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Probabilistic Powerdomains and Quasi-Continuous Domains

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Abstract

The probabilistic powerdomain $\mathbf{V}X$ on a space X is the space of all continuous valuations on X . We show that, for every quasi-continuous domain X , $\mathbf{V}X$ is again a quasi-continuous domain, and that the Scott and weak topologies then agree on $\mathbf{V}X$. This also applies to the subspaces of probability and subprobability valuations on X , in the first case under an assumption of pointedness. We also show that the Scott and weak topologies on $\mathbf{V}X$ may differ when X is not quasi-continuous, and we give a simple, compact Hausdorff counterexample.

1 Introduction

Continuous valuations are an alternative to measures, which are popular in computer science, and notably in the semantics of programming languages [13, 12]. The space of all continuous valuations on a topological space X is called the probabilistic powerdomain $\mathbf{V}X$ on X . It is known that the probabilistic powerdomain of a directed-complete partial order (dcpo) is a dcpo again, in short, \mathbf{V} preserves dcpos; similarly, \mathbf{V} preserves continuous dcpos, but fails to preserve complete lattices and bc-domains. All that was proved by Jones [13, 12]. It is unknown whether \mathbf{V} preserves RB-domains or FS-domains, except in special cases [14]. On the positive side, \mathbf{V} preserves stably compact spaces [14, 3], QRB-domains [8, 10], and coherent quasi-continuous dcpos [19]. (The latter two results are equivalent, since QRB-domains coincide with coherent quasi-continuous dcpos [17, 10], and also with Li and Xu's QFS-domains [18].)

Lyu and Kou [19] asked whether coherence was required, in other words, whether \mathbf{V} preserves quasi-continuous, not necessarily coherent, dcpos. We show that this is indeed the case, and that, in this case, the Scott and weak topologies agree on the probabilistic powerdomain.

We will present our main theorem and its proof in Section 2. We use a similar strategy as Lyu and Kou [19, Theorem 3.3]. The crucial new ingredient that allows us to get rid of coherence is a useful lemma on Choquet capacities, which we establish in Section 4 by appealing to von Neumann’s classical minimax theorem. We also use some material due to Heckmann on so-called point-continuous valuations, which will greatly simplify some arguments, and which we recall in Section 3, right after a first section on preliminaries (Section 2).

Once we have established the main theorem, we refine it to deal with the case of probability continuous valuations in Section 6.

Since every continuous dcpo is quasi-continuous, the coincidence of the Scott and weak topologies on $\mathbf{V}X$, where X is quasi-continuous, generalizes a result of Kirch [16, Satz 8.6], see also [20, Satz 4.10], according to which the Scott and weak topologies on $\mathbf{V}X$ agree for every continuous dcpo X . Alvarez-Manilla, Jung and Keimel asked whether they agree on $\mathbf{V}_{\leq 1}X$ for every stably compact space X [3, Section 5, second open problem]. We will show that this is not the case, through a simple, compact Hausdorff example in Section 7. Hence the situation with quasi-continuous domains is probably rather exceptional.

2 Preliminaries

We refer to [7, 9] on domain theory and point-set topology, specially non-Hausdorff topology. Compactness does not involve separation.

A *dcpo* (directed-complete partial order) is a poset P in which every directed family D has a supremum $\sup D$. A *Scott-open subset* of P is a subset U that is *upwards-closed* (for every $x \in U$ and every y such that $x \leq y$, y is in U) and is such that, for every directed family D in P , if $\sup D \in U$ then D intersects U . The Scott-open subsets of P form a topology called the *Scott topology*.

Every complete lattice is a dcpo. For example, $\overline{\mathbb{R}}_+ \stackrel{\text{def}}{=} \mathbb{R}_+ \cup \{\infty\}$, with the usual ordering that places ∞ above all non-negative real numbers, is a dcpo. The family of open subsets $\mathcal{O}X$ of a topological space is a dcpo, too.

A *Scott-continuous map* $f: P \rightarrow Q$ between dcpos is a monotonic map that preserves suprema of directed families. A map from P to Q is Scott-continuous if and only if it is continuous with respect to the Scott topologies on P and Q .

A *valuation* ν on a topological space X is a strict, modular, monotonic map from $\mathcal{O}X$ to $\overline{\mathbb{R}}_+$. That ν is *strict* means that $\nu(\emptyset) = 0$. That it is *modular* means that $\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$ for all open subsets U and V . A *continuous valuation* is a valuation that is Scott-continuous from $\mathcal{O}X$ to $\overline{\mathbb{R}}_+$.

Continuous valuations and measures are close cousins. Every τ -smooth Borel measure defines a continuous valuation by restricting it to $\mathcal{O}X$; and every Borel measure on a hereditary Lindelöf space is τ -smooth [1]. Conversely, every continuous valuation on an LCS-complete space extends to a measure on the Borel σ -algebra [5, Theorem 1.1]—an *LCS-complete* space is any subspace obtained as a G_δ subset of a locally compact sober space.

We write $\mathbf{V}X$ for the dcpo of all continuous valuations on X , ordered by the *stochastic ordering*: $\mu \leq \nu$ if and only if $\mu(U) \leq \nu(U)$ for every $U \in \mathcal{O}X$. $\mathbf{V}_{\leq 1}X$

(resp., \mathbf{V}_1X) is the subdcpo of all *subprobability* (resp., *probability*) continuous valuations ν , namely those such that $\nu(X) \leq 1$ (resp., $\nu(X) = 1$). We will usually write \mathbf{V}_*X to denote any of those dcpos, where $*$ stands for nothing, “ ≤ 1 ”, or “1”.

Unless said otherwise, we will equip \mathbf{V}_*X not with the Scott topology of \leq , rather with the *weak topology*, defined as the coarsest one that makes $[r \ll U]_* \stackrel{\text{def}}{=} \{\nu \in \mathbf{V}_*X \mid r \ll \nu(U)\}$ open for every $r \in \mathbb{R}_+$ and every $U \in \mathcal{O}X$. Here \ll is the so-called way-below relation on $\overline{\mathbb{R}}_+$; we have $r \ll s$ if and only if $r = 0$ or $r < s$. The sets $[U > r]_*$ with $U \in \mathcal{O}X$ and $r \in \mathbb{R}_+ \setminus \{0\}$ form another subbase of the weak topology, since $[U > r]_* = [r \ll U]_*$ if $r \neq 0$, and $[0 \ll U]_* = \mathbf{V}_*X$. The weak topology is coarser than the Scott topology of the stochastic ordering \leq .

Every topological space X has a *specialization preordering* \leq , defined by $x \leq y$ if and only if every open neighborhood of x contains y . A T_0 space is one such that \leq is an ordering. As examples, for every dcpo P , ordered by \leq , the specialization preordering of P with its Scott topology is \leq ; and the specialization preordering of \mathbf{V}_*X is the stochastic ordering.

For every point $x \in X$, the closure of $\{x\}$ coincides with the downward closure $\downarrow x \stackrel{\text{def}}{=} \{y \in X \mid y \leq x\}$ of x in the specialization preordering. In general, we write $\downarrow A$ for the downward closure of a set A , so that $\downarrow x = \downarrow \{x\}$.

A subset A of a space X is *saturated* if and only if it is equal to the intersection of its open neighborhoods, equivalently if it is upwards-closed with respect to the specialization preordering \leq . We write $\uparrow A$ for the upward closure of A .

For every compact subset K of X , $\uparrow K$ is compact saturated. This is the case in particular if K is finite: we call the sets of the form $\uparrow E$, with E finite, *finitary compact*. A space X is *locally finitary compact* if and only if it has a base consisting of interiors $\text{int}(\uparrow E)$ of finitary compact sets.

The standard definition of a quasi-continuous dcpo is through the notion of a so-called way-below relation between finite subsets. We will instead use the following characterization [9, Exercise 8.3.39]: the quasi-continuous dcpos are exactly the locally finitary compact, sober spaces. The topology is exactly the Scott topology of the specialization ordering \leq .

We have mentioned sober spaces a few times already. A closed subset C of a space X is *irreducible* if and only if it is non-empty and, for all closed subsets C_1 and C_2 of X , if $C \subseteq C_1 \cup C_2$ then $C \subseteq C_1$ or $C \subseteq C_2$. The closures $\downarrow x$ of points are always irreducible closed. A *sober space* is any T_0 space in which the only irreducible closed subsets are closures of points. $\overline{\mathbb{R}}_+$ is sober in its Scott topology. Every quasi-continuous dcpo is sober in its Scott topology (by our definition), every Hausdorff space is sober; also, $\mathbf{V}X$ is sober in the weak topology, for every space X [11, Proposition 5.1].

The sober subspaces Y of a sober space X are exactly those that are closed in the so-called *Skula*, or *strong* topology on X [15, Corollary 3.5]. That topology is the coarsest one that contains both the original open and the original closed sets as open sets. We note that $\mathbf{V}_{\leq 1}X$ is closed in $\mathbf{V}X$, being the complement of $[X > 1]$. Every closed set is Skula-closed, so $\mathbf{V}_{\leq 1}X$ is also a sober space. Also,

$\mathbf{V}_1 X$ is the intersection of the closed set $\mathbf{V}_{\leq 1} X$ with the open sets $[X > 1 - \epsilon]$, $\epsilon > 0$, hence is also Skula-closed and therefore sober as well.

The forgetful functor from the category of sober spaces and continuous maps to the category of topological spaces has a left adjoint called *sobrification*. Explicitly, this means that every topological space X has a sobrification X^s , which is a sober topological space; there is a continuous map $\eta_X: X \rightarrow X^s$, called the *unit*; and every continuous map $f: X \rightarrow Y$ where Y is sober extends to a unique continuous map $\hat{f}: X^s \rightarrow Y$, in the sense that $\hat{f} \circ \eta_X = f$. Concretely, X^s can be realized as the space of all irreducible closed subsets of X , with a suitable topology, and $\eta_X(x) \stackrel{\text{def}}{=} \downarrow x$. By Proposition 3.4 of [15], given any subspace Y of a sober space X , the Skula-closure $cl_s(Y)$ of Y in X is also a sobrification of Y , with η_Y defined as the inclusion map. In general, for a T_0 space Y , and a sober space X , together with a continuous map $f: Y \rightarrow X$, X is a sobrification of Y with unit f if and only if f is a topological embedding, with Skula-dense image [15, Proposition 3.2].

3 Simple and point-continuous valuations

Among all the continuous valuations that exist on a space X , the *simple valuations* are those of the form $\sum_{x \in A} a_x \delta_x$, where A is a finite subset of X , $a_x \in \mathbb{R}_+$, and δ_x is the Dirac valuation, defined by $\delta_x(U) \stackrel{\text{def}}{=} 1$ if $x \in U$, 0 otherwise. We let $\mathbf{V}_{*,f} X$ be the subspace of $\mathbf{V}_* X$ that consists of its simple valuations.

Heckmann characterized the sobrification of $\mathbf{V}_f X$ as being the space $\mathbf{V}_p X$ of so-called *point-continuous valuations* on X [11, Theorem 5.5], together with inclusion as unit. Those are the valuations ν on X that are continuous from $\mathcal{O}_p X$ to \mathbb{R}_+ . $\mathcal{O}_p X$ is the lattice of open subsets of X with the *point topology*, namely the coarsest topology that makes $\{U \in \mathcal{O} X \mid x \in U\}$ open for every point $x \in X$. We write $\mathbf{V}_{*,p} X$ for the usual variants. The point topology is coarser than the Scott topology, so every point continuous valuation is Scott-continuous. We equip $\mathbf{V}_{*,p} X$ with the subspace topology from $\mathbf{V}_* X$, and we call it the *weak topology* again.

Lemma 3.1. *Let X be a topological space. $\mathbf{V}_f X$ is Skula-dense in $\mathbf{V}_p X$.*

Proof. By Proposition 3.2 of [15], cited earlier: since $\mathbf{V}_p X$ is a sobrification of $\mathbf{V}_f X$, with unit given by the inclusion map i , the image of i must be Skula-dense. \square

Lemma 3.2. *Let X be a topological space and \mathcal{U} be an open subset of $\mathbf{V}_p X$. For every $\nu \in \mathcal{U}$, there is a simple valuation ν' in \mathcal{U} such that $\nu' \leq \nu$.*

Proof. $\mathcal{U} \cap \downarrow \nu$ is open in the Skula topology of $\mathbf{V}_p X$, and is non-empty, since it contains ν . Using Lemma 3.1, it must contain an element ν' of $\mathbf{V}_f X$. \square

Heckmann also showed that, when X is locally finitary compact, *all* continuous valuations are point-continuous, hence $\mathbf{V} X = \mathbf{V}_p X$ [11, Theorem 4.1]. Using that information, we obtain the following.

Lemma 3.3. *Let X be a locally finitary compact space, \mathcal{U} be an open subset of \mathbf{V}_*X , where $*$ is nothing or “ ≤ 1 ”. For every $\nu \in \mathcal{U}$, there is a simple valuation ν' in \mathcal{U} such that $\nu' \leq \nu$.*

Proof. When $*$ is nothing, this is Lemma 3.2, together with the fact that $\mathbf{V}X = \mathbf{V}_pX$.

When $*$ is “ ≤ 1 ”, we use the definition of the weak topology: ν is in some finite intersection $\bigcap_{i=1}^m [U_i > r_i]_{\leq 1}$ of subbasic open sets included in \mathcal{U} . Then ν is also in the corresponding finite intersection $\bigcap_{i=1}^m [U_i > r_i]$ of subbasic open sets of $\mathbf{V}X$. We have just seen that there is a simple valuation $\nu' \leq \nu$ in $\bigcap_{i=1}^m [U_i > r_i]$. Since $\nu' \leq \nu$, ν' is a subprobability valuation, so ν' is in $\bigcap_{i=1}^m [U_i > r_i]_{\leq 1}$, hence in \mathcal{U} . \square

4 Capacities

Capacities are a generalization of valuations (or measures) introduced by Choquet [4], where modularity is abandoned in favor of weaker properties. We will need the following kind.

Given a subset B of a topological space, the *unanimity game* $\mathbf{u}_B: \mathcal{O}X \rightarrow \overline{\mathbb{R}}_+$ maps every open set U to 1 if $B \subseteq U$, to 0 otherwise. When $B = \{x\}$, \mathbf{u}_B is simply the Dirac valuation δ_x , but in general \mathbf{u}_B is not modular.

We will consider functions κ of the form $\sum_{i=1}^n a_i \mathbf{u}_{B_i}$, where $n \in \mathbb{N}$, and for each i with $1 \leq i \leq n$, a_i is a number in \mathbb{R}_+ and B_i is a finite non-empty subset of X . We call such functions *simple capacities* here.

We compare capacities, and in general all functions from $\mathcal{O}X$ to $\overline{\mathbb{R}}_+$, by $\kappa \leq \nu$ if and only if $\kappa(U) \leq \nu(U)$ for every $U \in \mathcal{O}X$, extending the stochastic ordering from continuous valuations to all maps.

Below, the indices will really be taken from some finite subset A of X , and we will consider simple capacities written as $\sum_{x \in A} a_x \mathbf{u}_{B_x}$. In this setting, an element f of $\Sigma \stackrel{\text{def}}{=} \prod_{x \in A} B_x$ is a function that maps each point $x \in A$ to an element $f(x)$ in B_x . One can think of such functions f as *strategies* for picking an element of B_x for each $x \in A$. We let Δ_Σ be the set of all families $\vec{\beta} \stackrel{\text{def}}{=} (\beta_f)_{f \in \Sigma}$ of non-negative real numbers such that $\sum_{f \in \Sigma} \beta_f = 1$. Δ_Σ is simply the standard n -simplex $\Delta_n \stackrel{\text{def}}{=} \{(\beta_1, \dots, \beta_n) \in \mathbb{R}_+^n \mid \sum_{i=1}^n \beta_i = 1\}$, where n is the cardinality of Σ .

In order to show the following lemma, we will need to introduce the Choquet integral $\int_{x \in X} h(x) d\nu$ of a lower semicontinuous map $h: X \rightarrow \overline{\mathbb{R}}_+$ with respect to a monotone set function $\nu: \mathcal{O}X \rightarrow \overline{\mathbb{R}}_+$. By definition, this is equal to the Riemann integral $\int_0^\infty \nu(h^{-1}([t, \infty])) dt$. Note that this makes sense, because $h^{-1}([t, \infty])$ is open for every $t \in \mathbb{R}_+$, and because every non-increasing map is Riemann-integrable. In our setting, this form of the Choquet integral was introduced by Tix [20], and differs only slightly from Choquet’s original definition [4, Section 48]. Tix proved that, when ν is a continuous valuation, $\int_{y \in X} h(y) d\nu$ is linear and Scott-continuous in h [20, Satz 4.4]. It is an easy exercise to ver-

ify that $\int_{y \in X} \chi_U(y) d\nu = \nu(U)$ for every open subset U of X , where χ_U is the characteristic map of U . It follows that, when h is of the form $\sum_{j=1}^m \alpha_j \chi_{U_j}$, $\int_{y \in X} h(y) d\nu = \sum_{j=1}^m \alpha_j \nu(U_j)$.

For a simple capacity $\kappa \stackrel{\text{def}}{=} \sum_{x \in A} a_x \mathbf{u}_{B_x}$, we compute $\int_{y \in X} h(y) d\kappa$ as follows. For each $x \in A$, $\int_{y \in X} h(y) d\mathbf{u}_{B_x} = \int_0^\infty \mathbf{u}_{B_x}(h^{-1}([t, \infty])) dt$ by the Choquet formula. But $\mathbf{u}_{B_x}(h^{-1}([t, \infty])) = 1$ if and only if $B_x \subseteq h^{-1}([t, \infty])$, if and only if $\min_{y \in B_x} h(y) > t$. It follows that $\int_{y \in X} h(y) d\mathbf{u}_{B_x} = \min_{y \in B_x} h(y)$. Hence $\int_{y \in X} h(y) d\kappa = \sum_{x \in A} a_x \min_{y \in B_x} h(y)$.

Lemma 4.1. *Let X be a topological space, and $\kappa \stackrel{\text{def}}{=} \sum_{x \in A} a_x \mathbf{u}_{B_x}$ be a simple capacity on X . Let $\Sigma \stackrel{\text{def}}{=} \prod_{x \in A} B_x$.*

Let ν be any bounded continuous valuation on X . If $\kappa \leq \nu$, then, for some $\vec{\beta} \in \Delta_\Sigma$, $\sum_{f \in \Sigma, x \in A} \beta_f a_x \delta_{f(x)} \leq \nu$.

Proof. This is a consequence of von Neumann's original minimax theorem [21], which says that given any $n \times m$ matrix M with real entries, where $m, n \geq 1$,

$$\min_{\vec{\alpha} \in \Delta_m} \max_{\vec{\beta} \in \Delta_n} \vec{\beta} M \vec{\alpha}^\top = \max_{\vec{\beta} \in \Delta_n} \min_{\vec{\alpha} \in \Delta_m} \vec{\beta} M \vec{\alpha}^\top. \quad (1)$$

In particular: (†) if for every $\vec{\alpha} \in \Delta_m$, there is a $\vec{\beta} \in \Delta_n$ such that $\vec{\beta} M \vec{\alpha}^\top \geq 0$ (namely, if the left-hand side of (1) is non-negative), then there is a $\vec{\beta} \in \Delta_n$ such that, for every $\vec{\alpha} \in \Delta_m$, $\vec{\beta} M \vec{\alpha}^\top \geq 0$.

We first show that: (*) given finitely many open subsets U_1, \dots, U_m of X , we can find a $\vec{\beta} \in \Delta_\Sigma$ such that, for every j , $1 \leq j \leq m$, $\sum_{f \in \Sigma, x \in A} \beta_f a_x \delta_{f(x)}(U_j) \leq \nu(U_j)$. This is clear if $m = 0$, so let us assume $m \geq 1$.

Let κ denote $\sum_{x \in A} a_x \mathbf{u}_{B_x}$. Since $\kappa \leq \nu$, for every lower semicontinuous map h , $\int_{y \in X} h(y) d\kappa = \int_0^\infty \kappa(h^{-1}([t, \infty])) dt \leq \int_0^\infty \nu(h^{-1}([t, \infty])) dt = \int_{y \in X} h(y) d\nu$. In other words, $\sum_{x \in A} a_x \min_{y \in B_x} h(y) \leq \int_{y \in X} h(y) d\nu$.

For every $\vec{\alpha} \in \Delta_m$, we consider $h_{\vec{\alpha}} \stackrel{\text{def}}{=} \sum_{j=1}^m \alpha_j \chi_{U_j}$ for h . The inequality we have just shown can be rewritten as $\sum_{x \in A} a_x \min_{y \in B_x} h_{\vec{\alpha}}(y) \leq \sum_{j=1}^m \alpha_j \nu(U_j)$. For each $x \in A$, there is an element $y \in B_x$ that makes $h_{\vec{\alpha}}(y)$ minimal, and we call it $f_{\vec{\alpha}}(x)$. Therefore $\sum_{x \in A} a_x h_{\vec{\alpha}}(f_{\vec{\alpha}}(x)) \leq \sum_{j=1}^m \alpha_j \nu(U_j)$. By definition of $h_{\vec{\alpha}}$, and since $\chi_{U_j}(f_{\vec{\alpha}}(x)) = \delta_{f_{\vec{\alpha}}(x)}(U_j)$, this can be written equivalently as $\sum_{x \in A} \sum_{j=1}^m \alpha_j a_x \delta_{f_{\vec{\alpha}}(x)}(U_j) \leq \sum_{j=1}^m \alpha_j \nu(U_j)$. It follows that there is a vector $\vec{\beta}$ in Δ_Σ such that $\sum_{j=1}^m \alpha_j \nu(U_j) - \sum_{j=1}^m \alpha_j \sum_{f \in \Sigma, x \in A} \beta_f a_x \delta_{f(x)}(U_j) \geq 0$: namely, $\beta_f \stackrel{\text{def}}{=} 1$ if $f = f_{\vec{\alpha}}$, and $\beta_f \stackrel{\text{def}}{=} 0$ otherwise.

That can also be written as $\sum_{f \in \Sigma, 1 \leq j \leq m} \alpha_j \beta_f (\nu(U_j) - \sum_{x \in A} a_x \delta_{f(x)}(U_j)) \geq 0$, hence as $\vec{\beta} M \vec{\alpha}^\top \geq 0$ for some matrix M . Using (†), there is a vector $\vec{\beta} \in \Delta_\Sigma$ such that, for every $\vec{\alpha} \in \Delta_m$, $\vec{\beta} M \vec{\alpha}^\top \geq 0$, in other words $\sum_{j=1}^m \alpha_j \nu(U_j) - \sum_{j=1}^m \alpha_j \sum_{f \in \Sigma, x \in A} \beta_f a_x \delta_{f(x)}(U_j) \geq 0$. In particular, for each j with $1 \leq j \leq m$, taking $\vec{\alpha}$ such that $\alpha_j \stackrel{\text{def}}{=} 1$ and all its other components are 0, $\nu(U_j) \geq \sum_{f \in \Sigma, x \in A} \beta_f a_x \delta_{f(x)}(U_j)$. This proves (*).

For every finite family \mathcal{A} of open subsets of X , let $C_{\mathcal{A}}$ be the set of vectors $\vec{\beta} \in \Delta_{\Sigma}$ such that $\sum_{f \in \Sigma, x \in A} \beta_f a_x \delta_{f(x)}(U) \leq \nu(U)$ for every $U \in \mathcal{A}$. Claim (*) above states that $C_{\mathcal{A}}$ is non-empty. It is also a closed subset of Δ_{Σ} , equipped with the Euclidean topology. The family $(C_{\mathcal{A}})_{\mathcal{A} \in \mathbb{P}_{\text{fin}}(\mathcal{O}X)}$ then has the finite intersection property: given any finite collection of elements $\mathcal{A}_1, \dots, \mathcal{A}_k$ in $\mathbb{P}_{\text{fin}}(\mathcal{O}X)$, $\bigcap_{i=1}^k C_{\mathcal{A}_i} = C_{\bigcup_{i=1}^k \mathcal{A}_i}$ is non-empty. Since Δ_{Σ} is compact, the intersection $\bigcap_{\mathcal{A} \in \mathbb{P}_{\text{fin}}(\mathcal{O}X)} C_{\mathcal{A}}$ is non-empty. Let $\vec{\beta}$ be any vector in that intersection. For every $U \in \mathcal{O}X$, since $\vec{\beta}$ is in $C_{\{U\}}$, we have $\sum_{f \in \Sigma, x \in A} \beta_f a_x \delta_{f(x)}(U) \leq \nu(U)$, and we conclude. \square

5 The main theorem

We come to our main theorem. It applies in particular to every quasi-continuous dcpo X , namely to every locally finitary compact sober space, as we have announced; but sobriety is not needed. We spend the rest of the section proving it.

Theorem 5.1. *For every locally finitary compact space X , $\mathbf{V}X = \mathbf{V}_p X$ and $\mathbf{V}_{\leq 1} X = \mathbf{V}_{\leq 1, p}$ are compact, locally finitary compact, sober spaces. In particular, they are quasi-continuous dcpos and the weak topology coincides with the Scott topology.*

The sets of the form $\text{int}(\uparrow E)$, where E ranges over the finite non-empty sets of simple (resp., simple subprobability) valuations form a base of the topology.

Let $*$ be nothing or “ ≤ 1 ”. We recall that the equality $\mathbf{V}X = \mathbf{V}_p X$ holds for every locally finitary compact space X , as shown by Heckmann [11, Theorem 4.1]. The equality $\mathbf{V}_{\leq 1} X = \mathbf{V}_{\leq 1, p}$ immediately follows from it.

We also recall that the quasi-continuous dcpos are exactly the locally finitary compact sober spaces, and in particular that their topology must be the Scott topology. The fact that $\mathbf{V}_* X$ is compact follows from the fact that it has a least element in the stochastic ordering, namely the zero valuation: every open cover $(\mathcal{U}_i)_{i \in I}$ of $\mathbf{V}_* X$ must be such that some \mathcal{U}_i contains the zero valuation, and therefore coincide with the whole of since open sets are upwards-closed.

Finally, we recall that $\mathbf{V}_* X$ is sober.

Therefore, it remains to show that $\mathbf{V}_* X$ is locally finitary compact. In the rest of this section, we fix $\nu \in \mathbf{V}_* X$, and an open neighborhood \mathcal{U} of ν in the weak topology. Then ν is in some finite intersection $\bigcap_{i=1}^n [U_i > r_i]_*$ included in \mathcal{U} , where each U_i is open in X and $r_i \in \mathbb{R}_+ \setminus \{0\}$. We will find a finite set E of simple valuations and an open subset \mathcal{V} of $\mathbf{V}_* X$ such that $\nu \in \mathcal{V} \subseteq \uparrow E \subseteq \mathcal{U}$.

Let us simplify the problem slightly. By Lemma 3.3, there is a simple valuation $\nu' \leq \nu$ in $\bigcap_{i=1}^n [U_i > r_i]_*$. Hence, without loss of generality, we may assume that ν itself is a simple valuation $\sum_{x \in A} a_x \delta_x$, where A is a finite subset of X , and $a_x \in \mathbb{R}_+ \setminus \{0\}$ for every $x \in A$.

Since $\nu(U_i) > r_i$ for every i , $1 \leq i \leq n$, there is a number $a \in]0, 1[$ such that $a \cdot \nu(U_i) > r_i$ for every i . There is also a positive number s_i such that

$a \cdot \nu(U_i) > s_i > r_i$. We will need those numbers a and s_i only near the end of the proof.

Let us define a suitable open set \mathcal{V} . For each point $x \in A$, let $I_x \stackrel{\text{def}}{=} \{i \in I \mid x \in U_i\}$. Then $\bigcap_{i \in I_x} U_i \searrow \downarrow(A \searrow \uparrow x)$ is an open neighborhood of x .

Since X is locally finitary compact, for each $x \in A$, one can find a finite set B_x such that $x \in \text{int}(\uparrow B_x) \subseteq \uparrow B_x \subseteq \bigcap_{i \in I_x} U_i \searrow \downarrow(A \searrow \uparrow x)$. We will require a bit more, and we will make sure that B_x is also included in $\text{int}(\uparrow B_y)$ for every $y \in A$ such that $y < x$. This can be done by finding B_x in stages, starting from the lowest elements x of A and going up. Formally, since A is finite, we define B_x by course-of-values induction on the number of elements $y \in A$ such that $y < x$, as follows: for each $x \in A$, we simply find a finite set B_x such that $x \in \text{int}(\uparrow B_x) \subseteq \uparrow B_x \subseteq \bigcap_{i \in I_x} U_i \searrow \downarrow(A \searrow \uparrow x) \cap \bigcap_{y \in A, y < x} \text{int}(\uparrow B_y)$, where the last term is available, and an open neighborhood of $y \in A$ hence of x , by induction hypothesis.

We also define $V_x \stackrel{\text{def}}{=} \text{int}(\uparrow B_x)$. We note the following three facts.

Lemma 5.2. *For all $x, y \in A$ with $x < y$, $\uparrow B_y \subseteq V_x \subseteq \uparrow B_x$.*

If $x \leq y$ instead, then $\uparrow B_y \leq \uparrow B_x$. □

Lemma 5.3. *For every $x \in A$, for every $i \in I_x$, then $B_x \subseteq U_i$.* □

Lemma 5.4. *For all $x, y \in A$, $x \in V_y$ if and only if $y \leq x$.*

Proof. If $y \leq x$, and since V_y is an open neighborhood of y , and is in particular upwards-closed, x is also in V_y . If $y \not\leq x$, then x is in $A \searrow \uparrow y$, hence in $\downarrow(A \searrow \uparrow y)$. It follows that x cannot be in $\bigcap_{i \in I_y} U_i \searrow \downarrow(A \searrow \uparrow y)$, hence cannot be in the smaller set V_y . □

Definition 5.5 (\mathcal{V}). *Let $\mathbb{P}_{\uparrow} A$ denote the (finite) family of upwards-closed subsets of A . For each $B \in \mathbb{P}_{\uparrow} A$, let $V_B \stackrel{\text{def}}{=} \bigcup_{x \in B} V_x$. Let also $s_B \stackrel{\text{def}}{=} a \cdot \sum_{x \in B} a_x$. The open set \mathcal{V} is $\bigcap_{B \in \mathbb{P}_{\uparrow} A} [s_B \ll V_B]$.*

Recall that $\mu \in [s_B \ll V_B]$ if and only if $s_B \ll \mu(V_B)$, if and only if $s_B = 0$ or $s_B < \mu(V_B)$.

Lemma 5.6. $\nu \in \mathcal{V}$.

Proof. For every $B \in \mathbb{P}_{\uparrow} A$, we claim that $A \cap V_B = B$. For every $x \in B$, V_x is included in V_B , and since V_x is an open neighborhood of x , it follows that x is in V_B ; x is also in A , since $B \subseteq A$. Conversely, if $x \in A \cap V_B$, then x is in V_y for some $y \in B$. Both x and y are in A , so by Lemma 5.4, we obtain that $y \leq x$. Since B is upwards-closed, x is in B .

Let us verify that ν is in \mathcal{V} , namely that, for every $B \in \mathbb{P}_{\uparrow} A$, $s_B \ll \nu(V_B)$. Indeed, $\nu(V_B) = \sum_{x \in A \cap V_B} a_x = \sum_{x \in B} a_x$, since $A \cap V_B = B$. Now, since $a < 1$, $a \cdot \sum_{x \in B} a_x \ll \sum_{x \in B} a_x$. In other words, $s_B \ll \nu(V_B)$, as desired. □

Finding the finite set E is more difficult. As a first step in that direction, let $\kappa \stackrel{\text{def}}{=} a \cdot \sum_{x \in A} a_x u_{B_x}$, and let us consider the set \mathcal{Q} of all the continuous valuation $\mu \in \mathbf{V}_* X$ such that $\kappa \leq \mu$.

Lemma 5.7. $\mathcal{V} \subseteq \mathcal{Q}$.

Proof. Let μ be any element of \mathcal{V} . We must show that, for every open subset U of X , $a \cdot \sum_{x \in A, B_x \subseteq U} a_x \leq \mu(U)$.

Let $B \stackrel{\text{def}}{=} \{x \in A \mid B_x \subseteq U\}$. For every $x \in B$ and every $y \in A$ with $x \leq y$, we have $B_y \subseteq \uparrow B_x \subseteq U$ by Lemma 5.2, so y is in B . Hence B is upwards-closed in A .

Then the left-hand side $a \cdot \sum_{x \in A, B_x \subseteq U} a_x$ is just s_B . Since $\mu \in \mathcal{V}$, $s_B \ll \mu(V_B)$. We recall that $V_B = \bigcup_{x \in B} V_x$, that V_x is included in $\uparrow B_x$ for each x , and that (by the definition of B), $\uparrow B_x$ is included in U for every $x \in B$. Therefore $V_B \subseteq U$, and hence $\mu(V_B) \leq \mu(U)$, which concludes the proof. \square

Let $\Sigma \stackrel{\text{def}}{=} \prod_{x \in A} B_x$, and Δ_Σ be the associated standard simplex. Lemma 5.7, together with Lemma 4.1, immediately implies the following.

Lemma 5.8. *Every element μ of \mathcal{V} is above a simple valuation of the form $a \cdot \sum_{f \in \Sigma, x \in A} \beta_f a_x \delta_{f(x)}$, for some $\vec{\beta} \in \Delta_\Sigma$.*

Let E_0 be the set of simple valuations obtained this way, namely the set of simple valuations $a \cdot \sum_{f \in \Sigma, x \in A} \beta_f a_x \delta_{f(x)}$, where $\vec{\beta} \in \Delta_\Sigma$. We have just shown that every element μ of \mathcal{V} is above some element of E_0 .

Note that the elements ϖ of E_0 can all be written as $\sum_{z \in Z} c_z \delta_z$, where $Z \stackrel{\text{def}}{=} \bigcup_{x \in A} B_x$, and $c_z \in \mathbb{R}_+$. For each such ϖ , let $\overline{\varpi}$ be $\sum_{z \in Z} \frac{1}{N} \lfloor N c_z \rfloor \delta_z$, where N is a fixed, large enough (in particular, non-zero) natural number that we will determine shortly.

Definition 5.9 (E). *E is the set of all simple valuations $\overline{\varpi}$, where ϖ ranges over E_0 .*

Lemma 5.10. *E is a finite set.*

Proof. Z is finite and the coefficients $\frac{1}{N} \lfloor N c_z \rfloor$ are integer multiples of $\frac{1}{N}$ between 0 and $\sum_{x \in A} a_x$. \square

Lemma 5.11. $\mathcal{V} \subseteq \uparrow E$.

Proof. For every $z \in Z$, and every $c_z \in \mathbb{R}_+$, $\frac{1}{N} \lfloor N c_z \rfloor \leq c_z$. It follows that $\overline{\varpi} \leq \varpi$ for every $\varpi \in E_0$. Since every element of \mathcal{V} is above some element ϖ of E_0 by Lemma 5.8, it is also above the corresponding element $\overline{\varpi}$ of E . \square

Lemma 5.12. $\uparrow E \subseteq \mathcal{U}$.

Proof. We show that E is included in $\bigcap_{i=1}^n [U_i > r_i]_*$. For every $x \in A$, for every $y \in B_x$, we have $\delta_y \geq \mathbf{u}_{B_x}$, simply because every open neighborhood of B_x must contain x . Hence, for every $\varpi \in E_0$, say $\varpi = a \cdot \sum_{f \in \Sigma, x \in A} \beta_f a_x \delta_{f(x)}$, where $\vec{\beta} \in \Delta_\Sigma$, we have $\varpi \geq a \cdot \sum_{f \in \Sigma, x \in A} \beta_f a_x \mathbf{u}_{B_x} = a \cdot \sum_{x \in A} (\sum_{f \in \Sigma} \beta_f) a_x \mathbf{u}_{B_x} = a \cdot \sum_{x \in A} a_x \mathbf{u}_{B_x} = \kappa$. For every i , $1 \leq i \leq n$, Lemma 5.3 states that for every $x \in A$, if $x \in U_i$ then B_x is included in U_i . Therefore $\varpi(U_i) = a \cdot \sum_{x \in A, B_x \subseteq U_i} a_x \geq$

$a. \sum_{x \in A \cap U_i} a_x = a.\nu(U_i)$. We now remember that $a.\nu(U_i) > s_i > r_i$. In particular, $\varpi(U_i) > s_i$.

It is time we fixed the value of N . The values of c_z and of $\frac{1}{N} \lfloor Nc_z \rfloor$ differ by $\frac{1}{N}$ at most, so for any open set U , the values $\varpi(U)$ and $\overline{\varpi}(U)$ differ by $\frac{1}{N}|Z|$ at most, where $|Z|$ is the cardinality of Z . It follows that $\overline{\varpi}(U_i) > s_i - \frac{1}{N}|Z|$. By picking any non-zero natural number N larger than or equal to $\frac{|Z|}{s_i - r_i}$ for every i , $1 \leq i \leq n$, we therefore ensure that $\overline{\varpi}(U_i) > r_i$ for every i , hence that $\overline{\varpi}$ is in \mathcal{U} . Since that holds for every $\varpi \in E_0$, E is included in \mathcal{U} , hence also $\uparrow E$. \square

Hence, as promised, $\nu \in \mathcal{V}$ (Lemma 5.6) $\subseteq \uparrow E$ (Lemma 5.11) $\subseteq \mathcal{U}$ (Lemma 5.12), where \mathcal{V} is open (Definition 5.5) and E is finite (Lemma 5.10). This concludes the proof of Theorem 5.1. \square

6 The case of probability continuous valuations

We now apply the previous results to the space \mathbf{V}_1X of probability continuous valuations. A space X is *pointed* if and only if it has a least element \perp in its specialization preordering. We are not assuming X to be T_0 , so $\downarrow \perp$ is a closed subset that may be different from $\{\perp\}$. The open subsets of $X \searrow \downarrow \perp$ are just the proper open subsets of X .

The following is *Edalat's lifting trick*, which was introduced in [6, Section 3] for dcpos, and in [2, Section 7.4] for stably locally compact spaces. Every continuous valuation ν on X gives rise to a continuous valuation ν^- on $X \searrow \downarrow \perp$ by $\nu^-(U) \stackrel{\text{def}}{=} \nu(U)$ for every $U \in \mathcal{O}(X \searrow \downarrow \perp)$. If $\nu \in \mathbf{V}_1X$, then ν^- is in $\mathbf{V}_{\leq 1}X$, and we have much more, as we now show.

Lemma 6.1. *Let X be a pointed topological space, with least element \perp . The map $\nu \mapsto \nu^-$ is a homeomorphism of \mathbf{V}_1X onto $\mathbf{V}_{\leq 1}(X \searrow \downarrow \perp)$. Its inverse maps every subprobability continuous valuation μ on $X \searrow \downarrow \perp$ to μ^+ , defined by $\mu^+(U) \stackrel{\text{def}}{=} \mu(U \searrow \downarrow \perp) + (1 - \mu(X \searrow \downarrow \perp))\delta_{\perp}(U)$, for every $U \in \mathcal{O}(X \searrow \downarrow \perp)$.*

Proof. Let $\nu \in \mathbf{V}_1X$. For every $U \in \mathcal{O}X$, $(\nu^-)^+(U) = \nu^-(U \searrow \downarrow \perp) + (1 - \nu^-(X \searrow \downarrow \perp))\delta_{\perp}(U)$. If U is a proper open subset of X , then U does not contain \perp , so $U \searrow \downarrow \perp = U$, and $\delta_{\perp}(U) = 0$, so $(\nu^-)^+(U) = \nu^-(U) = \nu(U)$. If $U = X$, then $(\nu^-)^+(U) = \nu^-(X \searrow \downarrow \perp) + (1 - \nu^-(X \searrow \downarrow \perp)) = 1$, and this is equal to $\nu(U)$ since $U = X$ and $\nu \in \mathbf{V}_1X$.

For every $U \in \mathcal{O}(X \searrow \downarrow \perp)$, $(\mu^+)^-(U) = \mu^+(U) = \mu(U)$, since $U \searrow \downarrow \perp = U$, and \perp is not in U .

Hence the two maps $\nu \mapsto \nu^-$ and $\mu \mapsto \mu^+$ are inverses of each other.

For every open subset U of X and every $r \in \mathbb{R}_+ \setminus \{0\}$, the inverse image of $[U > r]_1$ by $\mu \mapsto \mu^+$ is equal to one of the following sets. If $U = X$ and $r < 1$, this is the whole of $\mathbf{V}_{\leq 1}X$. If $U = X$ and $r \geq 1$, this is empty. Finally, if U is a proper subset of X , hence does not contain \perp , then this is the set of all $\mu \in \mathbf{V}_{\leq 1}(X \searrow \downarrow \perp)$ such that $\mu^+(U) > r$, where $\mu^+(U) = \mu(U)$: hence this is $[U > r]_{\leq 1}$. In any case, that inverse image is open, so $\mu \mapsto \mu^+$ is continuous.

For every open subset U of $X \searrow \downarrow \perp$, for every $r \in \mathbb{R}_+ \setminus \{0\}$, the inverse image of $[U > r]_{\leq 1}$ by $\nu \mapsto \nu^-$ is $[U > r]_1$. Therefore $\nu \mapsto \nu^-$ is continuous. \square

Lemma 6.1 allows us to obtain the following corollary to Theorem 5.1.

Corollary 6.2. *For every locally finitary compact, pointed space X , $\mathbf{V}_1 X$ is compact, locally finitary compact, and sober. In particular, it is a quasi-continuous dcpo, and the weak topology coincides with the Scott topology.*

The sets of the form $\text{int}(\uparrow E)$, where E ranges over the finite non-empty sets of simple probability valuations form a base of the topology. \square

7 The Scott and weak topologies may differ

The Scott and weak topologies on $\mathbf{V}_* X$ seem to agree in many situations, and Alvarez-Manilla, Jung and Keimel asked whether they agree on $\mathbf{V}_{\leq 1} X$ for every stably compact space X [3, Section 5, second open problem]. We show that this is not the case.

Let $\alpha(\mathbb{N})$ be the one-point compactification of the discrete space \mathbb{N} . Its elements are the natural numbers, plus a fresh element ∞ . Its open subsets are the subsets of \mathbb{N} (not containing ∞), plus all the subsets $\alpha(\mathbb{N}) \setminus E$, where E ranges over the finite subsets of \mathbb{N} . A *discrete valuation* on $\alpha(\mathbb{N})$ is any valuation of the form $\sum_{n \in \mathbb{N}} a_n \delta_n + a_\infty \delta_\infty$, where each a_n and a_∞ are in $\overline{\mathbb{R}}_+$. They are all continuous valuations.

Lemma 7.1. *Let $*$ be “ ≤ 1 ” or “1”.*

- (i) *The continuous valuations ν on $\alpha(\mathbb{N})$ are exactly the discrete valuations.*
- (ii) *The function $f: \mathbf{V}_*(\alpha(\mathbb{N})) \rightarrow Y_*$ that maps $\sum_{n \in \mathbb{N}} a_n \delta_n + a_\infty \delta_\infty$ to $(a_x)_{x \in \alpha(\mathbb{N})}$ is an order-isomorphism onto the poset Y_* of families of non-negative real numbers whose sum is at most 1 (if $*$ is “ ≤ 1 ”) or exactly 1 (if $*$ is “1”), ordered pointwise.*
- (iii) *The set \mathcal{V} of families $(a_x)_{x \in \alpha(\mathbb{N})}$ of Y_* such that $a_\infty > 0$ is Scott-open in Y_* , but $f^{-1}(\mathcal{V})$ does not contain any basic open neighborhood $\bigcap_{i=1}^n [U_i > r_i]_*$ of δ_∞ .*

Proof. (i) Let ν be any continuous valuation over $\alpha(\mathbb{N})$. We recall that every continuous valuation on an LCS-complete space extends to a measure on the Borel σ -algebra [5, Theorem 1.1]. Every locally compact sober space is G_δ in itself, hence LCS-complete. Since every Hausdorff space is sober, and clearly locally compact, $\alpha(\mathbb{N})$ is LCS-complete, and therefore ν extends to a measure $\tilde{\nu}$ on the Borel σ -algebra of $\alpha(\mathbb{N})$. It is easy to see that the latter σ -algebra is the whole of $\mathbb{P}(\mathbb{N})$. We define $a_n \stackrel{\text{def}}{=} \tilde{\nu}(\{n\}) = \nu(\{n\})$, and $a_\infty \stackrel{\text{def}}{=} \tilde{\nu}(\{\infty\})$. By σ -additivity, for every (necessarily measurable) subset E of $\alpha(\mathbb{N})$, $\tilde{\nu}(E) = \sum_{x \in E} a_x$. In particular, for every open subset U of $\alpha(\mathbb{N})$, $\nu(U) = \sum_{x \in U} a_x = (\sum_{n \in \mathbb{N}} a_n \delta_n + a_\infty \delta_\infty)(U)$.

(ii) Let ν be any element of $\mathbf{V}_*(\alpha(\mathbb{N}))$, and $\tilde{\nu}$ be a measure that extends ν to the Borel σ -algebra. We define $f(\nu)$ as $(a_x)_{x \in \alpha(\mathbb{N})}$, where a_x is defined as $\tilde{\nu}(\{x\})$, as above, so that $\nu = \sum_{n \in \mathbb{N}} a_n \delta_n + a_\infty \delta_\infty$. This defines a bijection f of $\mathbf{V}_*(\alpha(\mathbb{N}))$ onto Y_* .

Since $\{\infty\} = \bigcap_{n \in \mathbb{N}} V_n$, where V_n is the open set $\{n, n+1, \dots, \infty\}$, and since $\tilde{\nu}$ is a bounded measure, $a_\infty = \tilde{\nu}(\{\infty\}) = \inf_{n \in \mathbb{N}} \tilde{\nu}(V_n) = \inf_{n \in \mathbb{N}} \nu(V_n)$. This implies that a_∞ grows as ν grows. It is clear that $a_n = \nu(\{n\})$ grows, too, as ν grows. Therefore f is monotonic, and its inverse is clearly monotonic as well. (This discussion is superfluous when $*$ is “1”, by the way, since in that case the ordering on $\mathbf{V}_1(\alpha(\mathbb{N}))$ and on Y_1 is just equality.)

(iii) \mathcal{V} is clearly Scott-open in Y_* . We now imagine that $f^{-1}(\mathcal{V})$ contains a basic open neighborhood $\bigcap_{i=1}^n [U_i > r_i]_*$ of δ_∞ , where each U_i is open in $\alpha(\mathbb{N})$, and $r_i \in \mathbb{R}_+ \setminus \{0\}$. Since $\delta_\infty \in [U_i > r_i]_*$, U_i must contain ∞ (and $r_i < 1$), so $U_i = \alpha(\mathbb{N}) \setminus E_i$ for some finite subset E_i of \mathbb{N} . Let n be a natural number that is not in any of the finite sets E_i , $1 \leq i \leq n$. Then $\delta_n(U_i) = 1 > r_i$, so δ_n is in $\bigcap_{i=1}^n [U_i > r_i]_*$, hence in $f^{-1}(\mathcal{V})$. However, $f(\delta_n)$ is the family $(a_x)_{x \in \alpha(\mathbb{N})}$ such that $a_x = 0$ for every $x \in \alpha(\mathbb{N})$ except for $a_n = 1$; in particular, $a_\infty = 0$, showing that $f(\delta_n)$ is not in \mathcal{V} : contradiction. \square

Theorem 7.2. *Let $*$ be nothing, “ ≤ 1 ” or “1”. The Scott topology on $\mathbf{V}_*(\alpha(\mathbb{N}))$ is strictly finer than the weak topology.*

Proof. We recall that the Scott topology on any space of the form \mathbf{V}_*X is always finer than the weak topology.

When $*$ is “ ≤ 1 ” or “1”, this is Lemma 7.1, item (iii): $f^{-1}(\mathcal{V})$ is a Scott-open neighborhood of δ_∞ in $\mathbf{V}_*(\alpha(\mathbb{N}))$ that is not open in the weak topology.

When $*$ is nothing, we notice that $\mathbf{V}_{\leq 1}(\alpha(\mathbb{N}))$ is Scott-closed in $\mathbf{V}(\alpha(\mathbb{N}))$. This easily implies that the Scott topology on $\mathbf{V}_{\leq 1}(\alpha(\mathbb{N}))$ is the subspace topology induced by the Scott topology on $\mathbf{V}(\alpha(\mathbb{N}))$. If the latter agreed with the weak topology, then the Scott topology on $\mathbf{V}_{\leq 1}(\alpha(\mathbb{N}))$ would be the subspace topology induced by the inclusion in $\mathbf{V}(\alpha(\mathbb{N}))$ (with the weak topology). But that subspace topology is just the weak topology on $\mathbf{V}_{\leq 1}(\alpha(\mathbb{N}))$, and we have just seen that it differs from the Scott topology. \square

The gap between the Scott and weak topologies on $\mathbf{V}_1(\alpha(\mathbb{N}))$ is really enormous. By Corollary 37 of [3], $\mathbf{V}_{\leq 1}X$ and \mathbf{V}_1X are stably compact, in their weak topologies, for any stably compact space X . This applies to $X \stackrel{\text{def}}{=} \alpha(\mathbb{N})$, since every compact Hausdorff space is stably compact. One checks easily (e.g., by using Lemma 7.1, item (ii)) that the stochastic ordering on $\mathbf{V}_1(\alpha(\mathbb{N}))$ is simply equality, hence that the Scott topology is the discrete topology. But the only discrete spaces that are (stably) compact are finite, and $\mathbf{V}_1(\alpha(\mathbb{N}))$ is far from finite.

The coincidence of the Scott and weak topologies of Theorem 5.1, and first observed by Kirch in the case where X is a continuous dcpo, is probably exceptional. We leave open the question of the exact characterization of those spaces X for which the weak and Scott topologies agree on \mathbf{V}_*X .

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