}_i, x_i \right) \right) \right| \\ &+ \mathbf{E} \left| \frac{1}{N} \sum_{i=1}^N \left(\phi \left(\bar{Z}_i, x_i \right) - \int \phi(\eta, x_i) P_{[0, T], \infty}(d\eta|x_i) \right) \right| \\ &+ \mathbf{E} \left| \frac{1}{N} \sum_{i=1}^N \int \phi(\eta, x_i) P_{[0, T], \infty}(d\eta|x_i) - \int \phi(\eta, x) P_{[0, T], \infty}(d\eta|x) \nu(dx) \right| \\ &:= A + B + C. \end{aligned}$$

The term A is treated easily with Theorems 3.10 or 3.12 : as ϕ is Lipschitz continuous and $\|\phi\|_L \leq 1$,

$$A \leq \frac{1}{N} \sum_{i=1}^N \mathbf{E} \left[d_0 \left(Z_i^{(N)}, \bar{Z}_i \right) \right] \leq \frac{1}{N} \sum_{i=1}^N \mathbf{E} \left[\sup_{t \leq T} \left| Z_i^{(N)}(t) - \bar{Z}_i(t) \right| \right] \xrightarrow{N \rightarrow \infty} 0.$$

To treat B , we set for each $i \in \llbracket 1, N \rrbracket$ $G_i := \phi(\bar{Z}_i, x_i)$, it is a random variable with expectation $\int \phi(\eta, x_i) P_{[0, T], \infty}(d\eta | x_i)$. We have then applying Lemma A.2

$$B \leq \frac{1}{N} \sqrt{\sum_{i=1}^N \text{Var}(G_i)}.$$

To calculate $\text{Var}(G_i)$, let $(\tilde{Z}_i(t))_{0 \leq t \leq T}$ be an independent copy of $(\bar{Z}_i(t))_{0 \leq t \leq T}$ and set $\tilde{G}_i := g(\tilde{Z}_i, x_i)$, then with \tilde{E} the expectation taken with respect to \tilde{G}_i , we have

$$\text{Var}(G_i) = \mathbf{E} \left[(G_i - \mathbf{E}[G_i])^2 \right] = \mathbf{E} \left[\tilde{E} \left[G_i - \tilde{G}_i \right]^2 \right] \leq \mathbf{E} \left[\tilde{E} \left[(G_i - \tilde{G}_i)^2 \right] \right]$$

by Jensen's inequality. We have, as $\|g\|_L \leq 1$:

$$\tilde{E} \left[(G_i - \tilde{G}_i)^2 \right] \leq \tilde{E} \left[d_0 \left(\bar{Z}_i, \tilde{Z}_i \right)^2 \right] \leq \tilde{E} \left[\left(\sup_{0 \leq t \leq T} \left| \bar{Z}_i(t) - \tilde{Z}_i(t) \right| \right)^2 \right] \leq 2\bar{Z}_i(T)^2 + 2\tilde{E} \left[\tilde{Z}_i(T)^2 \right]$$

as the processes are increasing. Thus we obtain

$$\text{Var}(G_i) \leq 4\mathbf{E} \left[\bar{Z}_i(T)^2 \right].$$

As $Z_i(T)$ is a Poisson random variable with rate $\int_0^T \lambda(t, x_i) dt$,

$$\mathbf{E} \left[\bar{Z}_i(T)^2 \right] = \text{Var} \left(\tilde{Z}_i(T) \right) + \left(\mathbf{E} \left[\tilde{Z}_i(T) \right] \right)^2 = \int_0^T \lambda(t, x_i) dt + \left(\int_0^T \lambda(t, x_i) dt \right)^2$$

which is finite as λ is bounded (Proposition 2.7). We have then shown that $B \xrightarrow{N \rightarrow \infty} 0$.

To treat C , note that it can be rewritten

$$C = \left| \int \int \phi(\eta, x) P_{[0, T], \infty}(d\eta | x) \left(\nu^{(N)}(dx) - \nu(dx) \right) \right|.$$

We denote by h the function bounded on I such that $h(x) = \int \phi(\eta, x) P_{[0, T], \infty}(d\eta | x)$. Under Scenarios (1), $C = \left| \frac{1}{N} \sum_{i=1}^N h(x_i) - \int_I h(x) \nu(dx) \right| \xrightarrow{N \rightarrow \infty} 0$ by the Law of Large Numbers.

Under Scenario (2), we recognise a Riemann sum with $C = \left| \frac{1}{N} \sum_{i=1}^N h\left(\frac{i}{N}\right) - \int_0^1 h(x) dx \right|$: it suffices to show that h is continuous to have $C \xrightarrow{N \rightarrow \infty} 0$. Fix x in I and consider a sequence (x_n) such that $x_n \xrightarrow{n \rightarrow \infty} x$. We have

$$|h(x_n) - h(x)| \leq \int |\phi(\eta, x_n) - \phi(\eta, x)| P_{[0, T], \infty}(d\eta | x_n) + \left| \int \phi(\eta, x) \left(P_{[0, T], \infty}(d\eta | x) - P_{[0, T], \infty}(d\eta | x_n) \right) \right|.$$

We deal with the first term : by the Lipschitz continuity of ϕ and the fact that $P_{[0, T], \infty}(\cdot | x_n)$ is a probability measure, we have $\int |\phi(\eta, x_n) - \phi(\eta, x)| P_{[0, T], \infty}(d\eta | x_n) \leq \|x - x_n\| \xrightarrow{n \rightarrow \infty} 0$. As x is fixed, to have the second term $\left| \int \phi(\eta, x) \left(P_{[0, T], \infty}(d\eta | x) - P_{[0, T], \infty}(d\eta | x_n) \right) \right| \xrightarrow{n \rightarrow \infty} 0$.

0, we show that for any function ψ with Lipschitz constant $\|\psi\|_L \leq 1$ defined on $\mathbb{D}([0, T], \mathbb{N})$, the fonction $\rho(y) := \int \psi(\eta) P_{[0, T], \infty}(d\eta|y)$ is continuous on I : let π be a random Poisson measure with intensity $dsdz$ on $\mathbb{R}_+ \times \mathbb{R}_+$, and for each $y \in I$ construct a Poisson point process \bar{Z}^y on $[0, T]$ with intensity $\lambda(\cdot, y)$ by taking $\bar{Z}^y(t) = \int_0^t \int_0^\infty \mathbf{1}_{z \leq \lambda(s, y)} \pi(ds, dz)$. Then, as ψ is Lipschitz continuous,

$$\begin{aligned} |\rho(x) - \rho(x_n)| &= |\mathbf{E} [\psi(\bar{Z}^{x_n}) - \psi(\bar{Z}^x)]| \leq \mathbf{E} [d_0(\bar{Z}^{x_n}, \bar{Z}^x)] \\ &\leq \mathbf{E} \left[\sup_{0 \leq t \leq T} |\bar{Z}^{x_n}(t) - \bar{Z}^x(t)| \right] \leq \mathbf{E} \left[\int_0^t |d(\bar{Z}^{x_n}(s) - \bar{Z}^x(s))| \right] \\ &\leq \mathbf{E} \left[\int_0^t \int_0^\infty |\mathbf{1}_{z \leq \lambda(s, x_n)} - \mathbf{1}_{z \leq \lambda(s, x)}| \pi(ds, dz) \right] \\ &\leq \int_0^t |\lambda(s, x_n) - \lambda(s, x)| ds \leq T \|x - x_n\| \xrightarrow[n \rightarrow \infty]{} 0, \end{aligned}$$

where we used (2.7). Then ρ is indeed continuous on I , and so is h and then $C = \left| \frac{1}{N} \sum_{i=1}^N h\left(\frac{i}{N}\right) - \int_0^1 h(x) dx \right| \xrightarrow[N \rightarrow \infty]{} 0$. We have then shown that for any function ϕ on S such that $\|\phi\|_{BL} \leq 1$, we have

$$\mathbf{E} \left| \int \phi(d\mu_N - d\mu_\infty) \right| \xrightarrow[N \rightarrow \infty]{} 0. \quad (8.1)$$

Step 2 - Approximation of any ϕ by a finite set of functions and conclusion. To have (3.19), we use an argument that can be found in Lemma 4.5 of [46] and Theorem 11.3.3 of [29].

For all $\varepsilon > 0$, there exists a compact set $K \in S$ with $\mu_\infty(K) > 1 - \varepsilon$. The set of functions $B := \{\phi|_K, \|\phi\|_{BL} \leq 1\}$, restricted to K is a compact set by Arzela-Ascoli Theorem, hence there exists $k \geq 1$ and k functions in B ϕ_1, \dots, ϕ_k such that for any ϕ satisfying $\|\phi\|_{BL} \leq 1$, there exists $j \leq k$ that verifies $\sup_{y \in K} |\phi(y) - \phi_j(y)| \leq \varepsilon$. We denote by $K^\varepsilon := \{z \in S, d_S(z, K) < \varepsilon\}$. Then $\sup_{z \in K^\varepsilon} |\phi(z) - \phi_j(z)| < 3\varepsilon$ as for any $z \in K^\varepsilon$, we can find $y_z \in K$ such that $d_S(z, y_z) < \varepsilon$ and

$$\begin{aligned} |\phi(z) - \phi_j(z)| &\leq |\phi(z) - \phi(y_z)| + |\phi(y_z) - \phi_j(y_z)| + |\phi_j(y_z) - \phi_j(z)| \\ &\leq \|\phi\|_L d_S(z, y_z) + \varepsilon + \|\phi_j\|_L d_S(z, y_z) \leq 3\varepsilon. \end{aligned}$$

We introduce the function on S : $g(z) = \max\left(0, 1 - \frac{d_S(z, K)}{\varepsilon}\right)$. Note that $\mathbf{1}_K \leq g \leq \mathbf{1}_{K^\varepsilon}$ and g is bounded and Lipschitz continuous. Then, integrating on μ_N , we obtain $\mu_N(K^\varepsilon) \geq \int g d\mu_N$. We put together all the precedent bounds to have, for any ϕ such that $\|\phi\|_{BL} \leq 1$:

$$\begin{aligned} \left| \int \phi(d\mu_\infty - d\mu_N) \right| &\leq \int |\phi - \phi_j|(d\mu_\infty + d\mu_N) + \left| \int \phi_j(d\mu_\infty - d\mu_N) \right| \\ &\leq \int_{K^\varepsilon} |\phi - \phi_j|(d\mu_\infty + d\mu_N) + \int_{S - K^\varepsilon} |\phi - \phi_j|(d\mu_\infty + d\mu_N) + \left| \int \phi_j(d\mu_\infty - d\mu_N) \right| \\ &\leq 3\varepsilon \cdot 2 + 2\mu_\infty(S - K^\varepsilon) + 2\mu_N(S - K^\varepsilon) + \left| \int \phi_j(d\mu_\infty - d\mu_N) \right|. \end{aligned}$$

Hence, taking the supremum on such function ϕ we obtain

$$\sup_{\phi, \|\phi\|_{BL} \leq 1} \left| \int \phi (d\mu_N - d\mu_\infty) \right| \leq 8\varepsilon + 2 \left(1 - \int g d\mu_N \right) + \max_{1 \leq j \leq k} \left| \int \phi_j (d\mu_\infty - d\mu_N) \right|.$$

Using (8.1), for N large enough $\mathbb{E} [\int g d\mu_N] > \int g d\mu_\infty - \varepsilon$ and as $\int g d\mu_\infty \geq \mu_\infty(K) \geq 1 - \varepsilon$, we have $\mathbb{E} [\int g d\mu_N] > 1 - 2\varepsilon$ and then

$$\mathbf{E} [d_{BL}(\mu_N, \mu_\infty)] \leq 12\varepsilon + \mathbf{E} \left[\max_{1 \leq j \leq k} \left| \int \phi_j (d\mu_\infty - d\mu_N) \right| \right]$$

and using (8.1), $\mathbf{E} [\max_{1 \leq j \leq k} |\int \phi_j (d\mu_\infty - d\mu_N)|] \xrightarrow{N \rightarrow \infty} 0$ as there is a finite number of functions considered, which concludes the proof of (3.19). \square

8.2. Proof of Proposition 3.17. We show the convergence of the spatial profil U_N , when the positions are regularly distributed on $[0, 1]$ and W continuous. We have

$$\begin{aligned} \mathbf{E} \left[\int_0^T \int_0^1 |U_N(t, x) - u(t, x)| dx dt \right] &\leq \mathbf{E} \left[\int_0^T \int_0^1 \left| \sum_{i=1}^N \mathbf{1}_{(\frac{i-1}{N}, \frac{i}{N}]}(x) (U_{i,N}(t) - u(t, x)) \right| dx dt \right] \\ &\leq \mathbf{E} \left[\int_0^T \frac{1}{N} \sum_{i=1}^N |U_{i,N}(t) - u(t, x_i)| dt \right] + \int_0^T \int_0^1 \left| \sum_{i=1}^N \mathbf{1}_{(\frac{i-1}{N}, \frac{i}{N}]}(x) (u(t, x_i) - u(t, x)) \right| dx dt. \end{aligned}$$

The first term is dealt with the proof of Theorem 3.10 : recall (7.9), we recognise

$$\int_0^T \mathbf{E} [|U_{i,N}(t) - u(t, x_i)|] dt \leq \left(\sum_{k=1}^5 A_{i,t,k}^{(N)} \right),$$

and we have showed that each $\frac{1}{N} \sum_{i=1}^N A_{i,t,k}^{(N)} \xrightarrow{N \rightarrow \infty} 0$ \mathbb{P} -almost surely. We then have

$$\mathbf{E} \left[\int_0^T \frac{1}{N} \sum_{i=1}^N |U_{i,N}(t) - u(t, x_i)| dt \right] \xrightarrow{N \rightarrow \infty} 0 \text{ } \mathbb{P}\text{-almost surely.}$$

The other term is treated easily : as u is continuous on the compact set $[0, T] \times [0, 1]$ it is uniformly continuous. Fix $\varepsilon > 0$, then there exists $\eta > 0$ such that if $\|t - t'\| + \|x - x'\| \leq \eta$, $|u(t, x) - u(t', x')| \leq \frac{\varepsilon}{T}$. We have then for N large enough (such that $\frac{1}{N} \leq \eta$) :

$$\int_0^T \int_0^1 \left| \sum_{i=1}^N \mathbf{1}_{(\frac{i-1}{N}, \frac{i}{N}]}(x) (u(t, x_i) - u(t, x)) \right| dx dt = \int_0^T \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} |u(t, x_i) - u(t, x)| dx dt \leq \int_0^T \frac{\varepsilon}{T} dt = \varepsilon,$$

which concludes the proof. \square

9. PROOFS : BEHAVIOR IN LARGE TIME LIMIT - LINEAR CASE

9.1. Proof of Theorem 4.4. We show that in the subcritical case, $\lambda(\cdot, x)$ has a large time limit that is the solution of (4.5). The subcritical case implies that there exists n_0 such that $\|h\|_1^{n_0} \|T_W^{n_0}\| < 1$.

Step 1 - We show existence and uniqueness of ℓ by applying Banach fixed-point Theorem. We consider the map defined on $\mathcal{C}_b(I, \mathbb{R}_+)$ (the set of bounded continuous functions defined on I taking non-negative values) :

$$F : g \longmapsto F(g) \text{ such that for all } x \in I,$$

$$F(g)(x) = u(x) + \|h\|_1 \int_I W(x, y)g(y)\nu(dy).$$

As u is bounded on I , $F(g)$ is bounded for any $g \in \mathcal{C}_b(I, \mathbb{R}_+)$ by $\|u\|_\infty + \|h\|_1 \|g\|_\infty C_{W_1} < \infty$.

We check now that for any g , $F(g)$ is continuous. Let $(x, z) \in I \times I$. We have as u is Lipschitz continuous and using (2.5), for any $g \in \mathcal{C}_b(I, \mathbb{R}_+)$:

$$|F(g)(x) - F(g)(z)| \leq |u(x) - u(z)| + \|h\|_1 \left| \int_I (W(x, y) - W(z, y)) g(y)\nu(dy) \right|$$

$$\leq \|u\|_L \|x - z\| + \|h\|_1 \|g\|_\infty \|x - z\|^t$$

We have then shown the existence of a constant C_g independent of the choice of (x, z) such that $|F(g)(x) - F(g)(z)| \leq C_g \phi(\|x - z\|)$.

Hence, $\mathcal{C}_b(I, \mathbb{R}_+)$ is stable by F . We are going to prove that F admits an **unique fixed point**, which is ℓ satisfying (4.5). To do it, we show that some iteration of F is contractive, and then the Banach fixed-point Theorem gives the result. Let g and \tilde{g} be two functions in $\mathcal{C}_b(I, \mathbb{R}_+)$. As $Fg = u + \|h\|_1 T_W g$, we have immediately that $F^{n_0} g = \sum_{k=0}^{n_0-1} \|h\|_1^k T_W^k u + \|h\|_1^{n_0} T_W^{n_0} g$. Then

$$\|F^{n_0} g - F^{n_0} \tilde{g}\| = \|h\|_1^{n_0} \|T_W^{n_0}(g - \tilde{g})\| \leq \|h\|_1^{n_0} \|T_W^{n_0}\| \|g - \tilde{g}\|_\infty.$$

As n_0 is chosen such that $\|h\|_1^{n_0} \|T_W^{n_0}\| < 1$, F^{n_0} is contractive, thus has an unique fixed point which is also the unique fixed point of F in $\mathcal{C}_b(I, \mathbb{R}_+)$ that we call ℓ , solution to (4.6).

Step 2 - Let us show that under the present hypotheses, $\sup_{t \geq 0} \sup_{x \in I} |\lambda(t, x)| < \infty$. As $\lambda(t, x) = u_0(t, x) + h * (T_W \lambda)(t, x)$, $T_W \lambda(t, x) = T_W u_0(t, x) + h * T_W^2 \lambda(t, x)$ and the iteration gives $\lambda(t, x) = \left(\sum_{k=0}^{n_0-1} h^{*k} * T_W^k u_0 \right)(t, x) + h^{*n_0} * T_W^{n_0} \lambda(t, x)$ for any $(t, x) \in \mathbb{R}_+ \times I$, hence $\|\lambda(t, \cdot)\|_\infty \leq C(u_0, h, W) + \|h\|_1^{n_0} \|T_W^{n_0}\| \|\lambda(t, \cdot)\|_\infty$ with $C(u_0, h, W)$ a positive constant. As we are in the subcritical case, it gives then $\sup_{t \geq 0} \|\lambda(t, \cdot)\|_\infty \leq \frac{C(u_0, h, W)}{1 - \|h\|_1^{n_0} \|T_W^{n_0}\|} < \infty$.

As λ is then continuous and bounded on $\mathbb{R}_+ \times I$, we can define its (temporal) Laplace transform : for any $x \in I$ and $z > 0$, let

$$\Lambda(z, x) := \int_0^\infty e^{-tz} \lambda(t, x) dt. \quad (9.1)$$

Let us study $z\Lambda(z, x)$. We have, for any $x \in I$ and $z > 0$, $z\Lambda(z, x) = \int_0^\infty z e^{-tz} \lambda(t, x) dt = \lambda(0, x) + \int_0^\infty e^{-tz} \frac{\partial \lambda}{\partial t}(t, x) dt$. Suppose that we are able to show that $I(x) := \int_0^\infty \left| \frac{\partial \lambda}{\partial t}(t, x) \right| < \infty$ for some x . Then, by dominated convergence theorem, $\int_0^\infty e^{-tz} \frac{\partial \lambda}{\partial t}(t, x) dt$ converges as $z \rightarrow \infty$ to the finite limit $\int_0^\infty \frac{\partial \lambda}{\partial t}(t, x) dt$. This implies in particular that $\lambda(t, x)$ has a finite limit as $t \rightarrow \infty$, and we have in this case $\lim_{z \rightarrow 0} \Lambda(z, x) = \lim_{t \rightarrow \infty} \lambda(t, x)$.

We have, by integrating by parts

$$\begin{aligned}\frac{\partial \lambda}{\partial t}(t, x) &= \frac{\partial u_0}{\partial t}(t, x) + h(0) \int_I W(x, y) \lambda(t, y) \nu(dy) + \int_0^t \int_I W(x, y) h'(t-s) \lambda(s, y) \nu(dy) ds \\ &= \frac{\partial u_0}{\partial t}(t, x) + \int_I W(x, y) h(t) \lambda(0, y) \nu(dy) + \int_0^t \int_I W(x, y) h(t-s) \frac{\partial \lambda}{\partial s}(s, y) \nu(dy) ds,\end{aligned}$$

where we used Proposition 2.7 for the regularity of $\frac{\partial \lambda}{\partial s}$. We also know from Proposition 2.7 that $(t, x) \rightarrow \frac{\partial \lambda}{\partial t}(t, x)$ is bounded of $[0, T] \times I$ for any $T > 0$, which implies that for any $A > 0$, $\sup_{x \in I} \int_0^A \left| \frac{\partial \lambda}{\partial t}(t, x) \right| dt < \infty$. Integrating on $[0, A]$, we have

$$\int_0^A \left| \frac{\partial \lambda}{\partial t}(t, x) \right| dt \leq 2\|u_0\|_\infty + \|h\|_1 \|\lambda\|_\infty D(x) + \int_0^A \int_0^t \int_I W(x, y) h(t-s) \left| \frac{\partial \lambda}{\partial s}(s, y) \right| dt ds \nu(dy).$$

Yet with a change of variable

$$\begin{aligned}\int_0^A \int_0^t \int_I W(x, y) h(t-s) \left| \frac{\partial \lambda}{\partial s}(s, y) \right| dt ds \nu(dy) &= \int_I W(x, y) \int_0^A \left(\int_s^A h(t-s) dt \right) \left| \frac{\partial \lambda}{\partial s}(s, y) \right| ds \nu(dy) \\ &\leq \|h\|_1 \int_I W(x, y) \int_0^A \left| \frac{\partial \lambda}{\partial t}(t, y) \right| dt \nu(dy).\end{aligned}$$

Setting $I_A(x) := \int_0^A \left| \frac{\partial \lambda}{\partial s}(s, x) \right|$ and $C = 2\|u_0\|_\infty + \|h\|_1 \|\lambda\|_\infty C_{W_1}$, we have shown that $I_A(x) \leq C + \|h\|_1 (T_W I_A)(x)$ and by iteration

$$I_A(x) \leq \sum_{k=0}^{n_0-1} C \|h\|_1^k C_{W_1}^k + \|h\|_1^{n_0} T_W^{n_0} I_A(x) \leq \sum_{k=0}^{n_0-1} C \|h\|_1^k C_{W_1}^k + \|h\|_1^{n_0} \|T_W^{n_0}\| \|I_A\|_\infty,$$

and then $\|I_A\|_\infty \leq \frac{C(h, u_0, W)}{1 - \|h\|_1^{n_0} \|T_W^{n_0}\|} = C'$, with C' a positive constant independent of A .

We can then make $A \rightarrow \infty$ to obtain $\sup_{x \in I} I(x) < \infty$. Hence by dominated convergence $\lim_{z \rightarrow 0} z\Lambda(z, x)$ exists for any $x \in I$ and is equal to $\lim_{t \rightarrow \infty} \lambda(t, x) =: \ell(x)$ that can now be defined.

Coming back to the definition on Λ , we do the same for u_0 and define for any $x \in I$ and $z > 0$ $U(z, x) := \int_0^\infty e^{-tz} u_0(t, x) dt$. As $u_0(t, x) \xrightarrow[t \rightarrow \infty]{} u(x)$, note that $\lim_{z \rightarrow 0} zU(z, x) = u(x)$. As h is integrable in this framework, we can also define its Laplace transform for any $z \geq 0$ by $H(z) := \int_0^\infty e^{-tz} h(t) dt$, with $H(0) = \|h\|_1$. Using the fact that the Laplace transform of a convolution is the product of the Laplace transforms, we have for any $x \in I$ and $z > 0$

$$z\Lambda(z, x) = u(x) + H(z) \int_I W(x, y) z\Lambda(z, y) \nu(dy). \quad (9.2)$$

Doing $z \rightarrow 0$ in (9.2), we obtain that ℓ is solution of the equation (4.5). \square

9.2. Proof of Proposition 4.7. We show that in the supercritical case, $\int_I \lambda(t, x)^2 \nu(dx) \xrightarrow[t \rightarrow \infty]{} \infty$.

We start by giving some properties on the operator T_W under Hypothesis 4.6. We consider now T_W from L^2 to L^2 : for any g in $L^2(I)$, using Cauchy Schwarz inequality

$$\begin{aligned} \|T_W g\|_2^2 &= \int \left(\int W(x, y) g(y) \nu(dy) \right)^2 \nu(dx) \\ &\leq \int \left(\int W(x, y)^2 \nu(dy) \right) \left(\int g(y)^2 \nu(dy) \right) \nu(dx) \leq \|g\|_2^2 C_{W_2}. \end{aligned}$$

As $T_W : L^2 \rightarrow L^2$ is compact, and so is T_W^2 . Let us show that $T_W^2 : L^\infty \rightarrow L^\infty$ is also compact. Consider $(f_n)_n$ a bounded sequence of L^∞ . It is then also bounded in L^2 , and as $T_W : L^2 \rightarrow L^2$ is compact, there exists a subsequence $(f_{\phi(n)})$ such that $T_W f_{\phi(n)}$ converges in L^2 to a certain g . Then for any $x \in I$,

$$|T_W^2 f_{\phi(n)} - T_W g|(x) \leq \int_I W(x, y) |T_W f_{\phi(n)}(y) - g(y)| dy \leq \sqrt{C_{w_2}} \|T_W f_{\phi(n)} - g\|_2 \xrightarrow[n \rightarrow \infty]{} 0,$$

hence $T_W^2 : L^\infty \rightarrow L^\infty$ is compact. Note that $T_W^2 : L^2 \rightarrow L^2$ and $T_W^2 : L^\infty \rightarrow L^\infty$ have the same spectral radius: denote by $\sigma_\infty(T_W^2)$ and $\sigma_2(T_W^2)$ their spectrum. Let $\mu \in \sigma_2(T_W^2) - \{0\}$, there exists $g \in L^2(I)$ such that $\mu g = T_W^2 g$. As

$$|T_W^2 g(x)| = \left| \int_I W'(x, y) \int_I W(y, z) g(z) \nu(dz) \nu(dy) \right| \leq C_{W_1} \sqrt{C_{W_2}} \|g\|_2 < \infty,$$

$g = \frac{1}{\mu} T_W^2 g \in L^\infty(I)$ and $\mu \in \sigma_\infty(T_W^2)$. Let $\mu \in \sigma_\infty(T_W^2) - \{0\}$, there exists $g \in L^\infty(I)$ such that $\mu g = T_W^2 g$. As $L^\infty \subset L^2$, $\mu \in \sigma_2(T_W^2)$. We denote now by $r(T_W^2) = r_\infty(T_W^2) = r_2(T_W^2)$. We have also that $T_W : L^\infty \rightarrow L^\infty$ is compact and as $\sqrt{r(T_W^2)} = \left(\lim_{n \rightarrow \infty} \|T_W^{2n}\|_n^{\frac{1}{n}} \right)^{\frac{1}{2}} = \lim_{n \rightarrow \infty} \|T_W^{2n}\|^{\frac{1}{2n}} = r_\infty(T_W) = r$, we similarly have $r_\infty = r_\infty(T_W) = r_2(T_W)$. With (4.13), T_W is moreover a self-adjoint compact operator.

Now, we focus on the proof of 4.7. First, we consider the case where $k = 1$ (recall (4.11)). Then, under Hypothesis 4.6, Jentzsch/Krein–Rutman Theorem (which can be found in [67] as Theorem 1 of and its consequences) gives that $r_\infty = r_\infty(T_W)$ is a single eigenvalue of T_W and every other eigenvalue μ verifies $|\mu| < r_\infty$, and there exists $h_0 \in L^2(I)$ eigenvector of r_∞ such that $h_0(x) > 0$ ν -almost everywhere. As $\|T_W h_0\|_\infty \leq \sqrt{C_{W_2}} \|h_0\|_2$ (using Cauchy-Schwarz inequality) and $h_0 = \frac{1}{r(T_W)} T h_0$, h_0 is bounded. With (2.5), it implies that $T_W h_0$ is continuous on I (it satisfies a Hölder condition), hence h_0 is a positive continuous function on I . As T_W is self-adjoint, we can complete h_0 in a Hilbert basis (h_0, h_1, \dots) of eigenvector functions of $L^2(I)$ associated of the eigenvalues $(r_\infty, r_1, r_2, \dots)$ with $\sup_{k \geq 1} |r_k| =: \tilde{r}(T_W) < r_\infty$. We denote by P_0 the projection on $\text{Vect}(h_0)$ and $P_1 = Id - P_0$: for any $g \in L^2(I)$, $P_0 g = \frac{\langle g, h_0 \rangle}{\|h_0\|_2^2} h_0 =: p_0(g) h_0$ (with $p_0(g) \in \mathbb{R}$) and

$$P_1 g = \sum_{n \geq 1} \frac{\langle g, h_n \rangle}{\|h_n\|_2^2} h_n.$$

Using (4.1), as T and P_0 commutes,

$$\begin{aligned} P_0\lambda(t, \cdot) &= P_0u_0(t, \cdot) + \int_0^t h(t-s)T_W P_0\lambda(s, \cdot)ds \\ &= p_0(u_0(t, \cdot))h_0 + \int_0^t h(t-s)T_W p_0(\lambda(s, \cdot))h_0ds \\ &= p_0(u_0(t, \cdot))h_0 + \int_0^t h(t-s)p_0(\lambda(s, \cdot))r_\infty h_0ds \\ p_0(\lambda(t, \cdot))h_0 &= p_0(u_0(t, \cdot))h_0 + r_\infty \int_0^t h(t-s)p_0(\lambda(s, \cdot))dsh_0. \end{aligned}$$

As $h_0(x) > 0$ everywhere (since continuous), we obtain that $p_0(\lambda(t, \cdot))$ solves the convolution equation in \mathbb{R}

$$p_0(\lambda(t, \cdot)) = p_0(u_0(t, \cdot)) + r_\infty \int_0^t h(t-s)p_0(\lambda(s, \cdot))ds. \quad (9.3)$$

We show that $p_0(\lambda(t, \cdot)) \sim_{t \rightarrow \infty} C e^{\sigma_r t}$ where $C > 0$ depends on the parameter functions and $\sigma_r > 0$ verifies $r_\infty \int_0^\infty e^{-\sigma_r t} h(t) dt = 1$. To do this, we apply Theorem 4 of [32] and its Remark with (4.10) : we need to have $t \rightarrow e^{-\sigma_r t} h(t)$ of bounded total variation and $e^{-\sigma_r t} p_0(u_0(t, \cdot)) \rightarrow 0$ as $t \rightarrow \infty$, and both conditions are true as we are under Hypothesis 4.1: h is piecewise continuously differentiable and u_0 is uniformly bounded.

Now, we project on the rest of the space and take the norm $L^2(I)$:

$$\begin{aligned} P_1\lambda(t, \cdot) &= P_1u_0(t, \cdot) + \int_0^t h(t-s)T_W P_1\lambda(s, \cdot)ds \\ \|P_1\lambda(t, \cdot)\|_2 &\leq \|P_1u_0(t, \cdot)\|_2 + \int_0^t h(t-s)\|T_W P_1\lambda(s, \cdot)\|_2 ds. \end{aligned}$$

As $T_W P_1\lambda(s, \cdot) = T_W \left(\sum_{n \geq 1} \frac{\langle P_1\lambda(s, \cdot), h_n \rangle}{\|h_n\|_2^2} h_n \right) = \sum_{n \geq 1} \frac{\langle P_1\lambda(s, \cdot), h_n \rangle}{\|h_n\|_2^2} r_n h_n$, we have

that $\|T_W P_1\lambda(s, \cdot)\|_2^2 = \sum_{n \geq 1} \frac{|\langle P_1\lambda(s, \cdot), h_n \rangle|^2}{\|h_n\|_2^2} |r_n|^2 \leq \tilde{r}(T_W)^2 \|P_1\lambda(s, \cdot)\|_2^2$ and then

$$\|P_1\lambda(t, \cdot)\|_2 \leq \|P_1u_0(t, \cdot)\|_2 + \tilde{r}(T) \int_0^t h(t-s)\|P_1\lambda(s, \cdot)\|_2 ds.$$

We compare $\|P_1\lambda(t, \cdot)\|_2$ to β solution of the equation $\beta_{\tilde{r}}(t) = \|P_1u_0(t, \cdot)\|_2 + 1 + \tilde{r}(T) \int_0^t h(t-s)\beta_{\tilde{r}}(s)ds$: Lemma A.4 gives $\|P_1\lambda(t, \cdot)\|_2 \leq \beta_{\tilde{r}}(t)$ for all $t \geq 0$. We want know to show that $\beta_{\tilde{r}}(t) = O(e^{\sigma_r t})$ when $t \rightarrow \infty$.

- If $\|h\|_1 \tilde{r}(T) > 1$, we apply (as done for P_0) Theorem 4 of [32] and obtain $\beta_{\tilde{r}}(t) \sim_{t \rightarrow \infty} \tilde{C} e^{\sigma_{\tilde{r}} t}$ where $\tilde{C} > 0$ depends on the parameter functions and $\sigma_{\tilde{r}} > 0$ verifies $\tilde{r}(T) \int_0^\infty e^{-\sigma_{\tilde{r}} t} h(t) dt = 1$. Note that in this case, $\sigma_{\tilde{r}} < \sigma_r$ as $\tilde{r}(T) < r_\infty$, and $\beta_{\tilde{r}}(t) = O(e^{\sigma_r t})$ follows.
- If $\|h\|_1 \tilde{r}(T) \leq 1$, as $\|h\|_1 r_\infty > 1$, we can find \bar{r} such that $\tilde{r}(T) < \bar{r} < r_\infty$ and $\|h\|_1 \bar{r} > 1$. Then, considering δ satisfying $\delta_{\bar{r}}(t) = \|P_1u_0(t, \cdot)\|_2 + 2 + \bar{r} \int_0^t h(t-s)\delta_{\bar{r}}(s)ds$, as done before Lemma A.4 gives $\beta_{\tilde{r}}(t) \leq \delta_{\bar{r}}(t)$ and Theorem 4 of [32] gives $\delta_{\bar{r}}(t) \sim_{t \rightarrow \infty} \bar{C} e^{\sigma_{\bar{r}} t}$ where $\bar{C} > 0$ depends on the parameter functions and

$\sigma_{\bar{r}} > 0$ verifies $\bar{r} \int_0^\infty e^{-\sigma_{\bar{r}} t} h(t) dt = 1$. We have then that $\beta_{\bar{r}}(t) \leq \delta_{\bar{r}}(t) \sim_{t \rightarrow \infty} \bar{C} e^{\sigma_{\bar{r}} t} = O(e^{\sigma_{\bar{r}} t})$.

In any case, we obtain $\|P_1 \lambda(t, \cdot)\|_2 = O(e^{\sigma_{\bar{r}} t})$, and as Parseval equality gives

$$\|\lambda(t, \cdot)\|_2^2 = \|P_0 \lambda(t, \cdot)\|_2^2 + \|P_1 \lambda(t, \cdot)\|_2^2,$$

it implies that $\|\lambda(t, \cdot)\|_2 \sim_{t \rightarrow \infty} C e^{\sigma_{\bar{r}} t} \xrightarrow{t \rightarrow \infty} +\infty$, with C a positive constant.

To treat the case $k > 1$, we deal with $k = 2$ and leave the generalisation to the reader. Hypothesis 4.6 (4.11) is then that the kernel of T_W^2 is positive. As $\lambda(t, x) = u_0(t, x) + \int_0^t h(t-s) T_W \lambda(s, \cdot)(x) ds$, we have $T_W \lambda(t, \cdot)(x) = T_W u_0(t, \cdot)(x) + \int_0^t h(t-s) T_W^2 \lambda(s, \cdot)(x) ds$ and

$$\begin{aligned} \lambda(t, x) &= u_0(t, x) + \int_0^t h(t-s) T_W u_0(s, \cdot)(x) ds + \int_0^t h(t-s) \int_0^s h(s-u) T_W^2 \lambda(u, \cdot)(x) du ds \\ &= v_0(t, x) + \int_0^t \tilde{h}(t-s) T_W^2 \lambda(s, \cdot)(x) ds. \end{aligned} \quad (9.4)$$

with $\tilde{h} = h * h = h^{*2}$ and $v_0(t, x) = u_0(t, x) + \int_0^t h(t-s) T_W u_0(s, \cdot)(x) ds$. As $\|\tilde{h}\|_1 = \int_0^\infty \int_0^t h(t-s) h(s) ds dt = \int_0^\infty h(s) \int_s^\infty h(t-s) dt ds = \|h\|_1^2$ and $\sqrt{r(T_W^2)} = \left(\lim_{n \rightarrow \infty} \|T_W^{2n}\|^{1/n} \right)^{\frac{1}{2}} = \lim_{n \rightarrow \infty} \|T_W^{2n}\|^{\frac{1}{2n}} = r(T_W)$, the condition (4.10) implies $\|\tilde{h}\|_1 r(T_W^2) = \|h\|_1^2 r(T_W)^2 > 1$. Then, we can apply the precedent case ($k = 1$) on λ satisfying (9.4). \square

APPENDIX A. USEFUL RESULTS

We remind here different useful results that we use frequently in this paper.

The proof of the following lemma can be found in Lemma 22 of [23].

Lemma A.1. *Let $\phi : [0, \infty[\rightarrow \mathbb{R}$ be locally integrable and $\alpha : [0, \infty[\rightarrow \mathbb{R}$ with finite variations on compact intervals such that $\alpha(0) = 0$. Then for all $t \geq 0$, we have*

$$\int_0^t \int_0^{s^-} \phi(s-u) d\alpha(u) ds = \int_0^t \int_0^s \phi(s-u) d\alpha(u) ds = \int_0^t \phi(t-s) \alpha(s) ds.$$

Lemma A.2. *Let $(X_i)_{1 \leq i \leq N}$ be a family of N independent random variables. Then*

$$\mathbf{E} \left[\left| \frac{1}{N} \sum_{i=1}^N (X_i - \mathbf{E}[X_i]) \right| \right] \leq \frac{1}{N} \sqrt{\sum_{i=1}^N \text{Var}(X_i)}.$$

Proof. We set $Y := \frac{1}{N} \sum_{i=1}^N X_i$, then $\mathbf{E}[Y] = \frac{1}{N} \sum_{i=1}^N \mathbf{E}[X_i]$ and $\text{Var}(Y) = \frac{1}{N^2} \sum_{i=1}^N \text{Var}(X_i)$ by independence. We have

$$\begin{aligned} \mathbf{E} \left[\left| \frac{1}{N} \sum_{i=1}^N (X_i - \mathbf{E}[X_i]) \right| \right] &= \mathbf{E} [|Y - \mathbf{E}[Y]|] = \mathbf{E} \left[\sqrt{(Y - \mathbf{E}[Y])^2} \right] \\ &\leq \sqrt{\mathbf{E} [(Y - \mathbf{E}[Y])^2]} = \sqrt{\text{Var}(Y)} \end{aligned}$$

using Jensen's inequality, and the result follows with the expression of $\text{Var}(Y)$. \square

Lemma A.3. *Let $\phi : [0, \infty[\rightarrow [0, \infty[$ be a locally integrable function and $g : [0, \infty[\rightarrow [0, \infty[$ a locally bounded function.*

(i) *Let u be a locally bounded nonnegative function such that for all $t \geq 0$:*

$$u(t) \leq g(t) + \int_0^t \phi(t-s)u(s)ds.$$

Then for all $T \geq 0$ there exists C_T (depending on T and ϕ) verifying

$$\sup_{[0, T]} u(t) \leq C_T \sup_{[0, T]} g(t).$$

(ii) *Let (u_n) be a sequence of locally bounded non-negative functions such that for all $t \geq 0$ and $n \geq 0$:*

$$u_{n+1}(t) \leq \int_0^t \phi(t-s)u_n(s)ds.$$

Then for all $T \geq 0$ there exists C_T (depending on T , ϕ and u_0) verifying

$$\sup_{[0, T]} \sum_{n \geq 0} u_n(t) \leq C_T.$$

(iii) *Let (u_n) be a sequence of locally bounded non-negative functions such that for all $t \geq 0$ and $n \geq 0$:*

$$u_{n+1}(t) \leq g(t) + \int_0^t \phi(t-s)u_n(s)ds.$$

Then for all $T \geq 0$ there exists C_T (depending on T , ϕ , u_0 and g) verifying

$$\sup_{[0, T]} \sup_{n \geq 0} u_n(t) \leq C_T.$$

Proof. See Lemma 23 of [23]. \square

Lemma A.4. *Let $r > 0$, h be a nonnegative locally integrable function, u , α and β be locally bounded nonnegative continuous functions such that for all $t \geq 0$:*

$$\begin{aligned}\alpha(t) &\leq u(t) + r \int_0^t h(t-s)\alpha(s)ds, \\ \beta(t) &= u(t) + 1 + r \int_0^t h(t-s)\beta(s)ds.\end{aligned}$$

Then $\alpha(t) \leq \beta(t)$ for all $t \geq 0$.

Proof. Let $t^* = \inf \{s > 0, \alpha(s) > \beta(s)\}$. Note that $t^* > 0$ as $\beta(0) - \alpha(0) \geq 1$. Suppose $t^* < \infty$, then $\beta(t^*) - \alpha(t^*) \geq 1 + \int_0^{t^*} h(t-s)(\beta(s) - \alpha(s))ds \geq 1$ which is impossible, then necessarily $t^* = +\infty$ and $\alpha(t) \leq \beta(t)$ for all $t \geq 0$. \square

Lemma A.5. *Let u and h be locally square integrable functions, u non-negative, $T > 0$ and α, β two constants. Assume that for any $t \in [0, T]$,*

$$u(t) \leq \alpha \int_0^t h(t-s)u(s)ds + \beta.$$

Then u satisfies the following Grönwall's inequality :

$$u(T) \leq \sqrt{2}\beta \exp(\alpha^2 \|h\|_{T,2}^2 T)$$

Proof. Using Cauchy-Schwarz inequality, $u(t)^2 \leq 2\alpha^2 \|h\|_{T,2}^2 \int_0^t u(s)^2 ds + 2\beta^2$. Then, applying standard Grönwall lemma to u^2 and taking the square root (since $u \geq 0$) concludes the proof. \square

Lemma A.6. *Fix $N > 1$ and $(Y_l)_{l=1,\dots,n}$ real valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that there exists $\nu > 0$ such that, almost surely, for all $l = 1, \dots, n-1$, $Y_l \leq 1$, $\mathbb{E}[Y_{l+1} | Y_l] = 0$ and $\mathbb{E}[Y_{l+1}^2 | Y_l] \leq \nu$. Then for all $x \geq 0$,*

$$\mathbb{P}(n^{-1}(Y_1 + \dots + Y_n) \geq x) \leq \exp\left(-nH\left(\frac{x+\nu}{1+\nu} \middle| \frac{\nu}{1+\nu}\right)\right),$$

where $H(p|q) = p \log(p/q) + (1-p) \log((1-p)/(1-q))$, for $p, q \in [0, 1]$.

Proof. See Corollary 2.4.7 of [25] and Lemma 4.2 of [46]. \square

Lemma A.7. *Fix $N \geq 1$, (p_1, \dots, p_N) in $[0, 1]$ and a sequence (v_1, \dots, v_N) such that $|v_l| \leq 1$ for any $l \in \llbracket 1, N \rrbracket$. Suppose that there exists $\kappa_N > 0$ and $w_N \in]0, 1]$ such that $p_l \leq w_N$ for any $l \in \llbracket 1, N \rrbracket$. Then, setting $\varepsilon_n := 32 \frac{\kappa_N^2 w_N}{N} \log(N)$ for (U_1, \dots, U_N) independent random variables with $U_l \sim \mathcal{B}(p_l)$, we have*

$$\mathbb{P}\left(\left|\frac{\kappa_N}{N} \sum_{l=1}^N (U_l - p_l) v_l\right| > \varepsilon_N\right) \leq 2 \exp\left(-16 \log(N) B\left(4\sqrt{2} \left(\frac{\log(N)}{N w_N}\right)^{\frac{1}{2}}\right)\right) \quad (\text{A.1})$$

with

$$B(u) := u^{-2}((1+u) \log(1+u) - u). \quad (\text{A.2})$$

Proof. Lemma 4.3 of [46], as a consequence of Lemma A.6. \square

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