Reachable states for the distributed control of the heat equation

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Abstract

We are concerned with the determination of the reachable states for the distributed control of the heat equation on an interval. We consider either periodic boundary conditions or homogeneous Dirichlet boundary conditions. We prove that for a L^2 distributed control, the reachable states are in the Sobolev space H^1 and that they have complex analytic extensions on squares whose horizontal diagonals are regions where no control is applied.

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1 Introduction

The null controllability of the heat equation has been investigated for a long time [3], and sharp results in dimension N were obtained in the nineties by using Carleman estimates [8, 4]. By contrast, sharp results for the exact controllability of the heat equation were obtained only recently. In [12], the authors noticed that for the boundary control of the heat equation on a real interval, the reachable space was sandwiched between the set of analytic functions on a ball and the set of analytic functions on a square. These results were improved in [2, 5], where the reachable space was sandwiched between two spaces of analytic functions on squares. The sharp result, derived in [14, 6], tells that the reachable space for the heat equation on the interval $(-1, 1)$ with two boundary controls at $x = \pm 1$ taken in the space $L^2(0, T)$, is exactly the Bergman space of the functions that are both analytic and square integrable on the square $\Omega = \{x+iy;\ |x|+|y| < 1\}$. See also [7] for the reachable spaces of semilinear parabolic equations, and [13] and [1] for the reachable spaces of the linear Korteweg-de Vries equation and the linear Zakharov-Kuznetsov equation, respectively.

The present paper is concerned with the determination of the reachable space for the *distributed control* of the heat equation on an interval. Roughly, it is proved that when using L^2 distributed controls, the reachable states are H^1 in the control region and analytic elsewhere. At the same time, we shall provide a short proof of the main result in [2] by using the method introduced in [9] to construct backward solutions of the heat equation.

Let us review the main results in this paper. Consider first periodic boundary conditions. Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and let $\chi_{(l_1,l_2)}(x) = 1$ if $x \in (l_1, l_2)$, 0 otherwise. We consider the control problem:

$$\begin{cases} w_t - w_{xx} = \chi_{(l_1,l_2)} u & (x,t) \in \mathbb{T} \times (0,T), \\ w(x,0) = 0 & x \in \mathbb{T}. \end{cases} \tag{1.1}$$

For $0 < L_1 < L_2 < 2\pi$, we introduce the open set

$$\mathcal{S}(L_1, L_2) = \{x + iy \in \mathbb{C};\ |y| < x - L_2 \text{ and } |y| < L_1 + 2\pi - x\}$$

which is a square with the interval $(L_2, L_1 + 2\pi)$ as a diagonal. Define

$$\mathcal{H}(L_1, L_2) = \{f \in H^1(\mathbb{T});\ f|_{(L_2, L_1+2\pi)} \text{ can be extended as an analytic function in } \mathcal{S}(L_1, L_2)\},$$
$$\mathcal{A}(L_1, L_2) = \{f \in \mathcal{H}(L_1, L_2);\ \int_{\mathcal{S}(L_1,L_2)} |f(x+iy)|^2 dxdy < \infty\}.$$

Note that any function $f \in \mathcal{H}(L_1, L_2)$ can be extended as an analytic function on $\cup_{k\in\mathbb{Z}} \mathcal{S}(L_1, L_2) + 2k\pi$, by 2π-periodicity of f. The following result is the first main result in this paper.

Theorem 1.1. *Let $T > 0$ and $0 < l_1 < l_2 < 2\pi$. Then*

(i) *for any $u \in L^2(0,T; L^2(\mathbb{T}))$, the solution w of system (1.1) satisfies $w(\cdot, T) \in \mathcal{A}(l_1, l_2)$;*

(ii) *for any $0 < \varepsilon < (l_2 - l_1)/2$ and any $w_T \in \mathcal{H}(l_1 + \varepsilon, l_2 - \varepsilon)$, there exists a control input $u \in L^2(0,T; L^2(\mathbb{T}))$ such that the solution w of system (1.1) satisfies $w(\cdot, T) = w_T$ in $\mathbb{T}$.*

Remark 1.1. *1. The above result is "almost sharp", for ε can be taken as small as desired in the inclusion*

$$\mathcal{H}(l_1 + \varepsilon, l_2 - \varepsilon) \subset \mathcal{A}(l_1, l_2).$$

2. Having in mind the characterisation of the reachable space for the boundary control of the heat equation in [6], it is natural to conjecture that the reachable space for system (1.1) is the Bergman space $\mathcal{A}(l_1, l_2)$.

Next, we consider distributed control systems on the interval $(0, 1)$ with homogeneous boundary conditions.[1]

Let $0 < l_1 < l_2 < 1$. For any given $u \in L^2(0, T, L^2(0, 1))$, let w denote the solution of the control system

$$\begin{cases} w_t - w_{xx} = \chi_{(l_1,l_2)} u & (x,t) \in (0,1) \times (0,T), \\ w(0,t) = w(1,t) = 0 & t \in (0,T), \\ w(x,0) = 0 & x \in (0,1). \end{cases} \tag{1.2}$$

[1]Homogeneous Neumann boundary conditions could be treated in a similar way.

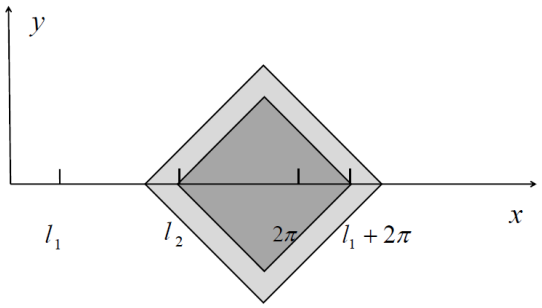

Figure 1: Reachable states for periodic boundary conditions.

For any $L > 0$, we introduce the set

$$\mathcal{S}(L) = \{x + iy \in \mathbb{C};\ |x| + |y| < L\},$$

and the spaces

$$\mathcal{H}(L) = \{f \in H^1(0, L);\ f \text{ can be extended as an odd analytic function on } \mathcal{S}(L)\},$$
$$\mathcal{A}(L) = \{f \in \mathcal{H}(L);\ \int_{\mathcal{S}(L)} |f(x + iy)|^2 dxdy < \infty\}.$$

The following result is the second main result in this paper.

Theorem 1.2. *Let $T > 0$ and $0 < l_1 < l_2 < 1$. Then*

(i) *for any $u \in L^2(0, T; L^2(0, 1))$, the solution w of system (1.2) satisfies $w(\cdot, T) \in H_0^1(0, 1)$, $w(\cdot, T) \in \mathcal{A}(l_1)$ and $w(1 - \cdot, T) \in \mathcal{A}(1 - l_2)$;*

(ii) *for any $0 < \varepsilon < (l_2 - l_1)/2$, for any $w_T \in H_0^1(0, 1)$ with $w_T \in \mathcal{H}(l_1 + \varepsilon)$ and $w_T(1 - \cdot) \in \mathcal{H}(1 - l_2 + \varepsilon)$, there exists a control function $u \in L^2(0, T; L^2(0, 1))$ such that the solution w of system (1.2) satisfies $w(\cdot, T) = w_T$ in $(0, 1)$.*

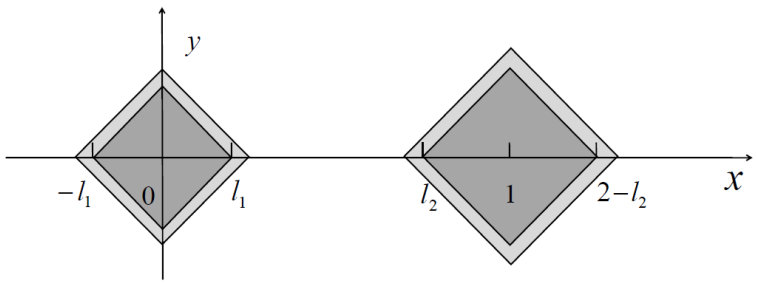

Figure 2: Reachable states for homogeneous Dirichlet boundary conditions.

Remark 1.2. *The result is again "almost sharp". We conjecture that for $0 < l_1 < l_2 < 1$, the reachable space for system (1.2) is the set of functions $w_T \in H_0^1(0,1)$ such that $w_T \in \mathcal{A}(l_1)$ and $w_T(1-\cdot) \in \mathcal{A}(1-l_2)$.*

The paper is outlined as follows. The proof of Theorem 1.1 (resp. Theorem 1.2) is given in Section 2 (resp. in Section 3). We provide in appendix a short proof of the main result in [2] which is used to prove Theorem 1.1.

2 Proof of Theorem 1.1

(i) Pick any $u \in L^2(0,T,L^2(\mathbb{T}))$ and let w denote the solution of (1.2). Let us first check that $w \in C([0,T],H^1(\mathbb{T}))$. Expand $\chi_{(l_1,l_2)}u$ as a Fourier series: $\chi_{(l_1,l_2)}(x)u(x,t) = \sum_{n\in\mathbb{Z}} u_n(t)e^{inx}$ with $\int_0^T (\sum_{n\in\mathbb{Z}} |u_n(t)|^2)dt < \infty$. Then one has by Duhamel formula for all $t \in [0,T]$

$$w(x,t) = \int_0^t \sum_{n\in\mathbb{Z}} e^{-n^2(t-\tau)}u_n(\tau)e^{inx}d\tau \quad \text{ in } L^2(\mathbb{T}).$$

Since for $n \neq 0$ and $t \in [0,T]$ we have, by Cauchy-Schwarz inequality,

$$\int_0^t |e^{-n^2(t-\tau)}u_n(\tau)|d\tau \leq (\int_0^t e^{-2n^2(t-\tau)}d\tau)^{\frac{1}{2}}(\int_0^t |u_n(\tau)|^2d\tau)^{\frac{1}{2}} \leq \frac{1}{\sqrt{2n^2}}(\int_0^T |u_n(\tau)|^2d\tau)^{\frac{1}{2}},$$

we infer that

$$\sum_{n\in\mathbb{Z}^*}\int_0^t |e^{-n^2(t-\tau)}u_n(\tau)|d\tau \leq \left(\sum_{n\in\mathbb{Z}^*}\frac{1}{2n^2}\right)^{\frac{1}{2}}\left(\sum_{n\in\mathbb{Z}^*}\int_0^T |u_n(\tau)|^2d\tau\right)^{\frac{1}{2}} < \infty \tag{2.1}$$

so that $w(x,t) = \sum_{n\in\mathbb{Z}}(\int_0^t e^{-n^2(t-\tau)}u_n(\tau)\,d\tau)e^{inx}$ for $(x,t) \in \mathbb{T}\times[0,T]$, and that

$$\sum_{n\in\mathbb{Z}^*}\left|n\int_0^t e^{-n^2(t-\tau)}u_n(\tau)d\tau\right|^2 \leq \frac{1}{2}\sum_{n\in\mathbb{Z}^*}\int_0^T |u_n(\tau)|^2d\tau < \infty. \tag{2.2}$$

Thus

$$w \in C([0,T],H^1(\mathbb{T})) \tag{2.3}$$

and $w(.,T) \in H^1(\mathbb{T})$. Introduce the function

$$v(x,t) = w(x,t) \quad \text{ for } (x,t) \in [l_2, l_1+2\pi]\times[0,T].$$

Then v satisfies the following system

$$\begin{cases} v_t - v_{xx} = 0 & (x,t) \in (l_2, l_1+2\pi)\times(0,T), \\ v(l_2,t) = w(l_2,t),\ v(l_1+2\pi,t) = w(l_1+2\pi,t) & t \in (0,T), \\ v(x,0) = 0 & x \in (l_2, l_1+2\pi), \end{cases}$$

where the boundary controls $w(l_2,.)$ and $w(l_1+2\pi,.)$ are in $C([0,T])$, by (2.3). Then $v(\cdot,T)$ can be extended as an analytic function in $\mathcal{S}(l_1,l_2)$ by [12, Theorem 2.1]. Furthermore, $w(\cdot,T)\in L^2(\mathcal{S}(l_1,l_2))$ by [5, Proposition 1.2]. Therefore $w(\cdot,T)\in\mathcal{A}(l_1,l_2)$.

(ii) We introduce a partition of unity. Let $\psi_1,\psi_2\in C^\infty(\mathbb{T})$ be such that

$$\begin{aligned}&\psi_1+\psi_2=1,\quad 0\le\psi_i\le 1,\quad i=1,2\\&\psi_1(x)=0,\quad x\in\mathbb{T}\setminus(l_1,l_2),\\&\psi_2(x)=0,\quad x\in(l_1+\frac{\varepsilon}{2},l_2-\frac{\varepsilon}{2}).\end{aligned}$$

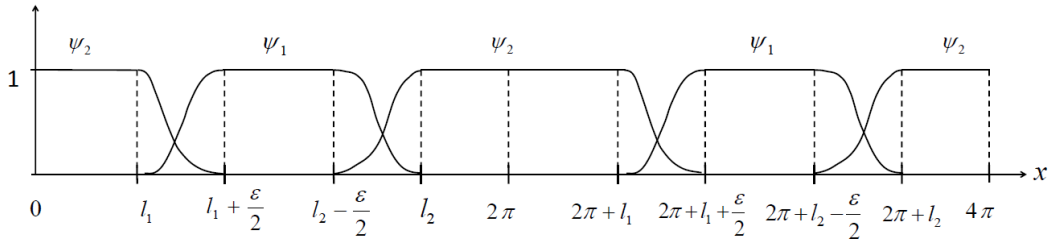

Figure 3: partition of unity

Consider a control problem with a distributed control supported in $\mathbb{T}$:

$$\begin{cases} w_{1t}-w_{1xx}=\widetilde{u}_1 & (x,t)\in\mathbb{T}\times(0,T),\\ w_1(x,0)=0 & x\in\mathbb{T},\end{cases}\tag{2.4}$$

where $\widetilde{u}_1$ is the control.

Lemma 2.1. *For any $T>0$ and any $w_T\in H^1(\mathbb{T})$, there exists a function $\widetilde{u}_1\in L^2(0,T;L^2(\mathbb{T}))$ such that the solution of (2.4) satisfies*

$$w_1(x,T)=w_T(x)\quad\forall x\in\mathbb{T}.$$

Proof. Expand the control input as a Fourier series

$$\widetilde{u}_1(x,t)=\sum_{n\in\mathbb{Z}}u_n(t)e^{inx}.$$

Then by Duhamel formula

$$w_1(x,t)=\int_0^t\sum_{n\in\mathbb{Z}}e^{-n^2(t-\tau)}u_n(\tau)e^{inx}d\tau.$$

Since $w_T\in H^1(\mathbb{T})$, we can expand w_T as $w_T(x)=\sum_{n\in\mathbb{Z}}a_ne^{inx}$ with $\sum_{n\in\mathbb{Z}}n^2|a_n|^2<\infty$.

Pick

$$u_n(t) = \begin{cases} \dfrac{a_0}{T} & \text{for } n = 0, \\ \dfrac{2n^2 e^{-n^2(T-t)}}{1 - e^{-2n^2 T}} a_n & \text{for } n \in \mathbb{Z}^*. \end{cases}$$

Then we obtain for $n \in \mathbb{Z}^*$

$$\begin{aligned} \int_0^T |u_n(t)|^2 dt &= \Big(\frac{2n^2}{1 - e^{-2n^2 T}}\Big)^2 |a_n|^2 \int_0^T e^{-2n^2(T-t)} dt \\ &= \frac{2n^2}{1 - e^{-2n^2 T}} |a_n|^2 \sim 2n^2 |a_n|^2 \quad \text{as } n \to +\infty. \end{aligned}$$

It follows that

$$\|\widetilde{u}_1\|^2_{L^2(0,T;L^2(\mathbb{T}))} = 2\pi \sum_{n \in \mathbb{Z}} \int_0^T |u_n^2(t)| dt < \infty.$$

On the other hand, we have

$$\int_0^T e^{-n^2(T-\tau)} u_n(\tau) d\tau = a_n \quad \forall n \in \mathbb{Z},$$

and hence $w_1(\cdot, T) = w_T$. The proof of Lemma 2.1 is complete. □

It follows from (2.4) that the function $\psi_1 w_1$ satisfies the system:

$$\begin{cases} (\psi_1 w_1)_t - (\psi_1 w_1)_{xx} = u_1 & (x,t) \in \mathbb{T} \times (0,T), \\ (\psi_1 w_1)(x,0) = 0 & x \in \mathbb{T}, \\ (\psi_1 w_1)(x,T) = (\psi_1 w_T)(x) & x \in \mathbb{T}, \end{cases} \tag{2.5}$$

with

$$\begin{aligned} u_1 =& \psi_1(w_{1t} - w_{1xx}) - 2\psi_{1x} w_{1x} - \psi_{1xx} w_1 \\ =& \psi_1 \widetilde{u}_1 - 2\psi_{1x} w_{1x} - \psi_{1xx} w_1. \end{aligned}$$

By (2.3) (still valid for $l_1 = 0$ and $l_2 = 2\pi$), we have that $w_1 \in C([0,T], H^1(\mathbb{T}))$. By construction of ψ_1, we have that $\partial_x^n \psi_1(x) = 0$ for all $x \in \mathbb{T} \setminus (l_1, l_2)$ and all $n \geq 0$, and hence

$$u_1 = \chi_{(l_1,l_2)} u_1 \qquad \text{in } L^2(0,T,L^2(\mathbb{T})).$$

Let $\mathrm{Hol}(\Omega)$ denote the space of (complex) analytic functions in Ω. A function $h \in C^\infty([0,T])$ is said to be *Gevrey of order 2*, and we write $h \in G^2([0,T])$, if there exist some positive constants C, R such that

$$|\partial_t^p h(t)| \leq C \frac{(p!)^2}{R^p} \quad \forall t \in [0,T], \ \forall p \geq 0.$$

The following result is needed.

Proposition 2.1. *Let $L > 1$, $T > 0$, and $\psi \in Hol(S(L))$. Then there exist $h_{-1}, h_1 \in G^2([0,T])$ such that the solution $w = w(x,t)$ of the control system*

$$w_t - w_{xx} = 0, \qquad (x,t) \in (-1,1) \times (0,T), \tag{2.6}$$

$$w(-1,t) = h_{-1}(t), \ w(1,t) = h_1(t), \qquad t \in (0,T), \tag{2.7}$$

$$w(x,0) = 0, \qquad x \in (-1,1), \tag{2.8}$$

satisfies $w \in C^\infty([-1,1] \times [0,T])$ and $w(x,T) = \psi(x)$ for $x \in [-1,1]$. If, in addition, ψ is odd, then we can require that $w(.,t)$ be odd for all $t \in [0,T]$, so that $h_{-1}(t) = -h_1(t)$ and $w(0,t) = 0$ for all $t \in [0,T]$.

Note that a similar result with $h_{-1}, h_1 \in C^\infty([0,T])$ was derived in [2, Theorem 5.2]. Here, we provide in Appendix a very short proof of Proposition 2.1 which is interesting in itself.

By Proposition 2.1, if $w_T \in \mathcal{H}(l_1 + \varepsilon, l_2 - \varepsilon)$, we can find $h_1, h_2 \in G^2([0,T])$ such that the solution w_2 of the following system

$$\begin{cases} w_{2t} - w_{2xx} = 0 & (x,t) \in (l_2 - \frac{\varepsilon}{2}, l_1 + \frac{\varepsilon}{2} + 2\pi) \times (0,T), \\ w_2(l_2 - \frac{\varepsilon}{2}, t) = h_1(t), \ w_2(l_1 + \frac{\varepsilon}{2} + 2\pi, t) = h_2(t) & t \in (0,T), \\ w_2(x,0) = 0 & x \in (l_2 - \frac{\varepsilon}{2}, l_1 + \frac{\varepsilon}{2} + 2\pi) \end{cases} \tag{2.9}$$

satisfies $w_2 \in C^\infty([l_2 - \frac{\epsilon}{2}, l_1 + \frac{\varepsilon}{2} + 2\pi] \times [0,T])$ and

$$w_2(x,T) = w_T(x), \quad \forall x \in (l_2 - \frac{\varepsilon}{2}, l_1 + \frac{\varepsilon}{2} + 2\pi).$$

Extend w_2 as a function in $C^\infty(\mathbb{T} \times [0,T])$ (i.e. as a function smooth in (x,t) and 2π-periodic in x) and still denote this function by w_2. Then we have

$$\begin{cases} (\psi_2 w_2)_t - (\psi_2 w_2)_{xx} = u_2 & (x,t) \in \mathbb{T} \times (0,T), \\ (\psi_2 w_2)(x,0) = 0 & x \in \mathbb{T}, \\ (\psi_2 w_2)(x,T) = (\psi_2 w_T)(x) & x \in \mathbb{T}, \end{cases} \tag{2.10}$$

where

$$u_2 = \psi_2(w_{2t} - w_{2xx}) - 2\psi_{2x} w_{2x} - \psi_{2xx} w_2.$$

It follows from the definition of ψ_2 and the first equation in (2.9) that $u_2(x,t) = 0$ for $x \in [l_2, l_1 + 2\pi]$ and $t \in [0,T]$. Thus we have

$$u_2 = \chi_{(l_1,l_2)} u_2 \quad \text{ in } L^2(0,T,L^2(\mathbb{T})).$$

Since $w_2 \in C^\infty(\mathbb{T} \times [0,T])$, we have $u_2 \in C^\infty(\mathbb{T} \times [0,T])$. Combining (2.5) and (2.10), if we take

$$u = \chi_{(l_1,l_2)} u_1 + \chi_{(l_1,l_2)} u_2,$$

then $w = \psi_1 w_1 + \psi_2 w_2$ satisfies (1.1) and

$$w(x,T) = (\psi_1 w_T)(x) + (\psi_2 w_T)(x) = w_T(x), \ \ \forall x \in \mathbb{T}.$$

The proof of Theorem 1.1 is complete. □

3 Dirichlet boundary conditions

In this section, we prove Theorem 1.2. The necessary conditions in (i) are obtained as in the proof of Theorem 1.1. Indeed, introducing $v_1 = w_{|(0,l_1)\times(0,T)}$ and $v_2 = w_{|(l_2,1)\times(0,T)}$ and applying [5, Proposition 5.1], we obtain the desired result.

To prove (ii), we use again a partition of unity. We pick some functions $\psi_1, \psi_2, \psi_3 \in C^\infty([0,1])$ such that

$$\begin{aligned}
&\psi_1 + \psi_2 + \psi_3 = 1, \quad 0 \leq \psi_i \leq 1, \quad i = 1,2,3,\\
&\psi_1(x) = 0 \quad x \in [0, l_1] \cup [l_2, 1],\\
&\psi_2(x) = 0 \quad x \in [l_1 + \frac{\varepsilon}{2}, 1],\\
&\psi_3(x) = 0 \quad x \in [0, l_2 - \frac{\varepsilon}{2}].
\end{aligned}$$

As in the proof of Lemma 2.1, we can show that for any $w_T \in H_0^1(0,1)$, there exists a function $\widetilde{u}_1 \in L^2(0,T;L^2(0,1))$ such that the solution w_1 of the system

$$\begin{cases}
w_{1t} - w_{1xx} = \widetilde{u}_1 & (x,t) \in (0,1)\times(0,T),\\
w_1(0,t) = w_1(1,t) = 0 & t \in (0,T),\\
w_1(x,0) = 0 & x \in (0,1),
\end{cases}$$

satisfies

$$w_1(x,T) = w_T(x), \quad \forall x \in (0,1).$$

Proceeding as for system (2.5), we see that there exists a control function $u_1 \in L^2(0,T;L^2(0,1))$ such that $\psi_1 w_1$ is the solution of the following system:

$$\begin{cases}
(\psi_1 w_1)_t - (\psi_1 w_1)_{xx} = \chi_{(l_1,l_2)} u_1 & (x,t) \in (0,1)\times(0,T),\\
(\psi_1 w_1)(0,t) = (\psi_1 w_1)(1,t) = 0 & t \in (0,T),\\
(\psi_1 w_1)(x,0) = 0 & x \in (0,1),\\
(\psi_1 w_1)(x,T) = (\psi_1 w_T)(x) & x \in (0,1).
\end{cases}$$

Since $w_T \in \mathcal{H}(l_1 + \varepsilon)$, there exists by Proposition 2.1 a function $h_1 \in G^2([0,T])$ such that the solution w_2 of

$$\begin{cases}
w_{2t} - w_{2xx} = 0 & (x,t) \in (0, l_1 + \frac{\varepsilon}{2})\times(0,T),\\
w_2(0,t) = 0,\ w_2(l_1 + \frac{\varepsilon}{2}, t) = h_1(t) & t \in (0,T),\\
w_2(x,0) = 0 & x \in (0, l_1 + \frac{\varepsilon}{2})
\end{cases}$$

satisfies $w_2 \in C^\infty([0, l_1 + \frac{\varepsilon}{2}]\times[0,T])$ and

$$w_2(x,T) = w_T(x), \quad \forall x \in (0, l_1 + \frac{\varepsilon}{2}).$$

We still denote by w_2 a smooth extension of w_2 to $[0,1]\times[0,T]$. Then there exists a function $u_2 \in L^2(0,T;L^2(0,1))$ such that $\psi_2 w_2$ solves

$$\begin{cases}
(\psi_2 w_2)_t - (\psi_2 w_2)_{xx} = \chi_{(l_1, l_1+\frac{\varepsilon}{2})} u_2 & (x,t) \in (0,1)\times(0,T),\\
(\psi_2 w_2)(0,t) = (\psi_2 w_2)(1,t) = 0 & t \in (0,T),\\
(\psi_2 w_2)(x,0) = 0 & x \in (0,1),\\
(\psi_2 w_2)(x,T) = (\psi_2 w_T)(x) & x \in (0,1).
\end{cases}$$

Similarly, since $w_T(1-\cdot)\in\mathcal{H}(1-l_2+\varepsilon)$, there exists a function $h_2\in G^2([0,T])$ such that the solution $\widetilde{w_3}$ of the system

$$\begin{cases} \widetilde{w}_{3t}-\widetilde{w}_{3xx}=0 & (x,t)\in(0,1-l_2+\frac{\varepsilon}{2})\times(0,T),\\ \widetilde{w}_3(0,t)=0,\ \widetilde{w}_3(1-l_2+\frac{\varepsilon}{2},t)=h_2(t) & t\in(0,T),\\ \widetilde{w}_3(x,0)=0 & x\in(0,1-l_2+\frac{\varepsilon}{2}) \end{cases}$$

satisfies $\widetilde{w}_3\in C^\infty([0,1-l_2+\frac{\varepsilon}{2}]\times[0,T])$ and

$$\widetilde{w}_3(x,T)=w_T(1-x),\qquad x\in(0,1-l_2+\frac{\varepsilon}{2}).$$

Let

$$w_3(x,t)=\widetilde{w}_3(1-x,t),\quad x\in(l_2-\frac{\varepsilon}{2},1),\ t\in(0,T).$$

Then we have

$$\begin{cases} w_{3t}-w_{3xx}=0 & (x,t)\in(l_2-\frac{\varepsilon}{2},1)\times(0,T),\\ w_3(l_2-\frac{\varepsilon}{2},t)=h_2(t),\ w_3(1,t)=0 & t\in(0,T),\\ w_3(x,0)=0 & x\in(l_2-\frac{\varepsilon}{2},1),\\ w_3(x,T)=w_T(x) & x\in(l_2-\frac{\varepsilon}{2},1). \end{cases}$$

We still denote by w_3 a smooth extension of w_3 to $[0,1]\times[0,T]$. It follows that there exists a control function $u_3\in L^2(0,T;L^2(0,1))$ such that ψ_3w_3 solves

$$\begin{cases} (\psi_3w_3)_t-(\psi_3w_3)_{xx}=\chi_{(l_2-\frac{\varepsilon}{2},l_2)}u_3 & (x,t)\in(0,1)\times(0,T),\\ (\psi_3w_3)(0,t)=(\psi_3w_3)(1,t)=0 & t\in(0,T),\\ (\psi_3w_3)(x,0)=0 & x\in(0,1),\\ (\psi_3w_3)(x,T)=(\psi_3w_T)(x) & x\in(0,1). \end{cases}$$

Finally, if we take

$$u=\chi_{(l_1,l_2)}u_1+\chi_{(l_1,l_1+\frac{\varepsilon}{2})}u_2+\chi_{(l_2-\frac{\varepsilon}{2},l_2)}u_3,$$

we infer that $w=\psi_1w_1+\psi_2w_2+\psi_3w_3$ is the solution of (1.2). On the other hand, we have that

$$w(x,T)=(\psi_1w_T)(x)+(\psi_2w_T)(x)+(\psi_3w_T)(x)=w_T(x),\quad\forall x\in(0,1).$$

The proof of Theorem 1.2 is complete. □

Appendix. Proof of Proposition 2.1

The proof is inspired by [9] where backward solutions of the heat equation were obtained by integrating the heat kernel along lines different from the real line. Introduce the following notations borrowed from [9]. For $\theta\in\mathbb{R}$ and $R>0$, let

$$\begin{aligned}\mathcal{O}(\theta,R)&:=\{z\in\mathbb{C};\ |z-Re^{i\theta}|<R\},\\ \Omega(\theta,R)&:=\{z\in\mathbb{C};\ \mathrm{dist}(z,e^{i\theta}\mathbb{R})<R\}.\end{aligned}$$

(See Figure 4.) Note that

$$S(1)=\Omega(\frac{\pi}{4},\frac{1}{\sqrt{2}})\cap\Omega(\frac{3\pi}{4},\frac{1}{\sqrt{2}}).$$

The first lemma is concerned with *separation of singularities.*

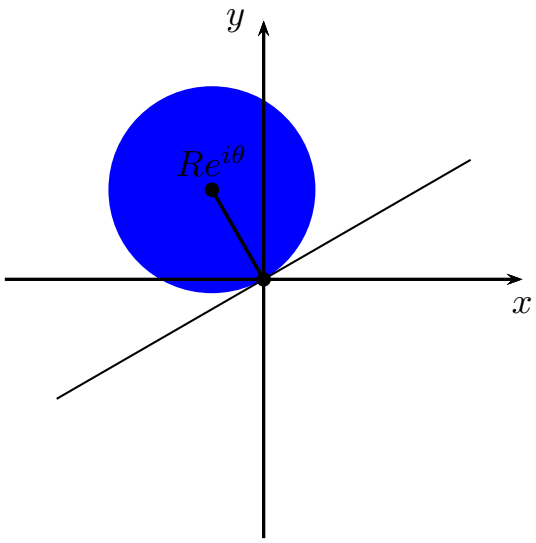

Figure 4: The set $\mathcal{O}(\theta, R)$ for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.

Lemma 3.1. *Let $1 < l < L$ and $\psi \in Hol(S(L))$. Then there exist $\theta_1 \in (\pi, \frac{3\pi}{2})$, $\theta_2 \in (\frac{\pi}{2}, \pi)$, $r \in (\frac{1}{\sqrt{2}}, +\infty)$, $\psi_1 \in Hol\big(\Omega(\frac{\theta_1}{2}, r)\big)$ and $\psi_2 \in Hol\big(\Omega(\frac{\theta_2}{2}, r)\big)$ such that*

$$\begin{aligned}
&\overline{S(l)} \subset \Omega(\frac{\theta_1}{2}, r) \cap \Omega(\frac{\theta_2}{2}, r),\\
&\partial_z^j \psi_i \in L^\infty(\Omega(\frac{\theta_i}{2}, r)), \quad i = 1, 2,\ j \in \mathbb{N},\\
&\psi = \psi_1 + \psi_2 \qquad \text{in } S(l).
\end{aligned}$$

Proof of Lemma 3.1. Pick any $\hat{l} \in (l, L)$. Let $\gamma_1(t) = (1 - t + it)\hat{l}$ for $t \in [0, 1]$, and let

$$\big(\gamma_2(t), \gamma_3(t), \gamma_4(t)\big) = (i\gamma_1(t), -\gamma_1(t), -i\gamma_1(t)) \quad \text{for } t \in [0, 1].$$

Let $\gamma : [0, 4] \to \mathbb{C}$ be defined by

$$\gamma(t) = \gamma_i(t - i + 1) \text{ for } i \in \{1, ..., 4\}, t \in [i - 1, i].$$

Note that $\gamma([0, 4]) = \partial S(\hat{l})$. We infer from Cauchy formula that for any $z \in S(\hat{l})$

$$\begin{aligned}
\psi(z) &= \frac{1}{2\pi i} \int_\gamma \frac{\psi(\zeta)}{\zeta - z}\, d\zeta\\
&= \frac{1}{2\pi i} \int_{\gamma_1 \cup \gamma_3} \frac{\psi(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_{\gamma_2 \cup \gamma_4} \frac{\psi(\zeta)}{\zeta - z}\, d\zeta\\
&=: \psi_1(z) + \psi_2(z)
\end{aligned}$$

where $\psi_1 \in \mathrm{Hol}(\mathbb{C} \setminus (\gamma_1 \cup \gamma_3))$ and $\psi_2 \in \mathrm{Hol}(\mathbb{C} \setminus (\gamma_2 \cup \gamma_4))$. Since

$$\overline{S(l)} \subset S(\hat{l}) = \Omega(\frac{\pi}{4}, \frac{\hat{l}}{\sqrt{2}}) \cap \Omega(\frac{3\pi}{4}, \frac{\hat{l}}{\sqrt{2}}),$$

there exist $r \in (\frac{1}{\sqrt{2}}, \frac{\hat{l}}{\sqrt{2}})$ and $\theta_1 \in (\pi, \frac{3\pi}{2})$, $\theta_2 \in (\frac{\pi}{2}, \pi)$ such that

$$\begin{aligned}
&\overline{S(l)} \subset \Omega(\frac{\theta_1}{2}, r) \cap \Omega(\frac{\theta_2}{2}, r) \subset S(\hat{l}),\\
&\psi_i \in \mathrm{Hol}\big(\Omega(\frac{\theta_i}{2}, r)\big) \qquad i = 1, 2,\\
&\partial_z^j \psi_i \in L^\infty(\Omega(\frac{\theta_i}{2}, r)), \quad i = 1, 2,\ j \in \mathbb{N}
\end{aligned}$$

(see Figure 5). The proof of Lemma 3.1 is complete. □

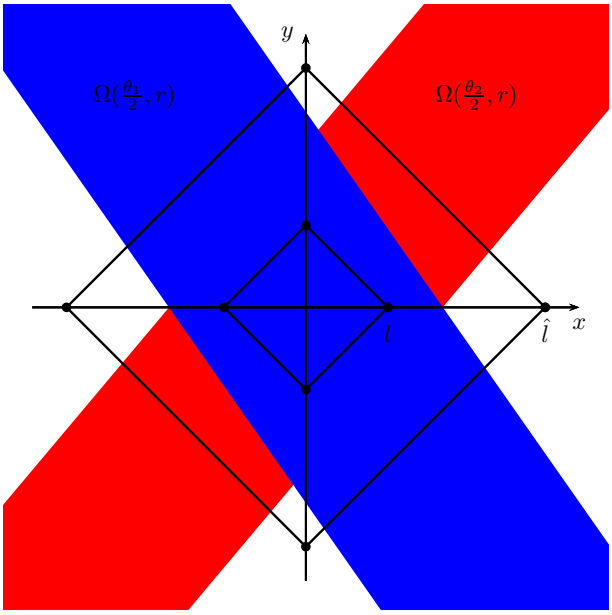

Figure 5: Separation of singularities.

The second lemma yields a backward solution of the heat equation for the initial data ψ_1 (resp. ψ_2).

Lemma 3.2. *Let $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$, $r > 1/\sqrt{2}$, and $\psi \in Hol\big(\Omega(\frac{\theta}{2}, r)\big) \cap L^\infty\big(\Omega(\frac{\theta}{2}, r)\big)$. Then the function*

$$v(z, \tau) := \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} e^{-\frac{\zeta^2}{4\tau}} \psi(z - \zeta)\, d\zeta$$

is well-defined and analytic in z and τ for $z \in \Omega(\frac{\theta}{2}, r)$ and $\tau \in \mathcal{O}(\theta, R)$ for any $R > 0$.[2] *Furthermore, v satisfies*

$$v_\tau - v_{zz} = 0, \quad z \in \Omega(\frac{\theta}{2}, r),\ \tau \in \mathcal{O}(\theta, R), \tag{3.1}$$

[2]We pick a branch of the argument function defined in $\mathbb{C} \setminus \mathbb{R}_- i$ (resp. in $\mathbb{C} \setminus \mathbb{R}_+ i$) if $\frac{\pi}{2} < \theta \leq \pi$ (resp. if $\pi < \theta \leq \frac{3\pi}{2}$).

$$\lim_{\tau\to 0^-} v(z,\tau) \;=\; \psi(z), \quad z\in\Omega(\frac{\theta}{2},r). \tag{3.2}$$

If, in addition, $\partial_z^j\psi\in L^\infty(\Omega(\frac{\theta}{2},r))$ for all $j\in\mathbb{N}$ and if $\overline{S(1)}\subset\Omega(\frac{\theta}{2},r)$, then $v\in C^\infty([-1,1]\times[-T,0])$ for all $T>0$ with $-T\in\mathcal{O}(\theta,R)$.

Proof of Lemma 3.2. For $s\in\mathbb{R}$ and $\zeta=e^{i\frac{\theta}{2}}s$, we have

$$\left|e^{-\frac{\zeta^2}{4\tau}}\right| = \left|e^{-\frac{s^2}{4}\frac{e^{i\theta}}{\tau}}\right| = e^{-\frac{s^2}{4}\mathrm{Re}\frac{e^{i\theta}}{\tau}}.$$

But if $\tau\in\mathcal{O}(\theta,R)$, $|\frac{\tau}{e^{i\theta}}-R|<R$, so that $\mathrm{Re}\,\frac{e^{i\theta}}{\tau}=\mathrm{Re}\,\frac{\tau}{e^{i\theta}}>0$. On the other hand

$$|\psi(z-\zeta)|\le\|\psi\|_{L^\infty(\Omega(\frac{\theta}{2},r))}, \qquad \text{for } z\in\Omega(\frac{\theta}{2},r) \text{ and } \ \zeta\in\mathbb{R}e^{i\frac{\theta}{2}}.$$

Straightforward calculations show that

$$\begin{aligned} v_{zz} &= \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} e^{-\frac{\zeta^2}{4\tau}}\psi''(z-\zeta)d\zeta \\ &= \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} \frac{d^2}{d\zeta^2}[e^{-\frac{\zeta^2}{4\tau}}]\psi(z-\zeta)d\zeta \\ &= \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} \left(-\frac{1}{2\tau}+\frac{\zeta^2}{4\tau^2}\right)e^{-\frac{\zeta^2}{4\tau}}\psi(z-\zeta)d\zeta, \end{aligned}$$

while

$$v_\tau=\frac{1}{\sqrt{4\pi\tau}}(-\frac{1}{2\tau})\int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} e^{-\frac{\zeta^2}{4\tau}}\psi(z-\zeta)d\zeta+\frac{1}{\sqrt{4\pi\tau}}(\frac{1}{4\tau^2})\int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}}\zeta^2 e^{-\frac{\zeta^2}{4\tau}}\psi(z-\zeta)d\zeta$$

and hence (3.1) holds in $\Omega(\frac{\theta}{2},r)\times\mathcal{O}(\theta,R)$. It remains to show that (3.2) is fulfilled. First, we notice that for $\frac{\pi}{2}<\theta<\frac{3\pi}{2}$ and $R>0$, there exists $T>0$ such that $\mathcal{O}(\theta,R)\cap\mathbb{R}=(-T,0)$ (see Figure 4). Therefore, taking the limit of $v(z,\tau)$ as $\tau\to 0^-$ is meaningful.

Claim 1. For $\theta\in(\frac{\pi}{2},\frac{3\pi}{2})$ and $\tau\in(-\infty,0)$, we have $\frac{1}{\sqrt{4\pi\tau}}\int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} e^{-\frac{\zeta^2}{4\tau}}d\zeta=1$.

Indeed, if $\theta\in(\frac{\pi}{2},\pi)$ (resp. $\theta\in(\pi,\frac{3\pi}{2})$), we have for $\kappa\in(\frac{\theta}{2},\frac{\pi}{2})$ (resp. $\kappa\in(\frac{\pi}{2},\frac{\theta}{2})$), $s\in\mathbb{R}$, $\tau\in(-\infty,0)$, and $\zeta:=se^{i\kappa}$,

$$|e^{-\frac{\zeta^2}{4\tau}}|=|e^{-\frac{s^2}{4\tau}\cos(2\kappa)}|\le e^{-\frac{s^2}{4|\tau|}|\cos(\theta)|}.$$

Thus

$$\lim_{|s|\to+\infty}\int_{\frac{\theta}{2}}^{\frac{\pi}{2}} e^{-\frac{(se^{i\kappa})^2}{4\tau}}sie^{i\kappa}d\kappa=0 \qquad \text{if } \theta\in(\frac{\pi}{2},\pi)$$

(resp.

$$\lim_{|s|\to+\infty}\int_{\frac{\pi}{2}}^{\frac{\theta}{2}} e^{-\frac{(se^{i\kappa})^2}{4\tau}}sie^{i\kappa}d\kappa=0 \qquad \text{if } \theta\in(\pi,\frac{3\pi}{2}).)$$

It follows from the residue theorem that

$$\frac{1}{\sqrt{4\pi\tau}}\int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} e^{-\frac{\zeta^2}{4\tau}}d\zeta = \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty e^{i\frac{\pi}{2}}}^{\infty e^{i\frac{\pi}{2}}} e^{-\frac{\zeta^2}{4\tau}}d\zeta = \frac{1}{i\sqrt{4\pi|\tau|}}\int_{-\infty}^{\infty} e^{-\frac{s^2}{4|\tau|}}ids = 1,$$

which completes the proof of Claim 1.
Finally, letting $\zeta = \sqrt{|\tau|}\xi$, we see that for any $z \in \Omega(\frac{\theta}{2}, r)$

$$v(z,\tau) - \psi(z) = \frac{1}{\sqrt{4\pi\frac{\tau}{|\tau|}}}\int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} e^{-\frac{\xi^2}{4\frac{\tau}{|\tau|}}}\big(\psi(z-\sqrt{|\tau|}\xi) - \psi(z)\big)d\xi$$

tends to 0 as $\tau \to 0^-$, by dominated convergence.

Finally, assume that $\partial_z^j\psi \in L^\infty(\Omega(\frac{\theta}{2}, r))$ for all $j \in \mathbb{N}$ and that $\overline{S(1)} \subset \Omega(\frac{\theta}{2}, r)$.
Claim 2. For all $j \in \mathbb{N}$, we have $\partial_z^j[v(z,\tau) - \psi(z)] \to 0$ as $\tau \to 0^-$ uniformly for $z \in [-1,1]$.
Indeed, for all $j \in \mathbb{N}$, we can write

$$\partial_z^j[v(z,\tau) - \psi(z)] = \frac{1}{\sqrt{4\pi\frac{\tau}{|\tau|}}}\int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} e^{-\frac{\xi^2}{4\frac{\tau}{|\tau|}}}\big(\psi^{(j)}(z-\sqrt{|\tau|}\xi) - \psi^{(j)}(z)\big)d\xi.$$

For given $\varepsilon > 0$, we pick a number $A > 0$ such that

$$\int_{|s|>A}\Big|e^{-\frac{(se^{i\frac{\theta}{2}})^2}{4\frac{\tau}{|\tau|}}}\Big|ds < \varepsilon.$$

Then

$$\left|\frac{1}{\sqrt{4\pi\frac{\tau}{|\tau|}}}\int_{\substack{\xi\,\in\,(-\infty,\,\infty)e^{i\frac{\theta}{2}},\\ |\xi|\,>\,A}} e^{-\frac{\xi^2}{4\frac{\tau}{|\tau|}}}\big(\psi^{(j)}(z-\sqrt{|\tau|}\xi) - \psi^{(j)}(z)\big)d\xi\right| \le \frac{2\|\psi^j\|_{L^\infty(\Omega(\frac{\theta}{2},r))}\varepsilon}{\sqrt{4\pi}}.$$

On the other hand, it follows from the uniform continuity of $\psi^{(j)}$ on some open neighborhood of $[-1,1]$ that

$$\frac{1}{\sqrt{4\pi\frac{\tau}{|\tau|}}}\int_{\substack{\xi\,\in\,(-\infty,\,\infty)e^{i\frac{\theta}{2}},\\ |\xi|\,\le\,A}} e^{-\frac{\xi^2}{4\frac{\tau}{|\tau|}}}\big(\psi^{(j)}(z-\sqrt{|\tau|}\xi) - \psi^{(j)}(z)\big)d\xi \to 0 \qquad \text{as } \tau \to 0^-,$$

uniformly for $z \in [-1,1]$. Claim 2 is proved.

Using the fact that $\partial_t^k\partial_z^j v = \partial_z^{2k+j}v$ for $k, j \in \mathbb{N}$, $\tau < 0$, and $z \in [-1,1]$, we infer that $v \in C^\infty([-1,1]\times[-T,0])$ for any $T > 0$ such that $-T \in \mathcal{O}(\theta, R)$. The proof of Lemma 3.2 is complete. □

Let us go back to the proof of Proposition 2.1. Pick ψ_1 and ψ_2 as given by Lemma 3.1, and let

$$v_j(z,\tau) = \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty e^{i\frac{\theta_j}{2}}}^{\infty e^{i\frac{\theta_j}{2}}} e^{-\frac{\zeta^2}{4\tau}}\psi_j(z-\zeta)d\zeta \quad \text{ for } j = 1,2,\ z \in \Omega(\frac{\theta_j}{2}, r) \text{ and } \tau \in \mathcal{O}(\theta_j, R).$$

Then

$$v_{1,\tau} - v_{1,zz} = 0, \qquad \text{in } \Omega(\frac{\theta_1}{2}, r) \times \mathcal{O}(\theta_1, R),$$
$$v_{2,\tau} - v_{2,zz} = 0, \qquad \text{in } \Omega(\frac{\theta_2}{2}, r) \times \mathcal{O}(\theta_2, R).$$

Let $v(z,\tau) = v_1(z,\tau) + v_2(z,\tau)$. Then v is well-defined and analytic in $D := [\Omega(\frac{\theta_1}{2}, r) \cap \Omega(\frac{\theta_2}{2}, r)] \times [\mathcal{O}(\theta_1, R) \cap \mathcal{O}(\theta_2, R)]$, and it fulfills

$$\begin{aligned} v_\tau - v_{zz} &= 0 \quad \text{in } D, \\ \lim_{\tau \to 0^-} v(z,\tau) = \psi_1(z) + \psi_2(z) &= \psi(z) \quad \text{in } \Omega(\frac{\theta_1}{2}, r) \cap \Omega(\frac{\theta_2}{2}, r). \end{aligned}$$

Furthermore, if $\hat{T} > 0$ is such that $[-\hat{T}, 0) \subset \mathcal{O}(\theta_1, R) \cap \mathcal{O}(\theta_2, R)$ and $\hat{T} \leq T$, then $v(., -\hat{T})$ is analytic in the open set $\Omega(\frac{\theta_1}{2}, r) \cap \Omega(\frac{\theta_2}{2}, r)$ which contains $\overline{S(1)}$.

To complete the proof, we proceed as in [15], combining the above construction with a null controllability result. Pick any $s \in (1,2)$ and any function $\rho \in G^s([-\hat{T}, 0])$ such that $\rho(t) = 1$ for $-\hat{T} \leq t \leq -\frac{\hat{T}}{2}$ and $\rho(t) = 0$ for $-\frac{\hat{T}}{4} \leq t \leq 0$. Let $g_0(t) := \rho(t) v(0,t)$ and $g_1(t) := \rho(t) \partial_\tau v(0,t)$ for $t \in [-\hat{T}, 0]$. Using the fact that $v(0,.), \partial_\tau v(0,.) \in \mathrm{Hol}(\mathcal{O}(\theta_1, R) \cap \mathcal{O}(\theta_2, R))$ and [12, Lemma 3.7], we infer that $g_0, g_1 \in G^s([-\hat{T}, 0])$. Therefore, for any $R > 1$ there exists some constant $C > 0$ such that

$$|\partial_t^j g_0(t)| + |\partial_t^j g_1(t)| \leq C \frac{(2j)!}{R^j} \quad \forall t \in [-\hat{T}, 0], \ \forall j \in \mathbb{N}.$$

According to [12, Proposition 3.1], the problem

$$\begin{aligned} \hat{v}_t - \hat{v}_{xx} = 0, &\qquad t \in (-\hat{T}, 0), \ x \in (-1,1), \\ \hat{v}(0,t) = g_0(t), \ \partial_x \hat{v}(0,t) = g_1(t), &\qquad t \in (-\hat{T}, 0) \end{aligned}$$

possesses a solution $\hat{v} \in C^\infty([-1,1] \times [-\hat{T}, 0])$. It follows then from the definition of ρ and Holmgren's theorem that

$$\begin{aligned} \hat{v}(x,t) = 0, &\qquad t \in (-\frac{\hat{T}}{4}, 0), \ x \in (-1,1), \\ \hat{v}(x,t) = v(x,t), &\qquad t \in (-\hat{T}, -\frac{\hat{T}}{2}), \ x \in (-1,1), \end{aligned}$$

so that $\hat{v}(.,0) = 0$ and $\hat{v}(., -\hat{T}) = v(., -\hat{T})$. Let

$$\begin{aligned} h_{\pm 1}(t) := v(\pm 1, t) - \hat{v}(\pm 1, t), &\qquad \text{for } t \in (-\hat{T}, 0), \\ w(x,t) := v(x,t) - \hat{v}(x,t), &\qquad \text{for } t \in (-\hat{T}, 0), \ x \in (-1,1). \end{aligned}$$

Then w satisfies

$$\begin{cases} w_t - w_{xx} = 0, & t \in (-\hat{T}, 0), \ x \in (-1,1), \\ w(x, -\hat{T}) = 0, & x \in (-1,1), \\ w(\pm 1, t) = h_{\pm 1}(t), & t \in (-\hat{T}, 0), \\ w(x,0) = \psi(x), & x \in (-1,1). \end{cases}$$

Extending w and $h_{\pm 1}$ by 0 for $t \leq -\hat{T}$, we obtain the main result in Proposition 2.1 on the interval $[-T, 0]$. A simple translation in time gives the result on the interval $[0, T]$.

Assume in addition that ψ be odd. Then it is easy to see that both ψ_1 and ψ_2 are odd, and that v_1 and v_2 are odd with respect to z. It follows that v and $\hat{v}$ are odd with respect to z. (Note that $g_0 \equiv 0$.) Therefore, w is odd with respect to z. □

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