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Local asymptotics of cross-validation in least-squares density estimation

Guillaume Maillard

June 17, 2021

Abstract

In model selection, several types of cross-validation are commonly used and many variants have been introduced. While consistency of some of these methods has been proven, their rate of convergence to the oracle is generally still unknown. Until now, an asymptotic analysis of cross-validation able to answer this question has been lacking. Existing results focus on the "pointwise" estimation of the risk of a single estimator, whereas analysing model selection requires understanding how the CV risk varies with the model. In this article, we investigate the asymptotics of the CV risk in the neighbourhood of the optimal model, for trigonometric series estimators in density estimation. Asymptotically, simple validation and "incomplete" V -fold CV behave like the sum of a convex function f_n and a symmetrized Brownian changed in time $W_{g_n/V}$. We argue that this is the right asymptotic framework for studying model selection.

1 Introduction

The problem of selecting a model, a hyperparameter or, more generally, an estimator, is ubiquitous in statistics and a large number of methods have been proposed. Cross-validation is certainly one of the most popular and most widely used in practice. However, there are many ways to perform cross-validation, depending on how the data is split: V -fold cross-validation, leave-p-out, Monte Carlo CV, etc. [Arlot and Celisse, 2010, Section 4.3]. Each of these methods has hyperparameters that must be set by the user, such as V for V -fold cross-validation and p for the leave-p-out. In addition, many variants and alternatives to CV have been proposed in the literature, including V -fold penalties [Arlot, 2008], AmCV [Lecué and Mitchell, 2012] and Aggregated hold-out [Maillard et al., 2019]. The practitioner is faced with the problem of choosing one of these methods and selecting its hyperparameters. Since these are already model selection methods, using data to select among them threatens to lead to an infinite regress. In order to make a principled choice, one must therefore find some way to compare their performance *à priori*.

However, theoretical results on model selection generally lack the precision necessary to compare the methods which are used in practice. As noted by Arlot and Lerasle [2016], when selecting a model $m \in \mathcal{M}$ to minimize the risk $\mathcal{L}(m)$ using estimators \hat{m}_1, \hat{m}_2 , existing results only compare \hat{m}_1, \hat{m}_2 at first order — that is to say, in terms of the ratio $\frac{\mathcal{L}(\hat{m}_1)}{\mathcal{L}(\hat{m}_2)}$ and its limit. This is insufficient

to distinguish the performance of V -fold from that of hold-p-out with $p = \frac{n}{V}$, for example, even though the first of these is generally much better in practice.

In order to perform fine comparisons between different CV methods, theoreticians have mainly focused on calculating the bias and variance of the CV risk estimator, in a variety of contexts (such as linear regression [Burman, 1989], kernel density estimation [Celisse, 2014] or k -nearest neighbour classification [Celisse and Mary-Huard, 2018]). An in-depth review can be found in Arlot and Celisse [2010]. Recently, asymptotic normality results have also been proved under stability assumptions [Austern and Zhou, 2020, Bayle et al., 2020]. However, this analyzes CV purely as a *risk estimator*, not as a *model selection* method. Risk estimation and model selection behave differently in theory and in practice. In practice, Breiman and Spector [1992] report that 10-fold CV is generally better than leave-one-out for model selection, whereas the reverse is true for risk estimation. In theory, as explained by Arlot and Lerasle [2016], what matters for model selection performance is that the *sign* of $\text{crit}(m_1) - \text{crit}(m_2)$ be the same as that of $\mathcal{L}(m_1) - \mathcal{L}(m_2)$, with errors occurring only for values of $\mathcal{L}(m)$ close to the optimum $\mathcal{L}(m_*)$. Moreover, the definition of \hat{m}, m_* implies the inequality

$$\mathcal{L}(\hat{m}) - \mathcal{L}(m_*) \leq |\text{crit}(\hat{m}) - \mathcal{L}(\hat{m}) - (\text{crit}(m_*) - \mathcal{L}(m_*))|.$$

As \hat{m} will typically concentrate around m_* (in some sense), this bound can be much smaller than $|\text{crit}(m_*) - \mathcal{L}(m_*)|$ and $|\text{crit}(\hat{m}) - \mathcal{L}(\hat{m})|$. Thus, an approach to model selection based on a central limit theorem for $\text{crit}(m) - \mathcal{L}(m)$ will not yield the right rate of convergence in general.

One potential solution is to perform a local approximation of crit in the vicinity of m_* or in other words, to replace the study of $\text{crit}(m)$ with that of a centered and rescaled process $\text{crit}(m_* + \alpha\Delta) - \text{crit}(m_*)$. In order to study the model selection problem, the scale Δ should reflect the order of magnitude of $\hat{m} - m_*$, where \hat{m} denotes the selected parameter, i.e the minimizer of crit .

This approach was successfully applied to kernel density estimation in a classical paper by Hall and Marron [1987], which establishes the asymptotic normality of the cross-validated kernel parameter \hat{h}_{CV} . The method used by these authors is, however, highly specific to kernel density estimation. As for parametric M-estimators (proof of [Vaart, 1998, Theorem 5.3], for example), they show that the CV criterion can be locally approximated by a process of the form $c(h - h_*)^2 + (h - h_*)Z$, where Z is a normal random variable and c is a non-negative constant. This crucially uses the differentiability of kernel methods with respect to their parameter. Moreover, to prove that the risk is locally quadratic, they assume that the true density s is twice differentiable and that the kernel K is such that $\int z^2 K(z) dz \neq 0$. This effectively sets the rate of convergence of the kernel method [Tsybakov, 2009, Proposition 1.6], whereas one of the main theoretical advantages of CV is that it *adapts* to various non-parametric convergence rates.

In model selection, these arguments do not apply. The set of models \mathcal{M} is discrete, so there is no differentiability; moreover, since one model m_* may be much better than all others, smooth, locally quadratic behaviour of $\mathcal{L}(m)$ around m_* cannot be expected in general. The point of the present article is to find a *local approximation* to cross-validation in a simple non-parametric model selection problem (least-squares density estimation using Fourier series),

while taking into account a variety of possible behaviours of the true density s (in particular, different convergence rates). The key assumption made about s is that its Fourier coefficients are non-increasing, which, roughly speaking, guarantees uniqueness of the optimum m_* . The program raises a number of technical issues. First, since the class of functions the risk of which must be estimated is random and grows with n , standard empirical process theory cannot be used. Moreover, it is by no means clear that the process, once appropriately centered and scaled, actually converges to a limit (in fact, we conjecture that this is false in general).

Instead, we prove that simple validation, at the relevant scale, is approximately the sum of a convex function f_n and a Brownian motion changed in time W_{g_n} . The same result holds for "incomplete" V -fold cross-validation (Definition 3), as long as V remains constant when $n \rightarrow +\infty$. The approximating process depends on n (this is not a limit), but several inequalities on f_n and g_n show that it does not become trivial when n tends to $+\infty$. Interestingly, in the case of simple validation, the process is independent from the training data constituting the "training sample" of the hold-out. As a consequence, for any fixed V , (incomplete) V -fold cross-validation behaves as if the folds were independent: in particular, the asymptotic variance is reduced by a factor V relative to the hold-out. When $V \rightarrow +\infty$, V -fold CV concentrates around the deterministic f_n , which suggests that the CV parameter concentrates faster than the hold-out parameter.

The results of this article allow to make use of the abundant theory available on Brownian motion in order to study the parameter selection step. They yield a (heuristic) formula for the rate of convergence of the CV parameter \hat{m} to m_* , and open the way for proving *second-order optimal* oracle inequalities for the hold-out and incomplete V -fold cross-validation, which will be the subject of future work.

2 L^2 density estimation

Let $s \in L^2([0; 1])$ be a probability density function. Given a sample X_1, \dots, X_n drawn according to the density s , the L^2 density estimation problem consists in constructing an estimator \hat{s}_n that approaches s in terms of the L^2 norm.

Although it is not obvious at first glance (this is not true for the other L^p norms), this non-parametric density estimation problem can be reformulated as a risk minimization problem, with a contrast function: $\gamma(t, x) = \|t\|^2 - 2t(x)$, which yields the *risk* $\mathbb{E}[\gamma(t, X)] = \|t\|^2 - 2 \int s(x)t(x)dx = \|t - s\|^2 - \|s\|^2$. It follows that s is indeed the minimizer of the risk corresponding to the γ contrast function, and furthermore the *excess risk* $\ell(s, t) := \mathbb{E}[\gamma(t, X)] - \mathbb{E}[\gamma(s, X)]$ coincides with the L^2 norm:

$$\ell(s, t) = \|t - s\|^2.$$

As a result, it is possible to construct an *empirical risk estimator*,

$$P_n \gamma(t) = \frac{1}{n} \sum_{i=1}^n \gamma(t, X_i) = \|t\|^2 - \frac{2}{n} \sum_{i=1}^n t(X_i),$$

which can in particular be used to perform cross-validation.

Here we will consider as a family of non-parametric estimators the *empirical orthogonal series* estimators [Efr, 1999, Section 3.1] on a trigonometric basis. To ease the presentation, we consider only cosine functions, which is equivalent to assuming that s is symmetrical with respect to $\frac{1}{2}$. This restriction is of no fundamental importance — it is reasonable to conjecture that the results remain valid with the complete trigonometric basis.

For every $j \in \mathbb{N}^*$, let $\psi_j : x \mapsto \sqrt{2} \cos(2\pi jx)$ and let $\psi_0 : x \mapsto 1$. The collection $(\psi_j)_{j \in \mathbb{N}}$ is an orthonormal basis of the subset of $L^2([0; 1])$ of functions symmetrical with respect to $\frac{1}{2}$.

Let $D_n = (X_1, \dots, X_n)$ be a sample. For any $n \in \mathbb{N}$ and any $T \subset \{1, \dots, n\}$, we will denote, for any real-valued measurable function t ,

$$P_n^T(t) = \frac{1}{|T|} \sum_{i \in T} t(X_i).$$

Consider the estimators defined as follows.

Definition 1 For all $k \in \mathbb{N}$ and all $T \subset \{1, \dots, n\}$,

$$\hat{s}_k^T = \sum_{j=0}^k P_n^T(\psi_j) \psi_j,$$

where $\psi_0 = 1$ and for all $j \geq 1$, $\psi_j(x) = \sqrt{2} \cos(2\pi jx)$.

The estimators \hat{s}_k^T are empirical risk minimizers on the *models*

$$E_k = \left\{ \sum_{j=0}^k v_j \psi_j : v \in \mathbb{R}^{k+1} \right\}.$$

The problem of parameter choice k is therefore a problem of *model selection* within the model collection $(E_k)_{k \geq 0}$. Here, the models are nested, meaning $E_k \subset E_{k'}$ for every $k \leq k'$.

3 Risk estimation for the hold-out

The larger k is, the better the approximation of s by the functions of E_k , but the more difficult it is to estimate the best approximation to s within E_k . The choice of k is therefore subject to a bias-variance trade-off which, if properly carried out, allows adaptation to the smoothness of s , simultaneously reaching the minimax risk on Lipschitz spaces of periodic functions [Efr, 1999].

3.1 Cross-validation

Since the risk, except for a constant, is expressed as the expectation of a contrast function

$$P\gamma(\hat{s}_k^T) := E_X [\gamma(\hat{s}_k^T, X)] = \|\hat{s}_k^T - s\|^2 - \|s\|^2,$$

it can be estimated by hold-out and cross-validation as in regression and classification.

This is the subject of the following definition.

Definition 2 Let D_n be an i.i.d sample drawn from the distribution $s(x)dx$. Let $n_t \in \{1, \dots, n-1\}$. Let $T \subset \{1..n\}$ be a subset with cardinality $|T| = n_t$. Then, for all $k \in \mathbb{N}$, we define the hold-out estimator of the risk of \hat{s}_k with training sample indices T by

$$HO_T(k) = \|\hat{s}_k^T\|^2 - 2P_n^{T^c}(\hat{s}_k^T).$$

$HO_T(\cdot)$ is indeed an estimator since the norm $\|\cdot\|$ is computed with respect to a known dominating measure (in this case the Lebesgue measure) and so does not depend on the distribution of X . Moreover,

$$HO_T(k) = \|\hat{s}_k^T - s\|^2 - 2(P_n^{T^c} - P)(\hat{s}_k^T) :$$

the hold-out risk estimator can be expressed as the sum of the excess risk and a centered empirical process.

The hold-out risk estimator depends on the choice of a subset T of $\{1, \dots, n\}$, but its distribution depends only on the cardinality of that subset. The precise choice of a subset T of cardinality n_t will thus play no role in the sequel. We will therefore denote by T any subset of $\{1, \dots, n\}$ of cardinality n_t .

Since the distribution of $HO_T(\cdot)$ only depends on T through its cardinality n_t , it is possible to construct an estimator with smaller variance by averaging several $HO_{T_i}(\cdot)$. This is the idea behind cross-validation. In the V -fold scheme presented below, the T_i are chosen such that the test sets T_i^c are disjoint.

Definition 3 Let n_t, V be integers such that $\frac{V-1}{V}n \leq n_t \leq n-1$. Let $(I_i)_{1 \leq i \leq V}$ be a collection of disjoint subsets of $\{1, \dots, n\}$ of equal cardinality $|I_i| = n - n_t$. For all $i \in \{1, \dots, V\}$, let $T_i = \{1, \dots, n\} \setminus I_i$. Let $\mathcal{T} = (T_1, \dots, T_V)$. The "incomplete" V -fold CV risk estimator is

$$CV_{\mathcal{T}}(k) = \frac{1}{V} \sum_{i=1}^V HO_{T_i}(k).$$

Similarly to the hold-out, the distribution of $CV_{\mathcal{T}}(\cdot)$ only depends on n_t, V and in the rest of this article, we will denote by \mathcal{T} any collection T_1, \dots, T_V which satisfies the assumptions of Definition 3. Compared to standard V -fold, the cross-validation scheme defined above retains the constraint that the I_i be disjoint and of equal size, but decouples the size of the test sets $|I_i| = n - n_t$ from the number of splits V . In particular, the hold-out (Definition 2) is a special case of Definition 3 (for $V = 1$). As the collection $(I_i)_{1 \leq i \leq V}$ may be "completed" into a partition by adding sets I_j , we shall call $CV_{\mathcal{T}}(k)$ "incomplete V -fold cross-validation".

Conditionally on $D_n^{T^c}$, $HO_T(k)$ is an unbiased, consistent estimator of $\|\hat{s}_k^T - s\|^2$, which "converges" to $\|\hat{s}_k^T - s\|^2$ at rate $\frac{1}{\sqrt{n-n_t}}$ by the central-limit theorem. $\|\hat{s}_k^T - s\|^2$ itself is known to concentrate around its expectation [Arlot and Lerasle, 2016, Lemma 14], so $HO_T(k)$ does too. $CV_{\mathcal{T}}(k)$ is an unbiased estimator of $\mathbb{E} \left[\|\hat{s}_k^T - s\|^2 \right]$, and concentrates at least as fast as $HO_T(k)$ by Jensen's inequality.

3.2 What CV estimates: the oracle

Since the purpose of cross-validation is to select the parameter k , we want to understand the behavior of $\text{HO}_T(\cdot)$, $\text{CV}_T(\cdot)$ in a region where their optima \hat{k}_T^{ho} , \hat{k}_T^{cv} can be found with high probability. The consistency and unbiasedness of $\text{HO}_T(k)$, $\text{CV}_T(k)$ suggest that \hat{k}_T^{ho} , \hat{k}_T^{cv} should be "close" to $\text{argmin}_{k \in \mathbb{N}} \mathbb{E} \left[\|\hat{s}_k^T - s\|^2 \right]$, the "optimal" parameter (for a sample of size n_t).

Under a few conditions, it is possible to give a simple, deterministic approximant for this optimal parameter. More precisely, let $n_t(n)$ be a sequence of integers such that, for all $n \in \mathbb{N}$, $1 \leq n_t(n) \leq n$, and define $n_v(n) = n - n_t(n)$. In the following, we shall denote $n_t = n_t(n)$ and $n_v = n_v(n)$ for a generic value of n . Whenever n, n_v, n_t appear in the same expression, it will be understood that $n_t = n_t(n)$ and $n_v = n_v(n) = n - n_t(n)$.

For any $j \in \mathbb{N}$, let $\theta_j = \langle s, \psi_j \rangle$ denote the Fourier coefficients of s on the cosine basis. The expected L^2 risk can be approximated as follows (see also claim 3):

$$\|\hat{s}_k^T - s\|^2 \sim \frac{k}{n_t} + \sum_{j=k+1}^{+\infty} \theta_j^2.$$

If the squared Fourier coefficients θ_j^2 form a non-increasing sequence, then the approximating function $k \mapsto \frac{k}{n_t} + \sum_{j=k+1}^{+\infty} \theta_j^2$ is convex and, in particular, has a unique minimizer $k_*(n_t)$. As a further consequence, the level sets of "near-optimal" values of k form a nested collection of intervals. These properties greatly simplify the analysis of the hold-out procedure by avoiding situations where the hold-out "jumps" between two widely separated regions. For this reason, in the remainder of the article, we shall always assume that the sequence θ_j^2 is non-increasing. We will discuss later how this assumption might be relaxed.

Assuming now that θ_j^2 is non-decreasing, it is possible to give simple approximate formulas for the argmin and minimum of the true risk, $\|\hat{s}_k^T - s\|^2$ and its expectation.

Definition 4 For all $n \in \mathbb{N}$, let

$$k_*(n) = \max \left\{ k \in \mathbb{N} : \theta_k^2 \geq \frac{1}{n} \right\}$$

and

$$\text{or}(n) = \inf_{k \in \mathbb{N}} \left\{ \sum_{j=k+1}^{+\infty} \theta_j^2 + \frac{k}{n} \right\}.$$

Equivalently,

$$k_*(n) = \max_{k \in \mathbb{N}} \text{argmin} \left\{ \sum_{j=k+1}^{+\infty} \theta_j^2 + \frac{k}{n} \right\}$$

and

$$\text{or}(n) = \sum_{j=k_*(n)+1}^{+\infty} \theta_j^2 + \frac{k_*(n)}{n}.$$

$k_*(n_t)$ and $\text{or}(n_t)$ are thus, approximately, the minimizer and the minimum in k of the L^2 risk of the estimators \hat{s}_k^T , which explains the name $\text{or}(n_t)$ (oracle).

Thus, it is to be expected that the minima of $\text{HO}_T(k)$, $\text{CV}_{\mathcal{T}}(k)$ lie close to $k_*(n_t)$. For this reason, $k_*(n_t)$ will be the most relevant value of k_* in the following, and we will often omit the argument n_t , with the understanding that $k_* = k_*(n_t)$.

Since $\text{CV}_{\mathcal{T}}(\cdot)$ can be expressed as an average of $\text{HO}_{T_i}(\cdot)$, we first focus on analyzing the hold-out risk estimator $\text{HO}_T(\cdot)$. Consequences for cross-validation will be derived in section 5.3. Assuming to simplify that k_* minimizes the L^2 risk,

$$\text{HO}_T(k) - \text{HO}_T(k_*(n_t)) = \|\hat{s}_k^T - s\|^2 - \|\hat{s}_{k_*(n_t)}^T - s\|^2 - 2(P_n^{T^c} - P)(\hat{s}_k^T - \hat{s}_{k_*(n_t)}^T) \quad (1)$$

is the sum of a non-negative term (the excess risk) and a centered empirical process. Both tend to 0 as k tends to k_* .

3.3 Scaling

The relevant scale at which to study the hold-out procedure, which minimizes $\text{HO}_T(k)$, is the scale of the fluctuations $\hat{k} - k_*(n_t)$ of the argmin \hat{k} of $\text{HO}_T(k)$. Since the asymptotic study of $\text{HO}_T(\cdot)$ precedes that of \hat{k} , the correct scaling must be deduced *a priori*.

Consider the following generic scaled and centered hold-out process:

$$\begin{aligned} \frac{1}{\epsilon} (\text{HO}_T(k_* + \alpha\Delta) - \text{HO}_T(k_*(n_t))) &= \frac{1}{\epsilon} \left(\|\hat{s}_{k_* + \alpha\Delta}^T - s\|^2 - \|\hat{s}_{k_*(n_t)}^T - s\|^2 \right) \\ &\quad - \frac{2}{\epsilon} (P_n^{T^c} - P)(\hat{s}_{k_* + \alpha\Delta}^T - \hat{s}_{k_*(n_t)}^T), \end{aligned} \quad (2)$$

where $\alpha \in \{\frac{k - k_*}{\Delta} : k \in \mathbb{N}\}$ and Δ, ϵ are values which may depend on s, n and n_t . The relative size of the excess risk and the empirical process in (2) depends on Δ . If the excess risk is much larger than the empirical process in equation (2), then the argmin of the process will concentrate around 0, which implies that $|\hat{k} - k_*| = o(\Delta)$: Δ is then too large. In contrast, if the centered empirical process is dominant in equation (2), then the argmin diverges, since the variance of $(P_n^{T^c} - P)(\hat{s}_k^T - \hat{s}_{k_*}^T)$ grows with $|k - k_*|$. This means that $\Delta = o(|\hat{k} - k_*|)$, so Δ is too small. Thus, Δ should be chosen based on the following principle: *The correct scaling Δ for $|\hat{k} - k_*|$ is such that the excess risk $\|\hat{s}_{k_* \pm \Delta}^T - s\|^2 - \|\hat{s}_{k_*}^T - s\|^2$ and the centered empirical process $2(P_n^{T^c} - P)(\hat{s}_{k_* \pm \Delta}^T - \hat{s}_{k_*}^T)$ have the same order of magnitude.* In other words, the bias of the rescaled process should be of the same order of magnitude as its standard deviation. ϵ should then be chosen so that bias and standard deviation both remain of order 1, in order to avoid divergence of the scaled process or convergence to 0 (which would be uninformative).

The appropriate choice of Δ, ϵ is given in the following definition.

Definition 5 For all $n \in \mathbb{N}$, let

$$\begin{aligned}\Delta_d(s, n_t, n) &= \max \left\{ l \in \mathbb{N} : \theta_{k_*(n_t)+l}^2 \geq \left[1 - \sqrt{\frac{n_t}{n-n_t} \frac{1}{\sqrt{l}}} \right] \frac{1}{n_t} \right\} \\ \Delta_g(s, n_t, n) &= \min \left\{ l \in \{0, \dots, k_*(n_t)\} : \theta_{k_*(n_t)-l}^2 \geq \left[1 + \sqrt{\frac{n_t}{n-n_t} \frac{1}{\sqrt{l}}} \right] \frac{1}{n_t} \right\} \\ \Delta(s, n_t, n) &= \max(\Delta_d(s, n_t, n), \Delta_g(s, n_t, n)) \\ \mathcal{E}(s, n_t, n) &= \frac{\Delta(s, n_t, n)}{n_t} \\ \mathfrak{e}(s, n_t, n) &= \sqrt{\frac{\mathcal{E}(s, n_t, n)}{n-n_t}}.\end{aligned}$$

Definition 5 also introduces the quantity $\mathcal{E}(s, n_t, n)$. This quantity appears often in the proofs, so it is helpful to have notation for it; it also has an interpretation as the order of magnitude of the fluctuations in the variance term,

$$\mathbb{E} \left[\|\hat{s}_k^T - \mathbb{E}[\hat{s}_k^T]\|^2 \right] - \mathbb{E} \left[\|\hat{s}_{k_*}^T - \mathbb{E}[\hat{s}_{k_*}^T]\|^2 \right],$$

and the bias term,

$$\|\mathbb{E}[\hat{s}_k^T] - s\|^2 - \|\mathbb{E}[\hat{s}_{k_*}^T] - s\|^2 = -\text{sign}(k - k_*) \sum_{j=k_* \wedge k+1}^{k \vee k_*} \theta_j^2,$$

of the estimators \hat{s}_k^T , for $k - k_*$ "of order" Δ (in a sense to be made precise later).

As the sequence $n_t(n)$ and the density s are considered to be fixed once and for all, the notation $\Delta(s, n_t, n)$, $\mathcal{E}(s, n_t, n)$, $\mathfrak{e}(s, n_t, n)$ will frequently be replaced by the abbreviations Δ , \mathcal{E} , \mathfrak{e} .

Definition 5 does not make clear how large Δ , \mathcal{E} and \mathfrak{e} are. Their order of magnitude may depend on the sequence $(\theta_j)_{j \in \mathbb{N}}$ of Fourier coefficients of s as well as on $n_t(n)$. However, the following inequalities always hold.

Lemma 3.1 For any density s such that the sequence $\theta_j^2 = \langle s, \psi_j \rangle^2$ is non-increasing,

$$\Delta \geq \frac{n_t}{n - n_t} \tag{3}$$

$$\mathcal{E} \geq \frac{1}{n - n_t} \tag{4}$$

$$\mathfrak{e} \geq \frac{1}{n - n_t} \tag{5}$$

$$\mathfrak{e} \leq \mathcal{E} \tag{6}$$

$$\mathcal{E} \leq 2\text{or}(n_t) + \frac{1}{n - n_t}. \tag{7}$$

This lemma is proved in section 7.1.1. The following two examples show that in extreme cases, lemma 3.1 may be optimal, at least up to constants.

Two examples

- Let $n_t(n)$ and u_n be two integer sequences, such that $\frac{n_t(n)}{n} \rightarrow 1$, $u_n \rightarrow +\infty$ and $u_n \leq \frac{\sqrt{n}}{2}$ for all n . Assume also that $\frac{n-n_t}{n_t} = o(\frac{u_{n_t}}{n_t})$. Let for all $j \in \mathbb{N}$

$$\theta_{j,n}^2 = \begin{cases} 1 & \text{if } j = 0 \\ \frac{1}{n_t} & \text{if } 1 \leq j \leq u_{n_t} \\ 0 & \text{if } j \geq u_{n_t} + 1, \end{cases} \quad (8)$$

corresponding for example to the pdf $s_n = 1 + \sum_{j=1}^{u_{n_t}} \sqrt{\frac{1}{n_t}} \psi_j$. Remark that equation (8) implies that $k_*(n_t) = u_{n_t}$. Then as $n \rightarrow +\infty$, $\mathcal{E}(s_n, n_t, n) \sim \frac{u_{n_t}}{n_t} \sim \text{or}(n_t)$ and $\frac{n-n_t}{n_t} = o(\text{or}(n_t))$, so $\epsilon(s_n, n_t, n) = o(\mathcal{E}(s_n, n_t, n))$.

- Let s be the pdf associated with the Fourier coefficients

$$\forall j \in \mathbb{N}, \langle s, \psi_j \rangle = \theta_j = \frac{1}{3^j}. \quad (9)$$

Let $n_t(n)$ be a sequence of integers such that $\frac{n_t(n)}{n} \rightarrow 1$. Then by Lemma 3.1, $\Delta \geq \frac{n_t}{n-n_t}$, but as

$$9^{-\frac{n_t}{n-n_t}} = o\left(1 - \sqrt{\frac{1}{1 + \frac{n-n_t}{n_t}}}\right),$$

it follows that $\Delta(s, n_t, n) \sim \frac{n_t}{n-n_t}$, hence

$$\mathcal{E}(s, n_t, n) \sim \frac{n_t}{(n-n_t)n_t} \sim \frac{1}{n-n_t}.$$

As a result, $\mathcal{E}(s, n_t, n) \sim \epsilon(s, n_t, n) \sim \frac{1}{n-n_t}$, and this for *any* sequence $n_t(n)$ such that $n_t(n) \sim n$.

Now that Δ, ϵ are defined, the hold-out process can be rescaled as in equation (6). More precisely, the rescaled hold-out process is given by Definition 6 below.

Definition 6 For all $j \in [-k_*; +\infty[\cap \mathbb{Z}$, let

$$\hat{R}_T^{ho} \left(\frac{j}{\Delta} \right) = \frac{1}{\epsilon} (HO_T(k_* + j) - HO_T(k_*)),$$

in other words (by definition 2)

$$\hat{R}_T^{ho} \left(\frac{j}{\Delta} \right) = \frac{1}{\epsilon} \left(\|\hat{s}_{k_*+j}^T - s\|^2 - \|\hat{s}_{k_*}^T - s\|^2 \right) - \frac{2}{\epsilon} \left(P_n^{Tc} - P \right) (\hat{s}_{k_*+j}^T - \hat{s}_{k_*}^T).$$

The \hat{R}_T^{ho} function is extended by linear interpolation to all $\alpha \in \left[\frac{-k_*(n_t)}{\Delta}; +\infty[\right]$.

Let $\hat{R}_{\mathcal{T}}^{cv}$ be defined in a similar manner, i.e

$$\hat{R}_{\mathcal{T}}^{cv} \left(\frac{j}{\Delta} \right) = \frac{1}{\epsilon} (CV_{\mathcal{T}}(k_* + j) - CV_{\mathcal{T}}(k_*))$$

for all $j \in [-k_*; +\infty[\cap \mathbb{Z}$, extended by linear interpolation to $\left[\frac{-k_*(n_t)}{\Delta}; +\infty[\right]$.

Note that by linearity of the interpolation operation, $\hat{R}_{\mathcal{T}}^{cv} = \frac{1}{V} \sum_{i=1}^V \hat{R}_{T_i}^{ho}$.

The extension of \hat{R}_T^{ho} , $\text{CV}_{\mathcal{T}}()$ by linear interpolation simplifies their approximation by a continuous process. Notice that any minimizer of \hat{R}_T^{ho} (resp. $\text{CV}_{\mathcal{T}}()$) on the grid $\frac{1}{\Delta}([-k_*(n_t); +\infty[\cap \mathbb{Z})$ remains a minimizer of \hat{R}_T^{ho} (resp. $\text{CV}_{\mathcal{T}}()$) on the interval $[-\frac{k_*(n_t)}{\Delta}; +\infty[$. In particular, this applies to the hold-out parameter obtained by minimisation of the hold-out risk estimator.

The process \hat{R}_T^{ho} can be expressed as the sum of the standardized excess risk,

$$\frac{1}{\epsilon} \left(\|\hat{s}_{k_*+j}^T - s\|^2 - \|\hat{s}_{k_*}^T - s\|^2 \right)$$

and a centered empirical process: $\frac{2}{\epsilon} (P_n^{T^c} - P) \left(\hat{s}_{k_*+j}^T - \hat{s}_{k_*}^T \right)$. Though the excess risk is a priori random (it depends on D_n^T), the proof will show that it concentrates around a deterministic function f_n , depending on n , which is given by definition 7 below.

Definition 7 For all $k \in \mathbb{N}$, let $R(k) = \sum_{j=k+1}^{+\infty} \theta_j^2$. Extend R to \mathbb{R}_+ by linear interpolation:

$$\forall x \in \mathbb{R}_+, R(x) = (1 + [x] - x)R([x]) + (x - [x])R([x] + 1).$$

$f_n :] -\frac{k_*(n_t)}{\Delta}; +\infty[\rightarrow \mathbb{R}_+$ is now defined by:

$$f_n(\alpha) = \frac{1}{\epsilon} \left(R(k_*(n_t) + \alpha\Delta) - R(k_*(n_t)) + \frac{\alpha\Delta}{n_t} \right). \quad (10)$$

Thus, for all $k \in \mathbb{N}$, $k \neq k_*(n_t)$,

$$\epsilon f_n \left(\frac{k - k_*(n_t)}{\Delta} \right) = \sum_{j=k \wedge k_*(n_t)+1}^{k \vee k_*(n_t)} \left| \theta_j^2 - \frac{1}{n_t} \right|. \quad (11)$$

It is clear by equation (11) that f_n reaches its minimum at 0, moreover the assumption that the sequence $(\theta_j^2)_{j \in \mathbb{N}}$ is non-increasing implies that f_n is convex. In particular, f_n is non-increasing on $]-\frac{k_*(n_t)}{\Delta}; 0]$ and non-decreasing on $[0; +\infty[$. Moreover, the definition of Δ and ϵ (Definition 5) implies the following bounds on the increments of f_n :

Lemma 3.2 For any $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1\alpha_2 \geq 0$ and $|\alpha_2| \geq |\alpha_1| \geq 1$,

$$f_n(\alpha_2) - f_n(\alpha_1) \geq |\alpha_2| - |\alpha_1|.$$

In particular, since $f_n(0) = 0$, for all $\alpha \in \mathbb{R}$,

$$f_n(\alpha) \geq (|\alpha| - 1)_+.$$

Moreover, using the notation from Definition 5,

- If $\Delta = \Delta_d$, then for any $\alpha_1, \alpha_2 \in [0; 1]$ such that $\alpha_1 \leq \alpha_2$, $f_n(\alpha_2) - f_n(\alpha_1) \leq \alpha_2 - \alpha_1$.
- If $\Delta = \Delta_g$, then for any $\alpha_1, \alpha_2 \in [-1; 0]$ such that $\alpha_1 \leq \alpha_2$, $f_n(\alpha_1) - f_n(\alpha_2) \leq \alpha_2 - \alpha_1$.

This lemma is proved in section 7.1.2. It guarantees that f_n remains in a sense of "finite order" and "non zero" as $n \rightarrow +\infty$, which means that f_n remains uniformly bounded on $[-1; 0]$ or on $[0; 1]$, and is lower-bounded on \mathbb{R} by the non-zero function $(|x| - 1)_+$.

4 Hypotheses

The main hypothesis of this article is that the squared Fourier coefficients θ_j^2 are non-increasing, which is admittedly strong, though a natural assumption in the context of this article, as discussed above. It seems likely that the desirable effects of this hypothesis can be retained under weaker conditions, as we will discuss later (section 6.2).

In addition to this "shape constraint" on the θ_j^2 , the main theorem of this article requires a number of more technical assumptions. First, approximating the process $HO_T(k)$, which is a sum over Fourier coefficients, inevitably involves bounding "tail sums" of the Fourier series, such as $\sum_{j=k+1}^{\infty} \theta_j^2$. To control these, we assume that the Fourier coefficients decay fast enough.

Hypothesis 1 *There exists constants $c_1 \geq 0$ and $\delta_1 \geq 0$ such that for all $k \in \mathbb{N}$, $\sum_{j=k+1}^{+\infty} \theta_j^2 \leq \frac{c_1}{k^{2+\delta_1}}$.*

This upper bound is satisfied for some c_1, δ_1 if and only if the smoothness assumption $s \in H^\beta$ holds for some $\beta > 1$, where H^β denotes the Sobolev Hilbert space. This implies in particular that $(k_*(n_t) \leq n_t)$ is satisfied for all sufficiently large n_t (thus also for all large enough n).

Secondly, since we seek to approximate the discrete process $HO_T(k)$ by a continuous one, it is necessary to make sure that the number of "points" $k \in [k_* - \alpha\Delta, k_* + \alpha\Delta]$ tends to infinity as $n \rightarrow +\infty$, for all α . Similarly, one must assume that $f_n(\frac{j}{\Delta}) \sim f_n(\frac{j-1}{\Delta})$, otherwise the continuous function f_n is not sufficiently close to its discrete version $j \mapsto f_n(\frac{j}{\Delta})$. For technical reasons, we will assume a polynomial growth rate, using the following three hypotheses.

Hypothesis 2 *There exists constants $c_2 \geq 0$, $\delta_2 \geq 0$ such that for all $k \in \mathbb{N}$, $\sum_{j=k+1}^{+\infty} \theta_j^2 \geq \frac{c_2}{k^{\delta_2}}$.*

This hypothesis states that the Fourier coefficients θ_j^2 cannot decay faster than polynomially, and excludes in particular analytic functions. This guarantees that $k_*(n_t) \rightarrow +\infty$ at a polynomial rate. Hypothesis 1 holds for example if s or one of its derivatives has a point of discontinuity.

Hypothesis 3 *There exists constants $c_3 > 0$, $\delta_3 > 0$ such that for all $k \geq 1$,*

$$\theta_{k+k^{\delta_3}}^2 \geq c_3 \theta_{k-k^{\delta_3}}^2.$$

Hypothesis 3 means that the sequence θ_j^2 cannot decrease too abruptly, excluding in particular a locally exponential decrease such as $\theta_{k_n+j}^2 = 2^{-j} w_n$, for $j \in \{1, \dots, \varepsilon \log k_n\}$ and $k_n \rightarrow +\infty$. Without this hypothesis, f_n may have "asymptotically sharp" discontinuities at ± 1 , violating the condition $f_n(\frac{j}{\Delta}) \sim f_n(\frac{j-1}{\Delta})$, see claim 3.2 and example (9). Hypothesis 3 is satisfied by polynomially decreasing sequences, $\theta_j^2 = \kappa j^{-\beta}$, with $\delta_3 = 1$, but also by sequences $\theta_j^2 = \kappa \exp(-j^\alpha)$, as long as $\alpha < 1$. Locally, θ_j^2 can thus decrease much faster than the polynomial lower bound given by hypothesis 2.

Together, hypotheses 1, 2, 3 basically mean that the Fourier coefficients of s on the cosine basis decrease polynomially. For example, they are satisfied if there are two strictly positive constants μ, L and a constant $\beta > 1$, such that

$$\forall k \in \mathbb{N}, \mu k^{-2\beta} \leq \sum_{j=k+1}^{+\infty} \theta_j^2 \leq L k^{-2\beta}.$$

The two remaining hypotheses 4 and 5 do not bear on s , but on the parameter n_t which is chosen by the statistician. They serve to establish upper and lower bounds on Δ using lemma 3.1.

Hypothesis 4 *There exists a constant $\delta_4 > 0$ such that $n - n_t \leq n^{1-\delta_4}$.*

By lemma 3.1, hypothesis 4 guarantees that Δ grows at least at a polynomial rate n^{δ_4} , as announced above. For technical reasons, it is also necessary to upper bound Δ , which is accomplished using hypothesis 5 below.

Hypothesis 5 *There exists a constant $\delta_5 > 0$ such that $n_v = n - n_t \geq n^{\frac{2}{3}+\delta_5}$.*

The statistician can always choose n_t such that hypotheses 4 and 5 hold. One should however check that this is compatible with good performance of the hold-out. The oracle inequalities of Arlot and Lerasle [2016] show that the risk of the hold-out in model selection for L^2 density estimation is (at most) of order $\frac{n}{n_t} \text{or}(n) + \frac{\log(n-n_t)}{n-n_t}$. If $\text{or}(n)$ decreases in n with rate $\frac{1}{n^\alpha}$ ($\alpha \in (0, 1)$), which is the case under assumptions 1 and 2, $n - n_t$ can be chosen within the interval $[\frac{1}{2}n^{\frac{2+\alpha}{3}}; n^{\frac{4+\alpha}{5}}]$ —so that assumptions 4 and 5 are satisfied—without changing the order of magnitude of the risk.

5 Main theorems

The purpose of this article is to find simple approximants $Y_{n,V}(u)$ to the rescaled CV estimators $\hat{R}_{\mathcal{T}}^{cv}(u)$ (including the hold-out when the number of splits $V = 1$).

5.1 Domain of approximation

In order to later derive results about the argmin $\hat{k}_{\mathcal{T}}^{cv}$ selected by CV, a uniform approximation is desirable; however it is unrealistic to expect a uniform approximation to the unbounded processes $\hat{R}_{\mathcal{T}}^{cv}$ on the whole real line. Instead, it is sufficient to uniformly approximate $\hat{R}_{\mathcal{T}}^{cv}$ on compact sets which contain the argmin with high probability. At first order, the probability for the argmin to lie in a given region should depend primarily on the size of f_n , which approximates the rescaled excess risk: we cannot expect $\hat{k}_{\mathcal{T}}^{cv}$ to lie where the excess risk is large, even if $|\hat{k}_{\mathcal{T}}^{cv} - k_*|$ is small. From a technical viewpoint, the size of the coefficients θ_j^2 matters for bounding the approximation error, and that can be taken into account through $f_n(\alpha)$. For these reasons, the "natural" sets on which to approximate $\hat{R}_{\mathcal{T}}^{cv}$ are the intervals $[a_x, b_x]$ defined below.

Definition 8 *Let $T \subset \{1 \dots n\}$ be a subset of cardinality n_t and let $k_* = k_*(n_t)$. For any $x > 0$, let*

$$a_x = \min\{\frac{j}{\Delta} : j \in \{-k_*(n_t), \dots, 0\}, f_n(\frac{j}{\Delta}) \leq x\}$$

$$b_x = \max\{\frac{j}{\Delta} : j \in \mathbb{N}, f_n(\frac{j}{\Delta}) \leq x\}.$$

Up to the constraint that $a_x, b_x \in \frac{1}{\Delta}\mathbb{Z}$, the sets $[a_x, b_x]$ are just the sublevel sets of f_n . Moreover, since $\hat{k}_{\mathcal{T}}^{cv} - k_* \in \mathbb{Z}$, $\frac{\hat{k}_{\mathcal{T}}^{cv} - k_*}{\Delta} \in [a_x, b_x]$ if and only if $f_n\left(\frac{\hat{k}_{\mathcal{T}}^{cv} - k_*}{\Delta}\right) \leq x$.

By lemma 3.2, for $x \geq 1$, $[a_x, b_x]$ either contains $[0, 1]$ (if $\Delta = \Delta_d$) or $[-1, 0]$ (if $\Delta = \Delta_g$). On the other hand, by lemma 7.1, $b_x - a_x \leq 2(1+x)$. This shows that the definition of a_x, b_x is consistent with the choice to center $\text{CV}_{\mathcal{T}}(\cdot)$ at k_* and rescale by Δ .

5.2 Simple validation

The following theorem shows that the process $\hat{R}_T^{ho}(\cdot)$ can be approximated on $[a_x, b_x]$ by the sum of f_n and a time-changed Brownian motion.

Theorem 1 *Assume that the assumptions of section 4 hold. There exists a non-decreasing function $g_n : [-\frac{k_*(n_t)}{\Delta}; +\infty[\rightarrow \mathbb{R}$ and for any $x > 0$, there exists a two-sided Brownian motion $(W_t)_{t \in [a_x; b_x]}$ independent from D_n^T such that, for any $y > 0$, with probability greater than $1 - e^{-y}$,*

$$\mathbb{E} \left[\sup_{u \in [a_x; b_x]} \left| \hat{R}_T^{ho}(u) - (f_n(u) - W_{g_n(u)}) \right| \middle| D_n^T \right] \leq \kappa_0 (1+y)^2 (1+x)^{\frac{3}{2}} n^{-u_1}, \quad (12)$$

where $u_1 > 0$ and $\kappa_0 \geq 0$ are two constants which depend only on $\delta_1, \delta_3, \delta_5, \delta_2$ and c_1, c_2, δ_1, c_3 , respectively. Moreover, g_n and W can be chosen so as to satisfy the following conditions.

1. $g_n(0) = 0, W_0 = 0,$
2. $\forall (\alpha_1, \alpha_2) \in [-\frac{k_*}{\Delta}; +\infty[^2, \alpha_2 < \alpha_1 \implies g_n(\alpha_1) - g_n(\alpha_2) \geq 4 \|s\|^2 [\alpha_1 - \alpha_2].$
3. For all $(\alpha_1, \alpha_2) \in [-\frac{k_*}{\Delta}; +\infty[^2$ such that $\alpha_1 < \alpha_2 < 0$ or $0 < \alpha_1 < \alpha_2,$

$$g_n(\alpha_2) - g_n(\alpha_1) \leq -\frac{8 \|s\|_{\infty}}{(n - n_t)\epsilon} [f_n(\alpha_2) - f_n(\alpha_1)] + \left(8 \|s\|_{\infty} + 4 \|s\|^2 \right) [\alpha_2 - \alpha_1]. \quad (13)$$

This theorem is proved in section 7. It states that the rescaled hold-out process, \hat{R}_T^{ho} , can be approximated uniformly in expectation on $[a_x, b_x]$ by a continuous process $Y_{n,1}$, which is the sum of a convex non-negative function f_n and a time-changed Brownian motion W_{g_n} . f_n and g_n depend on n_t and n , but not on the data (they are deterministic functions), while W depends on the data only through the test sample $D_n^{T^c}$. In particular, in this asymptotic setting, \hat{R}_T^{ho} doesn't depend on D_n^T , the training data.

The function g_n increases on its domain and has a Lipschitz-continuous inverse. By lemma 3.2 and equation (3), f_n, g_n are Lipschitz continuous either on $[-1, 0]$ or on $[0, 1]$ (depending on whether $\Delta = \Delta_d$ or $\Delta_g = \Delta$), with Lipschitz constants that depends only on $\|s\|^2, \|s\|_{\infty}$. In particular, f_n, g_n are both of constant order on $[a_x, b_x]$ for $x > 1$. Figure 1 illustrates the bounds that hold on f_n, g_n in a situation where $\Delta = \Delta_d$.

On the one hand, by definition of a_x, b_x , $0 \leq f_n \leq x$ on $[a_x, b_x]$ and by equation (3),

$$\sup_{\alpha \in [a_x, b_x]} |g_n(\alpha)| \leq 8 \|s\|_{\infty} x + \left(8 \|s\|_{\infty} + 4 \|s\|^2 \right) \max(|a_x|, |b_x|) \leq 20 \|s\|_{\infty} (1+x) \quad (14)$$

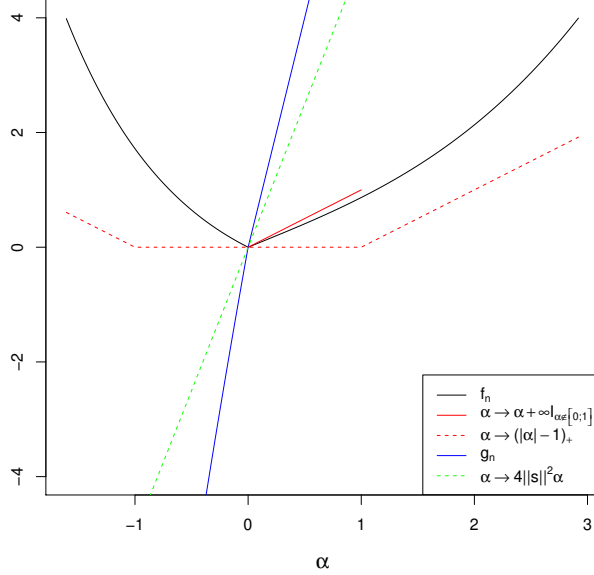


Figure 1: A plot of f_n, g_n on $[a_6; b_6]$ with upper and lower bounds, for $\|s\|^2 = 1.2$.

On the other hand, $|g_n(\alpha)| \geq 4\|s\|^2|\alpha|$ by equation (2) and $f_n(b_x) \geq x - o(1)$ by claim 7. Thus, the principle discussed in section 2 is satisfied on $[a_x, b_x]$: the mean and standard deviation of $Y_{n,1}$ are both of constant order.

It remains to see that the intervals $[a_x, b_x]$ are the "largest" on which this is true. Figure 2 gives an illustration of the situation for $x = 25$ and

$$f_n : \alpha \mapsto \begin{cases} e^{-\alpha} - 1 & \text{if } \alpha \leq 0 \\ \frac{8}{10}\alpha + \frac{8}{30}\alpha^3 & \text{if } \alpha \geq 0 \end{cases}$$

$$g_n : \alpha \mapsto \begin{cases} 7.8\alpha & \text{if } \alpha \geq 0 \\ 7.8\alpha - 3f_n(\alpha) & \text{if } \alpha \leq 0 \end{cases}$$

(which satisfy the properties of lemma 3.2 and Theorem 1 when $\|s\|^2 \leq 1.2$ and $\|s\|_\infty \leq 1.5$).

Figure 2 suggests that for large x , $\sqrt{g_n}$ and hence W_{g_n} become negligible compared to f_n outside the interval $[a_x, b_x]$. This intuition can be theoretically justified: if $\alpha \in \frac{1}{\Delta}\mathbb{Z}$ does not belong to $[a_x, b_x]$, then $f_n(\alpha) \geq x$ by definition of a_x, b_x , while on the other hand, by equation (13), there exists a constant κ ,

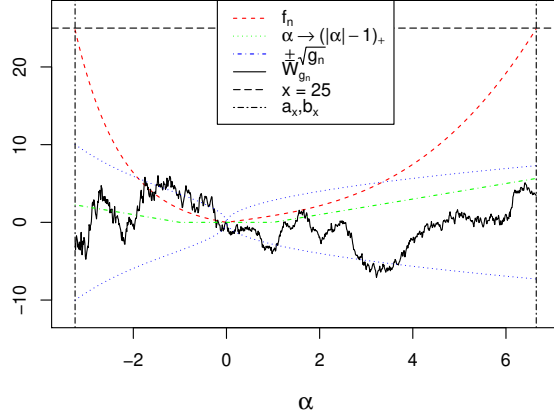


Figure 2: A plot of f_n, W_{g_n} on $[a_x; b_x]$, for $x = 25$, $g_n : \alpha \mapsto 7.8\alpha - 3f_n(\alpha)\mathbb{I}_{\alpha < 0}$.

depending only on $\|s\|_\infty, \|s\|^2$, such that

$$\begin{aligned} \frac{\sqrt{\text{Var}(W_{g_n(\alpha)})}}{f_n(\alpha)} &\leq \frac{\sqrt{g_n(\alpha)}}{f_n(\alpha)} \\ &\leq \frac{\sqrt{\kappa f_n(\alpha)}}{f_n(\alpha)} + \frac{\sqrt{\kappa|\alpha|}}{f_n(\alpha)}. \end{aligned}$$

By lemma 3.2, $f_n(\alpha) \geq (|\alpha| - 1)_+$ therefore $|\alpha| \leq 2f_n(\alpha)$ whenever $f_n(\alpha) \geq 1$, i.e for $\alpha \notin [a_1; b_1]$. This yields:

$$\forall x \geq 1, \forall \alpha \in \frac{1}{\Delta}\mathbb{Z} \setminus [a_x; b_x], \quad \frac{\sqrt{\text{Var}(W_{g_n(\alpha)})}}{f_n(\alpha)} \leq \sqrt{\frac{\kappa}{x}} + \sqrt{\frac{2\kappa}{x}} = \frac{\sqrt{\kappa}(1 + \sqrt{2})}{\sqrt{x}}.$$

Hence, for sufficiently large x , the random term W_{g_n} becomes negligible relative to the deterministic f_n outside the interval $[a_x; b_x]$.

5.3 Incomplete V -fold cross-validation

Since the cross-validation risk estimator $\text{CV}_{\mathcal{T}}(k)$ can be written as an average of hold-out risk estimators $\text{HO}_{T_i}(k)$, Theorem 1 has direct implications for CV.

Corollary 2 *Assume that the hypotheses of section 4 hold. Let f_n, g_n be as in definition 7 and theorem 1. For any $x > 0$,*

$$\mathbb{E} \left[\sup_{u \in [a_x; b_x]} \left| \hat{R}_{\mathcal{T}}^{\text{cv}}(u) - (f_n(u) - W_{\frac{g_n(u)}{\sqrt{x}}}) \right| \right] \leq 5\kappa_0(1+x)^{\frac{3}{2}}n^{-u_1},$$

with the same constants κ_0, u_1 as in Theorem 1.

Proof By integrating the bound of Theorem 1,

$$\mathbb{E} \left[\sup_{u \in [a_x; b_x]} \left| \hat{R}_{T_i}^{ho}(u) - (f_n(u) - W_{\frac{g_n(u)}{V}}^i) \right| \right] \leq 5\kappa_0(1+x)^{\frac{3}{2}}n^{-u_1}$$

for each $i \in \{1, \dots, V\}$, where the W^i are symmetrical BMs that are independent of $D_n^{T_i}$. We can construct W^i such that $W^i = H(D_n^{T_i^c}, U_i)$, where H is a measurable function and U_i is an auxiliary uniform random variable. Taking independent U_i yields i.i.d W^i , since the sets $T_i^c = I_i$ are disjoint. Let $\bar{W} = \frac{1}{V} \sum_{i=1}^V W^i$. By Jensen's inequality,

$$\mathbb{E} \left[\sup_{u \in [a_x; b_x]} \left| \hat{R}_{\mathcal{T}}^{cv}(u) - (f_n(u) - \bar{W}_{g_n(u)}) \right| \right] \leq 5\kappa_0(1+x)^{\frac{3}{2}}n^{-u_1}.$$

Conclude by noting that $(\bar{W}_t)_{t \in \mathbb{R}}$ is equal in distribution to $(W_{t/V})_{t \in \mathbb{R}}$, as a continuous random process. \blacksquare

Corollary 2 proves that cross-validation is effective at reducing the variance of risk estimation, compared to simple validation (the hold-out). The process approximating $\hat{R}_{\mathcal{T}}^{cv}$ is of the same form as that approximating \hat{R}_T^{ho} , but with its variance reduced by a factor V , as would be the case if the hold-out estimators $\text{HO}_{T_i}(\cdot)$ were independent. Importantly, this reduction in variance occurs for the rescaled process $\hat{R}_{\mathcal{T}}^{cv}$, and so can be expected to reflect the model selection performance. Corollary 2 is sharp when V is fixed as $n \rightarrow +\infty$, since in that case, the approximating process $f_n - W_{g_n/V}$ remains nontrivial (random and of bounded size) as $n \rightarrow +\infty$. When $V = V_n \rightarrow +\infty$, corollary 2 is still valid, but $\text{Var}(W_{g_n/V_n}) = g_n/V_n \rightarrow 0$, which means that $\hat{R}_{\mathcal{T}}^{cv}$ concentrates around the deterministic function f_n . However, corollary 2 cannot tell us the rate at which this convergence occurs, unless $V_n = o(n^{u_1})$.

6 Discussion

In this section, we interpret the results of the article and discuss how they could be extended.

6.1 Implications of our results

Theorem 1 provides an approximation to the rescaled hold-out process, with scale factor Δ given by Definition 5. The fact that the approximating process is asymptotically random, but trends away from zero at $\pm\infty$ (as discussed at length in section 5.2) supports the conjecture that $\hat{k}_T - k_*$ is of order Δ . In the case of "incomplete" cross-validation for fixed V , the process is of identical type, but with a smaller variance, which strongly suggests better model selection performance. If $V \rightarrow +\infty$, then rescaled cross-validation concentrates around the deterministic function f_n , which by the argument of section 3.3, suggests that the scale Δ is "asymptotically larger" than the fluctuations $\hat{k}_{\mathcal{T}}^{cv} - k_*$ of the CV parameter. This supports the belief that cross-validation improves when V is increased. Though $k_*(n_t)$ is not the oracle $k_*(n)$, the greater concentration of CV around $k_*(n_t)$ makes it possible to choose n_t closer to n than would be reasonable for the hold-out, resulting in improved overall performance.

6.2 Hypotheses

Theorem 1 relies on the assumptions of section 4, most importantly on the assumption that the sequence of squared Fourier coefficients θ_j^2 is non-increasing.

This could be weakened in various ways. First, given the local nature of our analysis, what is really required is that the coefficients θ_j^2 be non-increasing in a neighbourhood $[k_* - r_n, k_* + r_n]$ of k_* of radius r_n which dominates Δ ($\Delta = o(r_n)$). The only difference in that case is that the selected parameter $\hat{k}_{\mathcal{T}}^{cv}$ may lie outside $[k_* - r_n, k_* + r_n]$, which diminishes the usefulness of Theorem 1 for studying model selection.

Moreover, since the process $\hat{R}_{\mathcal{T}}^{cv}(\alpha)$ consists of sums $\sum_{j=1}^{k_*+\alpha\Delta}$, it is probably sufficient to replace the hypotheses on the individual coefficients θ_j^2 with hypotheses bearing on local averages $\bar{\theta}_j^2 = \frac{1}{2m_n} \sum_{r=j-m_n}^{j+m_n} \theta_r^2$, at some scale $m_n = o(\Delta)$. Depending on the scale m_n , the hypothesis that a smoothed sequence $\bar{\theta}_j^2$ is non-decreasing may be quite plausible, considering the fact that the Fourier coefficients tend to 0 at a prescribed rate for sufficiently smooth functions s .

6.3 Perspectives

Model selection The CV risk estimator is usually used to select a model, $\hat{k}_{\mathcal{T}}^{cv}$, which minimizes it. The final result of CV is then the estimator $\hat{s}_{\hat{k}_{\mathcal{T}}^{cv}}$, or $\hat{s}_{\hat{k}_{\mathcal{T}}^T}$ in the case of simple validation. Thus, what we are most interested in practice is the risk $\left\| \hat{s}_{\hat{k}_{\mathcal{T}}^{cv}} - s \right\|^2$ of this final estimator, and how it depends on the CV method used (at least when CV is used with a goal of *estimation*, as opposed to *identification* of the best model). There are several ways our results can contribute to answering these questions. First, since $\hat{R}_{\mathcal{T}}^{cv}(u)$ can be uniformly approximated by $f_n(u) - W_{g_n(u)/V}$, it is natural to approximate $\frac{\hat{k}_{\mathcal{T}}^{cv} - k_*}{\Delta}$ (the minimizer of $\hat{R}_{\mathcal{T}}^{cv}(u)$) by $\hat{\alpha}_{n,V}$ (the minimizer of $f_n - W_{g_n/V}$). The minima of $f_n - W_{g_n/V}$ can be studied using the theory of Wiener processes. Together with our results about f_n and g_n (in lemma 3.2 and Theorem 1), this makes the analysis of $\hat{\alpha}_{n,V}$ much easier than that of $\hat{k}_{\mathcal{T}}^{cv}$. Secondly, claim 3 of this article proves that the (excess) risk $\left\| \hat{s}_k^T - s \right\|^2 - \left\| \hat{s}_{k_*}^T - s \right\|^2$ concentrates around $\epsilon f_n\left(\frac{k-k_*}{\Delta}\right)$ for k "close enough" to k_* . This removes the dependency on the training sample D_n^T and reduces the analysis of $\left\| \hat{s}_k^T - s \right\|^2 - \left\| \hat{s}_{k_*}^T - s \right\|^2$ to that of the deterministic function $f_n\left(\frac{k-k_*}{\Delta}\right)$.

Other model selection methods In this article, we only considered a particular type of cross-validation. Corollary 2 relies on decomposing "incomplete" V -fold CV as a finite average of asymptotically independent hold-out estimators, which can be analysed more easily than general cross-validation because conditionally on their training data, they are empirical processes.

In general, for projection estimators in L^2 density estimation, the cross-validation risk estimator is not a (conditional) empirical process but a (weighted) U-statistic of order 2 (more precisely, a weighted sum of the terms $\psi_k(X_i)\psi_k(X_j)$). Thus, new methods are required to approximate general CV estimators by gaussian processes. We conjecture that more general cross-validation methods also behave locally like the sum of the (rescaled) excess risk and a time-changed

Wiener process, though we expect the scaling and the time-change g_n to be different for different versions of CV.

Theorem 1 can also shed light on the behaviour of other methods which use simple validation as a key ingredient, such as Aggregated hold-out [Maillard et al., 2019].

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7 Proofs

In this section, the term constant means a function of $\|s\|_\infty, \|s\|^2$ and the constants $c_1, c_2, c_3, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5$. which appear in the hypotheses of Theorem 1. Note that by hypothesis (1), $\|\theta\|_{\ell^1}, \|s\|_\infty, \|s\|^2$ are finite and can be bounded by functions of c_1, δ_1 . The letter u will denote strictly positive constants that only depend on $(\delta_i)_{1 \leq i \leq 5}$ (they will generally appear as exponents of $\frac{1}{n}$). The letter κ denotes a non-negative constant. The notation $n_v = n - n_t$ will also be used frequently.

7.1 Preliminary results

The results of this section are independent from the rest. They will be used in the rest of the proof of Theorem 1, as well as in the Appendix. Let's start by proving some basic properties of a_x, b_x and f_n that will be used repeatedly in the main proofs.

7.1.1 Proof of lemma 3.1

- By definition and non-negativity of θ_j^2 , $\sqrt{\frac{n_t}{n-n_t} \frac{1}{\Delta_d}} \leq 1$, therefore $\Delta \geq \Delta_d \geq \frac{n_t}{n-n_t}$.
- $\mathcal{E} = \frac{\Delta}{n_t} \geq \frac{1}{n-n_t}$.
- $\mathfrak{e} = \sqrt{\frac{\mathcal{E}}{n-n_t}} \geq \sqrt{\frac{1}{(n-n_t)^2}} = \frac{1}{n-n_t}$.
- $\frac{\mathcal{E}}{\mathfrak{e}} = \mathcal{E} \sqrt{\frac{n-n_t}{\mathcal{E}}} = \sqrt{(n-n_t)\mathcal{E}} \geq 1$.
- By definition, $\Delta_g \leq k_*$. Thus $\frac{\Delta_g}{n_t} \leq \frac{k_*}{n_t} \leq \text{or}(n_t)$. Moreover,

$$\Delta_d \left[1 - \sqrt{\frac{n_t}{n-n_t} \frac{1}{\Delta_d}} \right] \frac{1}{n_t} \leq \sum_{j=k_*+1}^{k_*+\Delta_d} \theta_j^2 \leq \sum_{j=k_*+1}^{+\infty} \theta_j^2 \leq \text{or}(n_t).$$

Thus

$$\begin{aligned}
n_t \text{or}(n_t) &\geq \Delta_d - \sqrt{\frac{n_t}{n-n_t}} \sqrt{\Delta_d} \\
&\geq \Delta_d - \frac{1}{2} \frac{n_t}{n-n_t} - \frac{1}{2} \Delta_d \\
&\geq \frac{1}{2} \Delta_d - \frac{1}{2} \frac{n_t}{n-n_t}.
\end{aligned}$$

It follows that

$$\Delta_d \leq 2n_t \text{or}(n_t) + \frac{n_t}{n-n_t},$$

so since $\frac{n_t}{n-n_t} \frac{1}{n_t} = \frac{1}{n-n_t}$,

$$\frac{\Delta_d}{n_t} \leq 2\text{or}(n_t) + \frac{1}{n-n_t},$$

which proves the result.

7.1.2 Proof of lemma 3.2

f_n is continuous and piecewise linear by definition 7. f_n is convex because the sequence θ_j^2 is non-increasing by assumption. Let $j \in \mathbb{Z}$ and $\alpha \in]\frac{j}{\Delta}; \frac{j+1}{\Delta}[$ be two numbers. By definition, f_n is linear on the interval $]\frac{j}{\Delta}; \frac{j+1}{\Delta}[$, in particular f_n is differentiable on this interval and

$$\begin{aligned}
f'_n(\alpha) &= \Delta [f_n(\frac{j+1}{\Delta}) - f_n(\frac{j}{\Delta})] \\
&= \frac{\Delta}{\mathbf{e}} \left[\frac{1}{n_t} - \theta_{k_*+j+1}^2 \right].
\end{aligned} \tag{15}$$

Because the sequence θ_j^2 is non-increasing, it follows from the definition of $k_*(n_t)$ that f_n is increasing on $]\frac{j}{\Delta}; \frac{j+1}{\Delta}[$ if $j \geq 0$ and non-increasing if $j < 0$. This implies that f_n reaches its minimum at $k_*(n_t)$. If $\alpha \geq 1$, then $j = \lfloor \alpha \Delta \rfloor \geq \Delta$, therefore by definition of $\Delta_d \leq \Delta$,

$$\begin{aligned}
f'_n(\alpha) &\geq \frac{\Delta}{\mathbf{e}} \left[\frac{1}{n_t} - \theta_{k_*+\Delta+1}^2 \right] \\
&\geq \frac{\Delta}{\mathbf{e}} \sqrt{\frac{n_t}{n-n_t}} \frac{1}{\sqrt{\Delta}} \frac{1}{n_t} \\
&= \Delta \sqrt{\frac{(n-n_t)n_t}{\Delta}} \sqrt{\frac{n_t}{n-n_t}} \frac{1}{\sqrt{\Delta}} \frac{1}{n_t} \\
&= 1.
\end{aligned}$$

In the same way, if $\alpha < -1$, then $j+1 = \lceil \alpha \Delta \rceil \leq -\Delta \leq -\Delta_g$, so

$$\begin{aligned}
f'_n(\alpha) &\leq \frac{\Delta}{\mathbf{e}} \left[\frac{1}{n_t} - \theta_{k_*-\Delta}^2 \right] \\
&\leq -\frac{\Delta}{\mathbf{e}} \sqrt{\frac{n_t}{n-n_t}} \frac{1}{\sqrt{\Delta}} \frac{1}{n_t} \\
&\leq -1.
\end{aligned}$$

Furthermore,

- If $\Delta = \Delta_d$, then for all $\alpha \in [0; 1]$, $j + 1 = \lceil \alpha \Delta \rceil \leq \Delta = \Delta_d$, therefore by definition of Δ_d ,

$$\begin{aligned} f'_n(\alpha) &\leq \frac{\Delta}{\epsilon} \left[\frac{1}{n_t} - \theta_{k_* + \Delta}^2 \right] \\ &\leq \frac{\Delta}{\epsilon} \sqrt{\frac{n_t}{n - n_t}} \frac{1}{\sqrt{\Delta}} \frac{1}{n_t} \\ &\leq 1. \end{aligned}$$

- If $\Delta = \Delta_g$, then for all $\alpha \in [-1; 0]$, $j = \lfloor \alpha \Delta \rfloor \geq -\Delta = -\Delta_g$, therefore by definition of Δ_g and since the sequence $(\theta_j^2)_{j \in \mathbb{N}}$ is non-increasing,

$$\begin{aligned} f'_n(\alpha) &\geq \frac{\Delta}{\epsilon} \left[\frac{1}{n_t} - \theta_{k_* - \Delta + 1}^2 \right] \\ &\geq -\frac{\Delta}{\epsilon} \sqrt{\frac{n_t}{n - n_t}} \frac{1}{\sqrt{\Delta}} \frac{1}{n_t} \\ &\geq -1. \end{aligned}$$

By continuity of f_n , this proves the lemma.

7.1.3 Properties of the interval $[a_x; b_x]$

Lemme 7.1 *Let a_x, b_x be as defined in Theorem 1. Then for all $x > 0$,*

$$\begin{aligned} [b_x - a_x] &\leq 2(1 + x) \\ \sum_{k_* + a_x \Delta}^{k_* + b_x \Delta} \theta_j^2 &\leq 4(1 + x)\mathcal{E}. \end{aligned}$$

Proof Either $b_x \leq 1$, or $b_x > 1$ and by lemma 3.2, $f_n(b_x) \geq b_x - 1$ which implies that $b_x \leq f_n(b_x) + 1 \leq x + 1$. In all cases, $b_x \leq x + 1$. In the same way, $a_x > -1 - x$. Thus $b_x - a_x \leq 2(1 + x)$. Moreover,

$$\begin{aligned} \sum_{k_* + a_x \Delta}^{k_* + b_x \Delta} \theta_j^2 &\leq \sum_{k_* + a_x \Delta}^{k_* + b_x \Delta} \left| \theta_j^2 - \frac{1}{n_t} \right| + \frac{(b_x - a_x)\Delta}{n_t} \\ &\leq \epsilon [f_n(a_x) + f_n(b_x)] + [b_x - a_x]\mathcal{E} \\ &\leq 2x\epsilon + 2[1 + x]\mathcal{E}. \end{aligned}$$

Since $\epsilon \leq \mathcal{E}$ by lemma 3.1,

$$\sum_{k_* + a_x \Delta}^{k_* + b_x \Delta} \theta_j^2 \leq (4x + 2)\mathcal{E}.$$

This proves lemma 7.1. ■

We now introduce some notation which will be used in the remainder of this chapter.

Definition 9 Let an i.i.d sample D_n be given, with distribution P and pdf s on $[0; 1]$. For all $j \in \mathbb{N}$ and any $T \subset \{1, \dots, n\}$, let

$$\begin{aligned}\theta_j &= P\psi_j = \langle s, \psi_j \rangle \\ \hat{\theta}_j^T &= P_n^T(\psi_j).\end{aligned}$$

This notation will be used very often in the remainder of the chapter.

The hold-out risk estimator can be expressed as the sum of two terms. Definition 10 below gives a name to each of these terms.

Definition 10 For all $j \in [-k_*(n_t); +\infty[\cap \mathbb{Z}$, let

$$L\left(\frac{j}{\Delta}\right) = \frac{1}{\epsilon} \left(\|\hat{s}_{k_*+j}^T - s\|^2 - \|\hat{s}_{k_*}^T - s\|^2 \right).$$

The function L is extended to the interval $[-\frac{k_*(n_t)}{\Delta}; +\infty[$ by linear interpolation. Let Z be the random function defined for all $j \in [-\frac{k_*(n_t)}{\Delta}; +\infty[\cap \mathbb{Z}$ by

$$Z\left(\frac{j}{\Delta}\right) = \frac{2}{\epsilon} \left(P_n^{T^c} - P \right) \left(\hat{s}_{k_*+j}^T - \hat{s}_{k_*}^T \right)$$

and extended by linear interpolation to the interval $[-\frac{k_*(n_t)}{\Delta}; +\infty[$, so that for all α , $\hat{R}_T^{ho}(\alpha) = L(\alpha) - Z_\alpha$.

Thus, L is the rescaled excess risk, and Z is a centered empirical process. These two terms will be approximated separately.

7.2 Approximation of the excess risk

Let $x > 0$ be fixed for the entirety of this section. We now prove the following claim.

Claim 3 Let L be the function introduced in definition 10, and f_n be given by definition 7. There exists a constant κ_1 such that, for any $y > 0$, with probability greater than $1 - e^{-y}$,

$$\sup_{\alpha \in [a_x; b_x]} |L(\alpha) - f_n(\alpha)| \leq \kappa_1(1+x)[\log(2+x) + y + \log n]^2 n^{-\min(\frac{1}{12}, \frac{\delta_4}{2})}.$$

Proof Let $j \in \{a_x\Delta, \dots, b_x\Delta\}$. Since $\hat{s}_k^T = \sum_{j=1}^k P_n^T(\psi_j)\psi_j = \sum_{j=1}^k \hat{\theta}_j^T \psi_j$,

$$\left\| \hat{s}_{k_*+j}^T - s \right\|^2 - \left\| \hat{s}_{k_*}^T - s \right\|^2 = \text{sgn}(j) \sum_{i=k_*(j)-+1}^{k_*(j)+} \left(\hat{\theta}_i^T - \theta_i \right)^2 - \theta_i^2.$$

It is known [Arlot and Lerasle, 2016, Lemma 14] [Lerasle, 2009, Proposition 6.3] that the process

$$\sum_{i=k_*(j)-+1}^{k_*(j)+} \left(\hat{\theta}_i^T - \theta_i \right)^2 = \sum_{i=k_*(j)-+1}^{k_*(j)+} (P^T - P)(\psi_j)^2$$

concentrates around its expectation, so that

$$\sum_{i=k_*(j)_-+1}^{k_*(j)_+} \left(\hat{\theta}_i^T - \theta_i \right)^2 \sim \sum_{i=k_*(j)_-+1}^{k_*(j)_+} \frac{\text{Var}(\psi_j)}{n_t}.$$

Furthermore, by lemma 8.1 in the appendix, $\text{Var}(\psi_j) \sim 1$, therefore

$$\sum_{i=k_*(j)_-+1}^{k_*(j)_+} \left(\hat{\theta}_i^T - \theta_i \right)^2 \sim \frac{|j|}{n_t}.$$

More precisely, proposition 8.3 in the appendix and a union bound show that, with probability greater than $1 - e^{-y}$, for any $j \in \mathbb{Z} \cap [a_x \Delta; b_x \Delta]$,

$$\left| \sum_{i=k_*(j)_-+1}^{k_*(j)_+} \left(\hat{\theta}_i^T - \theta_i \right)^2 - \frac{|j|}{n_t} \right| \leq \kappa_1 (y + \log n + \log((b_x - a_x) \Delta \wedge 1))^2 n^{-\min(\frac{1}{12}, \frac{\delta_4}{2})} \frac{j}{\Delta} \mathbf{e}.$$

Let $r_n = \kappa_1 (y + \log n + \log((b_x - a_x) \Delta \wedge 1))^2 n^{-\min(\frac{1}{12}, \frac{\delta_4}{2})}$. Then for any $j \geq 1$,

$$\begin{aligned} \|\hat{s}_{k_*+j}^T - s\|^2 - \|\hat{s}_{k_*}^T - s\|^2 &= - \sum_{i=k_*+1}^{k_*+j} \theta_i^2 + \sum_{i=k_*+1}^{k_*+j} \left(\hat{\theta}_i^T - \theta_i \right)^2 \\ &= - \sum_{i=k_*+1}^{k_*+j} \theta_i^2 + \frac{j}{n_t} \pm \frac{j}{\Delta} r_n \mathbf{e} \\ &= \sum_{i=k_*+1}^{k_*+j} \left[\frac{1}{n_t} - \theta_i^2 \right] \pm \frac{j}{\Delta} r_n \mathbf{e} \\ &= \mathbf{e} f_n \left(\frac{j}{\Delta} \right) \pm \frac{j}{\Delta} r_n \mathbf{e}. \end{aligned}$$

On this same event, for any $j \in \{-k_*(n_t), \dots, -1\}$,

$$\begin{aligned} \|\hat{s}_{k_*+j}^T - s\|^2 - \|\hat{s}_{k_*}^T - s\|^2 &= \sum_{i=k_*+j+1}^{k_*} \theta_i^2 - \sum_{i=k_*+j+1}^{k_*} \left(\hat{\theta}_i^T - \theta_i \right)^2 \\ &= \sum_{i=k_*+j+1}^{k_*} \theta_i^2 - \frac{|j|}{n_t} \pm \frac{|j|}{\Delta} r_n \mathbf{e} \\ &= \sum_{i=k_*+j+1}^{k_*} \left[\theta_i^2 - \frac{1}{n_t} \right] \pm \frac{j}{\Delta} r_n \mathbf{e} \\ &= \mathbf{e} f_n \left(\frac{j}{\Delta} \right) \pm \frac{j}{\Delta} r_n \mathbf{e}. \end{aligned}$$

Thus, since f_n and \hat{R}_T^{ho} are linear between the points of $\frac{1}{\Delta} \mathbb{Z}$,

$$\begin{aligned} \sup_{\alpha \in [a_x; b_x]} |L(\alpha) - f_n(\alpha)| &= \frac{1}{\mathbf{e}} \max_{a_x \Delta \leq j \leq b_x \Delta} \left| \|\hat{s}_{k_*+j}^T - s\|^2 - \|\hat{s}_{k_*}^T - s\|^2 - \mathbf{e} f_n \left(\frac{j}{\Delta} \right) \right| \\ &\leq \max(|a_x|, |b_x|) r_n. \end{aligned}$$

By lemma 7.1, $\max(|a_x|, |b_x|) \leq b_x - a_x \leq 2(1+x)$ so

$$\begin{aligned} \max(|a_x|, |b_x|)r_n &\leq 2(1+x)\kappa_1(y + \log n + \log(2(1+x)) + \log(1 \wedge \Delta))^2 n^{-\min(\frac{1}{12}, \frac{\delta_4}{2})} \\ &\leq \kappa(1+x)[\log(2+x) + \log n + y]^2 n^{-\min(\frac{1}{12}, \frac{\delta_4}{2})} \end{aligned}$$

for some constant κ , since by lemma 3.1 and hypothesis (5) of Theorem 1,

$$\begin{aligned} \Delta &= n_t \mathcal{E} \\ &\leq 2n_t \text{or}(n_t) + \frac{n_t}{n - n_t} \\ &\leq 2(\|s\|^2 - 1)n_t + n_t^{\frac{1}{3}}. \end{aligned}$$

This proves claim 3. ■

We will now seek to approximate the process Z given by definition 10.

7.3 Strong approximation of the hold-out process

Let us start by showing that the empirical process Z (definition 10) can be approximated by a gaussian process, uniformly on $[a_x; b_x]$. This is the purpose of the following result, which will be proven in this section.

Claim 4 *Let Z be the process given by definition 10. There exists a gaussian process $(Z_\alpha^1)_{\alpha \in [a_x; b_x]}$ with the same variance-covariance function as Z : for any $(\alpha_1, \alpha_2) \in [a_x; b_x]^2$, $\text{Cov}(Z_{\alpha_1}^1, Z_{\alpha_2}^1) = \text{Cov}(Z_{\alpha_1}, Z_{\alpha_2})$ and such that for all $n \geq 1$, for all $x > 0$, with probability greater than $1 - e^{-y}$,*

$$\mathbb{E} \left[\sup_{\alpha \in [a_x; b_x]} |Z_\alpha - Z_\alpha^1| \middle| D_n^T \right] \leq \kappa_5(c_1, \delta_5)(1+y)(1+x)^{\frac{3}{2}} n^{-\frac{\delta_5}{3}}.$$

Furthermore, Z^1 can be expressed as $Z^1 = H(Z, \nu)$, with ν a uniform random variable independent from D_n and H a measurable function on $C([0; 1], \mathbb{R})$.

Let $n_v = |T^c| = n - |T| = n - n_t$. Let $F : x \rightarrow \int_0^x s(t)dt$ be the cumulative distribution function of the given X_i . Let $F_{T^c} : x \rightarrow \frac{1}{n_v} \sum_{i \notin T} \mathbb{1}_{X_i \leq x}$ be the empirical cumulative distribution function of the sample $D_n^{T^c}$. By the Komlos-Major-Tusnady approximation theorem [Komlós et al., 1975, Theorem 3], there exist a universal constant C and a standard Brownian bridge process B_{T^c} such that for all $y > 0$, with probability greater than $1 - e^{-y}$, $\|B_{T^c} \circ F - \sqrt{n_v}(F_{T^c} - F)\|_\infty \leq \frac{C(\log n_v + y)}{\sqrt{n_v}}$ (remark that since F is continuous, $F(X_i) \sim \mathcal{U}([0; 1])$, which means that the result for general F follows from the result for the uniform distribution). Furthermore, B_{T^c} can always be realized as a measurable function of $D_n^{T^c}$ and an auxiliary, uniformly distributed random variable ν : $B_{T^c} = H(D_n^{T^c}, \nu)$, with ν independant from D_n . Let B^{T^c} be obtained in this way. From $B_{T^c} \circ F$, one can define an operator on the Sobolev space $W^1(\mathbb{R})$:

Definition 11 *For any function f such that $f' \in L^1([0; 1])$, let*

$$G_{T^c}(f) = - \int_0^1 f'(x) B_{T^c}(F(x)) dx.$$

G_{T^c} "approximates" the empirical process $\sqrt{n_v}(P_n^{T^c} - P)$ on the space W^1 . Lemma 7.2 below gives a bound on the error made with this approximation.

Lemma 7.2 For any function f such that $f' \in L^1([0; 1])$,

$$\left| G_{T^c}(f) - \sqrt{n_v}(P_n^{T^c} - P)(f) \right| \leq \|B_{T^c} - \sqrt{n_v}(F_{T^c} - F)\|_\infty \|f'\|_{L^1}.$$

Furthermore, for all functions f, g such that $f', g' \in L^1([0; 1])$,

$$\text{Cov}(G_{T^c}(f), G_{T^c}(g)) = P[f]P[g] - P[f]P[g] = \text{Cov}\left(\sqrt{n_v}(P_n^{T^c} - P)(f), \sqrt{n_v}(P_n^{T^c} - P)(g)\right).$$

Proof Let f be a function such that $f' \in L^1([0; 1])$. Then

$$\begin{aligned} (P_n^{T^c} - P)(f) &= \int f d(P_n^{T^c} - P) \\ &= \int [f - f(0)] d(P_n^{T^c} - P) \\ &= \int_0^1 \int_0^1 \mathbb{I}_{t < x} f'(t) dt d(F_{T^c} - F)(x) \\ &= \int_0^1 f'(t) (P_n^{T^c} - P)((t, +\infty)) \\ &= - \int_0^1 f'(t) (F_{T^c} - F)(t) dt. \end{aligned} \tag{16}$$

It follows that for all functions f such that $f' \in L^1([0; 1])$,

$$\begin{aligned} \left| G_{T^c}(f) - \sqrt{n_v}(P_n^{T^c} - P)(f) \right| &= \left| \int_0^1 f'(t) [\sqrt{n_v}(F_{T^c} - F) - B_{T^c} \circ F](t) dt \right| \\ &\leq \|f'\|_{L^1([0; 1])} \|B_{T^c} \circ F - \sqrt{n_v}(F_{T^c} - F)\|_\infty. \end{aligned}$$

By definition, it is clear that $\mathbb{E}[G_{T^c}(f)] = 0$. Thus,

$$\begin{aligned} \text{Cov}(G_{T^c}(f), G_{T^c}(g)) &= \mathbb{E}[G_{T^c}(f)G_{T^c}(g)] \\ &= \mathbb{E}\left[\int_0^1 \int_0^1 f'(u)g'(v)B_{T^c}(F(u))B_{T^c}(F(v))\right] \\ &= \int_0^1 \int_0^1 f'(u)g'(v)[F(u) \wedge F(v)][1 - F(u) \vee F(v)]dudv \\ &= \int_0^1 \int_0^1 f'(u)g'(v) (\mathbb{E}[\mathbb{I}_{X \leq u} \mathbb{I}_{X \leq v}] - \mathbb{E}[\mathbb{I}_{X \leq u}]\mathbb{E}[\mathbb{I}_{X \leq v}]) \\ &= n_v \int_0^1 \int_0^1 f'(u)g'(v) \mathbb{E}[(F_{T^c} - F)(u)(F_{T^c} - F)(v)] \\ &= \text{Cov}\left(\sqrt{n_v}(P_n^{T^c} - P)(f), \sqrt{n_v}(P_n^{T^c} - P)(g)\right) \text{ by equation (16)} \end{aligned}$$

■

Let the process Z^1 be defined for all $j \in \{a_x \Delta, \dots, b_x \Delta\}$ by

$$Z^1 \left(\frac{j}{\Delta} \right) = \frac{2}{\sqrt{n_v} \mathfrak{e}} G_{T^c} (\hat{s}_{k_*+j}^T - \hat{s}_{k_*}^T).$$

Z^1 is extended to the interval $[a_x; b_x]$ by linear interpolation, as for Z . By lemma 7.2, the variance-covariance function of Z^1 coincides with that of Z at the points $\frac{j}{\Delta}, j \in \mathbb{Z} \cap [a_x \Delta; b_x \Delta]$, and this property extends by bilinearity to the whole interval $[a_x; b_x]$. Furthermore,

$$\begin{aligned} \sup_{a_x \leq \alpha \leq b_x} |Z_\alpha^1 - Z_\alpha| &\leq \max_{j \in \mathbb{Z} \cap [a_x; b_x]} \left| Z^1 \left(\frac{j}{\Delta} \right) - Z \left(\frac{j}{\Delta} \right) \right| \\ &\leq \frac{4\sqrt{2}\pi}{\mathfrak{e}\sqrt{n_v}} \|B_{T^c} \circ F - \sqrt{n_v}(F_{n_v} - F)\|_\infty \times \max_{a_x \Delta \leq j \leq b_x \Delta} \left\| \sum_{i=k_*+1}^{k_*+j} i \hat{\theta}_i^T \sin(2i\pi \cdot) \right\|_1 \\ &\leq \frac{4\pi}{\mathfrak{e}\sqrt{n_v}} \|B_{T^c} \circ F - \sqrt{n_v}(F_{n_v} - F)\|_\infty \times \sqrt{\sum_{i=k_*+a_x \Delta+1}^{k_*+b_x \Delta} i^2 (\hat{\theta}_i^T)^2} \end{aligned}$$

By construction, the process $B_{T^c} \circ F - \sqrt{n_v}(F_{n_v} - F)$ is independent from D_n^T . As a result,

$$\begin{aligned} \mathbb{E} \left[\sup_{a_x \leq \alpha \leq b_x} |Z_\alpha^1 - Z_\alpha| | D_n^T \right] &\leq \frac{4\pi}{\mathfrak{e}\sqrt{n_v}} \mathbb{E} [\|B_{T^c} \circ F - \sqrt{n_v}(F_{n_v} - F)\|_\infty] \\ &\quad \times (k_* + b_x \Delta) \sqrt{\sum_{i=k_*+a_x \Delta+1}^{k_*+b_x \Delta} (\hat{\theta}_j^T)^2} \\ &\leq \frac{4\pi C \log n_v}{\mathfrak{e} n_v} \times (k_* + b_x \Delta) \sqrt{\sum_{j=k_*+a_x \Delta+1}^{k_*+b_x \Delta} 2\theta_j^2 + 2(\hat{\theta}_j^T - \theta_j)^2}. \end{aligned} \quad (17)$$

By proposition 8.3, there exists an event $E_1(y)$ of probability greater than $1 - e^{-y}$ such that, for all $D_n^T \in E_1(y)$,

$$\sum_{j=k_*+a_x \Delta+1}^{k_*+b_x \Delta} (\hat{\theta}_j^T - \theta_j)^2 \leq [b_x - a_x] \frac{\Delta}{n_t} + \kappa_1 (b_x - a_x) [\log n + y]^2 n^{-\min(\frac{1}{12}, \frac{\delta_4}{2})} \mathfrak{e}(n),$$

therefore by lemma 7.1 and equation (17), for all $D_n^T \in E_1(y)$,

$$\mathbb{E} \left[\sup_{a_x \leq \alpha \leq b_x} |Z_\alpha^1 - Z_\alpha| | D_n^T \right] \leq \frac{4\pi C \log n_v}{\mathfrak{e} n_v} \times (k_* + b_x \Delta) 2\sqrt{1+x} \left[2\sqrt{\mathcal{E}} + \sqrt{2\kappa_1} (\log n + y) n^{-\min(\frac{1}{12}, \frac{\delta_4}{2})} \sqrt{\mathfrak{e}} \right].$$

Since $\mathfrak{e} \leq \mathcal{E}$ and $n^{-\min(\frac{1}{12}, \frac{\delta_4}{2})} \log n \rightarrow 0$, there exists therefore a constant κ such that for all $D_n^T \in E_1(y)$:

$$\begin{aligned} \mathbb{E} \left[\sup_{a_x \leq \alpha \leq b_x} |Z_\alpha^1 - Z_\alpha| | D_n^T \right] &\leq \kappa \frac{\log n_v}{\sqrt{n_v}} \frac{\sqrt{\mathcal{E}}}{\mathfrak{e}\sqrt{n_v}} \times (k_* + b_x \Delta) (1+y) \sqrt{1+x} \\ &\leq \kappa \frac{\log n_v}{\sqrt{n_v}} \times (k_* + b_x \Delta) (1+y) \sqrt{1+x}. \end{aligned} \quad (18)$$

By lemma 3.1, $\mathcal{E} \leq 2\text{or}(n_t) + \frac{1}{n_v}$ therefore $\Delta \leq 2n_t\text{or}(n_t) + \frac{n_t}{n_v}$ and by definition of or , $k_*(n_t) = n_t \frac{k_*}{n_t} \leq n_t\text{or}(n_t)$ therefore $\frac{k_* + b_x \Delta}{\sqrt{n_v}} \leq (2b_x + 1) \frac{n_t\text{or}(n_t)}{\sqrt{n_v}} + \frac{b_x n_t}{n_v \sqrt{n_v}}$. By hypothesis 5 of section 4, $n_v \geq n^{\frac{2}{3} + \delta_5}$, so

$$\frac{k_* + b_x \Delta}{\sqrt{n_v}} \leq (2b_x + 1) \frac{n_t\text{or}(n_t)}{n^{\frac{1}{3} + \frac{\delta_5}{2}}} + \frac{b_x}{n^{\frac{3\delta_5}{2}}}.$$

Moreover, by hypothesis 1 of Theorem 1, $\sum_{j=k+1}^{+\infty} \theta_j^2 \leq \frac{c_1}{k^{2+\delta_1}}$, therefore

$$\text{or}(n_t) \leq \min_{k \in \mathbb{N}^*} \frac{c_1}{k^{2+\delta_1}} + \frac{k}{n_t} \leq 2 \inf_{x \geq 1} \frac{c_1}{x^{2+\delta_1}} + \frac{x}{n_t} \leq 3 \frac{c_1^{\frac{1}{3+\delta_1}}}{n_t^{\frac{2+\delta_1}{3+\delta_1}}},$$

whence (since $c_1 \geq 1$) $n_t\text{or}(n_t) \leq 3(c_1 n_t)^{\frac{1}{3+\delta_1}}$. It follows that:

$$\log n \frac{k_* + b_x \Delta}{\sqrt{n_v}} \leq 3(2b_x + 1) c_1^{\frac{1}{3+\delta_1}} \log n n^{-\frac{\delta_5}{2}} + \frac{b_x \log n}{n_t^{\frac{3\delta_5}{2}}}. \quad (19)$$

Since $\frac{\log n}{n^{\frac{\delta_5}{2}}} = o\left(n^{-\frac{\delta_5}{3}}\right)$, by equations (18), (19) and lemma 7.1, there exists a constant $\kappa(c_1, \delta_5)$ such that for any n , with probability greater than $1 - e^{-y}$,

$$\mathbb{E} \left[\sup_{a_x \leq \alpha \leq b_n} |Z_\alpha^1 - Z_\alpha| \middle| D_n^T \right] \leq \kappa(1+y)(1+x)^{\frac{3}{2}} n^{-\frac{\delta_5}{3}}.$$

7.4 Approximation of the covariance function

We will now seek to approximate the process Z^1 given by claim 4 by a time-changed Wiener process. To this end, we first approximate the variance-covariance function of Z^1 (which is the same as that of Z).

Claim 5 *There exists a function g_n satisfying the hypotheses of Theorem 1 and a constant $u_5 > 0$ such that, for all $x, y > 0$, with probability greater than $1 - e^{-y}$,*

$$\begin{aligned} \max_{(j_1, j_2) \in \{0, \dots, b_x \Delta\}^2} \left| \text{Cov} \left(Z \left(\frac{j_1}{\Delta} \right), Z \left(\frac{j_2}{\Delta} \right) \middle| D_n^T \right) - \min \left(g_n \left(\frac{j_1}{\Delta} \right), g_n \left(\frac{j_2}{\Delta} \right) \right) \right| &\leq \kappa_6 (1+x)^2 [y + \log n]^2 n^{-u_5} \\ \max_{(j_1, j_2) \in \{a_x \Delta, \dots, 0\}^2} \left| \text{Cov} \left(Z \left(\frac{j_1}{\Delta} \right), Z \left(\frac{j_2}{\Delta} \right) \middle| D_n^T \right) - \max \left(g_n \left(\frac{j_1}{\Delta} \right), g_n \left(\frac{j_2}{\Delta} \right) \right) \right| &\leq \kappa_6 (1+x)^2 [y + \log n]^2 n^{-u_5} \\ \max_{(j_1, j_2) \in \{a_x \Delta, \dots, 0\} \times \{0, \dots, b_x \Delta\}} \left| \text{Cov} \left(Z \left(\frac{j_1}{\Delta} \right), Z \left(\frac{j_2}{\Delta} \right) \middle| D_n^T \right) \right| &\leq \kappa_6 (1+x)^2 [y + \log n]^2 n^{-u_5}. \end{aligned}$$

We introduce the following definition.

Definition 12 *Let $(W_t)_{t \in \mathbb{R}}$ be a two-sided Wiener process such that $W_0 = 0$. For any function $g : I \rightarrow \mathbb{R}$, where I is an interval containing 0, let $K(g) : I^2 \rightarrow \mathbb{R}$ be defined for any $(s, t) \in I^2$ by*

$$K(g)(s, t) = \begin{cases} g(s \wedge t) & \text{if } (s, t) \in (I \cap \mathbb{R}_+)^2 \\ -g(s \vee t) & \text{if } (s, t) \in (I \cap \mathbb{R}_-)^2 \\ 0 & \text{else.} \end{cases} \quad (20)$$

For all $j \in \mathbb{Z} \cap [a_x \Delta; b_x \Delta]$, by definition 10 of Z :

$$Z\left(\frac{j}{\Delta}\right) = \frac{2}{\mathfrak{e}} \left(P_n^{T^c} - P \right) \left(\hat{s}_{k_*+j}^T - \hat{s}_{k_*}^T \right) = \begin{cases} 0 & \text{if } j = 0, \\ \frac{2}{\mathfrak{e}} \left(P_n^{T^c} - P \right) \sum_{i=k_*+1}^{k_*+j} \hat{\theta}_i^T \psi_i & \text{if } j > 0, \\ \frac{2}{\mathfrak{e}} \left(P_n^{T^c} - P \right) \sum_{i=k_*+j+1}^{k_*} \hat{\theta}_i^T \psi_i & \text{if } j < 0 \end{cases} \quad (21)$$

In other words, for all $j \in \mathbb{Z} \cap [a_x \Delta; b_x \Delta]$,

$$Z\left(\frac{j}{\Delta}\right) = \text{sgn}(j) \frac{2}{\mathfrak{e}} \sum_{i=k_*(j)_-+1}^{k_*(j)_+} \hat{\theta}_i^T \left(P_n^{T^c} - P \right) (\psi_i).$$

Let $n_v = |T^c| = n - n_t$. Thus, for any $(j_1, j_2) \in \{a_x \Delta, \dots, b_x \Delta\}^2$ and any variable X with distribution $s(x)dx$,

$$\begin{aligned} & \text{Cov} \left(Z\left(\frac{j_1}{\Delta}\right), Z\left(\frac{j_2}{\Delta}\right) \mid D_n^T \right) \\ &= \text{sgn}(j_1) \text{sgn}(j_2) \frac{4}{n_v \mathfrak{e}^2} \sum_{i_1=k_*(j_1)_-+1}^{k_*(j_1)_+} \sum_{i_2=k_*(j_2)_-+1}^{k_*(j_2)_+} \hat{\theta}_{i_1}^T \hat{\theta}_{i_2}^T \text{Cov}(\psi_{i_1}(X), \psi_{i_2}(X)). \end{aligned}$$

Let us now introduce the following definition.

Definition 13 Let $(m_1, m_2, m_3) \in \mathbb{N}^3$ be three integers. Let $m_{(1)} \leq m_{(2)} \leq m_{(3)}$ be their non-decreasing rearrangement. Define $E_{m_1, m_2, m_3} = 0$ if $m_{(1)} = m_{(2)}$ or $m_{(2)} = m_{(3)}$ and

$$E_{m_1, m_2, m_3} = \sum_{j_1=m_{(1)}+1}^{m_{(2)}} \sum_{j_2=m_{(2)}+1}^{m_{(3)}} \hat{\theta}_{j_1}^T \hat{\theta}_{j_2}^T \text{Cov}(\psi_{j_1}(X), \text{Cov}(\psi_{j_2}(X))) \quad (22)$$

if $m_{(1)} < m_{(2)} < m_{(3)}$.

Let $n_v = n - n_t$. The covariance can be broken down as follows: If $0 < j_1 \leq j_2$, conditionally on D_n^T .

$$\begin{aligned} \text{Cov} \left(Z\left(\frac{j_1}{\Delta}\right), Z\left(\frac{j_2}{\Delta}\right) \mid D_n^T \right) &= \text{Var} \left(Z\left(\frac{j_1}{\Delta}\right) \right) \\ &+ \frac{4}{n_v \mathfrak{e}^2} \sum_{i_1=k_*+1}^{k_*+j_1} \sum_{i_2=k_*+j_1+1}^{k_*+j_2} \hat{\theta}_{i_1}^T \hat{\theta}_{i_2}^T \text{Cov}(\psi_{i_1}(X), \psi_{i_2}(X)). \end{aligned}$$

If $j_1 \leq j_2 < 0$, symmetrically,

$$\begin{aligned} \text{Cov} \left(Z\left(\frac{j_1}{\Delta}\right), Z\left(\frac{j_2}{\Delta}\right) \mid D_n^T \right) &= \text{Var} \left(Z\left(\frac{j_2}{\Delta}\right) \right) \\ &+ \frac{4}{n_v \mathfrak{e}^2} \sum_{i_2=k_*+j_2+1}^{k_*} \sum_{i_1=k_*+j_1+1}^{k_*+j_2} \hat{\theta}_{i_1}^T \hat{\theta}_{i_2}^T \text{Cov}(\psi_{i_1}(X), \psi_{i_2}(X)). \end{aligned}$$

Finally, if $j_1 < 0 < j_2$,

$$\text{Cov} \left(Z\left(\frac{j_1}{\Delta}\right), Z\left(\frac{j_2}{\Delta}\right) \mid D_n^T \right) = \frac{-4}{n_v \mathfrak{e}^2} \sum_{i_1=k_*+j_1+1}^{k_*} \sum_{i_2=k_*+1}^{k_*+j_2} \hat{\theta}_{i_1}^T \hat{\theta}_{i_2}^T \text{Cov}(\psi_{i_1}(X), \psi_{i_2}(X)).$$

It follows from the previous equations that for any $(k_1, k_2) \in \mathbb{N}^2$,

$$\text{Cov}\left(Z\left(\frac{k_1 - k_*}{\Delta}\right), Z\left(\frac{k_2 - k_*}{\Delta}\right)\right) = \begin{cases} \text{Var}(Z\left(\frac{k_1 - k_*}{\Delta}\right)) + 4\frac{E_{k_*, k_1, k_2}}{n_v \epsilon^2} & \text{if } k_* < k_1 \leq k_2 \\ 4\frac{E_{k_*, k_1, k_2}}{n_v \epsilon^2} & \text{if } k_1 < k_* < k_2 \\ \text{Var}(Z\left(\frac{k_2 - k_*}{\Delta}\right)) + 4\frac{E_{k_*, k_1, k_2}}{n_v \epsilon^2} & \text{if } k_1 \leq k_2 < k_* \end{cases} \quad (23)$$

Let $I, J \subset \{a_x \Delta, \dots, b_x \Delta\}$. Assuming concentration around the expectation yields

$$\begin{aligned} \sum_{i \in I} \sum_{j \in J} \hat{\theta}_i^T \hat{\theta}_j^T \text{Cov}(\psi_i(X), \psi_j(X)) &\sim \sum_{i \in I} \sum_{j \in J} \theta_i \theta_j \text{Cov}(\psi_i(X), \psi_j(X)) \\ &+ \frac{1}{n_t} \sum_{i \in I} \sum_{j \in J} \text{Cov}(\psi_i(X), \psi_j(X))^2. \end{aligned}$$

Moreover, for any $(i_1, i_2) \in \mathbb{N}^2$,

$$\psi_{i_1}(X) \psi_{i_2}(X) = 2 \cos(2i_1 \pi X) \cos(2i_2 \pi X) = \cos(2(i_1 + i_2)\pi X) + \cos(2(i_1 - i_2)\pi X)$$

and by definition, for all $i \in \mathbb{N}^*$, $\psi_i = \sqrt{2} \cos(2i\pi X)$, while $\psi_0 = 1 = \cos(0\pi x)$.

As a result, $\psi_{i_1}(X) \psi_{i_2}(X) = \frac{\psi_{i_1+i_2}(X) + \psi_{|i_1-i_2|}(X)}{\sqrt{2}}$ if $i_1 \neq i_2$ and

$$\text{Cov}(\psi_{i_1}(X), \psi_{i_2}(X)) = \frac{\theta_{i_1+i_2}}{\sqrt{2}} + \left(\frac{1 - \delta_{i_1, i_2}}{\sqrt{2}} + \delta_{i_1, i_2} \right) \theta_{|i_2 - i_1|} - \theta_{i_1} \theta_{i_2}.$$

By assumption, the sequence $|\theta_k|$ tends to 0 with a polynomial rate of convergence, hence for sequences $i_1 \sim i_2$ tending to $+\infty$, $\theta_{|i_1 - i_2|}$ dominates $\theta_{i_1} \theta_{i_2}$ and $\theta_{i_1+i_2}$. Heuristically, it can thus be expected that

$$\begin{aligned} \sum_{i \in I} \sum_{j \in J} \hat{\theta}_i^T \hat{\theta}_j^T \text{Cov}(\psi_i(X), \psi_j(X)) &\sim \sum_{i \in I} \sum_{j \in J} \theta_i \theta_j \left(\frac{1 - \delta_{i,j}}{\sqrt{2}} + \delta_{i,j} \right) \theta_{|j-i|} \\ &+ \sum_{i \in I} \sum_{j \in J} \left(\frac{1 - \delta_{i,j}}{\sqrt{2}} + \delta_{i,j} \right)^2 \theta_{|j-i|}^2. \end{aligned}$$

This leads to the following proposition, the rigorous proof of which can be found in the appendix (proposition 8.6).

Proposition 7.3 *Let P be the probability measure with pdf s on $[0; 1]$, let $\theta_j = \langle s, \psi_j \rangle = P(\psi_j)$ and assume that the coefficients θ_j satisfy the hypotheses of section 4. Let $\hat{\theta}_j^T = P^T(\psi_j)$. Let $I_k^1, I_k^2 \subset \{k_* + a_x \Delta, \dots, k_* + b_x \Delta\}$ be two intervals. Then the statistics*

$$U_{I_k^1, I_k^2} = \sum_{i \in I_k^1} \sum_{j \in I_k^2} \hat{\theta}_i^T \hat{\theta}_j^T [P(\psi_i \psi_j) - P\psi_i P\psi_j]$$

can be approximated in the following way: there exists two constants κ_4 and $u_3 > 0$ such that, with probability greater than $1 - e^{-y}$,

$$\begin{aligned} U_{I_k^1, I_k^2} &= \frac{1}{2} \frac{|I_k^1 \cap I_k^2|}{n_t} + \left(1 - \frac{1}{\sqrt{2}}\right) \sum_{i \in I_k^1 \cap I_k^2} \theta_i^2 + \frac{1}{\sqrt{2}} \sum_{i \in I_k^1} \sum_{j \in I_k^2} \theta_i \theta_j \theta_{|i-j|} + \frac{1}{2n_t} \sum_{i \in I_k^1} \sum_{j \in I_k^2} \theta_{|i-j|}^2 \\ &\pm \kappa_4 (y + \log n)^2 (1+x) n^{-u_3} \mathcal{E}. \end{aligned}$$

It is now possible to show that the terms E_{k_*, k_1, k_2} which appear in equation (23) are negligible compared to \mathcal{E} . That is the point of the following claim.

Claim 6 *Under the assumptions of Theorem 1, there exists constants $\kappa_7 \geq 0$ and $u_4 > 0$ such that for all $n \in \mathbb{N}$, $x > 0$ and $(m_1, m_2, m_3) \in \{a_x \Delta, \dots, b_x \Delta\}^3$ such that $m_1 < m_2 < m_3$,*

$$\sum_{j_1=m_1+1}^{m_2} \sum_{j_2=m_2+1}^{m_3} \theta_{j_1} \theta_{j_2} \theta_{|j_1-j_2|} \leq \kappa_7 (1+x)^2 n^{-u_4} \mathcal{E} \quad (24)$$

$$\frac{1}{n_t} \sum_{j_1=m_1+1}^{m_2} \sum_{j_2=m_2+1}^{m_3} \theta_{|j_1-j_2|}^2 \leq \kappa_7 (1+x)^2 n^{-u_4} \mathcal{E}. \quad (25)$$

and moreover, for all $x > 0$, with probability greater than $1 - e^{-y}$, for any integers $(m_1, m_2, m_3) \in \{a_x \Delta, \dots, b_x \Delta\}^3$,

$$|E_{m_1, m_2, m_3}| \leq \kappa_7 (1+x)^2 (y + \log n)^2 n^{-u_4} \mathcal{E}. \quad (26)$$

Proof Assume without loss of generality that $m_1 < m_2 < m_3$. We start by proving equation (24). First, changing variables from j_1, j_2 to $i = j_1, r = j_2 - j_1$ yields

$$\begin{aligned} \sum_{j_1=m_1+1}^{m_2} \sum_{j_2=m_2+1}^{m_3} \theta_{j_1} \theta_{j_2} \theta_{|j_1-j_2|} &= \sum_{r \in \mathbb{N}} \theta_r \sum_{i=m_2+1-r}^{m_2} \mathbb{I}_{i \geq m_1+1} \mathbb{I}_{i+r \leq m_3} \theta_i \theta_{i+r} \\ &\leq \frac{1}{2} \sum_{r \leq r_0} |\theta_r| \sum_{i=(m_2+1-r) \vee (m_1+1)}^{m_2 \wedge (m_3-r)} \theta_i^2 + \theta_{i+r}^2 \\ &\quad + \frac{1}{2} \sum_{r > r_0} |\theta_r| \left[\sum_{j_1=m_1+1}^{m_2} \theta_{j_1}^2 + \sum_{j_2=m_2+1}^{m_3} \theta_{j_2}^2 \right] \\ &\leq \frac{\|\theta\|_{\ell^1}}{2} \max_{1 \leq r \leq r_0} \sum_{i=(m_2+1-r) \vee (m_1+1)}^{m_2 \wedge (m_3-r)} \theta_i^2 + \theta_{i+r}^2 \\ &\quad + \frac{1}{2} \sum_{r > r_0} |\theta_r| \sum_{i=a_x \Delta + 1}^{b_x \Delta} \theta_{k_* + i}^2. \end{aligned}$$

By claim 7 in appendix, for any $k \in \{m_1 + 1, \dots, m_3\} \subset \{a_x \Delta + 1, \dots, b_x \Delta\}$, $\theta_k^2 \leq \kappa_3 (1+x)^2 n^{-u_2} \mathbf{e}$, therefore

$$\sum_{j_1=m_1+1}^{m_2} \sum_{j_2=m_2+1}^{m_3} \theta_{j_1} \theta_{j_2} \theta_{|j_1-j_2|} \leq r_0 \frac{\|\theta\|_{\ell^1}}{2} \kappa_3 (1+x)^2 n^{-u_2} \mathbf{e} + \frac{1}{2} \sum_{r > r_0} |\theta_r| \sum_{i=a_x \Delta + 1}^{b_x \Delta} \theta_{k_* + i}^2.$$

By hypothesis 1 of section 4,

$$\sum_{j \geq r_0+1} |\theta_j| \leq \sum_{j=r_0+1}^{+\infty} \sqrt{\sum_{i=j}^{+\infty} \theta_i^2} \leq \sum_{j=r_0+1}^{+\infty} \frac{c_1}{(j-1)^{1+\frac{\delta_1}{2}}} \leq \frac{2c_1}{\delta_1} r_0^{-\frac{\delta_1}{2}}.$$

Thus, by lemma 7.1,

$$\sum_{j_1=m_1+1}^{m_2} \sum_{j_2=m_2+1}^{m_3} \theta_{j_1} \theta_{j_2} \theta_{|j_1-j_2|} \leq \frac{r_0 \|\theta\|_{\ell^1}}{2} \kappa_3 (1+x)^2 n^{-u_2} \mathcal{E} + \frac{2c_1}{\delta_1} r_0^{-\frac{\delta_1}{2}} 4(1+x) \mathcal{E}. \quad (27)$$

Let $r_0 = \lceil n^{\frac{2u_2}{2+\delta_1}} \rceil \leq 2n^{\frac{2u_2}{2+\delta_1}}$ and $u = \frac{\delta_1(u_2)}{2+\delta_1} > 0$. For all $n \geq 2$,

$$\sum_{j_1=m_1+1}^{m_2} \sum_{j_2=m_2+1}^{m_3} \theta_{j_1} \theta_{j_2} \theta_{|j_1-j_2|} \leq \left[\|\theta\|_{\ell^1} \kappa_3 + 8 \frac{c_1}{\delta_1} \right] (1+x)^2 n^{-u} \mathcal{E}, \quad (28)$$

which proves equation (24).

Moreover,

$$\begin{aligned} \sum_{j_1=m_1+1}^{m_2} \sum_{j_2=m_2+1}^{m_3} \theta_{|j_1-j_2|}^2 &= \sum_{r \in \mathbb{N}} \theta_r^2 |\{j_1 : (m_1+1 \leq j_1 \leq m_2) \wedge (m_2+1 \leq j_1+r \leq m_3)\}| \\ &\leq \sum_{r \in \mathbb{N}} \theta_r^2 [(m_3 - m_1) \wedge r] \\ &\leq r_0 \sum_{r=0}^{r_0} \theta_r^2 + (m_3 - m_1) \sum_{r>r_0} \theta_r^2 \\ &\leq r_0 \|s\|^2 + (b_x - a_x) \Delta \sum_{r>r_0} \theta_r^2 \\ &\leq r_0 \|s\|^2 + 2(1+x) \Delta \frac{c_1}{r_0^{2+\delta_1}}, \end{aligned}$$

by hypothesis 1 of Theorem 1 and lemma 7.1. Let now $r_0 = \lceil \Delta^{\frac{1}{3}} \rceil$. Since $\Delta \geq 1$, it follows that:

$$\begin{aligned} \frac{1}{n_t} \sum_{j_1=m_1+1}^{m_2} \sum_{j_2=m_2+1}^{m_3} \theta_{|j_1-j_2|}^2 &\leq \frac{\Delta^{\frac{1}{3}} + 1}{n_t} \|s\|^2 + 2c_1(1+x) \frac{\Delta}{n_t} (\Delta)^{-\frac{2}{3}} \\ &\leq \left[2 \frac{\|s\|^2}{(\Delta)^{\frac{2}{3}}} \mathcal{E} + 2c_1(1+x) \mathcal{E} (\Delta)^{-\frac{2}{3}} \right] \\ &\leq \left[2 \|s\|^2 + 2c_1(1+x) \right] \frac{\mathcal{E}}{(\Delta)^{\frac{2}{3}}}. \end{aligned}$$

On the other hand, $\Delta \geq \frac{n_t}{n-n_t} \geq n^{\delta_4}$ by hypothesis 4 of Theorem 1. There exists therefore $\kappa(c_1, \|s\|^2)$ such that, for any n ,

$$\frac{1}{n_t} \sum_{j_1=m_1+1}^{m_2} \sum_{j_2=m_2+1}^{m_3} \theta_{|j_1-j_2|}^2 \leq \kappa(1+x) n^{-\frac{2\delta_4}{3}} \mathcal{E}, \quad (29)$$

which proves equation (25). Since for all $x > 0$, $(b_x - a_x) \leq 2(1+x)\Delta \leq \kappa(1+x)n$, by proposition 7.3 and a union bound, there exists an event A of probability greater than $1 - e^{-y}$ and a constant κ such that, if $a_x \leq m_1 < m_2 < m_3 \leq b_x$,

then

$$\begin{aligned}
|E_{m_1, m_2, m_3}| &\leq \frac{1}{\sqrt{2}} \sum_{j_1=m_1+1}^{m_2} \sum_{j_2=m_2+1}^{m_3} \theta_{j_1} \theta_{j_2} \theta_{|j_1-j_2|} + \frac{1}{2n_t} \sum_{j_1=m_1+1}^{m_2} \sum_{j_2=m_2+1}^{m_3} \theta_{|j_1-j_2|}^2 \\
&\quad + \kappa(y + \log(2+x) + \log n)^2 (1+x)n^{-u_3} \mathcal{E}.
\end{aligned} \tag{30}$$

From equations (30), (28) and (29), equation (26) follows with $u_4 = \min\left(u_3, \frac{2\delta_4}{3}, \frac{\delta_1 u_2}{2+\delta_1}\right)$.
■

Let then $g_n^0 : \left[-\frac{k_*}{\Delta}; +\infty[\rightarrow \mathbb{R}$ be defined first for all $\alpha \in \left\{\frac{j}{\Delta} : j \in \mathbb{N} - k_*\right\}$ by

$$\forall \alpha \in \left\{\frac{j}{\Delta} : (j + k_*) \in \mathbb{N}\right\}, g_n^0(\alpha) = \text{sgn}(\alpha) \text{Var}(Z_\alpha), \tag{31}$$

then for all $\alpha \in \left[-\frac{k_*}{\Delta}; +\infty[$ by linear interpolation (hence in general, $g_n^0(\alpha) \neq \text{Var}(Z_\alpha)$). Let $K(g_n^0)$ be given by definition 12, then by equation (23) and claim 6, with probability greater than $1 - e^{-y}$, for any $x > 0$,

$$\begin{aligned}
&\max_{(j_1, j_2) \in \{a_x \Delta, \dots, b_x \Delta\}^2} \left| \text{Cov}\left(Z\left(\frac{j_1}{\Delta}\right), Z\left(\frac{j_2}{\Delta}\right)\right) - K\left(g_n^0, \frac{j_1}{\Delta}, \frac{j_2}{\Delta}\right) \right| \\
&\leq 4\kappa_7(1+x)^2 + (y + \log n)^2 n^{-u_4} \frac{\mathcal{E}}{n_v \epsilon^2} \\
&\leq 4\kappa_7(1+x)^2 (y + \log n)^2 n^{-u_4}.
\end{aligned} \tag{32}$$

Moreover, for any $j \in \mathbb{Z} \cap [a_x \Delta; b_x \Delta]$, by definition of Z ,

$$\begin{aligned}
\text{sgn}(j) g_n^0\left(\frac{j}{\Delta}\right) &= \text{Var}\left(Z\left(\frac{j}{\Delta}\right)\right) = \frac{4}{n_v \epsilon^2} \sum_{i_1=k_*(j)_-+1}^{k_*(j)_+} \sum_{i_2=k_*(j)_-+1}^{k_*(j)_+} \hat{\theta}_{i_1}^T \hat{\theta}_{i_2}^T \left[\frac{\theta_{i_1+i_2}}{\sqrt{2}} \right. \\
&\quad \left. + \left(\frac{1 - \delta_{i_1, i_2}}{\sqrt{2}} + \delta_{i_1, i_2} \right) \theta_{|i_1-i_2|} - \theta_{i_1} \theta_{i_2} \right].
\end{aligned} \tag{33}$$

Moreover, since $\epsilon^2 = \frac{\mathcal{E}}{n_v}$,

$$\begin{aligned}
\frac{4}{n_v \epsilon^2} \frac{1}{2} \frac{j}{n_t} &= \frac{4}{n_v \epsilon^2} \frac{1}{2} \frac{j}{\Delta} \frac{\Delta}{n_t} \\
&= \frac{\mathcal{E}}{n_v \epsilon^2} 2 \frac{j}{\Delta} \\
&= 2 \frac{j}{\Delta}.
\end{aligned}$$

Thus, by proposition 7.3, with probability greater than $1 - e^{-y}$,

$$\begin{aligned}
\operatorname{sgn}(j)g_n^0\left(\frac{j}{\Delta}\right) &= 2\frac{|j|}{\Delta} + \frac{4}{n_v\epsilon^2} \left(1 - \frac{1}{\sqrt{2}}\right) \sum_{i=k_*(j)_-+1}^{k_*(j)_+} \theta_i^2 \\
&\quad + \frac{4}{n_v\epsilon^2} \sum_{i_1=k_*(j)_-+1}^{k_*(j)_+} \sum_{i_2=k_*(j)_-+1}^{k_*(j)_+} \theta_{i_1}\theta_{i_2} \frac{\theta_{|i_1-i_2|}}{\sqrt{2}} \\
&\quad + \frac{4}{n_v\epsilon^2} \frac{1}{2n_t} \sum_{i_1=k_*(j)_-+1}^{k_*(j)_+} \sum_{i_2=k_*(j)_-+1}^{k_*(j)_+} \theta_{|i_1-i_2|}^2 \\
&\quad \pm 4\kappa_4(y + \log n)^2(1+x)n^{-u_3}.
\end{aligned} \tag{34}$$

Let g_n^1 be defined for all $\alpha = \frac{j}{\Delta}, j \in \mathbb{Z} \cap [-k_*(n_t); +\infty)$ by

$$\begin{aligned}
\operatorname{sgn}(j)g_n^1\left(\frac{j}{\Delta}\right) &= \frac{4}{n_v\epsilon^2} \sum_{i_1=k_*(j)_-+1}^{k_*(j)_+} \sum_{i_2=k_*(j)_-+1}^{k_*(j)_+} \theta_{i_1}\theta_{i_2} \frac{\theta_{|i_1-i_2|}}{\sqrt{2}} \\
&\quad + \frac{4}{n_v\epsilon^2} \left(1 - \frac{1}{\sqrt{2}}\right) \sum_{i=k_*(j)_-+1}^{k_*(j)_+} \theta_i^2 \\
&= \frac{4}{n_v\epsilon^2} \sum_{i_1=k_*(j)_-+1}^{k_*(j)_+} \sum_{i_2=k_*(j)_-+1}^{k_*(j)_+} \theta_{i_1}\theta_{i_2} \left(\frac{1 - \delta_{i_1, i_2}}{\sqrt{2}} + \delta_{i_1, i_2}\right) \theta_{|i_1-i_2|},
\end{aligned} \tag{35}$$

and for all $\alpha \in [-\frac{k_*}{\Delta}; +\infty[$ by linear interpolation.

We will now apply lemma 8.8 to g_n^1 . Let $x > 0$ and $(k_1, k_2) \in \{k_* + a_x\Delta, \dots, k_* + b_x\Delta\}^2$ be such that $k_1 < k_2$. Thus:

- If $k_* \leq k_1$,

$$\begin{aligned}
g_n^1\left(\frac{k_2 - k_*}{\Delta}\right) - g_n^1\left(\frac{k_1 - k_*}{\Delta}\right) &= \frac{4}{n_v\epsilon^2} \left(\sum_{i=k_*+1}^{k_2} \sum_{j=k_*+1}^{k_2} \theta_i\theta_j \left(\frac{1 - \delta_{i,j}}{\sqrt{2}} + \delta_{i,j}\right) \theta_{|i-j|} \right. \\
&\quad \left. - \sum_{i=k_*+1}^{k_1} \sum_{j=k_*+1}^{k_1} \theta_i\theta_j \left(\frac{1 - \delta_{i,j}}{\sqrt{2}} + \delta_{i,j}\right) \theta_{|i-j|} \right) \\
&= \frac{4}{n_v\epsilon^2} \left(\sum_{i=k_1+1}^{k_2} \sum_{j=k_1+1}^{k_2} \theta_i\theta_j \left(\frac{1 - \delta_{i,j}}{\sqrt{2}} + \delta_{i,j}\right) \theta_{|i-j|} \right. \\
&\quad \left. + 2 \sum_{i=k_1+1}^{k_2} \sum_{j=k_*+1}^{k_1} \theta_i\theta_j \frac{\theta_{|i-j|}}{\sqrt{2}} \right).
\end{aligned}$$

By lemma 8.7 in appendix,

$$0 \leq \sum_{i=k_1+1}^{k_2} \sum_{j=k_1+1}^{k_2} \theta_i\theta_j \left(\frac{1 - \delta_{i,j}}{\sqrt{2}} + \delta_{i,j}\right) \theta_{|i-j|} \leq \|s\|_\infty \sum_{i=k_1+1}^{k_2} \theta_i^2.$$

Thus by equation (24) from claim 6,

$$-\kappa_7(1+x)^2 n^{-u_3} \frac{4\sqrt{2}}{n_v \epsilon^2} \mathcal{E} \leq g_n^1 \left(\frac{k_2 - k_*}{\Delta} \right) - g_n^1 \left(\frac{k_1 - k_*}{\Delta} \right) \leq \frac{4 \|s\|_\infty}{n_v \epsilon^2} \sum_{i=k_1+1}^{k_2} \theta_i^2 + \kappa_7(1+x)^2 n^{-u_4} \frac{4\sqrt{2}}{n_v \epsilon^2} \mathcal{E}.$$

- If $k_2 \leq k_*$,

$$\begin{aligned} g_n^1 \left(\frac{k_2 - k_*}{\Delta} \right) - g_n^1 \left(\frac{k_1 - k_*}{\Delta} \right) &= \frac{4}{n_v \epsilon^2} \left(\sum_{i=k_1+1}^{k_*} \sum_{j=k_1+1}^{k_*} \theta_i \theta_j \left(\frac{1 - \delta_{i,j}}{\sqrt{2}} + \delta_{i,j} \right) \theta_{|i-j|} \right. \\ &\quad \left. - \sum_{i=k_2+1}^{k_*} \sum_{j=k_2+1}^{k_*} \theta_i \theta_j \left(\frac{1 - \delta_{i,j}}{\sqrt{2}} + \delta_{i,j} \right) \theta_{|i-j|} \right) \\ &= \frac{4}{n_v \epsilon^2} \left(\sum_{i=k_1+1}^{k_2} \sum_{j=k_1+1}^{k_2} \theta_i \theta_j \left(\frac{1 - \delta_{i,j}}{\sqrt{2}} + \delta_{i,j} \right) \theta_{|i-j|} \right. \\ &\quad \left. + 2 \sum_{i=k_1+1}^{k_2} \sum_{j=k_2+1}^{k_*} \theta_i \theta_j \frac{\theta_{|i-j|}}{\sqrt{2}} \right). \end{aligned}$$

In the same way, by lemma 8.7 and equation (24) from claim 6,

$$\begin{aligned} g_n^1 \left(\frac{k_2 - k_*}{\Delta} \right) - g_n^1 \left(\frac{k_1 - k_*}{\Delta} \right) &\geq -\kappa_7(1+x)^2 n^{-u_3} \frac{4\sqrt{2}}{n_v \epsilon^2} \mathcal{E}, \\ g_n^1 \left(\frac{k_2 - k_*}{\Delta} \right) - g_n^1 \left(\frac{k_1 - k_*}{\Delta} \right) &\leq \frac{4 \|s\|_\infty}{n_v \epsilon^2} \sum_{i=k_1+1}^{k_2} \theta_i^2 + \kappa_7(1+x)^2 n^{-u_4} \frac{4\sqrt{2}}{n_v \epsilon^2} \mathcal{E}. \end{aligned}$$

- If $k_1 \leq k_2 \leq k_*$,

$$g_n^1 \left(\frac{k_2 - k_*}{\Delta} \right) - g_n^1 \left(\frac{k_1 - k_*}{\Delta} \right) = g_n^1 \left(\frac{k_2 - k_*}{\Delta} \right) - g_n(0) + g_n(0) - g_n^1 \left(\frac{k_1 - k_*}{\Delta} \right),$$

therefore by the two previous cases,

$$0 \leq g_n^1 \left(\frac{k_2 - k_*}{\Delta} \right) - g_n^1 \left(\frac{k_1 - k_*}{\Delta} \right) \leq \frac{4 \|s\|_\infty}{n_v \epsilon^2} \sum_{i=k_1+1}^{k_2} \theta_i^2 + \kappa_7(1+x)^2 n^{-u_4} \frac{8\sqrt{2}}{n_v \epsilon^2} \mathcal{E}.$$

By definition $\epsilon^2 = \frac{\mathcal{E}}{n_v}$ therefore for any $x > 0$ and $(j_1, j_2) \in [a_x \Delta; b_x \Delta]^2$ such that $j_1 \leq j_2$,

$$-4\sqrt{2}\kappa_7(1+x)^2 n^{-u_4} \leq g_n^1 \left(\frac{j_2}{\Delta} \right) - g_n^1 \left(\frac{j_1}{\Delta} \right) \leq \frac{4 \|s\|_\infty}{n_v \epsilon^2} \sum_{i=j_1+1}^{j_2} \theta_{k_*+i}^2 + 8\sqrt{2}\kappa_7(1+x)^2 n^{-u_4}. \quad (36)$$

Moreover, for any j_1, j_2 such that $0 < j_1 < j_2$, $\theta_{k_*+j_i}^2 \leq \frac{1}{n_t}$ hence

$$\begin{aligned} \|s\|_\infty \sum_{i=j_1+1}^{j_2} \theta_{k_*+i}^2 &\leq \|s\|_\infty \sum_{j=k_*+j_1+1}^{k_*+j_2} \left[\theta_j^2 - \frac{1}{n_t} \right] + \frac{j_2 - j_1}{\Delta} \|s\|_\infty \frac{\Delta}{n_t} \\ &= -\|s\|_\infty \mathfrak{e} \left[f_n \left(\frac{j_2}{\Delta} \right) - f_n \left(\frac{j_1}{\Delta} \right) \right] + \frac{j_2 - j_1}{\Delta} \|s\|_\infty \mathcal{E}. \quad (37) \end{aligned}$$

For j_1, j_2 such that $j_1 < j_2 \leq 0$, $\theta_{k_*+j_i}^2 \geq \frac{1}{n_t}$ hence

$$\begin{aligned} \|s\|_\infty \sum_{i=j_1+1}^{j_2} \theta_{k_*+i}^2 &\leq \|s\|_\infty \sum_{j=k_*+j_1+1}^{k_*+j_2} \left[\theta_j^2 - \frac{1}{n_t} \right] + \frac{j_2 - j_1}{\Delta} \|s\|_\infty \frac{\Delta}{n_t} \\ &= \epsilon \|s\|_\infty \left[f_n \left(\frac{j_1}{\Delta} \right) - f_n \left(\frac{j_2}{\Delta} \right) \right] + \frac{j_2 - j_1}{\Delta} \|s\|_\infty \mathcal{E}. \end{aligned} \quad (38)$$

By equations (36), (37) and (37), it follows that, for any $(j_1, j_2) \in ([a_x \Delta; b_x \Delta] \cap \mathbb{Z})^2$,

$$\begin{aligned} g_n^1 \left(\frac{j_2}{\Delta} \right) - g_n^1 \left(\frac{j_1}{\Delta} \right) &\leq -4 \frac{\|s\|_\infty}{n_v \epsilon} \left[f_n \left(\frac{j_2}{\Delta} \right) - f_n \left(\frac{j_1}{\Delta} \right) \right] + 4 \|s\|_\infty \frac{j_2 - j_1}{\Delta} + 8\sqrt{2}\kappa_7(1+x)^2 n^{-u_4} \\ &\leq -4 \frac{\|s\|_\infty}{n_v \epsilon} \left[f_n \left(\frac{j_2}{\Delta} \right) - f_n \left(\frac{j_1}{\Delta} \right) \right] + 4 \|s\|_\infty \frac{j_2 - j_1}{\Delta} + 8\sqrt{2}\kappa_7(1+x)^2 n^{-u_4}. \end{aligned} \quad (39)$$

To extend the lower bound given by equation (36), notice that for any $(\alpha_1, \alpha_2) \in [a_x; b_x]^2$ such that $\alpha_1 < \alpha_2$,

- if $\lfloor \alpha_1 \Delta \rfloor = \lfloor \alpha_2 \Delta \rfloor \leq \alpha_1 \Delta < \alpha_2 \Delta \leq \lfloor \alpha_1 \Delta \rfloor + 1$, by linearity of g_n^1 on $[\lfloor \alpha_1 \Delta \rfloor; \lfloor \alpha_1 \Delta \rfloor + 1]$,

$$g_n^1(\alpha_2) - g_n^1(\alpha_1) \geq - \left[g_n^1 \left(\frac{\lfloor \alpha_1 \Delta \rfloor + 1}{\Delta} \right) - g_n^1 \left(\frac{\lfloor \alpha_1 \Delta \rfloor}{\Delta} \right) \right]_-,$$

- otherwise, $\lfloor \alpha_1 \Delta \rfloor + 1 \leq \lfloor \alpha_2 \Delta \rfloor$, therefore by linearity of $(u, v) \mapsto g_n^1(u) - g_n^1(v)$ on $\frac{1}{\Delta} [\lfloor \alpha_1 \Delta \rfloor; \lfloor \alpha_1 \Delta \rfloor + 1] \times \frac{1}{\Delta} [\lfloor \alpha_2 \Delta \rfloor; \lfloor \alpha_2 \Delta \rfloor + 1]$,

$$\begin{aligned} g_n^1(\alpha_2) - g_n^1(\alpha_1) &\geq \min \left\{ g_n^1(u) - g_n^1(v) : \right. \\ &\quad \left. (u, v) \in \left\{ \frac{\lfloor \alpha_2 \Delta \rfloor}{\Delta}; \frac{\lfloor \alpha_2 \Delta \rfloor + 1}{\Delta} \right\} \times \left\{ \frac{\lfloor \alpha_1 \Delta \rfloor}{\Delta}; \frac{\lfloor \alpha_1 \Delta \rfloor + 1}{\Delta} \right\} \right\}. \end{aligned}$$

In all cases,

$$g_n^1(\alpha_2) - g_n^1(\alpha_1) \geq - \max \left\{ \left[g_n^1 \left(\frac{j_2}{\Delta} \right) - g_n^1 \left(\frac{j_1}{\Delta} \right) \right]_- : j_1 \leq j_2, (j_1, j_2) \in \{a_x \Delta, \dots, b_x \Delta\}^2 \right\}. \quad (40)$$

Thus by equation (36), for any $x > 0$:

$$\forall (\alpha_1, \alpha_2) \in [a_x; b_x]^2, \alpha_1 < \alpha_2 \implies g_n^1(\alpha_2) - g_n^1(\alpha_1) \geq -4\sqrt{2}\kappa_7(1+x)^2 n^{-u_4}. \quad (41)$$

By the same argument applied to the function

$$\alpha \mapsto g_n^1(\alpha) + 4 \frac{\|s\|_\infty}{n_v \epsilon} f_n(\alpha) - 4 \|s\|_\infty \alpha,$$

which is piecewise linear on the partition $\left\{ \left[\frac{j}{\Delta}; \frac{j+1}{\Delta} \right] : j \in \{a_x \Delta, \dots, b_x \Delta\} \right\}$, equation (39) extends to $[a_x; b_x]$ for any $x > 0$:

$$\begin{aligned} \forall (\alpha_1, \alpha_2) \in [a_x; b_x]^2, g_n^1(\alpha_2) - g_n^1(\alpha_1) &\leq 4 \frac{\|s\|_\infty}{n_v \epsilon} [f_n(\alpha_1) - f_n(\alpha_2)] + 4 \|s\|_\infty [\alpha_2 - \alpha_1] \\ &\quad + 8\sqrt{2}\kappa_7(1+x)^2 n^{-u_4}. \end{aligned} \quad (42)$$

Let $\varepsilon_d : \alpha \mapsto \inf_{x \in \mathbb{R}_+ : b_x \geq \alpha} 8\sqrt{2}\kappa_7(1+x)^2 n^{-u_4}$. The function ε_d is non-decreasing by definition, and $\varepsilon_d(0) \geq 8\kappa_7 n^{-u_4} > 0$. Furthermore, by equations (41) and (42),

$$\begin{aligned} \forall(\alpha_1, \alpha_2) \in \mathbb{R}_+^2, \alpha_1 \leq \alpha_2 &\implies \\ -\varepsilon_d(\alpha_2) \leq g_n^1(\alpha_2) - g_n^1(\alpha_1) &\leq 4 \frac{\|s\|_\infty}{n_v \mathfrak{e}} [f_n(\alpha_1) - f_n(\alpha_2)] + 4 \|s\|_\infty [\alpha_2 - \alpha_1] + \varepsilon_d(\alpha_2). \end{aligned}$$

In this situation, lemma 8.8 applies with $g = g_n^1$, $h_+ = -\frac{8\|s\|_\infty}{n_v \mathfrak{e}} f_n + 8\|s\|_\infty \text{Id}$ and $\varepsilon = 2\varepsilon_d$. Note that h_+ is indeed non-decreasing since by equation (15), $f'_n \leq \frac{\Delta}{n_t \mathfrak{e}}$ on \mathbb{R}_+ which leads to $\frac{f'_n}{n_v \mathfrak{e}} \leq \frac{\Delta}{n_v n_t \mathfrak{e}^2} = 1$. This guarantees existence of a non-decreasing function $g_{n,+}^2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $g_{n,+}^2(0) = 0$,

$$\sup_{\alpha \in \mathbb{R}_+} \frac{|g_n^1(\alpha) - g_{n,+}^2(\alpha)|}{2\varepsilon_d(\alpha)} \leq 6$$

and for all α_1, α_2 such that $\alpha_1 \leq \alpha_2$,

$$g_{n,+}^2(\alpha_2) - g_{n,+}^2(\alpha_1) \leq 8 \frac{\|s\|_\infty}{n_v \mathfrak{e}} [f_n(\alpha_1) - f_n(\alpha_2)] + 8 \|s\|_\infty [\alpha_2 - \alpha_1]$$

Symmetrically, let $\varepsilon_g : \alpha \mapsto \inf_{x \in \mathbb{R}_+ : -a_x \geq \alpha} 8\sqrt{2}\kappa_7(1+x)^2 n^{-u_4}$, defined on $[0; \frac{k_*(n_t)}{\Delta}]$. ε_g is non-decreasing by definition. Furthermore, $\varepsilon_g(0) \geq 8\kappa_7 n^{-u_4} > 0$. By equations (41) and (42),

$$\begin{aligned} \forall(\alpha_1, \alpha_2) \in \mathbb{R}_+^2, \alpha_1 \leq \alpha_2 &\implies -\varepsilon_g(\alpha_2) \leq g_n^1(-\alpha_1) - g_n^1(-\alpha_2) \\ &\leq 4 \frac{\|s\|_\infty}{n_v \mathfrak{e}} [f_n(-\alpha_2) - f_n(-\alpha_1)] \\ &\quad + 4 \|s\|_\infty [\alpha_2 - \alpha_1] + \varepsilon_g(\alpha_2). \end{aligned}$$

In this situation, lemma 8.8 applies with $g = -g_n^1(\cdot)$, $h_+ = \frac{8\|s\|_\infty}{n_v \mathfrak{e}} f_n(\cdot) + 8\|s\|_\infty \text{Id}$, $\varepsilon = 2\varepsilon_g$. It guarantees existence of a function $g_{n,-}^2 : [0; \frac{k_*(n_t)}{\Delta}] \rightarrow \mathbb{R}_+$ such that $g_{n,-}^2(0) = 0$,

$$\sup_{\alpha \in [0; \frac{k_*(n_t)}{\Delta}]} \frac{|-g_n^1(-\alpha) - g_{n,-}^2(\alpha)|}{2\varepsilon_g(\alpha)} \leq 6$$

and for any α_1, α_2 such that $\alpha_1 \leq \alpha_2$,

$$g_{n,-}^2(\alpha_2) - g_{n,-}^2(\alpha_1) \leq 8 \frac{\|s\|_\infty}{n_v \mathfrak{e}} [f_n(-\alpha_2) - f_n(-\alpha_1)] + 8 \|s\|_\infty [\alpha_2 - \alpha_1].$$

Let then $g_n^2 : \alpha \mapsto g_{n,+}^2(\alpha) \mathbb{I}_{\alpha \geq 0} - g_{n,-}^2(-\alpha) \mathbb{I}_{\alpha < 0}$ and $\varepsilon(\alpha) = \varepsilon_d(\alpha) \mathbb{I}_{\alpha \geq 0} + \varepsilon_g(-\alpha) \mathbb{I}_{\alpha < 0}$, which yields

$$\left\| \frac{g_n^2 - g_n^1}{2\varepsilon} \right\|_\infty \leq 6 \quad (43)$$

and

$$\forall(\alpha_1, \alpha_2) \in \mathbb{R}^2, g_n^2(\alpha_2) - g_n^2(\alpha_1) \leq 8 \frac{\|s\|_\infty}{n_v \mathfrak{e}} [f_n(\alpha_1) - f_n(\alpha_2)] + 8 \|s\|_\infty [\alpha_2 - \alpha_1]. \quad (44)$$

By definition of ε , for any $x > 0$ and any $\alpha \in [a_x, b_x]$, $\varepsilon(\alpha) \leq 8\sqrt{2}\kappa_7(1+x)^2n^{-u_4}$, hence

$$\forall x > 0, \forall \alpha \in [a_x; b_x], |g_n^2(\alpha) - g_n^1(\alpha)| \leq 96\sqrt{2}\kappa_7(1+x)^2n^{-u_4}. \quad (45)$$

Let then:

$$g_n : \alpha \mapsto g_n^2(\alpha) + 4\|s\|^2\alpha. \quad (46)$$

Since g_n^2 is non-decreasing, $g_n(\alpha_2) - g_n(\alpha_1) \geq 4\|s\|^2[\alpha_2 - \alpha_1]$, which proves equation 2 of Theorem 1. Moreover, equation (44) yields equation (13) of Theorem 1.

Let now $x > 0$ be fixed until the end of this section. By definition of g_n^1 (equation (35)), equations (34), (45) and since the functions g_n^0 and $\alpha \mapsto 4\|s\|^2\alpha$ are piecewise linear on the partition $([\frac{j}{\Delta}, \frac{j+1}{\Delta}])_{a_x\Delta \leq j \leq b_x\Delta-1}$, with probability greater than $1 - e^{-y}$,

$$\begin{aligned} \|g_n^0 - g_n\|_\infty &\leq \|g_n^1 - g_n^2\|_\infty + \kappa_4(y + \log n)^2(1+x)n^{-u_3} \\ &\quad + \max_{a_x\Delta \leq j \leq b_x\Delta} \left| \frac{\text{sgn}(j)}{n_v\epsilon^2} \frac{4}{2n_t} \sum_{i_1=k_*(j)_-+1}^{k_*(j)_+} \sum_{i_2=k_*(j)_-+1}^{k_*(j)_+} \theta_{|i_1-i_2|}^2 - \frac{(4\|s\|^2-2)j}{\Delta} \right| \\ &\leq \max_{a_x\Delta \leq j \leq b_x\Delta} \left| \frac{4}{n_v\epsilon^2} \frac{1}{2n_t} \sum_{i_1=k_*(j)_-+1}^{k_*(j)_+} \sum_{i_2=k_*(j)_-+1}^{k_*(j)_+} \theta_{|i_1-i_2|}^2 - 4\left(\|s\|^2 - \frac{1}{2}\right) \frac{|j|}{\Delta} \right| \\ &\quad + 96\sqrt{2}\kappa_7(1+x)^2n^{-u_4} + \kappa_4(y + \log n)^2(1+x)n^{-u_3}. \end{aligned} \quad (47)$$

It remains to bound the max. By parity in j of the sum, one can assume $0 \leq j \leq \max(|a_x|, |b_x|)\Delta$ instead of $a_x\Delta \leq j \leq b_x\Delta$. Let therefore $j \in \{0, \dots, \max(|a_x|, |b_x|)\Delta\}$, then

$$\begin{aligned} \frac{1}{2n_t} \sum_{i_1=k_*+1}^{k_*+j} \sum_{i_2=k_*+1}^{k_*+j} \theta_{|i_1-i_2|}^2 &= \frac{|j|}{2n_t} + \frac{1}{2n_t} \sum_{r \in \mathbb{N}^*} 2|\{i : k_* \leq i \leq i+r \leq k_*+j\}| \theta_r^2 \\ &= \frac{|j|}{2n_t} + \frac{1}{n_t} \sum_{r=1}^{+\infty} (j-r)_+ \theta_r^2. \end{aligned} \quad (48)$$

Furthermore, for all $r_0 \in \mathbb{N}^*$,

$$\begin{aligned} \left| \frac{1}{n_t} \sum_{r=1}^{+\infty} (j-r)_+ \theta_r^2 - \frac{j}{n_t} (\|s\|^2 - 1) \right| &\leq \frac{1}{n_t} \sum_{r=1}^{+\infty} \theta_r^2 |(j-r)_+ - j| \\ &\leq \frac{r_0}{n_t} \sum_{r=1}^{r_0} \theta_r^2 + \frac{j}{n_t} \sum_{r=r_0+1}^{+\infty} \theta_r^2 \\ &\leq \|s\|^2 \frac{r_0}{n_t} + \max(|a_x|, |b_x|) \frac{\Delta}{n_t} \times \frac{c_1}{r_0^2}, \end{aligned}$$

by hypothesis 1 of section 4. By setting $r_0 = \lceil (\Delta)^{\frac{1}{3}} \rceil \leq 2(\Delta)^{\frac{1}{3}}$ (since $\Delta \geq 1$), it follows that

$$\begin{aligned} \left| \frac{1}{n_t} \sum_{r=1}^{+\infty} (j-r)_+ \theta_r^2 - \frac{j}{n_t} (\|s\|^2 - 1) \right| &\leq \left[2\|s\|^2 + c_1 \max(|a_x|, |b_x|) \right] \frac{(\Delta)^{\frac{1}{3}}}{n_t} \\ &\leq \left[2\|s\|^2 + 2c_1(1+x) \right] (\Delta)^{-\frac{2}{3}} \mathcal{E} \text{ by lemma 7.1.} \end{aligned}$$

Let $\kappa = \kappa(c_1, \|s\|)$. Since by hypothesis 4 of Theorem 1, $\Delta \geq \frac{n_t}{n-n_t} \geq n^{\delta_4}$,

$$\left| \frac{1}{n_t} \sum_{r=1}^{+\infty} (j-r)_+ \theta_r^2 - \frac{j}{n_t} (\|s\|^2 - 1) \right| \leq \kappa(1+x)n^{-\frac{2\delta_4}{3}} \mathcal{E}.$$

By equation (48) and since $\frac{j}{n_t} = \frac{j}{\Delta} \mathcal{E}$, for any $j \in [0; \max(|a_x|, |b_x|)\Delta]$.

$$\left| \frac{1}{2n_t} \sum_{i_1=k_*+1}^{k_*+j} \sum_{i_2=k_*+1}^{k_*+j} \theta_{|i_1-i_2|}^2 - \frac{j}{\Delta} \left(\|s\|^2 - \frac{1}{2} \right) \mathcal{E} \right| \leq \kappa(1+x)n^{-\delta_4} \mathcal{E}.$$

By equation (47) and since $\frac{\mathcal{E}}{n_v \epsilon^2} = 1$, it follows that, with probability greater than $1 - e^{-y}$,

$$\|g_n^0 - g_n\|_\infty \leq 96\sqrt{2}\kappa_7(1+x)^2 n^{-u_4} + \kappa_4(y + \log n)^2(1+x)n^{-u_3} + 4\kappa(1+x)n^{-\delta_4}. \quad (49)$$

Let $\kappa = 96\sqrt{2}\kappa_7 + \kappa_4 + 4\kappa$ and $u_5 = \min(u_4, u_3, \delta_4)$, it then follows from definition 12 of K that with probability greater than $1 - e^{-y}$,

$$\|K(g_n^0) - K(g_n)\|_\infty \leq \|g_n^0 - g_n\|_\infty \leq \kappa(1+x)^2(y + \log n)^2 n^{-u_5}.$$

By equation (32), it follows that that, with probability greater than $1 - e^{-y}$, for any $(j_1, j_2) \in \{a_x \Delta, \dots, b_x \Delta\}^2$,

$$\begin{aligned} \left| \text{Cov} \left(Z \left(\frac{j_1}{\Delta} \right), Z \left(\frac{j_2}{\Delta} \right) \right) - K(g_n) \left(\frac{j_1}{\Delta}, \frac{j_2}{\Delta} \right) \right| &\leq 4\kappa_7(1+x)^2(y + \log n)^2 n^{-u_3} \\ &\quad + \kappa(1+x)^2(y + \log n)^2 n^{-u_5} \\ &\leq \kappa(1+x)^2(y + \log n)^2 n^{-u_5} \end{aligned} \quad (50)$$

by setting $\kappa = \kappa + 4\kappa_7$ and since $u_5 \leq u_3$. Claim 5 follows. It remains to upper bound g_n on $[a_x; b_x]$ in order to check equation 14 of Theorem 1. This is the subject of the following lemma.

Lemma 7.4 *For any $\alpha \in \mathbb{R}$,*

$$|g_n(\alpha)| \leq 20 \|s\|_\infty f_n(\alpha) + 12 \|s\|_\infty \leq \max(40 \|s\|_\infty f_n(\alpha), 24 \|s\|_\infty).$$

In particular, for all $x > 0$, $\max(|g_n(a_x)|, |g_n(b_x)|) \leq 20 \|s\|_\infty (1+x)$.

Proof Since $\|s\|_\infty \geq \|s\|^2 \geq 1$ and $n_v \epsilon \leq 1$ by lemma 3.1, point 3 of Theorem 1 which we already proved implies that for any $\alpha \in \mathbb{R}$,

$$|g_n(\alpha)| \leq 8 \|s\|_\infty f_n(\alpha) + 12 \|s\|_\infty |\alpha|.$$

If $|\alpha| < 1$, then

$$|g_n(\alpha)| \leq 8 \|s\|_\infty |f_n(\alpha)| + 12 \|s\|_\infty \leq \max(16 \|s\|_\infty f_n(\alpha), 24 \|s\|_\infty),$$

else $|f_n(\alpha) - f_n(1)| \geq |\alpha| - 1$, therefore $|\alpha| \leq f_n(\alpha) + 1$, which yields

$$|g_n(\alpha)| \leq 20 \|s\|_\infty f_n(\alpha) + 12 \|s\|_\infty \leq \max(40 \|s\|_\infty f_n(\alpha), 24 \|s\|_\infty).$$

■

7.5 Construction of a Wiener process W such that $W \circ g_n$ approximates Z

Let E_y be the event of probability greater than $1 - e^{-y}$ on which the equations of claim 5 are satisfied. Let $x > 0$. Given $D_n^T \in E_y$, Z^1 is a piecewise linear gaussian process on the partition $([\frac{j}{\Delta}; \frac{j+1}{\Delta}])_{a_x \Delta \leq j \leq b_x \Delta}$, such that for any $j \in \{a_x \Delta, \dots, b_x \Delta\}$,

$$\max_{(j_1, j_2) \in \{0, \dots, b_x \Delta\}^2} |\text{Cov}(Z(\frac{j_1}{\Delta}), Z(\frac{j_2}{\Delta})) - K(g_n)(\frac{j_1}{\Delta}, \frac{j_2}{\Delta})| \leq \kappa_6 (1+x)^2 [y + \log n]^2 n^{-u_5}, \quad (51)$$

where $K(g_n)$ is given by definition 12. Since g_n is non-decreasing, $K(g_n)(s, t) = \text{Cov}(W_{g_n(s)}, W_{g_n(t)})$ for any two-sided Wiener process W on \mathbb{R} such that $W_0 = 0$. In particular, $K(g_n)$ is a positive-definite function. Furthermore, by definition, $\forall (\alpha_1, \alpha_2) \in [a_x; b_x]^2$,

$$K(g_n)(\alpha_1, \alpha_1) + K(g_n)(\alpha_2, \alpha_2) - 2K(g_n)(\alpha_1, \alpha_2) = |g_n(\alpha_2) - g_n(\alpha_1)|.$$

Moreover, for all $j \in \{a_x \Delta, \dots, b_x \Delta - 1\}$, since $n_v \epsilon \leq 1$,

$$\begin{aligned} |g_n(\frac{j+1}{\Delta}) - g_n(\frac{j}{\Delta})| &\leq 8 \|s\|_\infty |f_n(\frac{j+1}{\Delta}) - f_n(\frac{j}{\Delta})| + \frac{12 \|s\|_\infty}{\Delta} \text{ by equations (46) and (44)} \\ &\leq 8\kappa_3 \|s\|_\infty (1+x)^2 n^{-u_2} + 12 \|s\|_\infty \frac{n - n_t}{n_t} \text{ by claim 7} \\ &\leq 8\kappa_3 \|s\|_\infty (1+x)^2 n^{-u_2} + 12 \|s\|_\infty n^{-\delta_4} \text{ by hypothesis 4.} \end{aligned}$$

Finally, by lemma 7.4 and since g_n is non-decreasing,

$$\sup_{\alpha \in [a_x; b_x]} K(g_n)(\alpha, \alpha) \leq \max(|g_n(a_x)|, |g_n(b_x)|) \leq 20 \|s\|_\infty (1+x).$$

In this situation, proposition 8.9 in the appendix (applied to $Y = Z^1$, $K_X = K(g_n)$ with $h = g_n$) guarantees the existence of a continuous gaussian process $Z^2(D_n^T)$, with variance-covariance function $K(g_n)$ and such that for some constant κ and for $u = \min(u_5, u_2, \delta_4)$,

$$\forall y > 0, \forall D_n^T \in E_y, \mathbb{E} \left[\sup_{a_x \leq t \leq b_x} |Z^1(t) - Z^2(t)| |D_n^T \right] \leq \kappa (1+x)^{\frac{7}{6}} [y + \log n]^{\frac{2}{3}} \times n^{-\frac{u}{12}}. \quad (52)$$

Since the conditional distribution of $Z^2(D_n^T)$ given D_n^T is entirely determined by the function g_n which does not depend on D_n^T , Z^2 is independent from D_n^T . Moreover, since g_n increases, $W = Z^2 \circ g_n^{-1}$ is a continuous, centered gaussian process with covariance function

$$\text{Cov}(Z_s, Z_t) = K(g_n)(g_n^{-1}(s), g_n^{-1}(t)) = \begin{cases} s \wedge t & \text{if } 0 \leq s, t \\ -(s \vee t) & \text{if } s, t \leq 0 \\ 0 & \text{else,} \end{cases} \quad (53)$$

it is therefore a two-sided Wiener process on $[g_n(a_x); g_n(b_x)]$ taking value 0 at 0. W can be extended to \mathbb{R} by placing independent Wiener processes W_g, W_d on its left and on its right, by the equations $W(u) = W(g_n(a_x)) + W_g(u) - W_g(g_n(a_x))$

for $u < a_x$, $W(u) = W(g_n(b_x)) + W_d(u) - W_d(g_n(b_x))$ for $u > b_x$. Thus, by claim 4 and equation (52), with probability greater than $1 - 2e^{-y}$,

$$\begin{aligned} \mathbb{E} \left[\sup_{a_x \leq t \leq b_x} |Z(t) - W_{g_n(t)}| |D_n^T \right] &= \mathbb{E} \left[\sup_{a_x \leq t \leq b_x} |Z(t) - Z^2(t)| |D_n^T \right] \\ &\leq \mathbb{E} \left[\sup_{a_x \leq t \leq b_x} |Z^1(t) - Z(t)| |D_n^T \right] + \mathbb{E} \left[\sup_{a_x \leq t \leq b_x} |Z^1(t) - W_{g_n(t)}| |D_n^T \right] \\ &\leq \kappa(1+x)^{\frac{7}{6}} [y + \log n]^{\frac{2}{3}} n^{-\frac{u_2}{12}} + \kappa_5(c_1)(1+y)(1+x)^{\frac{3}{2}} n^{-\frac{\delta_5}{3}} \\ &\leq \kappa(1+y)(1+x)^{\frac{3}{2}} n^{-u}, \end{aligned}$$

for all $u < \min(\frac{u_5}{12}, \frac{u_2}{12}, \frac{\delta_4}{12}, \frac{\delta_5}{3})$ and a constant $\kappa(u)$. Finally, by claim 3, with probability greater than $1 - 3e^{-y}$,

$$\begin{aligned} &\sup_{\alpha \in [a_x; b_x]} \left| \hat{R}_T^{ho}(\alpha) - [f_n(\alpha) - W_{g_n(\alpha)}] \right| \\ &\leq \sup_{\alpha \in [a_x; b_x]} |L(\alpha) - f_n(\alpha)| + \sup_{\alpha \in [a_x; b_x]} |Z(\alpha) - W_{g_n(\alpha)}| \\ &\leq \kappa_1(1+x)[\log(n) + \log(2+x) + y]^2 n^{-\min(\frac{1}{12}, \frac{\delta_4}{2})} + \kappa(1+y)(1+x)^{\frac{3}{2}} n^{-u} \\ &\leq \kappa(1+y)^2(1+x)^{\frac{3}{2}} n^{-u_1}, \end{aligned}$$

for all $u_1 < \min(\frac{u_5}{12}, \frac{u_2}{12}, \frac{\delta_4}{12}, \frac{\delta_5}{3})$ and a constant κ . This proves Theorem 1.

8 Appendix

Lemma 8.1 *Let X be a random variable belonging to $[-1; 1]$, with pdf s . For all $j \in \mathbb{N}$, let $\theta_j = \langle s, \psi_j \rangle$. Then*

$$\begin{aligned} \text{Var}(\psi_j(X)) &\xrightarrow{j \rightarrow +\infty} 1 \\ \forall k_0 \leq k, \sum_{j=k_0}^k |\text{Var}(\psi_j) - 1| &\leq \|\theta\|_{\ell^1} = \sum_{j=0}^{+\infty} |\langle s, \psi_j \rangle|. \end{aligned}$$

Proof $\mathbb{E}[\psi_j(X)] = \int_0^1 \psi_j(x)s(x)dx = \theta_j$. Moreover, $\psi_j(X)^2 = 2 \cos^2(2\pi jX) = 1 + \cos(2\pi jX)$, therefore $\text{Var}(\cos(\pi jX)) = 1 + \frac{\theta_j}{\sqrt{2}} - \theta_j^2$, therefore since $|\theta_j| \leq \sqrt{2}$, $|\text{Var}(\cos(jX)) - 1| \leq \left| \sqrt{2} - \frac{1}{\sqrt{2}} \right| |\theta_j| \leq |\theta_j|$. \blacksquare

Lemma 8.2 *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a function, $g, h : \mathbb{R}_+ \rightarrow \mathbb{R}$ be two non-increasing functions. Then*

$$\inf_{x \in \mathbb{R}_+} \{f(x) + g(x) + h(x)\} \leq \inf_{x \in \mathbb{R}_+} \{f(x) + g(x)\} + \inf_{x \in \mathbb{R}_+} \{f(x) + h(x)\}.$$

Proof Let $\delta > 0$. Let x_g be such that $f(x_g) + g(x_g) \leq \delta + \inf_{x \in \mathbb{R}_+} \{f(x) + g(x)\}$. Let x_h be such that $f(x_h) + h(x_h) \leq \inf_{x \in \mathbb{R}_+} \{f(x) + h(x)\}$. Let $x_* = \max(x_g, x_h)$.

If $x_* = x_g$, then

$$\begin{aligned}
f(x_*) + g(x_*) + h(x_*) &\leq \inf_{x \in \mathbb{R}_+} \{f(x) + g(x)\} + \delta + h(x_*) \\
&\leq \inf_{x \in \mathbb{R}_+} \{f(x) + g(x)\} + \delta + h(x_h) \text{ since } h \text{ is non-increasing} \\
&\leq \inf_{x \in \mathbb{R}_+} \{f(x) + g(x)\} + \delta + f(x_h) + h(x_h) \\
&\leq \inf_{x \in \mathbb{R}_+} \{f(x) + g(x)\} + \inf_{x \in \mathbb{R}_+} \{f(x) + h(x)\} + 2\delta
\end{aligned}$$

Symmetrically, if $x_* = x_h$, then $f(x_*) + g(x_*) + h(x_*) \leq \inf_{x \in \mathbb{R}_+} \{f(x) + g(x)\} + \inf_{x \in \mathbb{R}_+} \{f(x) + h(x)\} + 2\delta$. As a result,

$$\begin{aligned}
\inf_{x \in \mathbb{R}_+} \{f(x) + g(x) + h(x)\} &\leq f(x_*) + g(x_*) + h(x_*) \leq \inf_{x \in \mathbb{R}_+} \{f(x) + g(x)\} \\
&\quad + \inf_{x \in \mathbb{R}_+} \{f(x) + h(x)\} + 2\delta.
\end{aligned}$$

Since no assumptions were made about $\delta > 0$, lemma 8.2 is proved. \blacksquare

Proposition 8.3 *For any integers $k_0 \leq k$, with probability greater than $1 - e^{-y}$:*

$$\left| \sum_{j=k_0+1}^k (\hat{\theta}_j^T - \theta_j)^2 - \frac{|k - k_0|}{n_t} \right| \leq \frac{\|\theta\|_{\ell^1}}{n_t} + C\sqrt{y + \log n} \left[\frac{\sqrt{|k - k_0|}}{n_t} + \frac{|k - k_0|}{n_t^{\frac{5}{4}}} \right].$$

In particular, there exists a constant $\kappa_1 = \kappa_1(\|s\|_{\infty}, c_1, \|\theta\|_{\ell^1})$ such that for any α_1, α_2 such that $(\alpha_1\Delta, \alpha_2\Delta) \in \mathbb{N}^2$ and $\alpha_1 < \alpha_2$, with probability greater than $1 - e^{-y}$,

$$\left| \sum_{j=k_*+\alpha_1\Delta}^{k_*+\alpha_2\Delta} (\hat{\theta}_j^T - \theta_j)^2 - [\alpha_2 - \alpha_1]\mathcal{E} \right| \leq \kappa_1(\alpha_2 - \alpha_1)[\log n + y]^2 \times n^{-\min(\frac{1}{12}, \frac{5}{2})} \mathbf{e}(n). \tag{54}$$

Proof Let $(k_0, k) \in \mathbb{N}^2$ be such that $k_0 < k$. The proof rests on lemma 14 of Arlot and Lerasle [2016] applied to $S_m = \langle \psi_{k_0+1}, \dots, \psi_k \rangle$. Let us compute $b_m = \sup_{u \in \mathbb{R}^{|k-k_0|}: \|u\| \leq 1} \sum_{j=k_0}^k u_j \psi_j(x) \leq \sup_x \sqrt{\sum_{j=k_0}^k \psi_j^2(x)} \leq \sqrt{|k - k_0|}$ and

$$\mathcal{D}_k = \sum_{j=k_0+1}^k \text{Var}(\psi_j(X)) = |k - k_0| \pm \frac{\|\theta\|_{\ell^1}}{n_t}$$

(by lemma 8.1). Furthermore, $\mathcal{D}_k \leq \sqrt{2}|k - k_0|$ since $\psi_j = \sqrt{2} \cos(2\pi j \cdot) : [0; 1] \rightarrow [-\sqrt{2}; \sqrt{2}]$. By [Arlot and Lerasle, 2016, lemma 14], with probability greater than $1 - e^{-y}$, for any $\varepsilon > 0$,

$$\left| \sum_{j=k_0+1}^k (\hat{\theta}_j^T - \theta_j)^2 - \frac{\mathcal{D}_k}{n_t} \right| \leq \varepsilon \frac{\mathcal{D}_k}{n_t} + \kappa \left(\frac{\|s\|_{\infty} [\log n + y]}{(\varepsilon \wedge 1)n_t} + \frac{|k - k_0| [\log n + y]^2}{(\varepsilon \wedge 1)^3 n_t^2} \right).$$

Let $\varepsilon_1 = \sqrt{\frac{\|s\|_\infty (\log n + y)}{|k - k_0|}} \wedge 1$. If $\varepsilon_1 = 1$, then $|k - k_0| \leq \|s\|_\infty (y + \log n)$ therefore $\varepsilon_1 \frac{|k - k_0|}{n_t} + \kappa \frac{\|s\|_\infty [\log n + y]}{(\varepsilon_1 \wedge 1) n_t} \leq (1 + \kappa) \frac{\|s\|_\infty (y + \log n)}{n_t}$. If $\varepsilon_1 < 1$, then $\varepsilon_1 \frac{|k - k_0|}{n_t} + \kappa \frac{\|s\|_\infty [\log n + y]}{(\varepsilon_1 \wedge 1) n_t} = (1 + \kappa) \sqrt{\|s\|_\infty (y + \log n)} \frac{\sqrt{|k - k_0|}}{n_t}$. In all cases, if $k > k_0$,

$$\varepsilon_1 \frac{|k - k_0|}{n_t} + \kappa \frac{\|s\|_\infty [\log n + y]}{(\varepsilon_1 \wedge 1) n_t} \leq (1 + \kappa) \|s\|_\infty (y + \log n) \frac{\sqrt{|k - k_0|}}{n_t}. \quad (55)$$

Let $\varepsilon_2 = \frac{\sqrt{\log n + y}}{n_t^{\frac{1}{4}}} \wedge 1$. If $\frac{\sqrt{y + \log n}}{n_t^{\frac{1}{4}}} \geq 1 = \varepsilon_2$, then $\varepsilon_2 \frac{|k - k_0|}{n_t} + \kappa \frac{|k - k_0| [\log n + y]^2}{(\varepsilon_2 \wedge 1)^3 n_t^2} \leq \sqrt{y + \log n} \frac{|k - k_0|}{n_t^{\frac{5}{4}}} + \kappa \frac{|k - k_0| (y + \log n)^2}{n_t^2} \leq (1 + \kappa) (y + \log n)^2 \frac{|k - k_0|}{n_t^{\frac{5}{4}}}$. If $\varepsilon_2 = \frac{\sqrt{y + \log n}}{n_t^{\frac{1}{4}}} < 1$, then

$$\begin{aligned} \varepsilon_2 \frac{|k - k_0|}{n_t} + \kappa \frac{|k - k_0| [\log n + y]^2}{(\varepsilon_2 \wedge 1)^3 n_t^2} &= \sqrt{y + \log n} \frac{|k - k_0|}{n_t^{\frac{5}{4}}} + \kappa (y + \log n)^2 \frac{|k - k_0|}{n_t^2} \frac{n_t^{\frac{3}{4}}}{(y + \log n)^{\frac{3}{2}}} \\ &\leq (1 + \kappa) \sqrt{y + \log n} \frac{|k - k_0|}{n_t^{\frac{5}{4}}}. \end{aligned}$$

In all cases,

$$\varepsilon_2 \frac{|k - k_0|}{n_t} + \kappa \frac{|k - k_0| [\log n + y]^2}{(\varepsilon_2 \wedge 1)^3 n_t^2} \leq (1 + \kappa) (y + \log n)^2 \frac{|k - k_0|}{n_t^{\frac{5}{4}}}. \quad (56)$$

By lemma 8.2,

$$\begin{aligned} \left| \sum_{j=k_0+1}^k (\hat{\theta}_j^T - \theta_j)^2 - \frac{\mathcal{D}_k}{n_t} \right| &\leq \inf_{\varepsilon \geq 0} \left\{ \varepsilon \frac{\mathcal{D}_k}{n_t} + \kappa \frac{\|s\|_\infty [\log n + y]}{(\varepsilon \wedge 1) n_t} \right\} \\ &\quad + \inf_{\varepsilon \geq 0} \left\{ \varepsilon \frac{\mathcal{D}_k}{n_t} + \kappa \frac{|k - k_0| [\log n + y]^2}{(\varepsilon \wedge 1)^3 n_t^2} \right\} \\ &\leq \varepsilon_1 \frac{|k - k_0|}{n_t} + \kappa \frac{\|s\|_\infty [\log n + y]}{(\varepsilon_1 \wedge 1) n_t} + \varepsilon_2 \frac{|k - k_0|}{n_t} \\ &\quad + \kappa \frac{|k - k_0| [\log n + y]^2}{(\varepsilon_2 \wedge 1)^3 n_t^2} + (\varepsilon_1 + \varepsilon_2) \frac{\|\theta\|_{\ell^1}}{n_t} \\ &\leq (1 + \kappa) \|s\|_\infty (y + \log n) \frac{\sqrt{|k - k_0|}}{n_t} \\ &\quad + (1 + \kappa) (y + \log n)^2 \frac{|k - k_0|}{n_t^{\frac{5}{4}}} + \frac{2 \|\theta\|_{\ell^1}}{n_t}, \end{aligned}$$

by equations (55), (56). In conclusion, on an event E_y of probability greater than $1 - e^{-y}$,

$$\begin{aligned} \left| \sum_{j=k_0+1}^k (\hat{\theta}_j^T - \theta_j)^2 - \frac{|k - k_0|}{n_t} \right| &\leq \left| \sum_{j=k_0+1}^k (\hat{\theta}_j^T - \theta_j)^2 - \frac{\mathcal{D}_k}{n_t} \right| + \frac{\|\theta\|_{\ell^1}}{n_t} \\ &\leq \frac{3 \|\theta\|_{\ell^1}}{n_t} + (1 + \kappa) \|s\|_\infty (y + \log n) \\ &\quad \times \left[\frac{\sqrt{|k - k_0|}}{n_t} + (y + \log n) \frac{|k - k_0|}{n_t^{\frac{5}{4}}} \right]. \quad (57) \end{aligned}$$

If $k_0 = k_* + \alpha_1 \Delta$ and $k = k_* + \alpha_2 \Delta$, then by hypothesis 4 of Theorem 1,

$$\frac{\sqrt{|k - k_0|}}{n_t} = \sqrt{\alpha_2 - \alpha_1} \sqrt{\frac{\Delta}{n_v n_t}} \sqrt{\frac{n_v}{n_t}} = \sqrt{\alpha_2 - \alpha_1} \sqrt{\frac{n - n_t}{n_t}} \mathbf{e} \leq \sqrt{\alpha_2 - \alpha_1} n^{-\frac{\delta_4}{2}} \mathbf{e}. \quad (58)$$

Furthermore,

$$\begin{aligned} \frac{|k - k_0|}{n_t^{\frac{5}{4}}} &= (\alpha_2 - \alpha_1) \frac{\mathcal{E}}{n_t^{\frac{1}{4}}} \\ &= (\alpha_2 - \alpha_1) \sqrt{\frac{\mathcal{E}}{n_v} \frac{\sqrt{n_v \mathcal{E}}}{n_t^{\frac{1}{4}}}} \\ &\leq (\alpha_2 - \alpha_1) \mathbf{e} \frac{\sqrt{2n_v \text{or}(n_t) + 1}}{n_t^{\frac{1}{4}}}. \end{aligned}$$

Let $k_1 = \lceil n_t^{\frac{1}{3+\delta_1}} \rceil$, so that $n_t^{\frac{1}{3+\delta_1}} \leq k_1 \leq 2n_t^{\frac{1}{3+\delta_1}}$. By hypothesis 1 of Theorem 1, $\sum_{j=k+1}^{+\infty} \theta_j^2 \leq \frac{c_1}{k^{2+\delta_1}}$ therefore

$$\text{or}(n_t) \leq \inf_{k \in \mathbb{N}^*} \frac{c_1}{k^{2+\delta_1}} + \frac{k}{n_t} \leq \frac{c_1}{k_1^{2+\delta_1}} + \frac{k_1}{n_t} \leq \frac{c_1}{n_t^{\frac{2}{3+\delta_1}}} + \frac{2n_t^{\frac{1}{3+\delta_1}}}{n_t} \leq \frac{2 + c_1}{n_t^{\frac{2}{3+\delta_1}}}.$$

Thus $1 + 2n_v \text{or}(n_t) \leq 1 + 2n_t \text{or}(n_t) \leq (5 + 2c_1) n_t^{\frac{1+\delta_1}{3+\delta_1}}$, hence

$$\begin{aligned} \frac{|k - k_0|}{n_t^{\frac{5}{4}}} &\leq (\alpha_2 - \alpha_1) \mathbf{e} \sqrt{5 + 2c_1} \frac{n_t^{\frac{1+\delta_1}{\delta_1+2\delta_1}}}{n_t^{\frac{1}{4}}} \\ &\leq (\alpha_2 - \alpha_1) \sqrt{5 + 2c_1} n_t^{-\frac{1}{12}} \mathbf{e} \\ &\leq (\alpha_2 - \alpha_1) \sqrt{5 + 2c_1} \frac{2^{\frac{1}{12}}}{n^{\frac{1}{12}}} \mathbf{e}. \end{aligned} \quad (59)$$

Finally, $\frac{\|\theta\|_{\ell^1}}{n_t} = \frac{n_v}{n_t} \frac{\|\theta\|_{\ell^1}}{n_v} \leq \|\theta\|_{\ell^1} \frac{n - n_t}{n_t} \mathbf{e} \leq \|\theta\|_{\ell^1} n^{-\delta_4}$. Equation (54) follows from equations (57), (58) and (59). ■

Lemme 8.4 *Let $(c_{i,j})_{(i,j) \in \mathbb{N}^2}$ be real coefficients. Let $I_1, I_2 \subset \mathbb{N}$ be two finite sets. Let $(\theta_j)_{j \in \mathbb{N}}$ be a sequence. Let $C = \max \left\{ \sup_{i \in I_1} \sum_{j \in I_2} |c_{i,j}|, \sup_{i \in I_2} \sum_{j \in I_1} |c_{i,j}| \right\}$. Then*

$$\sum_{i \in I_1} \left(\sum_{j \in I_2} c_{i,j} \theta_j \right)^2 \leq C^2 \sum_{j \in I_2} \theta_j^2$$

and

$$\left| \sum_{i \in I_1} \sum_{j \in I_2} \theta_i \theta_j c_{i,j} \right| \leq C \max \left\{ \sum_{i \in I_1} \theta_i^2, \sum_{j \in I_2} \theta_j^2 \right\}.$$

Proof Let $C_i = \sum_{j \in I_2} |c_{i,j}|$. Then

$$\begin{aligned}
\sum_{i \in I_1} \left(\sum_{j \in I_2} c_{i,j} \theta_j \right)^2 &= \sum_{i \in I_1} C_i^2 \left(\frac{1}{C_i} \sum_{j \in I_2} \operatorname{sgn}(c_{i,j}) |c_{i,j}| \theta_j \right)^2 \\
&\leq \sum_{i \in I_1} \frac{C_i^2}{C_i} \sum_{j \in I_2} |c_{i,j}| \theta_j^2 \text{ by l'inégalité of Jensen} \\
&\leq \left(\max_{i \in I_1} C_i \right) \sum_{j \in I_2} \theta_j^2 \sum_{i \in I_1} |c_{i,j}| \\
&\leq C^2 \sum_{j \in I_2} \theta_j^2.
\end{aligned}$$

This proves the first equation. Furthermore,

$$\begin{aligned}
\left| \sum_{i \in I_1} \sum_{j \in I_2} \theta_i \theta_j c_{i,j} \right| &\leq \sum_{i \in I_1} \sum_{j \in I_2} \frac{\theta_i^2 + \theta_j^2}{2} |c_{i,j}| \\
&= \frac{1}{2} \sum_{i \in I_1} \theta_i^2 \sum_{j \in I_2} |c_{i,j}| + \frac{1}{2} \sum_{j \in I_2} \theta_j^2 \sum_{i \in I_1} |c_{i,j}| \\
&\leq C \max \left\{ \sum_{i \in I_1} \theta_i^2, \sum_{j \in I_2} \theta_j^2 \right\},
\end{aligned}$$

which proves the second equation. ■

Lemma 8.5 *Under the assumptions of Theorem 1, there exists a constant $\kappa(c_1, c_2) > 0$ such that for any $x \geq 0$,*

$$k_* + a_x \Delta \geq \frac{\kappa}{(1+x)^{\frac{1}{\delta_2}}} n_t^{\frac{2}{3\delta_2}}.$$

Proof By hypothesis 2 of Theorem 1,

$$\begin{aligned}
c_2(k_* + a_x \Delta)^{-\delta_2} &\leq \sum_{j=k_*+a_x \Delta+1}^{+\infty} \theta_j^2 \\
&\leq \sum_{j=k_*+a_x \Delta+1}^{k_*} \left[\theta_j^2 - \frac{1}{n_t} \right] + |a_x| \mathcal{E} + \sum_{j=k_*+1}^{+\infty} \theta_j^2 \\
&\leq \epsilon f_n(a_x) + |a_x| \mathcal{E} + \operatorname{or}(n_t). \tag{60}
\end{aligned}$$

By definition, $f_n(a_x) \leq x$ and by lemma 7.1, $|a_x| \leq 2(1+x)$. Furthermore, by lemma 3.1, $\epsilon \leq \mathcal{E} \leq 2\operatorname{or}(n_t) + \frac{1}{n_v}$. Since by hypothesis 5 of Theorem 1, $n_v \geq n^{\frac{2}{3}+\delta_5}$, it follows that: $\mathcal{E} \leq 2\operatorname{or}(n_t) + \frac{1}{n^{\frac{2}{3}+\delta_5}}$. Equation (60) thus yields

$$c_2(k_* + a_x \Delta)^{-\delta_2} \leq 6(1+x) \left[\operatorname{or}(n_t) + \frac{1}{n^{\frac{2}{3}}} \right].$$

On the other hand, by hypothesis 1 of Theorem 1,

$$\text{or}(n_t) \leq \min_{k \in \mathbb{N}} \frac{c_1}{k^2} + \frac{k}{2n_t} \leq \frac{3c_1^{\frac{1}{3}}}{n_t^{\frac{2}{3}}}.$$

It follows finally that, for some constant $\kappa(c_1, c_2)$,

$$k_* + a_x \Delta \geq \frac{\kappa}{(1+x)^{\frac{1}{\delta_2}}} n_t^{\frac{2}{3\delta_2}}.$$

■

Claim 7 Let $u_2 = \min\left(\frac{2\delta_3}{3\delta_2}, \delta_4\right)$. Let x be a non-negative real number. Let a_x, b_x be such that $a_x \leq 0 \leq b_x$ and $\max(f_n(a_x), f_n(b_x)) \leq x$. Assume also that $a_x \Delta - 1 \geq \frac{-k_*}{\Delta}$. There exists a constant $\kappa_3 \geq 0$ such that for all $j \in [a_x \Delta; b_x \Delta + 1]$,

$$\left|f_n\left(\frac{j}{\Delta}\right) - f_n\left(\frac{j-1}{\Delta}\right)\right| \leq \kappa_3(1+x)^2 n^{-u_2} \quad (61)$$

$$\theta_{k_*+j}^2 \leq \kappa_3(1+x)^2 n^{-u_2} \mathbf{e}. \quad (62)$$

Proof By hypothesis 3 of Theorem 1, for all $k \geq 1$, $\theta_{k+k^{\delta_3}}^2 \geq c_3 \theta_{k-k^{\delta_3}}^2$. Thus, for all $k \geq 1$ and any $j \in [k - k^{\delta_3}; k + k^{\delta_3}]$,

$$\begin{aligned} \max\left(\theta_k^2, \frac{1}{n_t}\right) &\leq \max\left(c_3 \theta_j^2, \frac{1}{n_t}\right) \\ &\leq \frac{1+c_3}{n_t} + c_3 \left|\theta_j^2 - \frac{1}{n_t}\right|. \end{aligned} \quad (63)$$

Let $k \in [k_* + a_x \Delta; k_* + b_x \Delta + 1]$. Assume without loss of generality (up to a change in the constant κ_9) that $x \geq 1$. Thus by lemma 3.2, $\max(-a_x, b_x) \geq 1$.

- If $|b_x| \geq 1$, then two cases can be distinguished.

– If $k \leq k_* + \frac{\Delta}{2}$, then $k + k^{\delta_3} \wedge \frac{\Delta}{2} \leq k_* + \Delta \leq k_* + b_x \Delta$, therefore by definition of a_x, b_x ,

$$2x\mathbf{e} \geq \mathbf{e}[f_n(a_x) + f_n(b_x)] = \sum_{j=k_*+a_x\Delta+1}^{k_*+b_x\Delta} \left|\theta_j^2 - \frac{1}{n_t}\right| \geq \sum_{j=k+1}^{k+k^{\delta_3} \wedge \frac{\Delta}{2}} \left|\theta_j^2 - \frac{1}{n_t}\right|.$$

– If $k_* + \frac{\Delta}{2} < k \leq k_* + b_x \Delta + 1$, then $k - k^{\delta_3} \wedge \frac{\Delta}{2} \geq k_*$, therefore

$$2x\mathbf{e} \geq \sum_{j=k_*+a_x\Delta+1}^{k_*+b_x\Delta} \left|\theta_j^2 - \frac{1}{n_t}\right| \geq \sum_{j=k-k^{\delta_3} \wedge \frac{\Delta}{2}}^{k-1} \left|\theta_j^2 - \frac{1}{n_t}\right|.$$

- If $|a_x| \geq 1$, then we likewise consider two possibilities.

– If $k > k_* - \frac{\Delta}{2}$, then $k - k^{\delta_3} \wedge \frac{\Delta}{2} > k_* - \Delta \geq k_* + a_x \Delta$, therefore by definition of a_x, b_x ,

$$2x\mathbf{e} \geq \sum_{j=k_*+a_x\Delta+1}^{k_*+b_x\Delta} \left|\theta_j^2 - \frac{1}{n_t}\right| \geq \sum_{j=k-k^{\delta_3} \wedge \frac{\Delta}{2}}^{k-1} \left|\theta_j^2 - \frac{1}{n_t}\right|.$$

– If $k \leq k_* - \frac{\Delta}{2}$, then $k + k^{\delta_3} \wedge \frac{\Delta}{2} \leq k_*$, therefore

$$2x\epsilon \geq \sum_{j=k_*+a_x\Delta+1}^{k_*+b_x\Delta} \left| \theta_j^2 - \frac{1}{n_t} \right| \geq \sum_{j=k+1}^{k+k^{\delta_3} \wedge \frac{\Delta}{2}} \left| \theta_j^2 - \frac{1}{n_t} \right|.$$

In all cases, by equation (63),

$$\left(k^{\delta_3} \wedge \frac{\Delta}{2} \right) \max \left(\theta_k^2, \frac{1}{n_t} \right) \leq k^{\delta_3} \wedge \frac{\Delta}{2} \frac{1+c_3}{n_t} + 2x\epsilon,$$

in other words

$$\max \left(\theta_k^2, \frac{1}{n_t} \right) \leq \frac{1+c_3}{n_t} + \frac{2x\epsilon}{k^{\delta_3} \wedge \frac{\Delta}{2}}.$$

Furthermore, by hypothesis 4 of Theorem 1, $\Delta \geq \frac{n_t}{n-n_t} \geq n^{\delta_4}$, and by lemma 8.5,

$$k^{\delta_3} \geq (k_* + a_x\Delta)^{\delta_3} \geq \frac{\kappa}{(1+x)^{\frac{\delta_3}{\delta_2}}} n_t^{\frac{2\delta_3}{3\delta_2}}.$$

Let $u_2 = \min \left(\frac{2\delta_3}{3\delta_2}, \delta_4 \right)$. Since $\delta_3 \leq \delta_2$, there exists therefore a constant κ such that

$$\max \left(\theta_k^2, \frac{1}{n_t} \right) \leq \kappa(1+x)^2 n^{-u_2} \epsilon.$$

In conclusion, for all $j \in \{a_x\Delta, \dots, b_x\Delta + 1\}$,

$$\begin{aligned} \theta_{k_*+j}^2 &\leq \max \left(\theta_{k_*+j}^2, \frac{1}{n_t} \right) \leq \kappa(1+x)^2 n^{-u_2} \epsilon \\ |f_n \left(\frac{j}{\Delta} \right) - f_n \left(\frac{j-1}{\Delta} \right)| &= \frac{1}{\epsilon} \left| \theta_{j+k_*}^2 - \frac{1}{n_t} \right| \\ &\leq \frac{1}{\epsilon} \max \left(\theta_{k_*+j}^2, \frac{1}{n_t} \right) \\ &\leq \kappa(1+x)^2 n^{-u_2}. \end{aligned}$$

This proves claim 7. ■

Proposition 8.6 *Let P be the probability measure with pdf s on $[0; 1]$. Let $\theta_j = \langle s, \psi_j \rangle = P(\psi_j)$ and $\theta_j^2 = \theta_j^2$, and assume that they satisfy the hypotheses of Theorem 1. Let $\hat{\theta}_j^T = P^T \psi_j$. Let $I_k^1, I_k^2 \subset \{k_* + a_x\Delta + 1, \dots, k_* + b_x\Delta\}$ be two intervals. Then the statistics*

$$U_{I_k^1, I_k^2} = \sum_{i \in I_k^1} \sum_{j \in I_k^2} \hat{\theta}_i^T \hat{\theta}_j^T [P(\psi_i \psi_j) - P\psi_i P\psi_j]$$

can be approximated in the following way. There exists two constants κ_4 and $u_3 > 0$ such that, with probability greater than $1 - e^{-y}$,

$$\begin{aligned} U_{I_k^1, I_k^2} &= \frac{1}{2} \frac{|I_k^1 \cap I_k^2|}{n_t} + \left(1 - \frac{1}{\sqrt{2}} \right) \sum_{i \in I_k^1 \cap I_k^2} \theta_i^2 + \frac{1}{\sqrt{2}} \sum_{i \in I_k^1} \sum_{j \in I_k^2} \theta_i \theta_j \theta_{|i-j|} + \frac{1}{2n_t} \sum_{i \in I_k^1} \sum_{j \in I_k^2} \theta_{|i-j|}^2 \\ &\pm \kappa_4 (y + \log n)^2 (1+x) n^{-u_3} \mathcal{E}. \end{aligned}$$

Proof First, by lemma 7.1,

$$\max \left(\sum_{i \in I_k^1} \theta_i^2, \sum_{j \in I_k^2} \theta_j^2 \right) \leq \sum_{j=k_*+a_x\Delta+1}^{k_*+b_x\Delta} \theta_j^2 \leq 4(1+x)\mathcal{E}. \quad (64)$$

Let $c_{i,j} = \frac{\theta_{i+j}}{\sqrt{2}} + \left(\frac{1-\delta_{i,j}}{\sqrt{2}} + \delta_{i,j} \right) \theta_{|i-j|} - \theta_i \theta_j$. $U_{I_k^1, I_k^2}$ can be expressed as the sum of 6 terms: $U_{I_k^1, I_k^2} = V_1 + V_2 + V_3 + V_4 + V_5 + V_6$, where

$$\begin{aligned} V_1 &= \sum_{i \in I_k^1} \sum_{j \in I_k^2} \theta_i \theta_j \left[\frac{\theta_{i+j}}{\sqrt{2}} + \left(\frac{1-\delta_{i,j}}{\sqrt{2}} + \delta_{i,j} \right) \theta_{|i-j|} - \theta_i \theta_j \right] \\ V_2 &= (P^T - P) \sum_{i \in I_k^1} \psi_i \sum_{j \in I_k^2} \theta_j c_{i,j} \\ V_3 &= (P^T - P) \sum_{j \in I_k^2} \psi_j \sum_{i \in I_k^1} \theta_i c_{i,j} \\ V_4 &= \frac{1}{\sqrt{2}} \sum_{i \in I_k^1} \sum_{j \in I_k^2} (P^T - P) \psi_i (P^T - P) \psi_j \theta_{|i-j|} \\ V_5 &= \left(1 - \frac{1}{\sqrt{2}} \right) \sum_{j \in I_k^1 \cap I_k^2} \left(\hat{\theta}_j^T - \theta_j \right)^2 \\ V_6 &= \sum_{i \in I_k^1} \sum_{j \in I_k^2} (P^T - P) \psi_i (P^T - P) \psi_j \left[\frac{\theta_{i+j}}{\sqrt{2}} - \theta_i \theta_j \right] \end{aligned}$$

The first term is

$$V_1 = \left(1 - \frac{1}{\sqrt{2}} \right) \sum_{i \in I_k^1 \cap I_k^2} \theta_i^2 + \frac{1}{\sqrt{2}} \sum_{i \in I_k^1} \sum_{j \in I_k^2} \theta_i \theta_j \theta_{|i-j|} + \sum_{i \in I_k^1} \sum_{j \in I_k^2} \theta_i \theta_j \left[\frac{\theta_{i+j}}{\sqrt{2}} - \theta_i \theta_j \right].$$

For all $i \in I_k^1$,

$$\sum_{j \in I_k^2} \frac{|\theta_{i+j}|}{\sqrt{2}} + |\theta_i| |\theta_j| \leq 2 \sum_{j \geq k_*+a_x\Delta+1} |\theta_j|.$$

Furthermore, for all $k \geq 2$, by hypothesis 1 of Theorem 1,

$$\sum_{j \geq k} |\theta_j| \leq \sum_{j=k}^{+\infty} \sqrt{\sum_{i=j}^{+\infty} \theta_i^2} \leq \sum_{j=k}^{+\infty} \frac{c_1}{(j-1)^{1+\frac{\delta_1}{2}}} \leq \frac{2c_1}{\delta_1} (k-1)^{-\frac{\delta_1}{2}}. \quad (65)$$

Since $k_* + a_x\Delta \geq \frac{\kappa}{(1+x)^{\frac{1}{\delta_2}}} n_t^{\frac{2}{3\delta_2}}$ by lemma 8.5, there is a constant $\kappa(c_1, c_2)$ such that

$$\sum_{j \in I_k^2} \frac{|\theta_{i+j}|}{\sqrt{2}} + |\theta_i| |\theta_j| \leq \kappa \frac{(1+x)^{\frac{\delta_1}{2\delta_2}}}{n_t^{\frac{\delta_1}{3\delta_2}}}. \quad (66)$$

The same argument applies to $\sum_{i \in I_k^1} \frac{|\theta_{i+j}|}{\sqrt{2}} + |\theta_i| |\theta_j|$. Thus, by lemma 8.4,

$$\sum_{i \in I_k^1} \sum_{j \in I_k^2} \theta_i \theta_j [\theta_{i+j} - \theta_i \theta_j] \leq 2\kappa \frac{(1+x)^{\frac{\delta_1}{2\delta_2}}}{n_t^{\frac{\delta_1}{3\delta_2}}} \left[\sum_{i \in I_k^1} \theta_i^2 + \sum_{j \in I_k^2} \theta_j^2 \right].$$

By equation (64), it follows that for a certain constant $\kappa(c_1, c_2)$,

$$\sum_{i \in I_k^1} \sum_{j \in I_k^2} \theta_i \theta_j \left[\frac{\theta_{i+j}}{\sqrt{2}} - \theta_i \theta_j \right] \leq \kappa \frac{(1+x)^{1+\frac{\delta_1}{2\delta_2}}}{n_t^{\frac{\delta_1}{3\delta_2}}} \mathcal{E}.$$

Thus

$$V_1 = \left(1 - \frac{1}{\sqrt{2}}\right) \sum_{i \in I_k^1 \cap I_k^2} \theta_i^2 + \frac{1}{\sqrt{2}} \sum_{i \in I_k^1} \sum_{j \in I_k^2} \theta_i \theta_j \theta_{|i-j|} \pm \kappa \frac{(1+x)^{1+\frac{\delta_1}{2\delta_2}}}{n_t^{\frac{\delta_1}{3\delta_2}}} \mathcal{E} \quad (67)$$

Bernstein's inequality applies to V_2 and V_3 . By symmetry, let us only consider V_2 . Its variance satisfies the following inequality.

$$\begin{aligned} \text{Var} \left(\sum_{i \in I_k^1} \psi_i \sum_{j \in I_k^2} \theta_j c_{i,j} \right) &\leq \|s\|_\infty \left\| \sum_{i \in I_k^1} \psi_i \sum_{j \in I_k^2} \theta_j c_{i,j} \right\|^2 \\ &\leq \|s\|_\infty \sum_{i \in I_k^1} \left(\sum_{j \in I_k^2} \theta_j c_{i,j} \right)^2. \end{aligned}$$

Let us now apply lemma 8.4. For all $i \in I_k^1$,

$$\begin{aligned} \sum_{j \in I_k^2} |c_{i,j}| &\leq \frac{1}{\sqrt{2}} \sum_{j \in I_k^2} |\theta_{i+j}| + \frac{1}{\sqrt{2}} \sum_{j \in I_k^2} |\theta_{|i-j|}| + |\theta_i| \sum_{j \in I_k^2} |\theta_j| \\ &\leq \left(\sqrt{2} + \sup_{i \in \mathbb{N}} |\theta_i| \right) \sum_{r \in \mathbb{N}} |\theta_r| \\ &\leq 3 \|\theta\|_{\ell^1} \end{aligned}$$

In the same way, for all $j \in I_k^2$, $\sum_{i \in I_k^1} |c_{i,j}| \leq 3 \|\theta\|_{\ell^1}$, hence by lemma 8.4,

$$\begin{aligned} \text{Var} \left(\sum_{i \in I_k^1} \psi_i \sum_{j \in I_k^2} \theta_j c_{i,j} \right) &\leq 3 \|\theta\|_{\ell^1} \|s\|_\infty \sum_{j \in I_k^2} \theta_j^2 \\ &\leq 12 \|\theta\|_{\ell^1} \|s\|_\infty (1+x) \mathcal{E} \text{ by equation (64)}. \quad (68) \end{aligned}$$

As for the upper bound on the uniform norm, it follows from lemma 8.4 and the elementary upper bound $\|\psi_i\|_\infty \leq \sqrt{2}$ that

$$\begin{aligned} \sup_{x \in \mathbb{R}} \left| \sum_{i \in I_k^1} \psi_i(x) \sum_{j \in I_k^2} \theta_j c_{i,j} \right| &\leq \sqrt{\sum_{i \in I_k^1} \left(\sum_{j \in I_k^2} \theta_j c_{i,j} \right)^2} \sup_{x \in \mathbb{R}} \sqrt{\sum_{i \in I_k^1} \psi_i(x)^2} \\ &\leq 3 \|\theta\|_{\ell^1} \sqrt{2|I_k^1|} \sqrt{\sum_{j \in I_k^2} \theta_j^2} \\ &\leq 3 \|\theta\|_{\ell^1} \sqrt{2(b_x - a_x) \Delta} \sqrt{4(1+x) \mathcal{E}} \\ &\leq \kappa(1+x) \sqrt{\Delta \mathcal{E}} \text{ by lemma 7.1,} \quad (69) \end{aligned}$$

for some constant $\kappa = \kappa(\|\theta\|_{\ell^1})$. By Bernstein's inequality, there exists an event $E_2(y) \subset \mathbb{R}^{n_t}$ with probability $\mathbb{P}(D_n^T \in E_2(y)) \geq 1 - e^{-y}$ such that, for any $D_n^T \in E_2(y)$,

$$\begin{aligned} |V_2| &\leq \sqrt{\frac{2y}{n_t}} \sqrt{\text{Var} \left(\sum_{i \in I_k^1} \psi_i \sum_{j \in I_k^2} \theta_j c_{i,j} \right)} + \frac{y}{3n_t} \sup_{x \in \mathbb{R}} \left| \sum_{i \in I_k^1} \psi_i(x) \sum_{j \in I_k^2} \theta_j c_{i,j} \right| \\ &\leq \sqrt{24 \|\theta\|_{\ell^1} \|s\|_{\infty}} \sqrt{\frac{y(1+x)\mathcal{E}}{n_t}} + \frac{\kappa y}{3n_t} (1+x) \sqrt{\Delta \mathcal{E}} \text{ by (68), (69)}. \end{aligned}$$

Setting $\kappa = \max(\sqrt{24 \|\theta\|_{\ell^1} \|s\|_{\infty}}, \frac{\kappa}{3})$, it follows that on $E_2(y)$,

$$|V_2| \leq \kappa \sqrt{y(1+x)} \sqrt{\frac{n_v}{n_t}} \mathbf{e} + \kappa y (1+x) \sqrt{\frac{n_v}{n_t}} \mathbf{e} \sqrt{\mathcal{E}}.$$

By lemma 3.1, \mathcal{E} is uniformly bounded: $\mathcal{E} \leq 2 \sum_{j=1}^{n_t} \theta_j^2 + \frac{1}{n-n_t} \leq 1 + 2 \|s\|^2 \leq 1 + 2 \|s\|_{\infty}$. Furthermore, by hypothesis 4 of Theorem 1, $\sqrt{\frac{n_v}{n_t}} = \sqrt{\frac{n-n_t}{n_t}} \leq n^{-\frac{\delta_4}{2}}$. Thus, there exists a constant $\kappa(\|\theta\|_{\ell^1}, \|s\|_{\infty})$ such that, on $E_2(y)$,

$$|V_2| \leq \kappa y (1+x) n^{-\frac{\delta_4}{2}} \mathbf{e}. \quad (70)$$

Symmetrically, there exists an event $E_3(y)$ of probability greater than $1 - e^{-y}$, such that for any $D_n^T \in E_3(y)$,

$$|V_3| \leq \kappa y (1+x) n^{-\frac{\delta_4}{2}} \mathbf{e}. \quad (71)$$

Now consider V_4 . This term can be expressed as a finite sum of sums of squares:

$$\begin{aligned} V_4 &= \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z}} \sum_{i \in I_k^1 \cap (I_k^2 - r)} (P^T - P) \psi_i (P^T - P) \psi_{i+r} \theta_{|r|} \\ &= \frac{1}{4\sqrt{2}} \sum_{r \in \mathbb{Z}} \theta_{|r|} \sum_{i \in I_k^1 \cap (I_k^2 - r)} [(P^T - P)(\psi_i + \psi_{i+r})]^2 - [(P^T - P)(\psi_i - \psi_{i+r})]^2. \end{aligned}$$

Let $J_0 = \{j \in \mathbb{N} : \lfloor \frac{j}{r} \rfloor \text{ is even}\}$ and $J_1 = \{j \in \mathbb{N} : \lfloor \frac{j}{r} \rfloor \text{ is odd}\}$. Thus

$$V_4 = \frac{1}{4\sqrt{2}} \sum_{r \in \mathbb{Z}} \theta_{|r|} \sum_{(z, \varepsilon) \in \{0;1\} \times \{-1;1\}} \sum_{j \in J_z} \varepsilon (P^T - P)(\psi_i + \varepsilon \psi_{i+r})^2 \mathbb{I}_{I_k^1}(i) \mathbb{I}_{I_k^2}(i+r).$$

For any fixed $r \neq 0$, $(z, \varepsilon) \in \{0;1\} \times \{-1;1\}$, $\frac{1}{\sqrt{2}}(\psi_i + \varepsilon \psi_{i+r})_{i \in J_z}$ is an orthonormal collection of functions, since for any $(i, j) \in J_z^2$,

$$\begin{aligned} \langle \psi_i + \varepsilon \psi_{i+r}, \psi_j + \varepsilon \psi_{j+r} \rangle &= \langle \psi_i, \psi_j \rangle + \varepsilon \langle \psi_{i+r}, \psi_j \rangle + \varepsilon \langle \psi_i, \psi_{j+r} \rangle + \langle \psi_{i+r}, \psi_{j+r} \rangle \\ &= 2\delta_{i,j} + \varepsilon \langle \psi_{i+r}, \psi_j \rangle + \varepsilon \langle \psi_i, \psi_{j+r} \rangle \\ &= 2\delta_{i,j} \text{ since } i, j \in J_z \text{ and } i+r, j+r \in J_{1-z}. \end{aligned}$$

[Arlot and Lerasle, 2016, Lemma 14] applied to $S_m = \langle (\psi_i + \varepsilon\psi_{i+r})_{i \in J_z \cap I_k^1 \cap I_k^2} \rangle$ for all $(z, \varepsilon) \in \{0; 1\} \times \{-1; 1\}$, $r \in \{-n_t, \dots, n_t\}$ and a union bound yield an event $E_4(y)$ of probability $\mathbb{P}(D_n^T \in E_4(y)) \geq 1 - e^{-y}$ such that, for some absolute constant κ and for all $D_n^T \in E_4(y)$, $(z, \varepsilon) \in \{0; 1\} \times \{-1; 1\}$ and $r \in \mathbb{Z}$,

$$\begin{aligned} \sum_{i \in J_z \cap I_k^1 \cap (I_k^2 - r)} \varepsilon(P^T - P)(\psi_i + \varepsilon\psi_{i+r})^2 &= (1 \pm \delta) \frac{\varepsilon}{n_t} \sum_{i \in J_z \cap I_k^1 \cap (I_k^2 - r)} [\text{Var}(\psi_i) + \text{Var}(\psi_{i+r}) \\ &\quad + 2\varepsilon \text{Cov}(\psi_i, \psi_{i+r})] + \kappa \frac{\|s\|_\infty [\log(1+r) + \log n_t + y]}{(\delta \wedge 1)n_t} \\ &\quad + \kappa \frac{|I_k^1| [\log(1+r) + \log n_t + y]^2}{(\delta \wedge 1)^3 n_t^2}. \end{aligned}$$

By summing on $(r, z, \varepsilon) \in \mathbb{Z} \times \{0; 1\} \times \{-1; 1\}$ and since $\|\psi_i\|_\infty \leq \sqrt{2}$, it follows that for all $D_n^T \in E_4(y)$,

$$\begin{aligned} \left| V_4 - \frac{1}{n_t} \sum_{r \in \mathbb{Z}} \frac{\theta_{|r|}}{\sqrt{2}} \sum_{i \in I_k^1 \cap (I_k^2 - r)} c_{i, i+r} \right| &= \left| V_4 - \frac{1}{n_t} \sum_{r=-n_t}^{n_t} \frac{\theta_{|r|}}{\sqrt{2}} \sum_{i \in I_k^1 \cap (I_k^2 - r)} c_{i, i+r} \right| \\ &\leq \frac{\delta}{n_t \sqrt{2}} \sum_{r \in \mathbb{Z}} |\theta_{|r|}| \sum_{i \in I_k^1 \cap (I_k^2 - r)} [\text{Var}(\psi_i) + \text{Var}(\psi_{i+r})] \\ &\quad + \kappa \sum_{r \in \mathbb{Z}} \frac{|\theta_{|r|}|}{\sqrt{2}} \times \frac{\|s\|_\infty [\log n_t \log(1+r) + y]}{(\delta \wedge 1)n_t} \\ &\quad + \kappa \sum_{r \in \mathbb{Z}} \frac{|\theta_{|r|}|}{\sqrt{2}} \times \frac{|I_k^1| [\log n_t + \log(1+r) + y]^2}{(\delta \wedge 1)^3 n_t^2} \end{aligned}$$

By hypothesis 1, $|\theta_j| \leq \frac{\sqrt{c_1}}{j^{1+\frac{\delta_4}{2}}}$, hence the sum $\sum_{r \in \mathbb{Z}} |\theta_{|r|}| \log(1+r)^2$ converges to a finite value $\|\theta\|_{1, \log^2}$. Moreover, by lemma 7.1, $|I_k^1| \leq (b_x - a_x)\Delta \leq 2(1+x)\Delta$, hence

$$\begin{aligned} \left| V_4 - \frac{1}{n_t} \sum_{r \in \mathbb{Z}} \frac{\theta_{|r|}}{\sqrt{2}} \sum_{i \in I_k^1 \cap (I_k^2 - r)} c_{i, i+r} \right| &\leq 6 \|\theta\|_{\ell^1} (1+x) \frac{\delta}{n_t} \Delta + 2\kappa \|\theta\|_{1, \log^2} \frac{\|s\|_\infty [1+y]}{(\delta \wedge 1)n_t} \\ &\quad + 8\kappa(1+x) \|\theta\|_{1, \log^2} \frac{\Delta [1+y]^2}{(\delta \wedge 1)^3 n_t^2}. \end{aligned}$$

There exists therefore a constant $\kappa(\|\theta\|_{1, \log^2})$ such that, for all $D_n^T \in E_4(y)$,

$$\left| V_4 - \frac{1}{n_t} \sum_{r \in \mathbb{Z}} \frac{\theta_{|r|}}{\sqrt{2}} \sum_{i \in I_k^1 \cap (I_k^2 - r)} c_{i, i+r} \right| \leq \kappa \delta (1+x) \mathcal{E} + \frac{[\log n_t + y]}{(\delta \wedge 1)n_t} + \kappa(1+x) \frac{[\log n_t + y]^2}{(\delta \wedge 1)^3 n_t} \mathcal{E}.$$

Let now $\delta = \max \left\{ \frac{n-n_t}{n_t}, n^{-\frac{1}{3}} \right\}^{\frac{3}{4}}$. By hypothesis 4 of section 4, $\frac{n-n_t}{n_t} \leq n^{-\delta_4}$, therefore $\delta \mathcal{E} \leq n^{-\min(\frac{1}{4}, \frac{3\delta_4}{4})} \mathcal{E}$. Moreover, $\mathcal{E} \geq \frac{1}{n_v}$ therefore $\frac{1}{\delta n_t} \leq \left(\frac{n-n_t}{n_t} \right)^{\frac{1}{4}} \frac{1}{n_v} \leq$

$n^{-\frac{\delta_4}{4}} \mathcal{E}$. Finally, since $\delta \geq n^{-\frac{1}{4}}$ and $n_t \geq \frac{n}{2}$, $\frac{\mathcal{E}}{\delta^3 n_t} \leq 2n^{-\frac{1}{4}} \mathcal{E}$. Since $\delta_4 \leq 1$, there exists therefore a constant κ such that for all $D_n^T \in E_4(y)$,

$$\left| V_4 - \frac{1}{n_t \sqrt{2}} \sum_{i \in I_k^1} \sum_{j \in I_k^2} \theta_{|i-j|} c_{i,j} \right| \leq \kappa(1+x)[\log n_t + y]^2 n^{-\frac{\delta_4}{4}} \mathcal{E}. \quad (72)$$

Moreover, since $c_{i,j} = \frac{\theta_{i+j}}{\sqrt{2}} + \left(\frac{1-\delta_{i,j}}{\sqrt{2}} + \delta_{i,j} \right) \theta_{|i-j|} - \theta_i \theta_j$ and $\theta_0 = 1$,

$$\begin{aligned} \frac{1}{n_t} \sum_{i \in I_k^1} \sum_{j \in I_k^2} \frac{\theta_{|i-j|}}{\sqrt{2}} c_{i,j} &= \sum_{i \in I_k^1 \cap I_k^2} \frac{1}{\sqrt{2}} \left(1 - \frac{1}{\sqrt{2}} \right) \frac{1}{n_t} + \frac{1}{n_t} \sum_{i \in I_k^1} \sum_{j \in I_k^2} \frac{\theta_{|i-j|}^2}{2} \\ &\quad + \frac{1}{n_t} \sum_{i \in I_k^1} \sum_{j \in I_k^2} \frac{\theta_{|i-j|} \theta_{i+j}}{2} - \frac{1}{n_t} \sum_{i \in I_k^1} \sum_{j \in I_k^2} \frac{\theta_{|i-j|}}{\sqrt{2}} \theta_i \theta_j. \end{aligned}$$

Since for all $j \in \mathbb{N}$, $|\theta_j| \leq 1$,

$$\begin{aligned} \left| \frac{1}{n_t} \sum_{i \in I_k^1} \sum_{j \in I_k^2} \frac{\theta_{|i-j|}}{\sqrt{2}} c_{i,j} - \left(1 - \frac{1}{\sqrt{2}} \right) \frac{|I_k^1 \cap I_k^2|}{n_t \sqrt{2}} - \frac{1}{n_t} \sum_{i \in I_k^1} \sum_{j \in I_k^2} \frac{\theta_{|i-j|}^2}{2} \right| &\leq \frac{2}{n_t} \left(\sum_{r \in \mathbb{N}} |\theta_r| \right)^2 \\ &\leq 2 \frac{n - n_t}{n_t} \frac{1}{n_v} \|\theta\|_{\ell^1}^2 \\ &\leq 2 \|\theta\|_{\ell^1}^2 n^{-\delta_4} \mathcal{E} \end{aligned} \quad (73)$$

since $\mathcal{E} \geq \frac{1}{n_v}$ and $\frac{n - n_t}{n_t} \geq n^{-\delta_4}$, by hypothesis 4 of Theorem 1. From equations (72) and (73), it follows that, for some constant $\kappa(\|\theta\|_{1, \log^2})$,

$$\left| V_4 - \left(1 - \frac{1}{\sqrt{2}} \right) \frac{|I_k^1 \cap I_k^2|}{n_t \sqrt{2}} - \frac{1}{2n_t} \sum_{i \in I_k^1} \sum_{j \in I_k^2} \theta_{|i-j|}^2 \right| \leq \kappa(1+x)[\log n_t + y]^2 n^{-\frac{\delta_4}{4}} \mathcal{E}. \quad (74)$$

V_5 can be expressed as

$$V_5 = \left(1 - \frac{1}{\sqrt{2}} \right) \sum_{j \in I_k^1 \cap I_k^2} \left(\hat{\theta}_j^T - \theta_j \right)^2,$$

therefore by proposition 8.3, there exists an event $E_5(y)$ of probability greater than $1 - e^{-y}$ such that for all $D_n^T \in E_5(y)$,

$$V_5 = \left(1 - \frac{1}{\sqrt{2}} \right) \sum_{i \in I_k^1 \cap I_k^2} \frac{1}{n_t} \pm \left(1 - \frac{1}{\sqrt{2}} \right) \kappa_1 (b_x - a_x) [\log n + y]^2 n^{-\min(\frac{1}{12}, \frac{\delta_4}{2})} \mathbf{e}(n)$$

It follows by lemma 7.1 that on $E_5(y)$,

$$\left| V_5 - \left(1 - \frac{1}{\sqrt{2}} \right) \frac{|I_k^1 \cap I_k^2|}{n_t} \right| \leq 4\kappa_1(1+x)[\log n + y]^2 n^{-\min(\frac{1}{12}, \frac{\delta_4}{2})} \mathbf{e}(n). \quad (75)$$

Finally, V_6 can be bounded in the following manner.

$$\begin{aligned} V_6 &\leq \frac{1}{2} \sum_{i \in I_k^1} \sum_{j \in I_k^2} [(P^T - P)^2 \psi_i + (P^T - P)^2 \psi_j] \left[\frac{|\theta_{i+j}|}{\sqrt{2}} + |\theta_i| |\theta_j| \right] \\ &= \frac{1}{2} \sum_{i \in I_k^1} (P^T - P)^2 \psi_i \sum_{j \in I_k^2} \left[\frac{|\theta_{i+j}|}{\sqrt{2}} + |\theta_i| |\theta_j| \right] + \frac{1}{2} \sum_{j \in I_k^2} (P^T - P)^2 \psi_j \sum_{i \in I_k^1} \left[\frac{|\theta_{i+j}|}{\sqrt{2}} + |\theta_i| |\theta_j| \right]. \end{aligned}$$

Thus, by equation (66),

$$\begin{aligned} V_6 &\leq \kappa \frac{(1+x)^{\frac{\delta_1}{2\delta_2}}}{n_t^{\frac{\delta_1}{3\delta_2}}} \times \left[\frac{1}{2} \sum_{i \in I_k^1} (P^T - P)^2 \psi_i + \frac{1}{2} \sum_{j \in I_k^2} (P^T - P)^2 \psi_j \right] \\ &\leq \kappa \frac{(1+x)^{\frac{\delta_1}{2\delta_2}}}{n_t^{\frac{\delta_1}{3\delta_2}}} \times \sum_{j=k_*+a_x\Delta+1}^{k_*+b_x\Delta} (\hat{\theta}_j^T - \theta_j)^2. \end{aligned} \quad (76)$$

By proposition 8.3, there exists an event $E_6(y)$ of probability greater than $1 - e^{-y}$, such that for any $D_n^T \in E_6(y)$,

$$\begin{aligned} \sum_{j=k_*+a_x\Delta+1}^{k_*+b_x\Delta} (\hat{\theta}_j^T - \theta_j)^2 &\leq (b_x - a_x) \mathcal{E} + \kappa_1 (b_x - a_x) (y + \log n)^2 n^{-\min(\frac{1}{12}, \frac{\delta_1}{2})} \mathbf{e} \\ &\leq 2 \max(\kappa_1, 1) (b_x - a_x) (y + \log n)^2 \mathcal{E} \\ &\leq 8 \max(\kappa_1, 1) (1+x) (y + \log n)^2 \mathcal{E} \text{ by lemma 7.1.} \end{aligned}$$

It follows by equation (76) that on $E_6(y)$, for a certain constant $\kappa(\kappa_1, \delta_1, c_1, \kappa_6)$,

$$V_6 \leq \kappa [y + \log n]^2 \frac{(1+x)^{\frac{\delta_1}{2\delta_2}}}{n_t^{\frac{\delta_1}{3\delta_2}}} \mathcal{E}. \quad (77)$$

Combining equations (67), (70), (71), (74), (75), (77) on the event $\cap_{i=2}^6 E_i(\log 6 + y)$ yields the result. \blacksquare

Lemma 8.7 *Let $s \in L^\infty([0; 1])$ be a probability density function. For all $j \in \mathbb{N}$, let $\theta_j = \langle s, \psi_j \rangle$, where $\psi_0(x) = 1$ and $\psi_j(x) = \sqrt{2} \cos(2j\pi x)$ for all $j \in \mathbb{N}^*$. Thus for any finite set $I \subset \mathbb{N}$ and for all functions $u \in \mathbb{R}^I$,*

$$0 \leq \sum_{i \in I} \sum_{j \in I} u(i) u(j) \left(\frac{1 - \delta_{i,j}}{\sqrt{2}} + \delta_{i,j} \right) \theta_{|i-j|} \leq \|s\|_\infty \sum_{i \in I} u(i)^2.$$

Proof Let $X \sim s$ be a random variable with distribution $s(x)dx$ on $[0; 1]$. For any $x \in \mathbb{R}$, and any $i \neq j$,

$$\psi_i(x) \psi_j(x) = 2 \cos(2i\pi x) \cos(2j\pi x) = \cos(2(i+j)\pi x) + \cos(2(i-j)\pi x) = \frac{\psi_{i+j} + \psi_{|i-j|}}{\sqrt{2}}.$$

If $i \neq j$, then $\text{Cov}(\psi_i(X), \psi_j(X)) = \frac{\theta_{i+j} + \theta_{|i-j|}}{\sqrt{2}} - \theta_i \theta_j$. If $i = j$, $\text{Var}(\psi_i(X)) = 1 + \frac{\theta_{2i}}{\sqrt{2}} - \theta_i^2$. Let $u \in \mathbb{R}^I$, $k \in \mathbb{N}$ and $t_k = \sum_{i \in I} u(i) \psi_{i+k}$, then

$$\text{Var}(t_k(X)) = \sum_{i \in I} \sum_{j \in I} u(i) u(j) \left[\left(\frac{1 - \delta_{i,j}}{\sqrt{2}} + \delta_{i,j} \right) \theta_{|i-j|} + \frac{\theta_{i+j+2k}}{\sqrt{2}} - \theta_{i+k} \theta_{j+k} \right]$$

Furthermore, $\lim_{n \rightarrow +\infty} \theta_n = 0$, hence

$$\lim_{k \rightarrow +\infty} \text{Var}(t_k(X)) = \sum_{i \in I} \sum_{j \in I} u(i) u(j) \left(\frac{1 - \delta_{i,j}}{\sqrt{2}} + \delta_{i,j} \right) \theta_{|i-j|}.$$

It immediately follows that $\sum_{i \in I} \sum_{j \in I} u(i) u(j) \left(\frac{1 - \delta_{i,j}}{\sqrt{2}} + \delta_{i,j} \right) \theta_{|i-j|} \geq 0$. Moreover, for all $k \in \mathbb{N}$,

$$\begin{aligned} \text{Var}(t_k(X)) &\leq \mathbb{E} [t_k(X)^2] \\ &= \int_0^1 t_k(x)^2 s(x) dx \\ &\leq \|s\|_\infty \|t_k\|^2 \\ &\leq \|s\|_\infty \sum_{i \in I} u(i)^2. \end{aligned}$$

Thus

$$\sum_{i \in I} \sum_{j \in I} u(i) u(j) \left(\frac{1 - \delta_{i,j}}{\sqrt{2}} + \delta_{i,j} \right) \theta_{|i-j|} \leq \|s\|_\infty \sum_{i \in I} u(i)^2. \quad \blacksquare$$

Lemme 8.8 *Let $\varepsilon : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-decreasing function such that $\varepsilon(0) > 0$ and $h_+ : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuous, non-decreasing function. Let $g_0 : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a continuous function such that, for any $s < t$,*

$$-\varepsilon(\max(s, t)) \leq g_0(t) - g_0(s) \leq \max\{h_+(t) - h_+(s), \varepsilon(\max(s, t))\}.$$

Assume that $\varepsilon(0) > 0$. Then there exists a continuous, non-decreasing function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $g_0(0) = g(0)$,

$$\left\| \frac{g_0 - g}{\varepsilon} \right\|_\infty \leq 6,$$

and moreover

$$\forall x, y, |g(y) - g(x)| \leq |h_+(y) - h_+(x)|.$$

Proof Assume to begin with that ε is right-continuous. Let $r > 0, \delta > 0$. We define by induction a sequence $(x_i)_{i \in \mathbb{N}}$ and a function g on $[x_i; x_{i+1}]$. Let $x_0 = 0$

and $g(x_0) = g_0(x_0)$. For any $i \in \mathbb{N}$, assuming x_i and $g(x_i)$ have been defined, let

$$x_{i+1} = \inf \left\{ x \geq x_i : g_0(x) \geq g_0(x_i) + 2\varepsilon(x_i) \text{ ou } \varepsilon(x) \geq \frac{3}{2}\varepsilon(x_i) \right\}$$

$$\forall x \in (x_i, x_{i+1}], g(x) = \begin{cases} g(x_i) & \text{if } \varepsilon(x_{i+1}) \geq \frac{3}{2}\varepsilon(x_i) \\ g(x_i) + \frac{g_0(x_{i+1}) - g_0(x_i)}{h_+(x_{i+1}) - h_+(x_i)} [h_+(x) - h_+(x_i)] & \text{else.} \end{cases} \quad (78)$$

If $x_{i+1} = +\infty$, the above definitions still make sense and the induction stops. Notice first that for any $x \in [x_i; x_{i+1}]$, $g_0(x) - g_0(x_i) \leq [h_+(x) - h_+(x_i)] \vee \varepsilon(x) \leq [h_+(x) - h_+(x_i)] \vee \frac{3}{2}\varepsilon(x_i)$. Thus, by continuity of g_0 ,

$$g_0(x_{i+1}) - g_0(x_i) \leq [h_+(x_{i+1}) - h_+(x_i)] \vee \frac{3}{2}\varepsilon(x_i).$$

By assumption, ε is right-continuous, therefore if $\varepsilon(x_{i+1}) < \frac{3}{2}\varepsilon(x_i)$, it must be that $\inf\{x \geq x_i : \varepsilon(x) \geq \frac{3}{2}\varepsilon(x_i)\} > x_{i+1}$. Then by definition of x_{i+1} and continuity of g_0 , $g_0(x_{i+1}) = g_0(x_i) + 2\varepsilon(x_i)$, therefore

$$2\varepsilon(x_i) = g_0(x_{i+1}) - g_0(x_i) \leq [h_+(x_{i+1}) - h_+(x_i)] \vee \frac{3}{2}\varepsilon(x_i),$$

which implies that

$$0 < 2\varepsilon(x_i) = g_0(x_{i+1}) - g_0(x_i) \leq [h_+(x_{i+1}) - h_+(x_i)]. \quad (79)$$

This proves that g is well defined. g is non-decreasing and continuous since h_+ has these properties. If $\varepsilon(x_{i+1}) < \frac{3}{2}\varepsilon(x_i)$, then the previous equation implies that

$$\forall i \in \mathbb{N}, \forall (x, y) \in (x_i, x_{i+1}], x \leq y \implies g(y) - g(x) \leq h_+(y) - h_+(x),$$

else g is constant on $]x_i; x_{i+1}]$ and the above equation is trivially true. Hence, since g, h_+ are non-decreasing and continuous,

$$\forall (x, y) \in \mathbb{R}, x \leq y \implies g(y) - g(x) \leq h_+(y) - h_+(x).$$

We will now prove by induction that for all $i \in \mathbb{N}^*$,

$$0 \leq g_0(x_i) - g(x_i) \leq 6\varepsilon(x_{i-1}). \quad (80)$$

Base case: This equation is true for $i = 1$ since $x_0 = 0$ and $g(0) = g_0(0) = 0$, therefore by definition of g, x_1 , $0 \leq g(x_1) \leq g_0(x_1) \leq 2\varepsilon(x_0)$.

Inductive step: Assume that equation (80) is true for some $i \in \mathbb{N}$. Then by definition of x_{i+1} and g ,

- If $\varepsilon(x_{i+1}) \geq \frac{3}{2}\varepsilon(x_i)$, then $g(x_{i+1}) = g(x_i)$ therefore $g_0(x_{i+1}) - g(x_{i+1}) = g_0(x_{i+1}) - g_0(x_i) + g_0(x_i) - g(x_i)$. By the induction hypothesis and the definition of x_{i+1} ,

$$0 \leq g_0(x_{i+1}) - g(x_{i+1}) \leq 2\varepsilon(x_i) + 6\varepsilon(x_{i-1}) \leq 2\varepsilon(x_i) + 6 \times \frac{2}{3}\varepsilon(x_i) \leq 6\varepsilon(x_i),$$

which proves equation (80) for $i + 1$.

- Otherwise, by definition of g , $g(x_{i+1}) = g(x_i) + [g_0(x_{i+1}) - g_0(x_i)]$ therefore by the induction hypothesis and since ε is non-decreasing,

$$0 \leq g_0(x_{i+1}) - g(x_{i+1}) = g_0(x_i) - g(x_i) \leq 6\varepsilon(x_{i-1}) \leq 6\varepsilon(x_i).$$

This proves equation (80) for $i + 1$.

By induction, equation (80) is therefore true for all $i \in \mathbb{N}$ (such that $x_i < +\infty$). Let now $i \in \mathbb{N}$ and $x \in (x_i, x_{i+1}]$. By definition of g ,

$$g(x_i) \leq g(x) \leq g(x_i) + (g_0(x_{i+1}) - g_0(x_i))_+.$$

By equation (80) and definition of x_{i+1} ,

$$\begin{aligned} g(x) - g_0(x) &\leq g(x) - g_0(x_i) \\ &\leq g(x_i) - g_0(x_i) + (g_0(x_{i+1}) - g_0(x_i))_+ \\ &\leq 2\varepsilon(x_i) \\ &\leq 2\varepsilon(x). \end{aligned}$$

Moreover, by equation (80) and definition of the x_i ,

$$\begin{aligned} g(x) - g_0(x) &\geq g(x_i) - g_0(x_{i+1}) \\ &\geq g(x_i) - g_0(x_i) - [g_0(x_{i+1}) - g_0(x_i)] \\ &\geq -6\varepsilon(x_{i-1}) - 2\varepsilon(x_i) \\ &\geq -6 \times \frac{2}{3}\varepsilon(x_i) - 2\varepsilon(x_i) \\ &\geq -6\varepsilon(x_i) \\ &\geq -6\varepsilon(x). \end{aligned}$$

It has been proved that for all $i \in \mathbb{N}$ such that x_i is finite,

$$\forall x \in]x_i, x_{i+1}], |g(x) - g_0(x)| \leq 6\varepsilon(x).$$

It must now be proved that $\lim_{n \rightarrow +\infty} x_n = +\infty$. Since ε is non-decreasing and right-continuous, by definition of x_n , $g_0(x_{n+1}) \geq g_0(x_n) + 2\varepsilon(x_n) \geq g_0(x_n) + \varepsilon(0)$ ou $\varepsilon(x_{n+1}) \geq \frac{3}{2}\varepsilon(x_n)$. Since $\varepsilon(0) > 0$ by assumption, this implies that $\max(g_0, \varepsilon)(x_n) \rightarrow +\infty$. The function $\max(g_0, \varepsilon)$ is non-decreasing, thus it is bounded on every interval of the form $[0; x]$, which implies that $x_n \rightarrow +\infty$. This proves the proposition under the assumption that ε is right-continuous.

In the general case, let $\varepsilon_+ : x \mapsto \inf_{y>x} \varepsilon(y)$, which is non-decreasing and right-continuous. Since ε is non-decreasing, $\varepsilon_+ \geq \varepsilon$, therefore the assumptions of the proposition hold with ε_+ instead of ε . By the right-continuous case of the proposition, which we already proved, there exists a non-decreasing function g such that $\left\| \frac{g-g_0}{\varepsilon_+} \right\|_\infty \leq 6$ and

$$\forall x, y, x \leq y \implies g(y) - g(x) \leq h_+(y) - h_+(x).$$

Let $x \in \mathbb{R}_+$. For all $y < x$, $|g(y) - g_0(y)| \leq 6\varepsilon_+(y)$, therefore by continuity of g, g_0 ,

$$|g(x) - g_0(x)| \leq 6 \sup_{y<x} \varepsilon_+(y) = 6 \sup_{y<x} \inf_{y'>y} \varepsilon(y') \leq 6\varepsilon(x).$$

This proves the proposition in the general case. ■

Proposition 8.9 *Let $([x_i, x_{i+1}])_{1 \leq i \leq M-1}$ be a partition of the interval $[a, b]$. Let $Y : \{x_1, \dots, x_M\} \rightarrow \mathbb{R}$ be such that $(Y(x_j))_{1 \leq j \leq M}$ is a zero-mean gaussian vector. Abusing notation, we also denote by Y the extension of Y to $[a; b]$ by linear interpolation. Let $K_Y : [a; b]^2 \rightarrow \mathbb{R}$ be the variance-covariance function of Y . Let $h : [a; b] \rightarrow \mathbb{R}$ be a continuous, increasing function and let $K_X : [a; b]^2 \rightarrow \mathbb{R}$ be a positive semi-definite function such that:*

$$\forall (s, t) \in [a; b]^2, |K_X(s, s) + K_X(t, t) - 2K_X(s, t)| \leq |h(s) - h(t)|.$$

Assume that there exists constants $L > 0$ and $\varepsilon \in [0; 1]$ such that:

- $\sup_{t \in [a; b]} \sqrt{K_X(t, t)} \leq L$
- For any $i \in \{1, \dots, M-1\}$, $h(x_{i+1}) - h(x_i) \leq \varepsilon$
- $\max_{(i, j) \in \{1, \dots, M\}^2} |K_X(x_i, x_j) - K_Y(x_i, x_j)| \leq \varepsilon$.

There exists a universal constant κ and a measurable function $f : C([a; b], \mathbb{R}) \rightarrow C([a; b], \mathbb{R})$ such that for all random variables $\nu \sim \mathcal{U}([0; 1])$ independent from Y , $X = f(Y, \nu)$ is a zero-mean gaussian process with variance-covariance function K_X and moreover,

$$\mathbb{E} \left[\sup_{a \leq t \leq b} |X_t - Y_t| \right] \leq \kappa \sqrt{(1+L) \log M [(h(b) - h(a)) \vee 1] \varepsilon^{\frac{1}{12}}}.$$

Proof We assume without loss of generality that $h(b) - h(a) \geq 1$. We shall moreover use the following notation. For A, B two symmetric matrices, $A \prec B$ means that $B - A$ is positive definite. $\|A\|_{op}$ denotes the matrix operator norm corresponding to the euclidean norm, i.e $\|A\|_{op} = \sup_{x: \|x\|_2 \leq 1} \|Ax\|_2$. We will need the following lemmas:

Lemme 8.10 *For all $A \in \mathbb{R}^{m \times m}$, $\|A\|_{op} \leq m \max_{1 \leq i, j \leq m} |A_{i, j}|$.*

Proof Let $v \in \mathbb{R}^m$ be such that $\sum_{i=1}^m v_i^2 = 1$. By the Cauchy-Schwartz inequality,

$$\|Av\|^2 = \sum_{i=1}^m \left(\sum_{j=1}^m A_{i, j} v_j \right)^2 \leq \sum_{i=1}^m \sum_{j=1}^m A_{i, j}^2 \leq m^2 \max_{i=1, \dots, m} A_{i, j}^2.$$

This is true for any v , which proves lemma 8.10. ■

Lemma 8.11 below is a special case of Mc-Carthy's trace inequality [McCarthy, 1967, Lemma 2.6].

Lemme 8.11 *Let A, B be two symmetric, positive semi-definite matrices, then*

$$\text{Tr}(\sqrt{A+B}) \leq \text{Tr}(\sqrt{A}) + \text{Tr}(\sqrt{B}).$$

The hypotheses imply that h is bijective from $[a; b]$ to $[h(a); h(b)]$. Let $m \in \mathbb{N}$. For all $j \in \{1, \dots, m\}$, let

$$t_j = \max \left\{ x_i \mid i \in \{1, \dots, M\}, h(x_i) \leq h(a) + \frac{j-1}{m-1} [h(b) - h(a)] \right\}. \quad (81)$$

Let $K_{X,m} = (K_X(t_i, t_j))_{1 \leq i, j \leq m}$ and $K_{Y,m} = (K_Y(t_i, t_j))_{1 \leq i, j \leq m}$. The Wasserstein distance between two gaussian vectors is known [Olkin and Pukelsheim, 1982]: there exists a coupling \tilde{X}^m, \tilde{Y}^m of the distributions $\mathcal{N}(0, K_{X,m})$ and $\mathcal{N}(0, K_{Y,m})$ such that:

$$\mathbb{E} \left[\sum_{i=1}^m (\tilde{X}_i^m - \tilde{Y}_i^m)^2 \right] = \text{Tr} \left(K_{X,m} + K_{Y,m} - 2(K_{X,m}^{\frac{1}{2}} K_{Y,m} K_{X,m}^{\frac{1}{2}})^{\frac{1}{2}} \right).$$

Thus

$$\begin{aligned} K_{X,m}^2 &= K_{X,m}^{\frac{1}{2}} K_{Y,m} K_{X,m}^{\frac{1}{2}} + K_{X,m}^{\frac{1}{2}} (K_{Y,m} - K_{X,m}) K_{X,m}^{\frac{1}{2}} \\ &< K_{X,m}^{\frac{1}{2}} K_{Y,m} K_{X,m}^{\frac{1}{2}} + \|K_{Y,m} - K_{X,m}\|_{op} K_{X,m}. \end{aligned}$$

By lemma 8.11,

$$\text{Tr}(K_{X,m}) \leq \text{Tr} \left((K_{X,m}^{\frac{1}{2}} K_{Y,m} K_{X,m}^{\frac{1}{2}})^{\frac{1}{2}} \right) + \|K_{Y,m} - K_{X,m}\|_{op}^{\frac{1}{2}} \text{Tr} \left(K_{X,m}^{\frac{1}{2}} \right).$$

By the same argument (exchangeing X and Y),

$$\text{Tr}(K_{Y,m}) \leq \text{Tr} \left((K_{Y,m}^{\frac{1}{2}} K_{X,m} K_{Y,m}^{\frac{1}{2}})^{\frac{1}{2}} \right) + \|K_{Y,m} - K_{X,m}\|_{op}^{\frac{1}{2}} \text{Tr} \left(K_{Y,m}^{\frac{1}{2}} \right).$$

It follows that

$$\begin{aligned} \mathbb{E} \left[\sum_{i=1}^m (\tilde{X}_i^m - \tilde{Y}_i^m)^2 \right] &\leq \|K_{Y,m} - K_{X,m}\|_{op}^{\frac{1}{2}} \text{Tr} \left(K_{X,m}^{\frac{1}{2}} + K_{Y,m}^{\frac{1}{2}} \right) \\ &\leq m^{\frac{1}{2}} \|K_{Y,m} - K_{X,m}\|_{\infty}^{\frac{1}{2}} \left(\sqrt{\text{Tr}(K_{X,m})} + \sqrt{\text{Tr}(K_{Y,m})} \right) \\ &\leq m^{\frac{1}{2}} \|K_{Y,m} - K_{X,m}\|_{\infty}^{\frac{1}{2}} m^{\frac{1}{2}} \max_{1 \leq i, j \leq m} \left\{ \sqrt{K_X(t_i, t_i)} + \sqrt{K_Y(t_j, t_j)} \right\} \\ &\leq 2m\sqrt{\varepsilon}\sqrt{L^2 + \varepsilon}. \end{aligned} \tag{82}$$

By the transfer principle (Kallenberg, Theorem 5.10), there exists f_1 such that for all uniform random variables variables ν_1 independent from Y , $(\tilde{X}^m, \tilde{Y}^m) \sim (f_1(Y^m, \nu_1), Y^m)$.

Let X_0 be a gaussian process with variance-covariance function K_X . Let $W_0 = X_0 \circ h^{-1}$. For any $(s, t) \in [h(a); h(b)]^2$, $W_0(t) - W_0(s)$ is a centred gaussian random variable, hence for all $r > 0$, there exists a universal constant $C(r)$ such that

$$\begin{aligned} \mathbb{E}[(W_0(t) - W_0(s))^r] &\leq C(r)\mathbb{E}[(W_0(t) - W_0(s))^2]^{\frac{r}{2}} \\ &\leq C(r) \left(K_X(h^{-1}(t), h^{-1}(t)) + K_X(h^{-1}(s), h^{-1}(s)) \right. \\ &\quad \left. - 2K_X(h^{-1}(s), h^{-1}(t)) \right)^{\frac{r}{2}} \\ &\leq C(r)|t - s|^{\frac{r}{2}}. \end{aligned}$$

By the Kolmogorov continuity theorem [Revuz and Yor, 1999, Chapter 1, Theorem 2.1], applied to $x \mapsto \frac{W_0(h(a) + (h(b) - h(a))x)}{\sqrt{h(b) - h(a)}}$, there exists a continuous

version W_1 of W_0 such that for any $\theta \in [0; 1)$ and all $(s, t) \in [h(a); h(b)]^2$,

$$\mathbb{E} \left[\left(\sup_{s \neq t} \frac{|W_1(t) - W_1(s)|}{|t - s|^{\theta(\frac{1}{2} - \frac{1}{r})}} \right)^r \right] \leq \frac{(h(b) - h(a))^{\frac{r}{2}}}{(h(b) - h(a))^{\theta r(\frac{1}{2} - \frac{1}{r})}} B(\theta, r) < +\infty,$$

where $B(\theta, r)$ is a universal constant. Let $X_1 = W_1 \circ h$, which is still a gaussian process, with variance-covariance function K_X . Then, since by assumption $h(b) - h(a) \geq 1$, for all $r \geq 2$,

$$\mathbb{E} \left[\left(\sup_{s \neq t} \frac{|X_1(t) - X_1(s)|}{|h(t) - h(s)|^{\theta(\frac{1}{2} - \frac{1}{r})}} \right)^r \right] \leq [h(b) - h(a)]^{\frac{r}{2}} B(\theta, r) < +\infty. \quad (83)$$

The $C([a; b], \mathbb{R})$ -valued process X_1 induces a probability distribution Q on the Borel space $C([a; b], \mathbb{R})$. Furthermore, $(X_1(t_j))_{1 \leq j \leq m} \sim \tilde{X}^m \sim f_1(Y^m, \nu_1)$. By (Kallenberg, Theorem 5.10), there exists a measurable function f_2 such that for all uniform random variables ν_3 independent from Y^m, ν_1 , $(X_1, (X_1(t_j))_{1 \leq j \leq m}) \sim (f_2(f_1(Y^m, \nu_1), \nu_3), f_1(Y^m, \nu_1))$. Let $X = f_2(f_1(Y^m, \nu_1), \nu_3)$ and $X^m = (X(t_j))_{1 \leq j \leq m}$. Almost surely,

Claim 8 1. $X^m = (X(t_j))_{1 \leq j \leq m} = f_1(Y^m, \nu_1)$ *p.s.*, so

2. $(X^m, Y^m) \sim (\tilde{X}^m, \tilde{Y}^m)$, in particular by equation (82),

$$\mathbb{E} \left[\|X^m - Y^m\|^2 \right] \leq 2m\sqrt{\varepsilon}\sqrt{L^2 + \varepsilon}.$$

3. $X \sim X^1$ as a random continuous function, in particular by equation (83) with $\theta = \frac{3}{4}$ and $r = 6$,

$$\forall \delta > 0, \mathbb{E} \left[\sup_{(s, t) \in [a; b]: |h(t) - h(s)| \leq \delta} |X(t) - X(s)|^6 \right]^{\frac{1}{6}} \leq \sqrt{h(b) - h(a)} B(\frac{3}{4}, 6)^{\frac{1}{6}} \delta^{\frac{1}{4}}. \quad (84)$$

By abuse of notation, denote X^m, Y^m the random processes obtained by linear interpolation between the points (t_j, X_j^m) and (t_j, Y_j^m) , respectively. For all $t \in [a; b]$, there exists $j \in \{1, \dots, m\}$ such that $t_j \leq t \leq t_{j+1}$, therefore $|h(t) - h(t_j)| \leq h(t_{j+1}) - h(t_j)$, since h is non-decreasing. By definition of t_{j+1} (equation (81)), $h(t_{j+1}) \leq h(a) + \frac{j}{m-1}(h(b) - h(a))$ and furthermore, there exists $i \in \{1, \dots, M\}$ such that $x_i = t_j$. By equation (81) which defines t_j , $h(x_{i+1}) > h(a) + \frac{j-1}{m-1}(h(b) - h(a))$, which yields

$$\begin{aligned} |h(t) - h(t_j)| &\leq h(a) + \frac{j}{m-1}(h(b) - h(a)) - h(x_{i+1}) + h(x_{i+1}) - h(x_i) \\ &\leq \frac{h(b) - h(a)}{m-1} + h(x_{i+1}) - h(x_i). \end{aligned}$$

By assumption, $h(x_{i+1}) - h(x_i) \leq \varepsilon$, which yields

$$\forall t \in [a; b], \exists j(t) \in \{1, \dots, m\}, t_j \leq t \leq t_{j+1} \text{ and } |h(t) - h(t_{j(t)})| \leq \frac{h(b) - h(a)}{m} + \varepsilon. \quad (85)$$

Since Y is piecewise linear on the partition $([x_i, x_{i+1}])_{1 \leq i \leq M-1}$,

$$\begin{aligned} \sup_{a \leq t \leq b} |X(t) - Y(t)| &\leq \sup_{t \in [a; b]} |X(t) - X(t_j(t))| + \max_{j \in \{1, \dots, m\}} |X(t_j) - Y(t_j)| \\ &\quad + \sup_{t \in [a; b]} |Y(t) - Y(t_j(t))| \\ &\leq \sup_{(s, t): |h(s) - h(t)| \leq \varepsilon + \frac{h(b) - h(a)}{m}} |X(s) - X(t)| + \sqrt{\sum_{j=1}^m |X(t_j) - Y(t_j)|^2} \\ &\quad + \max_{j \in \{1, \dots, m\}} \max \{|Y(x_i) - Y(t_j)| : i \in \{1, \dots, M\} \cap [t_j; t_{j+1}]\}. \end{aligned}$$

Thus, by claim 8,

$$\begin{aligned} &\mathbb{E} \left[\sup_{a \leq t \leq b} |X(t) - Y(t)| \right] \\ &\leq \mathbb{E} \left[\sup_{(s, t): |h(s) - h(t)| \leq \varepsilon + \frac{h(b) - h(a)}{m}} |X(s) - X(t)| \right] + \mathbb{E} \left[\sum_{i=1}^m (X_i^m - Y_i^m)^2 \right]^{\frac{1}{2}} \\ &\quad + \mathbb{E} \left[\max_{j \in \{1, \dots, m\}} \max \{|Y(x_i) - Y(t_j)| : i \in \{1, \dots, M\} \cap [t_j; t_{j+1}]\} \right] \\ &\leq \sqrt{h(b) - h(a)} B\left(\frac{3}{4}, 6\right)^{\frac{1}{6}} \left(\varepsilon + \frac{h(b) - h(a)}{m} \right)^{\frac{1}{4}} + \left(2m\sqrt{\varepsilon} \sqrt{L^2 + \varepsilon} \right)^{\frac{1}{2}} \\ &\quad + \sqrt{2 \log M} \max_{j \in \{1, \dots, m\}} \max \left\{ \sqrt{\mathbb{E}[|Y(x_i) - Y(t_j)|^2]} : i \in \{1, \dots, M\} \cap [t_j; t_{j+1}] \right\}. \end{aligned} \tag{86}$$

Furthermore, for any $(i, j) \in [1; M]^2$,

$$\begin{aligned} \mathbb{E}[(Y(x_i) - Y(x_j))^2] &= K_Y(x_i, x_i) + K_Y(x_j, x_j) - 2K_Y(x_i, x_j) \\ &\leq K_X(x_i, x_i) + K_X(x_j, x_j) - 2K_X(x_i, x_j) \\ &\quad + 4 \max_{(r, s) \in [1; M]^2} |K_X(x_r, x_s) - K_Y(x_r, x_s)| \\ &\leq |h(x_i) - h(x_j)| + 4\varepsilon. \end{aligned}$$

Setting $\kappa = B\left(\frac{3}{4}, 6\right)^{\frac{1}{6}}$, it follows by equation (86) and the non-decreasing nature of h that

$$\begin{aligned} \mathbb{E} \left[\sup_{a \leq t \leq b} |X_t - Y_t| \right] &\leq \sqrt{h(b) - h(a)} B\left(\frac{3}{4}, 6\right)^{\frac{1}{6}} \left(\varepsilon + \frac{h(b) - h(a)}{m} \right)^{\frac{1}{4}} + \sqrt{2m(L+1)} \varepsilon^{\frac{1}{4}} \\ &\quad + \sqrt{2 \log M} \sqrt{h(t_{j+1}) - h(t_j)} + 4\varepsilon \\ &\leq \kappa \sqrt{h(b) - h(a)} \varepsilon^{\frac{1}{4}} + \kappa \frac{[h(b) - h(a)]^{\frac{3}{4}}}{m^{\frac{1}{4}}} + \sqrt{2m(L+1)} \varepsilon^{\frac{1}{4}} \\ &\quad + \sqrt{2 \log M} \sqrt{\frac{h(b) - h(a)}{m}} + 5\varepsilon \text{ by equation (85)}. \end{aligned}$$

Let now $m = \left\lceil \frac{h(b) - h(a)}{\varepsilon^{\frac{1}{3}}} \right\rceil$. Since by assumption $\varepsilon \leq 1$, $h(b) - h(a) \geq 1$ it follows finally, by keeping only the largest powers of $[h(b) - h(a)]$, ε , L and $\log M$, that

$$\mathbb{E} \left[\sup_{a \leq t \leq b} |X_t - Y_t| \right] \leq \kappa \sqrt{h(b) - h(a)} \sqrt{2(L+1) \log M} \varepsilon^{\frac{1}{12}}.$$

for an absolute constant κ . ■

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