Newton-type inertial algorithms for solving monotone equations governed by sums of potential and nonpotential operators

Samir Adly, Hedy Attouch, van Nam Vo

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Samir Adly* Hedy Attouch† Van Nam Vo‡

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Abstract. In a Hilbert space setting, we study a class of first-order algorithms which aim to solve structured monotone equations governed by sums of potential and nonpotential operators. Precisely, we are looking for the zeros of an operator $A = \nabla f + B$ where ∇f is the gradient of a differentiable convex function f, and B is a nonpotential monotone and cocoercive operator. Our study is based on the inertial autonomous dynamic previously studied by the authors to solve this type of problem, and which involves dampings which are respectively controlled by the Hessian of f, and by a Newton-type correction term attached to B. These geometric dampings attenuate the oscillations which occur with the inertial methods with viscous damping. Using Lyapunov analysis, we study the convergence properties of the proximal-gradient algorithms obtained by temporal discretization of this dynamic. These results open the door to the design of first-order accelerated algorithms in numerical optimization taking into account the specific properties of potential and nonpotential terms.

Mathematics Subject Classifications: 37N40, 46N10, 49M30, 65B99, 65K05, 65K10, 90B50, 90C25.

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1 Introduction and preliminary results

Let $\mathcal{H}$ be a real Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. Many situations coming from physics, biology, human sciences involve equations containing both potential and nonpotential terms. In human sciences, this comes from the presence of both cooperative and noncooperative aspects. In physics, this happens when the phenomena of diffusion and convection are both present. To describe such situations we will focus on solving the additively structured monotone problem

$$\text{Find } x \in \mathcal{H} \ : \ \nabla f(x) + B(x) = 0, \tag{1.1}$$

where ∇f is the gradient of a convex differentiable function $f : \mathcal{H} \to \mathbb{R}$ (that's the potential part), and $B : \mathcal{H} \to \mathcal{H}$ is an operator which is supposed to be monotone and cocoercive (that's the nonpotential part).

*Laboratoire XLIM, Université de Limoges, 123, avenue Albert Thomas, 87060 Limoges, France. E-mail: samir.adly@unilim.fr

†IMAG, Université Montpellier, CNRS, Place Eugène Bataillon, 34095 Montpellier CEDEX 5, France. E-mail: hedy.attouch@umontpellier.fr, Supported by COST Action: CA16228

‡Laboratoire XLIM, Université de Limoges, 123, avenue Albert Thomas, 87060 Limoges, France. E-mail: van-nam.vo@etu.unilim.fr

1.1 General presentation

Our study is based on the continuous inertial dynamic

$$\ddot{x}(t)+\gamma\dot{x}(t)+\nabla f(x(t))+B(x(t))+\beta_f\nabla^2 f(x(t))\dot{x}(t)+\beta_b B'(x(t))\dot{x}(t)=0,\ t\geq t_0 \qquad \text{(DINAM)}$$

previously studied by the authors in [3]. (DINAM) stands shortly for Dynamical Inertial Newton method for Additively structured Monotone problems. It involves geometric dampings which are respectively controlled by the Hessian of the potential f, and by a Newton-type correction term attached to B. In [3], it has been proved the existence and the uniqueness of the solution of the Cauchy problem, as well as the weak convergence of the generated trajectories towards the zeros of $\nabla f + B$. The introduction of geometric damping makes it possible to attenuate notably the oscillations which occur naturally with the inertial methods. Our objective is to analyze the convergence properties of the algorithms obtained by temporal discretization of this dynamic, and thus solve numerically the structured monotone equation (1.1). We will pay particular attention to the minimal assumptions which guarantee convergence of the algorithm, and which highlight the asymmetric role played by the two operators involved in the dynamic.

Throughout the paper we make the following standing assumptions:

$$\begin{cases} \text{(A1)}\ \ f:\mathcal{H}\to\mathbb{R} \text{ is convex, of class } \mathcal{C}^1,\ \nabla f \text{ is Lipschitz continuous on the bounded sets;} \\ \text{(A2)}\ \ B:\mathcal{H}\to\mathcal{H} \text{ is a } \lambda\text{-cocoercive operator for some } \lambda>0; \\ \text{(A3)}\ \ \gamma>0,\ \beta_f>0,\ \beta_b\geq 0 \text{ are given real damping parameters;} \\ \text{(A4)}\ \ \text{the solution set } S:=\{p\in\mathcal{H}:\ \nabla f(p)+B(p)=0\} \text{ is nonempty.} \end{cases}$$

Unless specified, we do not assume the gradient of f to be globally Lipschitz continuous. The cocoercivity assumption on the operator B plays a central role in our analysis. Recall that the operator $B:\mathcal{H}\to\mathcal{H}$ is said to be λ-cocoercive for some $\lambda>0$ if

$$\langle By-Bx,y-x\rangle\geq\lambda\|By-Bx\|^2,\ \forall x,y\in\mathcal{H}.$$

It is easy to check that B is λ-cocoercive implies that B is $1/\lambda$-Lipschitz continuous. The reverse implication holds true in the case where the operator is the gradient of a convex and differentiable function. Indeed, according to Baillon-Haddad's Theorem [19], ∇f is L-Lipschitz continuous implies that ∇f is a $1/L$-cocoercive operator (see [20, Corollary 18.16] for more details).

The following (DINAAM-split) algorithm is a model example of the splitting algorithms obtained by temporal discretization of the continuous dynamic (DINAM). The positive parameter h is the step size of the discretization.

(DINAAM-split):

Initialize: $x_0\in\mathcal{H}, x_1\in\mathcal{H}$

$$\alpha=\frac{1}{1+\gamma h},\quad s=\frac{h}{1+\gamma h},$$

$$y_k=x_k+\alpha(x_k-x_{k-1})+s\beta_b B(x_k)-s(h+\beta_f)\nabla f(x_k)+s\beta_f\nabla f(x_{k-1}),$$

$$x_{k+1}=\Big(\mathrm{Id}+s(h+\beta_b)B\Big)^{-1}(y_k).$$

Its convergence properties are analyzed in section 4, Theorem 4.1. Compared to the classical accelerated proximal gradient algorithms, it contains corrective terms where the potential and non-potential operators appear asymmetrically, and which make it possible to attenuate the oscillations. In section 5, Theorem 5.1, we consider a variant of this algorithm, where the role of the operators is reversed.

1.2 Historical aspects: the potential case

Let us first recall some classical results concerning the potential case ($B = 0$). The following inertial system with Hessian-driven damping

$$\ddot{x}(t) + \gamma\dot{x}(t) + \beta\nabla^2 f(x(t))\dot{x}(t) + \nabla f(x(t)) = 0,$$

was considered by Alvarez-Attouch-Peypouquet-Redont in [6]. Then, according to the continuous interpretation by Su-Boyd-Candès [29] of the accelerated gradient method of Nesterov [27], Attouch-Peypouquet-Redont [16] replaced the fixed viscous damping parameter γ by an asymptotic vanishing damping parameter $\frac{\alpha}{t}$, with $\alpha > 0$. At first glance, the presence of the Hessian may seem to entail numerical difficulties. However, this is not the case as the Hessian intervenes in the above ODE in the form $\nabla^2 f(x(t))\dot{x}(t)$, which is nothing but the derivative with respect to time of $\nabla f(x(t))$. So, the temporal discretization of these dynamics provides first-order algorithms of the form

$$\begin{cases} y_k = x_k + \alpha_k(x_k - x_{k-1}) - \beta_k\left(\nabla f(x_k) - \nabla f(x_{k-1})\right) \\ x_{k+1} = y_k - s\nabla f(y_k). \end{cases}$$

As a specific feature, and by comparison with the classical accelerated gradient methods, these algorithms contain a correction term which is equal to the difference of the gradients at two consecutive steps. While preserving the convergence properties of the accelerated gradient method, they provide fast convergence to zero of the gradients, and reduce the oscillatory aspects. Several recent studies have been devoted to this subject, see Attouch-Chbani-Fadili-Riahi [8], Boţ-Csetnek-László [22], Kim [24], Lin-Jordan [25], Shi-Du-Jordan-Su [28], and Alesca-Lazlo-Pinta [4] for an implicit version of the Hessian driven damping. Application to deep learning has been recently developed by Castera-Bolte-Févotte-Pauwels [23]. In [2], Adly-Attouch studied the finite convergence (finite number of iterations) of proximal-gradient inertial algorithms combining Coulomb-type dry friction with Hessian-driven damping.

1.3 Historical aspects: the non potential case

Let's come to the transposition of these techniques to the case of maximally monotone operators. Álvarez-Attouch [5] and Attouch-Maingé [12] studied the equation

$$\ddot{x}(t) + \gamma\dot{x}(t) + A(x(t)) = 0, \tag{1.2}$$

when $A : \mathcal{H} \to \mathcal{H}$ is a cocoercive (and hence maximally monotone) operator, (see also [21]). The cocoercivity assumption plays a crucial role in the study of (1.2), not only to ensure the existence of solutions, but also to analyze their long-term behavior. Assuming that the cocoercivity parameter λ and the damping coefficient γ satisfy the inequality $\lambda\gamma^2 > 1$, Attouch-Maingé [12] showed that each trajectory of (1.2) converges weakly to a zero of A, as $t \to +\infty$.
For general maximally monotone operators this property has been exploited by Attouch-Peypouquet [15], Attouch-Cabot [7] and by Attouch-Laszlo [10, 11]. The key property is that for $\lambda > 0$, the Yosida approximation operator A_λ associated with A is λ-cocoercive and $A_\lambda^{-1}(0) = A^{-1}(0)$. So the idea is to replace the operator A by its Yosida approximation A_λ, and to adjust the Yosida regularization parameter $\lambda > 0$.
Another related work has been done by Attouch-Maingé [12] who first consider the asymptotic behavior of the second-order dissipative evolution equation with $f : \mathcal{H} \to \mathbb{R}$ convex and $B : \mathcal{H} \to \mathcal{H}$ cocoercive

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nabla f(x(t)) + B(x(t)) = 0, \tag{1.3}$$

by combining potential (f convex) with nonpotential effects (B cocoercive). The novelty in (DINAM), compared to the previous works, is the introduction of the Hessian-driven term and the Newton-type correcting term into this dynamic. The convergence analysis of the associated algorithms will require some adjustments and new proofs.

1.4 Link with Newton-like methods for solving monotone inclusions

To overcome the ill-posed character of the continuous Newton method for a general maximally monotone operator A, the following first order evolution system was studied by Attouch-Svaiter [18],

$$\begin{cases} v(t) \in A(x(t)) \\ \gamma(t)\dot{x}(t) + \beta\dot{v}(t) + v(t) = 0. \end{cases}$$

This system can be considered as a continuous version of the Levenberg-Marquardt method, which acts as a regularization of the Newton method. Under a fairly general assumption on the regularization parameter $\gamma(t)$, this system is well posed and generates trajectories that converge weakly to equilibria (zeros of A). Parallel results have been obtained for the associated proximal algorithms obtained by implicit temporal discretization, see [1], [14], [17]. Formally, this system is written as

$$\gamma(t)\dot{x}(t) + \beta\frac{d}{dt}\left(A(x(t))\right) + A(x(t)) = 0.$$

Thus (DINAM) can be considered as an inertial version of this dynamical system for the structured monotone operator $A = \nabla f + B$, see also [13], [26]. Our study is also linked to the recent works by Attouch-László [10, 11] who considered the general case of monotone equations. By contrast with [10, 11], according to the cocoercivity of B, we don't use the Yosida regularization, and exhibit minimal assumptions involving only the nonpotential component.

1.5 Contents

After the introductory Section 1, we recall in Section 2 some of the results obtained in [3] concerning the continuous dynamics (DINAM). In Section 3, we analyze the convergence properties of the sequences generated by an inertial proximal algorithm obtained by implicit discretization of the continuous dynamics (DINAM). We highlight the interplay between the damping parameters β_f, β_b, γ and the cocoercivity parameter λ, which plays a significant role in our Lyapunov analysis. In Section 4, we analyze an inertial proximal-gradient splitting algorithm which makes use of the gradient of f and the resolvent of B. We also analyze the effect of errors, perturbations in the algorithm. In Section 5, we examine a variant of this proximal-gradient algorithm, where the role of the operators is reversed. In Section 6, we perform numerical experiments which show that the well-known oscillations for the heavy ball with friction are damped with the introduction of the geometric damping terms, and we compare numerically the algorithms. Applications to structured monotone equations involving a nonpotential operator are considered.

2 The continuous dynamic (DINAM)

In this section, we recall the main results obtained in [3] concerning the second-order differential equation (DINAM) that we recall below

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nabla f(x(t)) + B(x(t)) + \beta_f\nabla^2 f(x(t))\dot{x}(t) + \beta_b B'(x(t))\dot{x}(t) = 0,\ t \geq 0. \qquad \text{(DINAM)}$$

The following existence and uniqueness result for the Cauchy problem is proved in [3, Theorem 2.1].

Theorem 2.1 *Suppose that $\beta_f > 0$ and $\beta_b \geq 0$. Then, for any $(x_0, x_1) \in \mathcal{H} \times \mathcal{H}$, there exists a unique strong global solution $x : [0, +\infty[\to \mathcal{H}$ of the continuous dynamic* (DINAM) *which satisfies the Cauchy data $x(0) = x_0$, $\dot{x}(0) = x_1$.*

Before stating the asymptotic behavior of the solution trajectories of (DINAM), note that $B(p)$ is uniquely defined for $p \in S := \{p \in \mathcal{H} :\ \nabla f(p) + B(p) = 0\}$.

Lemma 2.1 *$B(p)$ is uniquely defined for $p \in S$, i.e. $p_1 \in S,\ p_2 \in S \Longrightarrow B(p_1) = B(p_2)$.*

The following Theorem establishes the asymptotic convergence properties of (DINAM), see [3].

Theorem 2.2 *Let $B : \mathcal{H} \to \mathcal{H}$ be a λ-cocoercive operator and $f : \mathcal{H} \to \mathbb{R}$ be a $\mathcal{C}^1$ convex function whose gradient is Lipschitz continuous on the bounded sets. Suppose that the parameters involved in (DINAM) satisfy $\beta_f > 0$ and*

$$\lambda\gamma > \frac{(\beta_b - \beta_f)^2}{4\beta_f} + \frac{1}{2}\left(\beta_b + \frac{1}{\gamma}\right) + \frac{1}{2}\sqrt{\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b - \beta_f)^2}{\gamma\beta_f}}.$$

Then, for any solution trajectory $x : [0, +\infty[\to \mathcal{H}$ of (DINAM) the following properties are satisfied:

(i) $x(t)$ converges weakly, as $t \to +\infty$, to an element of S.

(ii) Set $A := \nabla f + B$ and $p \in S$. Then,

$$\int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty, \int_0^{+\infty} \|\ddot{x}(t)\|^2 dt < +\infty,$$
$$\int_0^{+\infty} \|B(x(t)) - B(p)\|^2 dt < +\infty, \int_0^{+\infty} \left\|\frac{d}{dt}B(x(t))\right\|^2 dt < +\infty,$$
$$\int_0^{+\infty} \|A(x(t))\|^2 dt < +\infty, \text{ and } \int_0^{+\infty} \left\|\frac{d}{dt}A(x(t))\right\|^2 dt < +\infty.$$

(iii) $\lim_{t\to+\infty} \|\dot{x}(t)\| = 0$, $\lim_{t\to+\infty} \|B(x(t)) - B(p)\| = 0$, $\lim_{t\to+\infty} \|A(x(t))\| = 0$, where $B(p)$ is uniquely defined for $p \in S$.

The following lemmas will be useful, for a proof we refer to [3].

Lemma 2.2 *Let $T_1, T_2 : \mathcal{H} \to \mathcal{H}$ be two cocoercive operators with respective cocoercivity coefficients $\lambda_1, \lambda_2 > 0$. Then, $T := T_1 + T_2 : \mathcal{H} \to \mathcal{H}$ is $\lambda-$cocoercive with $\lambda = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}$.*

Lemma 2.3 *Let a, b, c be three real numbers. The quadratic form $q : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$*

$$q(X, Y) := a\|X\|^2 + 2b\langle X, Y\rangle + c\|Y\|^2$$

is positive definite if and only if $ac - b^2 > 0$ and $a > 0$. Moreover

$$q(X, Y) \geq \mu(\|X\|^2 + \|Y\|^2) \quad \text{for all} \quad X, Y \in \mathcal{H}$$

where the positive real number $\mu := \frac{1}{2}\left(a + c - \sqrt{(a-c)^2 + 4b^2}\right)$ is the smallest eigenvalue of the positive symetric matrix associated with q.

3 Inertial proximal algorithms associated with (DINAM)

Set $A := \nabla f + B$ and $A_\beta := \beta_f \nabla f + \beta_b B$. Consider the implicit finite-difference scheme for (DINAM):

$$\frac{1}{h^2}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\gamma}{h}(x_{k+1} - x_k) + \frac{1}{h}(A_\beta(x_{k+1}) - A_\beta(x_k)) + A(x_{k+1}) = 0, \tag{3.1}$$

where $h > 0$ is a fixed time step. After expanding (3.1), we obtain

$$x_{k+1} + \frac{h^2}{1+\gamma h} A(x_{k+1}) + \frac{h}{1+\gamma h} A_\beta(x_{k+1}) = x_k + \frac{1}{1+\gamma h}(x_k - x_{k-1}) + \frac{h}{1+\gamma h} A_\beta(x_k). \tag{3.2}$$

Set $s := \dfrac{h}{1+\gamma h}$ and $\alpha := \dfrac{1}{1+\gamma h}$. So we have

$$x_{k+1} + s\mathcal{A}_h(x_{k+1}) = y_k, \tag{3.3}$$

where

$$\mathcal{A}_h = (h+\beta_f)\nabla f + (h+\beta_b)B, \tag{3.4}$$

$$y_k = x_k + \alpha(x_k - x_{k-1}) + sA_\beta(x_k). \tag{3.5}$$

We get $x_{k+1} = (\mathrm{Id} + s\mathcal{A}_h)^{-1}(y_k)$. Thus, we obtain the following algorithm, where (DINAAM) stands for Dynamical Inertial Newton Algorithm for Additively structured Monotone problems.

(DINAAM):

Initialize: $x_0 \in \mathcal{H}, x_1 \in \mathcal{H}$

$\alpha = \dfrac{1}{1+\gamma h}, \quad s = \dfrac{h}{1+\gamma h},$

$y_k = x_k + \alpha(x_k - x_{k-1}) + sA_\beta(x_k),$

$x_{k+1} = (\mathrm{Id} + s\mathcal{A}_h)^{-1}(y_k).$

Note that (DINAAM) is not a splitting algorithm, since the computation of the resolvent of $\mathcal{A}_h = (h + \beta_f)\nabla f + (h+\beta_b)B$ is needed. Corresponding splitting algorithms will be examined in Sections 4 and 5.

3.1 Lyapunov analysis

Let us state the convergence properties of (DINAAM).

Theorem 3.1 *Let $B : \mathcal{H} \to \mathcal{H}$ be a λ-cocoercive operator and $f : \mathcal{H} \to \mathbb{R}$ be a convex differentiable function whose gradient is Lipschitz continuous on the bounded sets. Suppose that the positive parameters $\lambda, \gamma, \beta_b, \beta_f$ satisfy*

$$\beta_f > 0, \text{ and } \lambda\gamma > \frac{(\beta_b - \beta_f)^2}{4\beta_f} + \frac{1}{2}\left(\beta_b + \frac{1}{\gamma}\right) + \frac{1}{2}\sqrt{\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b - \beta_f)^2}{\gamma\beta_f}}. \tag{3.6}$$

Then, there exists h^* such that for all $0 < h < h^*$, the sequence (x_k) generated by the algorithm (DINAAM) has the following properties:

(i) (x_k) converges weakly to an element in S;

(ii) $\displaystyle\sum_{k=1}^{+\infty}\|x_k-x_{k-1}\|^2<+\infty,\quad \sum_{k=1}^{+\infty}\|A(x_k)\|^2<+\infty,$
$\displaystyle\sum_{k=1}^{+\infty}\|\nabla f(x_k)-\nabla f(x_{k-1})\|^2<+\infty,$ *and* $\displaystyle\sum_{k=1}^{+\infty}\|B(x_k)-B(x_{k-1})\|^2<+\infty;$

(iii) $\lim_{k\to\infty}\|x_{k+1}-x_k\|=0$, $\lim_{k\to+\infty}\|B(x_k)-B(p)\|=0$, *and* $\lim_{k\to+\infty}\|\nabla f(x_k)-\nabla f(p)\|=0.$

Proof. **The discrete energy.** Recall that $A:=\nabla f+B$ and $A_\beta:=\beta_f\nabla f+\beta_b B$. Take $p\in S$. Consider the sequence (V_k) defined for all $k\geq 1$ by the formula

$$V_k:=\frac{1}{2}\|(x_k-p)+c\left(\frac{1}{h}(x_k-x_{k-1})+A_\beta(x_k)-A_\beta(p)\right)\|^2+\frac{\delta}{2}\|x_k-p\|^2,$$

where c and δ are positive coefficients to adjust. For each $k\geq 1$, let us briefly write V_k as follows:

$$V_k=\frac{1}{2}\|v_k\|^2+\frac{\delta}{2}\|x_k-p\|^2,\text{ with } v_k=(x_k-p)+c\left(\frac{1}{h}(x_k-x_{k-1})+A_\beta(x_k)-A_\beta(p)\right).$$

By definition of v_k, we have $v_{k+1}=(x_{k+1}-p)+c\left(\frac{1}{h}(x_{k+1}-x_k)+A_\beta(x_{k+1})-A_\beta(p)\right)$.
Moreover, by using the formulation (3.1) of the algorithm (DINAAM), we have

$$\begin{aligned}v_k&=(x_{k+1}-p)+c\left(\frac{1}{h}(x_{k+1}-x_k)+\gamma(x_{k+1}-x_k)+A_\beta(x_{k+1})-A_\beta(p)+hA(x_{k+1})\right)\\&\quad-(x_{k+1}-x_k)\\&=v_{k+1}+(c\gamma-1)(x_{k+1}-x_k)+chA(x_{k+1}).\end{aligned}$$

Therefore, for $k\geq 1$, we have

$$\begin{aligned}&\frac{1}{2}\|v_{k+1}\|^2-\frac{1}{2}\|v_k\|^2\\&=-\frac{1}{2}(c\gamma-1)^2\|x_{k+1}-x_k\|^2-\frac{1}{2}c^2h^2\|A(x_{k+1})\|^2-hc(c\gamma-1)\langle x_{k+1}-x_k,A(x_{k+1})\rangle\\&-\left\langle(x_{k+1}-p)+c(\frac{1}{h}(x_{k+1}-x_k)+A_\beta(x_{k+1})-A_\beta(p)),(c\gamma-1)(x_{k+1}-x_k)+chA(x_{k+1})\right\rangle\\&=-\frac{1}{2}(c\gamma-1)^2\|x_{k+1}-x_k\|^2-\frac{1}{2}c^2h^2\|A(x_{k+1})\|^2-hc(c\gamma-1)\langle x_{k+1}-x_k,A(x_{k+1})\rangle\\&-(c\gamma-1)\langle x_{k+1}-p,x_{k+1}-x_k\rangle-ch\langle x_{k+1}-p,A(x_{k+1})\rangle-\frac{c(c\gamma-1)}{h}\|x_{k+1}-x_k\|^2\\&-c^2\langle x_{k+1}-x_k,A(x_{k+1})\rangle-c(c\gamma-1)\langle A_\beta(x_{k+1})-A_\beta(p),x_{k+1}-x_k\rangle\\&-c^2h\langle A_\beta(x_{k+1})-A_\beta(p),A(x_{k+1})\rangle.\end{aligned}\tag{3.7}$$

To write the above relation in a recursive form, we use the elementary identity

$$\frac{1}{2}\|x_{k+1}-p\|^2-\frac{1}{2}\|x_k-p\|^2=-\frac{1}{2}\|x_{k+1}-x_k\|^2+\langle x_{k+1}-x_k,x_{k+1}-p\rangle.\tag{3.8}$$

Write shortly $X_k:=x_{k+1}-x_k$, $Y_k:=B(x_{k+1})-B(p)$, $Z_k:=\nabla f(x_{k+1})-\nabla f(p)$ for $k\geq 0$.
Since $p\in S$, *i.e.*, $\nabla f(p)+B(p)=0$, we have $A(x_{k+1})=Y_k+Z_k$ for $k\geq 0$.
In the definition of (V_k), take $\delta=c\gamma-1$, which is assumed to be nonnegative, *i.e.*, $c\gamma\geq 1$.

According to (3.7), (3.8) and the definition of (V_k), we obtain after simplification

$$\begin{aligned}V_{k+1}-V_k =& -\frac{1}{2}(c\gamma-1)^2\|X_k\|^2-\frac{1}{2}c^2h^2\|Y_k+Z_k\|^2-hc(c\gamma-1)\langle X_k,Y_k+Z_k\rangle\\ &-\frac{1}{2}(c\gamma-1)\|X_k\|^2-ch\langle x_{k+1}-p,A(x_{k+1})\rangle-\frac{c(c\gamma-1)}{h}\|X_k\|^2\\ &-c^2\langle X_k,Y_k+Z_k\rangle-c(c\gamma-1)\langle\beta_bY_k+\beta_fZ_k,X_k\rangle-c^2h\langle\beta_bY_k+\beta_fZ_k,Y_k+Z_k\rangle.\end{aligned}$$

Using the fact that $p\in S$, ∇f is monotone, and B is λ-cocoercive, we have

$$\begin{aligned}-ch\langle x_{k+1}-p,A(x_{k+1})&=-ch\langle x_{k+1}-p,B(x_{k+1})-B(p)\rangle\\ &\quad-ch\langle x_{k+1}-p,\nabla f(x_{k+1})-\nabla f(p)\rangle\\ &\leq-ch\lambda\|B(x_{k+1})-B(p)\|^2.\end{aligned}$$

So, by combining the previous results, we get

$$\begin{aligned}&V_{k+1}-V_k+\left[\frac{1}{2}(c\gamma-1)^2+\frac{1}{2}(c\gamma-1)+\frac{c(c\gamma-1)}{h}\right]\|X_k\|^2\\ &\quad+\left[c(c\gamma-1)h+c^2+c(c\gamma-1)\beta_f\right]\langle X_k,Z_k\rangle\\ &\quad+\left[c(c\gamma-1)h+c^2+c(c\gamma-1)\beta_b\right]\langle X_k,Y_k\rangle+\left[ch\lambda+c^2h\beta_b+\frac{1}{2}c^2h^2\right]\|Y_k\|^2\\ &\quad+\left[c^2h\beta_f+\frac{1}{2}c^2h^2\right]\|Z_k\|^2+\left[c^2h(\beta_b+\beta_f)+c^2h^2\right]\langle Z_k,Y_k\rangle\leq 0.\end{aligned}\tag{3.9}$$

Let (Γ_k) be the sequence defined by

$$\Gamma_k=f(x_k)-f(p)-\langle\nabla f(p),x_k-p\rangle,\ \text{for } k\geq 0.$$

Since f is convex, we have $\Gamma_k\geq 0$, for all $k\geq 0$. Moreover,

$$\begin{aligned}\langle X_k,Z_k\rangle&=\langle x_{k+1}-x_k,\nabla f(x_{k+1})\rangle-\langle x_{k+1}-x_k,\nabla f(p)\rangle\\ &\geq f(x_{k+1})-f(x_k)+\Gamma_{k+1}-\Gamma_k+f(x_k)-f(x_{k+1})\\ &=\Gamma_{k+1}-\Gamma_k.\end{aligned}\tag{3.10}$$

For each $k\geq 1$, let us define

$$E_k=V_k+\left[c(c\gamma-1)h+c^2+c(c\gamma-1)\beta_f\right]\Gamma_k.$$

(E_k) will serve us as a discrete energy function. Indeed, it is clear that (E_k) is a sequence of nonnegative numbers. From (3.9), (3.10) and the definition of (E_k), we obtain

$$\begin{aligned}&E_{k+1}-E_k+\left[\frac{1}{2}(c\gamma-1)^2+\frac{1}{2}(c\gamma-1)+\frac{c(c\gamma-1)}{h}\right]\|X_k\|^2\\ &\quad+\left[c(c\gamma-1)h+c^2+c(c\gamma-1)\beta_b\right]\langle X_k,Y_k\rangle+\left[ch\lambda+c^2h\beta_b+\frac{1}{2}c^2h^2\right]\|Y_k\|^2\\ &\quad+\left[c^2h\beta_f+\frac{1}{2}c^2h^2\right]\|Z_k\|^2+\left[c^2h(\beta_b+\beta_f)+c^2h^2\right]\langle Z_k,Y_k\rangle\leq 0.\end{aligned}\tag{3.11}$$

Let us eliminate Z_k from this relation by using the elementary algebraic inequality

$$\left[c^2h\beta_f+\frac{1}{2}c^2h^2\right]\|Z_k\|^2+\left[c^2h(\beta_b+\beta_f)+c^2h^2\right]\langle Z_k,Y_k\rangle\geq-\frac{c^2h(\beta_b+\beta_f+h)^2}{4\beta_f+2h}\|Y_k\|^2.$$

Then, from (3.11), we deduce that

$$\begin{aligned} E_{k+1} - E_k + \left[\frac{1}{2}(c\gamma-1)^2 + \frac{1}{2}(c\gamma-1) + \frac{c(c\gamma-1)}{h}\right]\|X_k\|^2 \\ + \left[c(c\gamma-1)h + c^2 + c(c\gamma-1)\beta_b\right]\langle X_k, Y_k\rangle \\ + \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2 - \frac{c^2h(\beta_b+\beta_f+h)^2}{4\beta_f+2h}\right]\|Y_k\|^2 \leq 0. \end{aligned} \tag{3.12}$$

Equivalently,

$$E_{k+1} - E_k + \mathcal{S}_k \leq 0, \tag{3.13}$$

where

$$\begin{aligned} \mathcal{S}_k = \left[\frac{1}{2}(c\gamma-1)^2 + \frac{1}{2}(c\gamma-1) + \frac{c(c\gamma-1)}{h}\right]\|X_k\|^2 \\ + \left[c(c\gamma-1)h + c^2 + c(c\gamma-1)\beta_b\right]\langle X_k, Y_k\rangle \\ + \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2 - \frac{c^2h(\beta_b+\beta_f+h)^2}{4\beta_f+2h}\right]\|Y_k\|^2. \end{aligned}$$

We have $\mathcal{S}_k = q(X_k, Y_k)$ where $q : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ is the quadratic form

$$q(X_k, Y_k) := a\|X_k\|^2 + b\langle X_k, Y_k\rangle + g\|Y_k\|^2,$$

with

$$\begin{aligned} a &= \frac{1}{2}(c\gamma-1)^2 + \frac{1}{2}(c\gamma-1) + \frac{c(c\gamma-1)}{h} \\ b &= c(c\gamma-1)h + c^2 + c(c\gamma-1)\beta_b \\ g &= ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2 - \frac{c^2h(\beta_b+\beta_f+h)^2}{4\beta_f+2h}. \end{aligned}$$

According to Lemma 2.3, since $a > 0$, q is positive definite if and only if $4ag - b^2 > 0$. Equivalently

$$\begin{aligned} 4\left[\frac{1}{2}(c\gamma-1)^2 + \frac{1}{2}(c\gamma-1) + \frac{c(c\gamma-1)}{h}\right]\left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2 - \frac{c^2h(\beta_b+\beta_f+h)^2}{4\beta_f+2h}\right] \\ - \left[c(c\gamma-1)h + c^2 + c(c\gamma-1)\beta_b\right]^2 > 0. \end{aligned} \tag{3.14}$$

Our aim is to find c such that $c\gamma - 1 > 0$, and (3.14) is satisfied.
After development and simplification we obtain

$$\begin{aligned} 4\left[\frac{1}{2}(c\gamma-1)^2h + \frac{1}{2}(c\gamma-1)h + c(c\gamma-1)\right]\left[c\lambda + c^2\beta_b + \frac{1}{2}c^2h - \frac{c^2(\beta_b+\beta_f+h)^2}{4\beta_f+2h}\right] \\ - \left[c(c\gamma-1)h + c^2 + c(c\gamma-1)\beta_b\right]^2 > 0. \end{aligned} \tag{3.15}$$

Let us denote by $\mathcal{L}(h)$ the left handside of (3.15). We have

$$\lim_{h\to 0^+}\mathcal{L}(h) = 4c(c\gamma-1)\left[c\lambda + c^2\beta_b - \frac{c^2(\beta_b+\beta_f)^2}{4\beta_f}\right] - \left[c^2 + c(c\gamma-1)\beta_b\right]^2.$$

So, to guarantee the existence of $h > 0$ such that the quadratic form q is positive definite, it suffices to assume that

$$4c(c\gamma - 1)\left[c\lambda + c^2\beta_b - \frac{c^2(\beta_b+\beta_f)^2}{4\beta_f}\right] - \left[c^2 + c(c\gamma-1)\beta_b\right]^2 > 0.$$

The above inequality can be written equivalently as

$$4\lambda > \frac{\left[c^2 + c(c\gamma-1)\beta_b\right]^2}{c^2(c\gamma-1)} - 4c\beta_b + \frac{(\beta_b+\beta_f)^2}{\beta_f}c = \frac{\left[c + (c\gamma-1)\beta_b\right]^2}{c\gamma-1} + \frac{(\beta_b-\beta_f)^2}{\beta_f}c.$$

Let us formulate this inequation with the help of $\delta = c\gamma - 1 > 0$. Our aim is to find $\delta > 0$ such that

$$4\lambda > \frac{\left[\frac{\delta+1}{\gamma} + \delta\beta_b\right]^2}{\delta} + \frac{\delta+1}{\gamma}\frac{(\beta_b-\beta_f)^2}{\beta_f}.$$

After a few steps of algebraic calculation, we get

$$4\lambda > \frac{2}{\gamma}\left(\beta_b + \frac{1}{\gamma}\right) + \frac{1}{\gamma}\frac{(\beta_b-\beta_f)^2}{\beta_f} + \frac{1}{\gamma^2\delta} + \left[\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}\right]\delta.$$

Therefore, in order to ensure the existence of such δ, it is sufficient to assume that

$$4\lambda > \frac{2}{\gamma}\left(\beta_b + \frac{1}{\gamma}\right) + \frac{1}{\gamma}\frac{(\beta_b-\beta_f)^2}{\beta_f} + \inf_{\delta>0}\left(\frac{1}{\gamma^2\delta} + \left[\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}\right]\delta\right). \tag{3.16}$$

Elementary optimization argument gives that

$$\inf_{\delta>0}\left(\frac{C}{\delta} + D\delta\right) = 2\sqrt{CD}, \tag{3.17}$$

with C, D are positive constants. Combining (3.16) and (3.17), we end up with the condition

$$4\lambda > \frac{2}{\gamma}\left(\beta_b + \frac{1}{\gamma}\right) + \frac{1}{\gamma}\frac{(\beta_b-\beta_f)^2}{\beta_f} + \frac{2}{\gamma}\sqrt{\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}}.$$

When $\beta_b = \beta_f := \beta$, we recover the condition $\lambda\gamma > \beta + \dfrac{1}{\gamma}$. Therefore, under the above condition, and by taking h sufficiently small, there exists a positive real number μ such that for any $k \geq 1$,

$$E_{k+1} - E_k + \mu\|X_k\|^2 + \mu\|Y_k\|^2 \leq 0. \tag{3.18}$$

Estimates. According to (3.18), the sequence of non-negative numbers (E_k) is non-increasing and therefore converges. In particular, it is bounded. From this, we immediately deduce that

$$\sup_k \left\|(x_k - p) + c\left(\frac{1}{h}(x_k - x_{k-1}) + A_\beta(x_k) - A_\beta(p)\right)\right\|^2 < +\infty \tag{3.19}$$

$$\sup_k \|x_k - p\|^2 < +\infty. \tag{3.20}$$

Moreover, by summing the inequalities (3.18), we deduce that

$$\sum_{k=0}^{+\infty}\|X_k\|^2 < +\infty, \ \sum_{k=0}^{+\infty}\|Y_k\|^2 < +\infty.$$

Let us return to (3.11). Recall that

$$E_{k+1} - E_k + \left[\frac{1}{2}(c\gamma-1)^2 + \frac{1}{2}(c\gamma-1) + \frac{c(c\gamma-1)}{h}\right]\|X_k\|^2 + \left[c(c\gamma-1)h + c^2 + c(c\gamma-1)\beta_b\right]\langle X_k, Y_k\rangle + \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2\right]\|Y_k\|^2 + \left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\|Z_k\|^2 + \left[c^2h(\beta_b+\beta_f) + c^2h^2\right]\langle Z_k, Y_k\rangle \leq 0. \tag{3.21}$$

By using the estimates $\sum_{k=0}^{+\infty}\|X_k\|^2 < +\infty$ and $\sum_{k=0}^{+\infty}\|Y_k\|^2 < +\infty$, we obtain the existence of a constant $C > 0$ such that

$$\left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\sum_{k=0}^{+\infty}\|Z_k\|^2 \leq C + \left[c^2h(\beta_b+\beta_f) + c^2h^2\right]\sum_{k=0}^{+\infty}\|Z_k\|\|Y_k\|.$$

Therefore, for any $\epsilon > 0$, we have

$$\left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\sum_{k=0}^{+\infty}\|Z_k\|^2 \leq C + \left[c^2h(\beta_b+\beta_f) + c^2h^2\right]\left(\epsilon\sum_{k=0}^{+\infty}\|Z_k\|^2 + \frac{1}{4\epsilon}\sum_{k=0}^{+\infty}\|Y_k\|^2\right).$$

By taking $\epsilon > 0$ such that $c^2h\beta_f + \frac{1}{2}c^2h^2 > \epsilon\left[c^2h(\beta_b+\beta_f) + c^2h^2\right]$ which is always possible since $c^2h\beta_f + \frac{1}{2}c^2h^2 > 0$, we conclude that

$$\sum_{k=0}^{+\infty}\|Z_k\|^2 < +\infty.$$

Since $A(x_{k+1}) = Y_k + Z_k$, we immediately obtain

$$\sum_{k=1}^{+\infty}\|A(x_k)\|^2 < +\infty.$$

Furthermore, according to (3.20) the trajectory (x_k) is bounded. Set $R := \sup_{k\geq 0}\|x_k\|$. By assumption, ∇f is Lipschitz continuous on the bounded sets. Let $L_R < +\infty$ be the Lipschitz constant of ∇f on $\mathbb{B}(0,R)$. Since B is λ-cocoercive, it is $\frac{1}{\lambda}$-Lipschitz continuous. Therefore, A is L-Lipschitz continuous on the trajectory with $L := L_R + \frac{1}{\lambda}$. Thus,

$$\|A(x_{k+1}) - A(x_k)\| \leq L\|x_{k+1} - x_k\| \text{ for all } k \geq 0.$$

Therefore, $\sum_{k=1}^{+\infty}\|A(x_{k+1}) - A(x_k)\|^2 \leq \sum_{k=1}^{+\infty}L^2\|x_{k+1} - x_k\|^2 < +\infty$. Using the same argument, we get

$$\sum_{k=1}^{+\infty}\|B(x_{k+1}) - B(x_k)\|^2 < +\infty, \text{ and } \sum_{k=1}^{+\infty}\|\nabla f(x_{k+1}) - \nabla f(x_k)\|^2 < +\infty.$$

Since the general term of a convergent series goes to zero, we deduce that

$$\lim_{k\to+\infty}\|x_{k+1}-x_k\|=0,\ \lim_{k\to+\infty}\|A(x_k)\|=0,\ \text{and}\ \lim_{k\to+\infty}\|A(x_{k+1})-A(x_k)\|=0,$$
$$\lim_{k\to+\infty}\|B(x_{k+1})-B(x_k)\|=0,\ \lim_{k\to+\infty}\|\nabla(x_{k+1})-\nabla(x_k)\|=0. \tag{3.22}$$

Likewise, we also have

$$\lim_{k\to+\infty}\|B(x_k)-B(p)\|=0,\ \text{and}\ \lim_{k\to+\infty}\|\nabla f(x_k)-\nabla f(p)\|=0. \tag{3.23}$$

Convergence of (x_k). Let us first show that every weak cluster point of the sequence (x_k) belongs to S. Let x^* be a weak cluster point of (x_k) and consider a subsequence (x_{k_n}) of (x_k), such that $x_{k_n}\rightharpoonup x^*$, as $n\to+\infty$. We have

$$A(x_{k_n})\to 0 \text{ strongly in } \mathcal{H}\ \text{ and }\ x_{k_n}\rightharpoonup x^* \text{ weakly in } \mathcal{H}.$$

From the closedness property of the graph of the maximally monotone operator A in $w-\mathcal{H}\times s-\mathcal{H}$, we deduce that $A(x^*)=0$, that is $x^*\in S$. Since $\lim_k E_k$ exists, and according to the above strong convergence results, we deduce that there exists a constant r such that for any $p\in S$

$$\lim_{k\to\infty}\Big[\|x_k-p\|^2+r\left(f(x_k)-\langle\nabla f(p),x_k-p\rangle\right)\Big]\quad\text{exists.}$$

Suppose that the bounded sequence (x_k) has two weak limit points, let p and p'. By the above argument we have that p and p' belong to S. Therefore the following limits exist: $\lim_{k\to\infty}\Big[\|x_k-p\|^2+r\left(f(x_k)-\langle\nabla f(p),x_k-p\rangle\right)\Big]$ and $\lim_{k\to\infty}\Big[\|x_k-p'\|^2+r\left(f(x_k)-\langle\nabla f(p'),x_k-p'\rangle\right)\Big]$. By taking the difference, we deduce that $\lim_{k\to\infty}\|x_k-p\|^2-\|x_k-p'\|^2$ exists. Equivalently $\lim_{k\to\infty}\langle x_k,p-p'\rangle$ exists. By specializing this result to the subsequences defining p and p' we get

$$\langle p,p-p'\rangle=\langle p',p-p'\rangle,$$

that is $\|p-p'\|^2=0$, which gives $p=p'$. Therefore the bounded sequence (x_k) has a unique weak cluster point, and hence converges weakly. ■

3.2 Estimating the time step *h*

The preceding results are valid when h is taken small enough. For numerical reasons, it is important to specify this result, and find $h^*>0$ such that the convergence results hold true for all $h\in[0,h^*]$. So let's come back to (3.14), which is the key property for our Lyapunov analysis. After elementary calculation, it can be written as follows

$$2(c\gamma-1)(2+\gamma h)\left[\lambda+c\beta_b+\frac{1}{2}ch-\frac{c(\beta_b+\beta_f+h)^2}{4\beta_f+2h}\right]-[(c\gamma-1)h+c+(c\gamma-1)\beta_b]^2>0. \tag{3.24}$$

After dividing by $c\gamma-1>0$ and elementary calculation, we get

$$(2+\gamma h)\big((\lambda+c\beta_b+\frac{1}{2}ch)(4\beta_f+2h)-c(\beta_b+\beta_f+h)^2\big)-(c\gamma-1)(2\beta_f+h)\left[h+\frac{c}{c\gamma-1}+\beta_b\right]^2>0.$$

Let us develop the above expression. We obtain a third-order polynomial with respect to h, namely $P_c(h) = a_0 + a_1 h + a_2 h^2 + a_3 h^3$ with

$$\begin{aligned}
a_0 &= 2\Big(4\lambda\beta_f - c(\beta_b-\beta_f)^2 - \beta_f\frac{(c+\beta_b(c\gamma-1))^2}{c\gamma-1}\Big),\\
a_1 &= 4\lambda(1+\gamma\beta_f) - c\gamma(\beta_b-\beta_f)^2 - 4\beta_f\,(c+\beta_b(c\gamma-1)) - \frac{1}{c\gamma-1}\,(c+\beta_b(c\gamma-1))^2\,,\\
a_2 &= 2\lambda\gamma - 2c - 2(c\gamma-1)(\beta_b+\beta_f),\\
a_3 &= -(c\gamma-1).
\end{aligned}$$

We have observed that choosing adequately $c > 0$ with $c\gamma - 1 > 0$ gives that $a_0 > 0$ under the growth condition 3.6. Precisely we can take $c = c^*$ where $c^*\gamma - 1 = \frac{1}{\gamma}\left(\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}\right)^{-\frac{1}{2}}$.

Let us check that $P_{c^*}(0) > 0$. In fact, we consider

$$\frac{1}{2}\gamma a_0(c) = 4\lambda\gamma\beta_f - c\gamma(\beta_b-\beta_f)^2 - \beta_f\frac{(c\gamma+\beta_b\gamma(c\gamma-1))^2}{\gamma(c\gamma-1)}.$$

According to the growth condition

$$\lambda\gamma > \frac{(\beta_b-\beta_f)^2}{4\beta_f} + \frac{1}{2}\left(\beta_b+\frac{1}{\gamma}\right) + \frac{1}{2}\sqrt{\left(\beta_b+\frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}},$$

we have

$$\begin{aligned}
\frac{1}{2}\gamma a_0(c) > {}& (\beta_b-\beta_f)^2 + 2\beta_f\left(\beta_b+\frac{1}{\gamma}\right) + 2\beta_f\sqrt{\left(\beta_b+\frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}}\\
&- c\gamma(\beta_b-\beta_f)^2 - \beta_f\frac{(c\gamma+\beta_b\gamma(c\gamma-1))^2}{\gamma(c\gamma-1)}.
\end{aligned}$$

For short, we set

$$y = \sqrt{\left(\beta_b+\frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}}.$$

Now, take $c = c^*$ where $c^*\gamma - 1 = \frac{1}{\gamma}\left(\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}\right)^{-\frac{1}{2}}$. Then,

$$\begin{aligned}
\frac{1}{2}\gamma a_0(c^*) &> -\frac{1}{\gamma y}(\beta_b-\beta_f)^2 + 2\beta_f\left(\beta_b+\frac{1}{\gamma}\right) + 2\beta_f y - \beta_f y(1+\frac{1}{\gamma y}+\frac{\beta_b}{y})^2\\
&= -\frac{1}{\gamma y}(\beta_b-\beta_f)^2 + 2\beta_f y - \beta_f y - \frac{\beta_f}{y}(\frac{1}{\gamma}+\beta_b)^2\\
&= \frac{\beta_f}{y}\left[-\frac{1}{\gamma\beta_f}(\beta_b-\beta_f)^2 + 2y^2 - y^2 - (\frac{1}{\gamma}+\beta_b)^2\right]\\
&= 0.
\end{aligned}$$

Hence, $P_{c^*}(0) = a_0(c^*) > 0$. Note that for large h, $P_{c^*}(h) \sim -(c\gamma-1)h^3$, and so $P_{c^*}(h)$ is negative. Therefore, $h^* > 0$ is the smallest positive zero (which exists) of P_{c^*}. Its explicit determination is quite technical in our general setting. In practical situations, it is elementary numerical analysis. Let us emphasize the fact that h^* depends only on the parameters that enter (DINAAM) (not on f).

3.3 Case $\beta_b = \beta_f$

In the special and important case where the two coefficients β_b and β_f are equal, we have the following result. Set $\beta_b = \beta_f := \beta > 0$, and $A := \nabla f + B$. We thus consider the evolution system

$$\text{(DINAM)} \quad \ddot{x}(t) + \gamma\dot{x}(t) + A(x(t)) + \beta\frac{d}{dt}\left(A(x(t))\right) = 0, \quad t \geq 0.$$

By following the same lines as in the general case, we have the following algorithm

(DINAAM): $\beta_b = \beta_f = \beta$

Initialize: $x_0 \in \mathcal{H}, x_1 \in \mathcal{H}$

$$\alpha = \frac{1}{1+\gamma h}, \quad s = \frac{h(h+\beta)}{1+\gamma h},$$

$$y_k = x_k + \alpha(x_k - x_{k-1}) + h\alpha\beta A(x_k),$$

$$x_{k+1} = (\mathrm{Id} + sA)^{-1}(y_k).$$

The following result is a particular case of Theorem 3.1.

Corollary 3.1 *Let $B : \mathcal{H} \to \mathcal{H}$ be a λ-cocoercive operator and $f : \mathcal{H} \to \mathbb{R}$ be a $\mathcal{C}^1$ convex function whose gradient is Lipschitz continuous on the bounded sets. Suppose that $\beta_b = \beta_f := \beta > 0$ and that the parameters γ, λ, β satisfy the following conditions*

$$\gamma > 0, \ \beta > 0 \text{ and } \lambda\gamma > \beta + \frac{1}{\gamma}.$$

Then, there exists h^* such that for all $0 < h < h^*$, the sequence (x_k) generated by the algorithm (DINAAM) *has the following properties:*

(i) (x_k) converges weakly to an element in S;

(ii) $\sum_{k=1}^{+\infty} \|x_k - x_{k-1}\|^2 < +\infty, \ \sum_{k=1}^{+\infty} \|A(x_k)\|^2 < +\infty;$

(iii) $\lim_{k\to+\infty} \|x_{k+1} - x_k\| = 0, \ \lim_{k\to+\infty} \|A(x_k)\| = 0.$

4 An inertial proximal-gradient algorithm

In this section, we assume that f is a $\mathcal{C}^1$ function whose gradient is L-Lipschitz on the bounded sets. Set $A := \nabla f + B$ and $A_\beta := \beta_f \nabla f + \beta_b B$. We take a fixed time step $h > 0$, and consider the following finite-difference scheme for (DINAM):

$$\frac{1}{h^2}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\gamma}{h}(x_{k+1} - x_k) + \frac{\beta_b}{h}(B(x_{k+1}) - B(x_k)) + \frac{\beta_f}{h}(\nabla f(x_k) - \nabla f(x_{k-1})) + B(x_{k+1}) + \nabla f(x_k) = 0. \tag{4.1}$$

This scheme is implicit with respect to the nonpotential B and explicit with respect to the potential operator ∇f. Furthermore, the temporal discretization of the Hessian driven damping $\beta_f \nabla^2 f(x(t))\dot{x}(t)$ is taken

equal to $\frac{\beta_f}{h}(\nabla f(x_k) - \nabla f(x_{k-1}))$ instead of $\frac{\beta_f}{h}(\nabla f(x_{k+1}) - \nabla f(x_k))$.
After expanding (4.1), we obtain

$$x_{k+1} + \frac{h^2}{1+\gamma h}B(x_{k+1}) + \frac{h\beta_b}{1+\gamma h}B(x_{k+1}) = x_k + \frac{1}{1+\gamma h}(x_k - x_{k-1}) + \frac{h\beta_b}{1+\gamma h}B(x_k) \\ -\frac{h\beta_f}{1+h\gamma}(\nabla f(x_k) - \nabla f(x_{k-1})) - \frac{h^2}{1+h\gamma}\nabla f(x_k). \tag{4.2}$$

Set $s := \frac{h}{1+\gamma h}$ and $\alpha := \frac{1}{1+\gamma h}$. So we have

$$x_{k+1} + s\mathcal{B}_h(x_{k+1}) = y_k, \tag{4.3}$$

where $\mathcal{B}_h = (h+\beta_b)B$, and

$$y_k = x_k + \alpha(x_k - x_{k-1}) + s\beta_b B(x_k) - s(h+\beta_f)\nabla f(x_k) + s\beta_f \nabla f(x_{k-1}). \tag{4.4}$$

From (4.3) we get

$$x_{k+1} = (\mathrm{Id} + s\mathcal{B}_h)^{-1}(y_k). \tag{4.5}$$

Combining (4.4) and (4.5), we obtain the following algorithm, called (DINAAM-split).

(DINAAM-split):

Initialize: $x_0 \in \mathcal{H}, x_1 \in \mathcal{H}$

$$\alpha = \frac{1}{1+\gamma h}, \quad s = \frac{h}{1+\gamma h},$$

$$y_k = x_k + \alpha(x_k - x_{k-1}) + s\beta_b B(x_k) - s(h+\beta_f)\nabla f(x_k) + s\beta_f \nabla f(x_{k-1}),$$

$$x_{k+1} = (\mathrm{Id} + s\mathcal{B}_h)^{-1}(y_k).$$

4.1 Lyapunov analysis

Theorem 4.1 *Let $B : \mathcal{H} \to \mathcal{H}$ be a λ-cocoercive operator and $f : \mathcal{H} \to \mathbb{R}$ a $\mathcal{C}^1$ convex function whose gradient is $L-$Lipschitz continuous. Suppose that the positive parameters $\lambda, \gamma, \beta_b, \beta_f$ satisfy*

$$\beta_f > 0, \text{ and } \lambda\gamma > \frac{(\beta_b - \beta_f)^2}{4\beta_f} + \frac{1}{2}\left(\beta_b + \frac{1}{\gamma}\right) + \frac{1}{2}\sqrt{\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b - \beta_f)^2}{\gamma\beta_f}}.$$

Then, there exists h^* (depending on L) such that for all $0 < h < h^*$, the sequence (x_k) generated by the algorithm (DINAAM-split) has the following properties:

(i) (x_k) converges weakly to an element in S;

(ii) $\sum_{k=1}^{+\infty} \|x_k - x_{k-1}\|^2 < +\infty$, $\sum_{k=1}^{+\infty} \|A(x_k)\|^2 < +\infty$,
$\sum_{k=1}^{+\infty} \|\nabla f(x_k) - \nabla f(x_{k-1})\|^2 < +\infty$, *and* $\sum_{k=1}^{+\infty} \|B(x_k) - B(x_{k-1})\|^2 < +\infty$;

(iii) $\lim_{k\to+\infty} \|x_{k+1} - x_k\| = 0$, $\lim_{k\to+\infty} \|B(x_k) - B(p)\| = 0$, *and* $\lim_{k\to+\infty} \|\nabla f(x_k) - \nabla f(p)\| = 0$.

Proof. The structure of the proof is similar to that of Theorem 3.1. We therefore mainly focus on the Lyapunov analysis. Take $p \in S$. Let us consider the sequence (V_k) defined by, for each $k \geq 1$

$$V_k = \frac{1}{2}\|(x_k - p) + c\Big(\frac{1}{h}(x_k - x_{k-1}) + \beta_b B(x_k) + \beta_f \nabla f(x_{k-1}) - A_\beta(p)\Big)\|^2 + \frac{\delta}{2}\|x_k - p\|^2,$$

where c, δ are positive coefficients to adjust. For $k \geq 1$, let us briefly write V_k as follows

$$V_k = \frac{1}{2}\|v_k\|^2 + \frac{\delta}{2}\|x_k - p\|^2,$$

with $v_k = (x_k - p) + c\left(\frac{1}{h}(x_k - x_{k-1}) + \beta_b B(x_k) + \beta_f \nabla f(x_{k-1}) - A_\beta(p)\right)$.
By definition of (v_k), we have

$$v_{k+1} = (x_{k+1} - p) + c\left(\frac{1}{h}(x_{k+1} - x_k) + \beta_b B(x_{k+1}) + \beta_f \nabla f(x_k) - A_\beta(p)\right).$$

Moreover, according to the formulation of the algorithm (DINAAM-split), we have

$$\begin{aligned} v_k &= c\left(\frac{1}{h}(x_{k+1} - x_k) + \gamma(x_{k+1} - x_k) + \beta_b B(x_{k+1}) + \beta_f \nabla f(x_k) - A_\beta(p) + hB(x_{k+1}) + h\nabla f(x_k)\right) \\ &\quad + (x_{k+1} - p) - (x_{k+1} - x_k) \\ &= v_{k+1} + (c\gamma - 1)(x_{k+1} - x_k) + chB(x_{k+1}) + ch\nabla f(x_k). \end{aligned}$$

Set $X_k = x_{k+1} - x_k, Y_k = B(x_{k+1}) - B(p), Z_k = \nabla f(x_k) - \nabla f(p)$. Taking $\delta := c\gamma - 1$, we obtain

$$\begin{aligned} V_{k+1} - V_k = &-\frac{1}{2}(c\gamma - 1)^2\|X_k\|^2 - \frac{1}{2}c^2h^2\|Y_k + Z_k\|^2 - c(c\gamma - 1)h\langle X_k, Y_k + Z_k\rangle \\ &- \frac{1}{2}(c\gamma - 1)\|X_k\|^2 - ch\langle x_{k+1} - p, B(x_{k+1}) + \nabla f(x_k)\rangle - \frac{c(c\gamma - 1)}{h}\|X_k\|^2 \\ &- c^2\langle X_k, Y_k + Z_k\rangle - c(c\gamma - 1)\langle \beta_b Y_k + \beta_f Z_k, X_k\rangle - c^2h\langle \beta_b Y_k + \beta_f Z_k, Y_k + Z_k\rangle. \end{aligned}$$

Using the fact that $p \in S$, ∇f is monotone, and B is λ-cocoercive, we have

$$\begin{aligned} &-ch\langle x_{k+1} - p, B(x_{k+1}) + \nabla f(x_k)\rangle \\ &= -ch\langle x_{k+1} - p, B(x_{k+1}) - B(p)\rangle - ch\langle x_{k+1} - p, \nabla f(x_k) - \nabla f(p)\rangle \\ &\leq -ch\lambda\|B(x_{k+1}) - B(p)\|^2 - ch\langle x_{k+1} - x_k, \nabla f(x_k) - \nabla f(p)\rangle. \end{aligned}$$

By combining the previous results, we get

$$\begin{aligned} V_{k+1} - V_k &+ \left[\frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma - 1)}{h}\right]\|X_k\|^2 \\ &+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_f + ch\right]\langle X_k, Z_k\rangle \\ &+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b\right]\langle X_k, Y_k\rangle + \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2\right]\|Y_k\|^2 \\ &+ \left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\|Z_k\|^2 + \left[c^2h(\beta_b + \beta_f) + c^2h^2\right]\langle Z_k, Y_k\rangle \leq 0. \end{aligned} \tag{4.6}$$

Let (Γ_k) be the sequence defined by

$$\Gamma_k = f(x_k) - f(p) - \langle \nabla f(p), x_k - p\rangle, \text{ for } k \geq 0.$$

Since f is convex, we have $\Gamma_k \geq 0$, for all $k \geq 0$. According to the gradient descent lemma, and since ∇f is L-Lipschitz, we have

$$\begin{aligned}\langle X_k, Z_k\rangle &= \langle x_{k+1} - x_k, \nabla f(x_k)\rangle - \langle x_{k+1} - x_k, \nabla f(p)\rangle \\ &\geq f(x_{k+1}) - f(x_k) - \frac{L}{2}\|x_{k+1} - x_k\|^2 + \Gamma_{k+1} - \Gamma_k + f(x_k) - f(x_{k+1}) \\ &= \Gamma_{k+1} - \Gamma_k - \frac{L}{2}\|X_k\|^2. \end{aligned} \tag{4.7}$$

Let us define

$$E_k = V_k + \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_f + ch\right]\Gamma_k,$$

for $k \geq 1$. Indeed, (E_k) will serve us as a discrete energy function. Indeed, it is clear that (E_k) is a sequence of nonnegative numbers. From (4.6), (4.7) and the definition of (E_k), we obtain

$$\begin{aligned} E_{k+1} - E_k &+ \left[\frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma - 1)}{h} - \frac{L}{2}\left(c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_f + ch\right)\right]\|X_k\|^2 \\ &+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b\right]\langle X_k, Y_k\rangle + \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2\right]\|Y_k\|^2 \\ &+ \left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\|Z_k\|^2 + \left[c^2h(\beta_b + \beta_f) + c^2h^2\right]\langle Z_k, Y_k\rangle \leq 0. \end{aligned} \tag{4.8}$$

Let us eliminate Z_k from this relation by using the elementary algebraic inequality

$$\left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\|Z_k\|^2 + \left[c^2h(\beta_b + \beta_f) + c^2h^2\right]\langle Z_k, Y_k\rangle \geq -\frac{c^2h^2(\beta_b + \beta_f + h)^2}{4h\beta_f + 2h^2}\|Y_k\|^2.$$

Then, from (4.8) we deduce that

$$\begin{aligned} E_{k+1} - E_k &+ \left[\frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma - 1)}{h} - \frac{L}{2}\left(c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_f + ch\right)\right]\|X_k\|^2 \\ &+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b\right]\langle X_k, Y_k\rangle \\ &+ \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2 - \frac{c^2h^2(\beta_b + \beta_f + h)^2}{4h\beta_f + 2h^2}\right]\|Y_k\|^2 \leq 0. \end{aligned}$$

Equivalently

$$E_{k+1} - E_k + \mathcal{S}_k \leq 0,$$

where

$$\begin{aligned} \mathcal{S}_k &= \left[\frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma - 1)}{h} - \frac{L}{2}\left(c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_f + ch\right)\right]\|X_k\|^2 \\ &+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b\right]\langle X_k, Y_k\rangle \\ &+ \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2 - \frac{c^2h^2(\beta_b + \beta_f + h)^2}{4h\beta_f + 2h^2}\right]\|Y_k\|^2. \end{aligned}$$

We have $\mathcal{S}_k = q(X_k, Y_k)$ where $q : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is the quadratic form

$$q(X_k, Y_k) := a\|X_k\|^2 + b\langle X_k, Y_k\rangle + g\|Y_k\|^2,$$

with

$$\begin{aligned}
a &= \frac{1}{2}(c\gamma-1)^2+\frac{1}{2}(c\gamma-1)+\frac{c(c\gamma-1)}{h}-\frac{L}{2}\left(c(c\gamma-1)h+c^2+c(c\gamma-1)\beta_f+ch\right)\\
b &= c(c\gamma-1)h+c^2+c(c\gamma-1)\beta_b\\
g &= ch\lambda+c^2h\beta_b+\frac{1}{2}c^2h^2-\frac{c^2h^2(\beta_b+\beta_f+h)^2}{4h\beta_f+2h^2}.
\end{aligned}$$

The above coefficients differ from those involved in the Lyapunov analysis of Theorem 3.1 only by a, where L enters. Since, for h small, $a\sim\dfrac{c(c\gamma-1)}{h}$ it is immediate to verify that $a>0$ for h sufficiently small (depending now on L). Moreover the term with coefficient L induces a negligable perturbation. So, by using the same argument as the proof of Theorem 3.1, under the condition

$$4\lambda>\frac{2}{\gamma}\left(\beta_b+\frac{1}{\gamma}\right)+\frac{1}{\gamma}\frac{(\beta_b-\beta_f)^2}{\beta_f}+\frac{2}{\gamma}\sqrt{\left(\beta_b+\frac{1}{\gamma}\right)^2+\frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}}.$$

there exists c such that $c\gamma-1>0$, and $4ag-b^2>0$ is satisfied for h sufficiently small. Therefore, there exists a positive real number μ such that for any $k\geq 1$,

$$E_{k+1}-E_k+\mu\|X_k\|^2+\mu\|Y_k\|^2\leq 0. \tag{4.9}$$

The rest of the proof is similar to that of Theorem 3.1, so we omit it. ■

4.2 Errors, perturbations

Let us examine the effect of the introduction of perturbations, errors in the algorithm (DINAAM-split). Let us start from the perturbed version of (DINAM)

$$\ddot{x}(t)+\gamma\dot{x}(t)+\nabla f(x(t))+B(x(t))+\beta_f\nabla^2 f(x(t))\dot{x}(t)+\beta_b B'(x(t))\dot{x}(t)=e(t), \quad \text{(DINAM-pert)}$$

where the right-handside $e(\cdot)$ takes into account perturbations, errors. A similar discretization as before gives

$$\begin{aligned}
&\frac{1}{h^2}(x_{k+1}-2x_k+x_{k-1})+\frac{\gamma}{h}(x_{k+1}-x_k)+\frac{\beta_b}{h}(B(x_{k+1})-B(x_k))\\
&\qquad+\frac{\beta_f}{h}(\nabla f(x_k)-\nabla f(x_{k-1}))+B(x_{k+1})+\nabla f(x_k)=e_k.
\end{aligned}\tag{4.10}$$

After expanding (4.10), we obtain

$$\begin{aligned}
x_{k+1}+\frac{h(\beta_b+h)}{1+\gamma h}B(x_{k+1}) \;=\; & x_k+\frac{1}{1+\gamma h}(x_k-x_{k-1})+\frac{h\beta_b}{1+\gamma h}B(x_k)\\
&-\frac{h\beta_f}{1+\gamma h}(\nabla f(x_k)-\nabla f(x_{k-1}))-\frac{h^2}{1+\gamma h}\nabla f(x_k)+\frac{h^2}{1+\gamma h}e_k.
\end{aligned}\tag{4.11}$$

Set $s:=\dfrac{h}{1+\gamma h}$ and $\alpha:=\dfrac{1}{1+\gamma h}$. So we have

$$x_{k+1}+s\mathcal{B}_h(x_{k+1})=y_k, \tag{4.12}$$

where $\mathcal{B}_h=(h+\beta_b)B$, and

$$y_k=x_k+\alpha(x_k-x_{k-1})+s\beta_b B(x_k)-s(h+\beta_f)\nabla f(x_k)+s\beta_f\nabla f(x_{k-1})+she_k. \tag{4.13}$$

From (4.12) we get $x_{k+1} = (\mathrm{Id} + s\mathcal{B}_h)^{-1}(y_k)$. Combining the above results, we obtain the algorithm

(DINAAM-split-pert):

Initialize: $x_0 \in \mathcal{H}, x_1 \in \mathcal{H}$

$$\alpha = \frac{1}{1+\gamma h}, \quad s = \frac{h}{1+\gamma h},$$

$$y_k = x_k + \alpha(x_k - x_{k-1}) + s\beta_b B(x_k) - s(h+\beta_f)\nabla f(x_k) + s\beta_f \nabla f(x_{k-1}) + she_k,$$

$$x_{k+1} = (\mathrm{Id} + s\mathcal{B}_h)^{-1}(y_k).$$

Theorem 4.2 *Let us make the assumptions of Theorem 4.1, and suppose that the sequence (e_k) of perturbations, errors satisfies:*

$$\sum_{k=1}^{+\infty} \|e_k\| < +\infty.$$

Then, there exists h^* such that for all $0 < h < h^*$, the sequence (x_k) generated by the algorithm (DINAAM-split-pert) has the following properties:

(i) (x_k) converges weakly to an element in S;

(ii) $\displaystyle\sum_{k=1}^{+\infty} \|x_k - x_{k-1}\|^2 < +\infty, \sum_{k=1}^{+\infty} \|A(x_k)\|^2 < +\infty,$
$\displaystyle\sum_{k=1}^{+\infty} \|\nabla f(x_k) - \nabla f(x_{k-1})\|^2 < +\infty,$ *and* $\displaystyle\sum_{k=1}^{+\infty} \|B(x_k) - B(x_{k-1})\|^2 < +\infty;$

(iii) $\lim_{k\to+\infty} \|x_{k+1} - x_k\| = 0, \lim_{k\to+\infty} \|B(x_k) - B(p)\| = 0,$ *and* $\lim_{k\to+\infty} \|\nabla f(x_k) - \nabla f(p)\| = 0.$

Passing from the Lyapunov analysis in the unperturbed case to the perturbed case is classical procedure, see [9] for example. It is based on a similar Lyapunov analysis and the use of the following discrete version of the Gronwall Lemma, see [9, Lemma A.9.].

Lemma 4.1 *Let a be a positive real number and $(y_k), (g_k)$ be nonnegative sequences such that for all $k \geq 0$, we have*

$$\frac{1}{2}y_k^2 \leq \frac{1}{2}a^2 + \sum_{0\leq i<k} g_i y_i.$$

Then, the following inequality holds for all $k \geq 0$

$$y_k \leq a + \sum_{0\leq i<k} g_i.$$

Proof. (of Lemma 4.1) For any $\varepsilon > 0$, let us define the sequence $(z_k(\varepsilon))$ given by

$$z_k(\varepsilon) = \frac{1}{2}(a+\varepsilon)^2 + \sum_{0\leq i<k} g_i y_i.$$

We have $z_{k+1}(\varepsilon) - z_k(\varepsilon) = g_k y_k$ and $\frac{1}{2}y_k^2 \leq z_k(\varepsilon)$ for $k \geq 0$. Thus,

$$z_{k+1}(\varepsilon) - z_k(\varepsilon) \leq \sqrt{2} g_k \sqrt{z_k(\varepsilon)}. \tag{4.14}$$

Moreover, by the definition of $(z_k(\varepsilon))$, we deduce that $(z_k(\varepsilon))$ is a nondecreasing sequence as well. Hence,

$$\sqrt{z_{k+1}(\varepsilon)} - \sqrt{z_k(\varepsilon)} = \frac{z_{k+1}(\varepsilon) - z_k(\varepsilon)}{\sqrt{z_{k+1}(\varepsilon)} + \sqrt{z_k(\varepsilon)}} \leq \frac{z_{k+1}(\varepsilon) - z_k(\varepsilon)}{2\sqrt{z_k(\varepsilon)}}. \tag{4.15}$$

From (4.14) and (4.15), we obtain

$$\sqrt{z_{k+1}(\varepsilon)} - \sqrt{z_k(\varepsilon)} \leq \frac{1}{\sqrt{2}} g_k. \tag{4.16}$$

That implies

$$\sqrt{z_k(\varepsilon)} \leq \sqrt{z_0(\varepsilon)} + \frac{1}{\sqrt{2}} \sum_{0 \leq i < k} g_i,$$

for all $k \geq 0$. Then,

$$y_k \leq \sqrt{2z_k(\varepsilon)} \leq \sqrt{2z_0(\varepsilon)} + \sum_{0 \leq i < k} g_i = a + \varepsilon + \sum_{0 \leq i < k} g_i.$$

Taking $\varepsilon \to 0$, we obtain

$$y_k \leq a + \sum_{0 \leq i < k} g_i.$$

This completes the proof. ■

Proof. (of Theorem 4.2) The proof is similar to that of Theorem 4.1. It uses the following sequence (E_k) as a discrete energy function

$$E_k = V_k + \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_f\right] \Gamma_k,$$

where $c > \frac{1}{\gamma}$ is coefficient to adjust, and

$$\begin{aligned} V_k &= \frac{1}{2}\|(x_k - p) + c(\frac{1}{h}(x_k - x_{k-1}) + A_\beta(x_k) - A_\beta(p))\|^2 + \frac{c\gamma - 1}{2}\|x_k - p\|^2, \\ \Gamma_k &= f(x_k) - f(p) - \langle \nabla f(p), x_k - p \rangle. \end{aligned}$$

By setting $X_k = x_{k+1} - x_k, Y_k = B(x_{k+1}) - B(p), Z_k = \nabla f(x_k) - \nabla f(p)$ for $k \geq 0$ and following the same argument as in the proof of Theorem 4.1, we have

$$\begin{aligned} E_{k+1} - E_k + \left[\frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma - 1)}{h}\right] \|X_k\|^2 \\ + \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b\right] \langle X_k, Y_k \rangle + \left[ch\lambda + c^2 h\beta_b + \frac{1}{2}c^2h^2\right] \|Y_k\|^2 \\ + \left[c^2 h\beta_f + \frac{1}{2}c^2h^2\right] \|Z_k\|^2 + \left[c^2h(\beta_b + \beta_f) + c^2h^2\right] \langle Z_k, Y_k \rangle \leq \epsilon_k. \end{aligned} \tag{4.17}$$

Here,

$$\begin{aligned} \epsilon_k = &- \frac{1}{2}c^2h^2\|e_k\|^2 + c^2h^2\langle Y_k + Z_k, e_k\rangle + \left[c(c\gamma - 1)h + c^2\right] \langle X_k, e_k\rangle \\ &+ ch\langle x_{k+1} - p, e_k\rangle + c^2h\langle \beta_b Y_k + \beta_f Z_k, e_k\rangle. \end{aligned} \tag{4.18}$$

According to an elementary inequality, we have that

$$\langle X_k, e_k\rangle \leq \frac{1}{2\eta}\|X_k\|^2 + \frac{\eta}{2}\|e_k\|^2, \tag{4.19}$$

holds for any $\eta > 0$. Moreover, by using Cauchy-Schwarz inequality, and the fact that $B, \nabla f$ are Lipschitz, we have

$$\langle Y_k, e_k\rangle \leq \|Y_k\|.\|e_k\| \leq \frac{1}{\lambda}\|x_{k+1} - p\|.\|e_k\|, \tag{4.20}$$
$$\langle Z_k, e_k\rangle \leq \|Z_k\|.\|e_k\| \leq L\|x_{k+1} - p\|.\|e_k\|. \tag{4.21}$$

Combining (4.18)-(4.21), we obtain

$$\begin{aligned}\epsilon_k \leq &-\frac{1}{2}c^2h^2\|e_k\|^2 + \frac{c(c\gamma-1)h + c^2}{2\eta}\|X_k\|^2 + \frac{\eta}{2}[c(c\gamma-1)h + c^2]\|e_k\|^2\\ &+\left[ch + \frac{c^2h^2 + c^2h\beta_b}{\lambda} + (c^2h^2 + c^2h\beta_f)L\right]\|x_{k+1} - p\|\|e_k\|.\end{aligned} \tag{4.22}$$

From (4.17) and (4.22), we deduce that

$$\begin{aligned}E_{k+1} - E_k &+ \left[\frac{1}{2}(c\gamma-1)^2 + \frac{1}{2}(c\gamma-1) + \frac{c(c\gamma-1)}{h} - \frac{c(c\gamma-1)h + c^2}{2\eta}\right]\|X_k\|^2\\ &+ \left[c(c\gamma-1)h + c^2 + c(c\gamma-1)\beta_b\right]\langle X_k, Y_k\rangle\\ &+ \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2\right]\|Y_k\|^2\\ &+ \left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\|Z_k\|^2\\ &+ \left[c^2h(\beta_b + \beta_f) + c^2h^2\right]\langle Z_k, Y_k\rangle \leq \epsilon'_k,\end{aligned} \tag{4.23}$$

with

$$\epsilon'_k = \frac{\eta}{2}[c(c\gamma-1)h + c^2]\|e_k\|^2 + \left[ch + \frac{c^2h^2 + c^2h\beta_b}{\lambda} + (c^2h^2 + c^2h\beta_f)L\right]\|x_{k+1} - p\|\|e_k\|.$$

Let us eliminate Z_k from this relation by using the elementary algebraic inequality

$$\begin{aligned}\left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\|Z_k\|^2 &+ \left[c^2h(\beta_b + \beta_f) + c^2h^2\right]\langle Z_k, Y_k\rangle\\ &\geq -\frac{c^2h^2\left(\beta_b + \beta_f + h\right)^2}{4\left(h\beta_f + \frac{1}{2}h^2\right)}\|Y_k\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}E_{k+1} - E_k &+ \left[\frac{1}{2}(c\gamma-1)^2 + \frac{1}{2}(c\gamma-1) + \frac{c(c\gamma-1)}{h} - \frac{c(c\gamma-1)h + c^2}{2\eta}\right]\|X_k\|^2\\ &+ \left[c(c\gamma-1)h + c^2 + c(c\gamma-1)\beta_b\right]\langle X_k, Y_k\rangle\\ &+ \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2 - \frac{c^2h^2\left(\beta_b + \beta_f + h\right)^2}{4\left(h\beta_f + \frac{1}{2}h^2\right)}\right]\|Y_k\|^2 \leq \epsilon'_k.\end{aligned} \tag{4.24}$$

Equivalently

$$E_{k+1} - E_k + \mathcal{S} \leq \epsilon'_k, \tag{4.25}$$

where

$$\begin{aligned}\mathcal{S} = &\left[\frac{1}{2}(c\gamma-1)^2+\frac{1}{2}(c\gamma-1)+\frac{c(c\gamma-1)}{h}-\frac{c(c\gamma-1)h+c^2}{2\eta}\right]\|X_k\|^2\\ &+\left[c(c\gamma-1)h+c^2+c(c\gamma-1)\beta_b\right]\langle X_k,Y_k\rangle\\ &+\left[ch\lambda+c^2h\beta_b+\frac{1}{2}c^2h^2-\frac{c^2h^2\left(\beta_b+\beta_f+h\right)^2}{4\left(h\beta_f+\frac{1}{2}h^2\right)}\right]\|Y_k\|^2.\end{aligned}$$

Similarly, we have $\mathcal{S}_k = q(X_k, Y_k)$ where $q : \mathcal{H}\times\mathcal{H}\to\mathbb{R}$ is the quadratic form

$$q(X_k,Y_k) := a\|X_k\|^2+b\langle X_k,Y_k\rangle+g\|Y_k\|^2,$$

with

$$\begin{aligned}a &= \frac{1}{2}(c\gamma-1)^2+\frac{1}{2}(c\gamma-1)+\frac{c(c\gamma-1)}{h}-\frac{c(c\gamma-1)h+c^2}{2\eta},\\ b &= c(c\gamma-1)h+c^2+c(c\gamma-1)\beta_b,\\ g &= ch\lambda+c^2h\beta_b+\frac{1}{2}c^2h^2-\frac{c^2h^2\left(\beta_b+\beta_f+h\right)^2}{4\left(h\beta_f+\frac{1}{2}h^2\right)}.\end{aligned}$$

We choose $\eta>0$ such that $a>0$. That means

$$\eta > \frac{c(c\gamma-1)h^2+c^2h}{(c\gamma-1)^2h+(c\gamma-1)h+c(c\gamma-1)}.$$

Since the time step h will be taken small, there exists $\eta_0>0$ such that $\eta<\eta_0$.
Again, thanks to Lemma 2.3, we have that q is positive definite if and only if $4ag-b^2>0$.
By using the same argument as in the proof of Theorem 3.1, we have the existence of c such that $\mathcal{S}>0$.
To ensure the existence of such c, we need

$$4\lambda > \frac{2}{\gamma}\left(\beta_b+\frac{1}{\gamma}\right)+\frac{1}{\gamma}\frac{(\beta_b-\beta_f)^2}{\beta_f}+\frac{2}{\gamma}\sqrt{\left(\beta_b+\frac{1}{\gamma}\right)^2+\frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}}.$$

Therefore, there exists positive real number μ such that for any $k\geq 1$,

$$E_{k+1}-E_k+\mu\|X_k\|^2+\mu\|Y_k\|^2\leq\epsilon_k'. \tag{4.26}$$

From (4.26) we deduce that

$$E_{k+1}\leq E_1+\sum_{1\leq i<k+1}\epsilon_i'.$$

Taking into account the form of the energy sequence (E_k), we obtain

$$\frac{c\gamma-1}{2}\|x_{k+1}-p\|^2\leq E_1+\sum_{1\leq i<k+1}\epsilon_i'. \tag{4.27}$$

According to the assumption $\sum_{k=1}^{\infty}\|e_k\|<+\infty$, this implies that $\sum_{k=1}^{\infty}\|e_k\|^2<+\infty$. Therefore, there exists

$C > 0$ such that

$$\begin{aligned}\sum_{1\leq i<k+1}\epsilon_i' &= \left[ch + \frac{c^2h^2 + c^2h\beta_b}{\lambda} + (c^2h^2 + c^2h\beta_f)L\right]\sum_{1\leq i<k+1}\|x_{i+1}-p\|\|e_i\| \\ &\quad + \frac{\eta}{2}[c(c\gamma-1)h + c^2]\sum_{1\leq i<k+1}\|e_k\|^2 \\ &\leq \left[ch + \frac{c^2h^2 + c^2h\beta_b}{\lambda} + (c^2h^2 + c^2h\beta_f)L\right]\sum_{1\leq i<k+1}\|x_{i+1}-p\|\|e_i\| + C. \end{aligned} \tag{4.28}$$

From (4.27) and (4.28), we deduce that

$$\frac{c\gamma-1}{2}\|x_{k+1}-p\|^2 \leq E_1 + C + \left[ch + \frac{c^2h^2 + c^2h\beta_b}{\lambda} + (c^2h^2 + c^2h\beta_f)L\right]\sum_{1\leq i<k+1}\|e_i\|\|x_{i+1}-p\|.$$

More precisely, we have

$$\frac{1}{2}\|x_{k+1}-p\|^2 \leq \frac{1}{2}C_0^2 + c_0\sum_{1\leq i<k+1}\|e_i\|\|x_{i+1}-p\|, \tag{4.29}$$

where

$$C_0 = \sqrt{\frac{2(E_1+C)}{c\gamma-1}}, \quad c_0 = ch + \frac{c^2h^2 + c^2h\beta_b}{\lambda} + (c^2h^2 + c^2h\beta_f)L.$$

Now, applying Lemma 4.1 to (4.29), we obtain

$$\|x_{k+1}-p\| \leq C_0 + c_0\sum_{1\leq i<k+1}\|e_i\| < +\infty. \tag{4.30}$$

Therefore, $(\|x_{k+1}-p\|)$ and consequently $(\|x_k\|)$ is a bounded sequence.
Returning to (4.28), according to the boundedness of $(\|x_{k+1}-p\|)$ and the assumption of (e_k), we obtain

$$\sum_{k=1}^{\infty}\epsilon_k' < +\infty.$$

The rest of the proof is similar to that of Theorem 4.1, so we omit here. The above inequality allows us to estimate $\sum_{k=1}^{\infty}\|X_k\|^2$ and $\sum_{k=1}^{\infty}\|Y_k\|^2$. ■

5 A variant of the proximal-gradient algorithm

In this section, we consider a variant of the previous proximal-gradient algorithm, where the role of the operators is reversed. This allows us to weaken the hypothesis on f, *i.e.*, we suppose that f is a $\mathcal{C}^1$ convex function whose gradient is Lipschitz on the bounded sets (instead of globally Lipschitz). We consider the following implicit finite-difference scheme for (DINAM):

$$\begin{aligned}&\frac{1}{h^2}(x_{k+1}-2x_k+x_{k-1}) + \frac{\gamma}{h}(x_{k+1}-x_k) + \frac{\beta_b}{h}(B(x_k)-B(x_{k-1})) \\ &\quad + \frac{\beta_f}{h}(\nabla f(x_{k+1})-\nabla f(x_k)) + B(x_k) + \nabla f(x_{k+1}) = 0. \end{aligned} \tag{5.1}$$

The temporal discretization of the Hessian driven damping term $\beta_b B(x(t))\dot{x}(t)$ is taken equal to $\frac{\beta_b}{h}(B(x_k)-B(x_{k-1}))$ instead of $\frac{\beta_f}{h}(B(x_{k+1})-B(x_k))$. After expanding (5.1), we obtain

$$x_{k+1}+\frac{h^2}{1+\gamma h}\nabla f(x_{k+1})+\frac{h\beta_f}{1+\gamma h}\nabla f(x_{k+1})=x_k+\frac{1}{1+\gamma h}(x_k-x_{k-1})+\frac{h\beta_f}{1+\gamma h}\nabla f(x_k)$$
$$-\frac{h\beta_b}{1+h\gamma}(B(x_k)-B(x_{k-1}))-\frac{h^2}{1+h\gamma}B(x_k). \tag{5.2}$$

Set $s:=\dfrac{h}{1+\gamma h}$ and $\alpha:=\dfrac{1}{1+\gamma h}$. So we have

$$x_{k+1}+s\mathcal{F}_h(x_{k+1})=y_k, \tag{5.3}$$

where

$$\mathcal{F}_h=(h+\beta_f)\nabla f, \tag{5.4}$$
$$y_k=x_k+\alpha(x_k-x_{k-1})+s\beta_f\nabla f(x_k)-s(h+\beta_b)B(x_k)+s\beta_b B(x_{k-1}). \tag{5.5}$$

From (5.3) we get $x_{k+1}=(\mathrm{Id}+s\mathcal{F}_h)^{-1}(y_k)$, which gives the following algorithm

(DINAAM-split-var):

Initialize: $x_0\in\mathcal{H}, x_1\in\mathcal{H}$

$\alpha=\dfrac{1}{1+\gamma h},\quad s=\dfrac{h}{1+\gamma h},$

$y_k=x_k+\alpha(x_k-x_{k-1})+s\beta_f\nabla f(x_k)-s(h+\beta_b)B(x_k)+s\beta_b B(x_{k-1}),$

$x_{k+1}=(\mathrm{Id}+s\mathcal{F}_h)^{-1}(y_k)=\mathrm{prox}_{s(h+\beta_f)f}(y_k).$

Theorem 5.1 *Let $B:\mathcal{H}\to\mathcal{H}$ be a λ-cocoercive operator and $f:\mathcal{H}\to\mathbb{R}$ be a $\mathcal{C}^1$ convex function whose gradient is Lipschitz continuous on the bounded sets. Suppose the positive parameters $\lambda,\gamma,\beta_b,\beta_f$ satisfy*

$$\beta_f>0,\ \text{and}\ \lambda\gamma>\frac{(\beta_b-\beta_f)^2}{4\beta_f}+\frac{1}{2}\left(\beta_b+\frac{1}{\gamma}\right)+\frac{1}{2}\sqrt{\left(\beta_b+\frac{1}{\gamma}\right)^2+\frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}}.$$

Then, there exists h^* such that for all $0<h<h^*$, the sequence (x_k) generated by the algorithm (DINAAM-split-var) has the following properties:

(i) (x_k) converges weakly to an element in S;

(ii) $\displaystyle\sum_{k=1}^{+\infty}\|x_k-x_{k-1}\|^2<+\infty,\ \sum_{k=1}^{+\infty}\|A(x_k)\|^2<+\infty,$
$\displaystyle\sum_{k=1}^{+\infty}\|\nabla f(x_k)-\nabla f(x_{k-1})\|^2<+\infty,\ \text{and}\ \sum_{k=1}^{+\infty}\|B(x_k)-B(x_{k-1})\|^2<+\infty;$

(iii) (pointwise estimates)

$$\lim_{k\to+\infty}\|x_{k+1}-x_k\|=0,\ \lim_{k\to+\infty}\|B(x_k)-B(p)\|=0,\ \text{and}\ \lim_{k\to+\infty}\|\nabla f(x_k)-\nabla f(p)\|=0.$$

Proof. Let us consider the sequence (V_k) given by, for each $k \geq 1$

$$V_k = \frac{1}{2}\|(x_k - p) + c(\frac{1}{h}(x_k - x_{k-1}) + \beta_f \nabla f(x_k) + \beta_b B(x_{k-1}) - A_\beta(p))\|^2 + \frac{\delta}{2}\|x_k - p\|^2,$$

where c, δ are positive coefficients to adjust. For $k \geq 1$, let us briefly write V_k as follows

$$V_k = \frac{1}{2}\|v_k\|^2 + \frac{\delta}{2}\|x_k - p\|^2,$$

with $v_k = (x_k - p) + c\left(\frac{1}{h}(x_k - x_{k-1}) + \beta_f \nabla f(x_k) + \beta_b B(x_{k-1}) - A_\beta(p)\right)$.
Using successively the definition of (v_k), we obtain

$$v_{k+1} = (x_{k+1} - p) + c\left(\frac{1}{h}(x_{k+1} - x_k) + \beta_f \nabla f(x_{k+1}) + \beta_b B(x_k) - A_\beta(p)\right).$$

Moreover, by using the formulation of the algorithm, we have that

$$\begin{aligned} v_k &= c\big(\frac{1}{h}(x_{k+1} - x_k) + \gamma(x_{k+1} - x_k) + \beta_f \nabla f(x_{k+1}) + \beta_b B(x_k) - A_\beta(p) + h\nabla f(x_{k+1}) + hB(x_k)\big) \\ &\quad + (x_{k+1} - p) - (x_{k+1} - x_k) = v_{k+1} + (c\gamma - 1)(x_{k+1} - x_k) + ch\nabla f(x_{k+1}) + chB(x_k). \end{aligned}$$

Set $X_k = x_{k+1} - x_k, Y_k = B(x_k) - B(p), Z_k = \nabla f(x_{k+1}) - \nabla f(p)$. Taking $\delta := c\gamma - 1$, we get

$$\begin{aligned} V_{k+1} - V_k = &-\frac{1}{2}(c\gamma - 1)^2\|X_k\|^2 - \frac{1}{2}c^2h^2\|Y_k + Z_k\|^2 - c(c\gamma - 1)h\langle X_k, Y_k + Z_k\rangle \\ &- \frac{1}{2}(c\gamma - 1)\|X_k\|^2 - ch\langle x_{k+1} - p, \nabla f(x_{k+1}) + B(x_k)\rangle - \frac{c(c\gamma - 1)}{h}\|X_k\|^2 \\ &- c^2\langle X_k, Y_k + Z_k\rangle - c(c\gamma - 1)\langle \beta_b Y_k + \beta_f Z_k, X_k\rangle - c^2h\langle \beta_b Y_k + \beta_f Z_k, Y_k + Z_k\rangle. \end{aligned}$$

Using the fact that $p \in S$, ∇f is monotone, and B is λ-cocoercive, we have

$$\begin{aligned} &-ch\langle x_{k+1} - p, \nabla f(x_{k+1}) + B(x_k)\rangle \\ &= -ch\langle x_{k+1} - p, \nabla f(x_{k+1}) - \nabla f(p)\rangle - ch\langle x_{k+1} - p, B(x_k) - B(p)\rangle \\ &\leq -ch\lambda\|B(x_k) - B(p)\|^2 - ch\langle x_{k+1} - x_k, B(x_k) - B(p)\rangle. \end{aligned}$$

By combining the previous results, we get

$$\begin{aligned} V_{k+1} - V_k &+ \left[\frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma - 1)}{h}\right]\|X_k\|^2 \\ &+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_f\right]\langle X_k, Z_k\rangle \\ &+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b + ch\right]\langle X_k, Y_k\rangle + \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2\right]\|Y_k\|^2 \\ &+ \left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\|Z_k\|^2 + \left[c^2h(\beta_b + \beta_f) + c^2h^2\right]\langle Z_k, Y_k\rangle \leq 0. \end{aligned} \tag{5.6}$$

Let (Γ_k) be a sequence defined by

$$\Gamma_k = f(x_k) - f(p) - \langle \nabla f(p), x_k - p\rangle, \text{ for } k \geq 0.$$

Since f is convex, we have $\Gamma_k \geq 0$, for all $k \geq 0$. Moreover,

$$\begin{aligned}\langle X_k, Z_k\rangle &= \langle x_{k+1} - x_k, \nabla f(x_{k+1})\rangle - \langle x_{k+1} - x_k, \nabla f(p)\rangle \\ &\geq f(x_{k+1}) - f(x_k) + \Gamma_{k+1} - \Gamma_k + f(x_k) - f(x_{k+1}) \\ &= \Gamma_{k+1} - \Gamma_k. \end{aligned} \tag{5.7}$$

Let us define

$$E_k = V_k + \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_f\right]\Gamma_k,$$

for $k \geq 1$. (E_k) will serve us as a discrete energy function. Indeed, it is clear that (E_k) is a sequence of nonnegative numbers. From (5.6), (5.7) and the definition of (E_k), we obtain

$$\begin{aligned} E_{k+1} - E_k &+ \left[\frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma - 1)}{h}\right]\|X_k\|^2 \\ &+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b + ch\right]\langle X_k, Y_k\rangle + \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2\right]\|Y_k\|^2 \\ &+ \left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\|Z_k\|^2 + \left[c^2h(\beta_b + \beta_f) + c^2h^2\right]\langle Z_k, Y_k\rangle \leq 0. \end{aligned} \tag{5.8}$$

Let us eliminate Z_k from this relation by using the elementary algebraic inequality

$$\left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\|Z_k\|^2 + \left[c^2h(\beta_b + \beta_f) + c^2h^2\right]\langle Z_k, Y_k\rangle \geq -\frac{c^2h^2(\beta_b + \beta_f + h)^2}{4h\beta_f + 2h^2}\|Y_k\|^2.$$

From (5.8) we deduce that $E_{k+1} - E_k + \mathcal{S}_k \leq 0$, where

$$\begin{aligned}\mathcal{S}_k &= \left[\frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma - 1)}{h}\right]\|X_k\|^2 \\ &+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b + ch\right]\langle X_k, Y_k\rangle \\ &+ \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2 - \frac{c^2h^2(\beta_b + \beta_f + h)^2}{4h\beta_f + 2h^2}\right]\|Y_k\|^2. \end{aligned}$$

We have $\mathcal{S}_k = q(X_k, Y_k)$ where $q : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is the quadratic form

$$q(X_k, Y_k) := a\|X_k\|^2 + b\langle X_k, Y_k\rangle + g\|Y_k\|^2,$$

with

$$\begin{aligned} a &= \frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma - 1)}{h} \\ b &= c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b + ch \\ g &= ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2 - \frac{c^2h^2(\beta_b + \beta_f + h)^2}{4h\beta_f + 2h^2}. \end{aligned}$$

By using the same argument as the proof of Theorem 4.1, we obtain the existence of c such that $\mathcal{S}_k > 0$. To ensure the existence of such c, we need

$$4\lambda > \frac{2}{\gamma}\left(\beta_b + \frac{1}{\gamma}\right) + \frac{1}{\gamma}\frac{(\beta_b - \beta_f)^2}{\beta_f} + \frac{2}{\gamma}\sqrt{\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b - \beta_f)^2}{\gamma\beta_f}}.$$

Therefore, there exists positive real number μ such that for any $k \geq 1$,

$$E_{k+1} - E_k + \mu\|X_k\|^2 + \mu\|Y_k\|^2 \leq 0. \tag{5.9}$$

The rest of the proof is similar to that of Theorem 4.1, so we omit it. ■

6 Numerical illustrations

Remark 6.1 A general method to generate monotone cocoercive operators which are not gradients of convex functions is to start from a linear skew symmetric operator A, and then take its Yosida approximation A_λ. For example, starting from A equal to the counterclockwise rotation of angle $\pi/2$ in the plane, we obtain that, for any $\lambda > 0$, the following operator is λ-cocoercive

$$A_\lambda = \frac{1}{1+\lambda^2}\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}.$$

Example 6.1 Take $\mathcal{H} = \mathbb{R}^2$ equipped with the Euclidean structure. Let us consider the linear operator B whose matrix in the canonical basis of $\mathbb{R}^2$ is defined by $B = A_\lambda$ for $\lambda = 5$. According to Remark 6.1, B is a nonpotential operator B which is λ-cocoercive with $\lambda = 5$. In [3], we observed the classical oscillations, in the heavy ball with friction, when $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x_1, x_2) = 50x_2^2.$$

We set $\gamma = 0.9$. It is clear that f is convex but not strongly convex. We study 3 cases: (1) $\beta_b = 1, \beta_f = 0.5$, (2) $\beta_b = 0.5, \beta_f = 1$ and (3) $\beta_b = \beta_f = 0.5$. As a straight application of Theorem 3.1 and 4.1, we obtain that the sequences (x_k) generated by (DINAAM) and (DINAAM-split) converge to x_∞, where $x_\infty \in S = (B + \nabla f)^{-1}(0) = \{0\}$. The trajectory obtained by using Matlab is depicted in Figure 1 in [3]. In order to compare the two algorithms, we observe the norm of $x_k - x_\infty$. In Figure 1, we can see that the two algorithms give almost the same numerical results. The difference between them is the use or not of the resolvent operator of the sum of B and ∇f.

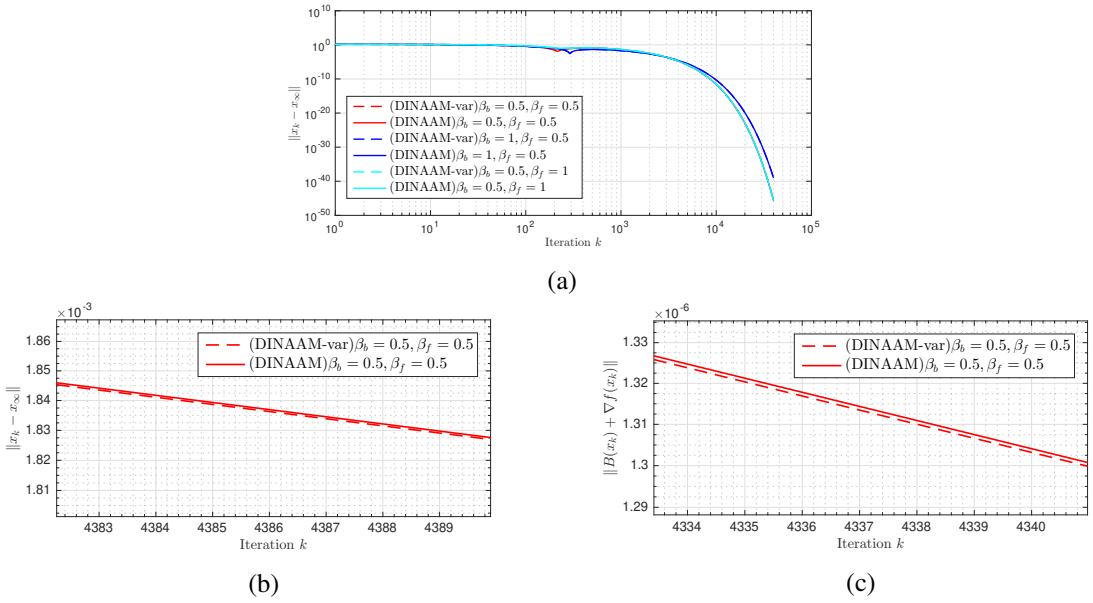

Figure 1: A comparison between (DINAAM) and (DINAAM-split).

In [3], we discuss an application of our model to dynamical games. Now we study another example to see how our algorithm can be applied to find the zeros of $\nabla f + B$.

Example 6.2 Non-potential version of sparse logistic regression. Let us recall the following sparse logistic regression problem for binary classification:

$$\min_{x\in\mathbb{R}^n} \frac{1}{m}\sum_{i=1}^{m} \log(1+e^{-v_i u_i^\top x}) + \mu\|x\|_1,$$

where $(u_i, v_i)_{1\leq i\leq m}$ is the training set with $u_i \in \mathbb{R}^n$ is the feature vector of each data sample, and $v_i \in \{-1, 1\}$ is the binary label. Here $\mu > 0$ is a regularization parameter. We set

$$f(x) = \frac{1}{m}\sum_{i=1}^{m} \log(1+e^{-v_i u_i^\top x}).$$

The gradient of f is given by $\nabla f(x) = -\frac{1}{m}A^\top(1_m - q(x))$, with $A^\top = \begin{pmatrix} v_1u_1 & v_2u_2 & \dots & v_mu_m \end{pmatrix} \in \mathbb{R}^{n\times m}$, $1_m = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix} \in \mathbb{R}^m$ and $q(x) = 1_m./(1+e^{-Ax})$ (./ denotes the componentwise division). Consider the following problem: Find $x \in \mathbb{R}^n$ such that $B_n(x) + \nabla f(x) = 0$, where

$$B_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Let us show that B_n is positive definite for all $n \geq 2$. Let us denote by y_k the k-th leading principal minor of a matrix B_n which is the determinant of its upper-left $k \times k$ sub-matrix. We have

$$y_k = \det(B_k), \text{ for } 1 \leq k \leq n.$$

For simplify, we define $B_1 = 2$. By the definition of B_n, we have that

$$\det(B_{n+1}) = 2\det(B_n) - \det(B_{n-1}),$$

for $n \geq 2$. An elementary calculation gives $\det(B_{n+1}) = n + 2$ for $n \geq 1$. Thus, $y_k = k + 1 > 0$ for all $1 \leq k \leq n$. Hence, B_n is positive definite. Furthermore, B_n is cocoercive. Indeed, for any $x, y \in \mathbb{R}^n$, there exists $\lambda > 0$ such that

$$\langle B_nx - B_ny, x - y\rangle \geq \lambda\|B_nx - B_ny\|^2. \tag{6.1}$$

Since $B_n, B_n^\top B_n$ are positive (semi)definite, for any $x, y \in \mathbb{R}^n$ we have

$$\langle B_nx - B_ny, x - y\rangle \geq \lambda_{min}(B_n)\|x - y\|^2, \tag{6.2}$$

and

$$\lambda_{max}(B_n^\top B_n)\|x - y\|^2 \geq \|B_nx - B_ny\|^2. \tag{6.3}$$

Here, $\lambda_{min}(B_n), \lambda_{max}(B_n^\top B_n)$ are the smallest eigenvalue of B_n and the greatest eigenvalue of $B_n^\top B_n$ respectively. For instance, take $\lambda = \dfrac{\lambda_{min}(B_n)}{\lambda_{max}(B_n^\top B_n)}$, from (6.2) and (6.3), we deduce that (6.1) holds.
Let us check that ∇f is Lipschitz continuous. In fact, for any $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned} m\|\nabla f(x) - \nabla f(y)\| &= \|A^\top q(x) - A^\top q(y)\| \\ &= \Big\|\sum_{i=1}^{m} \frac{v_i}{1+e^{-v_i u_i^\top x}}u_i - \frac{v_i}{1+e^{-v_i u_i^\top y}}u_i\Big\| \\ &\leq \sum_{i=1}^{m} \|u_i\|.|v_i|.\Big|\frac{1}{1+e^{-v_i u_i^\top x}} - \frac{1}{1+e^{-v_i u_i^\top y}}\Big| \\ &\leq \frac{1}{4}\sum_{i=1}^{m} \|u_i\|.|u_i^\top x - u_i^\top y| \leq \frac{1}{4}\sum_{i=1}^{m} \|u_i\|^2.\|x - y\|. \end{aligned}$$

Therefore, $\|\nabla f(x) - \nabla f(y)\| \leq \dfrac{1}{4m}\|x-y\| \sum_{i=1}^{m} \|u_i\|^2$.

For example, we take $n = 3, m = 2$. Then, B_3 is $\dfrac{1}{4}$-cocoercive. Setting $\gamma = 4, \beta_b = \beta_f = 0.5$, and $x_0 = 0_n, \dot{x}_0 = 1_n$ as initial conditions. According to Theorem 4.1, we can conclude that the sequence (x_k) generated by (DINAAM-split) converges to the zeros of $\nabla f + B_3$. Implementing the algorithm (DINAAM-split) in Matlab, we obtain the plot of k versus the norm of $\nabla f(x_k) + B_3(x_k)$, see Figure 2. Here, the training set is taken randomly for numerical test purposes.

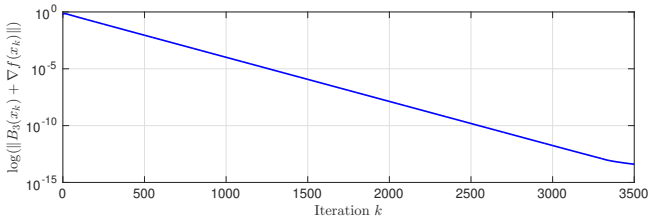

Figure 2: The plot of k versus the norm of $\nabla f(x_k) + B_3(x_k)$ obtained by (DINAAM-split).

Remark 6.2 In Example 6.2, since the resolvent operator $(\mathrm{Id} + s\mathcal{A}_h)^{-1}$ can not be computed easily, we used the algorithm (DINAAM-split) instead of (DINAAM). Then, our algorithm requires to compute $(\mathrm{Id} + s\mathcal{B}_h)^{-1}$ and in this situation it is easier to operate.

Example 6.3 Let us return to Example 6.1 and consider the effect of the introduction of perturbations, errors. With the same value of the parameters, we just add the errors $e_k = \dfrac{1}{k^2}$ and $\bar{e}_k = \dfrac{1}{\sqrt{k}}$. It is easy to check that the error $e(t) = e_1$ satisfies Theorem 4.2 while $e(t) = e_2$ does not. Running algorithm (DINAAM-split-pert) in Matlab, the plot of $\|x_k - x_\infty\|$ versus k is depicted in Figure 3. We observe that if the perturbed term e_k satisfies the assumptions of Theorem 4.2, then algorithm (DINAAM-split-pert) behaves as well as the nonperturbed version.

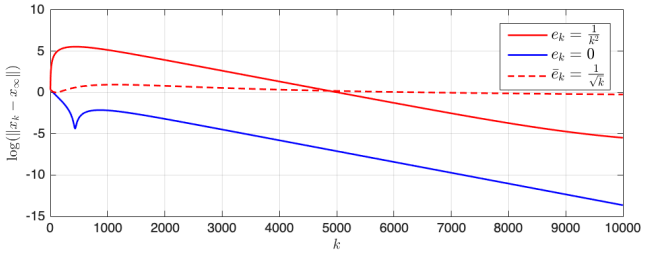

Figure 3: The effect of perturbations, errors in the algorithm (DINAAM-split).

7 Conclusion, perspectives

The crucial role played by the Hessian-driven damping in the convergence properties of inertial algorithms in convex optimization has been well documented in a series of recent papers. While keeping the convergence rates attached to the Nesterov accelerated gradient method, it provides fast convergence towards zeros of the gradients, and notably attenuates the oscillations. The corresponding notion for general monotone inclusions is the so called Newton's correction term. Our contribution is to bring together these two

aspects within the same algorithms, in order to design inertial algorithms for structured monotone inclusions involving potential (gradient) terms and nonpotential terms (skew symmetric operators for example). This is fundamental both for numerical reasons and for modeling in decision sciences and engineering. Indeed many decision-making processes involve cooperative and noncooperative aspects. Our Lyapunov analysis highlights the nonsymmetrical role played by the two operators. This is an important step compared to previous studies where the two operators were treated globally. Among perspectives, treating the case where B is a general maximally monotone operator (for example linear skew symmetric) is a central issue for dealing with primal-dual methods. In this regard, the approximation of B by its Yosida approximation (which is a cocoercive operator) allows us to come back to the situation studied in our article. It is an interesting subject for further studies. A similar technique can also be envisaged to deal with the case of a vanishing viscous damping, so as to cover the case of the accelerated gradient method of Nesterov.

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