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Abstract. We give explicit formulas for finding the complex (phased) scattering amplitude at fixed frequency and angles from absolute values of the scattering wave function given at $2n$ points in dimension $d = 3$, where distances between these points are fixed and satisfy a multiplicity condition. These formulas are asymptotic and their convergence rate is proportional to n. Our further results include: similar $2n$–point formulas without the aforementioned multiplicity condition in dimension $d = 3$ and in dimension $d = 2$ for the linearised case; related multipoint formulas without the multiplicity condition in dimension $d = 3$ and in dimension $d = 2$ for the general non–linearised case, but with somewhat more slow convergence. In particular, we continue studies going back to [Novikov, Bull. Sci. Math. 139(8), 923-936, 2015].

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1 Introduction

We consider monochromatic scattering modelled using the equation

$$-\Delta\psi + v(x)\psi = E\psi,\ x \in \mathbb{R}^d,\ d \geq 1,\ E > 0, \tag{1}$$

where

$$\begin{aligned} &v \in L^\infty(D),\ \ v \equiv 0 \text{ on } \mathbb{R}^d \setminus D, \\ &D \text{ is an open bounded domain in } \mathbb{R}^d. \end{aligned} \tag{2}$$

We assume that v is complex-valued. The regularity assumption that $v \in L^\infty(D)$ is just for simplicity and can be somewhat relaxed.

Equation (1) arises in quantum mechanics as the Schrödinger equation at fixed energy and in acoustics and electrodynamics as the Helmholtz equation at fixed frequency. Under assumption (2), the coefficient v describes a scatterer contained in D. The number E is related to the time-harmonic frequency and corresponds to the energy in the framework of the Schrödinger equation. In addition, v may depend on E, at least, in acoustics and electrodynamics. See, for example, [4], [6], [8], [12].

For equation (1) we consider the solutions $\psi^+(x,k)$, $k \in \mathbb{R}^d$, $k^2 = E$, specified by the following asymptotic as $|x| \to \infty$:

$$\psi^+(x,k) = e^{ikx} + \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f_1(k, |k|\frac{x}{|x|}) + \mathcal{O}\left(\frac{1}{|x|^{(d+1)/2}}\right), \tag{3}$$

for some a priori unknown f_1. The solutions $\psi^+ = \psi^+(x,k)$ are the scattering solutions, or scattering wave functions, for equation (1). These solutions describe scattering of the incident plan waves described by e^{ikx} on the scatterer described by v. In particular, the second term on the right-hand side of (3) describes the leading scattered spherical waves. The coefficient f_1 arising in (3) is a function defined on

$$\mathcal{M}_E = \{k, l \in \mathbb{R}^d : k^2 = l^2 = E\} = \mathbb{S}^{d-1}_{\sqrt{E}} \times \mathbb{S}^{d-1}_{\sqrt{E}}. \tag{4}$$

The function f_1 is the scattering amplitude, or far field pattern, for equation (1).

In order to study ψ^+ and f_1 one can use, in particular, the Lippmann–Schwinger integral equation (16) for ψ^+ and formulas (20)–(22) for f_1; see Subsection 2.1.

We recall that in quantum mechanics the complex values of the functions ψ^+ and f_1 have no direct physical sense, whereas the phaseless values of $|\psi^+|^2$ and $|f_1|^2$ have probabilistic interpretation (according to the Born's rule) and can be directly measured. See [5] and, for example, [11], for details. In turn, in acoustics or electrodynamics the complex values of ψ^+ and f_1 can be directly measured, at least, in principle. However, in electro–magnetic wave propagation at very high frequencies (as for X–rays and lasers) only phaseless values of $|\psi^+|^2$ and $|f_1|^2$ can be measured in practice by modern technical devices; see, e.g., [16] and references therein.

For equation (1) under assumptions (2), we consider, in particular, the following problems:

Problem 1.1. Reconstruct potential v from its scattering amplitude f_1.

Problem 1.2. Reconstruct potential v from its phaseless scattering data $|\psi^+|^2$ appropriately given outside of D.

Problem 1.3. Find f_1 from $|\psi^+|^2$ appropriately given outside of D.

A recent survey on these problems is given in [29]. Actually, in the present work we continue studies of [10], [25]–[30], [32] on Problem 1.3. These studies on Problem 1.3 and results on Problem 1.1 admit straightforward applications to Problem 1.2. For other possible approaches to Problem 1.2, see, for example, [38], [39], [21], [9], [15], [22], [23], [18], [34].

In particular, in the present work we give, for fixed $(k,l) \in \mathcal{M}_E$, $l \neq k$, for $d \geq 2$,

$$\begin{aligned} &\text{formulas for finding } f_1(k,l) \text{ up to } \mathcal{O}(s^{-n}) \text{ as } s \to +\infty, \\ &\text{from } |\psi^+(x,k)|^2 \text{ given at } m \text{ points } x = x_1(s), ..., x_m(s), \end{aligned} \tag{5}$$

where m depends linearly on n,

$$\begin{aligned} &x_j(s) = (s+\tau_j)\hat{l},\ j = 1, ..., m,\ \hat{l} = l/|l|, \\ &s > 0,\ \tau_1 = 0,\ \tau_{j_1} < \tau_{j_2},\ j_1 < j_2. \end{aligned} \tag{6}$$

These formulas are explicit and are presented in detail below in Introduction and in Section 3, where our precise assumptions on $m = m(n)$ and on τ_1, ..., τ_m are specified.

Let

$$a(x,k) = |x|^{(d-1)/2}(|\psi^+(x,k)|^2 - 1), \tag{7}$$

where $x, k \in \mathbb{R}^d \setminus \{0\}$.

Assume in (5), (6) that $d = 3$, $m = 2n$, $n \in \mathbb{N} = \{1, 2, 3, ...\}$, and that

$$\tau_j = \begin{cases} (j-1)\tau,\ j = 1, ..., n, \\ \sigma + (j-1-n)\tau,\ j = n+1, ..., 2n, \end{cases} \tag{8}$$

$$\tau = \tau(k,l) = \frac{2\pi}{\kappa},\ 0 < \sigma \neq 0\,(mod\,\frac{\pi}{\kappa}),\ \kappa = \kappa(k,l) = |k| - k\hat{l}. \tag{9}$$

Then our formulas (5), (6) are as follows:

$$f_1(k,l) = \frac{e^{-i(s+\sigma)\kappa}a_1(s) - e^{-is\kappa}a_2(s) + \mathcal{O}(s^{-n})}{-2i\sin(\sigma\kappa)},\ s \to +\infty, \tag{10}$$

$$a_1(s) = a_1(k,l,s) = \sum_{j=1}^{n} \frac{(-1)^{n-j}(s+\tau_j)^{n-1}a(x_j(s),k)}{(j-1)!(n-j)!\tau^{n-1}}, \tag{11}$$

$$a_2(s) = a_2(k,l,s) = \sum_{j=n+1}^{2n} \frac{(-1)^{j}(s+\tau_j)^{n-1}a(x_j(s),k)}{(j-1-n)!(2n-j)!\tau^{n-1}}, \tag{12}$$

where $(k,l) \in \mathcal{M}_E$, $l \neq k$, $d = 3$, $a(x,k)$ is defined by (7), $x_j(s)$ are defined in (6), (8), (9), τ, τ_j, σ, κ are the numbers of (8), (9).

Formulas (10)–(12) are new for $n \geq 2$; for $n = 1$ these formulas were given in [25], [27].

Somewhat more general version of formulas (10)–(12) is given as Theorem 3.1; see Subsection 3.1.

One can see that formulas (10)–(12) and formulas of Theorem 3.1 are completely explicit ! However, a possible inconvenience of these formulas is that the differences between the points $x_j(s)$, $j = 1, ..., n$, or $x_j(s)$, $j = n+1, ..., 2n$, in (10)–(12) (and between the related points in Theorem 3.1) are multiple to $\tau = 2\pi/\kappa$, where $\kappa = \kappa(k,l)$ is defined in (9). Besides, these formulas are valid for $d = 3$ but are not valid for $d = 2$. Therefore, in the present work we also give the following further results without the aforementioned multiplicity condition for the differences between $x_j(s)$.

We give an explicit version of formulas (5), (6) for the case of the linearised Problem 1.3 (near $v = 0$) for $d \geq 2$, $m = 2n$, $n \in \mathbb{N}$, where

$$\tau_j = (j-1)\tau,\ j = 1, ..., 2n,\ \tau > 0,\ \tau \neq 0\,(mod\,\frac{\pi}{\kappa}), \tag{13}$$

$\kappa = \kappa(k,l)$ in defined in (9); see Theorem 3.2 of Subsection 3.2.

We give explicit versions of formulas (5), (6) for the general non–linearised case for $d = 3$, $m = 3n-1$, and for $d = 2$, $m = 3n$, where

$$\tau_j = (j-1)\tau,\ j = 1, ..., m,\ \tau > 0, \tau \neq 0\,(mod\,\frac{\pi}{\kappa}), \tag{14}$$

$\kappa = \kappa(k,l)$ in defined in (9); see Proposition 3.1 of Subsection 3.3 and Proposition 3.2 of Subsection 3.4. For the general non-linearized case, finding such formulas for $m = 2n$ is an open question for $n > 1$, for $d = 3$ or $d = 2$, under assumption (14). For $n = 1$, $d = 3$, this question was solved in [25], [27]; for $n = 1$, $d = 2$, this question was solved in Section 9 of [28].

Note that τ in (13), (14) is fixed and independent of k, l (except the property that $\tau \neq 0\,(mod\,2\pi/\kappa)$) in contrast with τ in (9).

The results of the present work are obtained proceeding from methods developed in [25], [28], [30]. In particular, for fixed $(k,l) \in \mathcal{M}_E$, $l \neq k$, for $d = 3$ or $d = 2$, the work [28] gives a version of formulas (5), where

$$\begin{aligned} &x_j(s) = r_j(s)\hat{l},\ j = 1, ..., 2n,\ \hat{l} = l/|l|, \\ &r_{2i-1}(s) = \lambda_i s,\ r_{2i}(s) = \lambda_i s + \tau,\ i = 1, ..., n, \\ &\lambda_1 = 1,\ \lambda_{i_1} < \lambda_{i_2} \text{ for } i_1 < i_2,\ \tau > 0. \end{aligned} \tag{15}$$

Formulas (5), (15) realized in [28] are recurrent in n. For $n = 1$, $d = 3$, these formulas were given in [25].

Advantages of formulas (5), (6) (realized in the present work) in comparison with formulas (5), (15) (realized in [28]) can be summarized as follows:

(a) The geometry of $x_j(s)$ in (5), (6) is essentially simpler in the sense that the distances between all these points are fixed and are independent of $s \to +\infty$.

(b) Formulas (5), (6) (realized as formulas (10)–(12), (52)–(54), (63)–(65), (71)–(74), (82)–(84)) are drastically more explicit for large n.

In addition, the results of the present work essentially use the technique of [30], where [30] gives explicit asymptotic multipoint formulas for finding f from ψ^+.

Note also that explicit estimates on the reminder $\mathcal{O}(s^{-n})$ in our formulas (5), (6) can be given proceeding from methods developed in [30], [32].

In addition to Problem 1.3 and Problem 1.2, there are also other possible formulations of phase retrieval and phaseless inverse scattering problems for equation (1) and for other equations of wave propagations. In connection with such other formulations and related results, see, for

example, [7], [12], [14], [16], [17], [19], [27], [29], [33], [35], [36], [37], [40] and references therein. Note that formulas of the present work can be also used for Problems 1.3 and 1.2 when coefficient v in equation (1) is replaced, e.g., by an impenetrable obstacle (see, e.g., [8] for definition of impenetrable obstacles).

The further structure of the present article is as follows. In Section 2 we recall, in particular, some results on direct scattering for equation (1) under assumptions (2) and some formulas of [28] and [30]. The results of the present work on Problem 1.2, consisting in realizations of formulas (5), (6), are given in Section 3. These results are proved in Sections 4, 5, and 6. In Section 7, for completeness of presentation, we give a sketch of proof of formulas (23)–(25) for the higher scattering amplitudes f_j, $j \geq 2$.

2 Preliminaries

2.1 Asymptotics of the scattering solutions

We recall that the scattering solutions ψ^+ satisfy the following Lippmann-Schwinger integral equation:

$$\begin{aligned} &\psi^+(x,k) = e^{ikx} + \int_D G^+(x-y,k)v(y)\psi^+(y,k)dy, \\ &G^+(x,k) := -(2\pi)^{-d}\int_{\mathbb{R}^d} \frac{e^{i\xi x}d\xi}{\xi^2 - k^2 - i\cdot 0} = G_0^+(|x|,|k|), \end{aligned} \tag{16}$$

where $x, k \in \mathbb{R}^d$, $k^2 = E$; see, for example, [4], [8], [29].

Note also that

$$\begin{aligned} &G^+(x,k) = \frac{e^{i|k||x|}}{2i|k|},\ d=1, \\ &G^+(x,k) = -\frac{i}{4}H_0^1(|x||k|),\ d=2,\ G^+(x,k) = -\frac{e^{i|k||x|}}{4\pi|x|},\ d=3, \end{aligned} \tag{17}$$

where H_0^1 is the Hankel function of the first type.

Actually, in the present work, in addition to (2), we assume that, for fixed $E > 0$,

$$\text{equation (16) is uniquely solvable for } \psi^+(\cdot,k) \in L^\infty(D). \tag{18}$$

In addition, if v satisfies (2) and is real-valued, then (18) is fulfilled automatically.

Proceeding from (16) one can show that the scattering solutions ψ^+ have the following asymptotics:

$$\psi^+(x,k) = e^{ikx} + \frac{e^{i|k||x|}}{|x|^{(d-1)/2}}\left(\sum_{j=1}^{N}\frac{f_j(k,|k|\frac{x}{|x|})}{|x|^{j-1}} + \mathcal{O}\left(\frac{1}{|x|^N}\right)\right),\ |x|\to+\infty,\ N\in\mathbb{N}, \tag{19}$$

where the coefficients f_j arising in (19) are functions defined on $\mathcal{M}_E$; see [24], [28], [30].

Formula (19) for $N = 1$ reduces to (3). We say that the functions f_j for $j \geq 2$ are the higher scattering amplitudes for equation (1).

It is well-known that

$$f_1(k,l) = c(d,|k|)f(k,l), \tag{20}$$

$$c(d,|k|) = -\pi i(-2\pi i)^{(d-1)/2}|k|^{(d-3)/2},\ \text{ for } \sqrt{-2\pi i} = \sqrt{2\pi}e^{-i\pi/4}, \tag{21}$$

$$f(k,l) = (2\pi)^{-d}\int_D e^{-ily}v(y)\psi^+(y,k)dy, \tag{22}$$

where $(k,l) \in \mathcal{M}_E$; see, for example, [28].

It is also well-known that $f_j \equiv 0$ for $j \geq 2$, $d = 1$, under assumptions (2), (18).

Besides, in the present work we also use the following formulas, for $d \geq 2$:

$$f_j(k,l) = (2\pi)^{-d} c(d,|k|) \int_D \varphi_j(y,l) v(y) \psi^+(y,k) dy,\ j \geq 1, \tag{23}$$

$$\varphi_1(y,l) = e^{-ily}, \tag{24}$$

$$\varphi_j(y,l) = \frac{(2d + 1/4 - d^2/4 + j^2 - j - 2)\varphi_{j-1}(y,l) + \Delta_S \varphi_{j-1}(y,l)}{2i|l|(j-1)},\ j \geq 2, \tag{25}$$

where Δ_S is the Beltrami-Laplace operator on the unit sphere $\mathbb{S}^{d-1}$, acting with respect to $\hat{l} = l/|l|$. For completeness of the presentation, a sketch of proof of formulas (23)–(25), for $j \geq 2$, is given in Section 7. Note that the precise form of the recurrent relations (25) is not essential for the main results of the present work.

2.2 Asymptotic formulas for $a(x,k)$

Consider the function $a = a(x,k)$ defined by (7). Let $\kappa = \kappa(k, |k|\frac{x}{|x|})$ be defined as in (9). Let

$$f_j = f_j(k, |k|\frac{x}{|x|}), \tag{26}$$

where f_j are the functions arising in (19), $j \in \mathbb{N}$, $x, k \in \mathbb{R}^d \setminus \{0\}$. Then $a(x,k)$ can be presented as follows (see [28] for $d = 3$ or $d = 2$):

$$a(x,k) = a_N(x,k) + \delta_N a(x,k), \tag{27}$$

$$a_N(x,k) = a_N^1(x,k) + a_N^2(x,k), \tag{28}$$

$$a_N^1(x,k) = \sum_{j=1}^{N} \frac{e^{i|x|\kappa} f_j}{|x|^{j-1}} + \sum_{j=1}^{N} \frac{e^{-i|x|\kappa} \overline{f}_j}{|x|^{j-1}}, \tag{29}$$

$$a_N^2 = \sum_{j=1}^{N-[(d-1)/2]} \frac{h_j}{|x|^{j-1+(d-1)/2}},\ \ h_j = \sum_{\alpha=1}^{j} f_\alpha \overline{f}_{j-\alpha+1}, \tag{30}$$

$$\delta_N a(x,k) = \mathcal{O}(|x|^{-N}),\ \text{as } |x| \to +\infty, \tag{31}$$

where $x \in \mathbb{R}^d \setminus \{0\}$, $k \in \mathbb{R}^d$, $k^2 = E > 0$, $d \geq 2$, $[\cdot]$ stands for the integer part.

Note that formulas (7), (19) imply formulas (27)–(31) as follows:

$$a(x,k) = a_N^1(x,k) + \mathcal{O}(|x|^{-N}) + \frac{1}{|x|^{(d-1)/2}} \left(\sum_{j_1=1}^{N} \frac{f_{j_1}}{|x|^{j_1-1}} + \mathcal{O}(|x|^{-N}) \right) \left(\sum_{j_2=1}^{N} \frac{\overline{f}_{j_2}}{|x|^{j_2-1}} + \mathcal{O}(|x|^{-N}) \right) =$$

$$= a_N^1(x,k) + \mathcal{O}(|x|^{-N}) + \frac{1}{|x|^{(d-1)/2}} \left(\sum_{j_1,j_2=1}^{N} \frac{f_{j_1} \overline{f}_{j_2}}{|x|^{j_1-1}|x|^{j_2-1}} + \mathcal{O}(|x|^{-N}) \right) =$$

$$= [j = j_1 + j_2 - 1] = a_N^1(x,k) + \mathcal{O}(|x|^{-N}) + \frac{1}{|x|^{(d-1)/2}} \left(\sum_{j=1}^{N} \sum_{j_1=1}^{j} \frac{f_{j_1} \overline{f}_{j-j_1+1}}{|x|^{j-1}} + \mathcal{O}(|x|^{-N}) \right) =$$

$$= a_N^1(x,k) + \sum_{j=1}^{N-[(d-1)/2]} \frac{h_j}{|x|^{j-1+(d-1)/2}} + \mathcal{O}(|x|^{-N}). \tag{32}$$

Consider also the case of the Born approximation for small potentials. Suppose that potential v is small, for example, in the sense of the norm $\|\cdot\|_{L^\infty(D)}$, for fixed D. Then in formulas (27)–(31) the quadratic term a_N^2 is negligible in comparison with the linear term a_N^1. Such a situation arises

in many applications (see, for example, [5], [12], [15], [22], [23], [38], [39]). Thus, in the Born approximation for small potentials formula (28) reduces to the formula

$$a_N(x,k) \approx a_N^1(x,k). \tag{33}$$

The aforementioned smallness assumption on v can be specified as

$$\|v\|_{L^\infty(D)} = \mathcal{O}(\varepsilon), \text{ where } \varepsilon \to 0. \tag{34}$$

Then using (16), (20)–(25), (34) one can show that

$$\|f_j\|_{C(\mathcal{M}_E)} = \mathcal{O}(\varepsilon), \text{ for each } j \in \mathbb{N}. \tag{35}$$

In turn, formulas (28), (30) imply that

$$a_N(x,k) - a_N^1(x,k) = |x|^{-(d-1)/2}\mathcal{O}(\varepsilon^2), \tag{36}$$

where $N \in \mathbb{N}$.

Formula (36) specifies (33) under assumption (34).

2.3 Some results of [30]

We recall that the work [30] gives, in particular, explicit asymptotic multipoint formulas for finding f_1 from ψ^+ for $d \geq 2$. This work proceeds from formula (19) and considers, in particular, the functions $z = z(s)$, $s \in [r, +\infty)$, of the form

$$z(s) = \sum_{j=1}^{N} \frac{f_j}{s^{j-1}} + \mathcal{O}(s^{-N}), \text{ as } s \to +\infty, \tag{37}$$

where f_j, $j = 1, ..., n$, are the complex numbers.

Functions of the form (37) arise in the second term on the right-hand side of (19), where $s = |x|$. Functions of the form (37) also arise in the framework of direct and inverse scattering at high energies for equation (1) with smooth v; see [24] and [31].

For functions z satisfying (37) the work [30] considers, in particular, the problem of finding f_1 from $z(s)$ given at n points $s_j \in [r, +\infty)$, $j = 1, ..., n$, of the form

$$\begin{aligned} &s_j = s_j(s) = s + \tau_j,\ j = 1, ..., n, \\ &s > r,\ \tau_1 = 0,\ \tau_{j+1} > \tau_j,\ j = 1, ..., n-1, \\ &\vec{\tau} = (\tau_1, ..., \tau_n). \end{aligned} \tag{38}$$

Suppose that $N \geq 2n - 1$. Then the following formulas of [30] hold:

$$f_1 = \sum_{j=1}^{n} y_j(s, \vec{\tau}) z(s + \tau_j) + \mathcal{O}(s^{-n}), \text{ as } s \to +\infty, \tag{39}$$

where $y_j(s, \vec{\tau})$ are defined by

$$\sum_{j=1}^{n} \frac{y_j}{(s + \tau_j)^{i-1}} = \begin{cases} 1, & \text{for } i = 1, \\ 0, & \text{for } i = 2, ..., n; \end{cases} \tag{40}$$

in addition:

$$y_j(s, \vec{\tau}) = \frac{(-1)^{n-j}(s + \tau_j)^{n-1}}{\alpha_j(\vec{\tau})\beta_{n,j}(\vec{\tau})},\ 1 \leq j \leq n, \tag{41}$$

$$\alpha_j(\vec{\tau}) = \Pi_{k=1}^{j-1}(\tau_j - \tau_k),\ \beta_{n,j}(\vec{\tau}) = \Pi_{k=j+1}^{n}(\tau_k - \tau_j), \tag{42}$$

$$\sum_{j=1}^{n} \frac{y_j(s, \vec{\tau})}{(s + \tau_j)^{i-1}} = \mathcal{O}(s^{-n}) \text{ as } s \to +\infty, \text{ for } n < i < 2n. \tag{43}$$

3 Main results

3.1 Formulas for finding f_1 from a at $2n$ points for $d = 3$

For $d = 3$, the function $a(x, k)$ of formulas (7), (27)–(31) at fixed k and $\hat{x} = x/|x|$ can be written as

$$a(s) = a(s, \hat{x}, k) := a(s\hat{x}, k), \tag{44}$$

$$a(s) = \sum_{j=1}^{N} \frac{e^{is\kappa} f_j + e^{-is\kappa}\overline{f}_j + h_j}{s^{j-1}} + \mathcal{O}(s^{-N}),\ s \to +\infty,\ h_1 = 0, \tag{45}$$

$$h_j = \sum_{k=1}^{j-1} f_k \overline{f}_{j-k},\ j = 1, ..., N, \tag{46}$$

where $s = |x|$, $\kappa = |k| - k\hat{x}$, and $\kappa > 0$ if $\hat{x} = x/|x| \neq \hat{k} = k/|k|$.

Proceeding from this motivation we consider arbitrary functions $a = a(s)$, $s \in [r, +\infty), r > 0$, such that formula (45) holds for some fixed $\kappa > 0$ and some complex numbers f_j, $j = 1, ..., N$, and h_j, $j = 2, ..., N$.

We consider $2n$ points $s_{1,j}$, $s_{2,j} \in [r, +\infty)$, $j = 1, ..., n$, of the form

$$s_{1,j} = s_{1,j}(s) = s + \tau_{1,j},\ j = 1, ..., n, \tag{47}$$

$$s_{2,j} = s_{2,j}(s) = s + \sigma + \tau_{2,j},\ j = 1, ..., n, \tag{48}$$

$$\tau_{1,j} = \nu_{1,j}\tau,\ j = 1, ..., n,\ \tau = \frac{2\pi}{\kappa},\ 0 = \nu_{1,1} < \nu_{1,2} < ... < \nu_{1,n},\ \{\nu_{1,j}\} \in \mathbb{N}, \tag{49}$$

$$\tau_{2,j} = \nu_{2,j}\tau,\ j = 1, ..., n,\ \tau = \frac{2\pi}{\kappa},\ 0 = \nu_{2,1} < \nu_{2,2} < ... < \nu_{2,n}, \{\nu_{2,j}\} \in \mathbb{N}, \tag{50}$$

$$s > r,\ \sigma > 0,\ \sigma \neq 0\ (mod\ \frac{\pi}{\kappa}). \tag{51}$$

Theorem 3.1. *Let $a = a(s)$ satisfy (45) for some $N \geq 2n - 1$, $n \in \mathbb{N}$. Then $a(s)$ at $2n$ points $s_{1,j}$, $s_{2,j}$ of (47)–(51) approximately determines f_1 as follows:*

$$f_1 = \frac{e^{-i(s+\sigma)\kappa} a_1(s) - e^{-is\kappa} a_2(s) + \mathcal{O}(s^{-n})}{-2i \sin(\sigma\kappa)},\ s \to +\infty, \tag{52}$$

$$a_1(s) := \sum_{j=1}^{n} \frac{(-1)^{n-j}(s_{1,j}(s))^{n-1} a(s_{1,j}(s))}{\tau^{n-1}\Pi_{k=1}^{j-1}(\nu_{1,j} - \nu_{1,k})\Pi_{k=j+1}^{n}(\nu_{1,k} - \nu_{1,j})}, \tag{53}$$

$$a_2(s) := \sum_{j=1}^{n} \frac{(-1)^{n-j}(s_{2,j}(s))^{n-1} a(s_{2,j}(s))}{\tau^{n-1}\Pi_{k=1}^{j-1}(\nu_{2,j} - \nu_{2,k})\Pi_{k=j+1}^{n}(\nu_{2,k} - \nu_{2,j})}. \tag{54}$$

Remark 3.1. *Suppose that $\nu_{1,j} = \nu_{2,j} = j - 1$, $j = 1, ..., n$. Then formulas (53), (54) reduce to the formulas*

$$a_1(s) = \sum_{j=1}^{n} \frac{(-1)^{n-j}(s_{1,j}(s))^{n-1} a(s_{1,j}(s))}{(j-1)!(n-j)!\tau^{n-1}}, \tag{55}$$

$$a_2(s) = \sum_{j=1}^{n} \frac{(-1)^{n-j}(s_{2,j}(s))^{n-1} a(s_{2,j}(s))}{(j-1)!(n-j)!\tau^{n-1}}. \tag{56}$$

Theorem 3.1 is proved in Section 4.

Formulas (52)–(56) of Theorem 3.1 and Remark 3.1 are completely explicit ! But a possible inconvenience of these formulas is that the differences between the points $s_{1,j}$, $j = 1, ..., n$, or

$s_{2,j}, j = 1, ..., n$, are multiple to $2\pi/\kappa$. Below in Subsections 3.2, 3.3, we give approaches for relaxing this multiplicity condition.

Formulas (7), (44), (47)–(56) realize (5), (6) for $d = 3$. In addition, formulas (7), (44), (47)–(56) remain valid for any odd $d \geq 3$. The reason is that $a(s)$ defined in (44) satisfies (45) for any odd $d \geq 3$, for some h_j, where $h_1 = ... = h_{(d-1)/2} = 0$.

3.2 Formulas for finding f_1 from a at $2n$ points for small potentials

In the Born approximation for small potentials v formula (28) reduces to formula (33).

In these framework, the function $a(x, k)$ of (27)–(31) at fixed k and $\hat{x} = x/|x|$, for $d \geq 2$, can be written as

$$a(x, k) \approx a(s), \tag{57}$$

where

$$a(s) = \sum_{j=1}^{N} \frac{e^{is\kappa} f_j + e^{-is\kappa} \overline{f}_j}{s^{j-1}} + \mathcal{O}(s^{-N}),\ s \to +\infty, \tag{58}$$

where $s = |x|$, $\kappa = |k| - k\hat{x}$, and $\kappa > 0$ if $\hat{x} = x/|x| \neq \hat{k} = k/|k|$. Formulas (57), (58) follow from (27), (29), (31), (33).

Proceeding from this motivation, we consider arbitrary functions $a = a(s)$, $s \in [r, +\infty)$, $r > 0$, such that formula (58) holds for some fixed $\kappa > 0$ and some complex numbers f_j, $j = 1, ..., N$.

We consider $2n$ points $s_j \in [r, +\infty)$, $j = 1, ..., 2n$, of the form

$$s_j = s_j(s) = s + \tau_j,\ j = 1, ..., 2n, \tag{59}$$

$$\tau_j = (j-1)\tau,\ j = 1, ..., 2n, \tag{60}$$

$$s > r,\ \tau > 0,\ \tau \neq 0\,(mod\,\frac{\pi}{\kappa}). \tag{61}$$

Let $\Sigma_{n,\tau}$ be the operator acting on functions u on $[r, +\infty)$ and defined by the formula

$$\Sigma_{n,\tau} u(s) = \sum_{j=1}^{n} \frac{(-1)^{n-j} (s_j(s))^{n-1} u(s_j(s))}{(j-1)!(n-j)!\tau^{n-1}},\ s \in [r, +\infty), \tag{62}$$

where s_j are defined according to (59)–(61), $j = 1, ..., n$.

Theorem 3.2. *Let $a = a(s)$ satisfy (58) for some $N \geq 2n-1$, $n \in \mathbb{N}$. Let $\Sigma_{n,\tau}$ be defined by (62). Then $a(s)$ at $2n$ points s_j of (59)–(61) approximately determines f_1 as follows:*

$$f_1 = C(\kappa, \tau, n)(\tau^{n-1}(e^{-2i\tau\kappa} a_2(s) - a_2(s+\tau)) + \mathcal{O}(s^{-n})),\ s \to +\infty, \tag{63}$$

$$a_2(s) := \Sigma_{n,\tau} a_1(s),\ a_1(s) := \frac{e^{-2is\kappa}}{s^{n-1}} \Sigma_{n,\tau}(e^{is\kappa} a)(s), \tag{64}$$

$$C(\kappa, \tau, n) := \frac{(n-1)!}{(e^{2i\tau\kappa} - 1)^{n-1}(e^{-2i\tau\kappa} - 1)}. \tag{65}$$

Remark 3.2. *For $\Sigma_{n,\tau}$ defined by (62) and $2n$ points s_j of (59)–(61), we have that*

$$\Sigma_{n,\tau} w_2 \Sigma_{n,\tau} w_1 u(s) =$$
$$= \sum_{1 \leq j_1,\, j_2 \leq n} \frac{(-1)^{j_1+j_2} (s_{j_1}(s))^{n-1} (s_{j_1+j_2-1}(s))^{n-1} w_2(s_{j_1}(s)) w_1(s_{j_1+j_2-1}(s))}{(j_1-1)!(n-j_1)!(j_2-1)!(n-j_2)!\tau^{2n-2}} u(s_{j_1+j_2-1}(s)), \tag{66}$$

where w_1, w_2 are fixed functions on $[r, +\infty)$, and u is a test function on $[r, +\infty)$. In addition,

$$s_{j_1+j_2-1}(s+\tau) = s_{j_1+j_2}(s),\ 1 \leq j_1, j_2 \leq n. \tag{67}$$

Formulas (66), (67) explain that formula (63) involves $a(s)$ exactly in $2n$ points s_j of (59)–(61), $j = 1, ..., 2n$. Actually, formula (66) explains that computing $a_2(s)$ via (64) requires $a(s)$ in $2n-1$ points s_j of (59)–(61), $j = 1, ..., 2n-1$; formulas (67), (66), (63) explain that computing $a_2(s+\tau)$ requires $a(s)$ in shifted $2n-1$ points s_j of (59)–(61), $j = 2, ..., 2n$.

Theorem 3.2 is proved in Section 5.

Note that formulas (63)–(65) for finding f_1 from $a(s)$ coincide with formulas (52)–(54) for the case when $N = n = 1$, $\sigma = \tau$.

Note also that Theorem 3.2 is used in the proofs of Propositions 3.1 and 3.2; see Subsections 3.3, 3.4 and Section 6.

3.3 Formulas for finding f_1 from a at $3n-1$ points for $d = 3$

In this Subsection we give formulas for finding f_1 up to $\mathcal{O}(s^{-n})$, as $s \to +\infty$, from $a(s_j(s))$, $j = 1, ..., 3n-1$, where $a(s)$ is defined by (44), $s_j = s_j(s)$ are the points of (68)–(70). We use that $a(s)$ satisfies (45). In constrast to Theorem 3.1, we do not assume that differences between some of these points are multiple to π/κ. In addition, in constrast to Theorem 3.2, we do not assume that a is reduced to the linearised a of the form (58). On the other hand, the disadvantage of approximate finding f_1 mentioned above, is that this finding uses a at $3n-1$ points, in contrast to $2n$ points of Theorems 3.1 and 3.2.

We consider $3n-1$ points $s_j \in [r, +\infty)$, $j = 1, ..., 3n-1$, of the form

$$s_j = s_j(s) = s + \tau_j,\ j = 1, ..., 3n-1, \tag{68}$$

$$\tau_j = (j-1)\tau,\ j = 1, ..., 3n-1, \tag{69}$$

$$s > r,\ \tau > 0,\ \tau \neq 0\,(mod\,\frac{\pi}{\kappa}). \tag{70}$$

Proposition 3.1. *Let $a = a(s)$ satisfy (45) for some $N \geq 2n-1$, $n \in \mathbb{N}$. Let $\Sigma_{n,\tau}$ be defined by (62). Then $a(s)$ at $3n-1$ points s_j of (68)–(70) approximately determines f_1 as follows:*

$$f_1 = C_1(\kappa, \tau, n)(\tau^{n-1}(e^{-2i\tau\kappa}a_2(s) - a_2(s+\tau)) + \mathcal{O}(s^{-n})),\ s \to +\infty, \tag{71}$$

$$a_2(s) := \Sigma_{n,\tau}a_1(s),\ a_1(s) := \frac{e^{-2is\kappa}}{s^{n-1}}\Sigma_{n,\tau}(e^{is\kappa}a_0)(s), \tag{72}$$

$$a_0(s) := \frac{1}{s^{n-1}}\Sigma_{n,\tau}a(s), \tag{73}$$

$$C_1(\kappa, \tau, n) = \frac{(n-1)!\tau^{n-1}}{(e^{i\tau\kappa}-1)^{n-1}}C(\kappa, \tau, n), \tag{74}$$

where $C(\kappa, \tau, n)$ is given by (65).

Remark 3.3. *For $\Sigma_{n,\tau}$ defined by (62) and $3n-1$ points s_j of (68)–(70), we have that*

$$\Sigma_{n,\tau}w_3\Sigma_{n,\tau}w_2\Sigma_{n,\tau}w_1u(s) =$$
$$= \sum_{1\leq j_1, j_2, j_3 \leq n} \frac{(-1)^{n+j_1+j_2+j_3}(s_{j_1}(s))^{n-1}(s_{j_1+j_2-1}(s))^{n-1}(s_{j_1+j_2+j_3-2}(s))^{n-1}}{(j_1-1)!(n-j_1)!(j_2-1)!(n-j_2)!(j_3-1)!(n-j_3)!\tau^{3n-3}} \times$$
$$w_3(s_{j_1}(s))w_2(s_{j_1+j_2-1}(s))w_1(s_{j_1+j_2+j_3-2}(s))u(s_{j_1+j_2+j_3-2}(s)), \tag{75}$$

where w_1, w_2, w_3 are fixed functions on $[r, +\infty)$, and u is a test function on $[r, +\infty)$. In addition,

$$s_{j_1+j_2+j_3-2}(s+\tau) = s_{j_1+j_2+j_3-1}(s),\ 1 \leq j_1, j_2, j_3 \leq n. \tag{76}$$

Formulas (75), (76) explain that formula (71) involves $a(s)$ exactly in $3n-1$ points s_j of (68)–(70), $j = 1, ..., 3n-1$. Actually, formulas (72), (73) and (75) explain that computing $a_2(s)$ via (72), (73) involves $a(s)$ in $3n-2$ points s_j of (68)–(70), $j = 1, ..., 3n-2$; formulas (75), (76)

explain that computing $a_2(s+\tau)$ via formulas (72), (73) requires $a(s)$ in shifted $3n-2$ points s_j of (68)–(70), $j=2,...,3n-1$.

Proposition 3.1 is proved in Section 6.

Note that formulas (44), (68)–(74) for finding f_1 from a remain valid for any odd $d \geq 3$, in a similar way with formulas (44), (47)–(56).

Note also that formulas (71)–(74) for finding f_1 from $a(s)$ coincide with formulas (52)–(54) and with formulas (63)–(65) for the case when $N=n=1$, $\sigma=\tau$. For this case these formulas were given in [25], [27].

3.4 Formulas for finding f_1 from a at $3n$ points for $d=2$

For $d=2$, the function $a(x,k)$ of formulas (7), (27)–(31) at fixed k and $\hat{x}=x/|x|$ can be written as $a(s)$ defined according to (44), where

$$a(s)=\sum_{j=1}^{N}\frac{e^{is\kappa}f_j+e^{-is\kappa}\overline{f}_j}{s^{j-1}}+\sum_{j=1}^{N}\frac{h_j}{s^{j-1/2}}+\mathcal{O}\left(\frac{1}{s^N}\right),\ s\to+\infty, \tag{77}$$

$$h_j=\sum_{k=1}^{j}f_k\overline{f}_{j-k+1},\ j=1,...,N, \tag{78}$$

where $s=|x|$, $\kappa=|k|-k\hat{x}$, and $\kappa>0$ if $\hat{x}=x/|x|\neq\hat{k}=k/|k|$.

Proceeding from this motivation, we consider arbitrary functions $a=a(s)$, $s\in[r,+\infty)$, $r>0$, such that formula (77) holds for some fixed $\kappa>0$ and some complex numbers f_j, $j=1,...,N$, and h_j, $j=1,...,N$.

In this Subsection we give formulas for finding f_1 up to $\mathcal{O}(s^{-n})$, as $s\to+\infty$, from $a(s_j(s))$, $j=1,...,3n$, where $a(s)$ is of the form (77), $s_j=s_j(s)$ are the points of (79)–(81). In constrast to Theorem 3.2, we do not assume that a is reduced to the linearised a of the form (58). On the other hand, the disadvantage of approximate finding f_1 mentioned above, is that this finding uses a at $3n$ points, in contrast to $2n$ points of Theorem 3.2.

We consider $3n$ points $s_j\in[r,+\infty)$, $j=1,...,3n$ of the form

$$s_j(s)=s+\tau_j,\ j=1,...,3n, \tag{79}$$

$$\tau_j=(j-1)\tau,\ j=1,...,3n, \tag{80}$$

$$s\geq r>0,\ \tau>0,\tau\neq 0\ (mod\,\frac{\pi}{\kappa}). \tag{81}$$

Proposition 3.2. *Let $a(s)$ satisfy (77) for some $N\geq 2n+1$, $n\in\mathbb{N}$. Let $\Sigma_{n,\tau}$ be defined by (62). Then $a(s)$ at $3n$ points s_j of (79)–(81) approximately determines f_1 as follows:*

$$f_1=C_2(\kappa,\tau,n)(\tau^{n-1}(e^{-2i\tau\kappa}a_3(s)-a_3(s+\tau))+\mathcal{O}(s^{-n})),\ s\to+\infty, \tag{82}$$

$$a_3(s):=\Sigma_{n,\tau}a_2(s),\ a_2(s):=\frac{e^{-2is\kappa}}{s^{n-1}}\Sigma_{n,\tau}(e^{is\kappa}a_1)(s),\ a_1(s)=\frac{1}{s^{n-1/2}}\Sigma_{n+1,\tau}a_0(s),\ a_0(s)=a(s)/\sqrt{s}, \tag{83}$$

$$C_2(\kappa,\tau,n)=\frac{n!(n-1)!\tau^n}{(e^{i\tau\kappa}-1)^n(e^{2i\tau\kappa}-1)^{n-1}(e^{-2i\tau\kappa}-1)}. \tag{84}$$

Remark 3.4. *For $\Sigma_{n,\tau}$ defined by (62) and $3n$ points s_j of (79)–(81), we have that*

$$\begin{aligned}&\Sigma_{n,\tau}w_3\Sigma_{n,\tau}w_2\Sigma_{n+1,\tau}w_1u(s)=\\&=\sum_{1\leq j_1,\,j_2\leq n}\ \sum_{1\leq j_3\leq n+1}\frac{(-1)^{n+j_1+j_2+j_3}(s_{j_1}(s))^{n-1}(s_{j_1+j_2-1}(s))^{n-1}(s_{j_1+j_2+j_3-2}(s))^{n-1}}{(j_1-1)!(n-j_1)!(j_2-1)!(n-j_2)!(j_3-1)!(n-j_3)!\tau^{3n-3}}\times\\&\times w_3(s_{j_1}(s))w_2(s_{j_1+j_2-1}(s))w_1(s_{j_1+j_2+j_3-2}(s))u(s_{j_1+j_2+j_3-2}(s)),\end{aligned} \tag{85}$$

where w_1, w_2, w_3 are fixed functions on $[r,+\infty)$, and u is a test function on $[r,+\infty)$. In addition,

$$s_{j_1+j_2+j_3-2}(s+\tau)=s_{j_1+j_2+j_3-1}(s),\ 1\le j_1,j_2,j_3\le n. \tag{86}$$

Formulas (85), (86) explain that formulas (82) involves $a(s)$ exactly in $3n$ points s_j of (79)–(81), $j=1,...,3n$. Actually, formulas (85) and (83) explain that computing $a_2(s)$ via (83) involves $a(s)$ in $3n-1$ points s_j of (79)–(81), $j=1,...,3n-1$; formulas (85), (86) explain that computing $a_2(s+\tau)$ via formulas (83) involves $a(s)$ in $3n-1$ shifted points s_j of (79)–(81), $j=2,...,3n$.

Proposition 3.2 is proved in Section 6.

Note that formulas (44), (79)–(84) for finding f_1 from a remain valid for any even $d\ge 2$, in a similar way with formulas (44), (47)–(56), (68)–(74) for odd $d\ge 3$.

4 Proof of Theorem 3.1

Let

$$\vec{\tau}_1=(\tau_{1,1},...,\tau_{1,n}),\ \vec{\tau}_2=(\tau_{2,1},...,\tau_{2,n}). \tag{87}$$

Let $y_j(s,\vec{\tau}_1), y_j(s,\vec{\tau}_2)$, $j=1,...,n$, be defined according to (40), (41). We have that

$$a_1(s)=\sum_{j=1}^{n} y_j(s,\vec{\tau}_1)a(s_{1,j}(s)),\ a_2(s)=\sum_{j=1}^{n} y_j(s,\vec{\tau}_2)a(s_{2,j}(s)). \tag{88}$$

Note that

$$\begin{aligned} &e^{is_{1,j}(s)\kappa}=e^{is\kappa}e^{i\nu_{1,j}\tau\kappa}=e^{is\kappa},\ \ j=1,...,n,\\ &e^{is_{2,j}(s)\kappa}=e^{i(s+\sigma)\kappa}e^{i\nu_{2,j}\tau\kappa}=e^{i(s+\sigma)\kappa},\ \ j=1,...,n. \end{aligned} \tag{89}$$

Using formulas (40)–(43), (45), (88), (89) and the assumption that $N\ge 2n-1$, we obtain that:

$$\begin{aligned} &a_1(s)=\sum_{j=1}^{n} y_j(s,\vec{\tau}_1)\left(\sum_{m=1}^{N}\frac{e^{is_{1,j}(s)\kappa}f_m+e^{-is_{1,j}(s)\kappa}\overline{f}_m+h_m}{(s+\tau_{1,j})^{m-1}}+\mathcal{O}(s^{-N})\right)=\\ &=\sum_{m=1}^{N}(e^{is\kappa}f_m+e^{-is\kappa}\overline{f}_m+h_m)\sum_{j=1}^{n}\frac{y_j(s,\vec{\tau}_1)}{(s+\tau_{1,j})^{m-1}}+\mathcal{O}(s^{-n})=\\ &=e^{is\kappa}f_1+e^{-is\kappa}\overline{f}_1+\mathcal{O}(s^{-n}),\ \ \text{as } s\to+\infty; \end{aligned} \tag{90}$$

$$\begin{aligned} &a_2(s)=\sum_{j=1}^{n} y_j(s,\vec{\tau}_2)\left(\sum_{m=1}^{N}\frac{e^{is_{2,j}(s)\kappa}f_m+e^{-is_{2,j}(s)\kappa}\overline{f}_m+h_m}{(s+\sigma+\tau_{2,j})^{m-1}}+\mathcal{O}(s^{-N})\right)=\\ &=\sum_{m=1}^{N}(e^{i(s+\sigma)\kappa}f_m+e^{-i(s+\sigma)\kappa}\overline{f}_m+h_m)\sum_{j=1}^{n}\frac{y_j(s,\vec{\tau}_2)}{(s+\sigma+\tau_{2,j})^{m-1}}+\mathcal{O}(s^{-n})=\\ &=e^{i(s+\sigma)\kappa}f_1+e^{-i(s+\sigma)\kappa}\overline{f}_1+\mathcal{O}(s^{-n}),\ \ \text{as } s\to+\infty. \end{aligned} \tag{91}$$

We consider formulas (90), (91) as a linear system for approximate finding f_1, $\overline{f}_1$ from $a_1(s)$, $a_2(s)$, where s is sufficiently large.

If $\sigma>0$, $\sigma\ne 0\,(mod\ \pi/\kappa)$, then from (90), (91) we obtain that

$$\begin{pmatrix} f_1\\ \overline{f}_1\end{pmatrix}=\frac{1}{e^{-i\sigma\kappa}-e^{i\sigma\kappa}}\begin{pmatrix} e^{-i(s+\sigma)\kappa} & -e^{-is\kappa}\\ -e^{i(s+\sigma)\kappa} & e^{is\kappa}\end{pmatrix}\begin{pmatrix} a_1(s)+ & \mathcal{O}(s^{-n})\\ a_2(s)+ & \mathcal{O}(s^{-n})\end{pmatrix}. \tag{92}$$

Formula (52) follows from (92).

Theorem 3.1 is proved.

5 Proof of Theorem 3.2

Lemma 5.1. *Let $\Sigma_{n,\tau}$ be defined via (62). Let*

$$u(s) = \sum_{j=1}^{N} \frac{e^{is\kappa} f_j}{s^{j-1}},\ s \in [r, +\infty), \tag{93}$$

for some fixed $\kappa \in \mathbb{R}$ and some complex numbers $f_j, j = 1, ..., N$, where $N \geq 2n-1$. Then:

$$\Sigma_{n,\tau} u(s) = s^{n-1} \sum_{m=1}^{2n-1} \frac{e^{is\kappa} f_{m,\kappa}}{s^{m-1}} + \mathcal{O}(s^{-n}),\ s \in [r, +\infty), \tag{94}$$

$$f_{m,\kappa} = f_{m,\kappa,n,\tau}(f_1, ..., f_m) = \sum_{j_2=1}^{m} \frac{f_{j_2} C_{n-j_2}^{m-j_2}}{\tau^{n-1}} \sum_{j_1=1}^{n} \frac{(-1)^{n-j_1} \tau_{j_1}^{m-j_2} e^{i\tau_{j_1}\kappa}}{(j_1-1)!(n-j_1)!},\ \forall m = 1, ..., 2n-1, \tag{95}$$

where C_β^α denote the binomial coefficients,

$$f_{1,\kappa} = \frac{(e^{i\tau\kappa} - 1)^{n-1}}{(n-1)!\tau^{n-1}} f_1, \tag{96}$$

$$f_{n,\kappa} = \frac{(e^{i\tau\kappa} - 1)^{n-1}}{(n-1)!\tau^{n-1}} f_n \text{ if } f_1 = ... = f_{n-1} = 0. \tag{97}$$

Proof of Lemma 5.1. Using (41), (93), we obtain that

$$\Sigma_{n,\tau} u(s) = \sum_{j_1=1}^{n} \sum_{j_2=1}^{N} \frac{(-1)^{n-j_1}(s+\tau_{j_1})^{n-1}}{(j_1-1)!(n-j_1)!\tau^{n-1}} \frac{e^{i(s+\tau_{j_1})\kappa} f_{j_2}}{(s+\tau_{j_1})^{j_2-1}} =$$

$$= \frac{e^{is\kappa}}{\tau^{n-1}} \sum_{j_2=1}^{N} f_{j_2} \sum_{j_1=1}^{n} \frac{(-1)^{n-j_1}(s+\tau_{j_1})^{n-j_2} e^{i\tau_{j_1}\kappa}}{(j_1-1)!(n-j_1)!},\ s \in [r, +\infty). \tag{98}$$

$$\tag{99}$$

Note that

$$(s+\tau_{j_1})^{n-j_2} = \sum_{j_3=0}^{n-j_2} C_{n-j_2}^{j_3} s^{n-j_2-j_3} \tau_{j_1}^{j_3} = \sum_{j_3=0}^{2n-j_2-1} C_{n-j_2}^{j_3} \tau_{j_1}^{j_3} s^{n-j_2-j_3},\ n - j_2 \geq 0, \tag{100}$$

$$(s+\tau_{j_1})^{n-j_2} = \frac{1}{s^{j_2-n}} \sum_{j_3=0}^{\infty} C_{n-j_2}^{j_3} \tau_{j_1}^{j_3} s^{-j_3} = \sum_{j_3=0}^{2n-j_2-1} C_{n-j_2}^{j_3} \tau_{j_1}^{j_3} s^{n-j_2-j_3} + \mathcal{O}(s^{-n}), n - j_2 < 0, \tag{101}$$

where C_β^α are the binomial coefficients; we used in (100) that $C_\beta^\alpha = 0$, for $\alpha > \beta$, if $\beta \in \mathbb{N}$. Using (98), (100), (101) we obtain that

$$\frac{\Sigma_{n,\tau} u(s)}{s^{n-1}} = \frac{e^{is\kappa}}{\tau^{n-1}} \sum_{j_2=1}^{N} f_{j_2} \sum_{j_1=1}^{n} \frac{(-1)^{n-j_1} \sum_{j_3=0}^{2n-j_2-1} (C_{n-j_2}^{j_3} \tau_{j_1}^{j_3} / s^{j_2+j_3-1}) e^{i\tau_{j_1}\kappa}}{(j_1-1)!(n-j_1)!} + \mathcal{O}(s^{-2n+1}) =$$

$$= \frac{e^{is\kappa}}{\tau^{n-1}} \sum_{j_2=1}^{2n-1} f_{j_2} \sum_{j_1=1}^{n} \frac{(-1)^{n-j_1} \sum_{j_3=0}^{2n-j_2-1} (C_{n-j_2}^{j_3} \tau_{j_1}^{j_3} / s^{j_2+j_3-1}) e^{i\tau_{j_1}\kappa}}{(j_1-1)!(n-j_1)!} + \mathcal{O}(s^{-2n+1}) =$$

$$= \frac{e^{is\kappa}}{\tau^{n-1}} \sum_{j_3=0}^{2n-2} \sum_{j_2=1}^{2n-j_3-1} \frac{f_{j_2} C_{n-j_2}^{j_3}}{s^{j_2+j_3-1}} \sum_{j_1=1}^{n} \frac{(-1)^{n-j_1} \tau_{j_1}^{j_3} e^{i\tau_{j_1}\kappa}}{(j_1-1)!(n-j_1)!} + \mathcal{O}(s^{-2n+1}) = [m := j_2 + j_3]$$

$$= \sum_{m=1}^{2n-1} \frac{e^{is\kappa}}{s^{m-1}} \sum_{j_2=1}^{m} \frac{f_{j_2} C_{n-j_2}^{m-j_2}}{\tau^{n-1}} \sum_{j_1=1}^{n} \frac{(-1)^{n-j_1} \tau_{j_1}^{m-j_2} e^{i\tau_{j_1}\kappa}}{(j_1-1)!(n-j_1)!} + \mathcal{O}(s^{-2n+1}) = \sum_{m=1}^{2n-1} \frac{e^{is\kappa} f_{m,\kappa}}{s^{m-1}} + \mathcal{O}(s^{-2n+1}), \tag{102}$$

where

$$f_{m,\kappa} = \sum_{j_2=1}^{m} \frac{f_{j_2} C_{n-j_2}^{m-j_2}}{\tau^{n-1}} \sum_{j_1=1}^{n} \frac{(-1)^{n-j_1} \tau_{j_1}^{m-j_2} e^{i\tau_{j_1}\kappa}}{(j_1-1)!(n-j_1)!}, \tag{103}$$

$$f_{1,\kappa} = \frac{f_1 C_{n-1}^0}{\tau^{n-1}} \sum_{j=1}^{n} \frac{(-1)^{n-j} e^{i\tau(j-1)\kappa}}{(j-1)!(n-j)!} = \frac{(e^{i\tau\kappa}-1)^{n-1}}{(n-1)!\tau^{n-1}} f_1. \tag{104}$$

Formulas (94), (96) follows from (102), (103), (104).

In addition, if $f_1 = ... = f_{n-1} = 0$, then

$$f_{n,\kappa} = f_n \frac{C_0^0}{\tau^{n-1}} \sum_{j=1}^{n} \frac{(-1)^{n-j} e^{i(j-1)\tau\kappa}}{(j-1)!(n-j)!} = \frac{(e^{i\tau\kappa}-1)^{n-1}}{(n-1)!\tau^{n-1}} f_n. \tag{105}$$

Lemma 5.1 is proved.

The rest of the proof of Theorem 3.2 is as follows. Due to (58), we have that

$$\begin{aligned} &e^{is\kappa} a(s) = u(s) + z(s), \\ &u(s) = \sum_{j=1}^{N} \frac{e^{2is\kappa} f_j}{s^{j-1}}, \quad z(s) = \sum_{j=1}^{N} \frac{\overline{f}_j}{s^{j-1}} + \mathcal{O}(s^{-N}), \ \text{ as } s \to +\infty. \end{aligned} \tag{106}$$

Due to formulas (39), (41), (43), (62), and the formula for z in (106), we have that

$$\Sigma_{n,\tau} z(s) = \overline{f}_1 + \mathcal{O}(s^{-n}), \ \text{ as } s \to +\infty. \tag{107}$$

Due to Lemma 5.1 (with 2κ in place of κ) and the formula for u in (106), we have that

$$\begin{aligned} &\Sigma_{n,\tau} u(s) = s^{n-1} \sum_{m=1}^{2n-1} \frac{e^{2is\kappa} f_{m,2\kappa}}{s^{m-1}} + \mathcal{O}(s^{-n}), \ \text{ where} \\ &f_{1,2\kappa} = \frac{(e^{2i\tau\kappa}-1)^{n-1}}{(n-1)!\tau^{n-1}} f_1. \end{aligned} \tag{108}$$

Due to (106), (107), (108), we have that

$$\Sigma_{n,\tau}(e^{is\kappa} a)(s) = s^{n-1} \sum_{m=1}^{2n-1} \frac{e^{2is\kappa} f_{m,2\kappa}}{s^{m-1}} + \mathcal{O}(s^{-n}) + \overline{f}_1 + \mathcal{O}(s^{-n}), \ \text{ as } s \to +\infty. \tag{109}$$

The definition of a_1 in (64) and formula (109) imply that

$$\begin{aligned} &a_1(s) = u_1(s) + z_1(s), \ \text{ where} \\ &u_1(s) = \frac{e^{-2is\kappa}\overline{f}_1}{s^{n-1}}, \ z_1(s) = \sum_{m=1}^{2n-1} \frac{f_{m,2\kappa}}{s^{m-1}} + \mathcal{O}(s^{-2n+1}), \ \text{ as } s \to +\infty. \end{aligned} \tag{110}$$

Due to formulas (39), (41), (43), (62), and the formula for z_1 in (110), we have that

$$\Sigma_{n,\tau} z_1(s) = f_{1,2\kappa} + \mathcal{O}(s^{-n}), \ \text{ as } s \to +\infty. \tag{111}$$

Besides, the following formula holds,

$$\Sigma_{n,\tau} u_1(s) = e^{-2is\kappa}\overline{f}_{1,-2\kappa}, \quad \overline{f}_{1,-2\kappa} = \frac{(e^{-2i\tau\kappa}-1)^{n-1}}{(n-1)!\tau^{n-1}}\overline{f}_1. \tag{112}$$

Formula (112) follows from formulas (94), (95), (97) of Lemma 5.1, with (-2κ) in place of κ, $f_n = \overline{f}_1$, where $\overline{f}_1$ is the number in definition of u_1 in (110).

The definition of a_2 in (64) and formulas (108), (110)–(112) imply that

$$a_2(s) = f_{1,2\kappa} + e^{-2is\kappa}\overline{f}_{1,-2\kappa} + \mathcal{O}(s^{-n}) = \\ = \frac{(e^{2i\tau\kappa}-1)^{n-1}f_1 + e^{-2is\kappa}(e^{-2i\tau\kappa}-1)^{n-1}\overline{f}_1}{(n-1)!\tau^{n-1}} + \mathcal{O}(s^{-n}),\ s\to+\infty. \tag{113}$$

From (113) we obtain that

$$(e^{2i\tau\kappa}-1)^{n-1}f_1 + e^{-2is\kappa}(e^{-2i\tau\kappa}-1)^{n-1}\overline{f}_1 = (n-1)!\tau^{n-1}a_2(s) + \mathcal{O}(s^{-n}),\ s\to+\infty, \tag{114}$$
$$(e^{2i\tau\kappa}-1)^{n-1}f_1 + e^{-2i(s+\tau)\kappa}(e^{-2i\tau\kappa}-1)^{n-1}\overline{f}_1 = (n-1)!\tau^{n-1}a_2(s+\tau) + \mathcal{O}(s^{-n}),\ s\to+\infty.$$

We consider formulas (114) as a linear system for approximate finding f_1, $\overline{f}_1$ from $(n-1)!\tau^{n-1}a_2(s)$, $(n-1)!\tau^{n-1}a_2(s+\tau)$, where s is sufficiently large. From this system, for $\tau>0$, $\tau\neq 0\,(mod\,2\pi/\kappa)$, we obtain that

$$\begin{pmatrix} f_1 \\ \overline{f}_1 \end{pmatrix} = \frac{1}{(e^{2i\tau\kappa}-1)^{n-1}(e^{-2i\tau\kappa}-1)^{n-1}e^{-2is\kappa}(e^{-2i\tau\kappa}-1)}\times \\ \begin{pmatrix} e^{-2i(s+\tau)\kappa}(e^{-2i\tau\kappa}-1)^{n-1} & -e^{-2is\kappa}(e^{-2i\tau\kappa}-1)^{n-1} \\ -(e^{2i\tau\kappa}-1)^{n-1} & (e^{2i\tau\kappa}-1)^{n-1} \end{pmatrix} \begin{pmatrix} (n-1)!\tau^{n-1}a_2(s)+\mathcal{O}(s^{-n}) \\ (n-1)!\tau^{n-1}a_2(s+\tau)+\mathcal{O}(s^{-n}) \end{pmatrix},\ s\to+\infty. \tag{115}$$

Formula (115) implies that

$$f_1 = \frac{(n-1)!\tau^{n-1}(e^{-2i\tau\kappa}a_2(s) - a_2(s+\tau)) + \mathcal{O}(s^{-n})}{(e^{2i\tau\kappa}-1)^{n-1}(e^{-2i\tau\kappa}-1)},\ s\to+\infty. \tag{116}$$

This completes the proof of Theorem 3.2.

6 Proofs of Propositions 3.1 and 3.2

Proof of Proposition 3.1. The following formula holds:

$$\Sigma_{n,\tau}a(s) = s^{n-1}\sum_{m=1}^{n}\frac{e^{is\kappa}f_{m,\kappa} + e^{-is\kappa}\overline{f}_{m,-\kappa}}{s^{m-1}} + \mathcal{O}(s^{-n}),\ s\to+\infty, \tag{117}$$

where $f_{m,\kappa}$ and $\overline{f}_{m,-\kappa}$ are defined according to (95) in terms of $f_1,...,f_m$ and $\overline{f}_1,...,\overline{f}_m$, respectively. In particular, according to (96), we have that

$$f_{1,\kappa} = \frac{(e^{i\tau\kappa}-1)^{n-1}}{(n-1)!\tau^{n-1}}f_1,\ \overline{f}_{1,-\kappa} = \frac{(e^{-i\tau\kappa}-1)^{n-1}}{(n-1)!\tau^{n-1}}\overline{f}_1. \tag{118}$$

Formula (117) follows from formula (45), Lemma 5.1 and formulas (39), (41), (43), (62).

Formula (71), where a_2 is defined by (72), (73), follows from formulas (117), (118), and Theorem 3.2.

Proposition 3.1 is proved.

Proof of Proposition 3.2. Assume that

$$n\tau/s < 1,\ \text{where } s\geq r>0. \tag{119}$$

From the definition of a_0 and formula (77) we have that

$$a_0(s) = \sum_{j=1}^{N}\frac{e^{is\kappa}f_j}{s^{j-1/2}} + \sum_{j=1}^{N}\frac{e^{-is\kappa}\overline{f}_j}{s^{j-1/2}} + \left(\sum_{j=2}^{N}\frac{h_{j-1}}{s^{j-1}} + \mathcal{O}\left(\frac{1}{s^N}\right)\right) =: a_{0,1}(s) + a_{0,2}(s) + a_{0,3}(s). \tag{120}$$

Using that $N \geq 2n+1$ and formulas (37), (39) with $n+1$ in place of n, we obtain that

$$\Sigma_{n+1,\tau} a_{0,3}(s) = \mathcal{O}(s^{-(n+1)}),\ s \to +\infty. \tag{121}$$

We also have that

$$\Sigma_{n+1,\tau} a_{0,1}(s) = \sum_{j=1}^{n+1} \frac{(-1)^{n+1-j}(s+\tau_j)^n \sum_{k=1}^N e^{i(s+\tau_j)\kappa} f_k/(s+\tau_j)^{k-1/2}}{(j-1)!(n+1-j)!\tau^n} =$$
$$= e^{is\kappa} \sum_{j=1}^{n+1} \sum_{k=1}^{N} \frac{(-1)^{n+1-j}(s+\tau_j)^{n-k+1/2} e^{i(j-1)\tau\kappa} f_k}{(j-1)!(n+1-j)!\tau^n}. \tag{122}$$

Note that

$$\frac{(s+\tau_j)^{n-k+1/2}}{s^{n-1/2}} = \frac{(1+\tau_j/s)^{n-k+1/2}}{s^{k-1}}, \tag{123}$$
$$(1+\tau_j/s)^{n-k+1/2} = \sum_{l=0}^{2n} C^l_{n-k+1/2} \left(\frac{\tau_j}{s}\right)^l + \mathcal{O}(s^{-2n-1}),\ s \to +\infty, \tag{124}$$

where in (124) we used assumption (119) and that $\tau_j = (j-1)\tau$, $j = 1, ..., n+1$. Using that $N \geq 2n+1$ and formulas (122), (123), (124), we obtain that

$$\begin{aligned}
&\Sigma_{n+1,\tau} a_{0,1}(s)/s^{n-1/2} = e^{is\kappa} \sum_{j=1}^{n+1} \sum_{k=1}^{N} \frac{(-1)^{n+1-j}(1+(j-1)\tau/s)^{n-k+1/2} e^{i(j-1)\tau\kappa} f_k}{(j-1)!(n+1-j)!\tau^n s^{k-1}} = \\
&= \frac{e^{is\kappa}}{\tau^n} \sum_{j=1}^{n+1} \sum_{k=1}^{N} \left(\sum_{l=0}^{2n} \frac{(-1)^{n+1-j} C^l_{n-k+1/2} \left(\frac{(j-1)\tau}{s}\right)^l e^{i(j-1)\tau\kappa} f_k}{(j-1)!(n+1-j)! s^{k-1}} + \mathcal{O}(s^{-2n-1}) \right) = \\
&= \frac{e^{is\kappa}}{\tau^n} \sum_{j=0}^{n} \sum_{k=1}^{2n+1} \sum_{l=0}^{2n} \frac{(-1)^{n-j} C^l_{n-k+1/2} (j\tau)^l e^{ij\tau\kappa} f_k}{j!(n-j)! s^{k-1+l}} + \mathcal{O}(s^{-2n-1}) = \quad [l = m-k] \\
&= \frac{e^{is\kappa}}{\tau^n} \sum_{m=1}^{2n+1} \sum_{k=1}^{\min(m,2n+1)} \sum_{j=0}^{n} \frac{(-1)^{n-j} C^{m-k}_{n-k+1/2} (j\tau)^{m-k} e^{ij\tau\kappa}}{j!(n-j)! s^{m-1}} f_k + \mathcal{O}(s^{-2n-1}) = \\
&= \frac{1}{\tau^n} \sum_{m=1}^{2n+1} \frac{e^{is\kappa}}{s^{m-1}} \sum_{k=1}^{m} f_k C^{m-k}_{n-k+1/2} \sum_{j=0}^{n} \frac{(-1)^{n-j} (j\tau)^{m-k} e^{ij\tau\kappa}}{j!(n-j)!} + \mathcal{O}(s^{-2n-1}) = \\
&= \sum_{m=1}^{2n+1} \frac{e^{is\kappa} f_{m,\kappa}}{s^{m-1}} + \mathcal{O}(s^{-2n-1}),\ s \to +\infty,
\end{aligned} \tag{125}$$

where

$$f_{m,\kappa} = \frac{1}{\tau^n} \sum_{k=1}^{m} f_k C^{m-k}_{n-k+1/2} \sum_{j=0}^{n} \frac{(-1)^{n-j} (j\tau)^{m-k} e^{ij\tau\kappa}}{j!(n-j)!},\ m = 1, ..., 2n+1, \tag{126}$$
$$f_{1,\kappa} = \frac{f_1}{\tau^n} \sum_{j=0}^{n} \frac{(-1)^{n-j} e^{ij\tau\kappa}}{j!(n-j)!} = \frac{(e^{i\tau\kappa}-1)^n}{\tau^n n!} f_1, \quad \text{where } \frac{(e^{i\tau\kappa}-1)^n}{\tau^n n!} \neq 0. \tag{127}$$

In (127) we have that $e^{i\tau\kappa} \neq 1$ due to (81). In a similar way with (125) we also have that

$$\Sigma_{n+1,\tau} a_{0,2}(s)/s^{n-1/2} = \sum_{m=1}^{2n+1} \frac{e^{-is\kappa} \overline{f_{m,\kappa}}}{s^{m-1}} + \mathcal{O}(s^{-2n-1}),\ s \to +\infty. \tag{128}$$

From formulas (121), (125), (128) we obtain that

$$a_1(s) := \frac{\Sigma_{n+1,\tau} a_0(s)}{s^{n-1/2}} = \sum_{j=1}^{2n+1} \frac{e^{is\kappa} f_{j,\kappa} + e^{-is\kappa}\overline{f_{j,\kappa}}}{s^{j-1}} + \mathcal{O}(s^{-2n-1}) + \mathcal{O}(s^{-2n-1/2}) =$$
$$= \sum_{j=1}^{2n+1} \frac{e^{is\kappa} f_{j,\kappa} + e^{-is\kappa}\overline{f_{j,\kappa}}}{s^{j-1}} + \mathcal{O}(s^{-2n-1/2}) = \sum_{j=1}^{2n-1} \frac{e^{is\kappa} f_{j,\kappa} + e^{-is\kappa}\overline{f_{j,\kappa}}}{s^{j-1}} + \mathcal{O}(s^{-2n+1}),\ s \to +\infty. \tag{129}$$

From (129) one can see that the function $a_1(s)$ satisfies the assumptions of Theorem 3.2 for $N = 2n-1$. Therefore, applying Theorem 3.2 to the function $a_1(s)$ we reconstruct $f_{1,\kappa}$ via formulas

$$f_{1,\kappa} = C(\kappa,\tau,n)(\tau^{n-1}(e^{-2i\tau\kappa}a_3(s) - a_3(s+\tau)) + \mathcal{O}(s^{-n})),\ s \to +\infty, \tag{130}$$

$$a_3(s) := \Sigma_{n,\tau} a_2(s),\ a_2(s) := \frac{e^{-2is\kappa}}{s^{n-1}} \Sigma_{n,\tau}(e^{is\kappa} a_1)(s), \tag{131}$$

$$C(\kappa,\tau,n) := \frac{(n-1)!}{(e^{2i\tau\kappa}-1)^{n-1}(e^{-2i\tau\kappa}-1)}. \tag{132}$$

In addition, formula (127) implies that

$$f_1 = \frac{n!\tau^n f_{1,\kappa}}{(e^{i\tau\kappa}-1)^n}. \tag{133}$$

Formulas (82), (83), (84) follow from formulas (130)–(133) and the definitions of $a_1(s)$, $a_0(s)$. This completes the proof of Proposition 3.2.

7 Sketch of proof of formulas (23)–(25)

We will use that

$$\begin{aligned}
&G^+(x-y,k) = G_n^+(x-y,k) + \rho_n(y,x,|k|),\\
&G_n^+(x-y,k) = \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} \left(\sum_{j=1}^{n} \frac{g_j(y,\hat{x},|k|)}{|x|^{j-1}} \right),\\
&g_1(y,\hat{x},|k|) = e^{-i|k|y\hat{x}},\ \rho_n(y,x,|k|) := \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} \varepsilon_n(y,x,|k|),\\
&\varepsilon_n(y,x,|k|) = \mathcal{O}\left(\frac{1}{|x|^n}\right),\ |x| \to +\infty,\ \text{uniformly in } y \in D,
\end{aligned} \tag{134}$$

where G^+ is defined in (16), $\hat{x} = x/|x|$. The Lippman-Schwinger integral equation (16) and formula (134) imply formulas (19), (23), (24), where

$$(2\pi)^{-d} c(d,|k|)\varphi_j(y,l) = g_j(y,\hat{l},|k|),\ \hat{l} = l/|l|. \tag{135}$$

The recurrent relations (25) can be proved using also that:

$$(\Delta_x + k^2) G^+(x-y,k) = 0,\ \text{for } x \in \mathbb{R}^d \setminus D,\ y \in D; \tag{136}$$

$$\Delta_x = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_S, \tag{137}$$

where $r = |x|$, Δ_S is the Beltrami–Laplace operator with respect to $\hat{x} \in \mathbb{S}^{d-1}$, $\hat{x} = x/|x|$;

$$\Delta_x \rho_n(y,x,|k|) = \mathcal{O}\left(\frac{1}{|x|^{n+(d-1)/2}}\right),\ |x| \to +\infty. \tag{138}$$

We prove below the recurrent relations (25) proceeding from (134)–(138). Our approach consists in expanding equation (136) in $|x|^{-j}$, as $|x| \to +\infty$.

Note that

$$\frac{\partial}{\partial r}\frac{e^{i|k|r}}{r^{\frac{d-1}{2}+j}} = \frac{e^{i|k|r}}{r^{\frac{d-1}{2}}}\left(\frac{i|k|}{r^j} + \frac{(-\frac{d-1}{2}-j)}{r^{j+1}}\right), \tag{139}$$

$$\frac{\partial}{r\partial r}\frac{e^{i|k|r}}{r^{\frac{d-1}{2}+j-1}} = \frac{e^{i|k|r}}{r^{\frac{d-1}{2}}}\left(\frac{i|k|}{r^j} + \frac{(-\frac{d-1}{2}-j+1)}{r^{j+1}}\right), \tag{140}$$

$$\frac{1}{r^2}\Delta_S G^+(x-y,k) = \frac{e^{i|k|r}}{r^{(d-1)/2}}\left(\sum_{j=1}^{n}\frac{\Delta_S g_j(y,\hat{x},|k|)}{r^{j+1}}\right) + \mathcal{O}\left(\frac{1}{r^{n+(d+3)/2}}\right),\ r\to+\infty. \tag{141}$$

Using (134), (140) we obtain that

$$\begin{aligned}
&\frac{\partial G_n^+(x-y,k)}{r\partial r} = \sum_{j=1}^{n} g_j(y,\hat{x},|k|)\frac{\partial}{r\partial r}\frac{e^{i|k|r}}{r^{\frac{d-1}{2}+j-1}} = \\
&= \frac{e^{i|k|r}}{r^{(d-1)/2}}\sum_{j=1}^{n}\left(\frac{i|k|g_j(y,\hat{x},|k|)}{r^j} + (-\frac{d-1}{2}-j+1)\frac{g_j(y,\hat{x},|k|)}{r^{j+1}}\right) = \\
&= \frac{e^{i|k|r}}{r^{(d-1)/2}}\left(\sum_{j=2}^{n+1}\frac{i|k|g_{j-1}(y,\hat{x},|k|)}{r^{j-1}} + \sum_{j=3}^{n+2}(-\frac{d-1}{2}-j+3)\frac{g_{j-2}(y,\hat{x},|k|)}{r^{j-1}}\right) = \\
&= \frac{e^{i|k|r}}{r^{(d-1)/2}}(\frac{i|k|g_1(y,\hat{x},|k|)}{r} + \sum_{j=3}^{n+1}\frac{i|k|g_{j-1}(y,\hat{x},|k|) + (-\frac{d-1}{2}-j+3)g_{j-2}(y,\hat{x},|k|)}{r^{j-1}} + \\
&+ (-\frac{d-1}{2}-n+1)\frac{g_n(y,\hat{x},|k|)}{r^{n+1}}) = \frac{e^{i|k|r}}{r^{(d-1)/2}}\sum_{j=2}^{n}\frac{\theta_{j,d}}{r^{j-1}},
\end{aligned} \tag{142}$$

$$\theta_{j,d} = \begin{cases} i|k|g_{j-1}(y,\hat{x},|k|),\ j=2, \\ i|k|g_{j-1}(y,\hat{x},|k|) + (-\frac{d-1}{2}-j+3)g_{j-2}(y,\hat{x},|k|),\ j=3,...,n+1, \\ (-\frac{d-1}{2}-j+3)g_{j-2}(y,\hat{x},|k|),\ j=n+2. \end{cases} \tag{143}$$

Using (140) and the result of the second equality in (142) we obtain that:

$$\begin{aligned}
&\frac{\partial^2 G_n^+(x-y,k)}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{e^{i|k|r}}{r^{(d-1)/2}}\left(\sum_{j=1}^{n}\frac{i|k|g_j(y,\hat{x},|k|)}{r^{j-1}} + \sum_{j=2}^{n+1}(-\frac{d-1}{2}-j+2)\frac{g_{j-1}(y,\hat{x},|k|)}{r^{j-1}}\right)\right) = \\
&= \sum_{j=1}^{n} i|k|g_j(y,\hat{x},|k|)\frac{\partial}{\partial r}\frac{e^{i|k|r}}{r^{\frac{d-1}{2}+j-1}} + \sum_{j=2}^{n+1}(-\frac{d-1}{2}-j+2)g_{j-1}(y,\hat{x},|k|)\frac{\partial}{\partial r}\frac{e^{i|k|r}}{r^{\frac{d-1}{2}+j-1}} = \\
&= \sum_{j=1}^{n} i|k|g_j(y,\hat{x},|k|)\left(\frac{i|k|e^{i|k|r}}{r^{\frac{d-1}{2}+j-1}} + (-\frac{d-1}{2}-j+1)\frac{e^{i|k|r}}{r^{\frac{d-1}{2}+j}}\right) + \\
&+ \sum_{j=2}^{n+1}(-\frac{d-1}{2}-j+2)g_{j-1}(y,\hat{x},|k|)\left(\frac{i|k|e^{i|k|r}}{r^{\frac{d-1}{2}+j-1}} + (-\frac{d-1}{2}-j+1)\frac{e^{i|k|r}}{r^{\frac{d-1}{2}+j}}\right) = \frac{e^{i|k|r}}{r^{\frac{d-1}{2}}}\sum_{j=1}^{n}\frac{\omega_{j,d}}{r^{j-1}},
\end{aligned} \tag{144}$$

$$\omega_{j,d} = \begin{cases} -|k|^2 g_j(y,\hat{x},|k|),\ j=1, \\ -|k|^2 g_j(y,\hat{x},|k|) + i|k|(-d-2j+5)g_{j-1}(y,\hat{x},|k|),\ j=2, \\ -|k|^2 g_j(y,\hat{x},|k|) + i|k|(-d-2j+5)g_{j-1}(y,\hat{x},|k|) + \\ +((-\frac{d-1}{2}-j+1)^2 - \frac{d-1}{2}-j+1)g_{j-2}(y,\hat{x},|k|),\ j=3,...,n. \end{cases} \tag{145}$$

From (134), (136)–(138), (141), (142), (144) we have that

$$\frac{e^{i|k|r}}{r^{\frac{d-1}{2}}}\sum_{j=1}^{n}\frac{\theta_{j,d}}{r^{j-1}}+(d-1)\left(\frac{e^{i|k|r}}{r^{(d-1)/2}}\sum_{j=2}^{n}\frac{\omega_{j,d}}{r^{j-1}}\right)+\frac{e^{i|k|r}}{r^{(d-1)/2}}\sum_{j=3}^{n+2}\frac{\Delta_S g_{j-2}}{r^{j-1}}=$$
$$=-k^2\frac{e^{i|k|r}}{r^{(d-1)/2}}\sum_{j=1}^{n}\frac{g_j}{r^{j-1}}+\mathcal{O}\left(\frac{|k|^2}{r^{n+(d-1)/2}}\right),\ r\to+\infty, \tag{146}$$

where $\theta_{j,d}$, $\omega_{j,d}, g_j$ do not depend on r. Since n is arbitrary, we obtain, from (146), for $j\geq 3$, that

$$\theta_{j,d}+(d-1)\omega_{j,d}+\Delta_S g_{j-2}=-k^2 g_j. \tag{147}$$

Using definitions (143) for $\theta_{j,d}$, (145) for $\omega_{j,d}$ and equation (147) we obtain, for $j\geq 2$,

$$g_j(y,\hat{x},|k|)=\frac{1}{2i|k|(j-1)}\left(\left(\frac{8d+1-d^2}{4}+j^2-j-2\right)g_{j-1}(y,\hat{x},|k|)+\Delta_S g_{j-1}(y,\hat{x},|k|)\right). \tag{148}$$

Formulas (25) follow from (135), (148).

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