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Abstract

This paper deals with the design of output feedback event-triggered controllers equipped with generalized holding devices. Both emulation and co-design settings are explored. Specifically, a robust observer-based event-triggered controller with a dwell time logic is proposed to achieve closed-loop stability. The closed-loop system is modeled as a hybrid system and analyzed via Lyapunov theory for hybrid systems. Sufficient conditions in the form of matrix inequalities are given to ensure global exponential stability and input-to-state stability with respect to measurement disturbances for the closed-loop system. The proposed conditions enable the design of the controller gains, event-triggering mechanism, and of general holding devices, thereby including classical zero-order-holder devices. Convex optimization schemes address the implicit objective consisting of reducing the number of updates of the control input. The effectiveness of the conditions are illustrated through an illustrative example borrowed from the literature.

Keywords: Observer-based control, generalized holding devices, event-triggered control, hybrid dynamical systems, global exponential stability, linear matrix inequalities.

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1. Introduction

1.1. Background

An effective way to deal with computational and communication resources in control and network control systems consists of exploiting event-triggering mechanisms to reduce control updates and measurements transmissions. In particular, differently from traditional periodic implementations, the use of event-triggered schemes enables to update control inputs only when specific state-dependent events occur; see, e.g., [1], [2], [3], [4], [5] and references therein. One of the main challenges in the design of event-triggered control architectures consists of jointly design a continuous-time control law and a triggering rule with the objective of reducing the number of control input updates, while preserving stability and/or performance. Another fundamental challenge in the design of event-triggered architectures consists of ensuring a strictly positive lower bound in between transmission events, i.e., the so-called event separation property [6], which is paramount for implementation.

Due to their inherent nature, event-triggered control architectures heavily rely on the use of measurements. Therefore, when only limited information is available, as in the case of output feedback control, the problem of event-triggered control becomes visibly harder. In particular, when the plant state is not fully available, the avoidance of Zeno behaviors in event-triggered control architectures turns out to be a major problem. For this reason, the design of output feedback event-triggered controllers has attracted a lot of attention in the community; see, e.g., [7], [8], [9], [10], [11], [12], just to mention a few. In these papers, stability analysis is addressed via the use of Lyapunov theory for hybrid dynamical systems or impulsive systems. A common feature found in all the works mentioned so far is the use of zero-order-holder devices to generate inter-event control signals. In this paper, we propose an alternative route and consider the use of generalized holding devices that can be tuned to improve the behavior of the closed-loop system. The potential benefit of using generalized holding devices to enlarge inter-event times has been already highlighted in [13],

[14], [15]. Similar considerations appeared in our prior work on observer design in the presence of sporadic measurements [16].

1.2. Contributions

In this work, we present a systematic approach for the design of event-triggered control architectures with generalized holding devices and show how this may lead to a dramatic improvement in reducing the number of update events. More precisely, we consider a setup in which controller and sensors are co-located, while the communication between plant and controller occurs only at some time instances that need to be scheduled. Similar setups have been already considered in [17, 18, 19, 20]. We assume that the plant is equipped with a general holding device that is able to generate a control signal in between communications events. In this setting, we consider an event-triggered controller hinging upon a classical observer-based controller. The proposed triggering mechanism relies on the plant state estimate provided by the observer and on the state of a copy of the holding device at the plant end to generate control input transmission events. For the proposed architecture, our main contributions are as follows:

- We provide general sufficient conditions for closed-loop exponential stability. To this end, we make use of Lyapunov theory for hybrid systems in the framework in [21].
- By relying on the proposed general sufficient conditions, we provide quadratic conditions in the form of matrix inequalities to ensure closed-loop global exponential stability;
- We show that the satisfaction of the above quadratic conditions ensures that the closed-loop system is input-to-state stable with respect to measurement noise;
- Building upon the use of convex optimization techniques, some optimal designs are proposed for the considered controller with the objective of

reducing the number of transmission events. Both emulation and co-design scenarios are addressed.

The effectiveness of the methodology, as well as some interesting comparisons are illustrated in a numerical example borrowed from the literature.

1.3. Paper outline

The paper is organized as follows. Preliminaries on hybrid systems and the notation considered in the paper are included in Section 2. In Section 3, the class of systems and controllers under consideration together with the event-triggering mechanism are described. Both emulation and co-design problems are stated and then some structural properties of the closed-loop system are summarized. In Section 4, sufficient conditions for global exponential stability of the closed-loop system are presented and particularized to special case of quadratic Lyapunov functions. Section 5 proposes the main results to deal first with the emulation problem and next with the co-design problem. Section 6 shows that the closed-loop system enjoys some robustness properties in presence of measurement noise. In Section 7, the approach proposed is illustrated through an example of the literature, allowing to provide some interesting comparisons.

2. Notation and Preliminaries

2.1. Notation

The symbol $\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{R}_{\geq 0}$ represents the set of nonnegative real numbers, $\mathbb{R}^n$ is the n-dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ represents the set of the $n \times m$ real matrices. The symbol $\mathbb{S}^n_+$ stands for the set of $n \times n$ symmetric positive definite matrices, while $\mathbb{D}^n_+$ denotes the set of $n \times n$ diagonal positive definite matrices. The identity matrix is denoted by $\mathbf{I}$. For a matrix $A \in \mathbb{R}^{n\times m}$, A^T denotes the transpose of A, $\mathrm{spec}(A)$ is the spectrum of A, $\|A\|$ is the spectral norm of A, and, when $n = m$, $\mathrm{He}\, A = A + A^\mathsf{T}$. For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm. We use the equivalent notation for vectors $(x, y) = [x^\mathsf{T}\ y^\mathsf{T}]^\mathsf{T}$. For a symmetric matrix, A, positive definiteness

(negative definiteness) and positive semidefiniteness (negative semidefiniteness) are denoted, respectively, by $A \succ 0$ ($A \prec 0$) and $A \succeq 0$ ($A \preceq 0$). In partitioned symmetric matrices, the symbol $\bullet$ stands for symmetric blocks. The matrix $\bigoplus_{i=1}^{n} A_i$ is the block-diagonal matrix having $A_1, A_2, \ldots, A_n$ as diagonal blocks. The symbol $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$. Given a symmetric matrix $Q \in \mathbb{R}^{n \times n}$, we use the notation $\mathrm{cone}^{+}(Q) := \{x \in \mathbb{R}^n \colon x^{\mathsf{T}} Q x \geq 0\}$ and $\mathrm{cone}^{-}(Q) := \{x \in \mathbb{R}^n \colon x^{\mathsf{T}} Q x \leq 0\}$, while $\lambda_{\max}(Q)$ and $\lambda_{\min}(Q)$ stand, respectively, for the largest and smallest eigenvalue of Q. Given $x \in \mathbb{R}^n$ and a nonempty set $\mathcal{A} \subset \mathbb{R}^n$, the distance of x to $\mathcal{A}$ is defined as $|x|_{\mathcal{A}} = \inf_{y \in \mathcal{A}} |x - y|$. Given a set S, we denote by $\operatorname{co} S$ the convex-hull of S and by $\overline{S}$ its closure. The symbol $S_1 \times S_2$ denotes the Cartesian product of the sets S_1 and S_2. Given a set $S \subset X \times Y$, we define $\Pi_X(S) := \{x \in X \colon (x, y) \in S\}$. The symbol $T_S(x)$ stands for the tangent cone of the set $S \subset \mathbb{R}^n$ at $x \in \overline{S}$; see [21, Definition 5.12, page 103]. Given a locally Lipschitz continuous function $\mathscr{V} \colon \mathbb{R}^n \to \mathbb{R}$, we denote by $\partial \mathscr{V}(x) \subset \mathbb{R}^n$ the Clarke generalized gradient of $\mathscr{V}$ at x. The symbol $\mathcal{K}_\infty$ stands for set of functions from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$ that are continuous, zero at zero, strictly increasing, and unbounded. Let $\beta \colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we say that $\beta \in \mathcal{KL}$ if for all $r^\star, s^\star \in \mathbb{R}_{\geq 0}$ one has that: $r \mapsto \beta(r, s^\star)$ is nondecreasing, $s \mapsto \beta(r^\star, s)$ is nonincreasing, $\lim_{r \to 0^+} \beta(r, s^\star) = 0$, and $\lim_{s \to \infty} \beta(r^\star, s) = 0$.

2.2. Preliminaries on hybrid systems

We consider hybrid systems with state $x \in \mathbb{R}^n$ of the form

$$\mathcal{H} \begin{cases} \dot{x} &= f(x) & x \in \mathcal{C} \\ x^+ &= g(x) & x \in \mathcal{D} \end{cases} \tag{1}$$

In particular, we denote $f \colon \mathbb{R}^n \to \mathbb{R}^n$ as the *flow map*, $\mathcal{C} \subset \mathbb{R}^n$ as the *flow set*, $g \colon \mathbb{R}^n \to \mathbb{R}^n$ as the *jump map*, and $\mathcal{D} \subset \mathbb{R}^n$ as the *jump set*. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *hybrid time domain* if it is the union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the last interval (if existent) of the form $[t_j, T)$ with T finite or $T = \infty$. Given a hybrid time domain E, we denote $\sup_j E = \sup\{j \in \mathbb{N} \colon \exists t \in \mathbb{R}_{\geq 0} \text{ s.t. } (t, j) \in E\}$. A function $\phi \colon \operatorname{dom} \phi \to \mathbb{R}^n$ is

a hybrid arc if $\operatorname{dom}\phi$ is a hybrid-time domain and $\phi(\cdot, j)$ is locally absolutely continuous for each j. A hybrid arc ϕ is a solution to (1) if it satisfies the dynamics of (1). A solution ϕ to (1) is maximal if it cannot be extended and is complete if $\operatorname{dom}\phi$ is unbounded. In particular, given $S \subset \mathbb{R}^n$, we denote by $\mathcal{S}_{\mathcal{H}}(S)$ the set of maximal solutions ϕ to $\mathcal{H}$ with $\phi(0,0) \in S$; see [21] for more details on solutions to hybrid systems.

In this paper, we consider the following notion of exponential stability for the hybrid system $\mathcal{H}$ in (1).

Definition 2.1 (Global exponential stability [22])**.** *Let $\mathcal{A} \subset \mathbb{R}^n$ be closed. The set $\mathcal{A}$ is said to be globally exponentially stable (*GES*) for $\mathcal{H}$ if there exist positive real numbers κ, λ, such that every maximal solution ϕ to $\mathcal{H}$ is complete and*

$$|\phi(t,j)|_{\mathcal{A}} \leq \kappa e^{-\lambda(t+j)} |\phi(0,0)|_{\mathcal{A}} \qquad \forall (t,j) \in \operatorname{dom}\phi \tag{2}$$

$\lozenge$

In this paper, we also consider the following specific class of hybrid systems with inputs:

$$\mathcal{H}_\eta \begin{cases} \dot{x} &= f(x,\eta) & (x,\eta) \in \mathcal{C}_\eta \\ x^+ &= g(x) & (x,\eta) \in \mathcal{D}_\eta \end{cases} \tag{3}$$

where $\eta \in \mathbb{R}^{n_\eta}$, $f\colon \mathbb{R}^{n+n_\eta} \to \mathbb{R}^n$, $\mathcal{C}_\eta \subset \mathbb{R}^{n+n_\eta}$, $g\colon \mathbb{R}^{n+n_\eta} \to \mathbb{R}^n$, and $\mathcal{D}_\eta \subset \mathbb{R}^{n+n_\eta}$. We say that $\eta\colon E \to \mathbb{R}^{n_\eta}$ is a hybrid input if E is a hybrid time domain and for all j, $t \mapsto \eta(t,j)$ is Lebesgue measurable and locally essentially bounded. A hybrid arc ϕ and a hybrid input η with $\operatorname{dom}\eta = \operatorname{dom}\phi$ is a solution pair to (3) if it satisfies its dynamics; see [23] for a formal definition of solution pair to hybrid systems with inputs. We consider the following notion of input-to-state stability for (3).

Definition 2.2. *Let $\mathcal{A} \subset \mathbb{R}^n$ be closed. Hybrid system (3) is said to be input-to-state stable (*ISS*) with respect to η relatively to the set $\mathcal{A}$ if there exist $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$ such that every maximal solution pair (ϕ, η) to (3) is complete and for all $(t,j) \in \operatorname{dom}\phi$ it satisfies:*

$$|\phi(t,j)|_{\mathcal{A}} \leq \beta(|\phi(0,0)|_{\mathcal{A}}, t+j) + \mu(\|\eta\|_\infty) \tag{4}$$

$\lozenge$

The $\mathcal{L}_\infty$ norm of $(t,j) \mapsto \eta(t,j)$ is defined as:

$$\|\eta\|_\infty \coloneqq \lim_{T \to T^\star} \max \left\{ \operatorname*{ess.sup}_{(s,k)\in \operatorname{dom}\eta \setminus \Gamma(\eta), s+k\leq T} |\eta(s,k)| \;,\; \sup_{(s,k)\in\Gamma(\eta), s+k\leq T} |\eta(s,k)| \right\}$$

where $\Gamma(\eta) \coloneqq \{(t,j) \in \operatorname{dom}\eta \colon (t,j+1) \in \operatorname{dom}\eta\}$ and $T^\star \coloneqq \sup\{t+j \colon (t,j) \in \operatorname{dom}\eta\}$; see [23] for further details.

3. Modeling and problem formulation

3.1. Description of the setup and hybrid modeling

We consider the following LTI plant:

$$\mathcal{P} \begin{cases} \dot{x}_p = Ax_p + Bu_p \\ y_p = Cx_p \end{cases} \tag{5}$$

where $x_p \in \mathbb{R}^{n_p}$ is the plant state, $u_p \in \mathbb{R}^{n_u}$ is the plant control input, and $y_p \in \mathbb{R}^{n_y}$ a measured output. Matrices $A \in \mathbb{R}^{n_p \times n_p}$, $B \in \mathbb{R}^{n_p \times n_u}$, and $C \in \mathbb{R}^{n_y \times n_p}$ are assumed to be known. In this paper, we are interested in the event-triggered implementation of Luenberger-like observer-based controllers, that is, controllers of the form:

$$\mathcal{O} \begin{cases} \dot{\hat{x}}_p = A\hat{x}_p + By_c + L(u_c - C\hat{x}_p) \\ y_c = K\hat{x}_p \end{cases} \tag{6}$$

where $y_c \in \mathbb{R}^{n_u}$ is the controller output, $u_c \in \mathbb{R}^{n_y}$ is the controller input, and $K \in \mathbb{R}^{n_u \times n_p}$ and $L \in \mathbb{R}^{n_p \times n_y}$ are some gains to be designed to ensure closed-loop asymptotic stability. When the controller and the plant are colocated, the interconnection between plant and controller is realized by simply setting $u_p = y_c$ and $u_c = y_p$. In this paper, we consider a setup in which the controller and sensors are colocated, i.e., the controller has access to the plant output y_p, while the communication between plant and controller occurs only at some sporadic time instances to be scheduled. This prevents from continuously feeding the plant using the controller output y_c.

Remark 1. *The setup considered in this paper has been already studied in the literature; see, e.g., [17, 18, 19, 20], just to mention a few. This setting is relevant when, due to technological constraints, data from sensors need to be broadcast towards the controller through a data network. Possible practical applications in which this situation may occur is the case of mobile robotic systems in which controllers are located onboard, while measurements (for example of position and speed) are obtained via a camera and sent wirelessly to the robot. Another possible application pertains to large scale or distributed systems in which actuation and measuring do not happen at the same place.*

We assume that the plant is equipped with a holding device $\mathcal{S}$ that is able to generate a control signal in between communications events. Every time a transmission occurs, the state of $\mathcal{S}$ is updated with the estimate of the plant state $\hat{x}_p$ transmitted by the controller. More specifically

$$\mathcal{S} \begin{cases} \dot{\theta} = H\theta \\ \theta^+ = \hat{x}_p \quad \text{when a transmission occurs} \end{cases} \tag{7}$$

where $\theta \in \mathbb{R}^{n_p}$ is the state of the holding device and H is a parameter to be designed. In this setting, the interconnection between the plant and the controller is obtained by selecting

$$u_p = K\theta \tag{8}$$

and modifying the observer dynamics in (6) as follows:

$$\dot{\hat{x}}_p = A\hat{x}_p + BK\theta + L(y_p - C\hat{x}_p) \tag{9}$$

In particular, instead of considering $u_p = y_c = K\hat{x}_p$ as classically done in (6), we consider (8).

Remark 2. *As a matter of fact, the holding device is colocated with the plant, hence the controller does not have access to the state θ. On the other hand, a copy of the holding device can be embedded in the controller to generate an estimate of the signal θ to be fed to the observer. Every time a transmission occurs,*

both holding devices are updated with value of the state $\hat{x}$. Notice that, although the two devices may be potentially initialized to different values, after the first transmission occurs they synchronize. Therefore, for the sake of simplicity, we assume that both the holding devices are identically synchronized.

The closed-loop system under consideration in this paper is the interconnection of (5), (7), (8) and (9). Therefore, the interconnection can be modeled as:

$$\begin{cases} \dot{x}_p = Ax_p + BK\theta \\ \dot{\hat{x}}_p = A\hat{x}_p + BK\theta + LC(x_p - \hat{x}_p) \\ \dot{\theta} = H\theta \\ x_p^+ = x_p \\ \hat{x}_p^+ = \hat{x}_p \qquad\qquad \text{when a transmission occurs} \\ \theta^+ = \hat{x}_p \end{cases} \tag{10}$$

Our goal is to design the proposed architecture, that is, the gains K, L, and H and a transmission policy ensuring closed-loop stability while reducing the number of communications events.

Remark 3. *In this work, we assume that the plant is equipped with a generalized holding device parametrized via the matrix H. Obviously, if $H = 0$ the proposed holding device turns out to be a standard zero-order-holder (ZOH) device. In this sense, our approach encompasses more classical approaches, thereby providing more flexibility in the tuning of the controller. However, notice that the use of generalized holding device leads to an increased complexity in terms of implementation.*

Remark 4. *The proposed generalized holding device, up to a simple modification of the jump map, can be used to model more classical holders as, e.g., the so-called delayed First Order Holder.*

The last ingredient that needs to be introduced is the law triggering the communications events. To prevent from the occurrence of Zeno behaviors,

we follow the general approach in [19] and consider the following dwell-time event-triggered logic:

$$\begin{cases} \dot{\tau} = 1 - \mathrm{dz}\left(\frac{\tau}{\tau_D}\right) & \tau \in [0, \tau_D] \vee (\hat{x}_p, y_p, \theta) \in \mathcal{F} \\ \tau^+ = 0 & \tau \in [\tau_D, 2\tau_D] \wedge (\hat{x}_p, y_p, \theta) \in \mathcal{J} \end{cases} \tag{11}$$

where $\tau_D > 0$ is a design parameter and the function $\mathrm{dz}\colon \mathbb{R} \to \mathbb{R}$ is defined as $\mathrm{dz}(s) = 0$ if $s \in [-1, 1]$ and $\mathrm{dz}(s) = \mathrm{sign}(s)(|s| - 1)$ otherwise. Communication events are generated when $\tau \in [\tau_D, 2\tau_D]$ and $(\hat{x}_p, y_p, \theta) \in \mathcal{J}$. The sets $\mathcal{F}$ and $\mathcal{J}$ define the event-triggering policy and are part of the design parameters. The introduction of the timer τ prevents from the existence of Zeno solutions, by enforcing a minimum strictly positive dwell-time in between consecutive communication events, i.e., τ_D. Notice that the set $[0, 2\tau_D]$ is forward invariant for the τ-dynamics. Combining (10) and (11), the closed-loop system reads:

$$\begin{cases} \dot{x}_p = Ax_p + BK\theta \\ \dot{\hat{x}}_p = (A - LC)\hat{x}_p + LCx_p + BK\theta \\ \dot{\theta} = H\theta \\ \dot{\tau} = 1 - \mathrm{dz}\left(\frac{\tau}{\tau_D}\right) \end{cases} \quad (\hat{x}_p, y_p, \theta, \tau) \in \mathcal{C}$$

$$\begin{cases} x_p^+ = x_p \\ \hat{x}_p^+ = \hat{x}_p \\ \theta^+ = \hat{x}_p \\ \tau^+ = 0 \end{cases} \quad (\hat{x}_p, y_p, \theta, \tau) \in \mathcal{D} \tag{12}$$

where

$$\mathcal{C} := \left(\left\{(\hat{x}_p, x_p, \theta) \in \mathbb{R}^{3n_p} \colon (\hat{x}_p, Cx_p, \theta) \in \mathcal{F}\right\} \times [0, 2\tau_D]\right) \cup (\mathbb{R}^{3n_p} \times [0, \tau_D])$$

$$\mathcal{D} := \left\{(\hat{x}_p, x_p, \theta) \in \mathbb{R}^{3n_p} \colon (\hat{x}_p, Cx_p, \theta) \in \mathcal{J}\right\} \times [\tau_D, 2\tau_D]$$

Remark 5. *The proposed event-triggering policy depends only on the information actually available at the controller end, that is, the plant output y_p, the controller state $\hat{x}_p$, and the state of the holding device θ.*

To ensure that the closed-loop system has nontrivial solutions from any initial condition in $\mathbb{R}^{3n_p} \times [0, 2\tau_D]$, it has to be $\mathcal{F} \cup \mathcal{J} = \mathbb{R}^{2n_p + n_y}$. In this

paper, with the objective of devising a constructive approach for the design of the proposed event-triggered controller, we pick those sets as follows:

$$\mathcal{F} \coloneqq \text{cone}^{-}(Q) \quad \mathcal{J} \coloneqq \text{cone}^{+}(Q) = \overline{\mathbb{R}^{2n_p+n_y} \setminus \mathcal{F}}$$

where $Q = Q^{\mathsf{T}} \in \mathbb{R}^{(2n_p+n_y)\times(2n_p+n_y)}$ is to be designed.

At this stage, let us define the following change of variables $(e, \tilde{\theta}) \coloneqq (x_p - \hat{x}_p, \theta - \hat{x}_p)$. Then, by taking as state vector $x = (\hat{x}_p, e, \tilde{\theta}, \tau) =: (\zeta, \tau)$, the closed-loop system can be modeled as the following hybrid system in the framework [21] briefly recalled in Section 2.2:

$$\begin{cases} \dot{\zeta} = F_{cl}\zeta \\ \dot{\tau} = 1 - \text{dz}\left(\frac{\tau}{\tau_D}\right) \end{cases} \qquad \tau \in [0, \tau_D] \vee \zeta \in \widehat{\mathcal{F}}$$
$$\begin{cases} \zeta^{+} = G_{cl}\zeta \\ \tau^{+} = 0 \end{cases} \qquad \tau \in [\tau_D, 2\tau_D] \wedge \zeta \in \widehat{\mathcal{J}} \tag{13a}$$

where:

$$\widehat{\mathcal{F}} \coloneqq \text{cone}^{-}(\breve{C}^{\mathsf{T}} Q \breve{C}) \quad \widehat{\mathcal{J}} \coloneqq \overline{\mathbb{R}^{3n_p} \setminus \widehat{\mathcal{F}}} \tag{13b}$$

$$F_{cl} \coloneqq \begin{bmatrix} A + BK & LC & BK \\ 0 & A - LC & 0 \\ H - A - BK & -LC & H - BK \end{bmatrix}$$
$$\breve{C} \coloneqq \begin{bmatrix} \mathbf{I} & 0 & 0 \\ C & C & 0 \\ \mathbf{I} & 0 & \mathbf{I} \end{bmatrix}, G_{cl} \coloneqq \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{13c}$$

For easiness of exposition, we represent closed-loop system (13) in the following compact form:

$$\begin{cases} \dot{x} = f(x) & x \in \widehat{\mathcal{C}} \\ x^{+} = g(x) & x \in \widehat{\mathcal{D}} \end{cases} \tag{14a}$$

where

$$
\begin{aligned}
\widehat{\mathcal{C}} &:= (\widehat{\mathcal{F}} \times [0, 2\tau_D]) \cup (\mathbb{R}^{3n_p} \times [0, \tau_D]) \\
f(x) &:= \begin{bmatrix} F_{cl}\zeta \\ 1 - \mathrm{dz}\left(\frac{\tau}{\tau_D}\right) \end{bmatrix} && \forall x \in \widehat{\mathcal{C}} \\
\widehat{\mathcal{D}} &:= \overline{(\mathbb{R}^{3n_p} \times [0, 2\tau_D]) \setminus \widehat{\mathcal{C}}} \\
g(x) &:= \begin{bmatrix} G_{cl}\zeta \\ 0 \end{bmatrix} && \forall x \in \widehat{\mathcal{D}}
\end{aligned}
\tag{14b}
$$

Before concluding this subsection, next, we outline some structural properties for the closed-loop system (14).

Property 3.1. *For any $\xi \in \widehat{\mathcal{C}} \cup \widehat{\mathcal{D}}$, there exists a nontrivial solution ϕ to (14) such that $\phi(0,0) = \xi$. Furthermore, let ϕ be any maximal solution to (14). Then, the following items hold:*

(a) *ϕ is complete;*

(b) *ϕ is non Zeno. In particular, $(t, j) \in \operatorname{dom}\phi$ implies $j \leq \frac{1}{\tau_D} t + 1$.*

□

The proof of Property 3.1 is reported in Appendix A.

3.2. Problem formulation

The goal of this paper is to characterize exponential stability of the ζ substate of the closed loop (14), uniformly in τ. The fact that τ evolves in the compact set $[0, 2\tau_D]$ simplifies this task and allows stating it as a suitable stability property for the compact set (attractor):

$$
\mathcal{A} := \{0\} \times [0, 2\tau_D] \subset \mathbb{R}^{3n_p+1} \tag{15}
$$

More specifically, we provide sufficient conditions for the solution to the following two problems.

Problem 3.1 (Emulation-based Control Design)**.** *Given K and L, design H, τ_D, and Q such that the set $\mathcal{A}$ defined in (15) is GES for the closed-loop system (14) and the number of transmission events is minimized.*

Problem 3.2 (Co-Design)**.** *Design K, L, H, τ_D, and Q such that the set $\mathcal{A}$ defined in (15) is GES for the closed-loop system (14) and the number of transmission events is minimized.*

Remark 6. *Notice that hybrid system (14) is well-posed in the sense of [21, Definition 6.2]. Among other things, this implies that nominal global asymptotic stability of the set $\mathcal{A}$ (established in Theorem 4.1; see Section 4) is structurally robust with respect to small perturbations, in the sense of [21, Corollary 7.23]. This makes the use of our results appealing in practice.*

Remark 7. *In practice, the generalized holding device and the controller are likely to be implemented in a digital system. In this setting, the natural robustness of (14) highlighted in Remark 6 can be used to ensure a safe sampled-data implementation of the proposed controller; see, e.g., [24, 25] for similar issues.*

4. Closed-loop Stability Analysis

In this section, we provide sufficient conditions for global exponential stability of the closed-loop system. To this end, consider the following property, whose role is clarified later.

Property 4.1. *There exist continuously differentiable functions $V_1 : \mathbb{R}^{2n_p} \to \mathbb{R}$ and $V_2 : \mathbb{R}^{n_p} \to \mathbb{R}$ such that the following items hold:*

(i) there exist positive scalars ω_1, ω_2, β_1, and β_2 such that:

$$\omega_1 |(\hat{x}_p, e)|^2 \leq V_1(\hat{x}_p, e) \leq \omega_2 |(\hat{x}_p, e)|^2 \qquad \forall (\hat{x}_p, e) \in \mathbb{R}^{2n_p} \tag{16a}$$

$$\beta_1 |\tilde{\theta}|^2 \leq V_2(\tilde{\theta}) \leq \beta_2 |\tilde{\theta}|^2 \qquad \forall \tilde{\theta} \in \mathbb{R}^{n_p} \tag{16b}$$

(ii) Let

$$\begin{aligned} \mathscr{U}(x) &:= \mathscr{V}_1(\hat{x}_p, e) + \mathscr{V}_2(\tilde{\theta}) e^{\rho(\tau_D - \tau)} && \forall x \in \mathbb{R}^{3n_p+1} \\ \mathscr{S}(\zeta) &:= \mathscr{U}(x) \,|_{x=(\zeta, \tau_D)} && \forall \zeta \in \mathbb{R}^{3n_p} \end{aligned} \tag{16c}$$

There exist positive scalars ρ and λ such that[1]

$$\langle \nabla \mathscr{U}(x), f(x) \rangle \leq -\lambda \mathscr{U}(x) \qquad \forall x \in \mathbb{R}^{3n_p} \times [0, \tau_D] \tag{16d}$$

$$\langle \nabla \mathscr{S}(\zeta), F_{cl}\zeta \rangle \leq -\lambda \mathscr{S}(\zeta) \qquad \forall \zeta \in \widehat{\mathcal{F}} \tag{16e}$$

Next, we give the main result of this section.

Theorem 4.1. *Let Property 4.1 hold. Then, the set $\mathcal{A}$ defined in (15) is GES for hybrid system (14).*

Proof. The proof of the result hinges upon Theorem A.1 in the Appendix. In particular, for all $x \in \mathbb{R}^{3n_p+1}$ define:

$$\mathscr{W}(x) \coloneqq \mathscr{V}_1(\hat{x}_p, e) + \mathscr{V}_2(\tilde{\theta}) e^{\rho \max\{0, \tau_D - \tau\}} \tag{17}$$

We show that under the hypotheses of Property 4.1, (A.1) holds for the function $\mathscr{W}$ defined above. In particular, thanks to (16a) and (16b), (A.1a) holds with $p = 2$, $\alpha_1 \coloneqq \min\{\omega_1, \beta_1\}$, and $\alpha_2 \coloneqq \max\{\omega_2, \beta_2 e^{\rho \tau_D}\}$. Now observe that for all $x \in \mathbb{R}^{3n_p+1}$

$$\mathscr{W}(x) = \begin{cases} \mathscr{U}(x) & \text{if} \quad \tau \in [0, \tau_D) \\ \mathscr{S}(\zeta) & \text{elsewhere} \end{cases}$$

where, for all $x \in \mathbb{R}^{3n_p+1}$, $\mathscr{U}(x)$ and $\mathscr{S}(\zeta)$ are defined in (16c). Define

$$\dot{\mathscr{W}}(x) \coloneqq \sup_{\chi \in \partial \mathscr{W}(x), f \in \{f(x)\} \cap T_{\widehat{C}}(x)} \langle \chi, f \rangle$$

We show that the satisfaction of Property 4.1 ensures that (A.1b) holds with $\alpha_3 = \lambda$. To this end, first notice that for all $x \in \mathbb{R}^{3n_p+1}$:

$$\partial \mathscr{W}(x) = \begin{cases} \nabla \mathscr{U}(x) & \text{if} \quad \tau < \tau_D \\ (\nabla \mathscr{S}(\zeta), 0) & \text{if} \quad \tau > \tau_D \\ \text{co}\{\nabla \mathscr{U}(x), (\nabla \mathscr{S}(\zeta), 0)\} & \text{if} \quad \tau = \tau_D \end{cases}$$

To conclude this first part of the proof, we analyze the above three cases separately. Pick $x = (\zeta, \tau) \in \widehat{\mathcal{C}}$.

[1] There is no loss of generality in using the same scalar λ in both the inequalities (16d)-(16e).

- Case 1. Assume $\tau \in [0, \tau_D)$. Then, one has

$$\dot{\mathscr{W}}(x) = \langle \nabla \mathscr{U}(x), f(x) \rangle$$

Hence, using (16d) one gets $\dot{\mathscr{W}}(x) \leq -\lambda \mathscr{W}(x)$.

- Case 2. Assume $\tau \in (\tau_D, 2\tau_D]$. Then, $\zeta \in \widehat{\mathcal{F}}$ and $\dot{\mathscr{W}}(x) = \langle \nabla \mathscr{S}(\zeta), F_{cl}\zeta \rangle$. Using (16e), one gets

$$\dot{\mathscr{W}}(x) \leq -\lambda \mathscr{W}(x)$$

- Case 3. Assume $\tau = \tau_D$. Observe that in this case:

$$\partial \mathscr{W}(x) = \text{co}\{\nabla \mathscr{U}(x), (\nabla \mathscr{S}(\zeta), 0)\}$$

Now the following two sub-cases need to be considered:

(3.a) Assume $\zeta \in \widehat{\mathcal{F}}$. In this case, notice that:

$$\begin{aligned} \dot{\mathscr{W}}(x) = \sup_{\chi \in \partial \mathscr{W}(x)} \langle \chi, f(x) \rangle = \max\{ & \langle \nabla \mathscr{S}(\zeta), F_{cl}\zeta \rangle, \\ & \langle \nabla \mathscr{S}(\zeta), F_{cl}\zeta \rangle - \rho \mathscr{V}_2(\tilde{\theta}) \} \\ & = \langle \nabla \mathscr{S}(\zeta), F_{cl}\zeta \rangle \end{aligned}$$

Thus, thanks to (16e), $\dot{\mathscr{W}}(x) \leq -\lambda \mathscr{W}(x)$.

(3.b) Assume $\zeta \notin \widehat{\mathcal{F}}$. Then, from item (i) of Lemma A.1 in Appendix A, $T_{\widehat{\mathcal{C}}}(x) \subset \mathbb{R}^{3n_p} \times \mathbb{R}_{\leq 0}$. Thus, due to the definition of the flow map f, one has:

$$\{f(x)\} \cap T_{\widehat{\mathcal{C}}}(x) = \emptyset$$

which gives $\dot{\mathscr{W}}(x) = -\infty$.

The above analysis enables to conclude that (A.1b) holds with $\alpha_3 = \lambda$. We now ultimate the proof by showing that (A.1c) holds. To this end, it suffices to observe that for all $x \in \widehat{\mathcal{D}}$

$$\mathscr{W}(g(x)) = \mathscr{V}_1(\hat{x}_p, e), \qquad \mathscr{W}(x) = \mathscr{V}_1(\hat{x}_p, e) + \mathscr{V}_2(\tilde{\theta})$$

Thus, for all $x \in \widehat{\mathcal{D}}$

$$\mathscr{W}(g(x)) - \mathscr{W}(x) = -\mathscr{V}_2(\tilde{\theta}) \leq 0$$

namely, (A.1c) holds for $\mathscr{W}$. Hence, by recalling Property 3.1, all the assumptions of Theorem A.1 are fulfilled and the result is established. □

225 **Remark 8.** *The construction of the Lyapunov function $\mathscr{W}$ used in the proof of Theorem 4.1 draws inspiration from [19]. Nevertheless, the introduction of the locally Lipschitz term $\max\{0, \tau_D - \tau\}$ enables to get a condition on the decrease of $\mathscr{W}$ in the set $\widehat{\mathcal{F}}$ that is independent from the scalars ρ and τ_D.*

The result given next shows that under some mild conditions on the flow dynamics of the closed-loop system, it is always possible to find ρ and τ_D positive such that (16d) holds. For easiness of exposition, consider the following partitioning of the matrix F_{cl} defined in (13c):

$$\begin{bmatrix} A_c & B_c \\ B_s & A_s \end{bmatrix} := \left[\begin{array}{cc|c} A+BK & LC & BK \\ 0 & A-LC & 0 \\ \hline H-A-BK & -LC & H-BK \end{array}\right] \tag{18}$$

Proposition 4.1. *Suppose that there exist continuously differentiable functions $\mathscr{V}_1 \colon \mathbb{R}^{2n_p} \to \mathbb{R}$ and $\mathscr{V}_2 \colon \mathbb{R}^{n_p} \to \mathbb{R}$ and positive scalars ω_1, ω_2, β_1, β_2, ι_1, ι_2, γ_1, and γ_2 such that (16a) and (16b) hold and for all $(\hat{x}_p, e, \tilde{\theta}) \in \mathbb{R}^{3n_p}$:*

$$\begin{aligned} &\underbrace{\left\langle \nabla \mathscr{V}_1(\hat{x}_p, e), A_c \begin{bmatrix} \hat{x}_p \\ e \end{bmatrix} \right\rangle}_{\dot{\mathscr{V}}_1(\zeta)} \leq -\iota_1(\hat{x}_p^{\mathsf{T}} \hat{x}_p + e^{\mathsf{T}} e) \\ &\underbrace{\langle \nabla \mathscr{V}_2(\tilde{\theta}), A_s \tilde{\theta} \rangle}_{\dot{\mathscr{V}}_2(\zeta)} \leq \iota_2 \tilde{\theta}^{\mathsf{T}} \tilde{\theta} \end{aligned} \tag{19a}$$

$$\begin{aligned} |\nabla \mathscr{V}_1(\hat{x}_p, e)| &\leq \gamma_1 |(\hat{x}_p, e)| \\ |\nabla \mathscr{V}_2(\tilde{\theta})| &\leq \gamma_2 |\tilde{\theta}| \end{aligned} \tag{19b}$$

Then, there exist $\rho, \tau_D > 0$ such that (16d) holds.

Proof. Let $\mathscr{U}$ be defined as in (16c). Then, simple calculations show that for

all $(\zeta, \tau) \in \mathbb{R}^{3n_p} \times [0, \tau_D]$

$$\langle \nabla \mathscr{U}(x), f(x) \rangle = \dot{\mathscr{V}}_1(\zeta) + \langle \nabla \mathscr{V}_1(\hat{x}_p, e), B_c \tilde{\theta} \rangle$$
$$e^{\rho(\tau_D - \tau)} \left(\dot{\mathscr{V}}_2(\zeta) + \left\langle \nabla \mathscr{V}_2(\tilde{\theta}), B_s \begin{bmatrix} \hat{x}_p \\ e \end{bmatrix} \right\rangle - \rho \mathscr{V}_2(\tilde{\theta}) \right)$$

Hence, by using (16b) and (19a), for all $x \in \mathbb{R}^{3n_p} \times [0, \tau_D]$

$$\begin{aligned} \langle \nabla \mathscr{U}(x), f(x) \rangle \leq & -\iota_1 |(\hat{x}_p, e)|^2 + |\nabla \mathscr{V}_1(\hat{x}_p, e)||B_c \tilde{\theta}| \\ & + e^{\rho \tau_D} \left(\iota_2 |\tilde{\theta}|^2 + |\nabla \mathscr{V}_2(\tilde{\theta})| \left| B_s \begin{bmatrix} \hat{x}_p \\ e \end{bmatrix} \right| \right) \\ & - \rho \beta_1 |\tilde{\theta}|^2 \end{aligned}$$

which, thanks to (19b), yields for all $x \in \mathbb{R}^{3n_p} \times [0, \tau_D]$

$$\begin{aligned} \langle \nabla \mathscr{U}(x), f(x) \rangle \leq & -\iota_1 |(\hat{x}_p, e)|^2 + \gamma_1 |(\hat{x}_p, e)||B_c \tilde{\theta}| \\ & + \left(\iota_2 |\tilde{\theta}|^2 + \gamma_2 |\tilde{\theta}| \left| B_s \begin{bmatrix} \hat{x}_p \\ e \end{bmatrix} \right| \right) e^{\rho \tau_D} \\ & - \rho \beta_1 |\tilde{\theta}|^2 \end{aligned}$$

Let $\varpi_1, \varpi_2 > 0$ be two scalars to be selected later. From Young inequality, the latter inequality gives, for all $x \in \mathbb{R}^{3n_p} \times [0, \tau_D]$:

$$\begin{aligned} \langle \nabla \mathscr{U}(x), f(x) \rangle \leq & \overbrace{\left(-\iota_1 + \frac{\gamma_1^2}{2\varpi_1} + \frac{\varpi_2}{2} |B_s|^2 e^{\rho \tau_D} \right)}^{\varsigma_1} |(\hat{x}_p, e)|^2 \\ & + \underbrace{e^{\rho \tau_D} \left(\iota_2 + \frac{\gamma_2^2}{2\varpi_2} - e^{-\rho \tau_D} \rho \beta_1 + \frac{e^{-\rho \tau_D} \varpi_1}{2} |B_c|^2 \right)}_{\varsigma_2} |\tilde{\theta}|^2 \end{aligned} \tag{20}$$

Let

$$\sigma_1 \coloneqq \iota_2 + \frac{\gamma_2^2}{2\varpi_2} + \frac{\varpi_1}{2} \|B_c\|^2, \sigma_2 \coloneqq \left(\iota_1 - \frac{\gamma_1^2}{2\varpi_1} \right) \frac{2}{\varpi_2 \|B_s\|^2}$$

Select ϖ_1, ϖ_2, and ρ such that

$$1 < \sigma_2 < \frac{\beta_1}{\sigma_1} \rho \tag{21}$$

this is always possible by picking ρ and ϖ_1 large enough and ϖ_2 small enough. Select

$$\tau_D \in \left(0, \frac{1}{\rho}\ln(\sigma_2)\right) \tag{22}$$

The bound in (21) ensures that the interval in (22) is nonempty. Moreover, the bounds in (21) and (22) guarantee that ς_1 and ς_2 in (20) are strictly negative. Taking $\bar{\varsigma} = \min\{|\varsigma_1|, |\varsigma_2|\}$, from (20) it follows that, for all $(\zeta, \tau) \in \mathbb{R}^{3n_p} \times [0, \tau_D]$

$$\langle \nabla \mathscr{U}(x), f(x) \rangle \leq -\bar{\varsigma}(|(\hat{x}_p, e)|^2 + |\tilde{\theta}|^2)$$

Now using the fact that for all $(\zeta, \tau) \in \mathbb{R}^{3n_p} \times [0, \tau_D]$, $\mathscr{U}(\zeta, \tau) \leq \omega_2 |(\hat{x}_p, e)|^2 + \beta_2 |\tilde{\theta}|^2$, taking $\lambda = \frac{\bar{\varsigma}}{\max\{\omega_2, \beta_2\}}$, one has that for all $(\zeta, \tau) \in \mathbb{R}^{3n_p} \times [0, \tau_D]$

$$\langle \nabla \mathscr{U}(x), f(x) \rangle \leq -\lambda \mathscr{U}(x)$$

This concludes the proof. □

Remark 9. *When $A + BK$ and $A - LC$ are Hurwitz and $\mathscr{V}_1$ and $\mathscr{V}_2$ are positive definite quadratic functions, it is easy to check that the assumptions in Proposition 4.1 hold.*

4.1. Quadratic Conditions

The applicability of Theorem 4.1 requires one to find functions $\mathscr{V}_1$ and $\mathscr{V}_2$ fulfilling Property 4.1. This is in general a nontrivial task. To overcome this problem, we select $\mathscr{V}_1$ and $\mathscr{V}_2$ as quadratic functions and recast the conditions in Property 4.1 into some matrix inequalities. This approach is formalized in the result given below.

Theorem 4.2. *If there exist $\tau_D, \rho, \sigma > 0$, $X, Y \in \mathbb{S}_+^{n_p}$, $E \in \mathbb{R}^{n_p \times n_p}$, and $R \in \mathbb{S}_+^{n_p}$ such that:*

$$P \succ 0 \tag{23a}$$

$$\operatorname{He} \Psi(\tau_D) - \sigma \breve{C}^{\mathsf{T}} Q \breve{C} \prec 0 \tag{23b}$$

$$\operatorname{He} \Psi(0) - \Omega(0) \prec 0 \tag{23c}$$

$$\operatorname{He} \Psi(\tau_D) - \Omega(\tau_D) \prec 0 \tag{23d}$$

where for all $\tau \in [0, \tau_D]$*:*

$$\Psi(\tau) \coloneqq \begin{bmatrix} X(A+BK) & XLC + E(A-LC) & XBK \\ E^{\mathsf{T}}(A+BK) & E^{\mathsf{T}}LC + Y(A-LC) & E^{\mathsf{T}}BK \\ e^{\rho(\tau_D-\tau)}R(H-A-BK) & -e^{\rho(\tau_D-\tau)}RLC & e^{\rho(\tau_D-\tau)}R(H-BK) \end{bmatrix} \tag{24}$$

$$\Omega(\tau) \coloneqq \rho e^{\rho(\tau_D-\tau)}(0 \oplus 0 \oplus R) \tag{25}$$

and

$$P \coloneqq \begin{bmatrix} X & E \\ \bullet & Y \end{bmatrix} \tag{26}$$

Then, the set $\mathcal{A}$ *defined in* (15) *is GES for hybrid system* (14)*.*

Proof. The proof hinges upon Theorem 4.1. In particular, we show that under the hypotheses of the statement, all the inequalities in (16) hold with

$$\mathscr{V}_1(\hat{x}_p, e) \coloneqq \begin{bmatrix} \hat{x}_p \\ e \end{bmatrix}^{\mathsf{T}} P \begin{bmatrix} \hat{x}_p \\ e \end{bmatrix}, \quad \mathscr{V}_2(\tilde{\theta}) \coloneqq \tilde{\theta}^{\mathsf{T}} R \tilde{\theta} \tag{27}$$

where P is defined in (26). Notice that since $P, R \succ 0$, (16a) and (16b) hold with

$$\omega_1 = \lambda_{\min}(P), \quad \omega_2 = \lambda_{\max}(P)$$
$$\beta_1 = \lambda_{\min}(R), \quad \beta_2 = \lambda_{\max}(R)$$

Let $\mathscr{U}$ and $\mathscr{S}$ be defined as in (16c) with $\mathscr{V}_1$ and $\mathscr{V}_2$ as in (27). Then

$$\begin{aligned} &\langle \nabla \mathscr{U}(x), f(x) \rangle = \zeta^{\mathsf{T}} \operatorname{He} \Psi(\tau) \zeta - \rho e^{-\rho\tau} \mathscr{V}_2(\tilde{\theta}) = \zeta^{\mathsf{T}} \left(\operatorname{He} \Psi(\tau) - \Omega(\tau) \right) \zeta \quad \forall x \in \mathbb{R}^{3n_p} \times [0, \tau_D] \\ &\langle \nabla \mathscr{S}(\zeta), F_{cl} \zeta \rangle = \zeta^{\mathsf{T}} \operatorname{He} \Psi(\tau_D) \zeta \qquad \forall \zeta \in \mathbb{R}^{3n_p} \end{aligned} \tag{28}$$

where, for all $\tau \in [0, 2\tau_D]$, $\Psi(\tau)$ is defined in (24). Hence, thanks to the S-procedure, the satisfaction of (23b) implies that

$$\zeta^{\mathsf{T}} \operatorname{He} \Psi(\tau_D) \zeta < 0 \qquad \forall \zeta \in \widehat{\mathcal{F}} \setminus \{0\}$$

which in turn implies that there exists $\lambda_1 > 0$ small enough such that:

$$\zeta^{\mathsf{T}} \operatorname{He} \Psi(\tau_D) \zeta \leq -\lambda_1 \zeta^{\mathsf{T}} (P \oplus R) \zeta = -\lambda_1 \mathscr{S}(\zeta) \qquad \forall \zeta \in \widehat{\mathcal{F}} \tag{29a}$$

To conclude the proof, observe that there exists a function $\varkappa\colon [0,\tau_D] \to [0,1]$ such that for all $\tau \in [0,\tau_D]$

$$\operatorname{He}\Psi(\tau) - \Omega(\tau) = \varkappa(\tau)(\operatorname{He}\Psi(0) - \Omega(0)) + (1-\varkappa(\tau))(\operatorname{He}\Psi(\tau_D) - \Omega(\tau_D))$$

Therefore, from (23c) and (23d), it follows that there exist $\lambda_2 > 0$ small enough such that:

$$\zeta^{\mathsf{T}}(\operatorname{He}\Psi(\tau) - \Omega(\tau))\zeta \leq -\lambda_2\zeta^{\mathsf{T}}\zeta \qquad \forall x \in \mathbb{R}^{3n_p} \times [0,\tau_D]$$

At this stage recall that, from (16a) and (16b), for all $x \in \mathbb{R}^{3n_p} \times [0,\tau_D]$

$$\mathscr{U}(\zeta) = \mathscr{V}_1(x_p,e) + e^{\rho(\tau_D-\tau)}\mathscr{V}_2(\tilde{\theta}) \leq \alpha_2\zeta^{\mathsf{T}}\zeta$$

with $\alpha_2 \coloneqq \max\{\omega_2, \beta_2 e^{\rho\tau_D}\}$. Hence, combining the latter two relationships, one gets

$$\zeta^{\mathsf{T}}(\operatorname{He}\Psi(\tau) - \Omega(\tau))\zeta \leq -\frac{\lambda_2}{\alpha_2}\mathscr{U}(\zeta) \qquad \forall x \in \mathbb{R}^{3n_p} \times [0,\tau_D] \tag{29b}$$

Therefore, by taking $\lambda = \min\{\lambda_1, \lambda_2\alpha_2^{-1}\}$ and recalling (28), the satisfaction of (29a) and (29b) yields (16d) and (16e). This concludes the proof. □

Remark 10. *Following the discussion in Remark 9, it is easy to check that when $A+BK$ and $A-LC$ are Hurwitz, (23c) and (23d) are always feasible when τ_D is taken small enough. Therefore, since (23c) does not depend on τ_D either ρ, the satisfaction of (23c) with $A+BK$ and $A-LC$ Hurwitz is enough to ensure the feasibility of the conditions of Theorem 4.2. This fact follows from the selection of the Lyapunov function $\mathscr{W}$ in Theorem 4.1; see also Remark 8.*

5. Controller Design

Before stating the main results of this section, let us consider the following preliminary result that enables one to eliminate the scalar variable σ introduced by the S-procedure in (23b).

Lemma 5.1. *Let $\tau_D, \rho \in \mathbb{R}_{>0}$, and $P \in \mathbb{S}_+^{n_p}$ be given. The following items are equivalent:*

(*i*) *There exist* $Q = Q^\mathsf{T} \in \mathbb{R}^{(2n_p+n_y)\times(2n_p+n_y)}$ *and* $\sigma \in \mathbb{R}_{\geq 0}$ *such that* (23b) *holds;*

(*ii*) *There exists* $Q = Q^\mathsf{T} \in \mathbb{R}^{(2n_p+n_y)\times(2n_p+n_y)}$ *such that* (23b) *holds with* $\sigma = 1$.

Proof. The implication $(ii) \implies (i)$ is trivial. To show that $(i) \implies (ii)$, first note that if (23b) holds with $\sigma \geq 0$, then it must hold with $\sigma' \coloneqq \sigma + \sigma_\varepsilon > 0$ where $\sigma_\varepsilon > 0$ small enough exists due to the strict inequality in (23b). As a consequence, (ii) holds with $Q' = \sigma' Q$ and $\sigma = 1$. This concludes the proof. □

5.1. Sufficient Conditions for Emulation-based Design

In this first subsection, we provide sufficient conditions for the solution to Problem 3.1. In particular, under the assumption that the gains K and L are given, we provide sufficient conditions for the design of the event-triggering policy, i.e., the matrix Q and the holding device $\mathcal{S}$ in the form of some linear matrix inequalities.

Proposition 5.1. *Let* $K \in \mathbb{R}^{n_u\times n_p}$ *and* $L \in \mathbb{R}^{n_p\times n_y}$ *be given. Suppose that there exist* $\rho > 0$ *and matrices* $X, Y \in \mathbb{S}^{n_p}_+$, $E \in \mathbb{R}^{2n_p\times 2n_p}$, $R \in \mathbb{S}^{n_p}_+$, $Z \in \mathbb{R}^{n_p\times n_p}$, *and* $Q = Q^\mathsf{T} \in \mathbb{R}^{(2n_p+n_y)\times(2n_p+n_y)}$ *such that:*

$$\operatorname{He}\Phi(\tau_D) - \breve{C}^\mathsf{T} Q \breve{C} \prec 0 \tag{30a}$$

$$\operatorname{He}\Phi(0) - \Omega(0) \prec 0 \tag{30b}$$

$$\operatorname{He}\Phi(\tau_D) - \Omega(\tau_D) \prec 0 \tag{30c}$$

where $\breve{C}$ *is defined in* (13c) *and for all* $\tau \in [0, \tau_D]$, $\Omega(\tau)$ *is defined as in* (25) *and*

$$\Phi(\tau) \coloneqq \begin{bmatrix} X(A+BK) & XLC + E(A-LC) & XBK \\ E^\mathsf{T}(A+BK) & E^\mathsf{T}LC + Y(A-LC) & E^\mathsf{T}BK \\ e^{\rho(\tau_D-\tau)}Z & -e^{\rho(\tau_D-\tau)}RLC & e^{\rho(\tau_D-\tau)}(RA+Z) \end{bmatrix} \tag{30d}$$

Pick $H = A + BK + R^{-1}Z$. *Then, the set* $\mathcal{A}$ *defined in* (15) *is GES for the closed-loop system* (14).

Proof. The proof follows easily from Theorem 4.2 by observing that setting $Z = R(H - A - BK)$ in the expression of $\Phi(\tau)$ in (30d) gives the matrix $\Psi(\tau)$ in (24). □

Remark 11. *In some specific cases, due to technological reasons, it may be preferable to consider a zero-order-holder (ZOH) device, i.e., $H = 0$. This can be easily achieved by enforcing $Z = -R(A+BK)$ in Proposition 5.1. This shows that the methodology we propose in this paper encompasses the use of classical holding devices.*

5.2. Sufficient conditions for co-design

In this section, we show how the conditions proposed in this paper can be used to solve Problem 3.2.

Proposition 5.2. *Suppose that there exist $\rho, \tau_D \in \mathbb{R}_{>0}$ and matrices $W, Y \in \mathbb{S}_+^{n_p}$, $L \in \mathbb{R}^{n_p \times n_y}$, $F \in \mathbb{R}^{n_u \times n_p}$, $Z \in \mathbb{R}^{n_p \times n_p}$, and $\widetilde{Q} = \widetilde{Q}^\mathsf{T} \in \mathbb{R}^{(2n_p+n_y)\times(2n_p+n_y)}$ such that:*

$$\operatorname{He} \Pi(\tau_D) - \Upsilon^\mathsf{T} \widetilde{Q} \Upsilon \prec 0 \tag{31a}$$

$$\operatorname{He} \Pi(0) - \Sigma(0) \prec 0 \tag{31b}$$

$$\operatorname{He} \Pi(\tau_D) - \Sigma(\tau_D) \prec 0 \tag{31c}$$

where $\Upsilon \coloneqq \mathbf{I} \oplus C \oplus \mathbf{I}$ and for all $\tau \in [0, \tau_D]$

$$\Pi(\tau) \coloneqq \begin{bmatrix} AW + BF & LC & BF \\ 0 & Y(A - LC) & 0 \\ e^{\rho(\tau_D - \tau)} Z & -e^{\rho(\tau_D - \tau)} LC & (AW + Z) e^{\rho(\tau_D - \tau)} \end{bmatrix} \tag{31d}$$

$$\Sigma(\tau) \coloneqq \rho e^{\rho(\tau_D - \tau)} (0 \oplus 0 \oplus W)$$

Pick $K = FW^{-1}$, $H = A + BFW^{-1} + ZW^{-1}$, and

$$Q = \begin{bmatrix} W^{-1} & 0 & 0 \\ -C & \mathbf{I} & 0 \\ -W^{-1} & 0 & W^{-1} \end{bmatrix}^\mathsf{T} \widetilde{Q} \begin{bmatrix} W^{-1} & 0 & 0 \\ -C & \mathbf{I} & 0 \\ -W^{-1} & 0 & W^{-1} \end{bmatrix}$$

Then, the set $\mathcal{A}$ defined in (15) is GES for the closed-loop system (14).

Proof. To prove the claim, we show that the satisfaction of (31) implies that (23) holds with $E = 0$, $X = R = W^{-1}$, and L, K and H given as in the statement of the result.

Proof of (31a) $\Longrightarrow$ (30a). Let $\Xi \coloneqq W^{-1} \oplus \mathbf{I} \oplus W^{-1}$. Then, for all $\tau \in [0, \tau_D]$

$$\Xi\Pi(\tau)\Xi = \begin{bmatrix} W^{-1}(A + BFW^{-1}) & W^{-1}LC & W^{-1}BFW^{-1} \\ 0 & Y(A - LC) & 0 \\ e^{\rho(\tau_D-\tau)}W^{-1}ZW^{-1} & -e^{\rho(\tau_D-\tau)}W^{-1}LC & W^{-1}(A + ZW^{-1})e^{\rho(\tau_D-\tau)} \end{bmatrix} \tag{32}$$

Hence, thanks to the proposed selection for H and F, it follows that

$$\Xi\Pi(\tau)\Xi = \begin{bmatrix} W^{-1}(A + BK) & W^{-1}LC & W^{-1}BKW^{-1} \\ 0 & Y(A - LC) & 0 \\ e^{\rho(\tau_D-\tau)}W^{-1}(H - A - BK) & -e^{\rho(\tau_D-\tau)}W^{-1}LC & W^{-1}(H - BK)e^{\rho(\tau_D-\tau)} \end{bmatrix} \tag{33}$$

which reads as the matrix $\Psi(\tau)$ in (24) with $X = R = W^{-1}$ and $E = 0$. Moreover, using the relationship between Q and $\widetilde{Q}$ it follows that:

$$\widetilde{Q} = \begin{bmatrix} W & 0 & 0 \\ CW & \mathbf{I} & 0 \\ W & 0 & W \end{bmatrix}^{\mathsf{T}} Q \underbrace{\begin{bmatrix} W & 0 & 0 \\ CW & \mathbf{I} & 0 \\ W & 0 & W \end{bmatrix}}_{\Gamma}$$

Thus, by noticing that $\Upsilon\Xi = \Gamma^{-1}\breve{C}$, one gets

$$\Xi\Upsilon^{\mathsf{T}}\widetilde{Q}\Upsilon\Xi = \breve{C}^{\mathsf{T}}Q\breve{C} \tag{34}$$

Combining (33) and (34) and taking $X = R = W^{-1}$ and $E = 0$ in (24), yields:

$$\Xi(\Pi(\tau) - \Upsilon^{\mathsf{T}}\widetilde{Q}\Upsilon)\Xi = \Phi(\tau) - \breve{C}^{\mathsf{T}}Q\breve{C} \quad \forall \tau \in [0, \tau_D]$$

Thus, (31a) impies (30a), as to be proven.

Proof of (31b)-(31c) $\Longrightarrow$ (30b)-(30c). This implication can be easily proven by observing that, for all $\tau \in [0, \tau_D]$, when $R = W^{-1}$, $\Xi\Sigma(\tau)\Xi = \Omega(\tau)$. This concludes the proof. □

Notice that when L and ρ are fixed, the conditions provided by Proposition 5.2 are linear matrix inequalities. Therefore, provided that an observer gain is selected, Proposition 5.2 can be efficiently used to solve Problem 3.2 by performing a line search on the scalar ρ. In practice, this limitation can be easily overcome by selecting the gain L so to ensure suitable convergence properties for the estimation error e. This approach is illustrated in Section 7. It is worthwhile to mention that the design of observer-based controllers via linear matrix inequalities is an open problem also in the context of linear systems; see, e.g., [26].

5.3. Optimal Design of the Event-Triggered Controller

In this section, we show how the proposed approach can be embedded into computationally tractable optimization problems aimed at reducing the number of sampling events. Overall, the event-triggering policy is characterized by two parameters, the dwell-time parameter τ_D and the matrix Q defining the triggering policy itself. Obviously, the value of τ_D is directly connected with the number of expected sampling events. More precisely, the larger τ_D, the larger the time in between consecutive samplings. Therefore, the design of the triggering policy needs to be performed to maximize the value of τ_D.

Concerning the selection of the matrix Q, understanding how this affects the sampling frequency is in general a nontrivial problem. On the other hand, as unrevealed in the proof of Theorem 4.1, the event-triggering policy is designed to ensure the decrease of the function $\mathscr{W}$ defined in (17) in the flow set $\widehat{\mathcal{C}}$. As a matter of fact, as shown in Proposition 4.1, this decrease can be essentially ensured by properly selecting the matrix Q so to enforce a bound on the growth of the function $\mathscr{S}$, i.e., (16e). Namely, restricting the attention to the quadratic case analyzed in Section 4.1, the matrix Q should be designed to ensure that the following implication holds:

$$\zeta \in \operatorname{cone}^{-}(\breve{C}^{\mathsf{T}} Q \breve{C}) \setminus \{0\} \implies \mathscr{T}(\zeta) \coloneqq \zeta^{\mathsf{T}} \operatorname{He} \Psi(\tau_D) \zeta < 0 \tag{35}$$

where Ψ is defined (24). Obviously, the above bound may hold in a set $\mathcal{F}_S \supset$

cone$^-(\breve{C}^\mathsf{T} Q \breve{C})$ with $\mathcal{F}_S \cap \operatorname{int} \widehat{\mathcal{J}} \neq \emptyset$. When this happens, update events can be generated needlessly. Building upon this observation, our idea to decrease the number of sampling events consists of designing the event-triggering policy so to minimize $\|\Psi(\tau_D) - \breve{C}^\mathsf{T} Q \breve{C}\|$. Next, we show how suitable optimization problems can be associated to the design results in Section 5.1 and Section 5.2.

5.3.1. Optimal Emulation-based Design

When the controller gains K and L are given, the design conditions of the event-triggering policy given in Proposition 5.1 can be embedded into the following optimization problem:

$$\begin{aligned} \underset{\rho, X, Y, E, R, Z, Q, \gamma}{\text{minimize}} \quad & -\gamma \\ \text{subject to} \quad & (23) \\ & \Psi(\tau_D) - \breve{C}^\mathsf{T} Q \breve{C} \succeq \gamma \mathbf{I} \end{aligned} \tag{36}$$

In particular, notice that since the satisfaction of (30a) implies that $\Psi(\tau_D) - \breve{C}^\mathsf{T} Q \breve{C} \prec 0$, the formulation of the above minimization problem ensures that

$$\gamma \mathbf{I} \preceq \Psi(\tau_D) - \breve{C}^\mathsf{T} Q \breve{C} \prec 0$$

This shows that the maximization of γ leads to the minimization of $\|\Psi(\tau_D) - \breve{C}^\mathsf{T} Q \breve{C}\|$.

5.3.2. Optimal Co-Design

In the setting of Section 5.2, due to the congruence transformations and change of variables performed in Proposition 5.2, the formulation of an optimization problem similar to (36) is not straightforward. More specifically, in the case of Proposition 5.2, the relationship between the event-triggering policy parameters and the function $\mathscr{T}$ in (35) is not explicit. On the other hand, by retracing the steps presented in the proof of Proposition 5.2, it turns out that (35) can be formulated as:

$$\zeta \in \text{cone}^-(\Xi \Upsilon^\mathsf{T} \widetilde{Q} \Upsilon \Xi) \setminus \{0\} \implies \zeta^\mathsf{T} \Xi \Pi(\tau_D) \Xi \zeta < 0 \tag{37}$$

where $\Xi \coloneqq W^{-1} \oplus \mathbf{I} \oplus W^{-1}$ and, for all $\tau \in [0, \tau_D]$, Π is defined in (31d). This shows that in this setting, the design of the event-triggering law can be performed so to minimize $\|\Xi(\Pi(\tau_D) - \Upsilon^{\mathsf{T}}\widetilde{Q}\Upsilon)\Xi\|$. This naturally leads to the following optimization problem:

$$\begin{aligned} \underset{\rho,W,Y,Z,\widetilde{Q},F,L}{\text{minimize}} \quad & -\gamma \\ \text{subject to} \quad & (31) \\ & \Pi(\tau_D) - \Upsilon^{\mathsf{T}}\widetilde{Q}\Upsilon \succeq \gamma \mathbf{I} \\ & W \succeq \delta \mathbf{I} \end{aligned} \tag{38}$$

where $\delta > 0$ is a small scalar to be selected. Also in this case, the formulation of the above optimization problem assures that

$$\|\Xi(\Pi(\tau_D) - \Upsilon^{\mathsf{T}}\widetilde{Q}\Upsilon)\Xi\| \leq \delta^{-1}\|\Pi(\tau_D) - \Upsilon^{\mathsf{T}}\widetilde{Q}\Upsilon\| \leq \delta^{-1}|\gamma|$$

Thereby showing the relevance of the proposed optimization with respect to the considered objective.

6. Robustness to measurement noise

In this section, we show that under the assumptions of Theorem 4.2, the closed-loop system enjoys some appealing robustness properties with respect to measurement noise. In particular, let us assume that the plant output y_p is affected by a measurement noise η, i.e., $y_p = Cx_p + \eta$. In this case, by defining

$$\widehat{\mathcal{F}}_n \coloneqq \text{cone}^{-}\left(\begin{bmatrix} \breve{C}^{\mathsf{T}}Q\breve{C} & \breve{C}^{\mathsf{T}}JQ \\ \bullet & J^{\mathsf{T}}QJ \end{bmatrix}\right) \tag{39}$$

with $J \coloneqq \begin{bmatrix} 0 & \mathbf{I} & 0 \end{bmatrix}^{\mathsf{T}}$, the closed-loop system can be modeled via the following hybrid system with state $x = (\hat{x}_p, e, \tilde{\theta}, \tau)$ defined previously and input η:

$$\begin{cases} \dot{x} = f_n(x, \eta) & (x, \eta) \in \widehat{\mathcal{C}}_n \\ x^+ = g(x) & (x, \eta) \in \widehat{\mathcal{D}}_n \end{cases} \tag{40}$$

where for all $(x,\eta) \in \widehat{\mathcal{C}}_n \coloneqq (\widehat{\mathcal{F}}_n \times [0, 2\tau_D]) \cup (\mathbb{R}^{3n_p+n_y} \times [0, \tau_D])$ the flow map is defined as:

$$f_n(x,\eta) \coloneqq \begin{bmatrix} F_{cl}\zeta + \underbrace{\begin{bmatrix} L \\ -L \\ -L \end{bmatrix}}_{N_{cl}} \eta \\ 1 - \mathrm{dz}\left(\frac{\tau}{\tau_D}\right) \end{bmatrix}$$

the jump set is defined as $\widehat{\mathcal{D}}_n \coloneqq \overline{(\mathbb{R}^{3n_p+n_y} \times [0, 2\tau_D]) \setminus \widehat{\mathcal{C}}_n}$, and the jump map g being unchanged with respect to (14).

Robustness of the closed-loop system with respect to the exogenous input η is established in the result given next.

Proposition 6.1. *If there exist $\tau_D, \rho, \sigma > 0$, $X, Y \in \mathbb{S}^{n_p}_+$, $E \in \mathbb{R}^{n_p \times n_p}$, and $R \in \mathbb{S}^{n_p}_+$ such that (23) holds, then (40) is input-to-stable with respect to the input η relatively to the set $\mathcal{A}$ in (15).*

Proof. To prove the result, we show that under (23), all the conditions in Theorem A.2 hold with $\mathscr{W}$ defined as in (17) and $\mathscr{V}_1$ and $\mathscr{V}_2$ as in (27). In particular, since, as established in the proof of Theorem 4.1, (A.1a) and (A.3c) follows directly from the positive definiteness of $\mathscr{V}_1$ and $\mathscr{V}_2$, to complete the proof, it suffices to show that (A.3b) holds[2]. Let $\mathscr{U}$ and $\mathscr{S}$ be defined as in (16c). Then, for all $(x,\eta) \in \widehat{\mathcal{C}}_n$:

$$\langle \nabla U(x), f_n(x,\eta) \rangle = \begin{bmatrix} \zeta \\ \eta \end{bmatrix}^{\mathsf{T}} \underbrace{\begin{bmatrix} \mathrm{He}\,\Psi(\tau) - \Omega(\tau) & \widehat{P}N_{cl} \\ \bullet & 0 \end{bmatrix}}_{M_1(\tau)} \begin{bmatrix} \zeta \\ \eta \end{bmatrix} \quad \forall x \in \mathbb{R}^{3n_p} \times [0, \tau_D], \eta \in \mathbb{R}^{n_y}$$

$$\langle \nabla S(\zeta), F_{cl}\zeta + N_{cl}\eta \rangle = \begin{bmatrix} \zeta \\ \eta \end{bmatrix}^{\mathsf{T}} \underbrace{\begin{bmatrix} \mathrm{He}\,\Psi(\tau_D) & \widehat{P}N_{cl} \\ \bullet & 0 \end{bmatrix}}_{M_2} \begin{bmatrix} \zeta \\ \eta \end{bmatrix} \quad \forall \zeta \in \mathbb{R}^{3n_p}, \eta \in \mathbb{R}^{n_y}$$

[2]The addition of the input η in the jump map of (40) does not affect the change of the Lyapunov function $\mathscr{W}$ across jumps.

where, for all $\tau \in [0, \tau_D]$, $\Omega(\tau)$ and $\Psi(\tau)$ and are defined, respectively, in (25) and (24) and

$$\widehat{P} \coloneqq \begin{bmatrix} X & E \\ \bullet & Y \end{bmatrix} \oplus R$$

As a first step, we show that the satisfaction of (23c) and (23d) implies that for all $x \in \mathbb{R}^{3n_p} \times [0, \tau_D], \eta \in \mathbb{R}^{n_y}$

$$\begin{bmatrix} \zeta \\ \eta \end{bmatrix}^{\mathsf{T}} M_1(\tau) \begin{bmatrix} \zeta \\ \eta \end{bmatrix} \leq -\lambda_1 \zeta^{\mathsf{T}} \widehat{P} \zeta + \mu_1 \eta^{\mathsf{T}} \eta \tag{41}$$

for some positive scalars λ_1, μ_1. To this end, observe that, as shown in the proof of Theorem 4.2, the satisfaction of (23c) and (23d) assures that for all $\tau \in [0, \tau_D]$

$$M_1(\tau) - \underbrace{\begin{bmatrix} -\bar{\lambda}_1 \widehat{P} & \widehat{P} N_{cl} \\ \bullet & 0 \end{bmatrix}}_{M_1'} \preceq 0 \tag{42}$$

for some $\bar{\lambda}_1 > 0$. At this stage, pick $\lambda_1 \in (0, \bar{\lambda}_1)$ and

$$O_1 \coloneqq \frac{1}{\bar{\lambda}_1 - \lambda_1} N_{cl}^{\mathsf{T}} \widehat{P} N_{cl}$$

Then, via Schur complement, it can be shown that

$$M_1' - \begin{bmatrix} -\lambda_1 \widehat{P} & 0 \\ \bullet & O_1 \end{bmatrix} \preceq 0$$

Hence, combining the above bound with (42) shows that (41) holds with $\mu_1 = \lambda_{\max}(O_1)$. Now we show that the satisfaction of (23a) implies that for all $(\zeta, \eta) \in \widehat{\mathcal{F}}_s$

$$\begin{bmatrix} \zeta \\ \eta \end{bmatrix}^{\mathsf{T}} M_2 \begin{bmatrix} \zeta \\ \eta \end{bmatrix} \leq -\lambda_2 \zeta^{\mathsf{T}} \widehat{P} \zeta + \mu_2 \eta^{\mathsf{T}} \eta \tag{43}$$

for some positive scalars λ_2, μ_2. Observe that

$$M_2 - \sigma \underbrace{\begin{bmatrix} \breve{C}^{\mathsf{T}} Q \breve{C} & \breve{C}^{\mathsf{T}} J Q \\ \bullet & J^{\mathsf{T}} Q J \end{bmatrix}}_{Q'} = \underbrace{\begin{bmatrix} \mathrm{He}\, \Psi(\tau_D) - \sigma \breve{C}^{\mathsf{T}} Q \breve{C} & \widehat{P} N_{cl} - \sigma \breve{C}^{\mathsf{T}} J Q \\ \bullet & -\sigma J^{\mathsf{T}} Q J \end{bmatrix}}_{M_3}$$

From (23a), there exists $\bar{\lambda}_2 > 0$ such that

$$M_3 - \begin{bmatrix} -\bar{\lambda}_2 \widehat{P} & \widehat{P} N_{cl} - \sigma \breve{C}^{\mathsf{T}} J Q \\ \bullet & -\sigma J^{\mathsf{T}} Q J \end{bmatrix} \preceq 0 \tag{44}$$

Pick $\lambda_2 \in (0, \bar{\lambda}_2)$ and let

$$O_2 = -\sigma J^{\mathsf{T}} Q J + \frac{(\widehat{P} N_{cl} - \sigma \breve{C}^{\mathsf{T}} J Q)^{\mathsf{T}} \widehat{P}^{-1} (\widehat{P} N_{cl} - \sigma \breve{C}^{\mathsf{T}} J Q)}{\bar{\lambda}_2 - \lambda_2}$$

Then, by Schur complement, it can be easily shown that:

$$\begin{bmatrix} -\lambda_2 \widehat{P} & \widehat{P} N_{cl} - \sigma \breve{C}^{\mathsf{T}} J Q \\ \bullet & -\sigma J^{\mathsf{T}} Q J \end{bmatrix} - \begin{bmatrix} -\bar{\lambda}_2 \widehat{P} & 0 \\ \bullet & O_2 \end{bmatrix} \preceq 0$$

Hence, combining the above bound with (44), it follows that for all $(\zeta, \eta) \in \mathbb{R}^{3n_p + n_y}$

$$\begin{bmatrix} \zeta \\ \eta \end{bmatrix}^{\mathsf{T}} (M_2 - \sigma Q') \begin{bmatrix} \zeta \\ \eta \end{bmatrix} \leq -\lambda_2 \zeta^{\mathsf{T}} \widehat{P} \zeta + \eta^{\mathsf{T}} O_2 \eta$$

which, thanks to the S-procedure, yields:

$$\begin{bmatrix} \zeta \\ \eta \end{bmatrix}^{\mathsf{T}} M_2 \begin{bmatrix} \zeta \\ \eta \end{bmatrix} \leq -\lambda_2 \zeta^{\mathsf{T}} \widehat{P} \zeta + \eta^{\mathsf{T}} O_2 \eta \qquad \forall (\zeta, \eta) \in \widehat{\mathcal{F}}_n$$

The latter shows that (43) holds with $\mu_2 = \lambda_{\max}(O_2)$. To conclude the proof, it suffices to observe that, similarly as in the proof of Theorem 4.1, thanks to item (ii) of Lemma A.1 in Appendix A, the satisfaction of (41) and (43) assures that (A.3b) holds with $\mathscr{W}$ defined as in (17), $\mathscr{V}_1$ and $\mathscr{V}_2$ as in (27), and $\mu(s) := \max\{\mu_1 s, \mu_2 s\}$. Hence, by considering a straightforward extension of Property 3.1, all the assumptions of Theorem A.2 are fulfilled and the result is established. □

7. Numerical Example

7.1. Controller Design and Comparisons

We revisit [27, Example 2] in which the plant data is

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} -1 & 4 \end{bmatrix}$$

We consider the same observer gain in [27, Example 2], namely[3] $L = \begin{bmatrix} 0 & 1 \end{bmatrix}^{\mathsf{T}}$ and design τ_D, K, and Q via Proposition 5.2. Numerical tests show that in this case, the conditions in Proposition 5.2 are feasible for τ_D up to 1. with $\rho = 14$ and using (5.2), one obtains the following numerical values for controller parameters[4]:

$$K = \begin{bmatrix} -1.43 & -7.85 \end{bmatrix}, H = \begin{bmatrix} -2.93 \times 10^{-5} & 1 \\ -3.43 & -4.85 \end{bmatrix}$$

$$Q = \begin{bmatrix} -6330 & 25322 & -6330 & 0.004592 & -0.04636 \\ 25322 & -1.013 \times 10^5 & 25322 & -0.01696 & -0.1271 \\ -6330 & 25322.0 & -6330 & -0.002928 & -0.0123 \\ 0.004592 & -0.01696 & -0.002928 & -0.01171 & 0.01106 \\ -0.04636 & -0.1271 & -0.0123 & 0.01106 & 0.07965 \end{bmatrix} \quad (45)$$

It is interesting to remark that, although no specific structure is imposed to the matrix H, the solution to the proposed optimization is such that $H \approx A+BK$. Namely, the holding device mimics the dynamics of the plant controlled via the state feedback controller $u = Kx_p$. In other words, the designed holding device behaves as an intersample predictor of the controller plant. This shows that the use of model-based intersample predictors [18, 13, 30, 15] is somehow optimal in terms of reduction of the number of sampling events. On the other

[3] The output feedback controller in [27, Example 2] can be rewritten in the from $\dot{x}_c = (A_p + B_pK - LC_p)x_c + Ly_p, u = Kx_c$ with $K = \begin{bmatrix} 1 & -4 \end{bmatrix}$, thereby showing that it is an observer-based controller.

[4] LMIs are solved with YALMIP [28] and the solver SDPT3 [29].

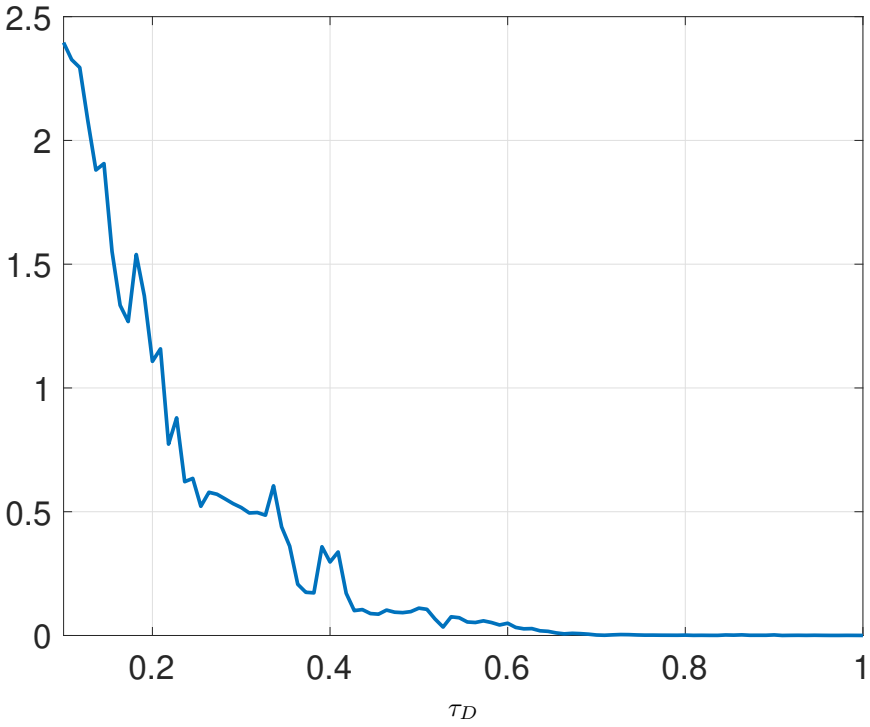

Figure 1: $\|H - A - BK\|$ vs value of the dwell time parameter τ_D.

hand, numerical experiments show that when the parameter τ_D is small, the matrix H may be unrelated to the "closed-loop" matrix $A + BK$. In Figure 1 we report the norm of the matrix $A + BK - H$ for different values of τ_D when K and H are designed via the solution to optimization problem (38). Figure 1 clearly points out that, in this example, when τ_D gets larger $A + BK \approx H$. It is interesting to notice that for $\tau_D = 0.1$, $\text{spec}(H) = \{0.18 \pm 1.5i\}$, i.e., the flow dynamics of the holding device are exponentially unstable.

To show the effectiveness of the proposed optimization strategy, we compare the above optimal controller and a controller obtained directly via the conditions in Proposition 5.2. To this end, we performed 100 simulations[5] over a horizon of

[5]Simulations of hybrid systems are performed in Matlab via the *Hybrid Equations (HyEQ) Toolbox* [31].

50 seconds. In these simulations, $x(0,0) = (\xi, 0, 0, 0)$ and ξ is selected randomly on a ball of radius 100. In Table 1, we report the average number of sampling events $\overline{N}_u$ and the average of the average inter-event time $\overline{T}_D$. Numerical results show that the selected nonoptimal controller leads to almost periodic sampling with an average sampling period $\overline{T}_D \approx \tau_D$. On the other hand, the controller designed via the proposed optimization leads to a larger average sampling period. Namely, in this example, the proposed optimal design enables one to decrease the average sampling rate about 73% with respect to the dwell-time τ_D.

Controller	$\overline{N}_u$	$\overline{T}_D$
Optimal	20	1.7339
Nonoptimal	39	1.1755

Table 1: Numerical values of the average dwell-time $\overline{T}_D$ and of the average number of sampling updates $\overline{N}_u$ for 100 simulations with random plant initial conditions.

7.2. Numerical Simulations

In Figure 2, we report a numerical simulation of a solution ϕ to the closed-loop system, for the controller parameters in (45). The picture clearly points out that the proposed design strategy leads to a dramatic reduction of the number of sampling events compared to a periodic implementation with sampling time $\tau_D = 1$.

To illustrate the results in Section 6, in Figure 3 we report a numerical simulation of a solution ϕ to hybrid system (40) in the presence of a constant noise signal for different amplitudes of the noise. As foreseen by our ISS analysis, $|\phi|_{\mathcal{A}}$ is bounded and converges in a neighborhood of zero whose size depends on the noise amplitude.

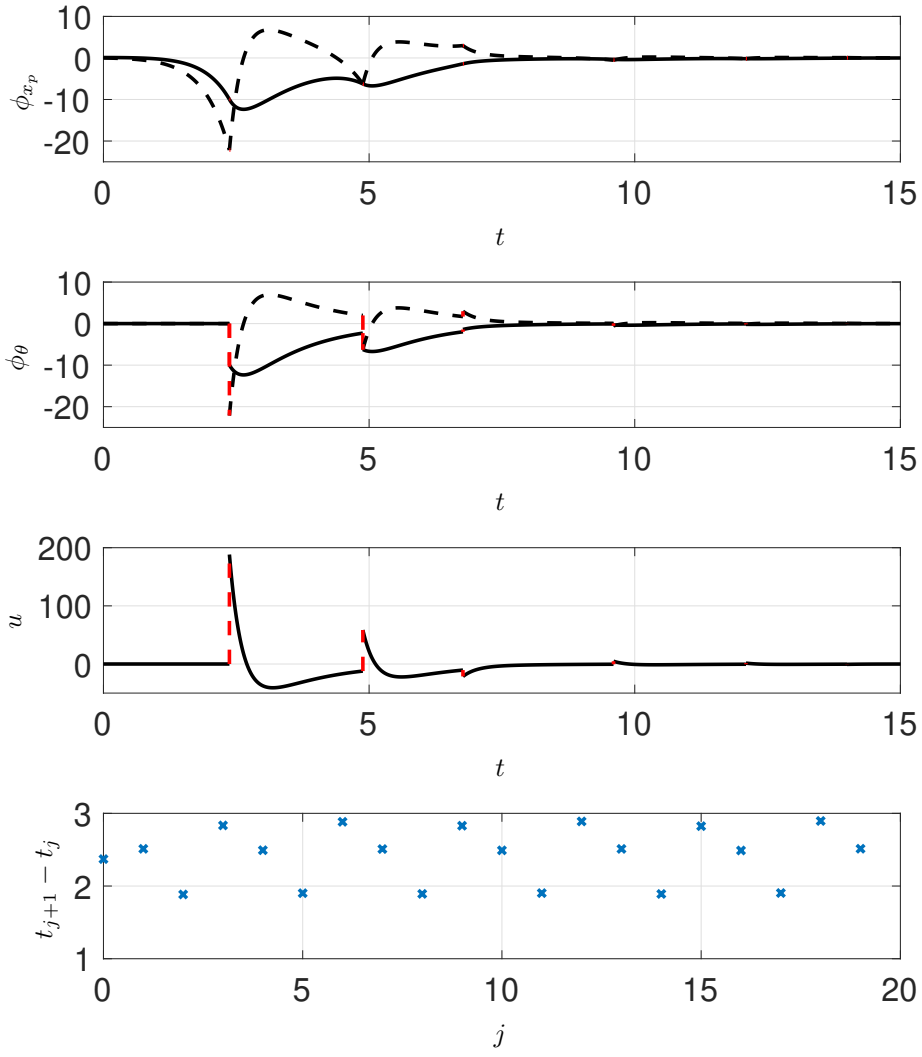

Figure 2: A numerical simulation of the closed-loop system from the initial condition $(0.0998, -0.0066, 0, 0, 0, 0)$ for the gains in (45). For each state trajectory, solid line indicates the first component and dashed line the second component. The plot at the bottom indicates the duration of each flow interval.

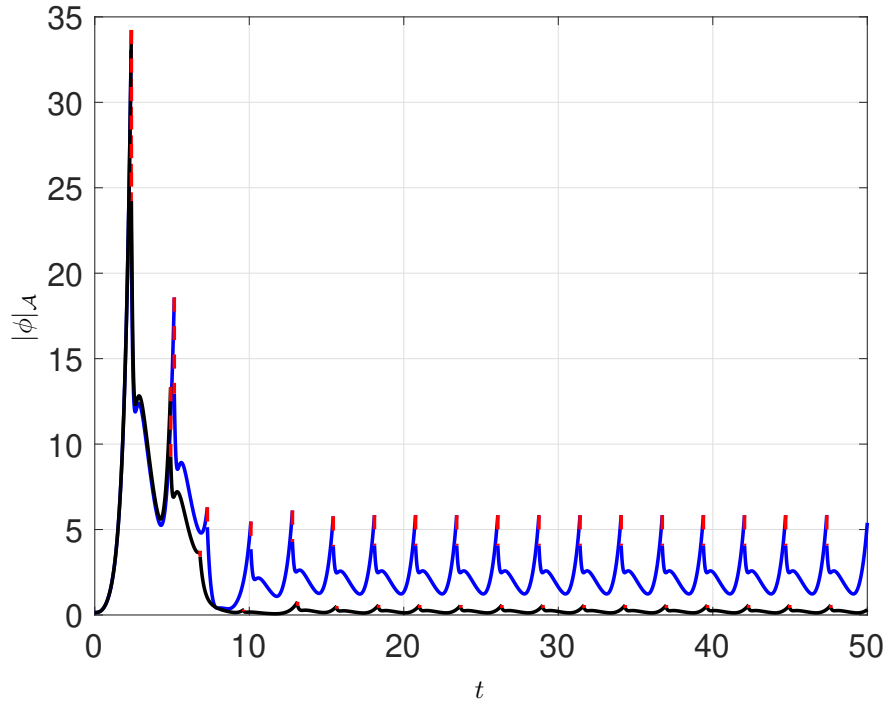

Figure 3: Numerical simulation of the closed-loop system from the initial condition $(0.0998, -0.0066, 0, 0, 0, 0)$ in the presence of measurement noise for different noise amplitudes: 0.01 (blue line) and 0.001 (black line).

8. Conclusion

In this paper, robust observer-based event-triggered control design has been investigated. Both emulation and co-design problems have been addressed by allowing tunable generalized holding devices. Such holding devices include classical zero-order-hold. Sufficient conditions for closed-loop global exponential stability are given in the form of matrix inequalities. With the objective of reducing the number of updates on the control input, the design of the proposed controller architecture, including the holding devices, is turned into the solution to some optimization problems. This leads to a systematic computationally affordable methodology for the optimal design of the considered event-triggered controller. The effectiveness of the methodology has been illustrated in an example borrowed from the literature. Future research directions include the extension of the proposed approach to more general output feedback dynamic controllers and to scenarios in which the communication between sensors and controllers happens in an event-triggered fashion.

A. Ancillary results and proofs

Proof of Property 3.1. Existence of nontrivial solutions to (14) from any initial condition in $\xi \in \widehat{\mathcal{C}} \cup \widehat{\mathcal{D}}$ follows from the satisfaction of property (VC) in [21, Proposition 2.10] for all $\xi \in \widehat{\mathcal{C}} \setminus \widehat{\mathcal{D}}$. Such a property can be easily proven by observing that flowing from any point in $\widehat{\mathcal{C}} \setminus \widehat{\mathcal{D}} = (\operatorname{int} \widehat{\mathcal{F}} \times [0, 2\tau_D]) \cup (\mathbb{R}^{3n_p} \times [0, \tau_D))$ is always possible. To prove item (a), first notice that hybrid system (14) satisfies the so-called *hybrid basic conditions*; see [21, Assumption 6.5]. Indeed, $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ are closed sets and f and g are continuous functions. Thus, since $g(\widehat{\mathcal{D}}) \subset \widehat{\mathcal{C}}$, from [21, Proposition 6.10], it follows that ϕ is either complete or blows up in finite time. On the other hand, since f is locally Lipschitz continuous, finite escape times cannot occur. This proves item (a). To conclude the proof, item (b) follows directly from the construction of the dynamics of the timer τ, which enforces a strictly positive uniform lower bound, i.e., τ_D on the length of any flow interval of ϕ. □

Theorem A.1. *Consider system (1). Let $\mathcal{A} \subset \mathbb{R}^n$ be closed. Suppose that there exist a locally Lipschitz continuous function $\mathscr{W} : \mathbb{R}^{3n_p+1} \to \mathbb{R}$ and positive scalars $\alpha_1, \alpha_2, \alpha_3$, and p such that:*

$$\alpha_1 |x|_{\mathcal{A}}^p \leq \mathscr{W}(x) \leq \alpha_2 |x|_{\mathcal{A}}^p \qquad \forall x \in \mathcal{C} \cup \mathcal{D} \tag{A.1a}$$

$$\sup_{\chi \in \partial \mathscr{W}(x), s \in \{f(x)\} \cap T_{\mathcal{C}}(x)} \langle \chi, s \rangle \leq -\alpha_3 \mathscr{W}(x) \qquad \forall x \in \mathcal{C} \tag{A.1b}$$

$$\mathscr{W}(g(x)) \leq \mathscr{W}(x) \qquad \forall x \in \mathcal{D} \tag{A.1c}$$

Assume that any maximal solution to (1) is complete. If there exist $N > 0$ and 405 *$\tau_D > 0$ such that, for each maximal solution ϕ to (1), $(t, j) \in \operatorname{dom} \phi$ implies $j \leq \frac{1}{\tau_D} t + N$. Then, $\mathcal{A}$ is globally exponentially stable for (1).*

Proof. The proof follows the same lines as [15, proof of Theorem 1]. In particular, let us define the following restriction of (1):

$$\mathcal{H}_r \begin{cases} \dot{x} \in \{f(x)\} \cap T_{\mathcal{C}}(x) & x \in \mathcal{C} \\ x^+ = g(x) & x \in \mathcal{D} \end{cases}$$

and observe that (1) and $\mathcal{H}_r$ have the same (nontrivial) solutions; see for example [32]. Pick any maximal solution to $\mathcal{H}_r$. Then, following the same steps as in [15, proof of Theorem 1] it can be shown that (A.1b) and (A.1c) give:

$$\mathscr{W}(\phi(t, j)) \leq e^{-\alpha_3 t} \mathscr{W}(\phi(0, 0)) \qquad \forall (t, j) \in \operatorname{dom} \phi$$

which thanks to (A.1a) yields

$$|\phi(t, j)|_{\mathcal{A}} \leq \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{p}} e^{-\frac{\alpha_3}{p} t} |\phi(0, 0)|_{\mathcal{A}} \quad \forall (t, j) \in \operatorname{dom} \phi \tag{A.2}$$

Now, pick $\gamma = \frac{\alpha_3 \tau_D}{1 + \tau_D}$ and $M = \gamma N$. Then, it can be easily shown that for any maximal solution ϕ to $\mathcal{H}$, $(t, j) \in \operatorname{dom} \phi$ implies:

$$-\alpha_3 t \leq M - \gamma (t + j)$$

Combining the above bound with (A.2) gives, for all $(t, j) \in \operatorname{dom} \phi$

$$|\phi(t, j)|_{\mathcal{A}} \leq e^M \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{p}} e^{-\frac{\alpha_3 \gamma (t+j)}{p}} |\phi(0, 0)|_{\mathcal{A}}$$

This concludes the proof. □

The proof of the result given next follows similar steps as in the proof of Theorem A.1 and so it is omitted.

Theorem A.2. *Consider system (3). Let $\mathcal{A} \subset \mathbb{R}^n$ be closed. Suppose that there exist a locally Lipschitz continuous function $\mathscr{W} : \mathbb{R}^{3n_p+1} \to \mathbb{R}$ positive scalars $\alpha_1, \alpha_2, \alpha_3$, and p, and $\mu \in \mathcal{K}_\infty$ such that:*

$$\alpha_1 |x|_{\mathcal{A}}^p \leq \mathscr{W}(x) \leq \alpha_2 |x|_{\mathcal{A}}^p \qquad \forall x \in \Pi_{\mathbb{R}^n}(\mathcal{C}_\eta) \cup \Pi_{\mathbb{R}^n}(\mathcal{D}_\eta) \tag{A.3a}$$

$$\sup_{\substack{\chi \in \partial \mathscr{W}(x) \\ s \in \{f(x,\eta)\} \cap T_{\Pi_{\mathbb{R}^n}(\mathcal{C}_\eta)}(x)}} \langle \chi, f \rangle \leq -\alpha_3 \mathscr{W}(x) + \mu(|\eta|) \quad \forall (x,\eta) \in \mathcal{C}_\eta \tag{A.3b}$$

$$\mathscr{W}(g(x)) \leq \mathscr{W}(x) \qquad \forall (x,\eta) \in \mathcal{D}_\eta \tag{A.3c}$$

Assume that any maximal solution pair to (3) is complete and that there exist $N > 0$ and $\tau_D > 0$ such that, for each maximal solution (ϕ, η) to (3), $(t,j) \in \operatorname{dom} \phi$ implies $j \leq \frac{1}{\tau_D} t + N$. Then, hybrid system $\mathcal{H}_\eta$ defined in (3) is ISS with respect to η relatively to the set $\mathcal{A}$. In particular, (4) holds with

$$\beta(s,k) = e^{N \frac{\alpha_3 \tau_D}{1+\tau_D}} \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{p}} e^{-\frac{\frac{\alpha_3^2 \tau_D}{1+\tau_D}}{p} k} s$$

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Lemma A.1. *Let $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{F}}_n$ be defined, respectively, as in (13b) and (39). The following items hold:*

(i) Let $x \in (\mathbb{R}^{3n_p} \setminus \widehat{\mathcal{F}}) \times \{\tau_D\}$. Then

$$T_{\widehat{\mathcal{C}}}(x) \subset \mathbb{R}^{3n_p} \times \mathbb{R}_{\leq 0}$$

(ii) Let $x \in (\Pi_{\mathbb{R}^{3n_p}}(\mathbb{R}^{3n_p+n_y} \setminus \widehat{\mathcal{F}}_n) \times \{\tau_D\})$. Then

$$T_{\Pi_{\mathbb{R}^{3n_p}}(\widehat{\mathcal{C}}_n)}(x) \subset \mathbb{R}^{3n_p} \times \mathbb{R}_{\leq 0}$$

Proof. We prove only item (i), the proof of item (ii) follows the same steps. By contradiction, assume that there exists $\omega = (\omega_\zeta, \omega_\tau) \in T_{\widehat{\mathcal{C}}}(x) \cap (\mathbb{R}^{3n_p} \times \mathbb{R}_{>0})$.

Then, by definition, there exist sequences $\{(\zeta_k, \tau_k)\} \subset \widehat{\mathcal{C}}$ and $\{\sigma_k\} \subset \mathbb{R}_{>0}$ with $x_k \coloneqq (\zeta_k, \tau_k) \to x$, $\sigma_k \to 0$ such that:

$$(\omega_\zeta, \omega_\tau) = \lim_{k\to\infty} \frac{x_k - x}{\sigma_k}$$

Now observe that since $\widehat{\mathcal{F}}$ is closed, without loss of generality, one can assume that $\{\zeta_k\} \subset \mathbb{R}^{3n_p} \setminus \widehat{\mathcal{F}} = \operatorname{int} \widehat{\mathcal{J}}$. Hence, since $\{x_k\} \subset \widehat{\mathcal{C}}$, it follows that $\{\tau_k\} \subset [0, \tau_D]$. Namely, for all k, $\tau_k - \tau_D \leq 0$, which yields:

$$\omega_\tau = \lim_{k\to\infty} \frac{\tau_k - \tau_D}{\sigma_k} \leq 0$$

This contradicts the fact that $\omega \in \mathbb{R}^{3n_p} \times \mathbb{R}_{>0}$. Thus, due to $T_{\widehat{\mathcal{C}}}(x)$ being nonempty, the proof is concluded. □

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