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Wavefront dislocations reveal insulator topology

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Wavefront dislocations are a fundamental and ubiquitous wave phenomenon that originates from an indetermination of the phase of a wavefield, where the amplitude of the wave vanishes. Since the seminal work of John Nye and Michael Berry in 1974 [1], it was realised that such topological defects could emerge in any wavefield irrespectively of its physical nature or its dispersion relation. Wavefront dislocations have been observed from the physics of fluids, sound, electromagnetism to oceanic tides and astronomy, and even led to the birth of a whole research field known as singular optics [2–7]. In quantum mechanics, wavefront dislocations have been predicted in connection to the Aharonov-Bohm effect [8, 9], and they have been observed recently in the electron density of graphene as a manifestation of the wave-function Berry phase [10, 11]. In parallel, topology also spread in condensed matter physics, giving rise to the field of topological phases of matter [12, 13] and its various classical analogs such as topological photonics [14]. In this context, topological properties are defined from singularities of the wave functions delocalised in the bulk of the material. Still, apart from the pioneering example of the quantised electric conductivity in the quantum Hall phase, their experimental manifestations are mainly indirect, through the existence of gapless excitations localised at the boundaries of the system. Despite this common underlying key role of phase singularities, topological phases and singular waves have remained two distinct fields [15]. Here we reconcile them by showing a wavefront dislocation as a direct evidence of the phase singularity of the delocalised wave functions and observe it through standing-wave interference in a 1D microwave photonic insulator. By bridging the bulk topology of insulators to a ubiquitous wave phenomenon, we open a promising route to investigate quantum and classical topological systems through real-space interference patterns.

Topological insulating systems are associated with integer-valued numbers (topological indices) that characterise the phase singularities of the bulk wave functions. A change of the topological index requires the spectral gap to close. Such a topological transition is also associated with a phase singularity of the eigenmodes. By including the parameter that controls the spectral gap, a topological transition of a D dimensional system is then described by a singular point in a $D + 1$ dimensional parameter space. The 1D case is particularly interesting as it allows us to describe topological transitions with vortices appearing in 2D parameter space. Vortices in real space are known to induce wavefront dislo-

cations onto an incident propagating wave [8]. Similarly, we reveal here that the vortex of the topological transition involves a quite analog phenomenon in parameter space. When a defect or an edge is included to a topological insulator, a defect-induced interference pattern of bulk wave functions emerges and abruptly changes at the singular band crossing point, then giving rise to a wavefront dislocation in the $D+1$ parameter space. We find that this wavefront dislocation is accessible experimentally through the local density of states (LDOS) and demonstrate its existence in a 1D microwave

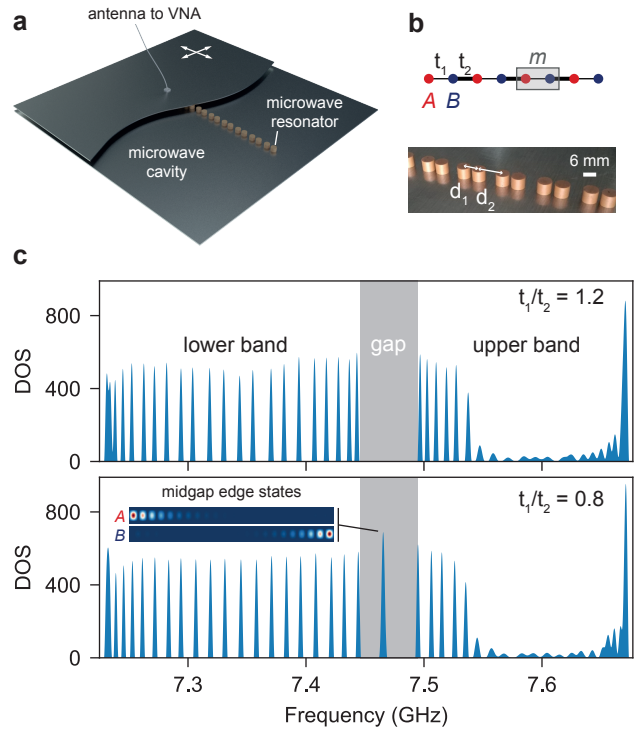


FIG. 1. **1D microwave photonic insulator.** **a**, Lattice of dielectric resonators inside the microwave cavity made of two metallic plates. The top plate, which is partially shown, is movable in the plane (white arrows), so that an antenna going through and connected to a vector network analyser (VNA) enables the generation and resolution of the microwave signal both spectrally and spatially (LDOS). **b**, Illustration of the SSH lattice (top) and picture of its realisation with dielectric resonators (bottom). **c** (top) DOS of a 44-resonator SSH lattice for $t_1/t_2 = 1.2$ ($w_> = 0$) showing two bands separated by a frequency gap, (bottom) DOS of a 44-resonator SSH lattice for $t_1/t_2 = 0.8$ ($w_< = -1$), revealing two midgap modes A and B. The resolution of their LDOS (inset) confirms that they are localised at the two ends of the lattice and fully sublattice polarised.

photonic insulator. Moreover, we show that the quantised charge of the vortex, which corresponds to the variation of the number of interference fringes at the transition, consistently coincides with the variation in the number of topologically-protected boundary modes inside the spectral gap. This leads us to the experimental demonstration of the pillar of the topological phases of matter, the bulk-boundary correspondence.

Description of the 1D photonic insulator

In 1D, the band topology of insulators may become nontrivial in the presence of chiral symmetry. For lattices with translational invariance, this concerns a class of Bloch Hamiltonians that are bipartite

$$H(k) = \begin{pmatrix} 0 & h(k) \\ h^\dagger(k) & 0 \end{pmatrix}, \quad (1)$$

where k is the dimensionless 1D quasi-momentum. An illuminating illustration of Hamiltonian (1) is found in the celebrated Su-Schrieffer-Heeger (SSH) model [16]. First introduced to describe conducting electrons in polyacetylene, it is involved in the physics of various chiral systems [17–22]. Here, we focus on an experimental realisation in a microwave photonic insulator. The system consists of a dimerised lattice of dielectric resonators in a microwave cavity (see Figs. 1a,b). Each cylindrical resonator is made of ZrSnTiO ceramics (radius $r = 3$ mm, height $h = 5$ mm, with an index of refraction $n_r \approx 6$) and supports a fundamental transverse-electric (TE₁) mode of bare frequency of 7.435 GHz. This mode spreads out evanescently, so that the coupling strength can be controlled by adjusting the separation distance between the resonators [23]. The lattice consists in two coupled sublattices A and B with staggered coupling strengths t_1 and t_2 , so that $h(k) = t_1 + t_2 e^{-ik}$ for the choice of unit cell in Fig. 1b. The corresponding resonator separations are denoted d_1 and d_2 . In our experiments, the coupling strengths $t_{1,2}$ can be typically adjusted from 10 MHz to 115 MHz which corresponds to separations $d_{1,2}$ of 16 mm and 7 mm respectively.

The SSH model is known to display a transition between two topologically distinct insulators when varying the coupling ratio t_1/t_2 . Its spectrum exhibits two bands given by $f_{\pm}(\mathbf{k}) = \pm|h(\mathbf{k})|$ and whose topology relies on the quantisation of the geometrical Zak phase of the Bloch wave function in the 1D Brillouin zone (BZ) [24]. This quantisation is characterised by the winding number $w = \oint_{\text{BZ}} \nabla_k \text{Arg}[h(k)] dk / 2\pi$, which leads to $w_{>} = 0$ and $w_{<} = -1$ in the two insulating regimes $t_1 \gtrless t_2$.

Topology from localised boundary modes

Before shedding new light on the topological transition, let us recall that in experiments, the band topology is mainly evidenced through the appearance/disappearance of midgap modes localised at the lattice boundaries, by virtue of the famous bulk-boundary correspondence [25]. Here, the bulk-boundary correspondence predicts the existence of $\mathcal{N}_A = -w_{\gtrless}$ ($\mathcal{N}_B = -w_{\gtrless}$) bound states with sublattice polarisation A (B) at

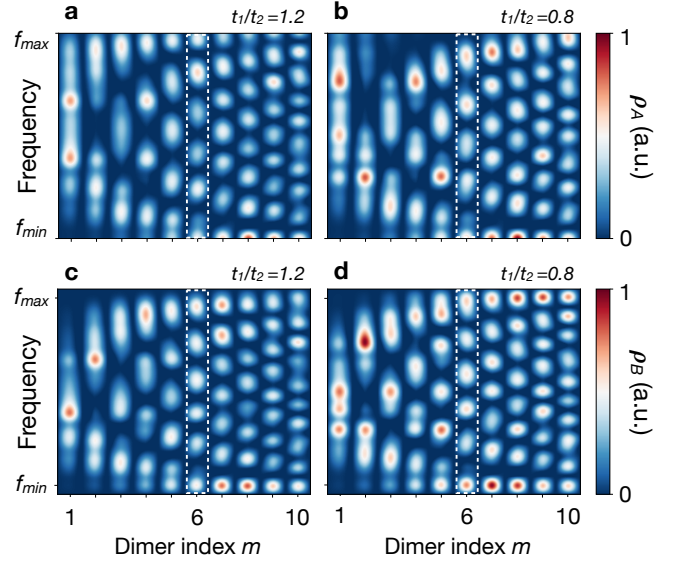


FIG. 2. **Experimental LDOS maps:** Resolution of the lower band between frequencies $f_{\min} = 7.23$ GHz and $f_{\max} = 7.45$ GHz (see Fig. 1c). **a**, LDOS of sublattice A for $t_1/t_2 = 1.2$. **b**, LDOS of sublattice A for $t_1/t_2 = 0.8$. The number N_A of LDOS maxima changes identically on each dimer from $N_A = m$ in panel a to $N_A = m - 1$ in panel b (see e.g. the in white dashed box at dimer index $m = 6$). **c**, LDOS of sublattice B for $t_1/t_2 = 1.2$. **d**, LDOS of sublattice B for $t_1/t_2 = 0.8$. The number N_B of LDOS maxima remains identical in panels c and d (see e.g. in the white dashed box at dimer index $m = 6$).

the leftmost (rightmost) edge of the crystal in Fig. 1b (See Supplementary Information). We report the observation of these midgap boundary modes in Figs. 1c,d. It shows the measured density of states (DOS) of the TE waves in a lattice of 44 microwave resonators (see Supplementary Information for details). For $t_1/t_2 = 1.2$, where the bulk winding number is $w_{>} = 0$, the sublattice structure produces two frequency bands of 22 modes each. In contrast, when $t_1/t_2 = 0.8$, the 1D winding number switches to $w_{<} = -1$ and we observe two modes pinned in the gap. We then confirm that they are sublattice polarised and spatially localised at the two ends of the crystal by resolving their LDOS (inset of Fig. 1d).

The observation of midgap modes *localised* at boundaries is commonly considered as the hallmark of topological transitions, as reported in mechanical, acoustic, photonic, microwave, cold-atomic, and electronic systems [26–33]. Nevertheless, the band topology is defined from the delocalised waves beyond the excitation gap. Now we present direct evidence of the topological transition through LDOS measurements of the *delocalised* waves.

LDOS interferences of delocalised waves

The delocalised waves correspond to resonance frequencies in the two bands f_{\pm} . Figures 2a-d represent the sublattice-resolved LDOS $\rho_{A,B}$ of the delocalised waves of the lower band f_{-} probed in the two topological regimes. Only the leftmost half of the photonic crystal is shown, for the second

half is inversion symmetric (see Supplementary Information). The LDOS maps consist of standing-wave interference patterns due to the lattice boundaries. We focus in particular on the number $N_{A(B)}$ of constructive-interference fringes in the LDOS of sublattice $A(B)$. For the sublattice A in Figs. 2a,b, we observe that N_A changes identically on each site m through the topological transition. For instance, there are six constructive-interference fringes on site $m = 6$ when $t_1/t_2 = 1.2$, whereas there are five of them when reducing the coupling ratio to $t_1/t_2 = 0.8$. More generally, it shows that $N_A = m$ for $t_1/t_2 = 1.2$ and $N_A = m - 1$ for $t_1/t_2 = 0.8$. In contrast, we do not observe such a change on sublattice B , where there are always $N_B = m$ constructive-interference fringes per site, regardless of the topological phase (Figs. 2c,d).

To explain this striking feature in the LDOS maps near the edge, we focus on a semi-infinite SSH chain and model the edge as an infinite potential barrier. Backscattering of the delocalised waves on the edge then leads to the LDOS [34]

$$\rho_A(m, k) \propto 1 + \cos(2km + \delta_A(k) + \pi) \quad (2)$$

$$\rho_B(m, k) \propto 1 + \cos(2km + \pi), \quad (3)$$

which reproduces very well the experimental LDOS maps in Fig. 2 (see Supplementary Information). The oscillating terms in the right-hand sides describe the LDOS fluctuations induced by the edge. The wavelength of the oscillations on both sublattices relates to the backscattering wavevector $2k$. Such $2k$ -wavevector oscillations are often referred to as (frequency-resolved) Friedel oscillations, with reference to the charge density oscillations that screen charged impurities in metals [35]. It varies with the frequency at which we probe the cavity through the dispersion relation $f_{\pm}(k)$. The wavelength of the oscillations is then a spectral measurement and does not imply the topology of the frequency band. For instance, the oscillations in ρ_B only depend on the backscattering wave-vector and give rise to similar interference patterns in the two topological regimes, as observed experimentally in Figs. 2c,d. In contrast, the oscillations in ρ_A imply the additional phase shift $\delta_A(k) = 2\text{Arg}[h(k)]$ (see Ref. [34]).

The phase shift δ_A further leads to dramatic modifications in the LDOS interference patterns. Let us focus, for instance, on the LDOS maxima in Eq. (2). They correspond to the specific wavefronts $2n_A\pi = 2km + \delta_A + \pi$, where n_A is an integer. When summing over the lower frequency band, we find that the number of constructive-interference fringes verifies

$$N_A = \int_{f_{\min}}^{f_{\max}} \frac{\partial n_A}{\partial f_{\pm}} df_{\pm} = \int_0^{\pi} \frac{dk}{2\pi} \left(2m + \frac{\partial \delta_A}{\partial k} \right) = m + w. \quad (4)$$

This sum rule shows that the scattering phase shift δ_A relates an observable quantity (N_A) of the delocalised waves to their topological winding w . In particular, the number of interference fringes depends on the site index m , but the topological invariant shifts the interference fringes identically on all sites. Remarkably, if the winding number w depends on the choice of unit cell [36, 37], this arbitrary choice is however included in the labelling of the dimers m , such that their sum yields the

observable quantity N_A . The sum rule then explains the uniform change of N_A observed in the LDOS maps in Figs 2a,b for the winding numbers $w_{>} = 0$ and $w_{<} = -1$. Therefore, the LDOS maps reveal direct evidence of the band topology of the delocalised waves in the 1D microwave insulator.

The phase shifts of wave functions also play a central role in scattering physics, because they relate to the number of (virtual) bound states at a potential barrier. Fundamental theorems, such as Levinson theorem or Friedel sum rule, show that, for given wave functions, the number of (virtual) bound states change with the depth of the barrier [35, 38, 39]. Similarly here, the bulk-boundary correspondence can be rephrased as a relation between the scattering phase shift δ_A and the number N_A of bound states localised at the potential barrier of the edge. Since $\delta_A(\pi) - \delta_A(0) = 2\pi w$ (see Eq. (4)), we readily find

$$\delta_A(0) - \delta_A(\pi) = 2\pi N_A. \quad (5)$$

The important difference, however, is that the number of bound states here changes with the topological transition, whereas the strength of the potential barrier remains the same. This change results from an intrinsic property of the delocalised waves and the potential barrier at the edge only acts as a natural interferometer that reveals their topology through the scattering phase shift. If the phase shift variation has been measured through N_A in the LDOS maps of sublattice A (Figs. 2a,b), we have also resolved the N_A midgap bound states localised at the edge (inset of Fig. 1d).

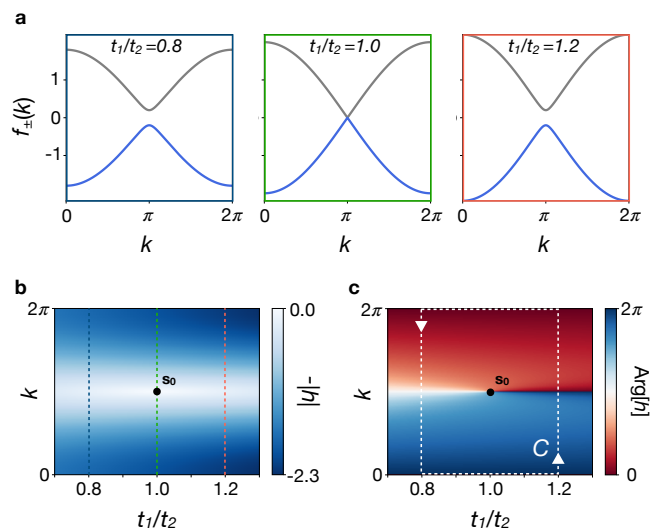


FIG. 3. **Band degeneracy and phase singularity:** **a**, Spectral bands $f_{\pm}(k) = \pm|h(k)|$ in the 1D BZ for various values of the parameter t_1/t_2 . The band crossing for $t_1/t_2 = 1$ marks the topological transition between the two insulating regimes. **b**, Energy band $f_{\pm}(s) = -|h(s)|$ represented in the 2D parameter space. The spectral degeneracy occurs at point $s_0 = (1, \pi)$. The colored vertical dashed lines correspond to the 1D spectral bands in panel a. **c**, Eigenstate phase $\theta(s) = \text{Arg}[h(s)]$ in the 2D parameter space. It is singular at the spectral degeneracy point s_0 .

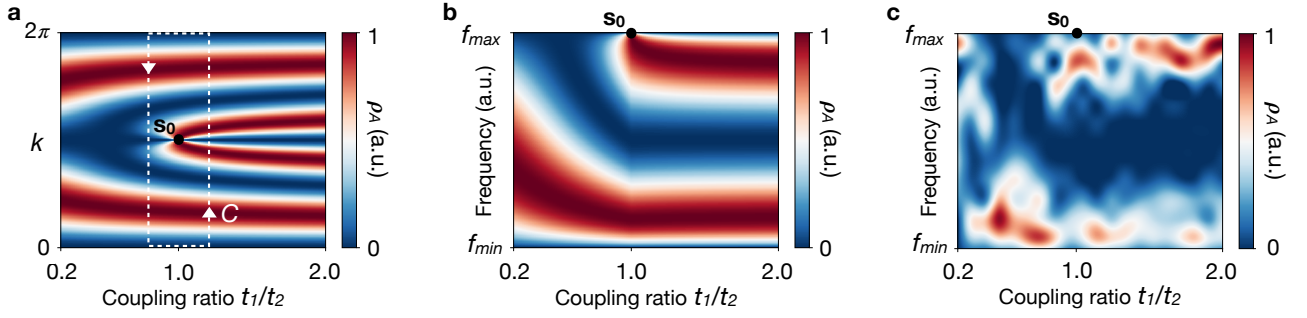


FIG. 4. **Wavefront dislocation in the LDOS:** **a**, Theoretical LDOS ρ_A in the 2D parameter space on site $m = 2$. The phase singularity in $\mathbf{s}_0 = (1, \pi)$ corresponds to a wavefront dislocation of strength $W_d = 2$ in the LDOS. **b**, Theoretical LDOS ρ_A on site $m = 2$ in the 2D space $(t_1/t_2, f_-(k))$, where $f_-(0) = f_{\min}$ and $f_-(\pi) = f_{\max}$. Only one interference fringe emerges from the dislocation core in this frequency representation of the LDOS. **c**, Experimental LDOS ρ_A resolved on site $m = 2$ for the lower frequency band. One constructive-interference fringe emerges in the LDOS near \mathbf{s}_0 , as expected from panel b.

Thus, both sides of Eq. (5) are observable independently, and our measurements also bring evidence of the bulk-boundary correspondence. This demonstrates an efficient method to test this key concept of gapped topological systems through the LDOS in the experiments.

Wavefront dislocations in the LDOS

Now we show that the change of N_A observed in the LDOS maps arises as a ubiquitous wave phenomenon and is the signature of a wavefront dislocation in the LDOS. Topological defects in waves rely on generic assumptions that do not involve the wave equation, and so they are ubiquitously involved in branches of physics as various as electromagnetism, optics, acoustics, fluid physics, astrophysics, and condensed matter physics [2–8, 10, 11, 40, 41]. The wavefront dislocations are associated with the topological phase singularities of wavefields in a space of at least dimension 2.

Here, the microwave photonic insulator is 1D and its topological transition relies on a spectral band crossing in the 1D BZ (Fig. 3a). Nevertheless, the topological transition is driven by the coupling ratio t_1/t_2 . Thus, it is fully characterised in a 2D space associated with the parameter $\mathbf{s} = (t_1/t_2, k)$. In this parameter space, the spectral bands are $f_{\pm}(\mathbf{s}) = \pm|h(\mathbf{s})|$ and the eigenstates can be chosen as $|u_{\pm}(\mathbf{s})\rangle \propto |A\rangle \pm e^{i\theta(\mathbf{s})}|B\rangle$, where $\theta(\mathbf{s}) = \text{Arg}[h(\mathbf{s})]$. The zeroes of $h(\mathbf{s})$ are points where i) the spectral band gap closes and ii) the eigenstate phase $\theta(\mathbf{s})$ becomes ill-defined. This phase singularity in 2D is nothing but a vortex that constrains the surrounding phase texture to wind. The vortex winding is then quantified by a topological index W_s , such that $\oint_C \nabla_s \theta \cdot d\mathbf{s} = 2\pi W_s$ along a closed circuit C enclosing the phase singularity. In the SSH model, $\mathbf{s}_0 = (1, \pi)$ is the only point where $h(\mathbf{s})$ vanishes (Fig. 3b). This leads to the singularity charge $W_s = 1$ for the counter-clockwise circuit C in Fig. 3c.

The phase singularity in the 2D parameter space is the source of a wavefront dislocation of strength $2W_s$ in the LDOS. We can evidence the dislocation by following the evolution of the LDOS interference patterns through the topological transition. Figure 4a shows the predicted LDOS evolution

on a given site of sublattice A ($m = 2$). It exhibits an edge dislocation with two constructive-interference fringes emerging from the core in \mathbf{s}_0 . Wavefront dislocations are known to occur as the phase singularity of a complex scalar field whose real (or imaginary) part represents a physical quantity [1]. Here, the physical quantity is the LDOS fluctuations defined as $\Delta\rho_A = -\text{Im}\Delta G_A/\pi$ (oscillating term in Eq. (2)), so that the complex scalar field is the scattering Green function ΔG_A that describes the delocalised waves backscattering on the edge of the microwave insulator

$$\Delta G_A(m, \mathbf{s}) \propto -ie^{i\varphi_A(\mathbf{s})}, \quad (6)$$

where $\varphi_A(\mathbf{s}) = 2km + \delta_A(\mathbf{s}) + \pi$. The scattering phase shift $\delta_A(\mathbf{s}) = 2\text{Arg}[h(\mathbf{s})]$ maps the phase singularity of the eigenstates into the phase of the scattering Green function. The latter effectively describes a plane wave (e^{i2km}) passing through a vortex ($e^{i\delta_A(\mathbf{s})}$) in the 2D parameter space. This effective vortex perturbs the surrounding phase of the wave in such a way that, for the counter-clockwise Burgers circuit C in Fig. 4a, the phase accumulated by the scattering Green function satisfies

$$2\pi W_d \equiv \oint_C \nabla_s \varphi_A(\mathbf{s}) \cdot d\mathbf{s} = 4\pi W_s \quad (7)$$

The phase variation is 2π -quantised because ΔG_A must be single valued to describe observable LDOS fluctuations along the circuit C . Thus, the number of additional interference fringes required to fulfil the phase variation along the Burgers circuit C is W_d . In analogy with Burgers' vectors whose length provides the dislocation strength for atomic planes in solids, W_d is the strength of the wavefront dislocation. Since $W_s = 1$ in the SSH model, there are $W_d = 2$ additional interference fringes emerging from the dislocation core, as shown in Fig. 4a. It is worth stressing that although the expression of the phase φ_A depends on the choice of the unit cell, its variation over C does not and is observable.

To confirm this prediction, we measure the LDOS on the site $m = 2$ of sublattice A for 20 values of the coupling ratio t_1/t_2 between 0.2 and 2.0. Since the LDOS is resolved as a

function of the frequency $f_-(k)$ instead of the wave vector k in our experiments, we do not expect W_d but $W_d/2$ interference fringes emerging from the dislocation core (Fig. 4b). Figure 4c experimentally confirms that the number N_A of constructive-interference fringes changes from one to two at the dislocation core. This is also in agreement with the LDOS change on site $m = 2$ shown in Figs. 2a,b. This observation reveals the wavefront dislocation at the topological transition that causes the uniform change in the number of fringes N_A in the LDOS interference pattern.

Conclusion

We have probed the band topology of a 1D photonic insulator through the standing-wave interference pattern in the LDOS resulting from backscattering on a boundary. We have shown that the uniform change in the number of interference fringes at the topological transition is a measurement of the dislocation strength and then of the eigenstate phase singularity. This 2D phase singularity constrains the 1D winding numbers of the two nonequivalent SSH insulators as $W_s = w_> - w_<$ (see Fig. 3). Although there is a gauge choice in the definition of the 1D winding numbers, their difference is gauge invariant and the uniform change in the number of interference fringes characterises unambiguously the change of band topology at the transition. Thus, the wavefront dislocation in the LDOS is an observable phenomenon that reveals the topological transition in 1D insulators.

The band topology of 1D insulators is also known to affect the electron response to external force fields through phenomena such as the electric polarisation and Bloch oscillations [42–44]. Nevertheless, these phenomena are observable in very specific systems. Bloch oscillations, for instance, are hardly observable with electrons in solids, where impurities are usually detrimental to phase coherence, and so they lead to band topology measurements in cold atoms [19]. In contrast, the LDOS is an observable routinely resolved in various kinds of systems [26–33]. Thus, we expect that topological defects in real-space LDOS interference can reveal the band topology in experiments involving propagating waves of very different natures.

In addition to the band topology through wavefront dislocations, the LDOS also leads to the resolution of midgap modes localised at boundaries. This enabled us to test of the bulk-boundary correspondence through a single observable and, thus, a single experiment. This efficient approach could then shed light into breakdowns of the bulk-boundary correspondence, as recently reported in systems where the number of bound states may no longer be provided by the bulk topological invariant [22, 45].

Methods

Each dielectric microwave resonator (see figures 1 a and b) is made of ZrSnTiO ceramics (radius $r = 3$ mm, height $h = 5$ mm, with an index of refraction $n_r \approx 6$) and supports a fundamental TE mode of bare frequency $\nu_0 = 7.435$ GHz. This mode spreads out evanescently, so that the coupling

strength can be controlled by adjusting the separation distance between the resonators [23]. As shown in Fig 1.b, the lattice consists of two coupled sublattices A and B with staggered coupling strengths t_1 and t_2 . The corresponding resonator separations are denoted d_1 and d_2 . In our experiments, the coupling strengths $t_{1,2}$ can be typically adjusted from 10 MHz to 115 MHz which corresponds to separations $d_{1,2}$ of 16 mm and 7 mm respectively.

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- [1] Nye, J. F. & Berry, M. V. Dislocations in wave trains. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* **336**, 165–190 (1974).
- [2] Berry, M. V. Making waves in physics. *Nature* **403**, 21–21 (2000).
- [3] Mawet, D., Riaud, P., Absil, O. & Surdej, J. Annular groove phase mask coronagraph. *The Astrophysical Journal* **633**, 1191 (2005).
- [4] Dennis, M. R., O’Holleran, K. & Padgett, M. J. Singular optics: optical vortices and polarization singularities. In *Progress in optics*, vol. 53, 293–363 (Elsevier, 2009).
- [5] Padgett, M. & Bowman, R. Tweezers with a twist. *Nature Photonics* **5**, 343–348 (2011).
- [6] Rafayelyan, M. & Brasselet, E. Bragg-Berry mirrors: reflective broadband q-plates. *Optics Letters* **41**, 3972–3975 (2016).
- [7] Al Dhaybi, A., Degert, J., Brasselet, E., Abraham, E. & Freydz, E. Terahertz vortex beam generation by infrared vector beam rectification. *Journal of the Optical Society of America B* **36**, 12–18 (2019).
- [8] Berry, M. V., Chambers, R. G., Large, M. D., Upstill, C. & Walmsley, J. C. Wavefront dislocations in the Aharonov-Bohm effect and its water wave analogue. *European Journal of Physics* **1**, 154 (1980).
- [9] Berry, M. Gauge-invariant Aharonov–Bohm streamlines. *Journal of Physics A: Mathematical and Theoretical* **50**, 43LT01 (2017).
- [10] Dutreix, C. *et al.* Measuring the Berry phase of graphene from wavefront dislocations in Friedel oscillations. *Nature* **574**, 219–222 (2019).
- [11] Zhang, Y., Su, Y. & He, L. Local Berry phase signatures of bilayer graphene in intervalley quantum interference. *Physical Review Letters* **125**, 116804 (2020).
- [12] Kosterlitz, J. M. Nobel lecture: Topological defects and phase transitions. *Reviews of Modern Physics* **89**, 040501 (2017).
- [13] Haldane, F. D. M. Nobel lecture: Topological quantum matter. *Reviews of Modern Physics* **89**, 040502 (2017).
- [14] Ozawa, T. *et al.* Topological photonics. *Reviews of Modern Physics* **91**, 015006 (2019).
- [15] Soskin, M., Boriskina, S. V., Chong, Y., Dennis, M. R. & Desyatnikov, A. Singular optics and topological photonics. *Journal of Optics* **19**, 010401 (2016).
- [16] Su, W., Schrieffer, J. R. & Heeger, A. J. Solitons in polyacetylene. *Physical Review Letters* **42**, 1698 (1979).
- [17] Ryu, S. & Hatsugai, Y. Topological origin of zero-energy edge

- states in particle-hole symmetric systems. *Physical Review Letters* **89**, 077002 (2002).
- [18] Delplace, P., Ullmo, D. & Montambaux, G. Zak phase and the existence of edge states in graphene. *Physical Review B* **84**, 195452 (2011).
- [19] Atala, M. *et al.* Direct measurement of the zak phase in topological bloch bands. *Nature Physics* **9**, 795–800 (2013).
- [20] St-Jean, P. *et al.* Lasing in topological edge states of a one-dimensional lattice. *Nature Photonics* **11**, 651–656 (2017).
- [21] Pardo, M. *et al.* Edge-mode lasing in 1d topological active arrays. *Physical Review Letters* **120**, 113901 (2018).
- [22] Downing, C. A., Sturges, T. J., Weick, G., Stobińska, M. & Martín-Moreno, L. Topological phases of polaritons in a cavity waveguide. *Physical Review Letters* **123**, 217401 (2019).
- [23] Bellec, M., Kuhl, U., Montambaux, G. & Mortessagne, F. Tight-binding couplings in microwave artificial graphene. *Physical Review B* **88**, 115437 (2013).
- [24] Zak, J. Berry's phase for energy bands in solids. *Physical Review Letters* **62**, 2747–2750 (1989).
- [25] Franz, M. & Molenkamp, L. *Topological insulators* (Elsevier, 2013).
- [26] Rechtsman, M. C. *et al.* Photonic Floquet topological insulators. *Nature* **496**, 196–200 (2013).
- [27] Hafezi, M., Mittal, S., Fan, J., Migdall, A. & Taylor, J. Imaging topological edge states in silicon photonics. *Nature Photonics* **7**, 1001–1005 (2013).
- [28] Poli, C., Bellec, M., Kuhl, U., Mortessagne, F. & Schomerus, H. Selective enhancement of topologically induced interface states in a dielectric resonator chain. *Nature communications* **6**, 1–5 (2015).
- [29] Nash, L. M. *et al.* Topological mechanics of gyroscopic metamaterials. *Proceedings of the National Academy of Sciences* **112**, 14495–14500 (2015).
- [30] Yang, Z. *et al.* Topological acoustics. *Physical Review Letters* **114**, 114301 (2015).
- [31] Goldman, N. *et al.* Direct imaging of topological edge states in cold-atom systems. *Proceedings of the National Academy of Sciences* **110**, 6736–6741 (2013).
- [32] Ménard, G. C. *et al.* Two-dimensional topological superconductivity in Pb/Co/Si (111). *Nature Communications* **8**, 1–7 (2017).
- [33] Schindler, F. *et al.* Higher-order topology in bismuth. *Nature Physics* **14**, 918–924 (2018).
- [34] Dutreix, C. & Delplace, P. Geometrical phase shift in friedel oscillations. *Physical Review B* **96**, 195207 (2017).
- [35] Friedel, J. XIV. the distribution of electrons round impurities in monovalent metals. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* **43**, 153–189 (1952).
- [36] Rhim, J.-W., Behrends, J. & Bardarson, J. H. Bulk-boundary correspondence from the intercellular Zak phase. *Physical Review B* **95**, 035421 (2017).
- [37] Guzman, M., Bartolo, D. & Carpentier, D. Geometry and topology tango in chiral materials. *arXiv:2002.02850* (2020).
- [38] Levinson, N. On the uniqueness of the potential in a Schrödinger equation for a given asymptotic phase. *Matematisk-fysiske meddelelser Kongelige Danske Videnskabskabernes Selskab* **25**, 1–29 (1949).
- [39] Friedel, J. Metallic alloys. *Il Nuovo Cimento (1955-1965)* **7**, 287–311 (1958).
- [40] Dutreix, C. & Katsnelson, M. I. Friedel oscillations at the surfaces of rhombohedral n-layer graphene. *Physical Review B* **93**, 035413 (2016).
- [41] Phong, V. T. & Mele, E. J. Obstruction and interference in low energy models for twisted bilayer graphene. *arXiv preprint arXiv:2003.00150* (2020).
- [42] King-Smith, R. D. & Vanderbilt, D. Theory of polarization of crystalline solids. *Physical Review B* **47**, 1651–1654 (1993).
- [43] Resta, R. Macroscopic polarization in crystalline dielectrics: the geometric phase approach. *Reviews Modern Physics* **66**, 899–915 (1994).
- [44] Xiao, D., Chang, M.-C. & Niu, Q. Berry phase effects on electronic properties. *Reviews of Modern Physics* **82**, 1959 (2010).
- [45] Tauber, C., Delplace, P. & Venaille, A. Anomalous bulk-edge correspondence in continuous media. *Physical Review Research* **2**, 013147 (2020).