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Version of Record: https://www.sciencedirect.com/science/article/pii/S0165011421002293
Manuscript_796844aab67bffd6e0dd8fe4a8307362

A New Methodology for Solving Fuzzy Systems of Equations: Thick Fuzzy Sets Based Approach

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Abstract

This paper presents a new method for solving fuzzy systems of equations (SoEs) where their parameters are represented by fuzzy intervals (FIs). A FI is a normal and convex fuzzy set (FS), where all its α-cuts are crisp intervals (CIs), i.e., conventional intervals. Due to the presence of uncertainty in the left-hand and right-hand sides of these fuzzy SoEs, the solutions are sought neither as FSs, nor as FIs or fuzzy boxes (FBs) —i.e., a Cartesian product of *n* FIs, but as uncertain FS. In this framework, an uncertain FS is regarded as a thick fuzzy set (TFS). A TFS is a new concept that is based on the joint use of thick sets (TSs) and the α-cuts principle. Therefore, a TS is an uncertain set and is represented by a pair of crisp sets (CSs), which describe its upper and lower bounds, i.e., a TS is an interval of CSs. Moreover, as a FS can be characterized by a family of nested CSs, a TFS can be represented by a family of nested TSs. Furthermore, a TFS can be regarded as an interval with FS boundaries. Nevertheless, in absence of uncertainty in the left-hand side of the fuzzy SoEs, the TFS solution becomes a FS solution. The proposed method is based on a set membership methodology according to paving and set projection techniques. The originality of the proposed approach resides in the fact that it applies whatever the form of the fuzzy system of equations (linear or nonlinear) and allows overcoming the approximation assumption of FS solutions by FIs (or FBs), often supposed in solving fuzzy SoEs. The proposed method has been validated using application examples that are issued from the literature.

Key words: Solving Fuzzy Systems of Equations, conventional Fuzzy Sets (FSs), Thick Sets (TSs) and Thick Fuzzy Sets (TFSs), Fuzzy Intervals (FIs), Fuzzy Boxes (FBs), Uncertainty.

1. Introduction

Solving SoEs are useful in numerous fields of applications such as engineering, physics, computer science, economics, etc. However, since many applications deal with data that are not deterministic, the parameters of these SoEs are often non-deterministic. Therefore, to rigorously solve these SoEs, it is necessary to consider uncertainty. One approach for uncertainty quantification is to consider crisp intervals (CIs) as an encoding of uncertainty, i.e., the parameters of SoEs are assumed to vary within prescribed CIs. In this framework, SoEs with CI parameters are frequently used to model linear problems subject to interval uncertainty (e.g. [41][54][65][66][70][71]). Usually, a CI is regarded as a set-valued data point, which represents an uncertain representation of the quantity of interest, i.e., a CI represents a piece of crisp but uncertain information that is not precisely measured or are hidden for confidentiality purposes. For instance, the linear SoEs $A \times x = b$ where the elements of the matrix A $(n \times n)$ and the elements of the vector b $(1 \times n)$ are CIs, is called a linear interval SoEs and is denoted by $[A] \times x = [b]$. This interval SoEs is regarded as a family of crisp linear SoEs $A \times x = b$ of the same structure with $A \in [A]$ and $b \in [b]$ [54][69].

In situations where degrees of confidence and/or flexibility are associated with CI parameters, fuzzy SoEs can be envisioned. Therefore, a fuzzy SoEs is based on the representation of its parameters by FSs. Usually and due to the simplicity of their interpretation and computer coding, fuzzy numbers [33][42][51] and FIs [25] are often used as an abstract representation of FSs. Therefore, in the literature, a normal FS of

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the real line with bounded support, whose α-cuts are CIs is referred to as a "fuzzy number", especially when their cores reduce to a point (for example, triangular fuzzy numbers). Philosophically, and as discussed in [24][30], a fuzzy number does not generalize the concept of a real number but rather the concept of a CI. Therefore, fuzzy arithmetic inherits algebraic properties of interval arithmetic, not of real numbers [30]. For instance, like CIs, fuzzy numbers do not have inverses for the addition and multiplication operations. In this framework, a fuzzy number should inherit the properties of CIs and not those of real numbers, which explains why the appellation "fuzzy interval – FI" is employed instead of "fuzzy number" throughout this study. A FI is a normal and convex FS, where all its α-cuts are CIs. A FI can be considered as a stack of nested CIs, defined by the α-cuts concept.

A fuzzy SoEs with FI parameters can be regarded as a generalization of an interval SoEs with CI parameters, in which the CI specificity has been enriched through the vertical dimension (the α-cuts dimension). Therefore, CI systems can be seen as a particular case of FI systems. In contrast to CI parameters where only a unique horizontal dimension is considered, FI parameters are represented by two dimensions (horizontal and vertical). If the horizontal dimension is similar to that which is used in CIs, the vertical dimension is related to the relevance degrees and is limited to the unit interval [0, 1]. These degrees of relevance can be interpreted as membership degrees, degrees of possibility, degrees of truth, degrees of flexibility, etc. For example, a linear SoEs where the elements of $[A]$ and $[b]$ are FIs is called a FI linear SoEs and is denoted by its α-cuts representation $[A(\alpha)]\times x = [b(\alpha)]$, $\forall\ \alpha\in[0, 1]$. Conceptually, a FI system has often been approached as a CI system *via* the concept of α-cuts. This finding is consistent with the works published in the literature where fuzzy SoEs has been treated as interval SoEs *via* the concept of α-cuts [7][8][20][29]. Such interval SoEs, especially in the linear case, have been studied since the mid-1960s [31][34][55][64]. Moreover, solving SoEs involving CIs and FIs are investigated for quite a long time (e.g. [6][7][15][16][17][23][29][33][43][51][62][71]). For instance, different numerical approaches for solving specific fuzzy linear SoEs where the elements of $[A]$ are crisp values and the elements of $[b]$ are FIs were proposed in [3][4][5][20][42]. Moreover, the situation where the elements of $[A]$ and $[b]$ are FIs have been addressed in [2][21][43][49]. Although linear fuzzy SoEs have been extensively studied and analyzed, only a few works have published for solving nonlinear fuzzy SoEs. For example, polynomial fuzzy SoEs are considered in [1][27][28]. Nowadays, research on solving interval and fuzzy SoEs has expanded and it is now difficult to draw up an exhaustive list of all the works that have been published in the literature. Unfortunately, many excellent works are missed and not mentioned in this paper.

Although significant advances were achieved in solving interval and fuzzy SoEs, two important considerations deserve special attention. The first consideration is related to the abstract interpretation where the solutions are often approximated by crisp boxes (CBs) and FBs. In a simplified way, a box is defined by the Cartesian product of n CIs and a FB can be regarded as the stacking of nested CBs according to vertical dimension $\alpha\in[0, 1]$. Thus, an FI is a one-dimensional (1D FB). If this abstraction (approximation) by CBs and FBs encapsulates the set of solutions and facilitates their computations, it generates a loss of information. Therefore, the solution set of an interval (resp. FI) SoEs is usually not a box (resp. not a FB) even though the elements of $[A]$ and $[b]$ are CIs (resp. FIs) [43]. Therefore, the obtained CBs (resp. FBs) solutions by numerous methodologies are only an abstract approximation of the solutions which are generally CSs (for interval SoEs) and FSs (for fuzzy SoEs). In this framework, outer [65][66] and inner [41][70] solutions are proposed using several algorithms [41][65][66][70]. An outer solution can be interpreted as the smallest box (resp. the FB) that encloses the CS (resp. the FS) solution. An inner solution (which is generally not unique) can be regarded as a box (resp. a FB) that fits inside the CS (resp. the FS) solution. In this paper, no abstraction of solutions is considered. The solutions are sought as uncertain CSs and uncertain FSs.

The second consideration concerns the meaning and the significance associated with the fuzzy SoEs $[A(\alpha)]\times x = [b(\alpha)]$, $\forall\ \alpha\in[0, 1]$ where strict equality between the left and right-hand sides is considered. Generally, the algebraic solution of such SoEs corresponds to the exact solution (sometimes called formal solution) where its substitution in the fuzzy SoEs leads to equality between the left and right-hand sides

[23][60][64]. This solution is usually too restrictive and sometimes even fails to exist. Furthermore, algebraic properties of CIs (or CBs) and FIs (or FBs) are often insufficient if we want to deal with inverse problems [10][13][44][45]. In fact, CIs (FIs) do not have inverses concerning the addition and multiplication operations [10][13][44][57]. To solve this deficiency, extended CIs are proposed [13][36][57]. Consequently, the set of proper CIs is completed by the set of improper CIs (a proper CI has a positive radius while an improper CI has a negative radius) to form the set of extended CIs. If this view can be useful for solving interval and FI systems, the results can sometimes be improper CIs (CBs) or improper FIs (FBs), which are not usable in practical applications. This methodology has been exploited in [29][61][67] where a general method for solving fuzzy linear SoEs using an embedding approach has been proposed. Another problem specific to fuzzy SoEs can also occur. As the solutions are computed at each level α, the resulting CIs (resp. CBs) can be non-nested according to the vertical dimension α. In this case, the result is not a FI (or a FB) but rather a gradual interval (a gradual box) [10][11][13][30]. A gradual interval (resp. a gradual box) can be regarded as the stacking of CIs (resp. CBs) that are not necessarily nested [10][30]. In this context, the fuzzy SoEs has no fuzzy solution. This result is in agreement with the criticisms formulated in [2] where it has been shown that the solution proposed in [29] is not a FI vector (a FB). In this framework, according to the epistemic view of intervals which is inherent to experimental scenarios, another way for understanding interval and fuzzy SoEs is to interpret them as an interval inclusion problem [8][43][66][69]. Therefore, the solution of an interval SoEs is defined as the solution for some realization of its CIs. This view leads to the concept of inclusion interval (or fuzzy) SoEs which has been used in [8] where the elements of $[A]$ are crisp values and the elements $[b]$ are FIs. Furthermore, this analysis is in concordance with the remark given in [62] where the authors have been stated that in modern approaches for solving fuzzy and interval SoEs, the strict equality of the left and right-hand sides is not an obligatory requirement [51].

From methodological perspectives, in an epistemic framework, solving the fuzzy SoEs according to the inclusion principle is more preferred. In this paper, we focus on solving fuzzy SoEs (linear and/or nonlinear) according to the interval inclusion concept where all the parameters are FIs. Usually, in solving interval SoEs, various solution sets are considered. The dominant solutions are the united solution set (USS), its subset which is named the tolerable solution set (TSS) and the controllable solution set (CSS) (e.g. [23][38][50][51][54][58][64][68]). These solutions have quantified formulations involving the universal quantifier ($\forall$) besides the existential quantifier ($\exists$) [64][66][67][68]. Furthermore, the united and the tolerable solutions are the most popular used concepts [23][63]. In the context and as discussed in [8][71], the USS and the TSS coincide with each other if $[A]$ is a crisp matrix, i.e., its elements are crisp real values. In the general case, the solutions USS and TSS are different but always subject to the constraint that TSS $\subset$ USS [71].

The motivation of this paper is to propose a new methodology of solving fuzzy SoEs using paving and set projection principles [18][35]. The proposed methodology is not based on the approximation of solutions by CBs or by FBs and it applies whatever the form of the SoEs: linear or nonlinear. In this framework, a new interpretation of the fuzzy solutions is proposed. As the parameters of the left-hand and right-hand sides of these SoEs are uncertain, the set of solutions should also be uncertain. Therefore, the set of solutions of an interval (resp. fuzzy) SoEs is not a CS $\mathbb{X}$ (resp. a FS $\mathbf{X}$) but an uncertain set (resp. an uncertain FS) which is represented by two lower and upper CS bounds $\mathbb{X}^{inf}$ and $\mathbb{X}^{sup}$ where $\mathbb{X}^{inf} \subset \mathbb{X}^{sup}$ (resp. upper FS bounds $\mathbf{X}^{\mathbf{inf}}$ and $\mathbf{X}^{\mathbf{sup}}$ with $\mathbf{X}^{\mathbf{inf}} \subset \mathbf{X}^{\mathbf{sup}}$). A FS $\mathbf{X}$ which is represented in bold is regarded as a family of stacked CSs according to the dimension α. The CSs $\mathbb{X}^{inf}$ and $\mathbb{X}^{sup}$ are constructed according to the quantifiers $\forall$ and $\exists$, respectively. Furthermore, two different semantics are associated with the solutions $\mathbb{X}^{inf}$ and $\mathbb{X}^{sup}$. Therefore, the CS $\mathbb{X}^{inf}$ (resp. the FS $\mathbf{X}^{\mathbf{inf}}$) which coincides with the TSS is regarded as the set of certain solutions. The CS $\mathbb{X}^{sup}$ (resp. the FS $\mathbf{X}^{\mathbf{sup}}$) which refers to the USS is regarded as a set of plausible solutions. The difference $\mathbb{X}^{?} = \mathbb{X}^{sup} \setminus \mathbb{X}^{inf}$ (between CSs) or $\mathbf{X}^{\mathbf{?}} = \mathbf{X}^{\mathbf{sup}} \setminus \mathbf{X}^{\mathbf{inf}}$ (between FSs) is interpreted as the set of ignorance (uncertainty) with plausible but not certain solutions. Furthermore, the uncertain CS composed of the bounds $\mathbb{X}^{inf}$ and $\mathbb{X}^{sup}$ is called a thick set (TS) and is formalized as an interval of CSs, i.e.,

$[\![\mathbb{X}]\!] = [\![\mathbb{X}^{\inf}, \mathbb{X}^{\sup}]\!]$. In the same way, the uncertain FS composed of the FSs bounds $\mathbf{X}^{\mathbf{inf}}$ and $\mathbf{X}^{\mathbf{sup}}$ is called a thick fuzzy set (TFS) and is formalized as an interval of FSs, i.e., $[\![\mathbf{X}]\!] = [\![\mathbf{X}^{\mathbf{inf}}, \mathbf{X}^{\mathbf{sup}}]\!]$. This formalism facilitates the propagation and the manipulation of the solutions *via* interval arithmetic tools and solvers.

In a fuzzy framework, as the TFS is composed of two FSs (lower $\mathbf{X}^{\mathbf{inf}}$ and upper $\mathbf{X}^{\mathbf{sup}}$ FSs) under the constraint $\mathbf{X}^{\mathbf{inf}} \subset \mathbf{X}^{\mathbf{sup}}$, it implies that a TFS could be regarded as a particular case of a type-2 fuzzy set. The lower bound $\mathbf{X}^{\mathbf{inf}}$ represents a FS which is certain. The FS bound $\mathbf{X}^{\mathbf{sup}}$ is an upper bound which delimits all the FSs that are plausible. The uncertainty is exhibited by the penumbra $\mathbf{X}^{\mathbf{sup}} \backslash \mathbf{X}^{\mathbf{inf}}$ which could be regarded as the FOU phenomenon in type-2 representation [46]. Furthermore, in the absence of uncertainties in the left-hand side of the fuzzy SoEs, the set of solutions becomes a FS, i.e., $\mathbf{X}^{\mathbf{inf}} = \mathbf{X}^{\mathbf{sup}} = \mathbf{X}$. For example, in a linear context, if a crisp matrix A implies a FS solution, a FI matrix A will implies an uncertain FS solution which is represented by a TFS.

The remainder of this paper is organized as follows: Section 2 is devoted to some preliminaries and notations. Section 3 investigates the essence and the representativeness of TSs and TFSs and their combination. The proposed approach for solving interval and fuzzy SoEs is given in section 4. Section 5 is dedicated to computational examples, issued from the literature. Finally, conclusions are given in section 6.

2. Preliminaries and notations

For the sake of rigor and clarity, let us define the basic concepts used in this paper. Furthermore, Table 1 shows the list of abbreviations used in the paper.

Abbreviations	Full meaning	Abbreviations	Full meaning
SoEs	System of equations	TI	Thick interval
CS	Crisp Set	TB	Thick box
CI	Crisp interval	TFS	Thick fuzzy set
CB	Crisp box	TFI	Thick fuzzy interval
FS	Fuzzy set	TFB	Thick fuzzy box
FI	Fuzzy interval	USS	United solution set
FB	Fuzzy Box	TSS	Tolerable solution set
TS	Thick set	CSS	Controllable solution set

Table 1: List of abbreviations

A CS $\mathbb{X}$ of $\Re^n$ is defined by the union of singletons $\boldsymbol{x} = (x_1, \ldots, x_n)$ it contains. A CI $[x] = [x^{\inf}, x^{\sup}] = \{x \in \Re \mid x^{\inf} \le x \le x^{\sup}\}$ is a special case of a CS, —i.e. a 1D CS. A CI vector $[\boldsymbol{x}]$ ($\boldsymbol{x}$ in bold), which is a particular case of a CS of $\Re^n$ is called a CB. It is defined by the Cartesian product of n CIs, i.e.:

$$[\boldsymbol{x}] = \Pi_{i=1}^{n}[x_i] = [x_1] \times [x_2] \times \ldots \times [x_n]; \text{ with: } [x_i] = [x_i^{\inf}, x_i^{\sup}] \tag{1}$$

Thanks to the representation theorem [52][59], any FS is decomposed into a system $\{\mathbb{X}(\alpha))\}_\alpha; \alpha \in [0,1]$ of its α-cuts under the constraint of monotonicity (consistency)—i.e., $\alpha_1 \ge \alpha_2 \Rightarrow \mathbb{X}(\alpha_1) \subseteq \mathbb{X}(\alpha_2)$.

An α-cut on a FS $\mathbf{X}$ (in bold) is CS $\mathbb{X}(\alpha) = \{\boldsymbol{x} \mid \mu_{\mathbf{X}}(\boldsymbol{x}) \ge \alpha\}$ where $\alpha \in [0, 1]$ and $\mu_{\mathbf{X}}$ is the membership function of $\mathbf{X}$. In this case, the FS $\mathbf{X}$ can be decomposed as:

$$\mathbf{X} = \bigcup_{\alpha \in [0,1]} \alpha \mathbb{X}(\alpha) \tag{2}$$

where $\mu_{\mathbf{X}}(\boldsymbol{x}) = \sup_{\boldsymbol{x} \in \mathrm{X}(\alpha)} \alpha$ and "sup" denotes the supremum. Furthermore, the membership function $\mu_{\mathbf{X}}(\boldsymbol{x})$ of $\mathbf{X}$ can be also obtained from the characteristic function of CSs by

$$\mu_{\mathbf{X}}(\boldsymbol{x}) = \sup_{\alpha \in [0,1]} \alpha \mu_{\mathrm{X}(\alpha)}(\boldsymbol{x}), \forall \boldsymbol{x} \in \Re^n. \tag{3}$$

A FI is a special case of a FS, i.e., a convex FS where all its α-cuts are CIs. For compatibility raisons with the CI notation, a FI $[X]$ is denoted by:

$$[X] = \bigcup_{\alpha \in [0,1]} \alpha [x(\alpha)]$$

where $[x(\alpha)]$ is a CI, representing an α-cut on $[X]$. The membership function of the FI $[X]$ is given by:

$$\mu_{[X]}(x) = \sup_{\alpha\in[0,1]} \alpha\mu_{[x(\alpha)]}(x);\ x\in\Re \tag{4}$$

A FI vector $[\boldsymbol{X}]$ (in bold) is called a FB. It is defined by the Cartesian product of n FIs, i.e.:

$$[\boldsymbol{X}] = \Pi_{i=1}^{n}\ [X_i] = [X_1]\times[X_2]\times\ldots\times[X_n] \tag{5}$$

At each α-cut, a FB $[\boldsymbol{X}]$ is a CB $[\boldsymbol{x}(\alpha)] = [x_1(\alpha)]\times\ldots\times[x_n(\alpha)]$, i.e., the Cartesian product of n α-cuts (CIs). The membership of a FB can be obtained from (3) and (4) as follows:

$$\mu_{[\boldsymbol{X}]}(\boldsymbol{x}) = \sup_{\alpha\in[0,1]} \alpha\mu_{[\boldsymbol{x}(\alpha)]}(\boldsymbol{x});\ \boldsymbol{x}\in\Re^n \tag{6}$$

We must note that the concept of FBs is very close to the concept of fuzzy n-cell numbers defined in [72], i.e., a FB can be regarded as a fuzzy n-cell number.

3. TSs and TFSs and their approximations

3.1. Tick Sets (TS), thick intervals (TIs) and thick boxes (TBs)

Usually, an ordered crisp set (CS), which is denoted by $\mathbb{X}$, has its boundary known with certainty. However, in some practical applications, the boundary may become uncertain. To tackle this problem, the concept of TSs has been proposed in [22]. In this framework and as illustrated in Fig. 1, a TS which is denoted by $[\![\mathbb{X}]\!]$ is represented by two CSs $\mathbb{X}^{\inf}$ and $\mathbb{X}^{\sup}$. The CS $\mathbb{X}^{\inf}$ that contains the elements that belong to $[\![\mathbb{X}]\!]$ with certainty and the CS $\mathbb{X}^{\sup}$ encompassing all the elements where their belonging to $[\![\mathbb{X}]\!]$ may be plausible. The difference $\mathbb{X}^{?} = \mathbb{X}^{\sup}\backslash\mathbb{X}^{\inf}$ that represents the uncertainty (ignorance) is called the penumbra. The elements of the penumbra $\mathbb{X}^{?}$ are plausible but not certain (perhaps possible).

When defining by $(\mathbb{P}(\Re^n), \subset)$ the power set of $\Re^n$ equipped with the inclusion order relation $\subset$, the set $\mathbb{P}(\Re^n)$ is a complete lattice with respect to $\subset$ (see [22] for more details). In fact, a TS $[\![\mathbb{X}]\!]$ of $\Re^n$ is an interval of $(\mathbb{P}(\Re^n), \subset)$. Consequently, if $[\![\mathbb{X}]\!]$ is a TS, there exist two CSs of $\Re^n$ baptized lower bound and upper bound such that (see Fig. 1):

$$[\![\mathbb{X}]\!] = [\![\mathbb{X}^{\inf}, \mathbb{X}^{\sup}]\!] = \{\mathbb{X}\in\mathbb{P}(\Re^n) \mid \mathbb{X}^{\inf}\subset\mathbb{X}\subset\mathbb{X}^{\sup}\} \tag{7}$$

Moreover, if $\mathbb{A}\in[\![\mathbb{X}]\!]$ and $\mathbb{B}\in[\![\mathbb{X}]\!]$, then $\mathbb{A}\cap\mathbb{B}\in[\![\mathbb{X}]\!]$ and $\mathbb{A}\cup\mathbb{B}\in[\![\mathbb{X}]\!]$. If $\mathbb{X}^{\inf} = \mathbb{X}^{\sup} = \mathbb{X}$, then $[\![\mathbb{X}]\!]$ is a CS of $\Re^n$—i.e., a singleton in $\mathbb{P}(\Re^n)$.

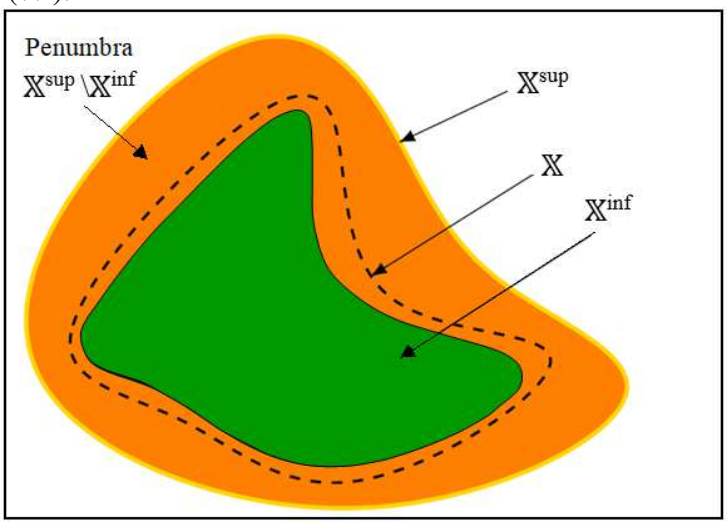


Fig. 1: Representation of a TS $[\![\mathbb{X}]\!]$

Unlike CSs where only two logic values are used, in the TS formalism, three logic values are necessary: 0 (False), ? (Perhaps) and 1 (True). The fundamental logical operations such as «*and*», «*or*» and «*not*» can be implemented using Kleene's ternary logic [40][47] and given in Fig. 2.

By analogy with a CS, the characteristic function of a TS $[\![\mathbb{X}]\!]$ can be defined as follows:

$$\mu_{\square X\square}:\Re^n\to\{0,?,1\};\ \boldsymbol{x}\mapsto\begin{cases}1 \text{ if } \boldsymbol{x}\in \mathrm{X}^{\inf}\\ 0 \text{ if } \boldsymbol{x}\notin \mathrm{X}^{\sup}\\ ?\ \text{otherwise}\end{cases} \tag{8}$$

When considering two characteristic functions of two TSs, they can be combined by the logical operators as illustrated in Fig. 2. Furthermore, the arithmetical and logical operators between CSs can be extended to TSs [22].

and (a, b)

$a \wedge b$		b		
		0	?	1
a	0	0	0	0
	?	0	?	?
	1	0	?	1

or (a, b)

$a \vee b$		b		
		0	?	1
a	0	0	?	1
	?	?	?	1
	1	1	1	1

not (a)

a	$\bar{a}$
0	1
?	?
1	0

Fig. 2: The operations and, or and not using Kleene's ternary logic

As a CI is a special case of a CS, a TI which is denoted by $[\![x]\!] = [\![[x^{\inf}], [x^{\sup}]]\!]$ is a 1D TS, —i.e., a special case of a TS [12]. If $[x^{\inf}] = [x^{\sup}]$, the TI $[\![x]\!]$ becomes a CI $[x]$. A TI vector which is defined by $[\![\boldsymbol{x}]\!] = [\![[\boldsymbol{x}^{\boldsymbol{\inf}}], [\boldsymbol{x}^{\boldsymbol{\sup}}]]\!]$ (in bold) is called a TB. It is a particular case of a TS of $\mathbb{P}(\Re^n)$. Since the bounds $[\boldsymbol{x}^{\boldsymbol{\inf}}]$ and $[\boldsymbol{x}^{\boldsymbol{\sup}}]$ are CBs of $\Re^n$, they can be expressed as the Cartesian product of n CIs:

$$[\boldsymbol{x}^{\boldsymbol{\inf}}] = [x_1^{\inf}]\times[x_2^{\inf}]\times\ldots\times[x_n^{\inf}];\ [\boldsymbol{x}^{\boldsymbol{\sup}}] = [x_1^{\sup}]\times[x_2^{\sup}]\times\ldots\times[x_n^{\sup}] \tag{9}$$

3.2. Thick fuzzy sets (TFSs), thick FIs (TFIs) and thick FBs (TFBs)

Based on TSs and FSs concepts, a TFS can be defined by a system of nested TSs $\{[\![\mathbb{X}(\alpha)]\!]\}_\alpha; \alpha\in[0,1]$ of its α-cuts under the monotonicity (consistency) constraint:

$$\alpha_1 \geq \alpha_2 \Rightarrow [\![\mathbb{X}(\alpha_1)]\!] \subseteq [\![\mathbb{X}(\alpha_2)]\!] \Leftrightarrow \alpha_1 \geq \alpha_2 \Rightarrow \mathbb{X}^{\inf}(\alpha_1) \subseteq \mathbb{X}^{\inf}(\alpha_2) \text{ and } \mathbb{X}^{\sup}(\alpha_1) \subseteq \mathbb{X}^{\sup}(\alpha_2)$$

An α-cut on a TFS $[\![\mathbb{X}]\!]$ (in bold) is TS $[\![\mathbb{X}(\alpha_1)]\!] = \{\boldsymbol{x} \mid \mu_{[\![\mathbb{X}]\!]}(\boldsymbol{x}) \geq \alpha\}$ where $\alpha\in[0, 1]$ and $\mu_{[\![\mathbb{X}]\!]}(\boldsymbol{x})$ is the membership function of $[\![\mathbb{X}]\!]$. In this case, the TFS $[\![\mathbb{X}]\!]$ can be decomposed as:

$$[\![\mathbb{X}]\!] = [\![\mathbb{X}^{\boldsymbol{\inf}}, \mathbb{X}^{\boldsymbol{\sup}}]\!] = \bigcup_{\alpha\in[0,1]} \alpha[\![\mathbb{X}(\alpha)]\!] = [\![\bigcup_{\alpha\in[0,1]} \alpha\, \mathbb{X}^{\inf}(\alpha),\ \bigcup_{\alpha\in[0,1]} \alpha\ \mathbb{X}^{\sup}(\alpha)\ ]\!] \tag{10}$$

Therefore, the TFS (10) which is considered as a family of α-cuts can be defined by the following membership function:

$$\mu_{[\![\mathbb{X}]\!]}(\boldsymbol{x}) = \sup_{\alpha\in[0,1]} \alpha\mu_{[\![\mathbb{X}(\alpha)]\!]}(\boldsymbol{x}) \left[\sup_{\alpha\in[0,1]} \alpha\mu_{\mathrm{X}^{\inf}(\alpha)}(\boldsymbol{x}),\ \sup_{\alpha\in[0,1]} \alpha\mu_{\mathrm{X}^{\sup}(\alpha)}(\boldsymbol{x})\right], \forall \boldsymbol{x}\in\Re^n.$$

From (10), it can be stated that $\mathbb{X}^{\boldsymbol{\inf}}$ and $\mathbb{X}^{\boldsymbol{\sup}}$ are FSs. Therefore, the TFS $[\![\mathbb{X}]\!]$ is represented by an interval of FSs where $\mathbb{X}^{\boldsymbol{\inf}}$ and $\mathbb{X}^{\boldsymbol{\sup}}$ represent its bounds. In situations when $\mathbb{X}^{\boldsymbol{\inf}} = \mathbb{X}^{\boldsymbol{\sup}} = \mathbb{X}$, the TFS $[\![\mathbb{X}]\!]$ becomes a FS $\mathbb{X}$. In this framework, as the TFS is composed of two FSs (lower $\mathbb{X}^{\boldsymbol{\inf}}$ and upper $\mathbb{X}^{\boldsymbol{\sup}}$ FSs) under the constraint $\mathbb{X}^{\boldsymbol{\inf}} \subset \mathbb{X}^{\boldsymbol{\sup}}$, it implies that a TFS could be regarded as a special case of a type-2 fuzzy set. The lower bound $\mathbb{X}^{\boldsymbol{\inf}}$ represents a FS, which is certain. The upper bound $\mathbb{X}^{\boldsymbol{\sup}}$, which delimits all the FSs that are plausible. The uncertainty is exhibited by the penumbra $\mathbb{X}^{\boldsymbol{\sup}}\backslash\mathbb{X}^{\boldsymbol{\inf}}$. This penumbra concept in the TFS representation could be regarded as the FOU phenomenon in type-2 representation [46].

A TFI is a special case of a TFS, i.e., a 1D TFS and is represented by:

$$[\![X]\!] = \bigcup_{\alpha\in[0,1]} \alpha[\![x(\alpha)]\!] = [\![[X^{\inf}], [X^{\sup}]]\!] = [\![\bigcup_{\alpha\in[0,1]} \alpha[x^{\inf}(\alpha)], \bigcup_{\alpha\in[0,1]} \alpha[x^{\sup}(\alpha)]]\!] \tag{11}$$

where $[\![x(\alpha)]\!] = [\![[x^{\inf}(\alpha)], [x^{\sup}(\alpha)]]\!]$ is an α-cut of $[\![X]\!]$ and is regarded as a TI. A TFI vector is called here a TFB and denoted by $[\![\boldsymbol{X}]\!] = [\![[\boldsymbol{X}^{\boldsymbol{\inf}}], [\boldsymbol{X}^{\boldsymbol{\sup}}]]\!]$. Since $[\boldsymbol{X}^{\boldsymbol{\inf}}]$ and $[\boldsymbol{X}^{\boldsymbol{\sup}}]$ are FBs it obtains:

$$[\boldsymbol{X}^{\mathbf{inf}}]=[X_1^{\inf}]\times[X_2^{\inf}]\times\ldots\times[X_n^{\inf}];\ [\boldsymbol{X}^{\mathbf{sup}}]=[X_1^{\sup}]\times[X_2^{\inf}]\times\ldots\times[X_n^{\sup}]$$

It is worth noting that a TFI can be regarded as a type-2 FI [12]. Furthermore, a TFB can be interpreted as an uncertain (thick) fuzzy n-cell number, that is defined in [72].

3.3. Combination and fusion operations of TSs and TFSs

According to the α-cuts principle, at each level α, a TFS is regarded as a TS. Therefore, operations on TFSs can be performed as operations on TSs. Let us consider a collection of CSs $\{\mathbb{X}_i\}_{i\in\Omega}$. The smallest TS which encloses all $\{\mathbb{X}_i\}_{i\in\Omega}$ is defined by:

$$\&\{\mathbb{X}_i,\ i\in\Omega\} = [\![\cap_{i\in\Omega}\mathbb{X},\ \cup_{i\in\Omega}\mathbb{X}]\!]$$

where & denotes the smallest TS. In this context, it is possible to extend the operators initially proposed for CSs to TSs as follows [22]:

$$[\![\mathbb{X}]\!]\bullet[\![\mathbb{Y}]\!] = \&\{\mathbb{T},\ \exists\,\mathbb{X}\in[\![\mathbb{X}]\!],\ \exists\,\mathbb{Y}\in[\![\mathbb{Y}]\!],\ \mathbb{T}=\mathbb{X}\bullet\mathbb{Y}\};\ \bullet\in\{\cap,\cup,\setminus,\ldots\} \tag{12}$$

For instance, according to the monotony property of the intersection, union, difference and addition operators, they are defined by the following expressions:

$$\text{Intersection: } [\![\mathbb{X}(\alpha)]\!]\cap[\![\mathbb{Y}(\alpha)]\!] = [\![\mathbb{X}^{\inf}(\alpha)\cap\mathbb{Y}^{\inf}(\alpha),\ \mathbb{X}^{\sup}(\alpha)\cap\mathbb{Y}^{\sup}(\alpha)]\!]$$

$$\text{Union: } [\![\mathbb{X}(\alpha)]\!]\cup[\![\mathbb{Y}(\alpha)]\!] = [\![\mathbb{X}^{\inf}(\alpha)\cup\mathbb{Y}^{\inf}(\alpha),\ \mathbb{X}^{\sup}(\alpha)\cup\mathbb{Y}^{\sup}(\alpha)]\!]$$

$$\text{Difference: } [\![\mathbb{X}(\alpha)]\!]\setminus[\![\mathbb{Y}(\alpha)]\!] = [\![\mathbb{X}^{\inf}(\alpha)\setminus\mathbb{Y}^{\sup}(\alpha),\ \mathbb{X}^{\sup}(\alpha)\setminus\mathbb{Y}^{\inf}(\alpha)]\!]$$

$$\text{Addition: } [\![\mathbb{X}(\alpha)]\!] + [\![\mathbb{Y}(\alpha)]\!] = [\![\mathbb{X}^{\inf}(\alpha)+\mathbb{Y}^{\inf}(\alpha),\ \mathbb{X}^{\sup}(\alpha)+\mathbb{Y}^{\sup}(\alpha)]\!]$$

More generally, for a given function $\boldsymbol{f}$ from $\Re^n$ to $\Re^m$, the image of the TS $[\![\mathbb{X}(\lambda)]\!] = [\![\mathbb{X}^{\inf}(\lambda),\ \mathbb{X}^{\sup}(\lambda)]\!]$ by $\boldsymbol{f}$ is evaluated by the following expression:

$$\boldsymbol{f}([\![\mathbb{X}(\alpha)]\!]) = \&\{\boldsymbol{f}(\mathbb{X}(\alpha)),\ \mathbb{X}(\alpha)\in[\![\mathbb{X}(\alpha)]\!]\} = [\![\boldsymbol{f}(\mathbb{X}^{\inf}(\alpha)),\ \boldsymbol{f}(\mathbb{X}^{\sup}(\alpha))]\!] \tag{13}$$

This extension of functions permits the propagation of TSs and TFSs solutions through linear and nonlinear models where the inputs, outputs, states and/or parameters can be represented by TFSs.

4. Solving systems of interval and fuzzy SoEs

4.1. Conventional approaches for solving interval and fuzzy SoEs

The major results presented in the literature relates not to a general nonlinear system but a linear interval SoEs. Let us consider the following conventional and crisp linear SoEs:

$$A\times x = b;\ \text{with: } A=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix};\ \ b=\begin{pmatrix} b_1 \\ \ldots \\ b_n \end{pmatrix}; x=\begin{pmatrix} x_1 \\ \ldots \\ x_n \end{pmatrix} \tag{14}$$

where A is a known $n\times n$ matrix of real numbers and b is a given $1\times n$ real-valued vector. The variable x is the unknown solutions set of real-valued vectors to be sought. The solution set of (14) exclusively takes one of three situations: the empty set, a singleton (vector) set, or a set with an infinite number of vectors (a region in $\Re^n$). Solving the system of equations (14) has a very long history. Many conventional algorithms such as Jacobi method, Gauss–Seidel method, Kaczmarz method, etc. [32] have been successfully developed. In several applications where the manipulated data are uncertain, the parameters of the system (14) are often uncertain and assumed to vary within prescribed CIs. Therefore, interval-based SoEs are usually used to model problems subject to interval uncertainty. In this context, the system (14) becomes:

$$[A]\times x = [b]\ ;\ \text{with: } [A]=\begin{pmatrix} [a_{11}] & \cdots & [a_{1n}] \\ \vdots & \ddots & \vdots \\ [a_{n1}] & \cdots & [a_{nn}] \end{pmatrix};\ \ [b]=\begin{pmatrix} [b_1] \\ \ldots \\ [b_n] \end{pmatrix} \tag{15}$$

where $[A]$ is a known $n\times n$ matrix of CIs and $[b]$ is a given $1\times n$ interval vector. In this framework, several methodologies have been proposed for solving the interval system (15) (e.g. [34][38][54][67]). The fundamental concept in solving (15) is the so-called algebraic solution, sometimes referred to as the formal

solution (or the exact solution). Therefore, if this solution is substituted in equation (15), equality between the left and right-hand sides is obtained. However, and as discussed in the paper introduction, this solution set is restrictive or even empty. Therefore, solving (15) using an embedded approach according to Kaucher interval arithmetic [36] can be turned out to be unrealistic when improper CIs are obtained. Another way for solving (15), which represents a dominant approach in the literature, is based on the treating of the interval SoEs (15) as a set of crisp SoEs whose parameters belong to the corresponding CIs (inclusion problem). In this case, the SoEs (15) is interpreted and understood as follows:

$$A \times x = b \text{ ; with: } A \in [A] \text{ and } b \in [b] \tag{16}$$

Therefore, the formulation (16) interprets (15) not as a strict equality between the left and right-hand interval sides but as a family of crisp linear SoEs $A \times x = b$ of the same structure with $A \in [A]$ and $b \in [b]$ [54][69]. That is the formulation and the approach adopted in this paper. In solving (16) the important and the popular ideas are the concepts of solutions: USS, TSS and CSS [23][38][50][51][54][58][64][68]. These solutions are formalized through the universal quantifier ($\forall$) besides the existential quantifier ($\exists$). Therefore, two solutions are often proposed [23][63]. The first solution is the USS which is defined by:

$$\mathbb{X}^{\text{USS}} = \{x = (x_1, ..., x_n) \in \Re^n, \ \exists A \in [A], \exists\, b \in [b] \mid A \times x = b\} \tag{17}$$

Equation (17) can be written as follows [71]:

$$\mathbb{X}^{\text{USS}} = \{\boldsymbol{x} = (x_1, ..., x_n) \in \Re^n, \ \exists A \in [A] \mid\mid A \times x \in [b]\} \tag{18}$$

This solution refers to the most understanding of what is a solution to the interval SoEs (16). The USS is the most studied solution in the literature where different methods have been proposed to estimate it. This USS refers to the set of solutions such that it exists at least one $A \in [A]$ for which the left-hand side: $A \times x$ falls into the right-hand side $[b]$. The second solution is the TSS which ensures strong compatibility between the parameters and the data [71]. Therefore, the TSS refers to the set of solutions for which the left-hand side $A \times x$ falls into the right-hand side $[b]$ for any $A \in [A]$. The TSS is defined by the expression:

$$\mathbb{X}^{\text{TSS}} = \{x = (x_1, ..., x_n) \in \Re^n, \ \forall A \in [A], \exists\, b \in [b] \mid A \times x = b\} \tag{19}$$

Equation (19) can be reformulated as follows:

$$\mathbb{X}^{\text{TSS}} = \{\boldsymbol{x} = (x_1, ..., x_n) \in \Re^n, \ \forall\, A \in [A] \mid A \times x \in [b]\} \tag{20}$$

There exists another solution which is named the CSS. Although the proposed methodology can be applied to determine the CSS, the latter is not addressed in this paper.

In situations when degrees of confidence are associated with the CI parameters, the interval SoEs (16) can be extended to the fuzzy context where the elements of $[A]$ and $[b]$ are FIs. In this case, the elements of $[A]$ and $[b]$ are expressed using their α-cuts representations as follows:

$$[A(\alpha)] = \begin{pmatrix} [a_{11}(\alpha)] & \dots & [a_{1n}(\alpha)] \\ \vdots & \ddots & \vdots \\ [a_{n1}(\alpha)] & \cdots & [a_{nn}(\alpha)] \end{pmatrix}; \ [b(\alpha)] = \begin{pmatrix} [b_1(\alpha)] \\ \dots \\ [b_n(\alpha)] \end{pmatrix}; \forall \alpha \in [0,1] \tag{21}$$

In (21), $[a_{ij}(\alpha)]$, $[b_i(\alpha)]$, $i = 1, \dots, n$ and $j = 1, \dots, n$ are α-cuts (CIs) of the FIs $[\boldsymbol{A_{ij}}]$ and $[\boldsymbol{B_i}]$, i.e.:

$$[\boldsymbol{A_{ij}}] = \bigcup_{\alpha \in [0,1]} \alpha [a_{ij}(\alpha)] \ ; \ [\boldsymbol{B_i}] = \bigcup_{\alpha \in [0,1]} \alpha [b_i(\alpha)]; \ i = 1, \dots, n \text{ and } j = 1, \dots, n \tag{22}$$

At each α-cut, a linear fuzzy SoEs is simply a linear interval SoEs. In this framework, all methodologies that have been developed for dealing with linear interval SoEs can be used in the fuzzy context.

4.2. The proposed methodology for solving interval and fuzzy equations

The proposed method applies whatever the form of the SoEs: linear or nonlinear. Let us consider the nonlinear crisp SoEs given in the following form:

$$\begin{cases} f_1(a, x) = b_1 \\ \dots \\ f_n(a, x) = b_n \end{cases} \text{ ; with: } a = \begin{pmatrix} a_1 \\ \dots \\ a_m \end{pmatrix}; \ x = \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix} \tag{23}$$

The system (23) can be written in the following concise form:

$$F(a, x) = b \text{ ; with: } F(a, x) = \begin{pmatrix} f_1(a, x) \\ \dots \\ f_n(a, x) \end{pmatrix}; \; b = \begin{pmatrix} b_1 \\ \dots \\ b_n \end{pmatrix} \tag{24}$$

where a and b are $1\times m$ and $1\times n$ real-valued vectors, respectively. In situations when the vectors of parameters a and b are subject to uncertainties that represented by CIs, the system (24) is regarded as a family of crisp nonlinear SOEs:

$$F(a, x) = b \text{ ; with } a\in[a] \text{ and } b\in[b] \text{ ; } [a] = \begin{pmatrix} [a_1] \\ \dots \\ [a_m] \end{pmatrix}; \; [b] = \begin{pmatrix} [b_1] \\ \dots \\ [b_n] \end{pmatrix}$$

In a fuzzy framework, the vectors of parameters a and b become FIs, that are represented by their α-cuts representations, i.e.,

$$[a(\alpha)] = \begin{pmatrix} [a_1(\alpha)] \\ \dots \\ [a_m(\alpha)] \end{pmatrix} \text{ and: } [b(\alpha)] = \begin{pmatrix} [b_1(\alpha)] \\ \dots \\ [b_n(\alpha)] \end{pmatrix}; \; \forall\alpha\in[0,1]$$

In this case, the fuzzy SoEs represents always a family of crisp SoEs $F(a, x) = b$ of the same structure with $a\in[a(\alpha)]$ and $b\in[b(\alpha)]$, $\forall\alpha\in[0, 1]$. Therefore, at each α-cut, as the parameters are uncertain and represented by CIs, the set of solutions should be an uncertain set, i.e., the solution of a SoEs with uncertainty must be a set with uncertainty. According to this principle, in the proposed approach, at each α-cut the set of solutions is an uncertain set and is represented by an interval of CSs where its bounds are two lower and upper bounds $\mathbb{X}^{\inf}$ and $\mathbb{X}^{\sup}$ with $\mathbb{X}^{\inf}\subset\mathbb{X}^{\sup}$. This uncertain solution is regarded as a TS and is represented by $[\![\mathbb{X}]\!] = [\![\mathbb{X}^{\inf}, \mathbb{X}^{\sup}]\!]$. The bounds $\mathbb{X}^{\inf}$ and $\mathbb{X}^{\sup}$ are given by the following expressions:

$$\mathbb{X}^{\inf}(\alpha) = \{x = (x_1, \dots, x_n)\in\Re^n, \; \forall a\in[a(\alpha)] \mid F(a, x)\in[b(\alpha)]\} \tag{25}$$

$$\mathbb{X}^{\sup}(\alpha) = \{\boldsymbol{x} = (x_1, \dots, x_n)\in\Re^n, \; \exists a\in[a(\alpha)] \mid F(a, x)\in[b(\alpha)]\} \tag{26}$$

The solution $\mathbb{X}^{\inf}(\alpha)$ can be regarded as a TSS and is associated with the quantifier $\forall$. This solution is regarded as a certain solution where $\forall a\in[a(\alpha)]$ it ensured that $F(a, x)\in[b(\alpha)]$. In the same way, the solution $\mathbb{X}^{\sup}(\alpha)$ can be interpreted as an USS. It is associated with the quantifier $\exists$ and is considered as the set of plausible solutions, i.e., it exists at least one $a\in[a(\alpha)]$ for which $F(a, x)\in[b(\alpha)]$. By stacking the solutions according to the α-dimension, the CSs $\mathbb{X}^{\inf}(\alpha)$ and $\mathbb{X}^{\sup}(\alpha)$ become FSs $\mathbb{X}^{\mathbf{inf}}$ and $\mathbb{X}^{\mathbf{sup}}$ and the TS $[\![\mathbb{X}(\alpha)]\!] = [\![\mathbb{X}^{\inf}(\alpha), \mathbb{X}^{\sup}(\alpha)]\!]$ turn out to be a TFS represented by $[\![\mathbb{X}]\!] = [\![\mathbb{X}^{\mathbf{inf}}, \mathbb{X}^{\mathbf{sup}}]\!]$. Furthermore, in absence of uncertainties in the left-hand side, i.e., $[a(\alpha)] = a$, the set of solutions becomes a FS $\mathbb{X}^{\mathbf{inf}} = \mathbb{X}^{\mathbf{sup}} = \mathbb{X}$, i.e., a CS $\mathbb{X}^{\inf} = \mathbb{X}^{\sup} = \mathbb{X}$ at each α-cut. It is worth noting that in some situations the solution $\mathbb{X}^{\mathbf{inf}}$ does not exist. In this context, the solution is limited to $\mathbb{X}^{\mathbf{sup}}$.

To lighten and to simplify the notations, in the sequel of this section, the parameter α is not associated with the equations. In this paper, the solutions $\mathbb{X}^{\inf}$ and $\mathbb{X}^{\sup}$ are computed by a new approach which is based on paving and projection operations [18][35]. To explain the proposed method, let us consider $\mathbb{W}(x, a) \subset \mathbb{X}\times\mathbb{A}$ as the relation generated by the constraint $\mathbb{W} = \{(x, a)\in \Re^{n+m} \mid F(a, x)\in[b]\}$. Let also denote by $[\boldsymbol{a}] = [a_1]\times\dots\times[a_m]$ and $[\boldsymbol{x}] = [x_1]\times\dots\times[x_n]$ two CBs in $\Re^m$ and $\Re^n$, respectively. In this case, $\mathbb{X}^{\inf}$ and $\mathbb{X}^{\sup}$ are computed from the projection on $\mathbb{X}$ of the intersection of $\mathbb{W}$ with the Cartesian product $\mathbb{X}\times[\boldsymbol{a}]$:

$$\mathbb{X}^{\sup} = \text{Proj}_{\mathbb{X}} \; \mathbb{W} \cap (\mathbb{X}\times[\boldsymbol{a}]) \text{ ; and : } \mathbb{X}^{\inf} = \overline{\text{Proj}_{\mathbb{X}} \; \overline{\mathbb{W}} \cap (\mathbb{X}\times[\boldsymbol{a}])} \tag{27}$$

To illustrate the computing principle using the projection principle, let us solve the nonlinear CI equation:

$$F(a, x) = (x_1 - a_1)^2 + (x_2 - a_2)^2 = b \text{ ; with: } a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}; \; x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{28}$$

The parameters a_1, a_2 and b are uncertain and represented by CIs, i.e., $a_1 \in [a_1]$, $a_2 \in [a_2]$, $b \in [b] = [0, 400]$. In this case, the constraint $\mathbb{W}$ is given by:

$$\mathbb{W} = \{(x, a) \in \Re^4 \,|\, F(a, x) \in [b]\} = \{(x, a) \in \Re^4 \,|\, (x_1 - a_1)^2 + (x_2 - a_2)^2 \in [0, 400]\} \quad (29)$$

For the sake of simplicity of representation, let us consider the situation where a_2 is constant (here $a_2 = 0$ and $[a_1] = [-10, 10]$). In such a case, the constraint (29) becomes:

$$\mathbb{W} = \{(x, a_1) \in \Re^3 \,|\, (x_1 - a_1)^2 + x_2^2 \in [0, 400]\} \quad (30)$$

The constraint (30) is a cylinder in the space (x_1, x_2, a_1). Fig. 3.a represents the envelope of the cylinder and the two planes which are the upper and lower bounds of the Cartesian product $\mathbb{X} \times [-10, 10]$. Figs. 3.b and Fig. 4 illustrate how $\mathbb{X}^{inf}$ and $\mathbb{X}^{sup}$ are obtained from the projections. The efficient computation of $\mathbb{X}^{inf}$ and $\mathbb{X}^{sup}$ can be approached using the method proposed in [18]. The key idea is to use interval computation to decide whether or not the projections of a box $[\boldsymbol{x}] \times [\boldsymbol{a}] \subset \mathbb{X} \times \mathbb{A}$ belong to $\mathbb{X}^{inf}$ or $\mathbb{X}^{sup}$. It is based on set properties for $\mathbb{X}^{inf}$ and $\mathbb{X}^{sup}$ when $[\boldsymbol{a}]$ is split into several parts. Therefore, if $[\boldsymbol{a}] = \bigcup_i [\boldsymbol{a}_i]$, then we have:

$$\mathbb{X}^{sup} = \mathrm{Proj}_X\ \mathbb{W} \cap (\mathbb{X} \times \bigcup_i [\boldsymbol{a}_i]) = \bigcup_i \mathrm{Proj}_X\ \mathbb{W} \cap (\mathbb{X} \times [\boldsymbol{a}_i]) = \bigcup_i \mathbb{X}_i^{sup} \quad (31)$$

$$\mathbb{X}^{inf} = \overline{\mathrm{Proj}_X\ \overline{\mathbb{W}} \cap (\mathbb{X} \times \bigcup_i [\boldsymbol{a}_i])} = \bigcup_i \overline{\mathrm{Proj}_X\ \overline{\mathbb{W}} \cap (\mathbb{X} \times [\boldsymbol{a}_i])} = \bigcap_i \overline{\mathrm{Proj}_X\ \overline{\mathbb{W}} \cap (\mathbb{X} \times [\boldsymbol{a}_i])} = \bigcap_i \mathbb{X}_i^{inf} \quad (32)$$

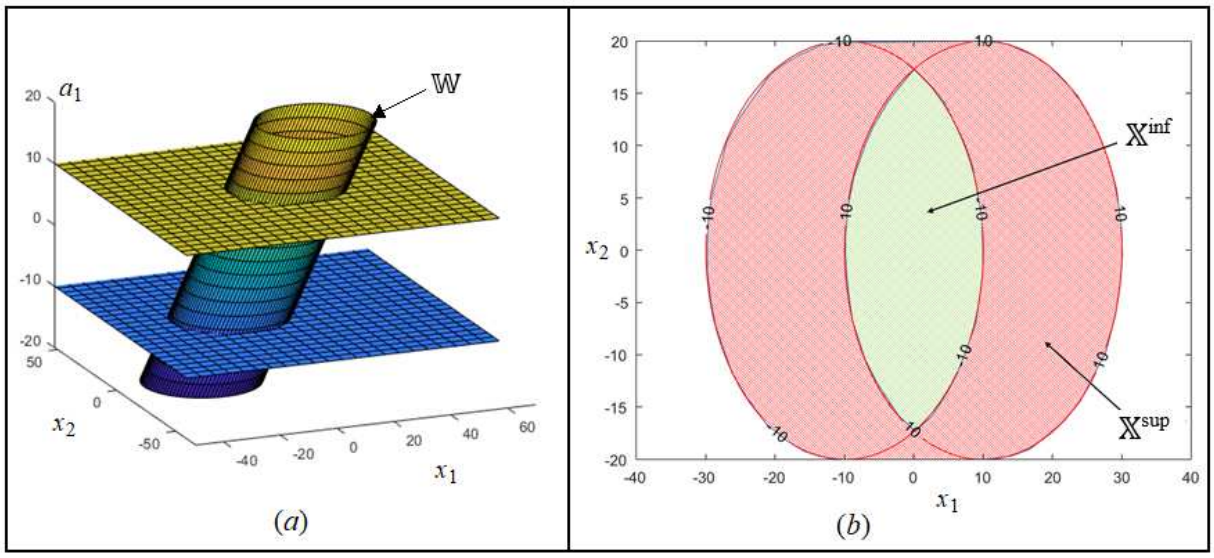


Figure 3: Illustration of the projection principle

These properties are illustrated in Fig. 4 with $\mathbb{X} \times [\boldsymbol{a}] = \mathbb{X} \times [-10, 2] \cup \mathbb{X} \times [2, 10]$.

It is worth noting that $\mathbb{X}^{inf}$ and $\mathbb{X}^{sup}$ of Fig. 3.b are obtained from $\mathbb{X}_1^{inf}$, $\mathbb{X}_2^{inf}$, $\mathbb{X}_1^{sup}$ and $\mathbb{X}_2^{sup}$ of Fig. 3.b, i.e., $\mathbb{X}^{inf} = \mathbb{X}_1^{inf} \cap \mathbb{X}_2^{inf}$ and $\mathbb{X}^{sup} = \mathbb{X}_1^{sup} \cup \mathbb{X}_2^{sup}$. The previous properties can be generalized to a paving of $\mathbb{X} \times [\boldsymbol{a}]$ by a set of CBs, i.e.,

$$\mathbb{X} \times [\boldsymbol{a}] = \bigcup_{i,j} [\boldsymbol{x}_i] \times [\boldsymbol{a}_j] \quad (33)$$

The paving algorithm, which is proposed in [18] leads to efficient computations using the interval-based solver PyIbex (https://www.ensta-bretagne.fr/desrochers/pyibex/docs/pyibex/). The PyIbex solver is a set of Python modules to solve nonlinear problems using interval arithmetic tools. The solutions $\mathbb{X}^{inf}$ and $\mathbb{X}^{sup}$ of Fig. 5 when $a_1 = [-10, 10]$ and $a_2 = [0, 0] = 0$ (i.e., $[\boldsymbol{a}] = [-10, 10] \times [0, 0]$) represent the solution obtained by PyIbex and represented by the visualization system VIBes (http://codac.io/manual/07-graphics/01-vibes.html) with and without paving illustration. In the paper sequel and for reasons of visibility, the paving is not illustrated in the figures. For ease of 3D representation, it has been assumed that a_2 is a real constant

value. Nevertheless, the principle is general and can be applied when a_2 belongs to an interval, i.e., (a_1, a_2) belongs to the 2D box $[\boldsymbol{a}] = [a_1]\times[a_2]$and $\mathbb{W}$ is the 4-dimensional relation given by (29).

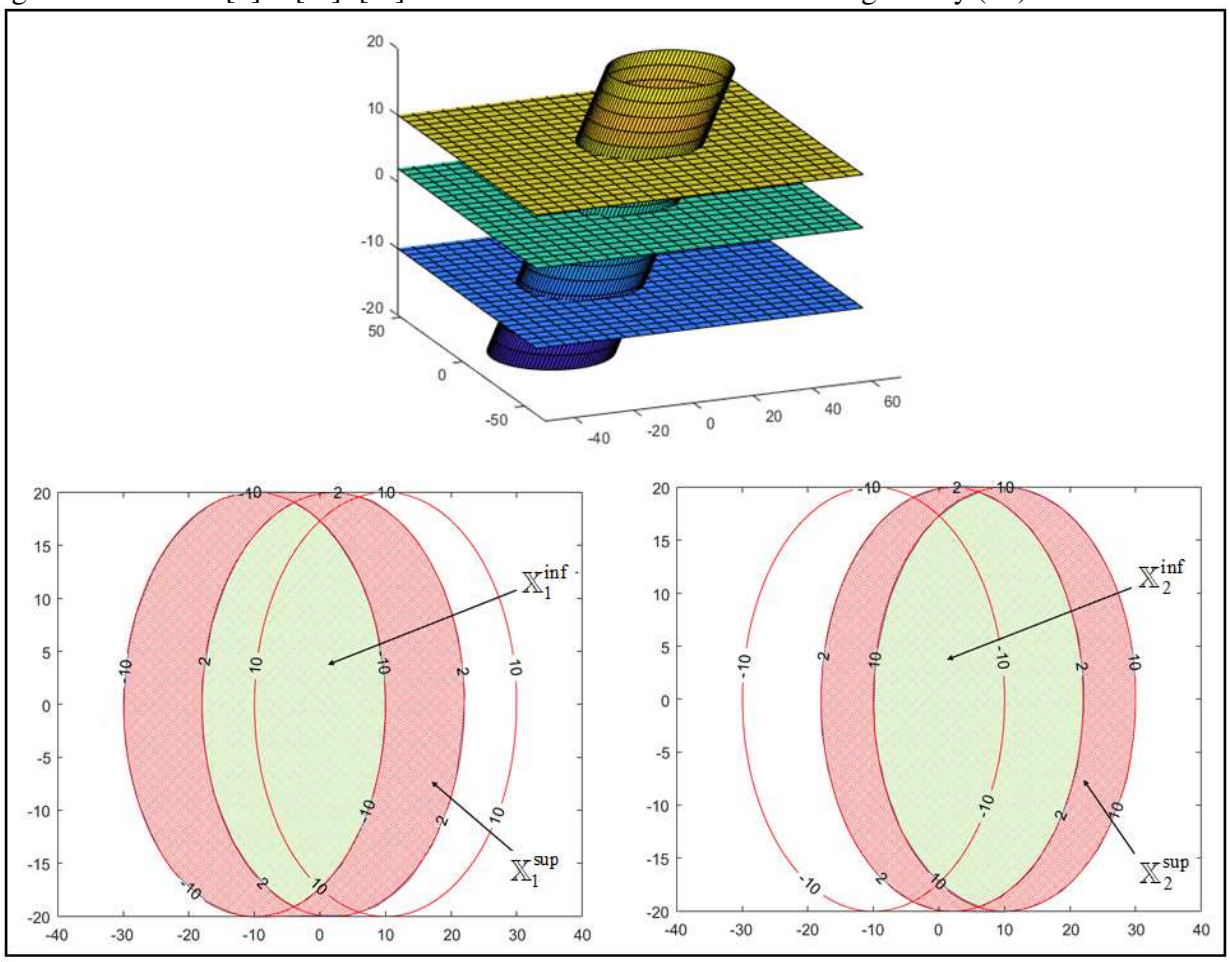
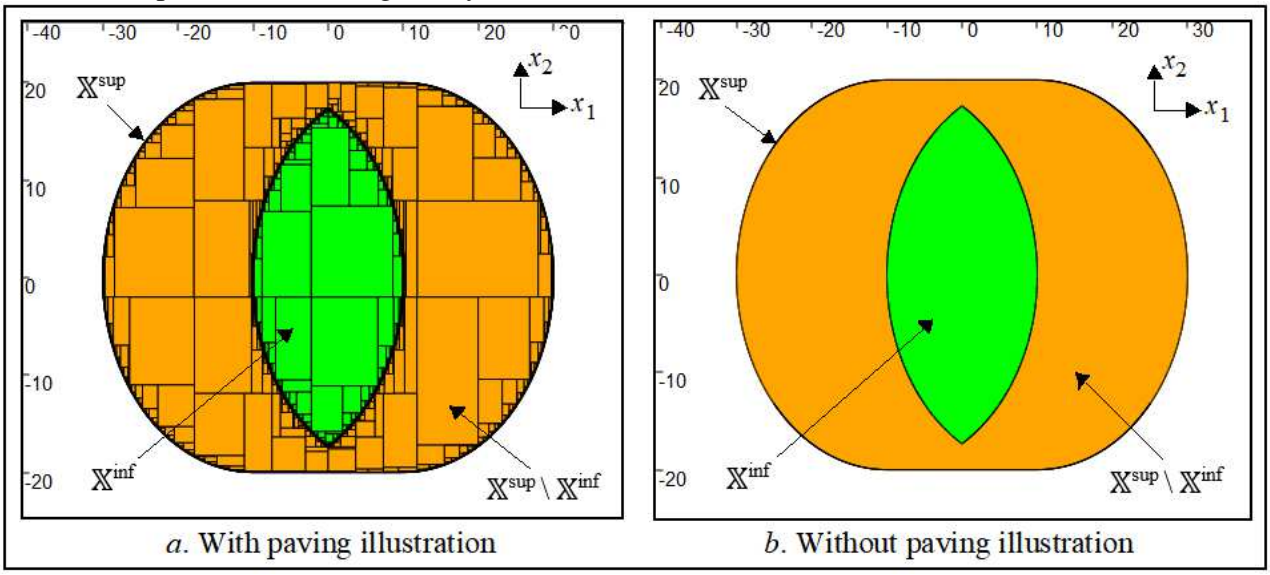


Figure 4: Projection illustration for computing $\mathbb{Z}^{inf}$ and $\mathbb{Z}^{sup}$

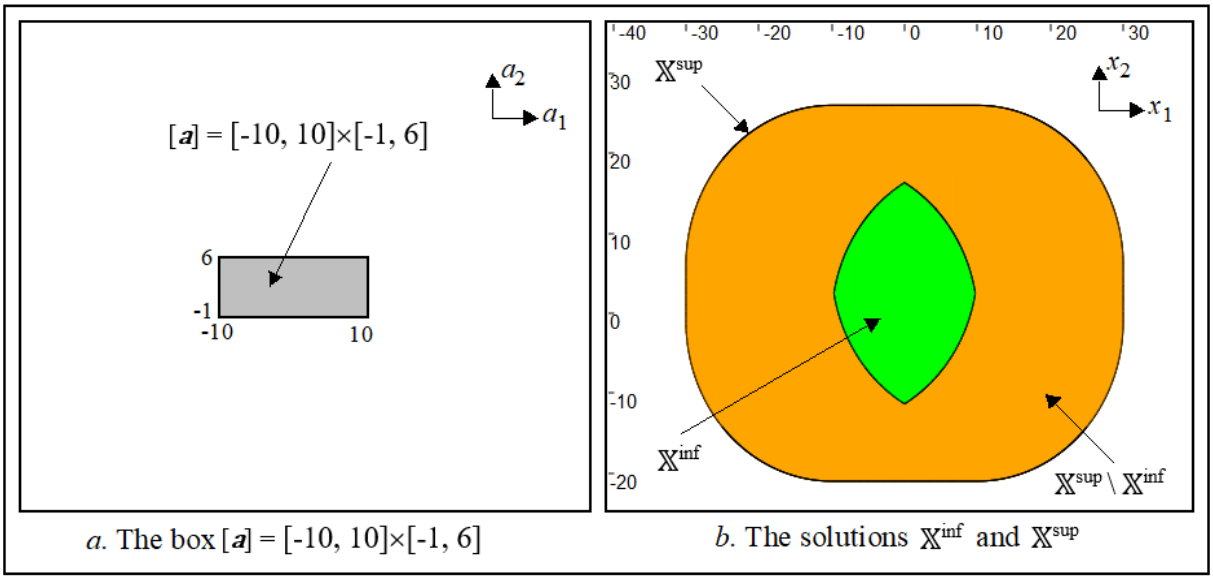
For instance, the application of the proposed method for $[\boldsymbol{a}] = [-10, 10]\times[-1, 6]$ leads to the solutions given in Fig. 6.

The formulations using intersection, projection, Cartesian product, and complement operations aim to facilitate the implementation using the PyIbex interval-based solver.



a. With paving illustration *b*. Without paving illustration

Figure 5: $\mathbb{X}^{inf}$ and $\mathbb{X}^{sup}$ using PyIbex for $[\boldsymbol{a}] = [-10, 10] \times [0, 0]$)



a. The box $[\boldsymbol{a}] = [-10, 10]\times[-1, 6]$ *b.* The solutions $\mathbb{X}^{inf}$ and $\mathbb{X}^{sup}$

Fig. 6: Uncertain solutions $\mathbb{X}^{inf}$ and $\mathbb{X}^{sup}$ for $[\boldsymbol{a}] = [-10, 10] \times [-1, 6]$)

The solutions of Fig. 6.b. are implemented using PyIbex as follows:

$$\begin{aligned}\mathbb{X}^{sup} &= \{x = (x_1, x_2) \in \Re^2,\ \exists a \in [a] \mid F(a, x) \in [b]\} \\ &= \{x = (x_1, x_2) \in \Re^2,\ \exists a \in [a] \mid (x_1 - a_1)^2 + (x_2 - a_2)^2 \in [0, 400]\} \\ &= \text{Proj}_x\{(x_1, x_2, a_1, a_2) \in \Re^4 \mid (x_1 - a_1]^2 + (x_2 - a_2)^2 \in [0, 400]\}\end{aligned}$$

$$\begin{aligned}\mathbb{X}^{inf} &= \{x = (x_1, x_2) \in \Re^2,\ \forall a \in [a] \mid F(a, x) \in [b]\} \\ &= \{x = (x_1, x_2) \in \Re^2,\ \forall a \in [a] \mid (x_1 - a_1)^2 + (x_2 - a_2)^2 \in [0, 400]\} \\ &= \overline{\{x = (x_1, x_2) \in \Re^2,\ \exists a \in [a] \mid (x_1 - a_1)^2 + (x_2 - a_2)^2 \notin [0, 400]\}} \\ &= \overline{\text{Proj}_x\{(x_1, x_2, a_1, a_2) \in \Re^4 \mid (x_1 - a_1)^2 + (x_2 - a_2)^2 \notin [0, 400]\}}\end{aligned}$$

Therefore, the solution of equation (28) is formalized as a TS $[\![\mathbb{X}]\!] = [\![\mathbb{X}^{inf}, \mathbb{X}^{sup}]\!]$, i.e., an interval of CS solutions where its bounds are $\mathbb{X}^{inf}$ and $\mathbb{X}^{sup}$. In the absence of uncertainties in $[a]$, i.e., $[a] = a$, the set of solutions of (28) will be a CS which is defined by the area of a circle of center a and a radius of 20. In this case, $\mathbb{X}^{sup} = \mathbb{X}^{inf} = \mathbb{X}$, i.e., $[\![\mathbb{X}]\!] = \mathbb{X}$.

Let us now consider a practical application that interprets this interval equation (28) and the semantic of its solutions. Therefore, let us assume that $x = (x_1, x_2)$ refers to the position of an autonomous vehicle V that is moving on a two-dimensional path. For its localization, the vehicle V needs to communicate with a transmitter T which is located at a position $a = (a_1, a_2)$. Due to the presence of obstacles and some disturbances, the vehicle cannot locate the position of T with precision. For example, the only information available for V is that T is located in a 2D box $[\boldsymbol{a}] = [a_1]\times[a_2] = [-10, 10]\times[-1, 6]$. This box corresponds to the uncertain location of a transmitter T. Furthermore, the interval equation (28) interprets the fact that if the vehicle V is at a distance less or equal to 20 m from T, it detects the signal from T and can communicate with it. In this framework, the solutions $\mathbb{X}^{inf}$ represents the set of all positions x of the vehicle V where the communication between V and T is certain regardless of the position a of T in the box $[\boldsymbol{a}]$. The solution $\mathbb{X}^{sup}$, which encompasses $\mathbb{X}^{inf}$, corresponds to the positions x where the communication between V and T is plausible—i.e., there is at least one position a in $[\boldsymbol{a}]$ where the communication between V and T is possible. Therefore, according to the meaning associated with the qualifiers $\forall$ and $\exists$, the solution is $\mathbb{X}^{inf}$ is called the certain solution, and $\mathbb{X}^{sup}$ is named the plausible solution. The difference $\mathbb{X}^{sup}\backslash\mathbb{X}^{inf}$ between the two solutions represents the uncertainty (the ignorance), which is interpreted as the set of plausible but not certain

solutions. In our example, $\mathbb{X}^{sup}\backslash\mathbb{X}^{inf}$ refers to all the positions x where the communication between V and T is plausible but not certain.

5. Applications examples

In this section, our approach has been confronted with several examples issued from the literature to demonstrate its interest, its contribution, its feasibility, and its specificity compared to the methods that are already published. For the sake of simplicity of 3D illustration for FSs and uncertain FSs, only SoEs with two variables are considered. However, the method remains applicable regardless the number of variables.

5.1 Example 1

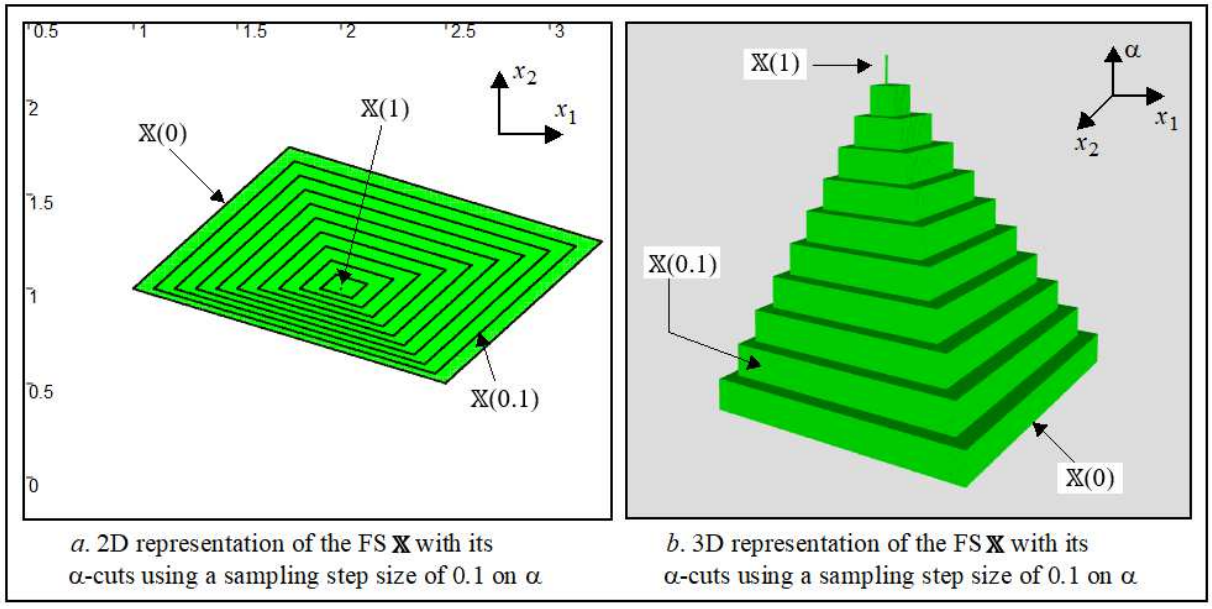
Let us consider the 2×2 fuzzy linear SoEs which has been considered in [9][26][29][42] where the elements of the matrix A are crisp values and the elements of $[b]$ are triangular FIs, which are represented by their α-cuts representation, i.e.,

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}=\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix};\ [b(\alpha)]=\begin{pmatrix} [b_1(\alpha)] \\ [b_2(\alpha)] \end{pmatrix}=\begin{pmatrix} [\alpha, 2-\alpha] \\ [4+\alpha, 7-2\alpha] \end{pmatrix};\ x=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{34}$$

In this case, as the matrix A has crisp elements (real numbers), at each α-cut, the set of solutions is a CS given by the following expression (see Fig. 7):

$$\mathbb{X}^{sup}(\alpha) = \mathbb{X}^{inf}(\alpha) = \mathbb{X}(\alpha) = \{x = (x_1, x_2) \in \Re^2 \mid A \times x \in [b(\alpha)]\} \tag{35}$$

Furthermore, at each α-cut level, the upper and lower solutions $\mathbb{X}^{sup}(\alpha)$ and $\mathbb{X}^{inf}(\alpha)$ are the same and equal to $\mathbb{X}(\alpha)$. By integrating the dimension $\alpha \in [0,1]$, the stacking of solutions $\mathbb{X}(\alpha)$ leads to the FS $\mathbb{X}$ (see equation (4)) which is illustrated in Fig. 7 according to a 2D representation in the plane (x_1, x_2) and a 3D view in space (x_1, x_2, α), when a sampling step size of 0.1 on α is used.



a. 2D representation of the FS $\mathbb{X}$ with its α-cuts using a sampling step size of 0.1 on α

b. 3D representation of the FS $\mathbb{X}$ with its α-cuts using a sampling step size of 0.1 on α

Fig. 7: 2D and 3D representations of the FS $\mathbb{X}$, solution of the system (34)

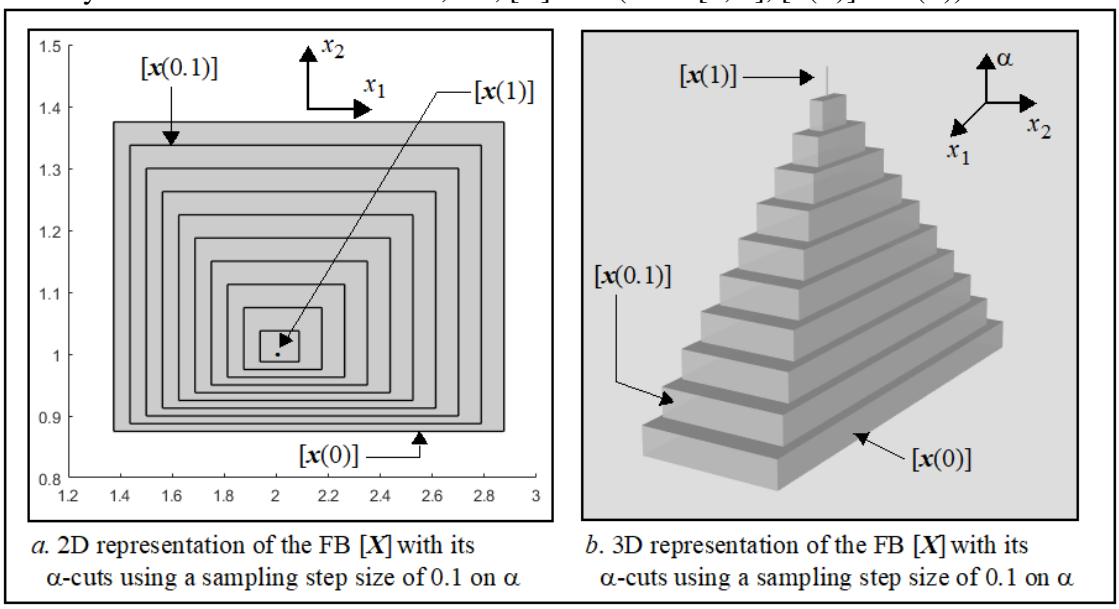
The algebraic solution (the exact solution) of the system (34) when the relation ∈ is replace by the equality stated in [9][26][29] is a FB $[\boldsymbol{X}]$ which is given by the following α-cut expression:

$$[\boldsymbol{X}] = \bigcup_{\alpha \in [0,1]} \alpha[\boldsymbol{x}(\alpha)]; \text{ where:}$$

$$[\boldsymbol{x}(\alpha)] = [x_1(\alpha)] \times [x_2(\alpha)] = [1.375+0.625\alpha,\ 2.875-0.875\alpha] \times [0.875+0.125\alpha,\ 1.375-0.375\alpha]$$

Therefore, the FB $[\boldsymbol{X}]$ which is obtained by stacking of CBs $[\boldsymbol{x}(\alpha)]$ is represented in Fig. 8 using a 2D and a 3D representation with a sampling step size of 0.1 on α.

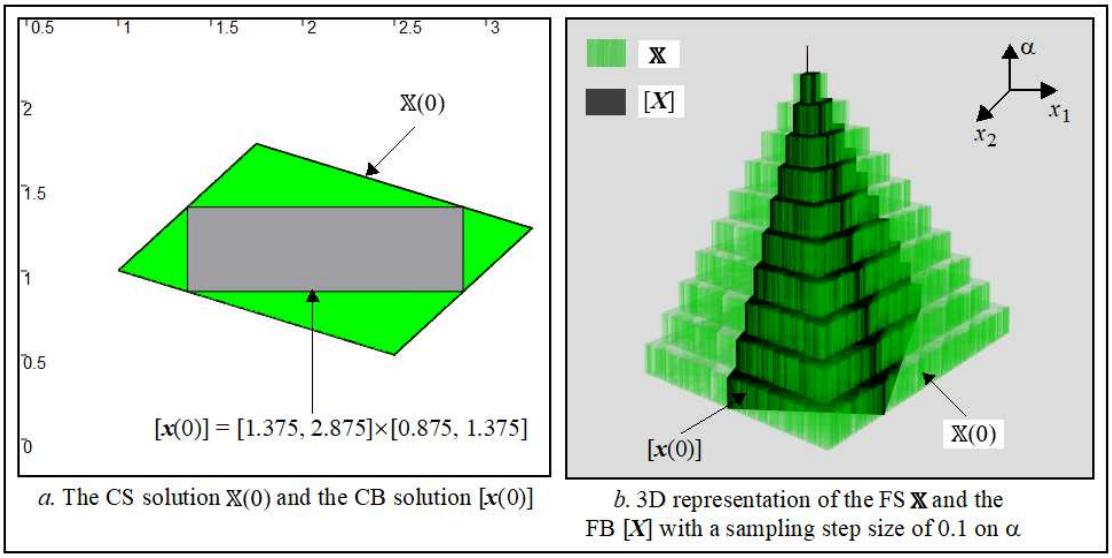
For comparison purposes and as illustrated in Fig. 9, it can be stated that when it exists, the algebraic solution is always included in the solution $\mathbb{X}$, i.e., $[\boldsymbol{X}] \subset \mathbb{X}$ ($\forall\ \alpha \in [0, 1]$, $[\boldsymbol{x}(\alpha)] \subset \mathbb{X}(\alpha)$).



a. 2D representation of the FB $[\boldsymbol{X}]$ with its α-cuts using a sampling step size of 0.1 on α

b. 3D representation of the FB $[\boldsymbol{X}]$ with its α-cuts using a sampling step size of 0.1 on α

Fig. 8: 2D and 3D representations of the FB $[\boldsymbol{X}]$, algebraic solution of the system (34)



a. The CS solution $\mathbb{X}(0)$ and the CB solution $[\boldsymbol{x}(0)]$

b. 3D representation of the FS $\mathbb{X}$ and the FB $[\boldsymbol{X}]$ with a sampling step size of 0.1 on α

Fig. 9: Compraison between the solutions $\mathbb{X}$ and $[\boldsymbol{X}]$ for the system (34)

Let us now consider the 2×2 fuzzy linear SoEs, which is taken in [2] as a counter-example of the approach proposed in [29]. In this case, the elements of A and $[b]$ are given by the following expression:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix};\ [b(\alpha)] = \begin{pmatrix} [b_1(\alpha)] \\ [b_2(\alpha)] \end{pmatrix};\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{36}$$

$$[b_1(\alpha)] = \begin{cases} [8\alpha - 14, -1 - 13\alpha];\ 0 \le \alpha \le 0.5 \\ [2\alpha - 11, -6 - 3\alpha];\ 0.5 \le \alpha \le 1 \end{cases};\quad [b_2(\alpha)] = \begin{cases} [12\alpha - 24, -18\alpha - 2];\ 0 \le \alpha \le 0.5 \\ [6\alpha - 21, -7 - 8\alpha];\ 0.5 \le \alpha \le 1 \end{cases}$$

The algebraic solution proposed in [2][29] is given by the following α-cuts expression:

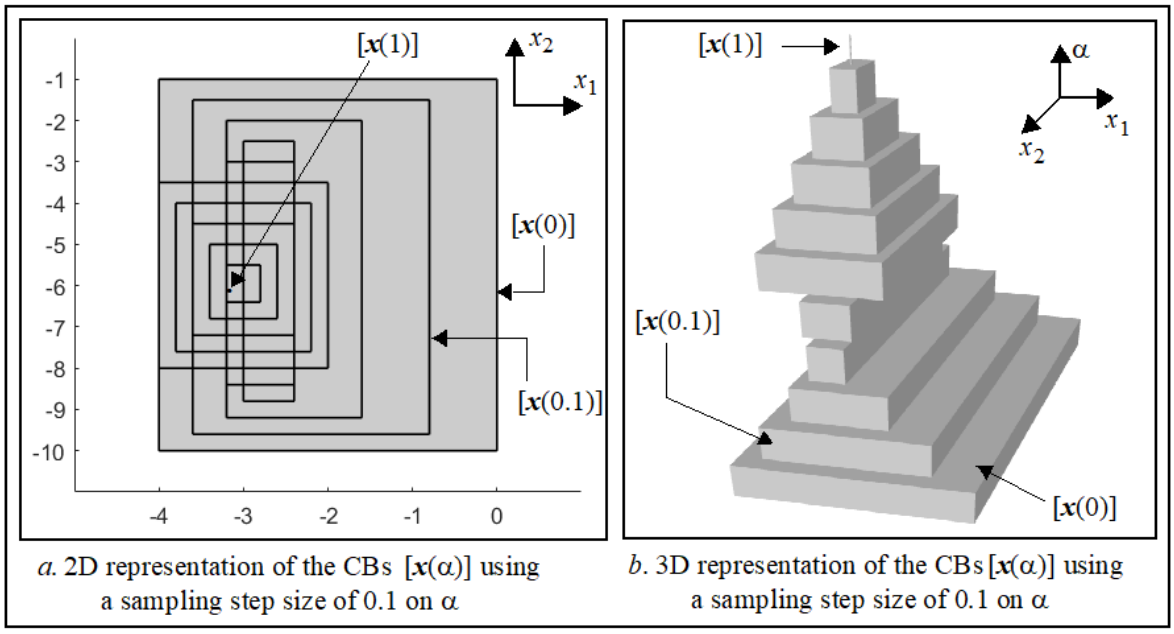
$$[\boldsymbol{x}(\alpha)] = [x_1(\alpha)] \times [x_2(\alpha)];\ \text{with: } x_2(\alpha) = [4\alpha - 10, -1 - 5\alpha] \ \text{ and:} \tag{37}$$

$$[x_1(\alpha)] = [x_1^{\inf}(\lambda), x_1^{\sup}(\lambda)], \text{ with } x_1^{\inf}(\alpha) = \begin{cases} 4\alpha - 4, \text{ if } 0 \le \alpha \le 1/4, \\ -3, \text{ if } 1/4 \le \alpha \le 3/8, \\ -8\alpha, \text{ if } 3/8 \le \alpha \le 1/2, \\ 2\alpha - 5, \text{ if } 1/2 \le \alpha \le 1, \end{cases} ; \; x_1^{\sup}(\alpha) = \begin{cases} -8\alpha, \text{ if } 0 \le \alpha \le 1/3, \\ 4\alpha - 4, \text{ if } 1/3 \le \alpha \le 1/2, \\ -8\alpha - 1, \text{ if } 1/2 \le \alpha \le 1, \end{cases}$$

The algebraic solution (37) is represented in Fig. 10 using 2D and 3D views according to a sampling step size of 0.1 on α. According to Fig. 10, it can be observed that the stacking of CBs $[\boldsymbol{x}(\alpha)]$ cannot be interpreted as a FB. Therefore, the CBs $[\boldsymbol{x}(\alpha)]$, $\alpha \in [0, 1]$ are not always nested according to the vertical dimension α, i.e., the monotonicity (consistency) condition:

$$\forall\, \alpha_1, \alpha_2 \in [0, 1]: \alpha_1 \le \alpha_2 \Leftrightarrow [\boldsymbol{x}(\alpha_2)] \subset [\boldsymbol{x}(\alpha_1)]$$

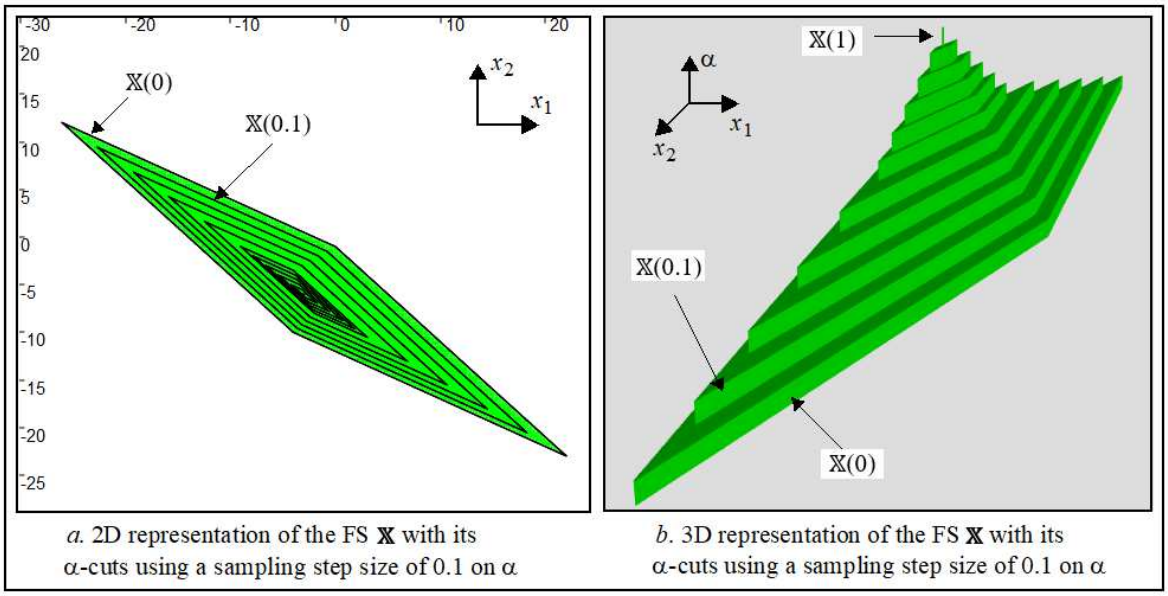
is not respected. In this case, a fuzzy solution of the fuzzy SoEs (36) does not exist. However, the solution given by (37) can be interpreted as a gradual box, i.e., the Cartesian product of two gradual intervals [11][12][13][30]. The application of the proposed approach leads the FS solution, which is illustrated in Fig. 11 using a 2D and a 3D representations using a sampling step size of 0.1 on α. Unlike the solution presented in Fig. 10, which is not a fuzzy set, our solution is a FS. Furthermore, at each α-cut, this solution corresponds to the conventional USS solution, often computed for linear interval SoEs. In a fuzzy framework, the proposed solution $\mathbb{X}$ can be regarded as a fuzzy USS.



a. 2D representation of the CBs $[\boldsymbol{x}(\alpha)]$ using a sampling step size of 0.1 on α

b. 3D representation of the CBs $[\boldsymbol{x}(\alpha)]$ using a sampling step size of 0.1 on α

Fig. 10 : 2D and 3D representations of the algebraic solution of the system (36)



a. 2D representation of the FS 𝕏 with its α-cuts using a sampling step size of 0.1 on α

b. 3D representation of the FS 𝕏 with its α-cuts using a sampling step size of 0.1 on α

Fig. 11: 2D and 3D representations of the FS 𝕏: solution of the fuzzy SoEs (36)

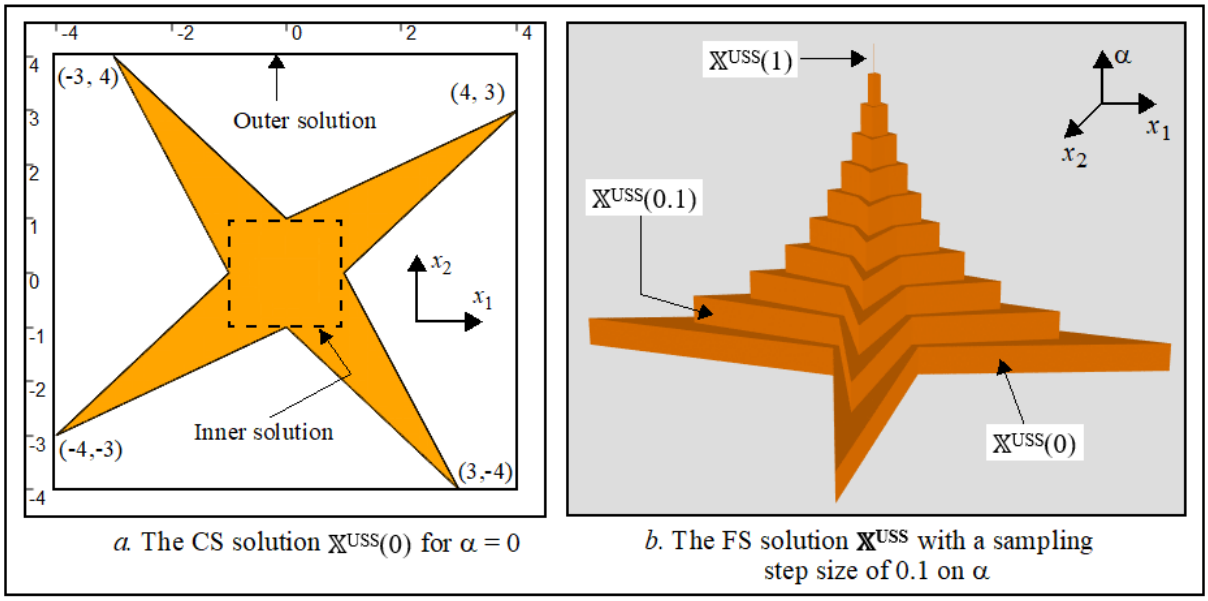
5.2 Example 2

Let us consider the 2×2 interval linear SoEs where the elements of $[A]$ and $[b]$ are FIs represented by their α-cuts representations:

$$[A(\alpha)] = \begin{pmatrix} [a_{11}(\alpha)] & [a_{12}(\alpha)] \\ [a_{21}(\alpha)] & [a_{22}(\alpha)] \end{pmatrix} = \begin{pmatrix} [2+\alpha, 4-\alpha] & [-2+1.5\alpha, 1-1.5\alpha] \\ [-1+1.5\alpha, 2-1.5\alpha] & [2+\alpha, 4-\alpha] \end{pmatrix} \quad (38)$$

$$[b(\alpha)] = \begin{pmatrix} [b_1(\alpha)] \\ [b_2(\alpha)] \end{pmatrix} = \begin{pmatrix} [-2+2\alpha, 2-2\alpha] \\ [-2+2\alpha, 2-2\alpha] \end{pmatrix};\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For α = 0, the system (38) corresponds to the popular CI linear SoEs which is repeatedly used by many authors (e.g. [23][33][39][43][48][68]). The set of solutions of (38) given in [33][39][43][48] (for α = 0) is a CS which is illustrated in Fig. 12.a. This solution of Fig. 12.a corresponds to the USS that produced by interval solvers like the Intlab solver [39][48] (see http://www.ti3.tu-harburg.de/intlab/). Furthermore, it can be stated that this solution is not a box even though the elements $[A]$ and $[b]$ are CIs. In this context to obtain box solutions, outer and inner solutions are often used. For instance, the box [−4, 4]×[−4, 4] can be regarded as an outer solution. In the same way, the box [−1, 1]×[−1, 1] can be estimated as a possible inner solution [43]. In our approach, no approximation of the set of solutions by CBs or FBs is considered. The stacking of the USSs 𝕏(α) leads the FS represented in Fig. 12.b with a sampling step size of 0.1 on α.



a. The CS solution $\mathbb{X}^{USS}(0)$ for $\alpha = 0$

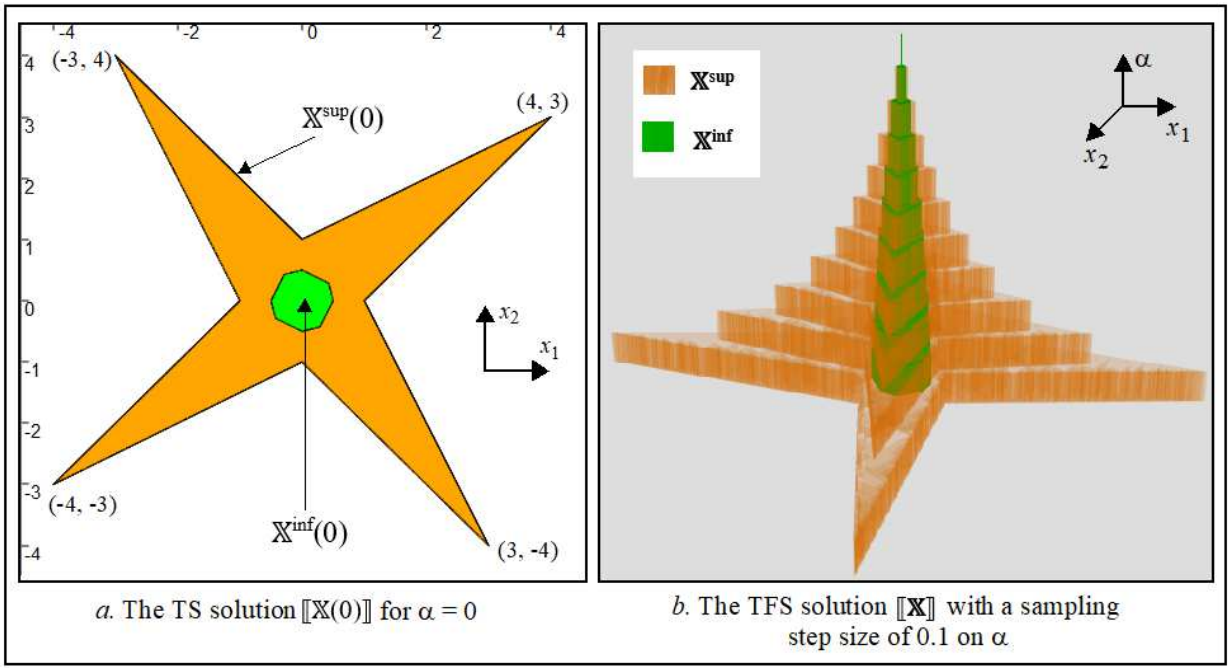
b. The FS solution $\mathbf{X}^{USS}$ with a sampling step size of 0.1 on α

Fig. 12: The solution of system of system of fuzzy equations (38) using Intlab solver

Using the proposed approach, for each level α, the solution of (38) is not regarded as a CS by as an uncertain set which is represented by a TS $[\![\mathbb{X}(\alpha)]\!] = [\![\mathbb{X}^{inf}(\alpha), \mathbb{X}^{sup}(\alpha)]\!]$, i.e., an interval of CSs. The solution $\mathbb{X}^{inf}(\alpha) = \mathbb{X}^{TSS}(\alpha)$, which is associated with the quantifier $\forall$, represents the set of certain solutions, and the solution $\mathbb{X}^{sup}(\alpha) = \mathbb{X}^{USS}(\alpha)$, which is related to the quantifier $\exists$, refers to the plausible solutions. These solutions are depicted in Fig. 13.a for $\alpha = 0$ and computed by the following expressions using the projection principle (see section 4.2):

$$\begin{aligned}\mathbb{X}^{sup}(\alpha) &= \{x = (x_1, x_2) \in \Re^2,\ \ \exists\, A \in [A(\alpha)] \mid A \times x \in [b(\alpha)]\} \\ &= \bigcap\nolimits_{i,j\in\{1,2\}} \{x = (x_1, x_2) \in \Re^2,\ \ \exists\, a_{ij} \in [a_{ij}(\alpha)] \mid A \times x \in [b(\alpha)]\} \\ \mathbb{X}^{inf}(\alpha) &= \{x = (x_1, x_2) \in \Re^2,\ \ \forall\, A \in [A(\alpha)] \mid A \times x \in [b(\alpha)]\} \\ &= \bigcap\nolimits_{i,j\in\{1,2\}} \{x = (x_1, x_2) \in \Re^2,\ \ \forall a_{ij} \in [a_{ij}(\alpha)] \mid A \times x \in [b(\alpha)]\} \\ &= \overline{\bigcap\nolimits_{i,j\in\{1,2\}} \{x = (x_1, x_2) \in \Re^2,\ \ \exists\, a_{ij} \in [a_{ij}(\alpha)] \mid A \times x \notin [b(\alpha)]\}}\end{aligned}$$

The solutions $\mathbb{X}^{sup}(\alpha)$ and $\mathbb{X}^{inf}(\alpha)$ are strictly equivalent to the USS and TSS solutions proposed in [23][68], respectively. The uncertainty in the solution is represented by the penumbra $\mathbb{X}^{sup}(\alpha) \backslash \mathbb{X}^{inf}(\alpha)$. By stacking the TSs $[\![\mathbb{X}(\alpha)]\!]$ according to vertical dimension α, the TFS $[\![\mathbb{X}]\!] = [\![\mathbf{X}^{inf}, \mathbf{X}^{sup}]\!]$ illustrated in Fig. 13.b is obtained. This TFS is composed of two upper and lower FSs $\mathbf{X}^{sup}$ and $\mathbf{X}^{inf}$, respectively (see Fig. 14). The TFS of Fig. 13.b can be regarded as a special case of a type-2 fuzzy set where the FOU is equivalent to the penumbra.



a. The TS solution $[\![\mathbb{X}(0)]\!]$ for $\alpha = 0$

b. The TFS solution $[\![\mathbb{X}]\!]$ with a sampling step size of 0.1 on α

Fig 13: The uncertain solutions of the fuzzy SoEs (38) using our approach

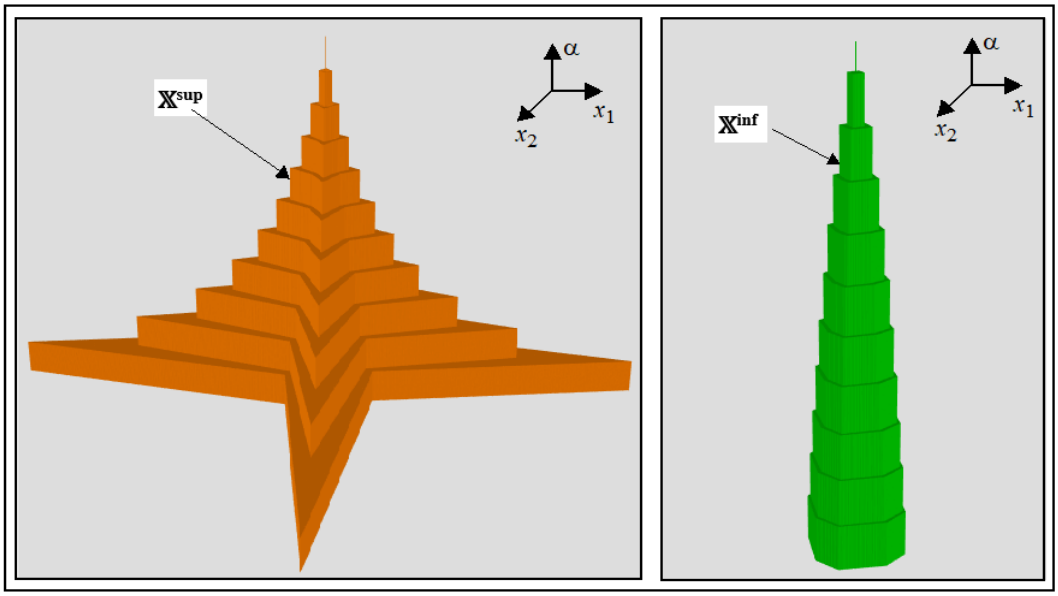


Fig 14: The upper and lower FSs $\mathbb{X}^{sup}$ and $\mathbb{X}^{inf}$

It is worth noting that in a fuzzy framework, if inner and outer solutions are desired, they can be regarded as FBs, i.e., CBs at each α-cut level. In this case, the joint consideration of inner and outer solutions leads to a TFB, which is composed of lower and upper FBs.

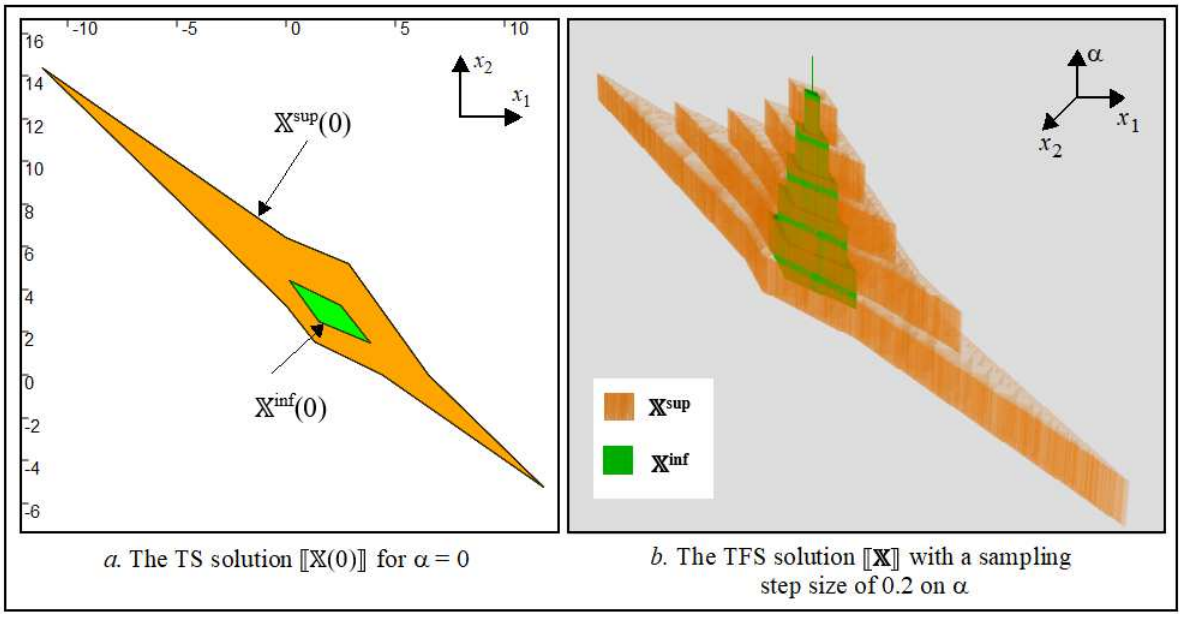
5.3 Example 3

Let us consider the 2×2 fuzzy linear SoEs, issued form [56] and given by:

$$[A(\alpha)] = \begin{pmatrix} [a_{11}(\alpha)] & [a_{12}(\alpha)] \\ [a_{21}(\alpha)] & [a_{22}(\alpha)] \end{pmatrix} = \begin{pmatrix} [2.5+0.5\alpha, 3.25-0.25\alpha] & [1.75+0.25\alpha, 2.5-0.5\alpha] \\ [0.75+0.25\alpha, 1.25-0.25\alpha] & [1.75+0.25\alpha, 2.5-0.5\alpha] \end{pmatrix} \quad (39)$$

$$[b(\alpha)] = \begin{pmatrix} [b_1(\alpha)] \\ [b_2(\alpha)] \end{pmatrix} = \begin{pmatrix} [8.125+3.875\alpha, 16.25-4.25\alpha] \\ [5.5+2.5\alpha, 11.25-3.25\alpha] \end{pmatrix}; \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Using the proposed approach, the solution of the fuzzy SoEs (39) is a TFS $[\![\mathbb{X}]\!] = [\![\mathbb{X}^{inf}, \mathbb{X}^{sup}]\!]$ which is illustrated in Fig. 15.b using a sampling step size of 0.2 on α. Therefore, at each α-cut, the solution is a TS $[\![\mathbb{X}(\alpha)]\!] = [\![\mathbb{X}^{inf}(\alpha), \mathbb{X}^{sup}(\alpha)]\!]$. For example, the TS solution $[\![\mathbb{X}(0)]\!]$ is depicted in Fig. 15.a.



a. The TS solution $[\![\mathbb{X}(0)]\!]$ for $\alpha = 0$

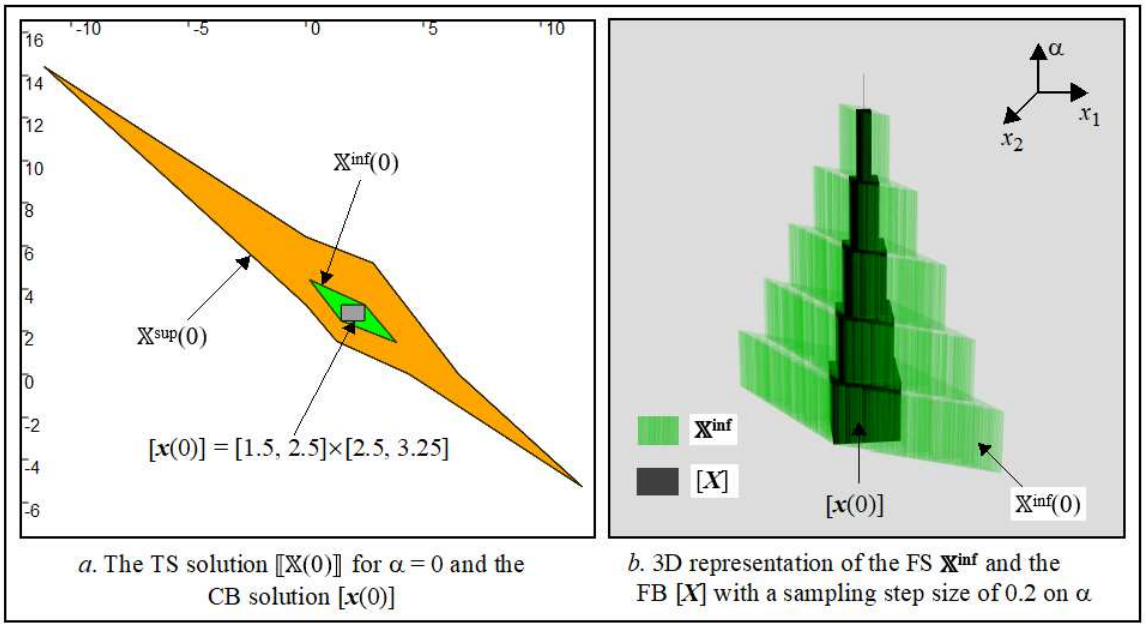
b. The TFS solution $[\![\mathbb{X}]\!]$ with a sampling step size of 0.2 on α

Fig. 15: The uncertain solutions of the fuzzy SoEs (39) using our approach

For comparaison purposes, let us consider the algebraic solution given in [56]. This solution is regarded as a FB $[\boldsymbol{X}]$ that is given by the following α-cuts representation:

$$[\boldsymbol{x}(\alpha)] = [x_1(\alpha)] \times [x_2(\alpha)] = [1.5+0.5\alpha,\ 2.5-0.5\alpha] \times [2.5+0.5\alpha,\ 3.25-0.25\alpha]$$

The solutions $[\![\mathbb{X}(0)]\!]$ and $[\boldsymbol{x}(0)]$ are illustrated in Fig. 16.a. Furthermore, the FB $[\boldsymbol{X}]$ is depicted in Fig. 16.b together with $\mathbb{X}^{inf}$.



a. The TS solution $[\![\mathbb{X}(0)]\!]$ for $\alpha = 0$ and the CB solution $[\boldsymbol{x}(0)]$

b. 3D representation of the FS $\mathbb{X}^{inf}$ and the FB $[\boldsymbol{X}]$ with a sampling step size of 0.2 on α

Fig. 16: Representations of the uncertain solution and the algebraic solution of (39)

In this case, it can be stated that $[\boldsymbol{X}] \subset \mathbb{X}^{inf}$. More generally, when it exists, the algebraic solution is always included the certain solution, i.e., the lower bound of the TFS.

5.4 Example 4

Let us consider the nonlinear SoEs proposed in [27] and given by the following representation:

$$\begin{cases} f_1(a_1,a_2,x_1,x_2) = a_1 \times x_1^2 + a_2 \times x_2^2 = b_1 \\ f_2(a_3,a_4,x_1,x_2) = a_3 \times x_1 + a_4 \times x_2 = b_2 \end{cases}; a = \begin{pmatrix} a_1 \\ \dots \\ a_4 \end{pmatrix}; b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}; x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (40)$$

The system (40) can be written in the following concise form:

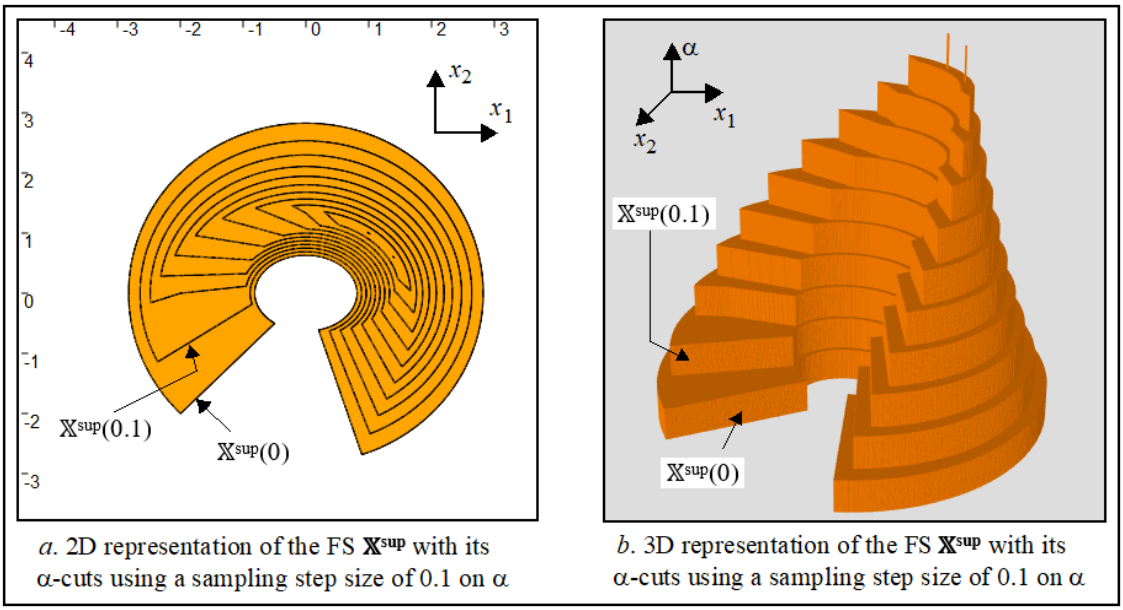
$$F(a, x) = b \text{ ; with: } F(a, x) = \begin{pmatrix} f_1(a,x) \\ f_2(a,x) \end{pmatrix} \quad (41)$$

In (41), the parameters are uncertain and represented by triangular FIs, i.e., $a_1 \in [a_1(\alpha)] = [1+\alpha, 3-\alpha]$; $a_2 \in [a_2(\alpha)] = [1+2\alpha, 5-2\alpha]$; $a_3 \in [a_3(\alpha)] = [-1+3\alpha, 3-\alpha]$; $a_4 \in [a_4(\alpha)] = [1+\alpha, 4-2\alpha]$; $b_1 \in [b_1(\alpha)] = [2+3\alpha, 8-3\alpha]$; $b_2 \in [b_2(\alpha)] = [4\alpha, 7-3\alpha]$. In this case, the fuzzy nonlinear SoEs is regarded as a family of crisp nonlinear SoEs $F(a, x) = b$ where $a = (a_1, \dots, a_2)^T \in [a(\alpha)]$ and $b = (b_1, b_2)^T \in [b(\alpha)]$.

Using our approach, at each α-cut, the solutions $\mathbb{X}^{sup}(\alpha)$ and $\mathbb{X}^{inf}(\alpha)$ are computed using the same methodology proposed for solving linear SoEs, i.e.,

$$\begin{aligned} \mathbb{X}^{sup}(\alpha) &= \{x = (x_1, x_2) \in \Re^2, \ \exists a \in [a(\alpha)] \mid F(a, x) \in [b(\alpha)]\} \\ &= \bigcap\nolimits_{i \in \{1,\dots,4\}} \{x = (x_1, x_2) \in \Re^2, \exists\, a_i \in [a_i(\alpha)] \mid F(a, x) \in [b(\alpha)]\} \\ \mathbb{X}^{inf}(\alpha) &= \{x = (x_1, x_2) \in \Re^2, \ \forall a \in [a(\alpha)] \mid F(a, x) \in [b]\} \\ &= \bigcap\nolimits_{i \in \{1,\dots,4\}} \{x = (x_1, x_2) \in \Re^2, \ \forall\, a_i \in [a_i(\alpha)] \mid F(a, x) \in [b(\alpha)]\} \\ &= \overline{\bigcap\nolimits_{i \in \{1,\dots,4\}} \{x = (x_1, x_2) \in \Re^2, \ \exists\, a_i \in [a_i(\alpha)] \mid F(a, x) \notin [b(\alpha)]\}} \end{aligned}$$

The system (41) with FI parameters only admits as certain solutions $\mathbb{X}^{inf}(\alpha)$ the two pairs $(x_1, x_2) = (1, 1)$ and $(x_1, x_2) = (7/5, 3/5)$ for $\alpha = 1$. Therefore, for the other values of α, $\mathbb{X}^{inf}(\alpha)$ is empty. These two solutions for $\alpha = 1$ correspond to the unique solutions given in [27] when an eigenvalue method is used. It can be stated that these two solutions refer to the algebraic solutions of a crisp nonlinear SoEs at $\alpha = 1$. Unlike the method proposed in [27], our method can provide a plausible solution which corresponds to $\mathbb{X}^{sup}(\alpha)$. For instance, the FS $\mathbb{X}^{sup}$ solution of the fuzzy SoEs is illustrated in Fig. 17 according to a 2D representation and a 3D view when a sampling step size of 0.1 on α is used.



a. 2D representation of the FS $\mathbb{X}^{sup}$ with its α-cuts using a sampling step size of 0.1 on α

b. 3D representation of the FS $\mathbb{X}^{sup}$ with its α-cuts using a sampling step size of 0.1 on α

Fig. 17: The solution $\mathbb{X}^{sup}$ of the SoEs (40) with FIs

6. Conclusions

In this paper, a new methodology for solving fuzzy SoEs with FI parameters is proposed. Due to the presence of uncertainty in the left and right-hand sides of the considered fuzzy SoEs, the fuzzy solutions are envisioned as uncertain FSs, which are represented by TFSs. Based on the TS and the α-cuts principles, a TFS is considered as a family of stacked TSs. Therefore, a TFS can be regarded as an interval of FSs where its lower and upper bounds are certain and plausible FSs, respectively. In absence of uncertainty in the left-hand side of the fuzzy SoEs, the TFS solution becomes a FS. The proposed methodology and its performances are illustrated using application examples that are issued from the literature, related to solving fuzzy SoEs. It is worth noting that the construction of a TFS requires that the TSs obtained by α-cuts are nested, i.e., the monotonicity (consistency) constraint must be respected for TSs. However, in some practical situations where the constraint of consistency can be relaxed, it is possible to obtain TSs, which are not necessarily nested. In this case, the lower and upper bounds are no longer FSs but gradual sets [24]. In addition to solving fuzzy SoEs, several other potential applications of the proposed approach can be envisioned in fuzzy control, in linear and nonlinear optimization problems, in decision-making problems, in uncertain fuzzy regressions, etc. Future work will be devoted to these interesting research perspectives.

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