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# A Robust Nonlinear Tracking MPC using qLPV Embedding and Zonotopic Uncertainty Propagation <sup>★</sup>

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## Abstract

In this paper, we present a Tracking Nonlinear Model Predictive Control (NMPC) formulation for piece-wise constant reference signals. In order to obtain a computationally efficient algorithm, we use quasi-Linear Parameter Varying (qLPV) embedding to describe the nonlinear dynamics. Accordingly, we develop a recursive extrapolation algorithm, which estimates the future values of the scheduling parameters along the prediction horizon with bounded estimation error. The proposed framework also avoids feasibility losses due to large set-point variations, which are tracked thanks to an artificial feasible target variable, whose distance to the real set-point is minimized through an additional offset cost. At each sampling period, an optimisation problem is solved based on linear (scheduled) predictions; the average numerical toughness is comparable to a quadratic program. Robust constraint satisfaction is achieved with zonotopes that propagate the uncertainty. These sets are computed with respect to the one-step-ahead bounds of the qLPV scheduling sequence estimation error, offering reduced conservatism. Closed-loop stability and recursive feasibility are provided with robust parameter-dependent terminal ingredients. A benchmark example illustrates the performances of the proposed method, which is able to ensure reference tracking with reduced numerical demand when compared to state-of-the-art techniques.

*Key words:* Tracking Model Predictive Control; Linear Parameter Varying Systems; Zonotopes.

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## 1 Introduction

### 1.1 Context

Model Predictive Control (MPC) is indisputably well established. Nevertheless, the application of robust Nonlinear MPC (NMPC) is not trivial and comes with increased numerical burden, which may be an impediment for real-time applications. The majority of stabilizing NMPC schemes ensure regulation of the closed-loop dynamics to a fixed target [1]. Accordingly, asymptotic stability and constraints satisfaction are usually guaranteed with terminal ingredients, which verify invariance conditions in the neighborhood of the operation target. Nevertheless, this design method is not valid for set-point changes, since feasibility may be lost [6].

Therefore, there has been an increasing focus on NMPC schemes for Tracking, considering time-varying set-points. We highlight the following concrete Tracking NMPC tools for piece-wise constant reference signals: (i) the use of artificial reference signals, as proposed by [8], which allows less conservative terminal constraints and ensures feasibility is not lost; and (ii) the generalisation of the prior for the case of periodic reference signals with the use of terminal equality constraints or online optimized terminal sets, as proposed by [7].

### 1.2 Motivations

In this paper, we also address the issue of tracking possibly unreachable output target signals using state-feedback NMPC, closely building upon these previous papers. We address the following disadvantages of the prior methods: (i) they require the online solution of Nonlinear Programs (NPs), which are numerically ex-

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<sup>★</sup> This paper was not presented at any IFAC meeting. Corresponding author M. M. Morato (*marcelomnm@gmail.com*).

pensive and not viable for time-critical applications<sup>1</sup>; and (ii) no prediction model mismatches are considered, which should be included in any real application.

We must mention that, in parallel to the theoretical establishment of NMPC, the Linear Parameter Varying (LPV) toolkit has been brought to focus [17]. LPV models can be used to represent nonlinear dynamics with the use of known, bounded scheduling parameters  $\rho$  [10]. Recent advances on NMPC algorithms have been presented by exploiting the quasi-LPV (qLPV) embedding, thus enabling fast implementation, e.g.[11] and references therein. The elegance of the qLPV approach is that the "full-blown" nonlinear predictions are replaced by linear laws.

### 1.3 Contributions

With respect to this context, we develop a NMPC algorithm for tracking, using nominal predictions obtained through qLPV embedding. Up to our best knowledge, qLPV-embedding NMPC has only been formalized for regulation purposes [13,3].

Accordingly, as suggested in [7], the proposed solution incorporates robustness qualities against model uncertainties and additive load disturbances, using constraint tightening under the framework of [15]. Based on the one-step-ahead disturbance propagation, we enforce the satisfaction of the performance requirements by binding the prediction error within zonotope extensions [2].

The main novelties of this work are:

- (1) A method for the recursive extrapolation of scheduling parameters is developed (Sec. 3). The extrapolation serves to compute the qLPV model at each sampling instant, with bounded prediction errors. Complementary, these bounds are used to build the zonotopes that bind the uncertainty propagation along the prediction horizon<sup>2</sup>.
- (2) We also offer robust parameter-dependent terminal ingredients, which ensure recursive feasibility and stability (Sec. 4). A complementary optimisation is proposed for the choice of the artificial reference with relived complexity.
- (3) Finally, using a benchmark example, we demonstrate that the numerical complexity of the proposed online algorithm is, on average, comparable to Quadratic Programs (QPs), being much faster than the NPs from [8,7] (Sec. 5). This property is

<sup>1</sup> We stress that the application results provided in [7] solve the online NPs very fast, but relying on a solver-based solution (CaSaDi), which approximates the solution of the NPs.

<sup>2</sup> The proposed method also considers bounded process disturbances. The uncertainty propagation zonotopes offer a direct extension for such case.

made viable due to the qLPV model realisation, which enables linear predictions at each sampling period.

Preliminary discussions and the overall problem setup is presented in Sec. 2. General conclusions are drawn in Sec. 6.

**Remark 1** *With respect to the Authors' previous works, we note that this paper enhances and generalises the robust regulation algorithm from [12] for the case of time-varying reference signals. The recursive extrapolation procedure is a novel adaptation of the method from [13], now with bounded residual.en*

**Notation.** The index set  $\mathbb{N}_{[a,b]}$  represents  $\{i \in \mathbb{N} \mid a \leq i \leq b\}$ , with  $0 \leq a \leq b$ . The identity matrix of size  $j$  is denoted as  $I_j$ ;  $I_{j,\{i\}}$  denotes the  $i$ -th row of  $I_j$ ;  $\text{col}\{\cdot\}$  denotes the vectorisation of the entries and  $\text{diag}\{v\}$  denotes the diagonal matrix generated with the line vector  $v$ .  $\mathbf{1}_{n \times m}$  stands for the  $n \times m$  vector of unit entries. The value of a given variable  $v(k)$  at time instant  $k+i$ , computed based on the information available at instant  $k$ , is denoted as  $v(k+i|k)$ .  $\mathcal{K}$  refers to the class of positive and strictly increasing scalar functions that pass through the origin. A  $\mathcal{C}^1$  function  $f$  is such that it is differentiable with continuous derivatives. In this case,  $\nabla^T f : \mathbb{R}^m \rightarrow \mathbb{R}^{n \times m}$  denotes its Jacobian matrix. Consider sets  $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^n$ ,  $\mathcal{C} \subset \mathbb{R}^m$  and a matrix  $R \in \mathbb{R}^{n \times m}$ . The Minkowski set addition is defined by  $A \oplus B := \{a+b \mid a \in A, b \in B\}$ , while the Pontryagin set difference is defined by  $A \ominus B := \{a \mid a \oplus B \subseteq A\}$ . A linear mapping is  $R\mathcal{A} = \{y \in \mathbb{R}^n : y = Ra, a \in \mathcal{A}\}$ , while the Cartesian product holds as  $\mathcal{A} \times \mathcal{C} = \{z \in \mathbb{R}^{n+m} : z = (a^T c^T)^T, a \in \mathcal{A}, c \in \mathcal{C}\}$ . The unitary  $m$ -dimensional box is denoted  $\mathcal{B}_{\infty}^m = \{\xi \in \mathbb{R}^m : \|\xi\|_{\infty} \leq 1\}$ . The set of real compact intervals is given by  $\mathbb{I} = \{[a, b], a, b \in \mathbb{R}, a \leq b\}$ . An interval matrix  $\mathbf{J} \in \mathbb{I}^{n \times m}$  has  $\text{mid}(\mathbf{J})$  and  $\text{rad}(\mathbf{J})$  denoting its middle point and radius, respectively.  $\|\cdot\|$  denotes the 2-norm.

## 2 Problem Statement

### 2.1 System Description

We consider the following qLPV-embedding of a nonlinear system:

$$\begin{aligned} x(k+1) &= A(\rho(k))x(k) + B(\rho(k))u(k) + w(k) \\ y(k) &= C(\rho(k))x(k) + D(\rho(k))u(k). \end{aligned} \quad (1)$$

where  $x \in \mathbb{R}^{n_x}$  represent the states,  $u \in \mathbb{R}^{n_u}$  the inputs, and  $y \in \mathbb{R}^{n_y}$  the outputs. We consider that the additive disturbance  $w \in \mathbb{R}^{n_x}$  is bounded to a compact set with the origin at its interior, in such a way that  $w_k \in \mathcal{W} \subseteq \mathbb{R}^{n_x}$ ,  $\|w\| \leq \bar{w}$ .

**Remark 2** We note that any nonlinear system can be embedded to a qLPV realisation, as long as (exact, linear, convex, convex-concave) differential inclusion is satisfied. This is detailed in [18,11] and experimentally exemplified in [4].

For set-operation simplicity, hard compact polyhedral constraints on state and input  $(x(k), u(k)) \in \mathcal{Z} = \mathcal{X} \times \mathcal{U}$  are considered, which define admissible operation of system given by:

$$\begin{aligned} \mathcal{X} &:= \{x \in \mathbb{R}^{n_x} : H_x x \leq h_x\} \\ \mathcal{U} &:= \{u \in \mathbb{R}^{n_u} : H_u u \leq h_u\}. \end{aligned} \quad (2)$$

Outer box-type constraints<sup>3</sup> are implied:  $|x_j| \leq \bar{x}_j, \forall j \in \mathbb{N}_{[1, n_x]}$  and  $|u_j| \leq \bar{u}_j, \forall j \in \mathbb{N}_{[1, n_u]}$ .  $\mathcal{Y} := \{y \in \mathbb{R}^{n_y} \mid H_y y \leq h_y\} = h(\mathcal{Z})$  is a compact and convex set which defines the possible outputs  $y$  mapped by  $h(x, u) = A(f_\rho(x))x + B(f_\rho(x))u$  with  $(x, u) \in \mathcal{Z}$ .

This qLPV model is scheduled by the state-dependent time-varying parameter  $\rho(k) = f_\rho(x(k)) \in \mathcal{P} \subseteq \mathbb{R}^{n_\rho}$ , which is bounded and measured online (known). The nonlinear scheduling map  $f_\rho(x)$  is algebraic, class  $\mathcal{C}^1$  for all  $x \in \mathcal{X}$ , and set- and vector-wise applicable<sup>4</sup>.

**Assumption 1** The nonlinear scheduling parameter map  $f_\rho : \mathcal{X} \rightarrow \mathcal{P}$  agrees to a local Lipschitz condition around any arbitrary point  $x \in \mathcal{X}$ , this is:

$$\|f_\rho(x) - f_\rho(\hat{x})\|_\infty \leq \gamma_\rho \|x - \hat{x}\|_\infty, \forall x, \hat{x} \in \mathcal{X}, \quad (3)$$

where the smallest constant  $\gamma_\rho$  that satisfies Eq. (3) is known as the Lipschitz constant for  $f_\rho(\cdot)$ .

We consider  $\mathcal{P} := \{\rho \in \mathbb{R}^{n_\rho} : \underline{\rho}_j \leq \rho_j \leq \bar{\rho}_j, j \in \mathbb{N}_{[1, n_\rho]}\}$ . We stress that the future scheduling parameters  $\rho(k+j), \forall j \in \mathbb{N}_{[1, \infty]}$  are unknown at time instant  $k$ . Therefore, for the application of MPC, estimates  $\hat{\rho}(k+j|k)$  are used, as further detailed in Sec. 3.

**Assumption 2** The scheduling variables  $\rho(k)$  exhibit a bounded rate of variation. This is:  $\delta\rho(k+1) = (\rho(k+1) - \rho(k)) \in \delta\mathcal{P}$ , with:

$$\delta\mathcal{P} := \left\{ \delta\rho_j \in \mathbb{R} : \underline{\delta\rho}_j \leq \delta\rho_j \leq \bar{\delta\rho}_j, \forall j \in \mathbb{N}_{[1, n_\rho]} \right\}. \quad (4)$$

**Remark 3** Bounds on scheduling parameters' variations  $\delta\mathcal{P}$  can be obtained based on their dependence on the states  $\rho(k) = f_\rho(x(k))$ . Specifically, this can be done by placing bounds<sup>5</sup> on  $\delta\rho = f_\rho(A(f_\rho(x))x + B(f_\rho(x))u +$

<sup>3</sup> The bounds  $\bar{x}_j$  and  $\bar{u}_j$  can be found from Eq. (2) through linear programming.

<sup>4</sup> It follows that  $\mathcal{P} \subseteq f_\rho(\mathcal{X})$ , as detailed in [3].

<sup>5</sup> For such, the system must be ultimately bounded.

$w) - f_\rho(x)$  for  $(x, u), (A(f_\rho(x))x + B(f_\rho(x))u + w, u) \in \mathcal{Z}$  and  $w \in \mathcal{W}$ , either by interval arithmetics or optimisation.

## 2.2 Closed-Loop Paradigm

Through the sequel, we assume that the states  $x(k)$  are measurable at each sampling instant. Therefore, in order to limit the disturbance propagation, we consider the following LPV state-feedback control law:

$$u(k) = v(k) + K(\rho(k))x(k), \quad (5)$$

where the virtual input  $v(k)$  is used for the MPC synthesis and the parameter-dependent feedback gain  $K(\rho(k))$  is chosen in order to attenuate the propagation of disturbances. We assume that this gain  $K(\rho(k))$  is structurally known<sup>6</sup>. The closed-loop dynamics are thus given by:

$$\begin{aligned} x(k+1) &= (A(\rho) + B(\rho)K(\rho))x(k) + B(\rho)v(k) + w(k) \\ &= A_\pi(\rho)x(k) + B(\rho)v(k) + w(k), \end{aligned} \quad (6)$$

and the process constraints  $(x(k), u(k)) \in \mathcal{Z}$  can be expressed in terms of state and virtual input by  $(x(k), v(k)) \in \mathcal{Z}_\pi$ , where:

$$\mathcal{Z}_\pi := \left\{ z \in \mathbb{R}^{n+n_u} : \begin{pmatrix} H_x & 0 \\ H_u K(\rho) & H_u \end{pmatrix} z \leq \begin{pmatrix} h_x \\ h_u \end{pmatrix} \right\}. \quad (7)$$

Given an initial condition  $x(k) \in \mathbb{R}^{n_x}$ , the closed-loop trajectories for the following  $j$  steps through the qLPV embedding are given by exploiting Eq. (6), which gives:

$$x(k+j) := \phi_j(x(k), \mathbf{v}_{[k+j-1]}, \mathbf{w}_{[k+j-1]}, \rho_{[k+j-1]}), \quad (8)$$

being  $\mathbf{v}_{[k+j-1]}$ ,  $\mathbf{w}_{[k+j-1]}$ , and  $\rho_{[k+j-1]}$  the vectors of future inputs, disturbances and scheduling parameters from sample  $k$  to  $k+j-1$ , respectively. Note that this is a closed-loop model due to the stabilising feedback  $K(\rho)x$ , with matrix  $A_\pi(\rho)$  structurally known, being  $v$  an external input (the MPC law).

Both the future disturbances (due to stochastic nature) and future scheduling parameters (due to future state-dependence) are unknown from the MPC viewpoint at the sampling instant  $k$ . Therefore, we consider the following nominal prediction model:

$$x(k+j|k) = \phi_j(x(k), \mathbf{v}_{[k+j-1]}, \mathbf{0}, \hat{\rho}_{[k+j-1]}^k), \quad (9)$$

where the future disturbances are presumably null and the scheduling parameters estimates  $\hat{\rho}_{[k+j-1]}^k =$

<sup>6</sup> Consider an affine gain  $K(\rho) = \sum_{i=1}^{n_\rho} K_i \rho_i$ . In this case,  $K_i$  are assumed to be known.

$\text{col}(r\hat{h}o(k|k), \dots, r\hat{h}o(k+j-1|k))$  are provided by the extrapolation algorithm from Sec. 3 at sampling instant  $k$ . This model is equivalent to using:  $x(k+1|k) = A_\pi(r\hat{h}o(k))x(k) + B(\rho)v(k|k)$ ,  $x(k+1|k) = A_\pi(\hat{\rho}(k+1|k))x(k+1|k) + B(\rho)v(k+1|k)$ , and so forth, with estimates  $\hat{\rho}$  generated by the extrapolation algorithm detailed in Sec. 3.

### 2.3 Disturbance Propagation

Given a prediction horizon  $N_p$ , we consider a group of compact sets  $\mathcal{E}(j)$ ,  $j \in \mathbb{N}_{[0, N_p-1]}$  that bounds the difference between the scheduling parameter predictions made in  $k$  and  $k+1$ . That is, we have  $\hat{\rho}(k+j|k+1) - \hat{\rho}(k+j|k) \in \mathcal{E}(j)$ ,  $\forall j \in \mathbb{N}_{[0, N_p-1]}$ <sup>7</sup>.

Due to the mismatch between nominal predictions in Eq. (9) and real system trajectories in Eq. (8), we consider one-step-ahead disturbance propagation sets  $\mathcal{S}(j)$ ,  $j \in \mathbb{N}_{[0, N_p]}$  [15] in order to guarantee recursive feasibility of the MPC strategy.

**Definition 1** *One-step-ahead disturbance propagation sets  $\mathcal{S}(j)$ ,  $j \in \mathbb{N}_{[0, N_p]}$  are compact sets that satisfy: (1)  $\mathcal{S}(0)$  contains  $\mathcal{W}$ , and (2) For all  $x_a, x_b \in \mathbb{R}^n$ ,  $v \in \mathbb{R}^{n_u}$ ,  $\rho_a, \rho_b \in \mathcal{P}$  and  $j \in \mathbb{N}_{[1, N_p]}$ , with  $(x_a, v) \in \mathcal{Z}_\pi \ominus (\mathcal{S}(j-1) \times \{0\})$ ,  $(x_b - x_a) \in \mathcal{S}(j-1)$ , and  $(\rho_b - \rho_a) \in \mathcal{E}(j-1)$ , it follows that  $(A_\pi(\rho_b)x_b + B(\rho_b)v) - (A_\pi(\rho_a)x_a + B(\rho_a)v) \in \mathcal{S}(j)$ .*

Taking  $x_a = x(k+1|k)$  and  $x_b = x(k+1)$ , we have, by induction from Def. 1, that  $x(k+j|k+1) \in x(k+j|k) \oplus \mathcal{S}(j-1)$ ,  $\forall j \in \mathbb{N}_{[1, N_p+1]}$  for any admissible sequence of inputs  $\mathbf{v}_{[k+j-1]}$  and scheduling parameter predictions  $\hat{\rho}_{[k+j-1]}^k$  and  $\hat{\rho}_{[k+j]}^{k+1}$ .

The sets  $\mathcal{S}(j)$  thus bound the difference between the predictions made in  $k$  and  $k+1$ , and therefore can be used to guarantee recursive feasibility and constraint satisfaction of the MPC based on nominal predictions.

In order to compute zonotopic reachable sets  $\mathcal{S}_z(j)$ ,  $\forall j \in \mathbb{N}_{[0, N_p]}$  that satisfy Def. 1, we develop Theorem 1, based on Lemma 1.

**Lemma 1** *Consider a centered zonotope  $X = MB_\infty^{n_g} \subseteq \mathbb{R}^m$ , an interval matrix  $\mathbf{J} \in \mathbb{I}^{n \times m}$ , and a zonotope family  $\mathbf{Z} = \mathbf{J}X = \{Jx, J \in \mathbf{J}, x \in X\}$ . A zonotopic inclusion is defined by:*

$$\diamond(\mathbf{Z}) := \text{mid}(\mathbf{J})X \oplus PB_\infty^{n_g}, \quad (10)$$

<sup>7</sup> Being  $N_p$  the prediction horizon, at time instant  $k$ , we only need to predict  $\rho(k+j|k)$ ,  $j \in \mathbb{N}_{[0, N_p-1]}$  in order to obtain  $x(k+j|k)$  for  $j \in [1, N_p]$ . We thus consider  $\hat{\rho}(k+N_p|k) = \hat{\rho}(k+N_p-1|k)$ ,  $\forall k \geq 0$ .

where  $P$  is a diagonal matrix such that:

$$P_{ii} = \sum_{j=1}^{n_g} \sum_{k=1}^m \text{rad}(\mathbf{J})_{ik} |M_{kj}|, \quad \forall i \in \mathbb{N}_{[1, n]}. \quad (11)$$

It holds that  $\mathbf{Z} \subseteq \diamond(\mathbf{Z})$ .

**Proof 1** Refer to [14]. □

**Theorem 1** *Consider  $Z \subseteq \mathbb{R}^{n_x+n_u}$ ,  $\mathcal{S}_0 \subseteq \mathbb{R}^{n_x}$  zonotopes and  $\mathbf{A}, \Delta_A(j) \in \mathbb{I}^{n_x \times n_x}$ ,  $\Delta_B(j) \in \mathbb{I}^{n_x \times n_u}$  interval matrices satisfying  $\mathcal{W} \subseteq \mathcal{S}_0$ ,  $\mathcal{Z}_\pi \subseteq Z$ ,  $A_\pi(\rho) \in \mathbf{A}$ ,  $A_\pi(\rho_b(j)) - A_\pi(\rho_a(j)) \in \Delta_A(j)$  and  $B(\rho_b(j)) - B(\rho_a(j)) \in \Delta_B(j)$  for all  $\rho, \rho_a(j), \rho_b(j) \in \mathcal{P}$  and  $j \in \mathbb{N}_{[1, N_p-1]}$ , with  $\rho_b(j) - \rho_a(j) \in \mathcal{E}(j-1)$ . The zonotopes  $\mathcal{S}_z(j)$ ,  $j \in \mathbb{N}_{[0, N_p]}$  defined recursively by  $\mathcal{S}_z(0) = \mathcal{S}_0$  and*

$$\mathcal{S}_z(j) = \mathcal{V}(j) \oplus \diamond(\mathbf{A}\mathcal{S}_z(j-1)), \quad j \in \mathbb{N}_{[1, N_p]}, \quad (12)$$

where  $\mathcal{V}(j)$  is given by  $\mathcal{V}(j) = \diamond\left(\left(\Delta_A(j) \ \Delta_B(j)\right) \mathcal{Z}_\pi\right)$ , satisfy Def. 1.

**Proof 2** *The first condition is satisfied by design. Considering  $x_a, x_b \in \mathbb{R}^{n_x}$ ,  $v \in \mathbb{R}^{n_u}$ ,  $\rho_a, \rho_b \in \mathbb{R}^{n_\rho}$  as given by the second condition for some  $j \in \mathbb{N}_{[1, N_p]}$ , and  $\Delta_j = (A_\pi(\rho_b)x_b + B(\rho_b)v) - (A_\pi(\rho_a)x_a + B(\rho_a)v)$ , we have:*

$$\begin{aligned} \Delta_j &= (A_\pi(\rho_b) - A_\pi(\rho_a))x_a + A_\pi(\rho_b)(x_b - x_a) \\ &\quad + (B(\rho_b) - B(\rho_a))v \\ &\in \left(\Delta_A(j) \ \Delta_B(j)\right) \mathcal{Z}_\pi \oplus \mathbf{A}\mathcal{S}_z(j-1) \\ &\subseteq \mathcal{V}(j) \oplus \diamond(\mathbf{A}\mathcal{S}_z(j-1)) = \mathcal{S}_z(j). \end{aligned}$$

Therefore, the sets  $\mathcal{S}(j)$  satisfy Def. 1. □

**Remark 4** *In the case of qLPV systems with  $A_\pi(\rho)$  and  $B(\rho)$  affine<sup>8</sup> on  $\rho$ , it follows that  $A_\pi(\rho_b(j)) - A_\pi(\rho_a(j)) = \overline{A}_\pi(\rho_b(j) - \rho_a(j))$  and  $B(\rho_b(j)) - B(\rho_a(j)) = \overline{B}(\rho_b(j) - \rho_a(j))$ , with  $\overline{A}_\pi(\cdot)$  and  $\overline{B}(\cdot)$  being linear mappings. Then, the interval matrices  $\Delta_A(j)$  and  $\Delta_B(j)$  can be computed directly from  $\mathcal{E}(j)$ . In the case of non-affine models, interval arithmetic or optimisation can be used to obtain the interval matrices  $\Delta_A(j)$  and  $\Delta_B(j)$  from  $\mathcal{P}$  and  $\mathcal{E}(j)$ .*

**Remark 5** *Due to the zonotope inclusion and Minkowsky sum, the number of generators of the zonotopes  $\mathcal{S}(j)$  increases for each iteration. Methods for complexity reduction can be used to restrict the number of generators of each  $\mathcal{S}(j)$  to a predefined value [16].*

<sup>8</sup> In this case,  $K$  must be parameter-independent such that  $A_\pi(\rho) = A(\rho) + B(\rho)K$  becomes affine.

**Remark 6** We stress that the qLPV model in Eq. (6) requires a linear nominal prediction model, as in Eq. (9), which changes at each sampling instant  $k$ , since it is based on an estimation of the future scheduling parameters. Therefore, the propagation of disturbances along the horizon is crucial, but may yield conservative sets  $\mathcal{S}(j)$  due to the changes in  $\hat{\rho}_{[k+j-1]}^k$ . Nonlinear predictions can result in less conservative sets  $\mathcal{S}(j)$ , but also result in a higher online computational burden.

**Remark 7** A trivial choice for  $\hat{\rho}_{[k+j-1]}^k$  would be  $\hat{\rho}_{[k+j-1]}^k = (\rho(k), \dots, \rho(k))$ , amounting to a frozen estimation of the scheduling parameters. This would result in  $\mathcal{E}(j) = \delta\mathcal{P}$ , simplifying calculations. Nonetheless, the application of extrapolation algorithms, as the one described in Sec. 3, allow for less variation between  $\hat{\rho}_{[k+j-1]}^k$  and  $\hat{\rho}_{[k+j]}^{k+1}$  and thus smaller sets  $\mathcal{E}(j)$ , reducing the conservatism of the disturbance propagation.

## 2.4 Admissible Equilibrium Points

In this paper, we consider a state-feedback MPC strategy in order to steer the system states to a steady-state condition  $x_r$  which implies in an output reference tracking goal  $y_r$ . Therefore, we use the MPC algorithm to minimize the variations of  $(x, u)$  from the desired set-point targets  $(x_r, u_r)$ , which implies in the output target  $y_r$ .

Following the lines of [8], we consider that there exists a linear (parameter varying) combination of the states  $x$  and inputs  $u$  that ensures  $y(k) \rightarrow y_r$ . Through the sequel, we assume that for all  $y_r \in \mathcal{Y}$  there exists a pair  $z_r = (x_r^T, u_r^T)^T$  which satisfies

$$\begin{bmatrix} (\mathbb{I} - A(f_\rho(z_r))) & -B(f_\rho(z_r)) \\ C(f_\rho(z_r)) & D(f_\rho(z_r)) \end{bmatrix} z_r = \begin{bmatrix} 0_{n_x} \\ y_r \end{bmatrix}. \quad (13)$$

We consider the set of admissible steady-state outputs  $\mathcal{Y}_a \subset \mathcal{Y}$  as the set of outputs generated by feasible states and inputs that satisfy the equilibrium condition, i.e., Eq. (13) is satisfied for some  $z_r \in \mathcal{Z}$ , with  $f_\rho(z_r) \in \mathcal{P}$ . More specifically, we have  $\mathcal{Y}_a := \{y \in \mathcal{Y} \mid y = [C(f_\rho(z_r)) D(f_\rho(z_r))]z_r, z_r \in \mathcal{Z} \ominus \mathcal{Z}_\xi, z_r = [A(f_\rho(z_r)) B(f_\rho(z_r))]z_r\}$ , with  $\mathcal{Z}_\xi$  being an auxiliary non-empty set with the origin in its interior, included so that the frontier of  $\mathcal{Z}$  is excluded ( $\mathcal{Z} \ominus \mathcal{Z}_\xi \subset \mathcal{Z}$ ).

We assume that Eq. (13) implies in a unique correspondence between the steady-state output goal  $y_r$  and the pair  $z_r$ . Thus, there exist locally Lipschitz continuous functions  $g_x : \mathcal{Y} \rightarrow \mathbb{R}^{n_x}$  and  $g_u : \mathcal{Y} \rightarrow \mathbb{R}^{n_u}$  such that  $x_r = g_x(y_r)$  and  $u_r = g_u(y_r)$  for all  $y_r \in \mathcal{Y}_a$ . Complementary, we assume that there exists a continuous functions  $g_\rho : \mathcal{Y} \rightarrow \mathbb{R}^{n_\rho}$  that maps the equilibrium scheduling parameter

$\rho_r = g_\rho(y_r)$ . Note that  $g_\rho(y_r) := f_\rho(g_x(y_r), g_u(y_r))$  and  $g_v(y_r) = g_u(y_r) - K(g_\rho(y_r))g_x(y_r)$ . For such unique correspondence to exist, the left hand-side matrix in Eq. (13) must be non-singular for all  $z_r \in \mathcal{Z}$  [8], which means there must be no redundant inputs or outputs as well as no integrator nodes in the qLPV dynamics in Eq. (1).

## 2.5 Tracking NMPC Design

Consider admissible time-varying piecewise constant output references  $y_r \in \mathcal{Y}_a$ . Then, the nonlinear process must be controlled in such way that the state trajectories are suitably steered to  $x_r$ , with a steady-state control input  $u_r$ , despite disturbances and the discrepancy between the qLPV prediction model (9) and the real system trajectories (8). We consider a prediction horizon of  $N_p$  steps.

As discussed in the prequel, we apply a robust NMPC scheme, for which contracted constraints are used. Considering an initial constraint set  $\mathcal{Z}_\pi(0) = \mathcal{Z}_\pi$ , the following sets for  $j \in \mathbb{N}_{[1, N_p]}$  are iteratively taken as:

$$\mathcal{Z}_\pi(j+1) = \mathcal{Z}_\pi(j) \ominus (S(j) \times \{0\}). \quad (14)$$

Therefore, at each sampling instant  $k$ , we measure the state  $x(k)$ , compute the scheduling parameter  $\rho(k)$ , estimate the scheduling sequence  $\hat{\rho}_{[k+N_p-1]}^k$ , and solve the following optimisation problem, which embeds the performance objectives of the system, as well as the operational constraints:

$$\begin{aligned} \min_{\mathbf{v}_{[k, k+N_p-1]}} & \sum_{j=0}^{N_p-1} \ell(x(k+j|k) - x_r, u(k+j|k) - u_r) \\ & + V(x(k+N_p|k) - x_r), \end{aligned} \quad (15)$$

$$\text{s.t. : } \begin{cases} x(k+j+1|k) = A_\pi(\hat{\rho}(k+j|k))x(k+j|k) \\ \quad + B(\hat{\rho}(k+j|k))v(k+j|k), j \in \mathbb{N}_{[0, N_p-1]}, \\ (x(k+j|k), v(k+j|k)) \in \mathcal{Z}_\pi(j), j \in \mathbb{N}_{[0, N_p-1]}, \\ x(k+N_p|k) \in \mathbf{X}_f, \end{cases}$$

where  $\ell(\cdot, \cdot)$  is a quadratic stage cost,  $V(\cdot)$  is a terminal cost,  $x_r = g_x(y_r)$ ,  $u_r = g_u(y_r)$  and  $\mathbf{X}_f$  is a terminal robust invariant set (more discussions on this matter are presented in Sec. 4). Note that  $u(k+j|k)$  is implicitly defined through Eq. (5). From the optimal solution of this problem  $\mathbf{v}_{[k, k+N_p-1]}^*$ , we apply the first entry  $v^*(k|k)$  to the process according to Eq. (5).

### 3 A Recursive Extrapolation Algorithm

In order to estimate the scheduling sequence  $\hat{\rho}_{[k+N_p-1]}$ , we use a rather simple extrapolation mechanism. The proposed method is an adaptation of the framework in [13], now using a first-order Taylor expansion of the scheduling proxy  $f_\rho(x(k))$  around the state deviation instead of the Least-Squares rule. The main novelty is that the generated residual is bounded.

We denote  $\delta x(k+j) = x(k+j+1) - x(k+j)$  as the incremental state deviation. This variable is bounded<sup>9</sup> to a compact and convex box-type set  $\delta\mathcal{X} := \{\delta x \in \mathbb{R}^{n_x} : |\delta x_j| \leq \bar{\delta}x_j, \forall j \in \mathbb{N}_{[1, n_x]}\}$ .

The method is as follows: consider that the static scheduling map  $f_\rho(x)$  can be approximated by:

$$f_\rho(x) = f_\rho(x)|_{\check{x}} + \left. \frac{\partial f_\rho(x)}{\partial x} \right|_{\check{x}} (x - \check{x}) + \xi_\rho, \quad (16)$$

being  $\check{x}$  the expansion point and  $\xi_\rho$  a residual noise which inherits the discrepancy between the real static map and its approximate.

The map  $f_\rho(x)$  is class  $\mathcal{C}^1$  in  $x$ , and we assume that  $\left. \frac{\partial f_\rho(x)}{\partial x} \right|_{\check{x}}$  is ultimately bounded for all  $\check{x} \in \mathcal{X}$ . Thus, from Eq. (16), we write:

$$f_\rho(x(k+j)) = f_\rho(x(k+j-1)) + \xi_\rho(k+j-1) + \underbrace{\left. \frac{\partial f_\rho}{\partial x(k+j)} \right|_{x(k+j-1)}}_{f_\rho^\partial(k+j-1)} \delta x(k+j-1). \quad (17)$$

Expanding Eq. (17) along the horizon gives:

$$\begin{aligned} \rho(k+1) &= \rho(k) + f_\rho^\partial(k)\delta x(k) + \xi_\rho(k), \\ &\vdots \\ \rho(k+N_p-1) &= \rho(k+N_p-2) + \xi_\rho(k+N_p-2) \\ &\quad + f_\rho^\partial(k+N_p-2)\delta x(k+N_p-2). \end{aligned}$$

It is a fact that  $\rho(k)$  and  $\delta x(k)$  are known variables at each instant  $k$ , whereas  $f_\rho^\partial(k)$  can be numerically evaluated. In practice,  $f_\rho^\partial(k+j)$  is unknown for  $j \in \mathbb{N}_{[1, N-2]}$ . Then, in order to construct the estimation mechanism, we consider that  $f_\rho^\partial(k+j) = f_\rho^\partial$ ,  $\forall j \in \mathbb{N}_{[1, N-2]}$ , where  $f_\rho^\partial$  denotes the partial derivative evaluated at instant  $k$ .

<sup>9</sup> Bounds can be found from Eqs. (1)-(2) through interval arithmetic.

By doing so, and disregarding the residual terms, it is implied that  $\hat{\rho}(k+j|k) = \hat{\rho}(k+j-1|k-1) + f_\rho^\partial \delta x(k+j-1|k)$ . This is: the estimate for the future scheduling variable  $\hat{\rho}(k+j|k)$  can be written as the sum of the estimate from the previous sample  $\hat{\rho}(k+j-1|k-1)$  with a adjustment term  $f_\rho^\partial \delta x(k+j-1|k)$ . Accordingly, we can write the vector-wise extrapolation in a recursive fashion:

$$\hat{\rho}_{[k+N_p-1]}^k = (\hat{\rho}_{[k+N_p-1]}^{k-1})^\dagger + f_\rho^\partial (\delta x_{[k+N_p-2]})^\dagger, \quad (18)$$

where the  $(\cdot)^\dagger$  operator indicates a correction of the vectors with the known terms<sup>10</sup>. If sought, a forgetting factor can be added to  $(\hat{\rho}_{[k+N_p-1]}^{k-1})^\dagger$ , multiplying the original vector by an exponentially decaying term, such as  $\mathbb{I}e^{-k/k_{max}}$ . Such forgetting factor can be used to attenuate the amount of mistaken information passed from the scheduling sequence estimate  $\hat{\rho}_{[k+N_p-1]}^{k-1}$  to the following  $\hat{\rho}_{[k+N_p-1]}^k$ . We note that  $(\delta x_{[k+N_p-1]})^\dagger$  can be computed thanks to the nominal prediction model of Eq. (9), using  $\hat{\rho}_{[k+N_p-1]}^{k-1}$  and  $\mathbf{v}_{[k+N_p-1]}$  as the solution of the previous MPC iteration.

We stress that the recursive extrapolation law in Eq. (18) does not ensure that the scheduling sequence estimates abide to the scheduling parameter set  $\mathcal{P}$ . Thus, each new extrapolation vector  $\hat{\rho}_{[k+N_p-1]}^k$  is ‘‘clipped’’ with respect to the bounds on  $\rho$  and  $\delta\rho$ .

**Lemma 2** *Assume that input-to-state stability is ensured, despite the biased prediction model. Then, the recursive extrapolation algorithm in Eq. (18) converges. This is, after a finite amount of steps  $k_c$  it holds that  $\lim_{k \rightarrow k_c} \hat{\rho}_{[k+N_p-1]}^k \rightarrow \rho_{[k+N_p-1]}^{k_c}$ .*

**Proof 3** *The convergence property can be demonstrated with the aid of the residual term  $\xi_\rho(k+j)$ , which should turn null. We demonstrate that this is true for  $j = 1$ ; the same holds for the following steps due to equivalency. Considering that  $\lim_{k \rightarrow \infty} x(k+1) = x(k)$ , take  $\xi_\rho(k) = f_\rho(x(k+1)) - f_\rho(x(k)) - f_\rho^\partial \delta x(k)$ . Due to the stabilisation implied by the MPC, it directly follows that  $\lim_{k \rightarrow \infty} f_\rho(x(k+1)) = \lim_{k \rightarrow \infty} f_\rho(x(k))$  and  $\lim_{k \rightarrow \infty} \delta x(k) = 0$ . Then,  $\lim_{k \rightarrow \infty} \xi_\rho(k) = -\lim_{k \rightarrow \infty} f_\rho^\partial \delta x(k) \rightarrow 0$ . This concludes the proof.  $\square$*

**Lemma 3** *The estimation error (residual) is ultimately bounded to a convex set  $\mathcal{E} \subset \mathcal{P}$ . It follows that  $\|\xi_\rho\|_\infty = (\gamma_\rho + \bar{f}_\rho^\partial) \bar{\delta}x$ .*

<sup>10</sup> Note that at, instant  $k$ , the scheduling variable  $\rho(k)$  and the state deviation  $\delta x(k)$  are known. Thus,  $(\hat{\rho}_{[k+N_p-1]}^{k-1})^\dagger$  stands for  $\hat{\rho}_{[k+N_p-1]}^{k-1}$  with the term  $\hat{\rho}(k|k-1)$  replaced by  $\rho(k)$ , whereas  $(\delta x_{[k+N_p-1]})^\dagger$  stands for  $\delta x_{[k+N_p-1]}$  with the term  $\delta x(k|k-1)$  replaced by  $\delta x(k)$ .

**Proof 4** The residual term from Eq. (17) is:

$$\xi_\rho(k+j+1) = f_\rho(x(k+j+1)) - f_\rho(x(k+j)) - f_\rho^\partial \delta x(k+j). \quad (19)$$

Using a triangular inequality, we have:

$$\|\xi_\rho(k+j+1)\|_\infty \leq \|f_\rho(x(k+j+1)) - f_\rho(x(k+j))\|_\infty + \|f_\rho^\partial \delta x(k+j)\|_\infty. \quad (20)$$

As of Assumption 1, we have:

$$\|\xi_\rho(k+j+1)\|_\infty \leq \gamma_\rho \|\delta x(k+j)\|_\infty + \|f_\rho^\partial \delta x(k+j)\|_\infty.$$

Since  $f_\rho^\partial$  is ultimately bounded, i.e.  $\|f_\rho^\partial\|_\infty \leq \bar{f}_\rho^\partial$ , it follows that:  $\|\xi_\rho\|_\infty = (\gamma_\rho + \bar{f}_\rho^\partial) \bar{\delta x}$ . Since  $\|\xi_\rho\|_\infty < \bar{\rho}$ ,  $\mathcal{E} \subset \mathcal{P}$ . This concludes the proof.  $\square$

**Remark 8** We note that even if  $f_\rho(\cdot)$  is not locally Lipschitz, different bounds for  $\|\xi_\rho\|_\infty$  can still be computed, since  $\xi_\rho$  converges to zero as of Lemma 2. The computation of these bounds is not the focus of this paper.

#### 4 Robust NMPC for Tracking

In this Section, we detail the proposed robust NMPC scheme for Tracking. The proposed strategy holds similarities to the tracking NMPCs from [8,7], but the method is conceived with regard to nominal predictions obtained through the qLPV model in Eq. (1). The use of these nominal predictions makes our method faster than the prior, since the resulting numerical toughness is much closer to that of a QP. As done in [15], we use constraint tightening, terminal cost and terminal constraints to ensure recursive feasibility and input-to-state stability properties.

Firstly, we make reference to Remark 9 from [7]: the NMPC propositions in previous references [8,7] provides exponential closed-loop stability and recursive feasibility guarantees in the case of no model mismatch. Nevertheless, this is rarely the case in any practical application. When considering the qLPV predictions based on an estimated scheduling sequence, as we do in this paper, model mismatches inherently emerge. Therefore, in order to ensure robust recursive feasibility and stability (despite bounded disturbances and prediction mismatches), the MPC problem should be adjusted using constraint tightening techniques [15]. This is what we perform in this paper with the use of the zonotopes detailed in Sec. 2.3. Through the sequel, we assume that robust constraint satisfaction is guaranteed thanks to the constraint tightening and the corresponding disturbance propagation zonotopes. Thence, we progress by detailing the proposed MPC setting for the case of possibly unreachable output reference signals.

In order to potentially increase the closed-loop domain of attraction and avoid loss of feasibility due to set-point changes, we benefit from the artificial referencing mechanism from [8]. Instead of tracking the time-varying output set-point  $y_r$  we consider an additional decision variable  $y_a$ . The tracking convergence is then ensured with the inclusion of an offset cost  $V_O(y_a - y_r)$ , which penalizes the deviation between the artificial output reference  $y_a$  and the real set-point  $y_r$ .

The terminal set in (5) is adjusted to the tracking case, instead of the origin, considering the equilibria defined by any triad  $(x_r, u_r, y_r)$  which verifies (13). Thus, we consider a terminal Tracking Robust Positive Invariant Set (TRPI set):

**Definition 2 (Parameter-dependent TRPI Set)** Consider a set  $\Gamma(\rho) \subseteq \mathbb{R}^{n_x+n_y}$  and a terminal control law  $u_t = \kappa_t(x, y_r)$ .  $\Gamma(\rho)$  is a TRPI set for system (1) if, for all  $(x, y_r) \in \Gamma(\rho)$  and  $w \in \mathcal{S}(N_p)$ ,  $(A(\rho)x + B(\rho)\kappa_t(x, y_r) + w, y_r) \in \Gamma(\rho)$ .

**Remark 9** The TRPI sets differ from the Tracking Positive Invariant (TPI) sets from [8] due to the robustness properties. The synthesis of TPI sets implicitly considers that the real system trajectories and the nominal predictions are identical.

The previous definition implies that once a given state  $x$  and output reference  $y_r$  are found inside the parameter-dependent set  $\Gamma(\rho)$ , the terminal control law  $\kappa_t: \mathbb{R}^{n_x+n_y} \rightarrow \mathbb{R}^{n_u}$  ensures that any subsequent state, with the same output reference, is also inside this set, regardless of the bounded load disturbance  $w \in \mathcal{W}$ .

Thus, based on the current state measurement  $x(k)$ , the estimated scheduling sequence  $\hat{\rho}_{[k+N_p-1]}^k$ , and the current output set-point  $y_r$ , the NMPC optimisation (15) is re-stated as follows, being  $g_x(\cdot)$  and  $g_u(\cdot)$  the non-linear maps that convert steady-state output references into state and input targets that satisfy Eq. (13), respectively:

$$\begin{aligned} \min_{\hat{v}, y_a} \quad & \sum_{j=0}^{N_p-1} \ell(x(k+j|k) - x_a, (v(k+j|k) + K(\hat{\rho}(k+j|k))x(k+j|k) - u_a) \\ & + V(x(k+N_p|k) - x_a) + V_O(y_a - y_r), \\ \text{s.t. :} \quad & \begin{cases} x(k+j+1|k) = A_\pi(\hat{\rho}(k+j|k))x(k+j|k) \\ + B(\hat{\rho}(k+j|k))v(k+j|k), j \in \mathbb{N}_{[0, N_p-1]}, \\ (x(k+j|k), v(k+j|k)) \in \mathcal{Z}_\pi(j), j \in \mathbb{N}_{[0, N_p-1]}, \\ x_a = g_x(y_a), u_a = g_u(y_a) \\ (x(k+N_p|k), y_a) \in \Gamma(\rho(k+N_p-1)) \end{cases} \end{aligned} \quad (21)$$

In this problem,  $\mathcal{Z}_\pi(j)$  represent the tightened con-

straints, as previously detailed, and  $\Gamma$  the terminal set. The set of states  $x(k) \in \mathcal{X}_a(N_p)$  such that this problem has a feasible solution is called the domain of attraction of the proposed controller<sup>11</sup>. We proceed by further detailing the MPC costs  $\ell(\cdot, \cdot)$ ,  $V(\cdot)$ , and the terminal TRPI set  $\Gamma(\cdot)$ .

**Assumption 3** (1) *The stage cost function is positive definite and uniformly continuous such that:*

$$\begin{aligned} \ell(x, u) &\geq \alpha_\ell(\|x\|) \\ |\ell(x_1, u_1) - \ell(x_2, u_2)| &\leq \lambda_x(\|x_1 - x_2\|) \\ &\quad + \lambda_u(\|u_1 - u_2\|), \end{aligned} \quad (22)$$

where  $\alpha_\ell$ ,  $\lambda_x$  and  $\lambda_u$  are  $\mathcal{K}$ -functions.

- (2) *The set of admissible artificial references  $\mathcal{Y}_t = \{y_a \in \mathbb{R}^{n_y} : (g_x(y_a), y_a) \in \Gamma(g_\rho(y_a))\}$  is a convex subset of  $\mathcal{Y}_a(N_p) = \{y_a \in \mathbb{R}^{n_y} : (g_x(y_a), g_v(y_a)) \in \mathcal{Z}_\pi(N_p)\}$ .*
- (3) *The offset cost function is positive definite, uniformly continuous and convex, thus assuring that the minimizer  $y_a^o = \arg \min_{y_a \in \mathcal{Y}_t} V_O(y_a - y_r)$  is unique, being  $V_O(\cdot)$  a weighted quadratic term. Furthermore, for any  $y_r \in \mathbb{R}^{n_y}$  and  $y_a \in \mathcal{Y}_t$ , we have  $V_O(y_a - y_r) - V_O(y_a^o - y_r) \geq \alpha_O(\|y_a - y_a^o\|)$ , where  $\alpha_O$  is a  $\mathcal{K}$ -function<sup>12</sup>.*

**Assumption 4** (1) *The terminal control law satisfies  $\kappa_t(x_a, y_a) = u_a$  for all admissible equilibrium points  $(x_a, u_a, y_a)$* <sup>13</sup>.

- (2) *The terminal set  $\Gamma$  is an admissible TRPI set. That is, we have  $\Gamma$  as a subset of  $\Lambda(N_p) = \{(x, y) \in \mathbb{R}^{n_x} \times \mathcal{Y}_t : (x, \kappa_t(x, y)) \in \mathcal{Z}(N_p)\}$ <sup>14</sup> satisfying Def. 2 for the control law  $u = \kappa_t(x, y)$  and disturbances  $w \in \mathcal{S}(N_p)$ .*
- (3) *The terminal cost function  $V(x - x_a)$  is a Lyapunov function for the unconstrained qLPV system (1) such that for all  $(x, y_a) \in \Gamma(\rho)$  there exist constants  $b > 0$  and  $\sigma > 1$  such that  $V(x - x_a) \leq b|x - x_a|^\sigma$ . Thus, we have:  $V(A(\rho)(x - x_a) + B(\rho)\kappa_t(x - x_a, y)) - V(x - x_a) \leq \ell(x - x_a, \kappa_t(x, y)x - u_a)$ , where  $x_a = g_x(y_a)$  and  $u_a = g_u(y_a)$ .*

Notice that the previous assumptions presented are similar to the ones in [15] and [8], which develop control strategies for robust regulation and nominal tracking respectively, but herein extended to the robust case, with a

<sup>11</sup> Notice that due to the freedom provided by the artificial reference  $y_a$ , the feasibility property becomes independent of  $y_r$  [8].

<sup>12</sup> Notice that if the target output is admissible ( $y_r \in \mathcal{Y}_t$ ), the minimiser is  $y_a^o = y_r$  and the previous inequality is reduced to  $V_O(y) \geq \alpha_O(\|y\|)$ .

<sup>13</sup> An evident explicit alternative for this terminal law is  $\kappa_t(x, y_a) = K_t(f_\rho(x))(x - g_x(y_a)) + g_u(y_a)$ .

<sup>14</sup> Note that  $\mathcal{Z}(N_p)$  is the set of points  $(x, u)$  such that  $u = v + K(f_\rho(x))x$  with  $(x, v) \in \mathcal{Z}_\pi(N_p)$ .

qLPV prediction model. Accordingly, we provide a computationally elegant solution that can be used to compute parameter-dependent qLPV terminal ingredients, through the solution of some LMIs. The following Theorems provide recursive feasibility and exponential stability guarantees for the qLPV system in Eq. (1) subject to the MPC control law from Eq. (5). For the sake of presentation clarity, the proofs are provided in Appendix A.

Consider parameter-dependent TRPI set  $\Gamma(\rho) := \mathbf{X}_f(\rho) \times \mathcal{Y}_t$ , being  $\mathbf{X}_f := \{x | x^T P(\rho)x \leq 1\}$  an ellipsoidal terminal set. Furthermore, consider the sub-level terminal cost  $V(x) = x^T P(\rho)x$  and the terminal qLPV feedback  $k_t(x, y) = K_t(\rho)(x - g_x(y)) + g_u(y)$ . Note that the terminal scheduling parameter  $\rho$  is, by definition,  $\rho = f_\rho(x)$ . Since we seek  $x \rightarrow g_x(y)$ , it is implicitly implied that the terminal scheduling parameter is  $\rho = g_\rho(y)$ . The NMPC optimisation can thus be implemented with the following terminal constraint:  $(x(k + N_p | k), y_a) \in \Gamma(y_a) := \Gamma(f_\rho(y_a))$ , where  $y_a$  is a free optimisation variable tuned through the offset cost  $V_O$  to track the actual set-point  $y_r$ .

In order to detach this nonlinear dependency, we synthesise  $\rho$ -dependent terminal ingredients, based on a positive-definite matrix  $P(\rho)$ . Likewise, we use a set  $\mathbf{X}_f(\rho)$  which is robust positively invariant regarding the closed-loop dynamics (Eq (6)), in such way that  $\Gamma(\rho)$  is a TRPI set.

Consider the tracking system  $e(k + j | k) = x(k + j | k) - g_x(y)$ , with  $g_x(y)$  (piece-wise) constant and  $x(k + j | k)$  being the nominal qLPV predictions from Eq. (9). We denote  $w^\dagger(k + j) := w(k + j) + \Delta_j$  as the resulting (bounded) uncertainties (load disturbance summed to the model-process mismatch due to the differences between  $\hat{\rho}(k + j | k)$  and  $\rho(k + j)$ ). Considering the nominal law  $u(k) = k_t(x, y)$ , it follows that:

$$\begin{aligned} e(k + j + 1 | k) &= A_\pi(\hat{\rho}(k + j))e(k + j | k) + \theta(k + j), \quad (23) \\ \theta(k + j) &= (A(\hat{\rho}(k + j)) - \mathbb{I}_{n_x})g_x(y) \\ &\quad + B(\hat{\rho}(k + j))g_u(y) + w^\dagger(k + j) \in \Theta. \quad (24) \end{aligned}$$

Accordingly, the following Theorems ensure that the error dynamics in Eq. (23) converge to the origin (i.e. tracking is ensured).

**Theorem 2** from [9]

*Consider Assumptions 3 and 4 hold. Suppose there exists a terminal control law  $u = K_t(\rho)(x - g_x(y)) + g_u(y)$ . Consider that the MPC is given by Eq. (21), with a terminal state set given by  $\mathbf{X}_f(\rho)$  and a terminal cost  $V(x, \rho)$ . Then, input-to-state stability is ensured if the following conditions hold  $\forall \rho \in \mathcal{P}$ :*

- (C1) *The origin  $e = 0$  lies in the interior of  $\mathbf{X}_f(\rho)$ ;*  
(C2) *Any consecutive error to  $e$ , in closed-loop given by  $e^+ = A_\pi(\rho)e + \theta$  lies within  $\mathbf{X}_f(\rho)$ ;*

(C3) The discrete Lyapunov equation is verified within this invariant set, this is,  $\forall x \in \mathbf{X}_f(\rho), \forall \rho \in \mathcal{P}$ , and  $\forall \delta\rho \in \delta\mathcal{P}: V(e^+, \rho + \delta\rho) - V(e, \rho) \leq -\ell(x, k_t(x, y) - g_u(y))$ .

(C4) The image of the terminal feedback lies within the admissible control domain:  $K_t(\rho)(x - g_x(y)) + g_u(y) \in \mathcal{U}, \forall \rho \in \mathcal{P}$ .

(C5) The terminal set  $\mathbf{X}_f(\rho)$  is a subset of  $\mathcal{X}$ .

Assuming that the initial solution of the MPC problem  $\mathbf{v}_{[0, N_p-1]}^*$  is feasible, then, the MPC is recursively feasible, steering  $e = (x - g_x(y))$  to the origin.

### Theorem 3 Tracking Robust Positive Invariant Set

Assume that there exists an ellipsoidal terminal set  $\mathbf{X}_f(\rho)$ .  $\mathbf{X}_f$  is a robust positively invariant set iff, for any  $e \in \mathbf{X}_f$  and  $\rho \in \mathcal{P}$ , i.e.  $e^T P(\rho)e \leq 1$ , it follows that  $(e^+)^T P(\rho + \delta\rho)e^+ \leq 1$ , i.e. the successor state  $e^+$  is also inside  $\mathbf{X}_f$ , which implies in:

$$(A_\pi(\rho)e + \theta)^T P(\rho + \delta\rho)(A_\pi(\rho)e + \theta) \leq 1. \quad (25)$$

Then,  $\Gamma(\rho)$  is a TRPI for system (1) as follows:

$$\Gamma := \{(x, y) \in \mathbb{R}^{n_x \times n_y} \mid (x - g_x(y)) \in \mathbf{X}_f(\rho), \quad (26)$$

$$h(x, K_t(\rho)(x - g_x(y)) + g_u(y)) \in \mathcal{Y}_t\}.$$

### Theorem 4 Terminal Ingredients

Conditions (C1)-(C5) of Theorem 2 and the inequality of Theorem 3 are satisfied if there exist a symmetric parameter-dependent positive definite matrix  $P(\rho) : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_x \times n_x}$ , a parameter-dependent rectangular matrix  $W(\rho) : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_u \times n_x}$ , and a scalar  $\lambda \in ]0, 1]$  such that  $Y(\rho) = (P(\rho))^{-1} > 0$ ,  $W(\rho) = K_t(\rho)Y(\rho)$  such that LMIs (27)-(28) and the BMI (29) hold for  $\rho^+ = \rho + \delta\rho$  for all  $\rho \in \mathcal{P}$  and  $\delta\rho \in \delta\mathcal{P}$ , and  $\theta$  and  $w^\dagger$  given as the vertices of  $\Theta$  and  $\mathcal{S}(N_p)$ , respectively, under the minimization of  $\log \det\{Y(\rho)\}$ .

**Remark 10** The BMI in Theorem 4 can be solved through simple bisection search over the optimization plane since  $0 < \lambda \leq 1$ , by construction, as argues [19].

**Remark 11** Theorem 4 provides infinite-dimensional inequalities, which must hold  $\forall \rho \in \mathcal{P}$  and  $\forall \delta\rho \in \delta\mathcal{P}$ . In practice, the solution can be found by enforcing the inequalities over a sufficiently dense grid of points  $(\rho, \delta\rho)$  along the  $\mathcal{P} \times \delta\mathcal{P}$  plane. Then, the solution can be verified over a denser grid. The parameter-dependency of  $P$  may be dropped if the system is quadratically stabilizable, but this may result in quite conservative performances.

**Proposition 1 (Recursive Feasibility)** Let there exist a solution  $Y(\rho)$  to Theorem 4. Then, given any  $x \in \mathcal{X}_a(N_p)$ ,  $y_r \in \mathbb{R}^{n_y}$  and  $u = \kappa(x, y_r)$ , we have  $x^+ = A(f_\rho(x))x + B(f_\rho(x))u + w \in \mathcal{X}_a(N_p), \forall w \in \mathcal{W}$ . Consider an optimal sequence  $\hat{\mathbf{v}}^* = (v_0^*, v_1^*, \dots, v_{N_p-1}^*)$  and

an optimal artificial target  $y_a^*(x, y_r) = y_a^*$ . Then  $\hat{\mathbf{v}}^c = (v_1^*, \dots, v_{N_p-1}^*, \kappa_t(x(k+N_p|k), y_a^*) - u_a)$  and  $y_a^c = y_a^*$  define feasible (candidate) solution of the MPC problem in Eq. (21) for any  $\bar{y}_r \in \mathbb{R}^{n_y}$  and  $w \in \mathcal{W}$ . This means that the NMPC optimisation in (21) is recursively feasible.

**Proposition 2 (Error ISS)** Let there exist a solution  $Y(\rho)$  to Theorem 4. Then, the qLPV system (1) in closed loop with the MPC input (5) has uniformly exponentially input-to-state stable error dynamics (as of Eq. (23)). That is, for any feasible initial condition  $x_0$  and constant set-point  $y_r \in \mathbb{R}^{n_y}$ , with  $w(k) \in \mathcal{W}$ , it is implied that:

$$\|x(k) - x_a(k)\| \leq \beta(\|x(0)\|, k) + \gamma(\bar{w}), \quad (30)$$

where  $\beta$  and  $\gamma$  are respectively a  $\mathcal{KL}$ -function and a  $\mathcal{K}$ -function and  $\bar{w}$  is such that  $\|w(k)\| \leq \bar{w}, \forall k$ .

**Remark 12** Note that for an admissible equilibrium state  $x_a$  the virtual control sequence  $\hat{\mathbf{v}} = (v_a, \dots, v_a)$ , where  $u_a = v_a + Kx_a$ , with artificial reference  $y_a$  is admissible, since it maintains the system at  $x_a$ . Therefore, the set of admissible equilibrium states  $\mathcal{X}_s(\mathcal{Z}_\pi(N_p), \mathcal{Y}_t) = \{x \in \mathbb{R}^{n_x} : \exists u_a \in \mathbb{R}^{n_u}, (x_a, u_a) \in \mathcal{Z}_s(\mathcal{Z}_\pi(N_p)), h(x_a, u_a) \in \mathcal{Y}_t\}$  is a subset of  $\mathcal{X}_a(N_p)$  and feasibility is not lost through set-point changes.

**Remark 13** The feasible candidate  $(\hat{\mathbf{v}}^c, y_a^c)$  can be used as a starting point for (21) in order to reduce computational load.

**Remark 14** In general, the functions  $g_x(\cdot)$  and  $g_u(\cdot)$  are not explicitly known. Therefore, rather than considering the constraints  $x_a = g_x(y_a)$  and  $u_a = g_u(y_a)$ , we can include  $x_a$  and  $u_a$  as additional decision variables and enforce the analogous constraints  $x_a = A(f_\rho(x_a))x_a + B(f_\rho(x_a))u_a, y_a = C(f_\rho(x_a))x_a + D(f_\rho(x_a))u_a$ .

#### 4.1 Artificial Reference Choice

The addition of the artificial reference  $y_a \in \mathcal{Y}_t$  prevents feasibility loss due to changes in set-point or non-admissible targets. However, it significantly increases the computational cost of the MPC optimization problem by introducing the new decision variable  $y_a$  and associated nonlinear constraints  $x_a = g_x(y_a)$  and  $u_a = g_u(y_a)$ .

In the sequel, a method for avoiding such additional computational burden is presented. First, consider the optimal admissible target  $y_a^o = \arg \min_{y_a \in \mathcal{Y}_t} V_O(y_a - y_r)$  and a feasible candidate artificial target<sup>15</sup>  $y_a^c$ . Let  $y_a^\alpha$  be a convex combination of  $y_a^c$  and  $y_a^o$ , i.e.  $y_a^\alpha = (1 - \alpha)y_a^c + \alpha y_a^o$ , for an  $\alpha \in [0, 1]$ . Note that, from the convexity of

<sup>15</sup> From the Recursive Feasibility property (Theorem 1), the previous artificial reference  $y_a^*(x_{k-1}, y_r)$  can be used as the candidate  $y_a^c$ .

$$\begin{bmatrix} Y(\rho) & (A(\rho)Y(\rho) + B(\rho)W(\rho))^T & Y(\rho) & W^T(\rho) & 0 & (A(\rho)Y(\rho) + B(\rho)W(\rho))^T & 0 & 0 \\ (A(\rho)Y(\rho) + B(\rho)W(\rho)) & Y(\rho^+) & 0 & 0 & \theta & Y(\rho^+) & 0 & 0 \\ Y(\rho) & 0 & Q^{-1} & 0 & 0 & 0 & \mathbb{I}_{n_x} & 0 \\ W(\rho) & 0 & 0 & R^{-1} & 0 & 0 & 0 & \mathbb{I}_{n_x} \\ \hline 0 & \theta^T & 0 & 0 & 0 & \theta^T & 0 & 0 \\ (A(\rho)Y(\rho) + B(\rho)W(\rho)) & Y(\rho^+) & 0 & 0 & \theta & Y(\rho^+) & 0 & 0 \\ 0 & 0 & \mathbb{I}_{n_x} & 0 & 0 & 0 & \mathbb{I}_{n_x} & 0 \\ 0 & 0 & 0 & \mathbb{I}_{n_x} & 0 & 0 & 0 & \mathbb{I}_{n_x} \end{bmatrix} \geq 0, \quad (27)$$

$$(a) \begin{bmatrix} (\bar{u}_i - I_{n_u, \{i\}} u_r)^2 & I_{n_u, \{i\}} W(\rho) \\ (I_{n_u, \{i\}} W(\rho))^T & Y(\rho) \end{bmatrix} \geq 0, \forall i \in \mathbb{N}_{[1, n_u]}, (b) \begin{bmatrix} (\bar{x}_i - I_{n_x, \{j\}} x_r)^2 & I_{n_x, \{j\}} Y(\rho) \\ (I_{n_x, \{j\}} Y(\rho))^T & Y(\rho) \end{bmatrix} \geq 0, \forall j \in \mathbb{N}_{[1, n_x]}, \quad (28)$$

$$\begin{bmatrix} \lambda Y(\rho) & (A(\rho)Y(\rho) + B(\rho)W(\rho))^T & 0 & (A(\rho)Y(\rho) + B(\rho)W(\rho))^T \\ (A(\rho)Y(\rho) + B(\rho)W(\rho)) & Y(\rho^+) & \theta & Y(\rho^+) \\ \hline 0 & \theta^T & (1 - \lambda) & \theta^T \\ (A(\rho)Y(\rho) + B(\rho)W(\rho)) & Y(\rho^+) & \theta & Y(\rho^+) \end{bmatrix} > 0. \quad (29)$$

$\mathcal{Y}_t, y_a^\alpha \in \mathcal{Y}_t$ . Thus, consider the following auxiliary optimization problem:

$$\begin{aligned} \max_{\hat{v}, \alpha} \quad & \alpha \quad (31) \\ \text{s.t. :} \quad & \begin{cases} x(k+j+1|k) = A_\pi(\hat{\rho}(k+j|k))x(k+j|k) \\ + B(\hat{\rho}(k+j|k))v(k+j|k), j \in \mathbb{N}_{[0, N_p-1]}, \\ (x(k+j|k), v(k+j|k)) \in \mathcal{Z}_\pi(j), j \in \mathbb{N}_{[0, N_p-1]}, \\ \alpha \in [0, 1] \\ (x_{k+N_p|k}, y_a^\alpha) \in \Gamma(y_a^\alpha) \end{cases} \end{aligned}$$

Based on the solution  $\alpha^*$  of (31), we can then solve the simpler optimization problem (15), with  $y_r = (1 - \alpha^*)y_a^c + \alpha^*y_a^o$ , to obtain the optimal control sequence  $\mathbf{v}_{[k, k+N_p-1]}^*$ . This two-step optimization is analogous to (21) with an arbitrarily large offset cost  $V_O(\cdot)$  (prioritizing the proximity of the artificial target from the optimum  $y_a^o$ ) and the added restriction that  $y_a$  must be inside the line segment  $[y_a^c, y_a^o]$ .

**Remark 15** Note that if, for some time instant  $k$ , the solution of (31) is  $\alpha^* = 1$ , we will have, assuming a constant reference,  $y_a^c = y_a^o$  for the future instants. The auxiliary problem (31) thus becomes irrelevant until there is a change on  $y_r$ .

**Remark 16** Since  $\alpha$  is a real scalar in  $[0, 1]$ , an alternative to the optimisation in Eq. (31) is to perform a bi-

section<sup>16</sup> mechanism with  $\alpha_k^c \in [0, 1]$ . By doing so, we test the feasibility of (15) for a given  $\alpha_k^c$ . If the problem is feasible, we apply the correspondent control law; otherwise, we take  $\alpha = 0$  (which is by definition feasible). A viable choice of candidates  $\alpha_k^c$ , in order to ensure the convergence of  $y_a$  to  $y_a^o$ , can be of  $\alpha_k^c = 1$  if  $\alpha_{k-1}^c$  was feasible and  $\alpha_k^c = \frac{1}{2}\alpha_{k-1}^c$  if not (bisection). This reduces the computational complexity to a maximum of two optimization problems given by (15), without the undesired nonlinearities.

## 5 Results

### 5.1 Application and qLPV Embedding

In this Section, we use a numeric<sup>17</sup> benchmark from the literature to demonstrate the properties and effectiveness of our method. We adapt the cascaded tank process from [5], considering the interconnection of two tanks, with an open hole at the bottom of the first tank, which leaks fluid to the second. The latter has a pump at its end, regulated by a local proportional controller. The

<sup>16</sup> In order to preserve the ISS property, this bisection search should also ensure that the terminal offset cost decays with the new artificial target, this is:  $V(x(k+1|k) - g_x(y_a^\alpha)) - V(x(k|k) - g_x(y_a^c)) \leq -\|x(k|k) - g_x(y_a^c)\|_Q$ .

<sup>17</sup> All the results presented in the sequel were obtained in a 2.4 GHz, 8 GB RAM Macintosh computer.

resulting nonlinear level dynamics are:

$$\begin{aligned}\frac{dh_1(t)}{dt} &= -\frac{a\sqrt{2gh_1(t)}}{A_1} + \frac{\gamma}{A_1}u(t), \\ \frac{dh_2(t)}{dt} &= \frac{a\sqrt{2gh_1(t)}}{A_2} - \frac{k}{A_2}h_2(t).\end{aligned}$$

Each  $h_i(t)$  represents the water level at the  $i$ -th tank;  $u(t)$  represents the tension applied for the main pump, for which the corresponding flow is  $\gamma u(t)$ . We are mainly interested with the dynamics of  $y(t) = h_1(t)$ . The tank cross sections  $A_i$  are of  $1 \text{ cm}^2$ , while the outlet hole cross section  $a$  is of  $0.05 \text{ cm}^2$ . The pump parameter  $\gamma$  is of  $1.4 \text{ cm}^3/\text{Vs}$ . The proportional coefficient  $k$  is of 1.1. The process constraints are:  $h_j \in [1, 10] \text{ cm}$ ,  $\forall j \in \mathbb{N}_{[1,2]}$  (level, process variable);  $u \in [0, 5] \text{ V}$  (pump flow, control input). This nonlinear model is Euler-discretized with  $T_s = 250 \text{ ms}$ , and qLPV-embedded to Eq. (1) with:

$$\begin{aligned}A(\rho) &= \begin{bmatrix} (1 - \frac{T_s a \sqrt{2g}}{A_1} \rho) & 0 \\ \frac{T_s a \sqrt{2g}}{A_2} \rho & (1 - \frac{T_s k}{A_2}) \end{bmatrix}, \quad B(\rho) = \begin{bmatrix} \frac{T_s \gamma}{A_1} \\ 0 \end{bmatrix}, \\ \rho = f_\rho(x) &= (x_1)^{-0.5} \in [0.3, 1] \text{ cm}^{-0.5}.\end{aligned}$$

In order to compute the bounds on  $\delta\rho$ , one can use either constrained optimisation or interval arithmetic methods as described in Remark 3. The obtained bounds are defined through  $\delta\rho := \{\delta\rho \in \mathbb{R}^1 : -0.034 \leq \delta\rho \leq 0.0052\}$ . The process is subject to bounded additive uncertainties  $\|w(k)\|_\infty \leq 0.05 \text{ cm}$ . In the sequel, we compare the proposed robust qLPV NMPC with the NMPC for Tracking algorithm from [8].

## 5.2 Terminal Ingredients

Consider the convex output tracking set  $\mathcal{Y}_t := [1, 10] \text{ cm}$ . The output steady-state condition from Eq. (13) implicitly defines  $g_x(y_r) = \left[ y_r \frac{a\sqrt{2gy_r}}{k} \right]^T$  and  $g_u(y_r) = \frac{a\sqrt{2gy_r}}{\gamma}$ . We use a prediction horizon of  $N_p = 4$  steps and the quadratic stage cost  $\ell(x, u) = \|x\|_Q^2 + \|u\|_R^2$  with  $Q = \mathbb{I}_{n_x}$  and  $R = 1$ .

In order to synthesise the terminal ingredients via Theorem 4, we partition the output set  $\mathcal{Y}_t$  into ten partitions, thus finding one parameter-dependent RPI set  $\Gamma(\rho, y_r)$ , for which  $y_r$  parametrises the partitions, i.e.  $y_r \in [1, 2]$  or  $(2, 3]$  or  $(3, 4]$ , and so on up to  $(9, 10]$ . For the NMPC algorithm, we use terminal ingredients synthesised through the procedure from [8, Appendix B].

Figure 1 shows the parameter-dependent TRPI sets (for frozen values of  $\rho \in \mathcal{P}$ ) used for the proposed algorithm and the quadratic TPI sets used for the NMPC algorithm [8]. All sets are translated to from the error coordinates  $(x - x_r)$  to the state coordinates  $x$ , centered

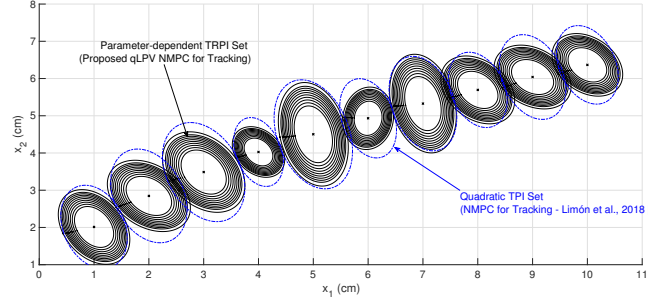


Fig. 1. Synthesised TRPI set partitions  $\Gamma(\rho)$  and the quadratic TPI set partitions.

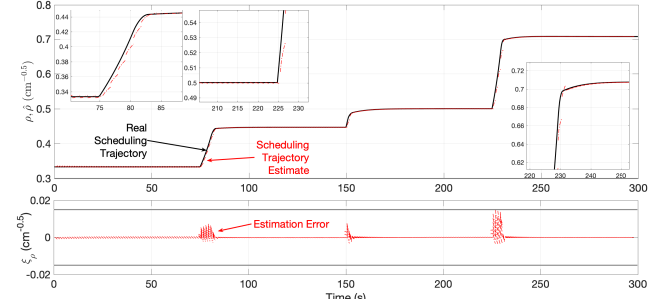


Fig. 2. Scheduling Sequence Estimates  $\hat{\rho}_{[k+N_p-1]}^k$  at different samples  $k$  and estimation error  $\xi_\rho$  (and bounds).

at the different state targets  $x_r$  of each partition. As we can see, Theo. 4 generates sufficiently a large terminal region, meaning that any reference point  $y_r \in \mathcal{Y}_t$  can be tracked.

## 5.3 Scheduling Sequence Estimation

Before presenting the actual control results, we provide the scheduling sequence extrapolation estimates obtained with the recursive method presented in Sec. 3. As detailed in Lemma 2, convergence is indeed verified. The obtained bounds for the estimation error are  $\|\xi_\rho\|_\infty \leq 0.015 \text{ cm}^{-0.5}$ , as shown in Fig. 2. The extrapolation mechanism offers very precise estimates  $\hat{\rho}_{[k+N_p-1]}^k$ , which means that the nominal qLPV predictions obtained through Eq. (9) are very close to the real system trajectories of Eq. (8) and thus the disturbance propagation along the horizon is reduced.

## 5.4 Disturbance Propagation

Now, we show the disturbance propagation reachable sets  $\mathcal{S}(j), \forall j \in \mathbb{N}_{[0, N_p]}$  for the proposed algorithm, considering the zonotope disturbance propagation paradigm. Note that  $\mathcal{S}(0)$  stands for the load disturbance set  $\mathcal{W}$ ; the following zonotopes comprise the propagation of the load disturbances and the model-process mismatches along the horizon. These sets are computed according to Theorem 1. In Fig. 3, we show the collection of sets  $\mathcal{S}(j)$  over the  $x_1 \times x_2$  plane (Def.

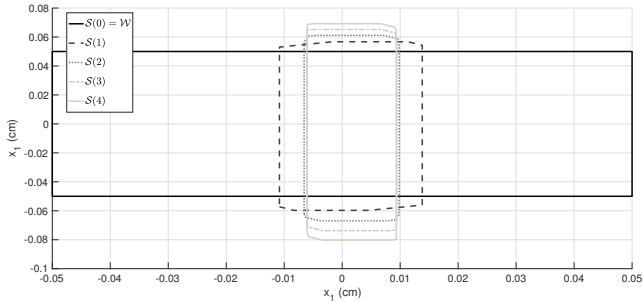


Fig. 3. Zonotopic sets  $\mathcal{S}(j)$ .

1). We recall that the vertices of  $\mathcal{S}(N_p)$  were used to construct the terminal ingredients through Theorem 4.

### 5.5 Simulation Scenarios

We consider two different simulation scenarios. Since the NMPC algorithm from [8] is not robust by design, we first consider and compare the obtained tracking performances of both algorithms, without the presence of load disturbances (i.e. with  $w$  nil). Then, we consider the disturbance rejection robust performances solely of the proposed robust algorithm.

### 5.6 Nominal Tracking Performances

Considering a step-like piece-wise constant output target signal which passes through  $y_r = 9, 5, 4$  and  $2$  cm, the obtained tracking performances with both algorithms are shown in Fig. 4: sub-figure (a) presents the resulting state, input and output trajectories, while sub-figure (b) shows the state phase plane and the terminal sets. Complementary, Fig. 5 provides the values for the artificial reference tuning variable  $\alpha$  for the proposed mechanism.

As we can see, the obtained tracking performances with both methods are offset-free steady-state output points. The proposed method ensures slightly faster convergence than the original NMPC for Tracking scheme from [8].

The main advantage resides in its simpler implementation, of QP-alike numerical burden, enable through the qLPV embedding. With the qLPV model, nonlinear mapping do not have to be solved internally by the optimisation procedure. The proposed qLPV NMPC mechanism requires only the operation of: one linear recursive law (Eq. (18)) and one QP problem (Eq. (21)). In the moments of reference changes, one bisection search (Remark 16) is also required, which increases the number of QPs to at most 5 iterations. In contrast, the original NMPC for Tracking requires the solution of an NP-hard optimisation problem per sampling instant, which is numerical-wise much harder.

In order to better compare the two tracking controllers, we assess the obtained performance results with performance indexes, presented in Table 1. We discuss them:

- Firstly, we note that there is an overall performance enhancement: there is a small decrease on the integral of absolute output tracking error (IAE) w.r.t. [8], of roughly 12%.
- Complementary, we stress that the generated control input is smoother with the proposed method: we obtain 43% reduced total variance (TV) of the control signal.
- Furthermore, the average computational time needed to solve the control problem ( $t_c$ ) is over 44% smaller with the proposed qLPV method. This is very significant, since this is a small system ( $n_x = 2$ ). The complexity of the NP solution from [8] grows exponentially with  $(N_p \times n_x)$ , which can a serious issue with time-critical systems. The proposed method has QP-alike burden, thus  $t_c$  grows linearly with  $(N_p \times n_x)$ , which means it is readily-conceived for embedded applications.
- Finally, note that the model-process discrepancies (differences between Eqs. (8) and (9)) are very well-handled with the zonotopic constraint-tightening approach, since the generated sets  $\mathcal{S}(j)$  are small (Fig. 3). This corroborates previous discussions in the literature indicating this option as a promising alternative, see e.g. [7]. We also stress that the terminal ingredients conceived with the proposed Theorem generate sufficiently large terminal sets, able to guarantee recursive feasibility for a rather larger output set.

### 5.7 Robust Tracking Performances

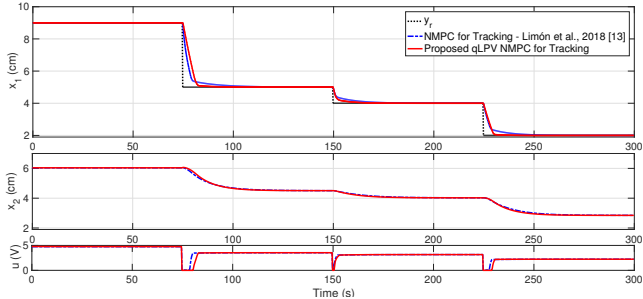
In order to illustrate the robustness properties of the proposed algorithm, we provide a second simulation scenario<sup>18</sup>. Consider three uniformly random disturbance sequences with unitary seeds, multiplied by decaying exponential terms, as illustrated in Fig. 6.

In Fig. 7, we show the state and output behaviours w.r.t. a step-like output target goal  $y_r$ . Clearly, robust stability is ensured: as the load disturbance sequences dissipate, the error trajectories  $(x - x_r)$  converge to the origin; moreover, while  $w(k)$  is non-null, the states stabilize at constant steady-states regimes, as close as possible to  $x_r$ . The average tracking error when disturbances are present is of 5.68%, which is arguably small.

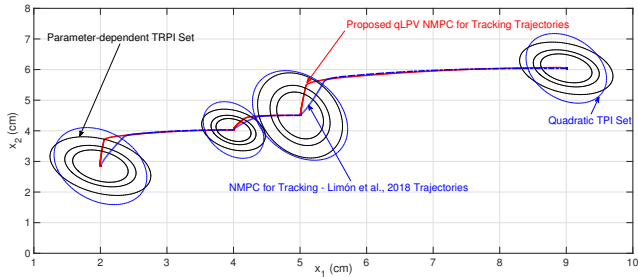
<sup>18</sup> We opt not to test the method from [8] against load disturbances since it is not a robust algorithm, which would in turn result in an unfair comparison.

Table 1  
Performance Comparison.

Method	IAE := $\sum_k \ y_r(k) - y(k)\ $	TV := $\sum_k \ u(k+1) - u(k)\ $	$t_c$
NMPC for Tracking [8]	108.23	35.93	90.53 ms
Proposed qLPV NMPC for Tracking	95.31	20.44	50.55 ms



(a) State, Output and Input Trajectories.



(b) State Phase Plane and TRPI sets.

Fig. 4. Nominal Performances: Proposed qLPV NMPC vs. NMPC for Tracking [8].

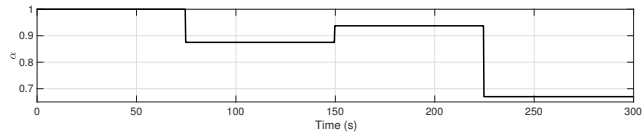


Fig. 5. Artificial reference choice variable.

## 6 Conclusion

In this paper, we provided a novel Tracking NMPC algorithm through qLPV embedding. The method is based on the recursive estimation of the future qLPV scheduling trajectories, made available through a simple Taylor expansion. The propagation of the model-process prediction mismatches along the NMPC horizon are bounded with zonotopes. Furthermore, we provide an LMI-solvable remedy for the case of bounded additive disturbances, which computes a robust LPV feedback gain and parameter-dependent terminal ingredients. The derived tracking robust positive invariant set ensures recursive feasibility of the optimisation procedure as well as input-to-state stability of the process. We compare our method against the nominal tracking NMPC framework from [8], considering a cascaded

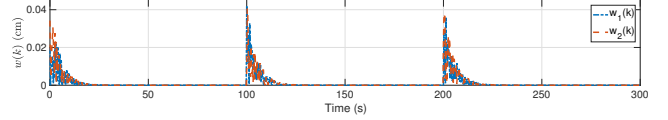


Fig. 6. Disturbance Scenario.

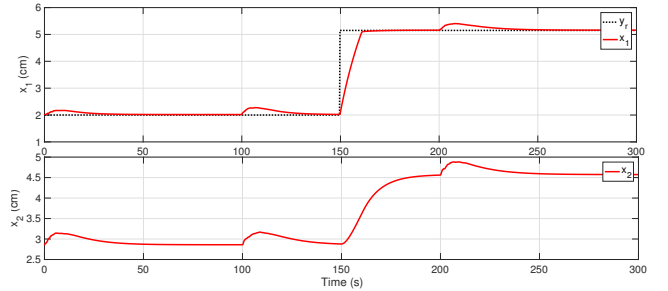


Fig. 7. Robust Performances: States and Output.

tanks benchmark system. We find very similar tracking performances with much smaller computational stress with the proposed scheme, which benefits from the linear predictions of the qLPV realisation. The method is ready for embedded applications (the online stress is similar to that of a QP) and offers robustness towards bounded load disturbances with reduced conservatism.

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## A Proofs of Theorems and Propostions

### A.1 Proof of Theorem 2

This proof is standard and basically gathers the sufficient conditions from Assumptions 3 and 4. Refer to [9].  $\square$

### A.2 Proof of Theorem 3

It is trivial to verify Eq. (25) by using  $x(k), x(k+1) \in \mathbf{X}_f, \forall k \in \mathbb{N}$  and  $y = h(x, K_t(\rho)(x - g_x(y)) + g_u(y)) \in \mathcal{Y}_t$ .  $\square$

### A.3 Proof of Theorem 4

We begin by showing the positive invariance of the ellipsoid. Applying the  $\mathcal{S}$ -procedure, with  $\lambda > 0$  to (25) and  $(1 - e^T P(\rho)e \geq 1)$ , we get:

$$1 - (A_\pi(\rho)e + \theta)^T P(\rho^+) (A_\pi(\rho)e + \theta) - \lambda (1 - e^T P(\rho)e) > 0,$$

which can be rewritten as:

$$\left( e^T P(\rho) \mathbb{I}_{n_x} \right) \underbrace{\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}}_N \begin{pmatrix} P(\rho)e \\ \mathbb{I}_{n_x} \end{pmatrix} > 0, \quad (\text{A.1})$$

with  $N > 0$  and:

$$N_{11} = \lambda Y(\rho) - A_\pi^T(\rho) Y^T(\rho) P(\rho^+) A_\pi(\rho) Y(\rho), \quad (\text{A.2})$$

$$N_{12} = -A_\pi^T(\rho) Y^T(\rho) P(\rho^+) \theta, \quad (\text{A.3})$$

$$N_{21} = -\theta^T P(\rho^+) A_\pi(\rho) Y(\rho), \quad (\text{A.4})$$

$$N_{22} = (1 - \lambda) - \theta^T P(\rho^+) \theta. \quad (\text{A.5})$$

Applying a Schur complement over  $P(\rho + \delta)$  for each  $N_{ij}$  leads to (29). This ensures Theorem 3.

Complementary, we proceed by demonstrating that the resulting  $P(\rho)$  satisfies all five conditions of Theorem 2. (C1) trivially holds due to the ellipsoidal form of  $\mathbf{X}_f$ . (C2) is verified due to the fact that  $\mathbf{X}_f$  is a sub-level set of the terminal cost  $V(\cdot)$ . Therefore, if condition (C3) is verified, (C2) is consequently ensured.

The discrete Ricatti condition (C3) is verified through the solution of LMI (27). Since  $Q^{-1} > 0, R^{-1} > 0$  and  $Y(\rho + \delta\rho) > 0$ , we apply two consecutive Schur, complements, which leads to:

$$\left( e^T P(\rho) \mathbb{I}_{n_x} \right) \underbrace{\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}}_M \begin{pmatrix} P(\rho)e \\ \mathbb{I}_{n_x} \end{pmatrix} \geq 0, \quad (\text{A.6})$$

with  $M \geq 0$  and:

$$M_{11} = Y(\rho) - Y^T(\rho)QY(\rho) - W^T(\rho)RW(\rho) - Y^T(\rho)A_\pi^T(\rho)P(\rho^+)A_\pi(\rho)Y(\rho), \quad (\text{A.7})$$

$$M_{12} = -Y(\rho)A_\pi^T(\rho)P(\rho^+)\theta, \quad (\text{A.8})$$

$$M_{21} = -\theta^T P(\rho^+)A_\pi(\rho)Y(\rho), \quad (\text{A.9})$$

$$M_{22} = -\theta^T P(\rho^+)\theta. \quad (\text{A.10})$$

Using  $e = eP(\rho)Y(\rho)$  and  $W(\rho) = K_t(\rho)Y(\rho)$  leads to:

$$(A_\pi(\rho)e + \theta)^T P(\rho^+)(A_\pi(\rho)e + \theta) - e^T P(\rho)e + e^T Qe + e^T K^T(\rho)RK_t(\rho)e \leq 0. \quad (\text{A.11})$$

This inequality is a sufficient condition for (C3) with  $V(\cdot)$  as a sub-level of  $\mathbf{X}_f$ .

The fourth and fifth conditions (C4-C5) are verified by the direct application of the Schur complement to Eq. (28a) and Eq. (28b), respectively, using  $W(\rho) = K_t(\rho)Y(\rho)$ . They lead, respectively, to:

$$\begin{aligned} (\bar{u}_i - I_{n_u, \{i\}}u_r)^2 &\geq \\ (I_{n_u, \{i\}}K_t(\rho))Y(\rho)P(\rho)Y(\rho)(I_{n_u, \{i\}}K_t(\rho))^T &. \end{aligned} \quad (\text{A.12})$$

Since the maximum normed  $Fe$  of an  $e$  that belongs to some ellipsoid  $e^T P e \leq 1$  is given by  $\sqrt{F(P^{-1})F^T}$ , it holds that the first inequality implies that the projection  $I_{n_u, \{i\}}K_t(\rho)e$  (i.e.  $i$ -th control signal) is upper-bounded, in norm, by  $\bar{u}_i - I_{n_u, \{i\}}u_r$ , which satisfies (C4). Analogously, LMI (28b) ensures that the projection  $I_{n_x, \{j\}}x$  (i.e.  $j$ -th state) is norm-bounded by  $\bar{x}_j - I_{n_x, \{j\}}x_r$ , which satisfies condition (C5). This concludes the proof.  $\square$

#### A.4 Proof of Proposition 1 (Recursive Feasibility)

Let Assumptions 3-4 hold. Consider there exists a solution  $Y(\rho)$  to Theorem 4. Then, from conditions (C1), (C2), (C4), and (C5) from Theorem 2, we can infer that the generated control signal provides recursively feasible solutions to the NMPC optimisation in (21). Take  $\kappa(x, y_r) = K_t(\rho)(x - g_x(y_r)) + g_u(y_r)$  with  $\rho = f_\rho(x)$  (implied implicitly through the qLPV prediction model). Then, the if the initial condition  $x(0)$  generates a feasible  $\hat{\mathbf{v}}$ , all future iterations of the optimisation will also be feasible: the generated control law control is admissible (condition (C4)) and all state variables  $x \in \mathbf{X}_f$  generate successor state variables  $x^+$  which are also inside  $\mathbf{X}_f$  (condition (C2)), which contains the origin (condition (C1), terminal condition for  $(x - x_r) = (x - g_x(y_r))$  and is a sub-set of  $\mathcal{X}$  (condition (C5), which ensures that  $x, x^+$  are admissible). This concludes the proof.  $\square$

#### A.5 Proof of Propostion 2 (Error ISS)

Let there be a terminal stage cost  $V(\cdot)$  such that Assumption 3 holds. Let Assumption 4 also hold and Theorem 1 be verified. Note that since  $\ell(x - x_a, u - u_a)$  is a quadratic stage cost  $\|x - x_a\|_Q^2 + \|u - u_a\|_R^2$ ,  $\alpha_\ell$ ,  $\gamma_x$  and  $\gamma_u$  indeed exists. Consider there exists a solution  $Y(\rho)$  to Theorem 4. Then, the closed-loop is stable due to (C3) of Theorem 2, which conversely ensures that

$$\begin{aligned} \delta V(k) &= V(x(k) - x_a(k)) \\ &\quad - V(x(k-1) - x_a(k-1)) \\ &\leq -\|x(k) - x_a(k)\|_Q + \gamma_V(\bar{w}). \end{aligned}$$

Assume that  $\lim_{k \rightarrow +\infty} y_a(k) \rightarrow y_a^o$ . Analogously, use  $\lim_{k \rightarrow +\infty} x_a(k) \rightarrow x_a^o := g_x(y_a^o)$ . Then, thanks to the error dynamics in Eq. (23), we obtain:

$$\|x(k) - x_a(k)\|_Q \leq \beta(\|x(0) - x_a(0)\|, k) + \gamma(\bar{w}).$$

Since  $Q > 0$  (and positive definite) and  $x_a(0) = 0$  (by design), we have  $\|x(k) - x_a(k)\|_Q \geq \|x(k) - x_a(k)\|$  and thus error-ISS is established.

Now we consider the convergence of  $V_O(y_a(k) - y_r)$  such that the limit  $\lim_{k \rightarrow +\infty} y_a(k) \rightarrow y_a^o$  holds. Let us define  $\hat{y}_a = (1 - \alpha)y_a(k) + \alpha y_a^o$ , where  $\alpha \in [0, 1]$  is the optimal solution from Eq. (31). From the convexity of  $V_O(\cdot)$ , we obtain:

$$V_O(\hat{y}_a - y_r) \leq (1 - \alpha)V_O(y_a(k) - y_r) + \alpha V_O(y_a^o - y_r).$$

We can use the Lipschitz continuity of the map  $x_r := g_x(y_r)$  in order to obtain  $\|x_a(k) - \hat{x}_a\| \leq L_x \|y_a(k) - \hat{y}_a\|$ , where  $L_x > 0$  is the Lipschitz constant of  $g_x(\cdot)$ . Consider  $(y_a(k) - \hat{y}_a) = \alpha(y_a(k) - y_a^o)$ .

Since the closed-loop is stable, it follows that the total MPC cost dissipates over time, which implies in:

$$\begin{aligned} V_O(y_a(k) - y_r) - V_O(\hat{y}_a - y_r) &\leq V(x_a(k) - \hat{x}_a) + \gamma_{V_o}(\bar{w}) \\ &\leq a_1 \|x_a(k) - \hat{x}_a\|^\sigma + \gamma_{V_o}(\bar{w}) \leq a_1 (L_x \|y_a(k) - \hat{y}_a\|)^\sigma \\ + \gamma_{V_o}(\bar{w}) &\leq a_1 L_x^\sigma \alpha^\sigma \|y_a(k) - y_a^o\|^\sigma + \gamma_{V_o}(\bar{w}). \end{aligned}$$

Then, from the convexity of  $V_O(\cdot)$ , we have:

$$V_O(\hat{y}_a - y_r) \leq (1 - \alpha)V_O(y_a(k) - y_r) + \alpha V_O(y_a^o - y_r),$$

Thus, since  $\frac{1}{\alpha} > 1$ , we obtain:

$$\begin{aligned} V_O(y_a(k) - y_r) &\leq a_1 L_x^\sigma \alpha^{\sigma-1} \|y_a(k) - y_a^o\|^\sigma + \gamma_{V_o}(\bar{w}) \\ &\quad + \left(\frac{1 - \alpha}{\alpha}\right) V_O(y_a(k) - y_r) + V_O(y_a^o - y_r). \end{aligned}$$

Finally, since  $\lim_{k \rightarrow +\infty} \left(\frac{1-\alpha}{\alpha}\right) V_O(y_a(k) - y_r) \rightarrow \gamma_y(\bar{w})$ , we obtain  $V_O(y_a(k) - y_r) - V_O(y_a^o - y_r) \leq a_1 L_x^\sigma \alpha^{\sigma-1} \|y_a(k) - y_a^o\|^\sigma + \gamma_n(\bar{w})$ , with  $\sigma > 1$  and  $a_1 > 0$  as a constant scalars. Taking the limit at both sides of this inequality leads to:

$$V_O\left(\left(\lim_{k \rightarrow +\infty} y_a(k)\right) - y_r\right) \leq V_O(y_a^o - y_r) + \gamma_n(\bar{w}).$$

Note that  $V_O(\cdot)$  is a weighted quadratic cost by definition, thus  $|\|y_a\| - \|y_r\|| \leq \|y_a - y_r\|_T \leq V_O(y_a - y_r)$ . Thus:

$$\begin{aligned} & \left| \left\| \lim_{k \rightarrow +\infty} y_a(k) \right\| - \|y_r\| \right| \leq \|y_a^o - y_r\|_T + \gamma_n(\bar{w}), \\ & -(\|y_a^o - y_r\|_T + \gamma_n(\bar{w})) + \|y_r\| \leq \lim_{k \rightarrow +\infty} \|y_a(k)\| \\ & \leq \underbrace{\|y_a^o - y_r\|_T + \gamma_n(\bar{w}) + \|y_r\|}_{\mu(y_a^o, y_r, \bar{w})}. \end{aligned}$$

Note that, in nominal conditions (reachable reference  $y_r \in \mathcal{Y}$  and null disturbances), we obtain  $\gamma_{V_o}(\bar{w}) = 0$  and  $y_a^o = y_r$ , and thus  $\lim_{k \rightarrow +\infty} V_O(y_a(k) - y_r) \rightarrow 0$ , which means the steady-state target is reached. In the case the reference isn't reachable and there are disturbances, we can only infer that  $\lim_{k \rightarrow +\infty} y_a(k)$  exists within  $(-\mu(y_a^o, y_r, \bar{w}), +\mu(y_a^o, y_r, \bar{w}))$ . Nevertheless, it is implied that  $\lim_{k \rightarrow +\infty} V_O(y_a(k) - y_r)$  is bounded, which means it converges and error-ISS holds. This concludes the proof.  $\square$