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DIVISORS IN THE MODULI SPACE OF DEBARRE–VOISIN VARIETIES

VLADIMIRO BENEDETTI AND JIEAO SONG

Abstract. Let V_{10} be a 10-dimensional complex vector space and let $\sigma \in \bigwedge^3 V_{10}^\vee$ be a non-zero alternating 3-form. One can define several associated degeneracy loci: the Debarre–Voisin variety $X_6^\sigma \subset \mathrm{Gr}(6, V_{10})$, the Peskine variety $X_1^\sigma \subset \mathbf{P}(V_{10})$, and the hyperplane section $X_3^\sigma \subset \mathrm{Gr}(3, V_{10})$. Their interest stems from the fact that the Debarre–Voisin varieties form a locally complete family of projective hyperkähler fourfolds of K3$^{[2]}$-type. We prove that when smooth, the varieties X_6^σ, X_1^σ, and X_3^σ share one same integral Hodge structure, and that X_1^σ and X_3^σ both satisfy the integral Hodge conjecture in all degrees. This is obtained as a consequence of a detailed analysis of the geometry of these varieties along three divisors in the moduli space. On one of the divisors, an associated K3 surface S of degree 6 can be constructed geometrically and the Debarre–Voisin fourfold is shown to be isomorphic to a moduli space of twisted sheaves on S, in analogy with the case of cubic fourfolds containing a plane.

1. Introduction

Let V_{10} be a 10-dimensional complex vector space and let $\sigma \in \bigwedge^3 V_{10}^\vee$ be a trivector, that is, an alternating 3-form on V_{10}. O. Debarre and C. Voisin showed in [DV] that for σ general, the locus $X_6^\sigma \subset \mathrm{Gr}(6, V_{10})$ where σ vanishes is smooth of dimension 4, and that by varying σ, these fourfolds form a locally complete family of projective hyperkähler varieties of deformation type K3$^{[2]}$. The second integral cohomology group of the hyperkähler fourfold X_6^σ carries the Beauville–Bogomolov–Fujiki quadratic form $\mathfrak{q}$, which provides a polarized Hodge structure on the primitive part.

Along with X_6^σ, there are several other degeneracy loci determined by the trivector σ that have interesting Hodge theoretical and categorical properties. To get a slightly unified notation, we will denote by X_k^σ a subvariety defined in the Grassmannian $\mathrm{Gr}(k, V_{10})$ as follows (see Section 2.1 for more precise definitions).

- The variety X_3^σ is the hyperplane section of $\mathrm{Gr}(3, V_{10})$ defined by σ. It has an interesting subcategory in its derived category, which is a K3 category (see the excellent lecture notes [MS2]).
- There is the variety X_1^σ, a 6-dimensional degeneracy locus in $\mathbf{P}(V_{10})$ also known as the Peskine variety. The Fano variety of lines of X_1^σ is related to another degeneracy locus X_7^σ in $\mathrm{Gr}(7, V_{10})$.
- One can also define a degeneracy locus X_2^σ inside $\mathrm{Gr}(2, V_{10})$.

All these varieties associated with σ are expected to have some common Hodge structures. We will prove the following result.

Theorem 1.1 (see Theorem 2.6 and Theorem 2.16)**.** *We have Hodge isometries*

$$\big(H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}}, \cdot\,\big) \simeq \big(H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}}, -\mathfrak{q}\big) \simeq \big(H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}, \cdot\,\big)$$

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given by algebraic correspondences between X_3^σ and X_6^σ, and between X_6^σ and X_1^σ, whenever they are smooth of expected dimension.

The isometry between the integral Hodge structures of X_3^σ and X_1^σ was already established using a different method by Bernardara–Fatighenti–Manivel [BFM]. Our method focuses on the geometry of these varieties along some special divisors in the moduli space for σ: we use the extra algebraic classes they admit to perform computations in order to show the isometries.

These Hodge isometries allow us to prove the integral Hodge conjecture for X_3^σ and X_1^σ, using the relevant result for 1-cycles on hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type developed by Mongardi–Ottem in [MO].

Theorem 1.2 (see Theorem 3.11 and Theorem 3.16)**.** *The integral Hodge conjecture holds for both X_3^σ and X_1^σ in all degrees, whenever they are smooth of expected dimension.*

We briefly review known results on the moduli space of Debarre–Voisin varieties. On the one hand, there is a 20-dimensional irreducible quasi-projective GIT moduli space

$$\mathcal{M} \coloneqq \mathbf{P}\big(\textstyle\bigwedge^3 V_{10}^\vee\big) /\!\!/ \mathrm{SL}(V_{10})$$

for the trivectors σ. On the other hand, there is a 20-dimensional irreducible quasi-projective moduli space $\mathcal{M}_{22}^{(2)}$ for Debarre–Voisin varieties, and the Torelli theorem for polarized hyperkähler manifolds tells us that the period map

$$\mathfrak{p}\colon \mathcal{M}_{22}^{(2)} \hookrightarrow \mathcal{P}$$

is an open immersion into the quasi-projective period domain $\mathcal{P}$ that parametrizes the corresponding Hodge structures. The Debarre–Voisin construction gives a rational map

$$\mathfrak{m}\colon \mathcal{M} \dashrightarrow \mathcal{M}_{22}^{(2)},$$

which was recently proved by O'Grady to be birational [O'G]. So one can talk about divisors in $\mathcal{M}$ coming from $\mathrm{SL}(V_{10})$-invariant divisors in $\mathbf{P}\big(\bigwedge^3 V_{10}^\vee\big)$, Noether–Lefschetz divisors in $\mathcal{M}_{22}^{(2)}$, and Heegner divisors in $\mathcal{P}$.

In this paper, we will study three Heegner divisors, $\mathcal{D}_{22}$, $\mathcal{D}_{24}$, and $\mathcal{D}_{28}$, labeled using their discriminant. All three come from $\mathrm{SL}(V_{10})$-invariant divisors in $\mathbf{P}\big(\bigwedge^3 V_{10}^\vee\big)$. This gives us alternative descriptions in terms of degeneracy conditions for the trivector σ. Note also that one can sometimes associate a K3 surface S to a Debarre–Voisin variety via Hodge theory or derived categories, just as in the case of cubic fourfolds. Such a phenomenon can only happen for special Heegner divisors in the period domain $\mathcal{P}$.

The first one, $\mathcal{D}_{22}$, was studied in the original article [DV]. Although the variety X_6^σ is not smooth in this case, its singular locus contains (and we will show that it coincides with) a K3 surface S of degree 22. It was proved in [DV] that X_6^σ is birational to the Hilbert square $S^{[2]}$. In particular, this means that S shares the same transcendental Hodge structure with X_6^σ.

For a very general member of the second divisor $\mathcal{D}_{24}$, we will give a geometric construction of a Brauer-twisted K3 surface (S, β) and we show that the Hodge structure of (S, β) embeds in that of X_6^σ. Moreover, we can recover X_6^σ as a moduli space of β-twisted sheaves on S. This case bears a lot of similarities with the case of cubic fourfolds containing a plane.

The third divisor $\mathcal{D}_{28}$ is related to the existence of Lagrangian planes on X_6^σ and is important for the study of the Hodge structures. There are however no associated K3 surfaces for very general members of this family.

In the following table, we sum up the results obtained concerning the Heegner divisors $\mathcal{D}_{22}$, $\mathcal{D}_{24}$, and $\mathcal{D}_{28}$.

Heegner divisor	$\mathcal{D}_{22}$	$\mathcal{D}_{24}$	$\mathcal{D}_{28}$
$\mathrm{SL}(V_{10})$-invariant divisor	$\mathcal{D}^{3,3,10}$	$\mathcal{D}^{1,6,10}$	$\mathcal{D}^{4,7,7}$
degree in $\mathbf{P}\big(\bigwedge^3 V_{10}^\vee\big)$	640	990	5500
distinguished flag in V_{10}	V_3	$V_1 \subset V_6$	$V_4 \subset V_7$
degeneracy condition on σ	$\sigma(V_3, V_3, V_{10}) = 0$	$\sigma(V_1, V_6, V_{10}) = 0$	$\sigma(V_4, V_7, V_7) = 0$
singular locus of X_1^σ	$\mathbf{P}(V_3)$	$\{[V_1]\}$	$\varnothing$
singular locus of X_3^σ	$\{[V_3]\}$	$\varnothing$	$\varnothing$
singular locus of X_6^σ	S_{22}	$\varnothing$	$\varnothing$
degree of associated K3	22	6 with a Brauer class of order 2	none
birational model of X_6^σ	$S_{22}^{[2]}$	$M(S_6, v, B)$	-

TABLE 1. Divisors in the moduli spaces

Let us explain the contents of the table and where to find the corresponding results. Each of the three divisors is given as the set of trivectors σ satisfying a vanishing condition with respect to a certain flag of subspaces of V_{10}. They are mapped to Heegner divisors with given discriminants in the period domain $\mathcal{P}$ via the rational map $\mathfrak{p} \circ \mathfrak{m}$ (see Section 2.2 for details). In Proposition 2.4 and Proposition 2.9, we study the singular loci of X_1^σ, X_3^σ, and X_6^σ, and identify the divisors $\mathcal{D}_{22}$ and $\mathcal{D}_{24}$ as the loci where these varieties turn singular. For a general member of the divisor $\mathcal{D}_{24}$, we define a twisted associated K3 surface (S_6, β) of degree 6 in Proposition 4.5, and we prove in Theorem 4.19 that the hyperkähler variety X_6^σ is isomorphic to a moduli space of twisted sheaves on (S_6, β). Notice that since $\mathcal{D}^{3,3,10}, \mathcal{D}^{1,6,10}$, and $\mathcal{D}^{4,7,7}$ are unirational, the Heegner divisors $\mathcal{D}_{22}, \mathcal{D}_{24}$, and $\mathcal{D}_{28}$ are unirational as well (see Remarks 2.5, 4.3, and 3.5).

Notation. *Grassmannians.* Throughout the paper, we will denote by U_n, V_n, or W_n an n-dimensional complex vector space. We denote by $\mathrm{Flag}(k_1, \dots, k_r, V_n)$ the flag variety parametrizing nested subspaces of V_n of dimensions $k_1, \dots, k_r$. We will denote by $\mathcal{U}_{k_i}$ the tautological vector subbundle of $V_n \otimes \mathcal{O}_{\mathrm{Flag}(k_1,\dots,k_r,V_n)}$ of rank k_i. When $r = 1$, we recover the ordinary Grassmannian, which we denote by $\mathrm{Gr}(k, V_n)$ (or $\mathbf{P}(V_n)$ if $k = 1$); when no confusion arises, $\mathcal{U}, \mathcal{Q}$ will denote respectively the tautological and the quotient vector bundles on $\mathrm{Gr}(k, V_n)$.

For a trivector $\omega \in \bigwedge^3 V_n$, its *rank* is defined as the dimension of the smallest subspace $V \subset V_n$ such that $\omega \in \bigwedge^3 V$. It is a $\mathrm{GL}(V_n)$-invariant. If $\sigma \in \bigwedge^3 V_n^\vee$ and $V_i \subset V_n$, we will denote by $\sigma(V_i, V_i, V_i)$ the restriction $\sigma|_{V_i} \in \bigwedge^3 V_i^\vee$. Similarly, if $V_i \subset V_j \subset V_k \subset V_n$, we use $\sigma(V_i, V_j, V_k)$ to denote the image of σ in $(V_i \wedge V_j \wedge V_k)^\vee$ (seen as a quotient of $\bigwedge^3 V_k^\vee$).

The notation for Schubert varieties inside a Grassmannian $\mathrm{Gr}(k, V_n)$ is as follows. Let us fix a complete flag $0 = V_0 \subset V_1 \subset \cdots \subset V_n$. For any sequence of integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ with $\lambda_1 \leq n - k$ and $\lambda_k \geq 0$, we define the Schubert variety

$$\Sigma_\lambda = \big\{ W \in \mathrm{Gr}(k, V_n) \;\big|\; \dim(W \cap V_{n-k-\lambda_j+j}) \geq j \text{ for } 1 \leq j \leq k \big\},$$

which is of codimension $\sum_i \lambda_i$ inside $\mathrm{Gr}(k, V_n)$. We let σ_λ be the Schubert class representing Σ_λ in cohomology.

Orbital degeneracy loci. Let G be an algebraic group and let V be a G-module. Given a principal G-bundle over an algebraic variety X, there is an associated vector bundle E with fiber V. Let Y be a G-stable subvariety of V. For a global section s of E, we will denote by

$$D_Y(s) = \{x \in X \mid s(x) \in Y \subset V \simeq E_x\}$$

the Y-degeneracy locus associated with s as defined in [BFMT1]. We will use basic properties of these loci that can be found in [BFMT1] and [BFMT2]. In particular, a Bertini-type theorem holds for orbital degeneracy loci: if E is globally generated, s is general, and $D_Y(s) \neq \varnothing$, then $\operatorname{codim}_X D_Y(s) = \operatorname{codim}_V Y$ and $\operatorname{Sing} D_Y(s) = D_{\operatorname{Sing} Y}(s)$. More generally, for arbitrary s, we have $\operatorname{codim}_X D_Y(s) \leq \operatorname{codim}_V Y$, and when equality holds, $D_{\operatorname{Sing} Y}(s) \subset \operatorname{Sing} D_Y(s)$. We will refer to $\operatorname{codim}_V Y$ (resp. $\dim X - \operatorname{codim}_V Y$) as the expected codimension (resp. expected dimension) of the orbital degeneracy locus $D_Y(s)$.

Lattices. By a lattice we shall mean a finitely generated free $\mathbf{Z}$-module L endowed with an integral quadratic form q. The following basic properties can be found in [BHPvV, Chapter I.2].

The *discriminant group* $D(L)$ of L is defined as $L^\vee/L$, where $L^\vee := \operatorname{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$ is the dual, and the order of $D(L)$ is called the *discriminant* of L and denoted by $d(L)$. If M denotes the Gram matrix of q in an integral basis of L, then we have $d(L) = |\det(M)|$. A lattice L is called *even* if $q(x) \in 2\mathbf{Z}$ for all $x \in L$, and *unimodular* if $d(L) = 1$. For a sublattice $A \subset L$ with $\operatorname{rank}(L) = \operatorname{rank}(A)$, its index $[L : A]$ is finite and satisfies $[L : A]^2 = d(A)/d(L)$.

A sublattice $A \subset L$ is called *saturated* if L/A is torsion-free. For any sublattice $A \subset L$, one can define the orthogonal sublattice $A^\perp \subset L$ with respect to q. If L is unimodular and A is a saturated sublattice, A and $A^\perp$ have isomorphic discriminant groups, hence the same discriminant $d(A) = d(A^\perp)$. In this case, the direct sum $A \oplus A^\perp$ has index $d(A)$ in L.

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2. General results for Debarre–Voisin varieties

In this section, we recall some basic results on Debarre–Voisin varieties and on the various degeneracy loci that we will study. We focus on properties that hold for general members of the moduli space instead of those belonging to special divisors, which will be the focus of later sections.

2.1. Degeneracy loci for a trivector.

Let $\sigma \in \bigwedge^3 V_{10}^\vee$ be a trivector. We introduce the various degeneracy loci that we will study.

By the Borel–Weil theorem, the space $\bigwedge^3 V_{10}^\vee$ can be identified with the space of global sections of certain vector bundles on the Grassmannians $\mathrm{Gr}(k, V_{10})$ for $k \in \{1, \ldots, 9\}$. More precisely, we have the following:

- When $k = 1$, we have $\bigwedge^3 V_{10}^\vee \simeq \Gamma\big(\mathbf{P}(V_{10}), (\bigwedge^2 \mathcal{Q}^\vee)(1)\big)$, and the evaluation of σ at a point $[V_1] \in \mathbf{P}(V_{10})$ is given by $\sigma(V_1, -, -) \in V_1^\vee \otimes \bigwedge^2 (V_{10}/V_1)^\vee$;

- When $k = 2$, we have $\bigwedge^3 V_{10}^\vee \simeq \Gamma\big(\mathrm{Gr}(2, V_{10}), \mathcal{Q}^\vee(1)\big)$, and the evaluation of σ at a point $[V_2] \in \mathrm{Gr}(2, V_{10})$ is given by $\sigma(V_2, V_2, -) \in (\bigwedge^2 V_2^\vee) \otimes (V_{10}/V_2)^\vee$;
- For $3 \le k \le 9$, we have $\bigwedge^3 V_{10}^\vee \simeq \Gamma\big(\mathrm{Gr}(k, V_{10}), \bigwedge^3 \mathcal{U}^\vee\big)$, and the evaluation of σ at a point $[V_k] \in \mathrm{Gr}(k, V_{10})$ is given by $\sigma(V_k, V_k, V_k) \in \bigwedge^3 V_k^\vee$.

We introduce the following degeneracy loci, all of which have already been studied in the literature:

- $X_1^\sigma \coloneqq \{[V_1] \in \mathbf{P}(V_{10}) \mid \operatorname{rank} \sigma(V_1, -, -) \le 6\}$. The skew-symmetric form $\sigma(V_1, -, -)$ always has rank less than or equal to 8, and X_1^σ is the locus where σ degenerates further. Its expected dimension is 6 and it is known as the *Peskine variety*.
- $X_2^\sigma \coloneqq \{[V_2] \in \mathrm{Gr}(2, V_{10}) \mid \sigma(V_2, V_2, -) = 0\}$. This variety is of expected dimension 8 and was thoroughly studied in [BFM].
- $X_3^\sigma \coloneqq \{[V_3] \in \mathrm{Gr}(3, V_{10}) \mid \sigma|_{V_3} = 0\}$. This is a Plücker hyperplane section of $\mathrm{Gr}(3, V_{10})$ and is irreducible of dimension 20 whenever $\sigma \neq 0$.
- $X_6^\sigma \coloneqq \{[V_6] \in \mathrm{Gr}(6, V_{10}) \mid \sigma|_{V_6} = 0\}$. This is known as the *Debarre–Voisin variety* and its expected dimension is 4.

The variety X_1^σ can also be realized as a subvariety of $\mathrm{Flag}(1, 4, V_{10})$ by mapping each point $[V_1]$ to the flag $[V_1 \subset K_4]$, where K_4 is the 4-dimensional kernel of the skew-symmetric form $\sigma(V_1, -, -)$. For general σ, we get an isomorphism onto the image. This alternative construction will be useful in several places.

We remark that for $k > 6$, the vanishing locus in $\mathrm{Gr}(k, V_{10})$ has negative expected dimension and is therefore empty for general σ. Nevertheless, for $k = 7$, we introduce the orbital degeneracy locus

$$X_7^\sigma \coloneqq \{[V_7] \in \mathrm{Gr}(7, V_{10}) \mid \operatorname{rank}(\sigma|_{V_7}) \le 5\}.$$

In other words, we consider those $[V_7]$ such that the restriction of σ on V_7 has rank ≤ 5, or equivalently, there exists $V_2 \subset V_7$ such that $\sigma(V_2, V_7, V_7) = 0$. We get a subvariety of expected dimension 6 in this way, which will play an important role in the picture. Finally, we suspect that the vanishing loci X_4^σ and X_5^σ (which can be defined similarly to X_6^σ) may also have interesting properties related to the geometry of Debarre–Voisin varieties.

2.2. **Moduli spaces.** It was proved in [DV] that for σ general, the variety X_6^σ is a smooth hyperkähler fourfold of deformation type K3$^{[2]}$. By the theory of hyperkähler manifolds, the second cohomology group $H^2(X_6^\sigma, \mathbf{Z})$ then carries a canonical integral quadratic form known as the *Beauville–Bogomolov–Fujiki form*, which we denote by $\mathfrak{q}$. It is non-degenerate of signature $(3, 19)$, with the property that for all $x_1, x_2, x_3, x_4 \in H^2(X_6^\sigma, \mathbf{Z})$, we have

$$\int_{X_6^\sigma} x_1 \cdot x_2 \cdot x_3 \cdot x_4 = \mathfrak{q}(x_1, x_2)\mathfrak{q}(x_3, x_4) + \mathfrak{q}(x_1, x_3)\mathfrak{q}(x_2, x_4) + \mathfrak{q}(x_1, x_4)\mathfrak{q}(x_2, x_3). \tag{1}$$

The lattice $(H^2(X_6^\sigma, \mathbf{Z}), \mathfrak{q})$ is isomorphic to the even lattice

$$\Lambda \coloneqq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle,$$

where U is the hyperbolic plane, $E_8(-1)$ is the E_8 lattice with negative definite form, and $\langle -2 \rangle$ is the lattice generated by one element with square -2. The discriminant of Λ is equal to 2. The Plücker line bundle coming from the ambient Grassmannian provides a polarization H on X_6^σ of square 22 and divisibility 2, that is $\mathfrak{q}(H, H) = 22$ and $\mathfrak{q}\big(H, H^2(X_6^\sigma, \mathbf{Z})\big) = 2\mathbf{Z}$. The primitive cohomology $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}}$ is defined as $H^\perp$, the sublattice orthogonal to H: it is a lattice of signature $(2, 19)$ and discriminant 11 and it carries a polarized integral Hodge

structure of type $(1, 20, 1)$. The square 22 and the divisibility 2 together determine a unique $\mathrm{O}(\Lambda)$-orbit, so we may fix one element h in this orbit. The *period domain* that parametrizes these Hodge structures is the normal quasi-projective variety

$$\mathcal{P} \coloneqq \{[x] \in \mathbf{P}(\Lambda_{\mathbf{C}}) \mid \mathfrak{q}(x,x) = \mathfrak{q}(x,h) = 0, \mathfrak{q}(x,\bar{x}) > 0\} / \mathrm{O}(\Lambda, h),$$

that is, the quotient of the domain of period points by the action of an orthogonal group (see the survey [Ma]). For each saturated sublattice $K \subset \Lambda$ of rank 2 and signature $(1,1)$ containing h, the orthogonal complement defines a codimension 2 subspace $\mathbf{P}(K^{\perp} \otimes \mathbf{C}) \subset \mathbf{P}(\Lambda_{\mathbf{C}})$, whose image in $\mathcal{P}$ is an irreducible algebraic hypersurface $\mathcal{D}_K$ called a *Heegner divisor*. The discriminant of $K^{\perp}$ is a positive even integer referred to simply as the *discriminant* of the Heegner divisor. Following Hassett and [DM], we will label each Heegner divisor using its discriminant. In our case, [DM, Proposition 4.1] shows that each Heegner divisor $\mathcal{D}_{2e}$ with given discriminant $2e$ is irreducible if non-empty. Moreover, in the proof of the same proposition, it is shown that the discriminant of K is always two times that of $K^{\perp}$, a fact that we will be needing later.

As mentioned in the introduction, we consider the GIT quotient

$$\mathcal{M} \coloneqq \mathbf{P}(\textstyle\bigwedge^3 V_{10}^{\vee}) /\!\!/ \mathrm{SL}(V_{10})$$

for the trivectors σ, and the irreducible coarse moduli space $\mathcal{M}_{22}^{(2)}$ of polarized hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type of degree 22 and divisibility 2 (see [D]). We have maps

$$\mathcal{M} \overset{\mathfrak{m}}{\dashrightarrow} \mathcal{M}_{22}^{(2)} \overset{\mathfrak{p}}{\hookrightarrow} \mathcal{P},$$

where $\mathfrak{m}$ is the modular map, defined over the locus $\mathcal{M}^{\mathrm{smooth}}$ of trivectors σ such that X_6^{σ} is smooth of dimension 4. It is dominant and was recently proved to be also birational ([O'G]). The map $\mathfrak{p}$ is the period map described above, which is an open immersion by the global Torelli theorem (proved by Verbitsky, Huybrechts, and Markman). Its image is the complement in $\mathcal{P}$ of the single irreducible Heegner divisor $\mathcal{D}_{22}$ ([DM, Theorem 6.1]).

The inverse image of each Heegner divisor $\mathcal{D}_K = \mathcal{D}_{2e}$ via the period map $\mathfrak{p}$ inside $\mathcal{M}_{22}^{(2)}$ is a *Noether–Lefschetz divisor* and is denoted by $\mathcal{C}_{2e}$. A very general member X_6 of each Noether–Lefschetz divisor has Picard number 2, and the algebraic part $H^2(X_6, \mathbf{Z})_{\mathrm{alg}} \coloneqq H^{1,1}(X_6^{\sigma}, \mathbf{Z})$ is precisely the sublattice of rank 2 defined by $K \subset \Lambda$. The transcendental sublattice $H^2(X_6, \mathbf{Z})_{\mathrm{trans}}$ is defined as the orthogonal complement, whose discriminant coincides with the discriminant of the divisor. Note that such X_6 might not always come from a trivector σ: Noether–Lefschetz divisors whose general members do not arise from the Debarre–Voisin construction are called *Hassett–Looijenga–Shah divisors* (HLS for short) and they are the main focus of the paper [DHO'GV]. These divisors correspond to $\mathrm{SL}(V_{10})$-orbits of higher codimension in $\mathcal{M}$ that need to be blown up in order to resolve the indeterminacy of $\mathfrak{m}$.

On the other hand, we will see that the complement in $\mathcal{M}$ of the locus $\mathcal{M}^{\mathrm{smooth}}$ is a divisor $\mathcal{D}^{3,3,10}$. It is induced by the $\mathrm{SL}(V_{10})$-invariant discriminant hypersurface in $\mathbf{P}\big(\bigwedge^3 V_{10}^{\vee}\big)$. We may extend the period map $\mathfrak{p} \circ \mathfrak{m}$ to an open subset of this divisor to get a birational map

$$\widetilde{\mathfrak{p}} \colon \mathcal{M} \dashrightarrow \mathcal{P}.$$

It can be shown that the divisor $\mathcal{D}^{3,3,10}$ is mapped onto the Heegner divisor $\mathcal{D}_{22}$ via this rational map $\widetilde{\mathfrak{p}}$. Since the period domain $\mathcal{P}$ is normal, it is regular in codimension 1 so the local ring at the generic point of each Heegner divisor $\mathcal{D}_{2e}$ is a discrete valuation ring. Any finite dominant map onto $\mathcal{D}_{2e}$ will thus be birational. Hence the divisor $\mathcal{D}^{3,3,10}$ is mapped

birationally onto the Heegner divisor $\mathcal{D}_{22}$ via $\widetilde{\mathfrak{p}}$. Note that the inverse image $\mathcal{C}_{22}$ in $\mathcal{M}_{22}^{(2)}$ is empty in this case:

$$\begin{tikzcd} \mathcal{M}^{\mathrm{smooth}} \arrow[r, "\mathfrak{m}"] & \mathcal{M}_{22}^{(2)} \arrow[r, hook, "\mathfrak{p}"] & \mathcal{P} \\ \sqcup & & \\ \mathcal{D}^{3,3,10} \arrow[rr, dashed, "\widetilde{\mathfrak{p}}", "\sim"'] & & \mathcal{D}_{22}. \arrow[uu, hook] \end{tikzcd}$$

We will also study two other divisors in $\mathcal{M}$ coming from $\mathrm{SL}(V_{10})$-invariant hypersurfaces that are birationally mapped onto some Heegner divisors $\mathcal{D}_{2e}$ via the extended period map $\widetilde{\mathfrak{p}}$. In principle, for each Heegner divisor in $\mathcal{P}$ that is not HLS, we could try to describe it in terms of divisors in $\mathcal{M}$. In the case of the three divisors that we study, this is done by imposing various degeneracy conditions on σ. Such descriptions also allow us to characterize these Heegner divisors as the loci where the varieties X_k^σ become singular.

Finally, we mention the recent results of G. Oberdieck [O] regarding Noether–Lefschetz numbers of a generic pencil of Debarre–Voisin varieties, obtained using Gromov–Witten techniques and modular forms. The three Heegner divisors $\mathcal{D}_{22}, \mathcal{D}_{24}$, and $\mathcal{D}_{28}$ that we will study are precisely the first three non-HLS divisors with lowest discriminants, and the corresponding Noether–Lefschetz numbers indeed coincide with the degrees of the $\mathrm{SL}(V_{10})$-invariant hypersurfaces that we will compute (see Table 1 and Section A.1).

2.3. **Smoothness of X_3^σ and X_6^σ.** We first state a lemma regarding the smoothness of X_6^σ.

Lemma 2.1. *Let $[V_6]$ be a point in X_6^σ. The Debarre–Voisin variety X_6^σ is not smooth of dimension 4 at $[V_6]$ if and only if there exists $V_3 \subset V_6$ such that $\sigma(V_3, V_3, V_{10}) = 0$.*

Proof. The Zariski tangent space $T_{X_6^\sigma,[V_6]}$ of the Debarre–Voisin variety X_6^σ at $[V_6]$ is given as the kernel of the differential

$$d\sigma\colon T_{\mathrm{Gr}(6,V_{10}),[V_6]} = \mathrm{Hom}(V_6, V_{10}/V_6) \longrightarrow \textstyle\bigwedge^3 V_6^\vee,$$

which maps $f \in \mathrm{Hom}(V_6, V_{10}/V_6)$ to the 3-form

$$d\sigma(f)\colon (v_1, v_2, v_3) \longmapsto \sigma(f(v_1), v_2, v_3) + \sigma(v_1, f(v_2), v_3) + \sigma(v_1, v_2, f(v_3)).$$

Therefore X_6^σ is not smooth of dimension 4 if and only if the differential is not surjective, or equivalently, if there exists some non-zero $\omega \in (\bigwedge^3 V_6^\vee)^\vee = \bigwedge^3 V_6$ such that $\omega|_{\mathrm{Im}(d\sigma)} = 0$, that is, for any $f \in \mathrm{Hom}(V_6, V_{10}/V_6)$ we have $d\sigma(f)(\omega) = 0$.

Suppose that $V_3 \subset V_6$ is a subspace satisfying the vanishing condition $\sigma(V_3, V_3, V_{10}) = 0$. Then a non-zero $\omega \in \bigwedge^3 V_3$ satisfies the above property, so X_6^σ is not smooth of dimension 4 at $[V_6]$.

Conversely, the orbit closures for the $\mathrm{GL}(V_6)$-action on $\bigwedge^3 V_6$ have long been classified: there are five of them, including $\{0\}$. So we study the four non-zero orbits case by case.

- If ω is completely decomposable, that is when $\omega = e_1 \wedge e_2 \wedge e_3$, consider a map f with $f(e_1) = f(e_2) = 0$: the property of ω shows that $\sigma(e_1, e_2, f(e_3)) = 0$, so by varying f we get $\sigma(e_1, e_2, V_{10}) = 0$. Similarly we have $\sigma(e_1, e_3, V_{10}) = \sigma(e_2, e_3, V_{10}) = 0$. So the subspace $V_3 = \langle e_1, e_2, e_3\rangle$ satisfies the vanishing condition $\sigma(V_3, V_3, V_{10}) = 0$.
- If ω is of rank 5, it can be written as $e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_4 \wedge e_5$. Let $V_1 = \langle e_1\rangle$ and $V_4 = \langle e_2, e_3, e_4, e_5\rangle$. We get $\sigma(V_1, V_1 + V_4, V_{10}) = 0$. Consider the map

$$\varphi_\sigma\colon \textstyle\bigwedge^2 V_4 \to (V_{10}/V_6)^\vee$$

induced by σ (note that $\sigma|_{V_6} = 0$). The kernel of φ_σ is a subspace of dimension at least 2. Note also that the subset in $\bigwedge^2 V_4$ of decomposable elements is the affine cone over the Grassmannian $\mathrm{Gr}(2, V_4)$, which is a quadric hypersurface. This shows that there is some decomposable element $u \wedge v$ in the kernel of φ_σ. The subspace $\langle e_1, u, v\rangle$ thus provides the V_3 we want. Moreover, without loss of generality, we may suppose that $u \wedge v = e_2 \wedge e_3$; then $\langle e_1, e_4, e_5\rangle$ gives another V_3 satisfying the vanishing condition.

- If ω is of type $e_1 \wedge e_2 \wedge e_4 + e_2 \wedge e_3 \wedge e_5 + e_1 \wedge e_3 \wedge e_6$, by considering a map f with $f(e_1) = f(e_2) = f(e_3) = 0$, we can see that $\langle e_1, e_2, e_3\rangle$ gives a V_3 such that $\sigma(V_3, V_3, V_{10}) = 0$.
- If ω is general, so of type $e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6$, both $\langle e_1, e_2, e_3\rangle$ and $\langle e_4, e_5, e_6\rangle$ give a V_3 such that $\sigma(V_3, V_3, V_{10}) = 0$.

Therefore, the Zariski tangent space is not of dimension 4 if and only if there exists a V_3 satisfying the vanishing condition. □

Inside $\mathbf{P}\big(\bigwedge^3 V_{10}^\vee\big)$, the set of trivectors σ admitting a V_3 such that $\sigma(V_3, V_3, V_{10}) = 0$ is the projective dual variety $\mathrm{Gr}(3, V_{10})^* = V(f)$, which is a hypersurface defined by an $\mathrm{SL}(V_{10})$-invariant polynomial f of degree 640 usually referred to as the *discriminant*. General elements in the hypersurface $V(f)$ are semi-stable with respect to the $\mathrm{SL}(V_{10})$-action, since otherwise the ring of invariants would be generated by one single polynomial f. Thus the unstable locus for the $\mathrm{SL}(V_{10})$-action is strictly contained in the hypersurface $V(f)$. In particular, there is a Zariski-dense open subset of the hypersurface $V(f)$ defining the irreducible subvariety

$$\mathcal{D}^{3,3,10} \coloneqq \{[\sigma] \in \mathcal{M} \mid \exists\, V_3 \quad \sigma(V_3, V_3, V_{10}) = 0\} \tag{2}$$

in the GIT quotient $\mathcal{M}$. By a parameter count, this is a divisor in $\mathcal{M}$. Any $[\sigma]$ in $\mathcal{M} \smallsetminus \mathcal{D}^{3,3,10}$ defines a smooth 4-dimensional hyperkähler X_6^σ, while general points in the divisor $\mathcal{D}^{3,3,10}$ admit a unique V_3 satisfying the degeneracy condition, which provides an ordinary double point on X_3^σ. In the latter case, due to Lemma 2.1 the singular locus of X_6^σ consists exactly of those V_6 containing V_3, so set-theoretically we get

$$\mathrm{Sing}(X_6^\sigma) = S_{22} \coloneqq \{[V_6] \in X_6^\sigma \mid V_6 \supset V_3\},$$

which generally is a degree-22 K3 surface living in the Grassmannian $\mathrm{Gr}(3, V_{10}/V_3)$. Hence the divisor $\mathcal{D}^{3,3,10}$ is exactly the complement of the locus $\mathcal{M}^{\mathrm{smooth}}$ and is mapped by the extended period map $\widetilde{\mathfrak{p}}$ onto the Heegner divisor $\mathcal{D}_{22} \subset \mathcal{P}$, originally studied in [DV] where X_6^σ was shown to be birational to the Hilbert square $S_{22}^{[2]}$. The algebraic sublattice $H^2(S_{22}^{[2]}, \mathbf{Z})$ of a general member has intersection matrix

$$\begin{pmatrix} 22 & 0 \\ 0 & -2 \end{pmatrix}$$

which is of discriminant 44, twice that of the Heegner divisor. Notice that in [DV], it was only proved that the K3 surface S_{22} is contained in the singular locus, instead of an equality. We have the following lemma concerning the type of singularity along S_{22}.

Lemma 2.2. *Let $[\sigma]$ be a general element in the divisor $\mathcal{D}^{3,3,10}$, so that there is a unique V_3 satisfying the vanishing condition $\sigma(V_3, V_3, V_{10}) = 0$. For all $[V_6] \in \mathrm{Sing}(X_6^\sigma)$, the Zariski tangent space $T_{X_6^\sigma,[V_6]}$ is of dimension 5.*

Proof. We retain the notation from the proof of Lemma 2.1: the Zariski tangent space is of dimension 5 if and only if there exists a unique (up to scalar) $\omega \in \bigwedge^3 V_6$ such that $\omega|_{\mathrm{Im}(d\sigma)} = 0$. Because of the uniqueness of the subspace V_3, we may conclude from the analysis of the four cases that the only possible ω are either the completely decomposable ones $e_1 \wedge e_2 \wedge e_3$, or the ones of the form $e_1 \wedge e_2 \wedge e_4 + e_2 \wedge e_3 \wedge e_5 + e_1 \wedge e_3 \wedge e_6$. In the latter case, the trivector σ not only need to satisfy the vanishing condition $\sigma(V_3, V_3, V_{10}) = 0$, but also the conditions

$$\begin{aligned}\sigma(V_{10}, e_2, e_4) + \sigma(V_{10}, e_3, e_6) &= 0\\ \sigma(e_1, V_{10}, e_4) + \sigma(V_{10}, e_3, e_5) &= 0\\ \sigma(e_2, V_{10}, e_5) + \sigma(e_1, V_{10}, e_6) &= 0.\end{aligned}$$

A parameter count shows that the trivectors satisfying these vanishing conditions form a subvariety of dimension 18 in the GIT quotient $\mathcal{M}$. Therefore, for $[\sigma]$ general in the divisor $\mathcal{D}^{3,3,10}$, the only $\omega \in \bigwedge^3 V_6$ satisfying the property $\omega|_{\mathrm{Im}(d\sigma)} = 0$ are the completely decomposable ones. So the Zariski tangent space is indeed of dimension 5. □

Remark 2.3. Consequently, the singularities along S_{22} are hypersurface singularities. It is very likely that we can obtain a smooth hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type by blowing up the singular locus, with the exceptional divisor being a $\mathbf{P}^1$-bundle over the K3 surface. But we were not able to prove this.

We summarize our results on the singular loci of X_3^σ and X_6^σ in the following proposition. Recall that $\widetilde{\mathfrak{p}}\colon \mathcal{M} \dashrightarrow \mathcal{P}$ is the birational map extending the composition $\mathfrak{p} \circ \mathfrak{m}$ of the modular map $\mathfrak{m}$ and the period map $\mathfrak{p}$ to its domain of definition.

Proposition 2.4. *The divisor $\mathcal{D}^{3,3,10}$ is the locus in $\mathcal{M}$ of trivectors $[\sigma]$ for which X_2^σ, X_3^σ, and X_6^σ become singular. Moreover, for a general element $[\sigma] \in \mathcal{D}^{3,3,10}$ such that $\sigma(V_3, V_3, V_{10}) = 0$, we have*

$$\begin{aligned}\mathrm{Sing}(X_2^\sigma) &= \{[V_2] \in \mathrm{Gr}(2, V_{10}) \mid V_2 \subset V_3\} \simeq \mathbf{P}(V_3^\vee),\\ \mathrm{Sing}(X_3^\sigma) &= \{[V_3]\},\\ \mathrm{Sing}(X_6^\sigma) &= \{[V_6] \in X_6^\sigma \mid V_6 \supset V_3\} = S_{22},\end{aligned}$$

where S_{22} is a degree-22 K3 surface. Finally, the divisor $\mathcal{D}^{3,3,10}$ is mapped birationally onto the Heegner divisor $\mathcal{D}_{22}$ via the birational map $\widetilde{\mathfrak{p}}\colon \mathcal{M} \dashrightarrow \mathcal{P}$.

Proof. The statement on the singular locus of X_3^σ is obvious. For X_6^σ, the statement follows from Lemma 2.1. For X_2^σ, we use [BFM, Lemma 11] to conclude that X_2^σ is singular exactly when $\sigma \in \mathcal{D}^{3,3,10}$, and the singular locus consists of points $[V_2] \in X_2^\sigma$ such that $V_2 \subset V_3$.

The birational map $\widetilde{\mathfrak{p}}$, when restricted to the divisor $\mathcal{D}^{3,3,10}$, gives a finite rational map that dominates the Heegner divisor $\mathcal{D}_{22}$. We can therefore conclude using the normality of the period domain $\mathcal{P}$. □

Remark 2.5. This result implies that $\mathcal{D}_{22}$ is unirational (which was already known from [DV]). Indeed, $\mathcal{D}^{3,3,10}$ can be seen as a quotient of the vector bundle $(\mathcal{U}_3 \wedge \mathcal{U}_3 \wedge V_{10})^\perp \subset \bigwedge^3 V_{10}^\vee \otimes \mathcal{O}_G$ over the Grassmannian $G = \mathrm{Gr}(3, V_{10})$ by the natural action of the group $\mathrm{SL}(V_{10})$.

2.4. **Hodge structures of X_3^σ.** In this section, we suppose that σ is such that X_3^σ and X_6^σ are both smooth of respective expected dimensions 20 and 4, which holds for $[\sigma]$ not in the divisor $\mathcal{D}^{3,3,10}$. We study the Hodge structures of X_3^σ. Note that the cohomology

ring $H^*(X_3^\sigma, \mathbf{Z})$ is torsion-free, thanks to the Lefschetz hyperplane theorem and the universal coefficient theorem.

We introduce one interesting Hodge structure on X_3^σ. Denote by $j\colon X_3^\sigma \to \mathrm{Gr}(3, V_{10})$ the canonical embedding. For a given coefficient ring R, the *vanishing cohomology*, studied in the original work [DV], is defined as

$$H^{20}(X_3^\sigma, R)_{\mathrm{van}} \coloneqq \ker\big(j_*\colon H^{20}(X_3^\sigma, R) \longrightarrow H^{22}(\mathrm{Gr}(3, V_{10}), R)\big).$$

When the coefficient ring is $\mathbf{Q}$, the vanishing cohomology can also be defined as the orthogonal complement of $j^*H^{20}(\mathrm{Gr}(3, V_{10}), \mathbf{Q})$ with respect to the cup-product on $H^{20}(X_3^\sigma, \mathbf{Q})$, hence there is an orthogonal decomposition

$$H^{20}(X_3^\sigma, \mathbf{Q}) = H^{20}(X_3^\sigma, \mathbf{Q})_{\mathrm{van}} \overset{\perp}{\oplus} j^*H^{20}(\mathrm{Gr}(3, V_{10}), \mathbf{Q}). \tag{3}$$

This decomposition does not work for $\mathbf{Z}$-coefficients, as the sum of the two sublattices is not saturated. In fact, $H^{20}(X_3^\sigma, \mathbf{Z})$ is a unimodular lattice and the lattice $j^*H^{20}(\mathrm{Gr}(3, V_{10}), \mathbf{Z})$ is generated by ten Schubert classes

$$j^*\sigma_{730}, j^*\sigma_{721}, j^*\sigma_{640}, j^*\sigma_{631}, j^*\sigma_{622}, j^*\sigma_{550}, j^*\sigma_{541}, j^*\sigma_{532}, j^*\sigma_{442}, j^*\sigma_{433},$$

with intersection product given by $j^*\alpha \cdot j^*\beta = \alpha \cdot \beta \cdot \sigma_{100}$. Thus we can explicitly write out the intersection matrix of $j^*H^{20}(\mathrm{Gr}(3, V_{10}), \mathbf{Z})$ as

$$\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}$$

which has determinant 11. Therefore, $j^*H^{20}(\mathrm{Gr}(3, V_{10}), \mathbf{Z})$ is a saturated sublattice of discriminant 11, and so is its orthogonal $H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}}$. The whole lattice $H^{20}(X_3^\sigma, \mathbf{Z})$ is not even (for example $(j^*\sigma_{730})^2 = 1$), while the vanishing cohomology is, as it is generated by the vanishing cycles, whose self-intersection is always 2 (see [V3, Chapter 2.3.3, Lemma 2.26 and Exercise 2]).

The Hodge structure on the vanishing cohomology $H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}}$ is of K3-type: it has Hodge numbers $h^{9,11} = h^{11,9} = 1$, $h^{10,10} = 20$, and the other Hodge numbers are all zero. For very general σ (those outside the union of all Noether–Lefschetz divisors), there are no Hodge classes of type $(10, 10)$, so this Hodge structure is simple (see for instance [V3, Theorem 3.27]).

To relate the varieties X_3^σ and X_6^σ, we define a diagram

$$\begin{tikzcd}
 & I_{3,6}^\sigma \coloneqq \{[V_3 \subset V_6] \in \mathrm{Flag}(3, 6, V_{10}) \mid \sigma|_{V_6} = 0\} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X_3^\sigma & & X_6^\sigma,
\end{tikzcd} \tag{4}$$

where q is a fibration with fibers isomorphic to $\mathrm{Gr}(3, 6)$. It is clear that the incidence variety $I_{3,6}^\sigma$ is smooth of expected dimension 13 whenever X_6^σ is smooth of dimension 4. Note

that the projection p is not surjective since X_3^σ has dimension 20. This correspondence induces a morphism

$$q_*p^* \colon H^{20}(X_3^\sigma, R) \longrightarrow H^2(X_6^\sigma, R).$$

When $R = \mathbf{Q}$, it was proven in [DV] that q_*p^* gives an isomorphism between the two $\mathbf{Q}$-Hodge structures $H^{20}(X_3^\sigma, \mathbf{Q})_{\mathrm{van}}$ and $H^2(X_6^\sigma, \mathbf{Q})_{\mathrm{prim}}$. We briefly recall the idea of the proof: the authors first proved that the morphism is not identically 0. Then, since the Hodge structure on $H^2(X_6^\sigma, \mathbf{Q})_{\mathrm{prim}}$ is simple for σ very general, the map q_*p^* is an isomorphism for such σ. Finally, since the topology does not change when we deform σ, the isomorphism holds whenever X_3^σ and X_6^σ are smooth.

We will show that q_*p^* also gives a Hodge isometry with $\mathbf{Z}$-coefficients. This is analogous to the result of Beauville–Donagi for cubic fourfolds ([BD]).

Theorem 2.6. *When σ is such that X_3^σ and X_6^σ are both smooth (that is, when $[\sigma] \notin \mathcal{D}^{3,3,10}$), the morphism q_*p^* gives an isomorphism of polarized integral Hodge structures*

$$q_*p^* \colon H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}} \xrightarrow{\sim} H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}}(-1), \tag{5}$$

where the Tate twist (-1) means that $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}}$ is endowed with the quadratic form $-\mathfrak{q}$.

Let us first state a lemma over $\mathbf{Q}$-coefficients.

Lemma 2.7. *The isomorphism of rational Hodge structures q_*p^* is a constant multiple of an isometry.*

Proof. The intersection form on $H^{20}(X_3^\sigma, \mathbf{Q})_{\mathrm{van}}$ transports to a second quadratic form $\mathfrak{q}'$ on $H^2(X_6^\sigma, \mathbf{Q})_{\mathrm{prim}}$, so we need to show that there exists λ such that $\mathfrak{q}' = \lambda\mathfrak{q}$. Let $\omega \in H^{2,0}(X_6^\sigma)$ be the class of a holomorphic symplectic form on X_6^σ. We have $\mathfrak{q}(\omega, \omega) = \mathfrak{q}'(\omega, \omega) = 0$, $\mathfrak{q}(\omega, \bar{\omega}) > 0$, and ω is orthogonal to $H^{1,1}(X_6^\sigma)$ for both $\mathfrak{q}$ and $\mathfrak{q}'$. Consider the following number

$$\lambda = \frac{\mathfrak{q}'(\omega, \bar{\omega})}{\mathfrak{q}(\omega, \bar{\omega})}.$$

We see that ω is also orthogonal to $\bar{\omega}$ for the form $\mathfrak{q}' - \lambda\mathfrak{q}$, therefore ω lies in the kernel $\ker(\mathfrak{q}' - \lambda\mathfrak{q})$. This form is thus degenerate so we have $\det(\mathfrak{q}' - \lambda\mathfrak{q}) = 0$. Since both $\mathfrak{q}$ and $\mathfrak{q}'$ have coefficients in $\mathbf{Q}$, the value λ is an algebraic number and therefore does not change when we deform σ in the moduli space. This means that the kernel $\ker(\mathfrak{q}' - \lambda\mathfrak{q})$ must contain all the period points, which, by the surjectivity of the local period map, span the entire $H^2(X_6^\sigma, \mathbf{Q})_{\mathrm{prim}}$. In other words, $\mathfrak{q}'$ is identically equal to $\lambda\mathfrak{q}$ on $H^2(X_6^\sigma, \mathbf{Q})_{\mathrm{prim}}$. □

Therefore we know that q_*p^* is a constant multiple of an isometry. If we can show that this constant is -1, then since the discriminants of the two lattices are the same, this isometry will also be onto, and we may conclude.

To determine this constant, we will use the argument of continuity: the constant is the same over the moduli space, so it suffices to compute its value over the Heegner divisor $\mathcal{D}_{28}$, where we have some explicit Hodge classes to work with. We postpone the proof of Theorem 2.6 to Section 3, where we study in detail the divisor $\mathcal{D}_{28}$. We will also prove the integral Hodge conjecture on $H^{20}(X_3^\sigma, \mathbf{Z})$ as a corollary (Corollary 3.10).

2.5. **Smoothness of the Peskine variety X_1^σ.** In this section, we give a criterion for the smoothness of the Peskine variety X_1^σ. Recall that this is the locus in $\mathbf{P}(V_{10})$ where the rank of $\sigma(V_1,-,-)$ drops to 6 or less. The orbits of skew-symmetric forms are entirely determined by their ranks. Denote by $\mathcal{O}_r^s$ the orbit of skew-symmetric 9×9-matrices of rank r, where r is even. Then X_1^σ is defined as the orbital degeneracy locus $D_{\mathcal{O}_6^s}(\sigma)$. By the theory of orbital degeneracy loci [BFMT1], when σ is general, the singular locus of X_1^σ is

$$D_{\mathrm{Sing}(\mathcal{O}_6^s)}(\sigma) = D_{\mathcal{O}_4^s}(\sigma) = \{[V_1]\in \mathbf{P}(V_{10}) \mid \operatorname{rank}\sigma(V_1,-,-)\le 4\},$$

which is empty for dimensional reasons since

$$\operatorname{codim}_{\mathbf{P}(V_{10})}(D_{\mathcal{O}_4^s}(\sigma)) = \operatorname{codim}_{\bigwedge^2\mathbf{C}^9}(\mathcal{O}_4^s) = 10 > \dim \mathbf{P}(V_{10}).$$

So X_1^σ is smooth for a general σ. Now we give a lemma that describes the situation for all σ.

Lemma 2.8. *Let $\sigma\in\bigwedge^3 V_{10}^\vee$ be such that X_1^σ is of dimension 6. A point $[V_1]\in X_1^\sigma$ is singular if and only if either $\sigma(V_1,-,-)$ is of rank ≤ 4 or there exists V_3 containing V_1 such that $\sigma(V_3,V_3,V_{10})=0$.*

Proof. Since the orbit $\mathcal{O}_6^s$ is singular along $\mathcal{O}_4^s$, the theory of orbital degeneracy loci tells us that $D_{\mathcal{O}_4^s}(\sigma)\subset \operatorname{Sing} D_{\mathcal{O}_6^s}(\sigma) = \operatorname{Sing} X_1^\sigma$, which means that X_1^σ is automatically singular where $\sigma(V_1,-,-)$ is of rank ≤ 4. Therefore, we only need to identify the singular locus of X_1^σ over the open subset $X_1^\sigma\smallsetminus D_{\mathcal{O}^4}(\sigma)$, that is, where $\sigma(V_1,-,-)$ is of rank 6.

We will study the singularities by using a desingularization $Z(\sigma)\subset \mathrm{Flag}(1,4,V_{10})$. The trivector $\sigma\in\bigwedge^3 V_{10}^\vee$ can be seen as a section of the vector bundle $(\mathcal{U}_1\wedge\mathcal{U}_4\wedge V_{10})^\vee$ over the flag variety $\mathrm{Flag}(1,4,V_{10})$ (a quotient of the trivial bundle $\bigwedge^3 V_{10}^\vee$). The zero-locus $Z(\sigma)$ of σ gives a desingularization of X_1^σ: since $Z(\sigma)$ parametrizes pairs $[V_1\subset V_4]$ such that $V_4\subset \ker\sigma(V_1,-,-)$, the projection $Z(\sigma)\to X_1^\sigma$ that maps $[V_1\subset V_4]$ to $[V_1]$ is an isomorphism over the open subset $X_1^\sigma\smallsetminus D_{\mathcal{O}_4^s}(\sigma)$ where σ is of rank 6. Therefore, it suffices to determine the singular locus of $Z(\sigma)$.

The variety $Z(\sigma)$ being not smooth of dimension 6 at $[V_1\subset V_4]$ means that the differential

$$d\sigma\colon T_{\mathrm{Flag}(1,4,V_{10}),[V_1\subset V_4]}\simeq \mathrm{Hom}(V_1,V_{10}/V_1)\oplus \mathrm{Hom}(V_4/V_1,V_{10}/V_4)\longrightarrow (V_1\wedge V_4\wedge V_{10})^\vee,$$

which maps $f\in \mathrm{Hom}(V_1,V_{10}/V_1)\oplus \mathrm{Hom}(V_4/V_1,V_{10}/V_4)$ to the 3-form

$$d\sigma(f)\colon (v_1,v_2,v_3)\longmapsto \sigma(f(v_1),v_2,v_3)+\sigma(v_1,f(v_2),v_3)+\sigma(v_1,v_2,f(v_3)),$$

is not surjective. Equivalently, there exists a non-zero $\omega\in((V_1\wedge V_4\wedge V_{10})^\vee)^\vee = V_1\wedge V_4\wedge V_{10}$ such that $\omega|_{\mathrm{Im}(d\sigma)}=0$, that is, for any $f\in \mathrm{Hom}(V_1,V_{10}/V_1)\oplus \mathrm{Hom}(V_4/V_1,V_{10}/V_4)$, we have $d\sigma(f)(\omega)=0$.

Modulo a change of coordinates, one can always suppose that $V_1=\langle e_1\rangle$, $V_4=\langle e_1,\dots,e_4\rangle$, and $V_{10}=\langle e_1,\dots,e_{10}\rangle$, and that

$$\omega = e_1\wedge\big(a(e_2\wedge e_3)+b(e_2\wedge e_5)+c(e_3\wedge e_6)+d(e_4\wedge e_7)\big)$$

for certain coefficients $a,b,c,d\in\mathbf{C}$. The proof is divided into three cases:

- If $d\neq 0$, consider a morphism f sending e_1,e_2,e_3 to 0. Then $d\sigma(f)(\omega)=0$ shows that $\sigma(e_1,e_7,f(e_4))=0$. By varying f, one gets $\sigma(V_1,V_4+\mathbf{C}e_7,V_{10})=0$, which implies that $\sigma(V_1,-,-)$ has rank at most 4.
- If $d=0$ and $b\neq 0$, consider a morphism f sending e_1,e_3,e_4 to 0. Then $d\sigma(f)(\omega)=0$ shows that $\sigma(e_1,e_5,f(e_2))=0$. By varying f, one gets $\sigma(V_1,V_4+\mathbf{C}e_5,V_{10})=0$, again implying that $\sigma(V_1,-,-)$ has rank at most 4. Similarly one can treat the case when $d=0$ and $c\neq 0$.

- If $b = c = d = 0$, consider a morphism f sending e_2, e_3, e_4 to 0. Then $d\sigma(f)(\omega) = 0$ shows that $\sigma(f(e_1), e_2, e_3) = 0$. By varying f and setting $V_3 = \langle e_1, e_2, e_3 \rangle$, one gets $\sigma(V_3, V_3, V_{10}) = 0$.

Therefore, for a singular point $[V_1 \subset V_4]$ in $Z(\sigma)$, either $\sigma(V_1, -, -)$ is of rank ≤ 4, or there exists $V_3 \supset V_1$ with $\sigma(V_3, V_3, V_{10}) = 0$. We may then conclude for the singular locus of X_1^σ. $\square$

As in the previous case of $\mathcal{D}^{3,3,10}$, we may define the subvariety

$$\mathcal{D}^{1,6,10} := \{[\sigma] \in \mathcal{M} \mid \exists [V_1 \subset V_6] \quad \sigma(V_1, V_6, V_{10}) = 0\} \tag{6}$$

in the GIT quotient $\mathcal{M}$. Again by a parameter count, this subvariety is an irreducible divisor of $\mathcal{M}$, and for σ general in this divisor, there exists a unique flag $[V_1 \subset V_6]$ satisfying the degeneracy condition. We can compute that the degree of the $\mathrm{SL}(V_{10})$-invariant hypersurface is equal to 990 (see Section A.1), therefore this divisor is indeed different from $\mathcal{D}^{3,3,10}$. These two divisors are related to the non-smoothness of X_1^σ.

Proposition 2.9. *The locus of trivectors $[\sigma]$ for which $X_1^\sigma \subset \mathbf{P}(V_{10})$ becomes singular is the union of two divisors $\mathcal{D}^{1,6,10} \cup \mathcal{D}^{3,3,10}$ in $\mathcal{M}$.*

- *If $[\sigma] \in \mathcal{D}^{1,6,10}$ is general such that $\sigma(V_1, V_6, V_{10}) = 0$, then $\mathrm{Sing}(X_1^\sigma) = \{[V_1]\}$.*
- *If $[\sigma] \in \mathcal{D}^{3,3,10}$ is general such that $\sigma(V_3, V_3, V_{10}) = 0$, then $\mathrm{Sing}(X_1^\sigma) = \mathbf{P}(V_3)$.*

We will see that the divisor $\mathcal{D}^{1,6,10} \subset \mathcal{M}$ maps birationally onto the Heegner divisor $\mathcal{D}_{24} \subset \mathcal{P}$, which we will study in detail in Section 4.

2.6. **Hodge structures of X_1^σ.** In this section, we will suppose that the trivector σ does not lie in $\mathcal{D}^{3,3,10} \cup \mathcal{D}^{1,6,10}$, so all three varieties X_1^σ, X_3^σ, and X_6^σ are smooth of expected dimension. The Peskine variety $X_1^\sigma \subset \mathbf{P}(V_{10})$ has many interesting geometric aspects. Firstly, it is a degree-15 sixfold which is Fano of index 3 (this is obtained by studying a resolution of the structure sheaf, for example see [BFM, Section 4.3]). Denote by h the natural polarization on X_1^σ. We have the following result from [H2].

Proposition 2.10 (Han)**.** *Let σ be a general trivector. For $[V_6]$ general in the Debarre–Voisin variety X_6^σ, the intersection of $\mathbf{P}(V_6)$ and X_1^σ inside $\mathbf{P}(V_{10})$ is a* Palatini threefold*, that is, a smooth degree-7 threefold in $\mathbf{P}^5$ which is a scroll over a smooth cubic surface. Otherwise stated, there is a 7-dimensional incidence variety $I_{1,6}^\sigma$ called the* universal Palatini variety *and a diagram*

$$\begin{tikzcd} & I_{1,6}^\sigma := \{[V_1 \subset V_6] \in \mathrm{Flag}(1,6,V_{10}) \mid [V_1] \in X_1^\sigma, [V_6] \in X_6^\sigma\} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X_1^\sigma & & X_6^\sigma, \end{tikzcd} \tag{7}$$

where the general fiber of q is a Palatini threefold.

We would like to use this correspondence to relate the Hodge structures of X_1^σ and X_6^σ. However, unlike the case of $I_{3,6}^\sigma$, there are some subtleties: the Proposition 2.10 only holds for σ general, in which case the fiber of q is a Palatini threefold only for general $[V_6] \in X_6$; *a priori* the incidence variety $I_{1,6}^\sigma$ might not be smooth, or not have the expected dimension at all. It turns out that it is *never* smooth but always has the expected dimension whenever X_1^σ and X_6^σ are smooth, so we can still use the correspondence $q_* p^*$ in families. More precisely, the incidence variety $I_{1,6}^\sigma$ can also be seen as a subvariety of $X_1^\sigma \times X_6^\sigma$, so when it has

the expected dimension, it will give a well-defined cohomology class $[I^\sigma_{1,6}] \in H^6(X^\sigma_1 \times X^\sigma_6)$. Thus by abuse of notation, we will write q_*p^* for $\mathrm{pr}_{2*}\big([I^\sigma_{1,6}] \cdot \mathrm{pr}_1^*(-)\big)$ and similarly p_*q^* for $\mathrm{pr}_{1*}\big([I^\sigma_{1,6}] \cdot \mathrm{pr}_2^*(-)\big)$.

First we show that $I^\sigma_{1,6}$ is not smooth for a general σ. We introduce a resolution of $I^\sigma_{1,6}$ that can be realized as the zero-locus of some section on a flag variety, whose smoothness can then be deduced for general σ. Consider the diagram

$$\begin{tikzcd}
 & \widetilde{I}^\sigma_{1,6} := \left\{ \begin{array}{c} [V_1 \subset V_3 \subset V_6] \\ \in \mathrm{Flag}(1,3,6,V_{10}) \end{array} \middle| \begin{array}{c} \sigma(V_1,V_3,V_{10}) = 0, \\ \sigma|_{V_6} = 0 \end{array} \right\} \arrow[d, "s"] \arrow[dl] \arrow[dr] & \\
X^\sigma_1 & I^\sigma_{1,6} \arrow[l, "p"'] \arrow[r, "q"] & X^\sigma_6.
\end{tikzcd} \tag{8}$$

In other words, apart from the pair $[V_1 \subset V_6]$, we introduce the extra information of a subspace V_3 contained in both V_6 and the kernel of the form $\sigma(V_1, -, -)$ (which immediately ensures that $\operatorname{rank} \sigma(V_1, -, -) \leq 6$).

Proposition 2.11. *Let σ be a general trivector with both X^σ_1 and X^σ_6 smooth. The variety $\widetilde{I}^\sigma_{1,6}$ defined in* (8) *is smooth of expected dimension* 7 *and the projection $s\colon \widetilde{I}^\sigma_{1,6} \to I^\sigma_{1,6}$ obtained by forgetting the subspace V_3 is an isomorphism exactly on the complement of a* 5*-dimensional subvariety of $\widetilde{I}^\sigma_{1,6}$. In particular $I^\sigma_{1,6}$ is not smooth.*

Proof. The variety $\widetilde{I}^\sigma_{1,6}$ is defined inside $\mathrm{Flag}(1,3,6,V_{10})$ as the zero-locus of σ viewed as a section of the vector bundle $\bigwedge^3 \mathcal{U}_6^\vee \oplus \big(\mathcal{U}_1 \wedge (\mathcal{U}_3/\mathcal{U}_1) \wedge (V_{10}/\mathcal{U}_6)\big)^\vee$. Therefore it is smooth of expected dimension 7 for a general σ.

The locus where the projection $\widetilde{I}^\sigma_{1,6} \to I^\sigma_{1,6}$ is not an isomorphism is precisely above those $[V_1 \subset V_6]$ where the kernel K_4 of $\sigma(V_1, -, -)$ is contained in V_6, in which case the fiber is a projective plane $\mathbf{P}\big((K_4/V_1)^\vee\big)$ parametrizing V_3 with $V_1 \subset V_3 \subset K_4$. We may look at the locus of such $[V_1 \subset K_4 \subset V_6]$ inside the flag variety $\mathrm{Flag}(1,4,6,V_{10})$: this is again the zero-locus of σ viewed as a section of a certain homogeneous bundle, so for general σ we get the 3-dimensional smooth subvariety

$$Z := \{[V_1 \subset K_4 \subset V_6] \mid \sigma(V_1, K_4, V_{10}) = 0, [V_6] \in X^\sigma_6\} \subset \mathrm{Flag}(1,4,6,V_{10}).$$

Since the kernel K_4 is uniquely determined by V_1, the subvariety Z projects injectively onto its image in $I^\sigma_{1,6} \subset \mathrm{Flag}(1,6,V_{10})$. The preimage in $\widetilde{I}^\sigma_{1,6}$ therefore has dimension 5. This means that the projection $\widetilde{I}^\sigma_{1,6} \to I^\sigma_{1,6}$ is a small contraction for general σ, so $I^\sigma_{1,6}$ cannot be smooth. □

Remark 2.12. For general σ, the 3-dimensional subvariety Z dominates a divisor in X^σ_6. This divisor has class $10H$, which can be shown by computing the degree of the pullback of the polarization H. Note that this is a canonically defined effective divisor in X^σ_6, which could be useful in constructing compactifications of the moduli space.

Now we show that $I^\sigma_{1,6}$ always has the expected dimension 7. We first state some lemmas.

Lemma 2.13. *Let $\sigma \in \bigwedge^3 V_{10}^\vee$ be a trivector. If there is a subspace V_7 such that $\sigma|_{V_7}$ vanishes, then σ is unstable with respect to the $\mathrm{SL}(V_{10})$-action.*

Proof. This can easily be verified using the Hilbert–Mumford criterion. Choose a basis $(e_1, \dots, e_{10})$ of V_{10} such that V_7 is the subspace $\langle e_1, \dots, e_7\rangle$ and consider the 1-parameter subgroup $\lambda\colon \mathbf{C}^* \to \mathrm{SL}(V_{10})$ given by

$$t \longmapsto \mathrm{diag}(t^3, t^3, t^3, t^3, t^3, t^3, t^3, t^{-7}, t^{-7}, t^{-7}).$$

Then σ has only negative weights with respect to this 1-parameter subgroup and is therefore unstable. □

Lemma 2.14. *Let $[\sigma] \in \mathcal{M}$. If the fiber of q above $[V_6]$ is not of dimension 3, there is a flag $V_4 \subset V_6 \subset V_7$ such that $\sigma(V_4, V_7, V_7) = 0$.*

Proof. This is a restatement of the results on Palatini threefolds obtained by Fania–Mezzetti in [FM]. For each $[V_6]$, since σ vanishes on V_6, we get an induced linear map

$$\varphi_\sigma\colon V_{10}/V_6 \longrightarrow \textstyle\bigwedge^2 V_6^\vee.$$

This map is injective: if some V_7/V_6 is mapped to 0, the trivector σ would vanish on V_7 which is impossible by Lemma 2.13.

Therefore we have a 4-dimensional subspace of $\bigwedge^2 V_6^\vee$, or equivalently a 3-dimensional projective subspace $\Delta \subset \mathbf{P}\big(\bigwedge^2 V_6^\vee\big)$. Inside $\mathbf{P}\big(\bigwedge^2 V_6^\vee\big)$, there are two $\mathrm{SL}(V_6)$-invariant orbits given by the discriminant hypersurface $\mathrm{Gr}(2, V_6)^*$—which is the Pfaffian cubic—and the Grassmannian $\mathrm{Gr}(2, V_6^\vee) \simeq \mathrm{Gr}(4, V_6)$, representing skew-symmetric forms of ranks ≤ 4 and ≤ 2 respectively. The fiber of q above $[V_6]$ is precisely the union of degeneracy loci in $\mathbf{P}(V_6)$ for the family of skew-symmetric forms parametrized by Δ. The classification result from [FM, Theorem 4.3 and Theorem 4.9] tells us that this degeneracy locus is of expected dimension 3 except in the following two cases

- Δ is entirely contained in the discriminant hypersurface $\mathrm{Gr}(2, V_6)^*$;
- Δ is not contained in $\mathrm{Gr}(2, V_6)^*$ but its intersection with the Grassmannian $\mathrm{Gr}(2, V_6^\vee)$ contains a line or a conic.

Moreover by a result of Manivel–Mezzetti [MM, Corollary 11], in the first case the projective 3-space Δ will necessarily intersect the Grassmannian $\mathrm{Gr}(2, V_6^\vee)$. Therefore in both cases, the 3-space Δ intersects the Grassmannian $\mathrm{Gr}(2, V_6^\vee)$, which means that there is a $V_7 \supset V_6$ whose image is decomposable: we have $\varphi_\sigma(V_7/V_6) = f_1 \wedge f_2$ where $f_1, f_2 \in V_6^\vee$ are linear forms. The common kernel $V_4 \subset V_6$ of f_1 and f_2 therefore satisfies the desired property $\sigma(V_4, V_7, V_7) = 0$. □

Note that such a flag $V_4 \subset V_7$ gives a plane $\mathbf{P}\big((V_7/V_4)^\vee\big)$ contained in X_6^σ, necessarily Lagrangian.

Proposition 2.15. *Let $[\sigma] \in \mathcal{M}$ be a trivector such that X_1^σ and X_6^σ are both smooth (that is, $[\sigma] \notin \mathcal{D}^{3,3,10} \cup \mathcal{D}^{1,6,10}$). The variety $I_{1,6}^\sigma$ defined in (7) has only one irreducible component of expected dimension 7, and this component is reduced.*

Proof. For a trivector $\sigma \notin \mathcal{D}^{3,3,10} \cup \mathcal{D}^{1,6,10}$, since X_6^σ is hyperkähler of dimension 4, it contains only finitely many planes of the form $\mathbf{P}\big((V_7/V_4)^\vee\big)$, since any such plane is necessarily Lagrangian hence rigid. We saw that for any $[V_6]$ away from these planes, the fiber is of expected dimension 3 and is generically smooth, so the irreducible component of $I_{1,6}^\sigma$ that dominates X_6^σ is reduced of expected dimension 7. On the other hand, the preimage of each

Lagrangian plane $\mathbf{P}\big((V_7/V_4)^\vee\big)$ has dimension ≤ 6: otherwise for any $[V_6] \in \mathbf{P}\big((V_7/V_4)^\vee\big)$ and any $V_1 \subset V_6$ we have $[V_1] \in X_1^\sigma$, then $\mathbf{P}(V_7)$ would be entirely contained in X_1^σ, which is impossible because X_1^σ is assumed to be smooth of dimension 6. □

Therefore whenever X_1^σ and X_6^σ are both smooth, the variety $I_{1,6}^\sigma$ has one unique reduced component of expected dimension 7. It defines a class on the product $X_1^\sigma \times X_6^\sigma$ with correct codimension, and we can thus talk about the morphisms q_*p^* and p_*q^* between Hodge structures given by this correspondence.

The degeneracy condition $\sigma(V_4, V_7, V_7) = 0$ from Lemma 2.14 defines a third divisor in the GIT moduli space $\mathcal{M}$ of trivectors: consider the subvariety

$$\mathcal{D}^{4,7,7} := \{[\sigma] \in \mathcal{M} \mid \exists [V_4 \subset V_7] \quad \sigma(V_4, V_7, V_7) = 0\}. \tag{9}$$

By a parameter count, it is an irreducible divisor in $\mathcal{M}$, and for a general σ in this divisor, there exists a unique flag $[V_4 \subset V_7]$ satisfying the degeneracy condition. Again we can compute the degree of the $\mathrm{SL}(V_{10})$-invariant hypersurface which is equal to 5500 (see Section A.1). Hence this divisor is different from both $\mathcal{D}^{3,3,10}$ and $\mathcal{D}^{1,6,10}$. It will be studied in detail in the next section.

We now begin the study of the Hodge structures of X_1^σ. The Hodge numbers of X_1^σ were recently computed in [BFM, Section 4.2], where the integral cohomology $H^*(X_1^\sigma, \mathbf{Z})$ is also shown to be torsion-free. Since all the cohomologies in odd degree vanish, we list only the even degree ones:

$$\begin{array}{c|ccccccc}
h^0 & & & & 1 & & & \\
h^2 & & & 0 & 1 & 0 & & \\
h^4 & & 0 & 1 & 22 & 1 & 0 & \\
h^6 & 0 & 0 & 1 & 22 & 1 & 0 & 0 \\
h^8 & & 0 & 1 & 22 & 1 & 0 & \\
h^{10} & & & 0 & 1 & 0 & & \\
h^{12} & & & & 1 & & &
\end{array} \tag{10}$$

We see that there are three Hodge structures of K3-type on different levels. They are related by the Lefschetz operator (see Lemma 3.15) and there is a polarization given by the cup product on $H^6(X_1^\sigma, \mathbf{Z})$. In [BFM], the authors showed that the Hodge structure of $H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}}$ can be mapped into each of the three Hodge structures of X_1^σ by using certain geometric constructions called *jumps* between the Grassmannians $\mathbf{P}(V_{10})$, $\mathrm{Gr}(2, V_{10})$, and $\mathrm{Gr}(3, V_{10})$. Here we show that this can also be done by using the incidence variety $I_{1,6}^\sigma$.

As in the case of X_3^σ, we first determine the suitable Hodge structure to study: we define the vanishing cohomology $H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}$ to be the orthogonal of the sublattice generated by h^3 and the class π of a Palatini threefold in X_1^σ (see Proposition 2.10). To check that these two classes generate a rank-2 sublattice, one can compute their intersection matrix as follows:

- the self-intersection number $h^3 \cdot h^3$ is the degree of X_1^σ, which is 15;
- the intersection number $h^3 \cdot \pi$ is the degree of the Palatini threefold, which is 7;
- to compute the self-intersection number $\pi \cdot \pi$, we take two general points $[V_6]$ and $[V_6']$ from X_6^σ. Their intersection $V_6 \cap V_6'$ is a 2-dimensional subspace V_2, and the sum $V_6 + V_6'$ is the whole V_{10}. So one obtains $\sigma(V_2, V_2, V_{10}) = 0$. In particular, this shows that $[V_2]$ is in the degeneracy locus X_2^σ. It defines then a 4-secant line $\mathbf{P}(V_2)$ of

the variety X_1^σ (see [H1, Section 3.1]). As the class π of the Palatini threefolds can be represented by both $\mathbf{P}(V_6) \cap X_1$ and $\mathbf{P}(V_6') \cap X_1$, its self-intersection number is 4.

The intersection matrix for $\mathbf{Z}h^3 + \mathbf{Z}\pi$ is therefore

$$\begin{pmatrix} 15 & 7 \\ 7 & 4 \end{pmatrix}.$$

This is a saturated rank-2 lattice of discriminant 11. Its orthogonal complement, the vanishing cohomology $H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}$—a polarized integral Hodge structure of type $(1, 20, 1)$—therefore also has discriminant 11.

For cohomologies in degree $k = 4, 8$, we first use the Lefschetz operator L_h over $\mathbf{Q}$ to identify $H^k(X_1^\sigma, \mathbf{Q})_{\mathrm{van}}$, then define the corresponding intersection with the integral cohomology to be $H^k(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}$. A priori, the Lefschetz operators might not remain isomorphisms over integral coefficients. We will clarify this in Lemma 3.15.

The following is the analogue of Theorem 2.6.

Theorem 2.16. *When σ is such that X_1^σ and X_6^σ are both smooth (that is, when $[\sigma] \notin \mathcal{D}^{3,3,10} \cup \mathcal{D}^{1,6,10}$), the morphism*

$$q_* p^* L_h \colon H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}} \xrightarrow{\sim} H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}}(-1) \tag{11}$$

is an isomorphism of polarized integral Hodge structures. Here $q_* p^*$ is the correspondence defined by $I_{1,6}^\sigma$ in the diagram (7), L_h is the Lefschetz operator given by cup product with h, and the Tate twist (-1) means that $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}}$ is endowed with the quadratic form $-\mathfrak{q}$.

The proof is essentially the same as the proof of Theorem 2.16 and involves the study of $\mathcal{D}_{28}$, so again we postpone it to Section 3.

Remark 2.17. One can also derive Theorem 2.16 directly from Theorem 2.6 and [BFM, Theorem 19] and, with the same line of ideas, one can obtain the analogous statement concerning X_2^σ. However, we decided to include a direct proof of Theorem 2.16 to illustrate once more how one can specialize $[\sigma]$ to divisors to obtain more precise information on the cohomologies.

2.7. **Fano variety of lines of X_1^σ and the variety X_7^σ.** The geometry of the universal Palatini variety $I_{1,6}^\sigma$ defined in (7) can be further studied by considering the Fano variety $F(X_1^\sigma)$ of lines of X_1^σ and the degeneracy locus X_7^σ. In this section, we will assume that σ is general in $\bigwedge^3 V_{10}^\vee$.

We first recall the definition of X_7^σ. Consider the space $\bigwedge^3 V_7^\vee$ of trivectors, a complex vector space of dimension 35. The orbit closures for the action of $\mathrm{GL}(V_7)$ have long been classified (see [Gu, Section 35.3]). We list the smallest ones:

$$\mathcal{O}_{20} \coloneqq \{\sigma \in \textstyle\bigwedge^3 V_7^\vee \mid \operatorname{rank} \sigma \le 5\} \quad \text{and} \quad \mathcal{O}_{13} \coloneqq \{\sigma \in \textstyle\bigwedge^3 V_7^\vee \mid \operatorname{rank} \sigma \le 3\},$$

which are of respective dimensions 20 and 13 and thus of codimensions 15 and 22 in $\bigwedge^3 V_7^\vee$. The two conditions can also be expressed equivalently as the existence of a subspace $V_k \subset V_7$ such that $\sigma(V_k, V_7, V_7) = 0$, with $k = 2$ and 4 in the two cases respectively. The variety X_7^σ can be defined as the orbital degeneracy locus

$$\begin{aligned} X_7^\sigma &\coloneqq \{[V_7] \in \mathrm{Gr}(7, V_{10}) \mid \operatorname{rank}(\sigma|_{V_7}) \le 5\} \\ &= \{[V_7] \in \mathrm{Gr}(7, V_{10}) \mid \sigma|_{V_7} \in \mathcal{O}_{20} \subset \textstyle\bigwedge^3 V_7^\vee\} = D_{\mathcal{O}_{20}}(\sigma). \end{aligned}$$

By the theory of orbital degeneracy loci [BFMT1], since σ is general, X_7^σ has codimension 15 and its singular locus is equal to

$$D_{\mathrm{Sing}(\mathcal{O}_{20})}(\sigma) = D_{\mathcal{O}_{13}}(\sigma) = \{[V_7] \in \mathrm{Gr}(7, V_{10}) \mid \sigma|_{V_7} \in \mathcal{O}_{13} \subset \textstyle\bigwedge^3 V_7^\vee\}.$$

However, this locus is empty for dimensional reasons, because

$$\mathrm{codim}_{\mathrm{Gr}(7,V_{10})}(D_{\mathcal{O}_{13}}(\sigma)) = \mathrm{codim}_{\bigwedge^3 \mathbf{C}^7}(\mathcal{O}_{13}) = 22 > \dim \mathrm{Gr}(7, V_{10}).$$

So for σ general, X_7^σ is a smooth variety and each point $[V_7] \in X_7^\sigma$ satisfies the condition $\mathrm{rank}(\sigma|_{V_7}) = 5$. We now show that X_7^σ is singular when $[\sigma]$ is in the divisor $\mathcal{D}^{4,7,7}$.

Proposition 2.18. *Let $[\sigma]$ be a general point in $\mathcal{D}^{4,7,7}$ such that $\sigma(V_4, V_7, V_7) = 0$. Then $[V_7] \in \mathrm{Sing}(X_7^\sigma)$.*

Proof. We already mentioned that the flag $V_4 \subset V_7$ is unique for such a trivector σ. For this unique V_7, we see that $\mathrm{rank}(\sigma|_{V_7}) = 3$, so $[V_7]$ belongs to

$$D_{\mathcal{O}_{13}}(\sigma) = \{[V_7] \in \mathrm{Gr}(7, V_{10}) \mid \sigma|_{V_7} \in \mathcal{O}_{13} \subset \textstyle\bigwedge^3 V_7^\vee\}.$$

As $\mathcal{O}_{13}$ is the singular locus of $\mathcal{O}_{20}$, we know by the theory of orbital degeneracy loci that $D_{\mathcal{O}_{13}}(\sigma)$ is contained in $\mathrm{Sing}(D_{\mathcal{O}_{20}}(\sigma))$ ($D_{\mathcal{O}_{20}}(\sigma)$ is of the expected dimension). Therefore, $[V_7]$ is a singular point of the variety $D_{\mathcal{O}_{20}}(\sigma) = X_7^\sigma$. Note that, since σ is not general in $\mathcal{M}$, one cannot a priori affirm that $\mathrm{Sing}(D_{\mathcal{O}_{20}}(\sigma)) = D_{\mathcal{O}_{13}}(\sigma)$. □

Remark 2.19. It is in principle possible to study the exact locus of trivectors σ for which X_7^σ becomes singular, as we did for X_2^σ, X_3^σ, and X_6^σ in Proposition 2.4, and for X_1^σ in Proposition 2.9. However, due to the large number of cases to study for X_7^σ, we decided not to include a more general statement.

When $[\sigma] \notin \mathcal{D}^{4,7,7}$, for each $[V_7] \in X_7^\sigma$, the property $\mathrm{rank}(\sigma|_{V_7}) = 5$ means that there is a unique $V_2 \subset V_7$ such that $\sigma(V_2, V_7, V_7) = 0$. It is evident that each $V_1 \subset V_2$ is in X_1^σ so we get a line $\mathbf{P}(V_2)$ contained in X_1^σ. This provides an injective morphism from X_7^σ to the Fano variety of lines $F(X_1^\sigma)$. For σ general, this morphism is in fact an isomorphism.

Theorem 2.20. *Let σ be a general trivector in the moduli space. Then the Fano variety of lines $F(X_1^\sigma)$ is isomorphic to X_7^σ. Moreover there exists a morphism $q_1 : X_7^\sigma \to X_6^\sigma$ which is a fibration in cubic surfaces.*

Proof. Since σ is supposed to be general, we may suppose that σ does not lie in the union of the three special divisors $\mathcal{D}^{3,3,10} \cup \mathcal{D}^{1,6,10} \cup \mathcal{D}^{4,7,7}$. In this case, we already obtained an injective morphism from X_7^σ to $F(X_1^\sigma)$ (because for a point $[V_7] \in X_7^\sigma$, $[V_2]$ satisfying $\sigma(V_2, V_7, V_7) = 0$ is uniquely defined by $[V_7]$). Moreover, by a Bertini-type argument, we may assume that X_7^σ and $F(X_1^\sigma)$ are both smooth. It will suffice now to show the surjectivity of the morphism $X_7^\sigma \to F_1(X_1^\sigma)$.

For each line $\mathbf{P}(V_2)$ contained in X_1^σ, we look for one V_7 such that $\sigma(V_2, V_7, V_7) = 0$. For each V_1 in V_2, we have $\mathrm{rank}\, \sigma(V_1, -, -) = 6$ as mentioned above, so there is a 4-dimensional kernel K_4 with the property that $\sigma(V_1, K_4, V_{10}) = 0$. We consider the sum

$$W := \sum_{V_1 \subset V_2} K_4.$$

Any two different V_1, V_1' generate V_2, so we have $\sigma(V_2, K_4, K_4') = 0$. Keeping V_1 fixed and summing over all V_1', we get $\sigma(V_2, K_4, W) = 0$. Summing again over all V_1, we get $\sigma(V_2, W, W) = 0$. Let us now prove that $\dim W = 7$. Suppose this is not the case.

- If $\dim W = 4$, all K_4 are the same and we get $\sigma(V_2, K_4, V_{10}) = 0$. Any 3-dimensional subspace of K_4 containing V_2 would give a V_3 with $\sigma(V_3, V_3, V_{10}) = 0$. But we have assumed that $[\sigma] \notin \mathcal{D}^{3,3,10}$.
- If $\dim W = 5$, for any V_1 and V_1' contained in V_2, the intersection $K_4 \cap K_4'$ has dimension 3, so there exists a fixed subspace V_3 with the property that $\sigma(V_2, V_3, V_{10}) = 0$. If $\sigma(V_2, V_2, V_{10}) = 0$, then V_2 is contained in every kernel K_4, thus $V_2 \subset V_3$ and V_3 would satisfy the condition $\sigma(V_3, V_3, V_{10}) = 0$. Otherwise, we have $V_2 \cap V_3 = \{0\}$. In this case we consider the kernel of the linear forms $\sigma(V_2, V_2, -)$ and $\sigma(V_3, V_3, -)$: since the second form has rank at most 2, we get a subspace of dimension at least 7. Denote by V_7 a 7-dimensional subspace of the kernel and by V_5 the sum $V_2 \oplus V_3$, we then have $\sigma(V_5, V_5, V_7) = 0$. This degeneracy condition implies the existence of a $V_4 \subset V_5$ such that $\sigma(V_4, V_7, V_7) = 0$: we look inside $\mathrm{Gr}(4, V_5) = \mathbf{P}(V_5^\vee)$ for the zero-locus of σ viewed as a section of the rank-4 bundle $\mathcal{U}_4^\vee \otimes \bigwedge^2 (V_7/V_5)^\vee$, whose top Chern class is equal to 1. But this cannot happen since we assumed that $[\sigma] \notin \mathcal{D}^{4,7,7}$.
- If $\dim W = 6$, we show that there exists some $V_1 \subset V_2$ satisfying the condition $\operatorname{rank} \sigma(V_1, -, -) \leq 4$ (which does not happen since $[\sigma] \notin \mathcal{D}^{1,6,10}$). Write W as W_6 for clarity. Pick one $V_1 \subset V_2$. Inside $\mathrm{Gr}(2, V_{10}/W_6)$, contracting σ with V_1 gives a section of $(W_6/K_4)^\vee \otimes \mathcal{U}_2^\vee \simeq (\mathcal{U}_2^\vee)^{\oplus 2}$ whose top Chern class is 1. So there exists a V_8 such that $\sigma(V_1, W_6, V_8) = 0$. Take another V_1'. Since V_1 and V_1' generate V_2, we get $\sigma(V_2, K_4', V_8) = 0$. Now summing over all V_1' we get $\sigma(V_2, W_6, V_8) = 0$. Then one may verify that there exists a $U_1 \subset V_2$ such that $\sigma(U_1, V_8, V_8) = 0$: we look inside $\mathbf{P}(V_2)$ for the zero-locus of σ viewed as a section of the line bundle $\mathcal{U}_1^\vee \otimes \bigwedge^2 (V_8/V_6)^\vee \simeq \mathcal{O}(1)$. This U_1 would then satisfy $\operatorname{rank} \sigma(U_1, -, -) \leq 4$, contradicts the fact that $[\sigma] \notin \mathcal{D}^{1,6,10}$.
- If $\dim W \geq 8$, any $V_1 \subset V_2$ satisfies $\operatorname{rank} \sigma(V_1, -, -) \leq 4$, again contradicts the fact that $[\sigma] \notin \mathcal{D}^{1,6,10}$.

Therefore the morphism $X_7^\sigma \to F(X_1^\sigma)$ that we described is an isomorphism of smooth varieties.

Let us now construct the fibration $q_1 \colon X_7^\sigma \to X_6^\sigma$. For $[V_7] \in X_7^\sigma$, let $[\mathbf{P}(V_2)] \in F(X_1^\sigma)$ be the line it determines, so that $V_2 \subset V_7$ with $\sigma(V_2, V_7, V_7) = 0$. Over $\mathrm{Gr}(4, V_7/V_2)$, the trivector σ gives a section of $\bigwedge^3 \mathcal{U}_4^\vee$, whose top Chern class is 1. We have therefore a V_6 such that $V_2 \subset V_6 \subset V_7$ and $\sigma|_{V_6} = 0$; it is easy to check that V_6 is unique provided that there exists no V_4 such that $\sigma(V_4, V_7, V_7) = 0$. This defines a morphism $q_1 \colon X_7^\sigma \to X_6^\sigma$. We can see that under this map, the preimage of a point $[V_6]$ is the Pfaffian cubic surface defined in $\mathbf{P}(V_{10}/V_6)$, that is, the set of $[V_7]$ where $\sigma(V_7/V_6, -, -)|_{V_6}$ degenerates. In fact, using the notation from the proof of Lemma 2.14, this is the intersection of the 3-space $\Delta = \mathbf{P}(V_{10}/V_6)$ with the cubic hypersurface $\mathrm{Gr}(2, V_6)^* \subset \mathbf{P}\big(\bigwedge^2 V_6^\vee\big)$. Since $[\sigma] \notin \mathcal{D}^{4,7,7}$, we see that Δ cannot be entirely contained in $\mathrm{Gr}(2, V_6)^*$, so we always get a cubic surface. $\square$

Remark 2.21. We still assume that the trivector σ is general and make some remarks on the geometry of the incidence variety I^σ between X_1^σ and its Fano variety of lines $F(X_1^\sigma)$.

Consider the diagram

$$\begin{tikzcd}
 & I^\sigma := \{(x,l) \mid x \in l \subset X_1^\sigma\} \arrow[dl] \arrow[dr, "\mathbf{P}^1\text{-bundle}"] & \\
X_1^\sigma & & F(X_1^\sigma) \simeq X_7^\sigma \arrow[d, "q_1"] \\
 & & X_6^\sigma
\end{tikzcd}$$

The fibers of the fibration $I^\sigma \to X_6^\sigma$ are isomorphic to Palatini threefolds. Moreover, the induced projection $r\colon I^\sigma \to I_{1,6}^\sigma$ is a birational morphism. In fact, consider the following subvariety

$$Z := \left\{[V_1 \subset V_6] \in I_{1,6}^\sigma \;\middle|\; \sigma(V_1, K_4, V_{10}) = 0, [V_6] \in X_6^\sigma, K_4 \subset V_6\right\}$$

from the proof of Proposition 2.11. We saw that it is precisely the singular locus of $I_{1,6}^\sigma$, it is itself smooth of dimension 3, and the projection $s\colon \widetilde{I}_{1,6}^\sigma \to I_{1,6}^\sigma$ is a small contraction with $\mathbf{P}^2$-fibers over Z. We now show that $r\colon I^\sigma \to I_{1,6}^\sigma$ gives a second resolution which is another small contraction with $\mathbf{P}^1$-fibers over Z.

First, note that the fiber of r over a pair $[V_1 \subset V_6] \in I_{1,6}^\sigma$ is the set of $V_7 \supset V_6$ such that $\sigma(V_1, V_7, V_7) = 0$: such V_7 defines a degenerate skew-symmetric form on V_6 so has a kernel V_2 containing V_1 such that $\sigma(V_2, V_7, V_7) = 0$. If a pair $[V_1 \subset V_6]$ does not lie in Z, that is, if the kernel K_4 of V_1 is not contained in V_6, then $K_4 + V_6$ provides the only such 7-dimensional subspace, so r^{-1} consists of the single point $[K_4 + V_6]$. On the other hand, for a pair $[V_1 \subset V_6]$ lying in Z, we get a unique 8-dimensional space $V_8 \supset V_6$ such that $\sigma(V_1, V_6, V_8) = 0$. Each $[V_7] \in \mathbf{P}(V_8/V_6)$ satisfies the vanishing condition $\sigma(V_1, V_7, V_7) = 0$, so the fiber of r over $[V_1 \subset V_6]$ is the line $\mathbf{P}(V_8/V_6)$.

Moreover, one can also compute that $K_{\widetilde{I}_{1,6}^\sigma}\mid_{s^{-1}(x)} \cong \mathcal{O}_{\mathbf{P}^2}(-1)$ and that $K_{I^\sigma}\mid_{r^{-1}(x)} \cong \mathcal{O}_{\mathbf{P}^1}(1)$; therefore we obtain the following flip

$$\begin{tikzcd}
\widetilde{I}_{1,6}^\sigma \arrow[rr, dashed, leftrightarrow, "\text{flip}"] \arrow[dr, "\mathbf{P}^2"] \arrow[ddr, "s"'] & & I^\sigma \arrow[dl, "\mathbf{P}^1"'] \arrow[ddl, "r"] \arrow[dd, "\mathbf{P}^1\text{-bundle}"] \\
 & Z \arrow[d, hook] & \\
 & I_{1,6}^\sigma \arrow[dl, "p"'] \arrow[dr, "q"] & F(X_1^\sigma) \simeq X_7^\sigma \arrow[d, "q_1"] \\
X_1^\sigma & & X_6^\sigma.
\end{tikzcd}$$

3. The Heegner divisor of degree 28

3.1. **The discriminant.** In Section 2.6, we defined the divisor $\mathcal{D}^{4,7,7}$ in $\mathcal{M}$ given by the degeneracy condition (9)

$$\exists [V_4 \subset V_7] \quad \sigma(V_4, V_7, V_7) = 0.$$

In Proposition 2.4, we showed that the complement of $\mathcal{M}^{\text{smooth}}$ is the divisor $\mathcal{D}^{3,3,10}$. Therefore, when $[\sigma]$ lies in $\mathcal{D}^{4,7,7} \smallsetminus \mathcal{D}^{3,3,10}$, the corresponding X_6^σ is a smooth hyperkähler fourfold.

Moreover, given the flag $[V_4 \subset V_7]$, we see that every V_6 in $\mathrm{Gr}(2, V_7/V_4) = \mathbf{P}\big((V_7/V_4)^\vee\big)$ is in X_6^σ. So the hyperkähler fourfold X_6^σ contains a plane P, necessarily Lagrangian.

Note that we have the following equivalent degeneracy condition

$$\exists [V_3 \subset V_7] \quad \sigma(V_3, V_7, V_7) = 0. \tag{12}$$

The equivalence can either be deduced directly from the descriptions of the $\mathrm{GL}(V_7)$-orbits in $\bigwedge^3 V_7^\vee$ from the last section, or be verified by looking inside $\mathbf{P}(V_7/V_3)$ for the vanishing condition $\sigma(U_{4/3}, V_7, V_7) = 0$, which is the zero locus of a section of the rank-3 vector bundle $\mathcal{U}_{4/3}^\vee \otimes \bigwedge^2 \mathcal{U}_{7/4}^\vee$ with top Chern class 1, therefore there exists some V_4 such that $\sigma(V_4, V_7, V_7) = 0$.

We will first determine the discriminant of the corresponding Noether–Lefschetz/Heegner divisor. In fact, we will show that a Debarre–Voisin fourfold X_6^σ containing a Lagrangian plane (that is, of degree 1 with respect to the Plücker polarization) is always in the family $\mathcal{C}_{28}$. This shows in particular that the divisor $\mathcal{D}_{4,7,7}$ is mapped onto $\mathcal{C}_{28}$ via the modular map $\mathfrak{m}$, and that *any* Lagrangian plane contained in a Debarre–Voisin fourfold is of the above form.

We begin by recalling the following general result of Hassett–Tschinkel [HT, Section 5] on Lagrangian planes contained in hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type.

Proposition 3.1 (Hassett–Tschinkel)**.** *Let X be a smooth hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type and let P be a Lagrangian plane contained in X. Let $l \in H^6(X, \mathbf{Z})$ be the class of a line contained in the plane P. Then there exists a unique class $\lambda \in H^2(X, \mathbf{Z})$ satisfying the property*

$$\forall\, x \in H^2(X, \mathbf{Z}) \quad 2x \cdot l = \mathfrak{q}(x, \lambda), \tag{13}$$

where $\mathfrak{q}$ is the Beauville–Bogomolov–Fujiki form. Moreover, the class λ is of square $\mathfrak{q}(\lambda, \lambda) = -10$ and divisibility 2, and we have the following relation

$$[P] = \tfrac{1}{20} q^\vee + \tfrac{1}{8}\lambda^2,$$

where $q^\vee \in H^4(X, \mathbf{Q})$ is the distinguished algebraic class $\frac{5}{6}c_2(X)$ that satisfies $q^\vee \cdot x_1 \cdot x_2 = 25 \cdot \mathfrak{q}(x_1, x_2)$ for all $x_1, x_2 \in H^2(X, \mathbf{Z})$, and $q^\vee \cdot q^\vee = 575$.

We prove an extra lemma.

Lemma 3.2. *Let X be a smooth hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type and $\lambda \in H^2(X, \mathbf{Z})$ be a class of square -10 and divisibility 2. Moreover, let H be a polarization on X. Then there is at most one plane P (that is, of degree 1 with respect to H) whose associated (-10)-class is equal to λ.*

Proof. Suppose that P and P' are distinct planes whose associated (-10)-classes are both λ. We may compute the intersection number

$$[P] \cdot [P'] = (\tfrac{1}{20} q^\vee + \tfrac{1}{8}\lambda^2)^2 = [P]^2 = 3.$$

On the other hand, since both P and P' are linearly embedded, their intersection can be empty, a point, or a line L. We verify that the last case is not possible in general: we have an exact sequence

$$0 \longrightarrow \mathcal{T}_P \longrightarrow \mathcal{T}_X|_P \longrightarrow \mathcal{N}_{P/X} \simeq \mathcal{T}_P^\vee \longrightarrow 0,$$

which, when restricted to L, gives

$$0 \longrightarrow \mathcal{T}_P|_L \simeq \mathcal{O}_L(1) \oplus \mathcal{O}_L(2) \longrightarrow \mathcal{T}_X|_L \longrightarrow \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-2) \longrightarrow 0.$$

If $P \cap P' = L$, the other normal bundle $\mathcal{N}_{P'/L} \simeq \mathcal{O}_L(1)$ should be a subbundle of the quotient, which is not possible. $\square$

Proposition 3.3.

(1) A smooth Debarre–Voisin fourfold X_6^σ containing a Lagrangian plane P is always in the family $\mathcal{C}_{28}$.

(2) Consequently, for $[\sigma]$ very general in the divisor $\mathcal{D}^{4,7,7}$, the corresponding transcendental sublattice $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{trans}}$ is of discriminant 28. The divisor $\mathcal{D}^{4,7,7}$ is mapped birationally onto the Noether–Lefschetz divisor $\mathcal{C}_{28}$ by the modular map $\mathfrak{m}$, and then onto the Heegner divisor $\mathcal{D}_{28} \subset \mathcal{P}$ by the period map $\mathfrak{p}$.

(3) A general X_6^σ in the family $\mathcal{C}_{28}$ contains exactly one Lagrangian plane.

(4) Finally, any Lagrangian plane P contained in a smooth Debarre–Voisin fourfold is of the form $\mathbf{P}\big((V_7/V_4)^\vee\big)$, for a flag $[V_4 \subset V_7]$ satisfying the degeneracy condition $\sigma(V_4, V_7, V_7) = 0$.

Proof. For statement (1), let σ be such that X_6^σ is smooth of dimension 4 and contains a Lagrangian plane P. Let H be the canonical polarization on X_6^σ of square 22 and divisibility 2. Let l be the class of a line contained in the plane P. Consider the class $\lambda \in H^2(X_6^\sigma, \mathbf{Z})$ given by Proposition 3.1. Since $H \cdot l = 1$, we have $\mathfrak{q}(H, \lambda) = 2$. Therefore the intersection matrix between H and λ is

$$\begin{pmatrix} 22 & 2 \\ 2 & -10 \end{pmatrix},$$

with discriminant $224 = 7 \cdot 2^5$. The saturation of the sublattice $\mathbf{Z}H + \mathbf{Z}\lambda$ can therefore have discriminant $224, 56$, or 14.

Since the discriminant of the lattice $H^2(X_6^\sigma, \mathbf{Z})$ is 2, and since both H and λ have divisibility 2, the images of $\frac{1}{2}H$ and $\frac{1}{2}\lambda$ in the discriminant group are equal, and the class $\frac{1}{2}(H + \lambda)$ is therefore integral. We may consider the sublattice generated by $\frac{1}{2}(H + \lambda)$ and λ, which has intersection matrix

$$\begin{pmatrix} 4 & -4 \\ -4 & -10 \end{pmatrix}. \tag{14}$$

Since $H^2(X_6^\sigma, \mathbf{Z})$ is an even lattice, this sublattice is now saturated: suppose that a class $a \cdot \frac{1}{2}(H + \lambda) + b\lambda$ with $\gcd(a, b) = 1$ is still divisible by 2, then after dividing by 2 the class has square $a^2 - 2ab - \frac{5}{2}b^2$, so b is even, a is odd, and the square would not be even. Therefore, $\mathbf{Z} \cdot \frac{1}{2}(H + \lambda) + \mathbf{Z}\lambda$ is a saturated sublattice of $H^2(X_6^\sigma, \mathbf{Z})$ of discriminant 56. Since its discriminant is always twice that of its orthogonal (*cf.* [DM, Proposition 4.1]), we get a member of the family $\mathcal{C}_{28}$.

For each $[\sigma]$ in $\mathcal{D}^{4,7,7} \smallsetminus \mathcal{D}^{3,3,10}$, the corresponding Debarre–Voisin variety X_6^σ is smooth of dimension 4 and contains a Lagrangian plane $\mathbf{P}\big((V_7/V_4)^\vee\big)$. We may thus conclude that $\mathcal{D}^{4,7,7}$ is mapped onto $\mathcal{C}_{28}$ in the moduli space and $\mathcal{D}_{28}$ in the period domain. This shows the statement (2).

In particular, $\mathcal{D}^{4,7,7}$ being mapped birationally onto $\mathcal{C}_{28}$ shows that a very general member X_6^σ of the family $\mathcal{C}_{28}$ indeed contains a plane. Moreover, since its algebraic sublattice $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{alg}}$ is of rank 2 and has intersection matrix as in (14), there is only one class λ satisfying $\mathfrak{q}(H, \lambda) = 2$ and $\mathfrak{q}(\lambda, \lambda) = -10$. By Lemma 3.2, this shows that a very general X_6^σ contains exactly one Lagrangian plane, which is of the form $\mathbf{P}\big((V_7/V_4)^\vee\big)$.

Finally, for each Lagrangian plane P contained in a smooth Debarre–Voisin fourfold, we may consider a generic deformation which preserves the Lagrangian plane. Since the class of a Lagrangian plane is $[P] = \frac{1}{20}q^\vee + \frac{1}{8}\lambda^2$, which remains algebraic as long as λ is algebraic, we may choose the deformation such that very general members have Picard rank 2, using

the results of Voisin [V2] on deformations of Lagrangian subvarieties. In this case, for very general members of the deformation, the Lagrangian plane is indeed of the form $\mathbf{P}\big((V_7/V_4)^\vee\big)$ for a certain flag $[V_4 \subset V_7]$. As this is a deformation of the pair (X_6^σ, P), the original plane P in the central fiber is necessarily also of this form, which concludes the proof. □

Remark 3.4. Another type of plane contained in $\mathrm{Gr}(6, V_{10})$ is one of form $\mathbf{P}(V_8/V_5)$ for a flag $V_5 \subset V_8 \subset V_{10}$. The above characterization would suggest that a Debarre–Voisin variety X_6^σ containing such a plane is not smooth. This is indeed the case: for such a trivector σ, we have the degeneracy condition $\sigma(V_5, V_5, V_8) = 0$. We may look for a singular V_3 contained in V_5: we study inside $\mathrm{Gr}(3, V_5)$ the vanishing condition $\sigma(U_3, U_3, V_{10}/V_8) = 0$, which is the zero locus of a section of the rank-6 bundle $\big(\bigwedge^2 \mathcal{U}_3^\vee\big)^{\oplus 2}$ with top Chern class 1 and is therefore non-empty.

Remark 3.5. This result implies that $\mathcal{D}_{28}$ is unirational. Indeed, $\mathcal{D}^{4,7,7}$ can be seen as the quotient of the vector bundle $(\mathcal{U}_4 \wedge \mathcal{U}_7 \wedge \mathcal{U}_7)^\perp \subset \bigwedge^3 V_{10}^\vee \otimes \mathcal{O}_F$ over the flag variety $F \coloneqq \mathrm{Flag}(4, 7, V_{10})$ by the natural action of the group $\mathrm{SL}(V_{10})$.

Concerning the possibility of associated K3 surfaces, by reduction modulo 7, we see that the lattice in (14) does not represent 28, so there is no associated K3 surface of degree 28. This last conclusion can also be obtained using [DHO'GV, Theorem 3.1].

3.2. **The correspondence $I_{3,6}^\sigma$.** We proceed to the proof of the Hodge isometries in Theorem 2.6 and Theorem 2.16. In order to prove Theorem 2.6, we will use the correspondence $X_3^\sigma \xleftarrow{p} I_{3,6}^\sigma \xrightarrow{q} X_6^\sigma$ from (4). The key point is to show that q_*p^* sends the intersection product to $-\mathfrak{q}$, as explained by Lemma 2.7 and the remarks thereafter. For this, it is enough to prove

$$\exists\, x \in H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}} \smallsetminus \{0\} \quad x^2 = -\mathfrak{q}(q_*p^*x, q_*p^*x).$$

By a continuity argument, we may specialize to the case of a general $[\sigma]$ in the divisor $\mathcal{D}^{4,7,7}$, for which X_3^σ and X_6^σ remain smooth.

Let us begin with some preliminary results. For $[\sigma] \in \mathcal{D}^{4,7,7}$ with $\sigma(V_4, V_7, V_7) = 0$, denote by l the class of a line contained in the plane $P = \mathbf{P}\big((V_7/V_4)\big)^\vee$. Such a line can be expressed as

$$l = \{[V_6] \in X_6^\sigma \mid V_5 \subset V_6 \subset V_7\},$$

where V_5 is a subspace such that $V_4 \subset V_5 \subset V_7$. The class $z \coloneqq p_*q^*l \in H^{20}(X_3^\sigma, \mathbf{Z})$ is represented by the subvariety

$$Z \coloneqq \{[V_3] \in X_3^\sigma \mid V_3 \subset V_7,\ \dim(V_3 \cap V_5) \geq 2\}. \tag{15}$$

We may decompose the class z as the sum of its vanishing part $z_0 \in H^{20}(X_3^\sigma, \mathbf{Q})_{\mathrm{van}}$ and its Schubert part $z_1 \in j^*H^{20}(\mathrm{Gr}(3, V_{10}), \mathbf{Q})$ according to the decomposition (3).

Lemma 3.6. *In the notation above, the Schubert part z_1 of the class $z \in H^{20}(X_3^\sigma, \mathbf{Z})$ has square $z_1^2 = \frac{5}{11}$.*

Proof. The class j_*z is the Schubert class σ_{443} on $\mathrm{Gr}(3, V_{10})$ represented by Z. We can compute $z \cdot j^*\sigma_{433} = 1$ while $z \cdot j^*\sigma_{abc} = 0$ for the rest of the Schubert classes. The intersection numbers allow us to completely determine z_1 in terms of the basis $j^*\sigma_{abc}$: we get

$$\begin{aligned} z_1 = \tfrac{1}{11}(&j^*\sigma_{730} - 3j^*\sigma_{721} - j^*\sigma_{640} + 2j^*\sigma_{631} + 3j^*\sigma_{622} \\ &+ j^*\sigma_{550} - j^*\sigma_{541} - 5j^*\sigma_{532} + 6j^*\sigma_{442} + 5j^*\sigma_{433}). \end{aligned} \tag{16}$$

We may then compute its self-intersection number and find $\frac{5}{11}$. □

To compute z_0^2, we will specialize the trivector further so that X_6^σ, while still smooth, contains two disjoint planes P and P'. Denote by λ and λ' their corresponding (-10)-classes as defined in Proposition 3.1. We have the following result.

Lemma 3.7. *If X_6^σ is smooth and contains two disjoint planes P, P', then either $\mathfrak{q}(\lambda,\lambda') = 2$ or $\mathfrak{q}(\lambda,\lambda') = -2$.*

Proof. As the two planes P, P' are disjoint, we have

$$0 = [P]\cdot[P'] = \left(\tfrac{1}{20}q^\vee + \tfrac{1}{8}\lambda^2\right)\cdot\left(\tfrac{1}{20}q^\vee + \tfrac{1}{8}\lambda'^2\right),$$

by Proposition 3.1. Using $q^\vee\cdot q^\vee = 575$, $q^\vee\cdot\lambda^2 = 25\cdot\mathfrak{q}(\lambda,\lambda) = -250 = q^\vee\cdot\lambda'^2$, and $\lambda^2\cdot\lambda'^2 = 2\mathfrak{q}(\lambda,\lambda')^2 + \mathfrak{q}(\lambda,\lambda)\cdot\mathfrak{q}(\lambda',\lambda')$, we find $\mathfrak{q}(\lambda,\lambda')^2 = 4$, therefore $\mathfrak{q}(\lambda,\lambda') = \pm 2$. □

Following [DV, Section 2], we will consider the following two situations for two disjoint planes P, P' contained in X_6^σ:

Case 1. We have $V_4 \subset V_7$ and $V_4' \subset V_7'$ with $\dim(V_7\cap V_7') = 4$ and $V_4\cap V_4' = \{0\}$. For a suitable choice of basis $(e_0,\dots,e_9)$, we may set $V_7 = \langle e_0,\dots,e_6\rangle$, $V_7' = \langle e_3,\dots,e_9\rangle$, $V_4 = \langle e_1,e_2,e_3,e_4\rangle$, and $V_4' = \langle e_5,e_6,e_7,e_8\rangle$. Note that $\dim(V_4\cap V_7') = \dim(V_4'\cap V_7) = 2$.

Case 2. We have $V_4 \subset V_7$ and $V_4' \subset V_7'$ with $\dim(V_7\cap V_7') = 4$ but $V_4\cap V_4'$ one-dimensional. In this case, we may set $V_7 = \langle e_0,\dots,e_6\rangle$, $V_7' = \langle e_3,\dots,e_9\rangle$, $V_4 = \langle e_0,e_1,e_2,e_3\rangle$, and $V_4' = \langle e_3,e_7,e_8,e_9\rangle$.

In both cases, the planes $P := \mathbf{P}\big((V_7/V_4)^\vee\big)$ and $P' := \mathbf{P}\big((V_7/V_4')^\vee\big)$ are disjoint.

Remark 3.8. Note that the existence of such σ was not proved in [DV], although it can be verified using a computer: we choose random trivectors σ that satisfy the vanishing conditions as above, and check the smoothness of the hyperplane section X_3^σ. For example, the following trivectors with coefficients in $\{0,\pm1\}$ suffice in the two cases:

$$\begin{gathered}[056]+[037]-[237]+[047]+[157]+[257]+[267]-[018]-[128]-[148]\\ -[058]+[258]+[168]-[078]-[129]+[249]+[349]+[059]+[269]-[289]\end{gathered}$$

and

$$\begin{gathered}[456]+[017]+[027]+[147]-[057]+[067]+[167]-[267]+[018]\\ +[138]+[238]-[148]-[258]+[039]+[149]+[169]+[189],\end{gathered}$$

where $[ijk]$ stands for the form $e_i^\vee\wedge e_j^\vee\wedge e_k^\vee$.

We are now ready to prove Theorem 2.6.

Proof of Theorem 2.6. For $[\sigma]$ very general in the divisor $\mathcal{D}^{4,7,7}$, the Debarre–Voisin fourfold X_6^σ has Picard rank 2. Therefore the space $H^2(X_6^\sigma,\mathbf{Z})_{\mathrm{prim}}\cap H^{1,1}(X_6^\sigma)$ of primitive algebraic classes has rank 1. Using the intersection matrix (14), we see that it is generated by the class $H - 11\lambda$. As proved in [DV], the map

$$q_*p^*\colon H^{20}(X_3^\sigma,\mathbf{Q})_{\mathrm{van}} \longrightarrow H^2(X_6^\sigma,\mathbf{Q})_{\mathrm{prim}}$$

is an isomorphism of rational Hodge structures, so there is some rational number $c\in\mathbf{Q}$ such that $q_*p^*z_0 = c(H-11\lambda)$.

We now specialize the trivector σ so that X_6^σ contains two disjoint planes P and P'. We have two (-10)-classes λ,λ' in $H^2(X_6^\sigma,\mathbf{Z})$, and two classes z,z' in $H^{20}(X_3^\sigma,\mathbf{Z})$ represented by the subvarieties Z and Z' defined in (15). Since both Z and Z' are Schubert varieties

of type Σ_{443}, the two classes z and z' share the same Schubert part $z_1 = z'_1$, which can be determined explicitly as in (16) of Lemma 3.6 and has square $\frac{5}{11}$.

Let us suppose that we are in **Case 1** above. Since $Z \cap Z' = \varnothing$, we have $0 = z \cdot z' = z_0 \cdot z'_0 + z_1^2$ so

$$z_0 \cdot z'_0 = -z_1^2 = -\tfrac{5}{11}.$$

Moreover, we have

$$z_0 \cdot z'_0 = z \cdot z'_0 = p_* q^* l \cdot z'_0 = l \cdot c(H - 11\lambda') = c(1 - \tfrac{11}{2}\mathfrak{q}(\lambda, \lambda')).$$

By Lemma 3.7, $\mathfrak{q}(\lambda, \lambda')$ has two possible values ± 2. If $\mathfrak{q}(\lambda, \lambda') = 2$, we get $c = \frac{1}{22}$ while if $\mathfrak{q}(\lambda, \lambda') = -2$, we get $c = -\frac{5}{132}$. We may compute $z^2 = z_0^2 + \frac{5}{11} = 56c + \frac{5}{11}$ which is equal to 3 or $-\frac{5}{3}$ in the two cases. Since Z is integral, the latter is absurd, so we may conclude that $\mathfrak{q}(\lambda, \lambda') = 2$, $c = \frac{1}{22}$, and $z^2 = 3$. Finally, we get

$$z_0^2 = \tfrac{28}{11}, \quad \mathfrak{q}(q_* p^* z_0, q_* p^* z_0) = \mathfrak{q}\left(\tfrac{1}{22}(H - 11\lambda), \tfrac{1}{22}(H - 11\lambda)\right) = -\tfrac{28}{11},$$

which proves what we need. □

Remark 3.9. By Lemma 3.7, we know that $\mathfrak{q}(\lambda, \lambda') = \pm 2$. In the proof of the theorem, we saw that in **Case 1**, we have $\mathfrak{q}(\lambda, \lambda') = 2$. We could also have used **Case 2**, in which case one obtains $\mathfrak{q}(\lambda, \lambda') = -2$ instead.

In the proof, we showed that $z^2 = 3$. This allows us to write out the full intersection matrix of the sublattice $\mathbf{Z}z + j^* H^{20}(\mathrm{Gr}(3, V_{10}), \mathbf{Z})$, whose discriminant can then be computed to be 28. Since the middle cohomology $H^{20}(X_3^\sigma, \mathbf{Z})$ is a unimodular lattice, the orthogonal complement $H^{20}(X_3^\sigma, \mathbf{Z})^{\perp z}_{\mathrm{van}}$ has the same discriminant. This last lattice is mapped via $q_* p^*$ onto the transcendental sublattice $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{trans}}$, so we may again conclude that $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{trans}}$ is of discriminant 28.

Another consequence of the theorem is the integral Hodge conjecture for $H^{20}(X_3^\sigma, \mathbf{Z})$, following ideas of Mongardi–Ottem [MO] for cubic fourfolds.

Corollary 3.10. *The integral Hodge conjecture holds for $H^{20}(X_3^\sigma, \mathbf{Z})$.*

Proof. The maps in diagram (4) define an injective morphism

$$q_* p^* \colon H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}} \xrightarrow{\sim} H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}} \hookrightarrow H^2(X_6^\sigma, \mathbf{Z})$$

of abelian groups. The key point is that, by definition of the primitive cohomology, it has saturated image, that is, the cokernel is torsion-free. Therefore the dual gives a surjective morphism

$$\begin{aligned} p_* q^* \colon H^6(X_6^\sigma, \mathbf{Z}) \simeq H^2(X_6^\sigma, \mathbf{Z})^\vee &\twoheadrightarrow \left(H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}}\right)^\vee \\ c &\longmapsto (x \longmapsto p_* q^* c \cdot x). \end{aligned}$$

Moreover, as the middle cohomology $H^{20}(X_3^\sigma, \mathbf{Z})$ is self-dual, we get, by restriction to the saturated subgroup $H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}}$, another surjective morphism

$$\begin{aligned} H^{20}(X_3^\sigma, \mathbf{Z}) \simeq H^{20}(X_3^\sigma, \mathbf{Z})^\vee &\twoheadrightarrow \left(H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}}\right)^\vee \\ a &\longmapsto (x \longmapsto a \cdot x) \end{aligned}$$

whose kernel is the orthogonal complement $j^* H^{20}(\mathrm{Gr}(3, V_{10}), \mathbf{Z})$. So we have an isomorphism

$$H^{20}(X_3^\sigma, \mathbf{Z}) / j^* H^{20}(\mathrm{Gr}(3, V_{10}), \mathbf{Z}) \xrightarrow{\sim} \left(H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}}\right)^\vee.$$

Comparing these two maps we see that

$$p_*q^*\colon H^6(X_6^\sigma,\mathbf{Z}) \longrightarrow H^{20}(X_3^\sigma,\mathbf{Z})/j^*H^{20}(\mathrm{Gr}(3,V_{10}),\mathbf{Z})$$

is surjective. In other words, we have

$$p_*q^*H^6(X_6^\sigma,\mathbf{Z}) + j^*H^{20}(\mathrm{Gr}(3,V_{10}),\mathbf{Z}) = H^{20}(X_3^\sigma,\mathbf{Z}).$$

Since the integral Hodge conjecture holds for $H^6(X_6^\sigma,\mathbf{Z})$ by [MO, Theorem 0.1], and since the map p_*q^* is given by an integral correspondence, every integral $(10,10)$-class on X_3^σ is therefore algebraic. □

Theorem 3.11. *When σ is such that X_3^σ is smooth (that is, when $[\sigma]\notin\mathcal{D}^{3,3,10}$), the integral Hodge conjecture holds for X_3^σ in all degrees.*

Proof. Since X_3^σ is a hyperplane section of the Grassmannian $\mathrm{Gr}(3,V_{10})$, the cohomology classes in degrees 2 to 18 all come from Schubert classes thanks to the Lefschetz hyperplane theorem.

The degree-20 case is settled in Corollary 3.10.

For degree 22 to degree 38, by using Poincaré duality, we can verify that the Schubert classes also produce all the cohomology classes, except in degree 22, where they only generate a subgroup of $H^{22}(X_3^\sigma,\mathbf{Z})$ of index 3. However, we have an extra algebraic class g represented by the Grassmannian $\mathrm{Gr}(3,U_6)$ for any $[U_6]\in X_6^\sigma$. It is easy to see that the class g only intersects the Schubert class $j^*\sigma_{333}\in H^{18}(X_3^\sigma,\mathbf{Z})$ with intersection number 1. This allows us to verify that $j^*H^{22}(\mathrm{Gr}(3,V_{10}),\mathbf{Z})+\mathbf{Z}g$ generates $H^{18}(X_3^\sigma,\mathbf{Z})^\vee\simeq H^{22}(X_3^\sigma,\mathbf{Z})$. □

Remark 3.12. The extra algebraic class g can be seen as $p_*q^*[*]$, where $[*]\in H^8(X_6^\sigma,\mathbf{Z})$ is the class of a point in X_6^σ. We see that Schubert classes only produce the class $3g$. This phenomenon reappears below for the variety X_1^σ: if $\pi=p_*q^*[*]$ is the class of a Palatini threefold, Schubert classes—in particular the Lefschetz operator—only produce the class 3π.

3.3. **The correspondence $I_{1,6}^\sigma$.** We will now prove the second Hodge isometry stated in Theorem 2.16. We will use the correspondence $X_1^\sigma \xleftarrow{p} I_{1,6}^\sigma \xrightarrow{q} X_6^\sigma$ from (7). Recall that $h\in H^2(X_1^\sigma,\mathbf{Z})$ is the polarization on X_1^σ, the class $\pi\in H^6(X_1^\sigma,\mathbf{Z})$ is the class of a Palatini threefold, and $H^6(X_1^\sigma,R)_{\mathrm{van}}$ is defined as $\langle h^3,\pi\rangle^\perp$ for $R=\mathbf{Q},\mathbf{Z}$. The idea of the proof is exactly the same.

Proof of Theorem 2.16. We will first show that q_*p^* is an isomorphism of $\mathbf{Q}$-vector spaces. Following the same idea as in [DV], it suffices to show that the map

$$p_*q^*\colon H^6(X_6^\sigma,\mathbf{Q}) \longrightarrow H^4(X_1^\sigma,\mathbf{Q})$$

has rank greater than 1. Indeed, if this is the case, the restriction to the primitive part $H^6(X_6^\sigma,\mathbf{Q})_{\mathrm{prim}}\simeq\big(H^2(X_6^\sigma,\mathbf{Q})_{\mathrm{prim}}\big)^\vee$ cannot be zero, so the dual map

$$q_*p^*\colon H^8(X_1^\sigma,\mathbf{Q}) \longrightarrow H^2(X_6^\sigma,\mathbf{Q})_{\mathrm{prim}}$$

is also non-zero. We may then use the simplicity of the Hodge structure $H^2(X_6^\sigma,\mathbf{Q})_{\mathrm{prim}}$ for a very general σ and a deformation argument to conclude.

As in the proof of Theorem 2.6, we consider a general $[\sigma]$ in the divisor $\mathcal{D}^{4,7,7}$, so that X_6^σ contains a unique plane $P=\mathbf{P}\big((V_7/V_4)^\vee\big)$. Denote by l the class of a line contained in P, and consider the class $z:=p_*q^*l\in H^4(X_1^\sigma,\mathbf{Z})$. We would like to show that the class z is represented by the intersection

$$\mathbf{P}(V_7)\cap X_1^\sigma\subset X_1^\sigma$$

and in particular only depends on V_7. In fact, it is easy to see that $\mathbf{P}(V_4)$ is always contained in X_1^σ, therefore this intersection is not irreducible, so we will use Z to denote the other component of $\mathbf{P}(V_7) \cap X_1^\sigma$ and show that when it is of expected dimension 4, it has class $p_* q^* l \in H^4(X_1^\sigma, \mathbf{Z})$.

We first describe the geometry of the incidence variety $I_{1,6}^\sigma$. Fibers of the map q above $[V_6] \in P$ are degenerate Palatini threefolds having $\mathbf{P}(V_4)$ as one irreducible component. The preimage $q^{-1}(P)$ therefore consists of two components Y and Y': the map p projects the first component Y onto $\mathbf{P}(V_4) \subset X_1^\sigma$, and the second component Y' onto Z. The fibers of $p\colon Y \to \mathbf{P}(V_4)$ are just copies of the plane P, while the fibers of $p\colon Y' \to Z$ away from $\mathbf{P}(V_4)$ can be described as follows: each $[V_1]$ not lying in $\mathbf{P}(V_4)$ spans a 5-dimensional subspace $V_1 \oplus V_4$, and since $\sigma(V_4, V_7, V_7) = 0$, each $[V_6]$ in the line $\mathbf{P}(V_7/(V_1 \oplus V_4))$ lies in X_6^σ. Therefore the generic fibers of $p\colon Y' \to Z$ are lines contained in P. For a fixed line $l \subset P$, the generic fibers of $p\colon q^{-1}(l) \cap Y' \to Z$ are therefore intersections of two lines in P, so this is a birational map, and we may conclude that the class $p_* q^* l \in H^4(X_1^\sigma, \mathbf{Z})$ is indeed represented by the class of Z with multiplicity 1. The geometry can be summarized in the following diagram

$$\begin{tikzcd}
 & Y \cup Y' \arrow[r, hook] \arrow[dl] \arrow[dr] & I_{1,6}^\sigma \arrow[dl, "p"] \arrow[dr, "q"] & \\
\mathbf{P}(V_4) \cup Z \arrow[r, hook] & X_1^\sigma & P \arrow[r, hook] & X_6^\sigma.
\end{tikzcd}$$

Now we consider again the two special cases where X_6^σ contains two planes P and P', and get two subvarieties Z and Z' and their classes z and z'. We would like to compute the intersection number $z \cdot z' \cdot h^2$. It will suffice to determine the intersection $\mathbf{P}(V_7) \cap \mathbf{P}(V_7') \cap X_1^\sigma$: in **Case 1**, it is a quadric surface and some lower-dimensional components, and in **Case 2**, it is a cubic surface and some lower-dimensional components.

To be more precise, in the basis $(e_0, \ldots, e_9)$ of **Case 1** described above, the intersection is defined inside $\mathbf{P}(V_7 \cap V_7') = \mathbf{P}(\langle e_3, e_4, e_5, e_6 \rangle)$ as the locus where the 10×10 skew-symmetric matrix

$$\left(\begin{array}{ccccc|ccccc}
 & & & & & a_{056}x_6 & -a_{056}x_5 & f_{07} & f_{08} & f_{09} \\
 & & & & & & & f_{17} & f_{18} & f_{19} \\
 & & & & & & & f_{27} & f_{28} & f_{29} \\
 & & & & & & & & & -a_{349}x_4 \\
 & & & & & & & & & a_{349}x_3 \\
\hline
-a_{056}x_6 & & & & & & & & & \\
a_{056}x_5 & & & & & & & & & \\
-f_{07} & -f_{17} & -f_{27} & & & & & & & \\
-f_{08} & -f_{18} & -f_{28} & & & & & & & \\
-f_{09} & -f_{19} & -f_{29} & a_{349}x_4 & -a_{349}x_3 & & & & &
\end{array}\right)$$

has rank ≤ 6. Here we only write down the non-zero entries: each $a_{ijk} := \sigma(e_i, e_j, e_k)$ is a constant, and each f_{ij} is the restriction of the linear form $\sigma(e_i, e_j, -)$ to $\langle e_3, e_4, e_5, e_6 \rangle$, a polynomial in x_3, x_4, x_5, x_6 of degree 1. So the locus where the rank drops is the union of the quadric surface defined by $f_{17}f_{28} - f_{27}f_{18}$ and the two lines $x_3 = x_4 = 0$ and $x_5 = x_6 = 0$.

In **Case 2**, the matrix is instead the following

$$\left(\begin{array}{ccc|c|ccc|ccc}
 & & & & & & & f_{07} & f_{08} & f_{09} \\
 & & & & & & & f_{17} & f_{18} & f_{19} \\
 & & & & & & & f_{27} & f_{28} & f_{29} \\
\hline
 & & & 0 & & & & & & \\
\hline
 & & & & & ax_6 & -ax_5 & & & \\
 & & & & -ax_6 & & ax_4 & & & \\
 & & & & ax_5 & -ax_4 & & & & \\
\hline
-f_{07} & -f_{17} & -f_{27} & & & & & & & \\
-f_{08} & -f_{18} & -f_{28} & & & & & & & \\
-f_{09} & -f_{19} & -f_{29} & & & & & & &
\end{array}\right).$$

Again $a = a_{456} = \sigma(e_4, e_5, e_6)$ is a constant and each f_{ij} is the restriction of the linear form $\sigma(e_i, e_j, -)$ to $\langle e_3, e_4, e_5, e_6\rangle$. The locus where the rank is ≤ 6 is the union of the cubic surface defined by $\det(f_{ij}) = 0$ and the point $x_4 = x_5 = x_6 = 0$.

Consequently, we see that $z \cdot z' \cdot h^2$ equals to 2 or 3 in the two cases respectively. Now if p_*q^* were to be of rank 1, then z and z' would always be proportional. Since by deforming in the divisor $\mathcal{D}^{4,7,7}$, we can map the class $z \cdot h$ to $z' \cdot h$ under some isometry, these two classes must then either be equal or opposite. So the above intersection numbers are not possible and we get a contradiction. Therefore we may conclude that the map q_*p^* as in (5) is indeed an isomorphism over $\mathbf{Q}$.

Once we know the isomorphism over $\mathbf{Q}$, to show that the scalar is -1, it suffices, as before, to find some class $x \in H^6(X_1^\sigma, \mathbf{Z})_{\text{van}}$ such that $x^2 \neq 0$ and

$$x^2 = -\mathfrak{q}(q_*p^*(x \cdot h), q_*p^*(x \cdot h)).$$

By intersecting z with h we get a class $z \cdot h \in H^6(X_1^\sigma, \mathbf{Z})$ which we can write as a sum $z \cdot h = x_0 + x_1$, where $x_0 \in H^6(X_1^\sigma, \mathbf{Q})_{\text{van}}$ and $x_1 \in \mathbf{Q}h^3 + \mathbf{Q}\pi$. For σ very general in the divisor $\mathcal{D}^{4,7,7}$, $H - 11\lambda$ generates the space of primitive algebraic classes, so there is a rational number $c \in \mathbf{Q}$ such that $q_*p^*(x_0 \cdot h) = c(H - 11\lambda)$. Again we specialize to the two cases where X_6^σ contains two planes. By Remark 3.9, we know that $\mathfrak{q}(\lambda, \lambda')$ equals to 2 in **Case 1** and -2 in **Case 2**. So we have

$$x_0 \cdot x_0' = z \cdot h \cdot x_0' = p_*q^*l \cdot h \cdot x_0' = l \cdot q_*p^*(h \cdot x_0') = l \cdot c(H - 11\lambda') = c - c \cdot \tfrac{11}{2}\mathfrak{q}(\lambda, \lambda')$$

which equals $-10c$ in **Case 1** and $12c$ in **Case 2**. As we have shown that $z \cdot z' \cdot h^2 = x_0 \cdot x_0' + x_1^2$ is equal to 2 or 3 in the two cases respectively, we get $c = \frac{1}{22}$ and $x_1^2 = \frac{27}{11}$. Finally we carry out the same calculation

$$x_0^2 = z \cdot h \cdot x_0 = p_*q^*l \cdot h \cdot x_0 = l \cdot q_*p^*(h \cdot x_0) = l \cdot c(H - 11\lambda) = \tfrac{1}{22}(1 - \tfrac{11}{2}\mathfrak{q}(\lambda, \lambda)) = \tfrac{28}{11},$$

where we used the fact that $\mathfrak{q}(\lambda, \lambda) = -10$. So the class x_0 satisfies the desired property:

$$x_0^2 = \tfrac{28}{11} \quad \text{while} \quad \mathfrak{q}\left(q_*p^*(x_0 \cdot h), q_*p^*(x_0 \cdot h)\right) = \mathfrak{q}\left(\tfrac{1}{22}(H - 11\lambda), \tfrac{1}{22}(H - 11\lambda)\right) = -\tfrac{28}{11}.$$

This allows us to conclude the proof. Also note that $(z \cdot h)^2 = x_0^2 + x_1^2 = 5$, which is an integer as one would expect. □

Since $x_1 \in \mathbf{Q}h^3 + \mathbf{Q}\pi$ with $x_1^2 = \frac{27}{11}$, one may easily verify that $x_1 = \pm\frac{3}{11}(h^3 + \pi)$. Using the effectiveness of the class z, we may conclude that $x_1 = \frac{3}{11}(h^3 + \pi)$. We can then compute

the intersection matrix for h^3, π, and $z \cdot h$, and find

$$\begin{pmatrix} 15 & 7 & 6 \\ 7 & 4 & 3 \\ 6 & 3 & 5 \end{pmatrix}. \tag{17}$$

In particular, the discriminant group of the lattice generated by these three classes is $\mathbf{Z}/28\mathbf{Z}$. So this is also the discriminant group of the orthogonal $H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}^{\perp z\cdot h}$ and that of the transcendental lattice $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{trans}}$.

Remark 3.13. As a side note, since the projective space $\mathbf{P}(V_4)$ is contained in X_1^σ, its class should be a linear combination of the three classes h^3, π, and $z \cdot h$. One may check that $[\mathbf{P}(V_4)] = \pi - z \cdot h$ by using the intersection numbers.

As in the case of X_3^σ, we can obtain the integral Hodge conjecture for X_1^σ. From the Hodge diamond (10) and the fact that X_1^σ contains lines, we see that the only non-trivial cases are in degrees 4, 6, and 8. First we treat the case of the middle cohomology.

Corollary 3.14. *When σ is such that X_1^σ is smooth (that is, when $[\sigma] \notin \mathcal{D}^{3,3,10} \cup \mathcal{D}^{1,6,10}$), the integral Hodge conjecture holds for $H^6(X_1^\sigma, \mathbf{Z})$.*

Proof. The proof is similar to that of Corollary 3.10: by considering the dual, we get a surjective morphism

$$\begin{aligned} L_h p_* q^* \colon H^6(X_6^\sigma, \mathbf{Z}) &\twoheadrightarrow \left(H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}\right)^\vee \\ c &\longmapsto (x \longmapsto p_* q^* c \cdot h \cdot x) \end{aligned}$$

and from the self-duality of $H^6(X_1^\sigma, \mathbf{Z})$, we get

$$\begin{aligned} H^6(X_6^\sigma, \mathbf{Z})/(\mathbf{Z}h^3 + \mathbf{Z}\pi) &\xrightarrow{\sim} \left(H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}\right)^\vee \\ a &\longmapsto (x \longmapsto a \cdot x). \end{aligned}$$

This gives us the equality

$$L_h p_* q^* H^6(X_6^\sigma, \mathbf{Z}) + (\mathbf{Z}h^3 + \mathbf{Z}\pi) = H^6(X_1^\sigma, \mathbf{Z}),$$

so we may conclude from the integral Hodge conjecture for $H^6(X_6^\sigma, \mathbf{Z})$. □

The proof for degrees 4 and 8 is a bit more involved, due to the following.

Lemma 3.15. *Suppose that X_1^σ is smooth. The image of the Lefschetz operator*

$$L_h \colon H^6(X_1^\sigma, \mathbf{Z}) \longrightarrow H^8(X_1^\sigma, \mathbf{Z})$$

is a subgroup of index 3. By duality, the same is true for

$$L_h \colon H^4(X_1^\sigma, \mathbf{Z}) \longrightarrow H^6(X_1^\sigma, \mathbf{Z}).$$

When restricted to the vanishing parts, both Lefschetz operators become isomorphisms.

Proof. We want to determine for which classes in $H^6(X_1^\sigma, \mathbf{Z})$ the image by L_h is divisible. We will do this by studying the image of $q_* p^* L_h$. Since this is a topological property, it suffices to study the case of a general σ in the divisor $\mathcal{D}^{4,7,7}$, so we retain the notation of $l \in H_2(X_6^\sigma, \mathbf{Z})$, $z \in H^4(X_1^\sigma, \mathbf{Z})$, and $z \cdot h = x_0 + x_1$, where we have already shown that the algebraic part x_1 is equal to $\frac{3}{11}(h^3 + \pi)$. Note that since $z \cdot h$ does not lie in the direct

sum $H^6(X_1^\sigma,\mathbf{Z})_{\mathrm{van}} \oplus (\mathbf{Z}h^3+\mathbf{Z}\pi)$ which is a sublattice of $H^6(X_1^\sigma,\mathbf{Z})$ of index 11, by adding the class $z\cdot h$ we get the entire lattice so

$$H^6(X_1^\sigma,\mathbf{Z}) = H^6(X_1^\sigma,\mathbf{Z})_{\mathrm{van}} + \mathbf{Z}z\cdot h + (\mathbf{Z}h^3+\mathbf{Z}\pi).$$

Using the intersection matrix (17), we may compute $q_*p^*(h^4)\cdot l = h^4\cdot z = 6$ and $q_*p^*(\pi\cdot h)\cdot l = \pi\cdot h\cdot z = 3$. As h^4 and $\pi\cdot h$ are always algebraic, while for very general σ, the only algebraic classes in $H^2(X_6^\sigma,\mathbf{Z})$ are the multiples of H, we may conclude that

$$q_*p^*(h^4) = 6H \quad \text{and} \quad q_*p^*(\pi\cdot h) = 3H.$$

Consequently, the image of the class $x_1\cdot h = \frac{3}{11}(h^3+\pi)\cdot h$ is

$$q_*p^*(x_1\cdot h) = q_*p^*\big(\tfrac{3}{11}(h^4+\pi\cdot h)\big) = \tfrac{27}{11}H,$$

and for the class $z\cdot h^2 = (x_0+x_1)\cdot h$ we get

$$q_*p^*(z\cdot h^2) = q_*p^*(x_0\cdot h + x_1\cdot h) = \tfrac{1}{22}(H-11\lambda) + \tfrac{27}{11}H = 3H - \tfrac{1}{2}(H+\lambda).$$

By Theorem 2.16, we see that $H^6(X_1^\sigma,\mathbf{Z})_{\mathrm{van}}$ is mapped isomorphically onto $H^2(X_6^\sigma,\mathbf{Z})_{\mathrm{prim}}$ via $q_*p^*L_h$, so we now have a complete description of the image of $q_*p^*L_h$. We may thus verify that any primitive class $x\in H^6(X_1^\sigma,\mathbf{Z})$ such that $L_h(x)$ becomes divisible in $H^8(X_1^\sigma,\mathbf{Z})$ must lie in $\mathbf{Z}h^3+\mathbf{Z}\pi$. Take one such class $x = ah^3+b\pi$ with $\gcd(a,b)=1$. We have $x\cdot h^3 = 15a+7b$ and $x\cdot z\cdot h = 6a+3b$. One verifies easily that for $L_h(x) = x\cdot h$ to be divisible, the only possibility is when $(a,b)=(1,0)$ and the divisibility is 3. In other words, only the class h^4 is potentially divisible by 3 in $H^8(X_1^\sigma,\mathbf{Z})$, so the index of the image $L_hH^6(X_1^\sigma,\mathbf{Z})$ in $H^8(X_1^\sigma,\mathbf{Z})$ is either 1 or 3.

To show that we have indeed index 3, we embed X_1^σ into the flag variety $\mathrm{Flag}(1,4,V_{10})$ by mapping $[V_1]$ to the pair $[V_1\subset K_4]$, where K_4 is the kernel of the skew-symmetric form $\sigma(V_1,-,-)$. This allows us to exhibit explicit algebraic classes on X_1^σ using (relative) Schubert classes in `Macaulay2` (see Section A.2 where we provide the code for all the following computations). In particular we construct algebraic classes $s\coloneqq\sigma_{200}\in H^4(X_1^\sigma,\mathbf{Z})$ and $t\coloneqq\sigma_{400}\in H^8(X_1^\sigma,\mathbf{Z})$. One may verify using the intersection numbers that

$$\pi = \tfrac{1}{3}(6h^2-s)\cdot h \quad \text{and} \quad t = \tfrac{17}{3}h^4 - 7\pi\cdot h.$$

The second equality shows that h^4 is indeed divisible by 3 in $H^8(X_6^\sigma,\mathbf{Z})$, which allows us to conclude. For the degree-4 case, notice that we have the degree of a Palatini threefold $(6h^2-s)\cdot\frac{1}{3}h^4 = 7$, so the class $6h^2-s$ is primitive while its image $L_h(6h^2-s) = 3\pi$ is divisible by 3.

Finally, for both Lefschetz operators, the extra 3-divisible class lies in the algebraic part. So when we restrict to the vanishing parts, we get isomorphisms. □

Theorem 3.16. *When σ is such that X_1^σ is smooth (that is, when $[\sigma]\notin\mathcal{D}^{3,3,10}\cup\mathcal{D}^{1,6,10}$), the integral Hodge conjecture holds for X_1^σ in all degrees.*

Proof. For $H^4(X_1^\sigma,\mathbf{Z})$, we see that the image of $6h^2-s$ is divisible by 3. The subgroup $\mathbf{Z}h^2+\mathbf{Z}s$ is therefore saturated in $H^4(X_1^\sigma,\mathbf{Z})$. As before, we have a surjective map

$$p_*q^*\colon H^6(X_6^\sigma,\mathbf{Z}) \simeq \big(H^2(X_6^\sigma,\mathbf{Z})\big)^\vee \twoheadrightarrow (H^8(X_1^\sigma,\mathbf{Z})_{\mathrm{van}})^\vee,$$

and by considering the duality between $H^4(X_1^\sigma,\mathbf{Z})$ and $H^8(X_1^\sigma,\mathbf{Z})$, we get a surjective map

$$H^4(X_1^\sigma,\mathbf{Z}) \simeq \big(H^8(X_1^\sigma,\mathbf{Z})\big)^\vee \twoheadrightarrow (H^8(X_1^\sigma,\mathbf{Z})_{\mathrm{van}})^\vee,$$

whose kernel is precisely $\mathbf{Z}h^2 + \mathbf{Z}s$ and is therefore generated by algebraic classes. Comparing the two, we get

$$p_*q^*H^6(X_6^\sigma, \mathbf{Z}) + (\mathbf{Z}h^2 + \mathbf{Z}s) = H^4(X_1^\sigma, \mathbf{Z}),$$

which allows us to conclude.

For $H^8(X_1^\sigma, \mathbf{Z})$, we could proceed similarly by showing that the orthogonal of $H^4(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}$ is generated by algebraic classes. But since we have already obtained the integral Hodge conjecture for $H^6(X_1^\sigma, \mathbf{Z})$, in view of the above lemma it suffices to show that $\frac{1}{3}h^4$ is algebraic. By looking at the integral algebraic class $t = \frac{17}{3}h^4 - 7\pi \cdot h$, we have $\frac{1}{3}h^4 = 6h^4 - 7\pi \cdot h - t$ so it is indeed algebraic. This concludes the proof. □

4. The Heegner divisor of degree 24

In the GIT moduli space $\mathcal{M}$ of trivectors, we have the divisor $\mathcal{D}^{1,6,10}$ given by trivectors satisfying the degeneracy condition $\sigma(V_1, V_6, V_{10}) = 0$ as in (6), which is also the locus where the Peskine variety X_1^σ becomes singular and generically admits an isolated singularity at $[V_1]$. In this last section, we will study the geometry along this divisor. Notably we will give the geometric construction of a K3 surface S of degree 6 and a divisor D in X_6^σ ruled over S.

First we show that the divisor $\mathcal{D}^{1,6,10}$ is mapped to the Noether–Lefschetz divisor $\mathcal{C}_{24}$ under the modular map $\mathfrak{m}$ and further to the Heegner divisor $\mathcal{D}_{24}$ by the period map $\mathfrak{p}$. We state a lemma which gives an alternative description for $\mathcal{D}^{1,6,10}$.

Lemma 4.1. *For a trivector σ, there is a flag $V_1 \subset V_6$ such that $\sigma(V_1, V_6, V_{10}) = 0$ if and only if there is a flag $V_1 \subset V_8$ such that $\sigma(V_1, V_8, V_8) = 0$.*

Proof. If we have a flag $V_1 \subset V_8$ as above, the skew-symmetric 2-form $\sigma(V_1, -, -)$ is of rank at most 4, so there is a 6-dimensional V_6 in the kernel.

Conversely, if we have a flag $V_1 \subset V_6$ as in the lemma, the set of V_8 in $\mathrm{Gr}(2, V_{10}/V_6)$ such that $\sigma(V_1, V_8, V_8) = 0$ is exactly the set of subspaces which are isotropic with respect to $\sigma(V_1, -, -)$, which is a linear section of the quadric $\mathrm{Gr}(2, V_{10}/V_6)$. □

Proposition 4.2. *The divisor $\mathcal{D}^{1,6,10}$ is mapped birationally onto the Noether–Lefschetz divisor $\mathcal{C}_{24}$ via the moduli map $\mathfrak{m}$, and then to the Heegner divisor $\mathcal{D}_{24}$ via the period map $\mathfrak{p}$.*

Proof. As with the other two divisors, it suffices to compute the discriminant.

The degeneracy condition $\sigma(V_1, V_8, V_8) = 0$ shows that there is a Grassmannian $\mathrm{Gr}(2, 7) = \mathrm{Gr}(2, V_8/V_1)$ contained in X_3^σ. Notice that the choice of V_8 is not canonical, as generally these V_8 are parametrized by a 3-dimensional quadric for a fixed flag $V_1 \subset V_6$, and so are the Grassmannians $\mathrm{Gr}(2, 7)$ contained in X_3^σ.

If we fix one $Z = \mathrm{Gr}(2, 7)$ contained in X_3^σ and look at its class $z \in H^{20}(X_3^\sigma, \mathbf{Z})$, we may compute the self-intersection number

$$z^2 = c_{10}(\mathcal{N}_{Z/X_3^\sigma}) = 2.$$

Indeed, using the two normal sequences

$$0 \longrightarrow \mathcal{T}_Z \longrightarrow \mathcal{T}_{X_3^\sigma}|_Z \longrightarrow \mathcal{N}_{Z/X_3^\sigma} \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{T}_{X_3^\sigma} \longrightarrow \mathcal{T}_{\mathrm{Gr}(3,V_{10})}|_{X_3^\sigma} \longrightarrow \mathcal{O}_{X_3^\sigma}(1) \longrightarrow 0,$$

the normal bundle $\mathcal{N}_{Z/X_3^\sigma}$ can be expressed in terms of homogeneous vector bundles $\mathcal{U}_2$ and $\mathcal{Q}_5$ on $Z = \mathrm{Gr}(2, 7)$, so we may calculate explicitly its Chern classes using Schubert calculus (see Section A.3).

Moreover, we see that $j_*z = \sigma_{722}$ is a Schubert class. So we can compute the full intersection matrix for the lattice $\mathbf{Z}z + j^*H^{20}(\mathrm{Gr}(3, V_{10}), \mathbf{Z})$ and find that its determinant is 24. It is possible to show that the discriminant group is $\mathbf{Z}/24\mathbf{Z}$ by explicitly finding a vector of divisibility 24. □

Remark 4.3. This result implies that $\mathcal{D}_{24}$ is unirational. Indeed, $\mathcal{D}^{1,6,10}$ can be seen as a quotient of the vector bundle $(\mathcal{U}_1 \wedge \mathcal{U}_6 \wedge V_{10})^\perp \subset \bigwedge^3 V_{10}^\vee \otimes \mathcal{O}_F$ over the variety $F := \mathrm{Flag}(1, 6, V_{10})$ by the natural action of the group $\mathrm{SL}(V_{10})$.

4.1. **Review: cubic fourfolds containing a plane.** Before studying the geometry of the Debarre–Voisin hyperkähler manifold X_6^σ along the divisor $\mathcal{D}^{1,6,10}$, we first briefly review results for cubic fourfolds containing a plane, originally considered by Voisin in her thesis [V1], with later studies on their derived aspects by Kuznetsov [K], and moduli aspects by Macrì–Stellari [MS1] and Ouchi [Ou]. We will see that analogous results hold in our case.

Let X be a cubic fourfold in $\mathbf{P}(V_6)$ that contains a plane $\mathbf{P}(V_3)$ for some $V_3 \subset V_6$. As shown in [V1], the blow up $\mathrm{Bl}_{\mathbf{P}(V_3)} X$ projects onto the plane $\mathbf{P}^2 = \mathbf{P}(V_6/V_3)$ and the fibers are quadric surfaces which are generically smooth. The discriminant locus, that is, the locus where the quadrics are singular, is a sextic curve in $\mathbf{P}^2$. Let S be the variety parametrizing rulings of lines in these fibers: as the fibers are quadric surfaces, the projection $S \to \mathbf{P}^2$ is a generically 2-to-1 morphism, ramified along the discriminant curve, and S is therefore a K3 surface of degree 2.

The variety $F \subset \mathrm{Gr}(2, V_6)$ of lines contained in X is a hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type by [BD]. The lines in the fibers of $\mathrm{Bl}_{\mathbf{P}(V_3)} X \to \mathbf{P}^2$ form a uniruled divisor D in F. Alternatively, it can also be defined as the closure of the set of lines in X that intersect $\mathbf{P}(V_3)$ at one point. Clearly, D admits a $\mathbf{P}^1$-fibration over S. In [V1], it was shown that the transcendental part $H^2(F, \mathbf{Z})_{\mathrm{trans}}$ (of discriminant 8 and Hodge type $(1, 19, 1)$) embeds as a sublattice of index two into the primitive cohomology $H^2(S, \mathbf{Z})_{\mathrm{prim}}$ (of discriminant 2). This sublattice is closely related to the Brauer class β induced by the $\mathbf{P}^1$-fibration $D \to S$, so it should be considered as the "primitive cohomology" of the twisted K3 surface (S, β) (see [vG]). For a general X containing a plane, the class β is non-trivial and is related to rationality questions (see [Ha]). Finally, it was proved in [MS1] that for a general X containing a plane, the hyperkähler variety F can be recovered (birationally) as a moduli space of β-twisted sheaves on S.

4.2. **Construction of a K3 surface and a uniruled divisor.** From now on, we let $\sigma \in \bigwedge^3 V_{10}^\vee$ be a general trivector in the divisor $\mathcal{D}^{1,6,10}$, so there is a unique distinguished flag $[V_1 \subset V_6]$ such that $\sigma(V_1, V_6, V_{10}) = 0$. We study the geometry of the Debarre–Voisin variety X_6^σ, which resembles a lot that of a cubic fourfold containing a plane. Notably, we will construct a K3 surface S of degree 6 and a uniruled divisor D in X_6^σ that admits a $\mathbf{P}^1$-fibration over S. In later sections, we will compare the Hodge structures of X_6^σ and of the K3 surface S. The $\mathbf{P}^1$-fibration defines a non-trivial Brauer class $\beta \in \mathrm{Br}(S)$, and we will show that X_6^σ can be recovered as a moduli space of β-twisted sheaves on S (which is proved in a purely Hodge theoretical way).

Let W_7 be a complex vector space of dimension 7. We begin by recalling some properties on $\mathrm{GL}(W_7)$-orbit closures inside $\bigwedge^3 W_7$ that we will need later. Let $Y \subset \bigwedge^3 W_7^\vee$ be the unique $\mathrm{GL}(W_7)$-invariant hypersurface. It can also be characterized as the affine cone over the projective dual variety $\mathrm{Gr}(3, W_7)^*$ embedded in $\mathbf{P}(\bigwedge^3 W_7^\vee)$, which is a hypersurface of degree 7. In other words, the polynomial f defining Y lives inside $\mathrm{Sym}^7(\bigwedge^3 W_7^\vee)^\vee$, which

is usually referred to as the *discriminant* or the *hyperdeterminant*. Equivalently, $\mathbf{C}f$ is the unique one-dimensional $\mathrm{GL}(W_7)$-subrepresentation of $\mathrm{Sym}^7 \bigwedge^3 W_7$. Since all one-dimensional representations of $\mathrm{GL}(W_7)$ are of the form $\det(W_7)^{\otimes i}$ for $i \in \mathbf{Z}$, weight invariance with respect to the torus $\mathbf{C}^* \mathrm{Id} \subset \mathrm{GL}(W_7)$ implies that we have $\mathbf{C}f \simeq \det(W_7)^{\otimes 3} \subset \mathrm{Sym}^7 \bigwedge^3 W_7$. This also means that we can canonically define the discriminant $\operatorname{disc} y$ of each $y \in \bigwedge^3 W_7^\vee$ as an element of $\det(W_7^\vee)^{\otimes 3}$.

Remark 4.4. The orbit closure $Y \subset \bigwedge^3 W_7^\vee$ admits a nice desingularization (a *Kempf collapsing*, see [BFMT1]). Consider the vector bundle

$$\mathcal{W} \coloneqq \bigwedge^2 \mathcal{Q}_4^\vee \wedge W_7^\vee$$

over $\mathrm{Gr}(3, W_7)$, where $\mathcal{Q}_4$ is the tautological quotient bundle. The bundle $\mathcal{W}$ is a subbundle of the trivial bundle $(\bigwedge^3 W_7^\vee) \otimes \mathcal{O}_{\mathrm{Gr}(3,W_7)}$, whose total space is just $\bigwedge^3 W_7^\vee \times \mathrm{Gr}(3, W_7)$. By projecting onto the first factor, we obtain a morphism $\mathrm{Tot}(\mathcal{W}) \to \bigwedge^3 W_7^\vee$; the image of this morphism is exactly the discriminant hypersurface Y, and it is birational onto its image [KW]. This should be interpreted in the following way: for any general element $y \in Y$, there exists a unique subspace $W_3 \subset W_7$ such that $y \in \bigwedge^2 (W_7/W_3)^\vee \wedge W_7^\vee$ or equivalently, $y(W_3, W_3, V_{10}) = 0$.

One can check that this implies that $[W_3]$ is a singular point of the hyperplane section defined by $y \in \bigwedge^3 W_7^\vee$ inside $\mathrm{Gr}(3, W_7)$, thus providing the description of $\mathbf{P}(Y) \subset \mathbf{P}(\bigwedge^3 W_7^\vee)$ as the projective dual of $\mathrm{Gr}(3, W_7)$.

Let us now return to our trivector σ.

Proposition 4.5. *Suppose the trivector $\sigma \in \bigwedge^3 V_{10}^\vee$ is general in the divisor $\mathcal{D}^{1,6,10}$, that is, we have $\sigma(V_1, V_6, V_{10}) = 0$ for a unique flag $V_1 \subset V_6 \subset V_{10}$. Then it defines a smooth K3 surface S of degree 6 inside $\mathrm{Gr}(2, V_{10}/V_6)$, where the polarization is given by the Plücker line bundle.*

Proof. The Grassmannian $\mathrm{Gr}(2, V_{10}/V_6)$ is a 4-dimensional quadric. The K3 surface S will be the intersection of a linear section and a cubic section of this quadric, hence the Plücker line bundle will be of degree 6. For clarity, we denote by $\mathcal{U}_{8/6}$ the tautological subbundle and by $\mathcal{Q}_{10/8}$ the quotient bundle on $\mathrm{Gr}(2, V_{10}/V_6)$ respectively.

The linear section is given by $\sigma(V_1, V_8, V_8) = 0$: since $\sigma(V_1, V_6, V_{10}) = 0$, this is equivalent to the condition $\sigma(V_1, V_8/V_6, V_8/V_6) = 0$, which can be seen as the vanishing of a general section of the line bundle $\bigwedge^2 \mathcal{U}_{8/6}^\vee \simeq \mathcal{O}(1)$. The zero-locus is therefore a 3-dimensional quadric S'.

Now for each $[V_8/V_6] \in S'$, since we have $\sigma(V_1, V_8, V_8) = 0$, the form σ induces an element of $\bigwedge^3 (V_8/V_1)^\vee$. In the relative setting, by letting $\mathcal{W}_7 \coloneqq V_6/V_1 \oplus \mathcal{U}_{8/6}$ where V_6/V_1 is the trivial bundle $(V_6/V_1) \otimes \mathcal{O}_{\mathrm{Gr}(2,V_{10}/V_6)}$, we get a global section σ' of the vector bundle $\bigwedge^3 \mathcal{W}_7^\vee$. So we may define the orbital degeneracy locus

$$S \coloneqq D_Y(\sigma') = \left\{ [V_8/V_6] \in S' \;\middle|\; \sigma'|_{V_8/V_6} \in Y \subset \bigwedge^3 (V_8/V_1)^\vee \simeq \left(\bigwedge^3 \mathcal{W}_7^\vee\right)_{[V_8/V_6]} \right\}.$$

As we have already seen, the hypersurface Y is defined in $\bigwedge^3 W_7^\vee$ by the vanishing of the discriminant. Therefore S is the hypersurface in S' defined by the vanishing of $\operatorname{disc} \sigma'$, which is a section of $\det(\mathcal{W}_7^\vee)^{\otimes 3} \simeq \mathcal{O}_{S'}(3)$. As σ is general, so is σ' among sections of $\bigwedge^3 \mathcal{W}_7^\vee$. Moreover, the hypersurface Y is smooth in codimension 2, so by a Bertini-type theorem for orbital degeneracy loci (see [BFMT1]), the zero-locus S is a smooth surface obtained as the intersection of a quadric and a cubic, that is, a degree-6 K3 surface. □

Remark 4.6. As pointed out in Remark 4.4, any general element $y \in Y$ defines a unique point $[W_3] \in \mathrm{Gr}(3, W_7)$. In the relative setting, this implies that a general point $[V_8/V_6]$ of S defines a 3-dimensional subspace of V_8/V_1, in other words a 4-dimensional subspace V_4 with $V_1 \subset V_4 \subset V_8$ such that

$$\sigma'(V_4/V_1, V_4/V_1, V_8/V_1) = 0 \text{ or equivalently, } \sigma(V_4, V_4, V_8) = 0.$$

It might be useful to keep in mind that the relative setting over S' is formally analogous to the affine setting inside $\bigwedge^3 W_7^\vee$ by replacing W_7 with the vector bundle $\mathcal{W}_7$.

In conclusion, having fixed the flag $V_1 \subset V_6$ and the trivector $\sigma \in \mathcal{D}^{1,6,10}$, the K3 surface S can also be defined as the set

$$S = \left\{ [V_4 \subset V_8] \;\middle|\; \begin{array}{c} V_1 \subset V_4,\ \sigma(V_1, V_8, V_8) = 0, \\ \text{and } \sigma(V_4, V_4, V_8) = 0 \end{array} \right\} \subset \mathrm{Flag}(4, 8, V_{10}). \tag{18}$$

The advantage of this description is that S can now be characterized as the zero-locus of a section of some vector bundle on a flag variety. In particular, we get two more tautological bundles $\mathcal{U}_{4/1} := \mathcal{U}_4/V_1$ and $\mathcal{U}_{8/4} := \mathcal{U}_8/\mathcal{U}_4$ on S induced by the inclusion $S \subset \mathrm{Flag}(4, 8, V_{10})$. By construction, the line bundle $\det(\mathcal{U}_{8/6}^\vee)$ gives the degree-6 polarization $\mathcal{O}_S(1)$ on S. One may verify numerically that $\det(\mathcal{U}_{4/1}^\vee) \simeq \mathcal{O}_S(3)$ and $\det(\mathcal{U}_{8/4}) \simeq \mathcal{O}_S(2)$ so no new polarizations are produced this way (see Section A.4).

In fact, we will see later that the family of polarized K3 surfaces of degree 6 parametrized by $\mathcal{D}_{24}$ is a locally complete family, as a consequence of the study of their Hodge structures. Hence we have $\mathrm{Pic}(S) \simeq \mathbf{Z}$ for a very general member of the family.

Next, we construct a uniruled divisor D in X_6^σ.

Proposition 4.7. *For a general σ in the divisor $\mathcal{D}^{1,6,10}$, the set*

$$D := \{[U_6] \in X_6^\sigma \mid \exists [V_4 \subset V_8] \in S \quad V_4 \subset U_6 \subset V_8\}$$

defines a divisor in X_6^σ which has a smooth conic fibration $\pi\colon D \to S$ over the K3 surface S.

Proof. First we construct the morphism $\pi\colon D \to S$ by showing that for each $[U_6] \in D$, the corresponding $[V_4 \subset V_8] \in S$ is unique. We claim that each $[U_6] \in D$ satisfies $\dim(U_6 \cap V_6) = 4$. Otherwise, suppose that there exists some U_6 with $\dim(U_6 \cap V_6) \geq 5$. Then we may look for a V_3 with $V_1 \subset V_3 \subset U_6 \cap V_6$ such that $\sigma(V_3, V_3, V_{10}) = 0$: this is a codimension-4 condition on $\mathrm{Gr}(2, U_6 \cap V_6/V_1)$ so such a V_3 must exist. For a general σ in $\mathcal{D}^{1,6,10} \smallsetminus \mathcal{D}^{3,3,10}$ this will not happen, so we have $\dim(U_6 \cap V_6) = 4$. We may then recover V_8 as the sum $U_6 + V_6$ and get a morphism $\pi\colon D \to S$.

Now we show that this morphism $\pi\colon D \to S$ is a smooth conic fibration. We first study the fiber $\{[U_6] \in X_6^\sigma \mid V_4 \subset U_6 \subset V_8\}$ above each $[V_4 \subset V_8] \in S$. This fiber can be seen as the locus

$$\{[U_6] \in \mathrm{Gr}(2, V_8/V_4) \mid \sigma|_{U_6} = 0\}.$$

The trivector σ, when restricted to V_8, becomes a section of $(V_4/V_1)^\vee \otimes \bigwedge^2 \mathcal{U}_2^\vee = \mathcal{O}(1)^{\oplus 3}$. So we get three hyperplane sections, whose intersection in $\mathrm{Gr}(2, V_8/V_4)$ is generically a conic.

In the relative setting, the fibers are defined inside the projectivization $\mathbf{P}_S(\mathcal{E})$ of a rank-3 vector bundle $\mathcal{E}$ over S, which is realized as the kernel

$$\mathcal{E} \hookrightarrow \bigwedge^2 \mathcal{U}_{8/4} \xrightarrow{\sigma} \mathcal{U}_{4/1}^\vee \longrightarrow 0.$$

Here the last arrow is surjective for a general σ: otherwise we would get a subspace $U_2 \supset V_1$ such that $\sigma(U_2, V_8, V_8) = 0$; then all the U_1 contained in U_2 will have $\operatorname{rank} \sigma(U_1, -, -) \leq 4$ which does not happen for σ general. The quadratic form q on $\mathcal{E}$ is given by

$$\operatorname{Sym}^2 \mathcal{E} \hookrightarrow \operatorname{Sym}^2 \bigwedge^2 \mathcal{U}_{8/4} \xrightarrow{q} \mathcal{L} := \det \mathcal{U}_{8/4}$$

where it takes value in the line bundle $\mathcal{L}$. We have $\det \mathcal{E} \simeq \mathcal{O}_S(3)$ while $\mathcal{L} \simeq \mathcal{O}_S(2)$. The discriminant locus is defined by a section of the line bundle $(\det \mathcal{E}^\vee)^{\otimes 2} \otimes \mathcal{L}^{\otimes \operatorname{rank} \mathcal{E}} \simeq \mathcal{O}_S$ and is therefore empty. Thus the conic fibration is everywhere smooth. □

We also have an alternative description of D.

Proposition 4.8. *For a general σ in the divisor $\mathcal{D}^{1,6,10}$, let $[V_1 \subset V_6]$ be the distinguished flag. A point $[U_6] \in X_6^\sigma$ is contained in D if and only if U_6 contains V_1. In other words, we have*

$$D = \{[U_6] \in X_6^\sigma \mid U_6 \supset V_1\}.$$

Proof. Since any $[U_6]$ in D contains a subspace $V_4 \supset V_1$, one direction is evident.

Suppose now that U_6 contains V_1 and $\sigma|_{U_6} = 0$. Then U_6 is isotropic with respect to $\sigma(V_1, -, -)$ and is contained inside a maximal isotropic subspace V_8 of dimension eight. Let us consider a point $[V_4] \in \operatorname{Gr}(3, U_6/V_1)$. As $[U_6] \in X_6^\sigma$, any such V_4 satisfies $\sigma(V_4, V_4, U_6) = 0$. Therefore, the condition $\sigma(V_4, V_4, V_8) = 0$ is a codimension-6 condition, and there exists exactly one point $[V_4] \in \operatorname{Gr}(3, U_6/V_1)$ satisfying it since the bundle $(\bigwedge^2 \mathcal{U}_3^\vee) \otimes (V_8/U_6)^\vee$ has top Chern class 1. This tells us that $[V_4 \subset V_8]$ is a point of S as in (18) and therefore $[U_6] \in D$. □

4.3. **Hodge structures.** Denote by $i\colon D \hookrightarrow X_6^\sigma$ the embedding of the divisor D constructed above. By abuse of notation, we denote the class $[D] \in H^2(X_6^\sigma, \mathbf{Z})$ also by D. We first compute the intersection matrix for the classes H and D under the Beauville–Bogomolov–Fujiki form $\mathfrak{q}$. Note that for σ very general in $\mathcal{D}^{1,6,10}$, the algebraic sublattice $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{alg}}$ is generated by H and D over $\mathbf{Q}$.

Lemma 4.9. *The intersection matrix between H and D is*

$$\begin{pmatrix} 22 & 2 \\ 2 & -2 \end{pmatrix}.$$

So H and D span a lattice of discriminant 48 and generate the algebraic sublattice $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{alg}}$ over $\mathbf{Z}$.

Proof. By the adjunction formula and the fact that X_6^σ has trivial canonical bundle, the canonical class K_D of the divisor D is the restriction i^*D. One can then compute explicitly the intersection numbers using Schubert calculus in `Macaulay2` (see Section A.5 for the code):

$$D^4 = 12,\ D^3 \cdot H = -12,\ D^2 \cdot H^2 = -36,\ D \cdot H^3 = 132,\ H^4 = 1452.$$

Then we use the property (1) of the Beauville–Bogomolov–Fujiki form to obtain the desired numbers.

Since the divisors $\mathcal{D}^{1,6,10}$ is mapped to the Heegner divisor $\mathcal{D}_{24}$ by the period map, we may again use the fact that the discriminant of the algebraic lattice is twice that of its orthogonal (*cf.* [DM, Proposition 4.1]) to conclude that the algebraic sublattice is generated by H and D. □

We have the following useful result.

Corollary 4.10. *The class D has divisibility 1, that is, there exists $C \in H^2(X_6^\sigma, \mathbf{Z})$ such that $\mathfrak{q}(C, D) = 1$.*

Note that the class C is not algebraic for very general σ in the family.

Proof. Suppose that D has divisibility 2, then the class $[D/2]$ would have order 2 in the discriminant group $D(\Lambda) := \Lambda^\vee/\Lambda$, which is isomorphic to $\mathbf{Z}/2\mathbf{Z}$. In particular, the class $[D/2]$ would coincide with $[H/2]$. This shows that the class $[(H + D)/2]$ is trivial in $D(\Lambda)$, so $\frac{1}{2}(H + D)$ is integral, which contradicts the fact that $\mathbf{Z}H + \mathbf{Z}D$ is saturated in Λ. □

Now we would like to compare the Hodge structures on $H^2(X_6^\sigma, \mathbf{Z})$ and $H^2(S, \mathbf{Z})$. Consider the diagram

$$\begin{array}{ccc} H^2(X_6^\sigma, \mathbf{Z}) & \xrightarrow{i^*} & H^2(D, \mathbf{Z}) \\ & & \uparrow{\scriptstyle \pi^*} \\ & & H^2(S, \mathbf{Z}). \end{array}$$

The idea is to make the comparison inside $H^2(D, \mathbf{Z})$.

Lemma 4.11. *For σ general in the divisor $\mathcal{D}^{1,6,10}$, there exists a class $\zeta \in H^2(D, \mathbf{Z})$ such that*

$$H^2(D, \mathbf{Z}) = \pi^* H^2(S, \mathbf{Z}) \oplus \mathbf{Z}\zeta.$$

Let $H^2(X_6^\sigma, \mathbf{Z})^{\perp D} \subset H^2(X_6^\sigma, \mathbf{Z})$ denote the orthogonal of D with respect to $\mathfrak{q}$. Then

$$i^*(H^2(X_6^\sigma, \mathbf{Z})^{\perp D}) \subset \pi^* H^2(S, \mathbf{Z}).$$

Proof. As we saw in Proposition 4.7, the natural projection $\pi\colon D \to S$ is a smooth conic fibration over the K3 surface S. Denote by $l \in H_2(D, \mathbf{Z})$ the class of a fiber of π. We have $i^*H \cdot l = 2$ since π is a conic fibration, and $i^*D \cdot l = -2$ since i^*D is the canonical divisor of D.

Consider $i_*l \in H_2(X_6^\sigma, \mathbf{C})$ as the class of a rational curve on X_6^σ which is of type $(3, 3)$. There exists a unique element $y \in H^2(X_6^\sigma, \mathbf{Q})$ such that

$$\forall\, x \in H^2(X_6^\sigma, \mathbf{Z}) \quad \mathfrak{q}(x, y) = x \cdot i_*l.$$

Moreover y must be of type $(1, 1)$ so it is a $\mathbf{Q}$-linear combination of H and D. Since

$$\mathfrak{q}(D, D) = -2 = i^*D \cdot l = D \cdot i_*l,$$
$$\mathfrak{q}(H, D) = 2 = i^*H \cdot l = H \cdot i_*l,$$

we see that $y = D$. By Corollary 4.10, there exists a class $C \in H^2(X_6^\sigma, \mathbf{Z})$ such that $\mathfrak{q}(C, D) = 1$. We have $i^*C \cdot l = C \cdot i_*l = \mathfrak{q}(C, D) = 1$, so the class i^*C restricts to $\mathcal{O}(1)$ on each fiber l of π. By the Leray–Hirsch theorem, the classes 1 and i^*C generate $H^*(D, \mathbf{Z})$ as a $\pi^*H^*(S, \mathbf{Z})$-module, hence we have

$$H^*(D, \mathbf{Z}) = \pi^* H^*(S, \mathbf{Z}) \oplus \pi^* H^*(S, \mathbf{Z})(i^*C),$$

and in particular

$$H^2(D, \mathbf{Z}) = \pi^* H^2(S, \mathbf{Z}) \oplus \mathbf{Z}(i^*C).$$

We may choose i^*C as the class ζ that we want. For each class in $H^2(D, \mathbf{Z})$, its coefficient before i^*C is simply its intersection number with the fiber l.

Any class $x \in H^2(X_6^\sigma, \mathbf{Z})$ with $\mathfrak{q}(x, D) = 0$ must satisfy $i^*x \cdot l = x \cdot i_*l = 0$. This shows that $i^*(H^2(X_6^\sigma, \mathbf{Z})^{\perp D})$ is indeed contained in $\pi^* H^2(S, \mathbf{Z})$. □

Let $\mathfrak{q}_S$ denote the intersection product on S pulled back to $\pi^* H^2(S, \mathbf{Z})$ via π^*. By the previous lemma, it also induces a form on $H^2(X_6^\sigma, \mathbf{Z})^{\perp D}$ via i^* and we can compare it with $\mathfrak{q}$.

Proposition 4.12. *Let σ be general in the divisor $\mathcal{D}^{1,6,10}$. For any $x \in H^2(X_6^\sigma, \mathbf{Z})^{\perp D}$, we have*

$$\mathfrak{q}(x, x) = \mathfrak{q}_S(i^*x, i^*x).$$

As a consequence, the morphism i^* is injective.

Proof. Consider the class $i^*D \in H^2(D, \mathbf{Z})$. By Lemma 4.11, since $i^*D \cdot l = -2$, we have $i^*D = -2\zeta + \pi^*y$ for some $y \in H^2(S, \mathbf{Z})$. So the intersection number $i^*D \cdot i^*x \cdot i^*x$ is equal to $-2\mathfrak{q}_S(i^*x, i^*x)$. On the other hand, we have $i^*D \cdot i^*x \cdot i^*x = \int_{X_6^\sigma} D^2x^2$ which is equal to $\mathfrak{q}(D, D)\mathfrak{q}(x, x) + 2\mathfrak{q}(x, D)^2$. Since $\mathfrak{q}(x, D) = 0$ and $\mathfrak{q}(D, D) = -2$, we get the equality $\mathfrak{q}(x, x) = \mathfrak{q}_S(i^*x, i^*x)$.

This shows that i^* is injective when restricted to $H^2(X_6^\sigma, \mathbf{Z})^{\perp D}$. But any x such that $i^*x = 0$ will satisfy $i^*x \cdot l = 0$ so we have $\mathfrak{q}(x, D) = 0$ and hence $x = 0$. Thus i^* itself is injective. □

We see that the lattice $\big(H^2(X_6^\sigma, \mathbf{Z})^{\perp D}, \mathfrak{q}\big)$ embeds isometrically inside $\big(H^2(S, \mathbf{Z}), \cdot\big)$. It remains to determine the index of the embedding.

Theorem 4.13. *For σ very general in the divisor $\mathcal{D}^{1,6,10}$, there is an embedding of integral Hodge structures*

$$\iota\colon \big(H^2(X_6^\sigma, \mathbf{Z})^{\perp D}, \mathfrak{q}\big) \hookrightarrow \big(H^2(S, \mathbf{Z}), \cdot\big)$$

as a sublattice of index 2. We have

$$\iota(H + D) = 2h,$$

where h is the polarization on S of degree 6. Restricted to the transcendental part, we get

$$\iota\colon \big(H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{trans}}, \mathfrak{q}\big) \hookrightarrow \big(H^2(S, \mathbf{Z})_{\mathrm{prim}}, \cdot\big)$$

again of index 2.

Proof. By Corollary 4.10, the class D is of divisibility 1 in $H^2(X_6^\sigma, \mathbf{Z})$. As its orthogonal, the sublattice $H^2(X_6^\sigma, \mathbf{Z})^{\perp D}$ is of discriminant 4. On the other hand, $H^2(S, \mathbf{Z})$ is unimodular. Hence by comparing discriminants, the first statement follows.

Since the $(1, 1)$ part of $H^2(X_6^\sigma, \mathbf{Z})^{\perp D}$ is generated by the class $H + D$ with square $\mathfrak{q}(H + D, H + D) = 24$, while the negative generator $-(H + D)$ is not effective, it is clear that $i^*(H + D)$ must be equal to $2\pi^*h$, so $\iota(H + D) = 2h$.

Finally, the second embedding follows by looking at the respective orthogonals of these two classes, while the index 2 is again obtained by comparing discriminants. □

It is possible to get a more precise description of the index-2 sublattice. We first define a class A in $H^2(S, \mathbf{Z})$ as follows. Consider the class C as in Corollary 4.10. Since $\mathfrak{q}(H - 2C, D) = 0$, we define A to be the image $\iota(H - 2C) \in H^2(S, \mathbf{Z})$. Notice that, since $\mathfrak{q}(H, H) = 22$ and $\mathfrak{q}(C, C)$ is even,

$$A \cdot A = \mathfrak{q}(H - 2C, H - 2C) \equiv 6 \pmod 8,$$

so A is not divisible by 2.

Proposition 4.14. *For σ very general in the divisor $\mathcal{D}^{1,6,10}$, the lattice $H^2(X_6^\sigma, \mathbf{Z})^{\perp D}$ can be identified via the embedding ι as the sublattice*

$$\Lambda_{\frac{1}{2}A} \coloneqq \big\{u \in H^2(S, \mathbf{Z}) \;\big|\; u \cdot A \in 2\mathbf{Z}\big\},$$

while the sublattice $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{trans}}$ can be identified as the sublattice

$$\Lambda_{\frac{1}{2}A,\mathrm{prim}} \coloneqq \big\{u \in H^2(S, \mathbf{Z})_{\mathrm{prim}} \;\big|\; u \cdot A \in 2\mathbf{Z}\big\}.$$

Proof. For each class $x \in H^2(X_6^\sigma, \mathbf{Z})^{\perp D}$, the intersection number

$$\iota(x) \cdot A = \mathfrak{q}(x, H - 2C) = \mathfrak{q}(x, H) - 2\mathfrak{q}(x, C)$$

is always even, because $\mathrm{div}(H) = 2$. So we get the inclusion in one direction. For the other direction: since the index is 2, it suffices to show that $\Lambda_{\frac{1}{2}A,\mathrm{prim}}$ is a proper sublattice of $H^2(S, \mathbf{Z})_{\mathrm{prim}}$, and the sublattice $\Lambda_{\frac{1}{2}A}$ will then also be proper in $H^2(S, \mathbf{Z})$.

Thus we search for a class $v \in H^2(S, \mathbf{Z})_{\mathrm{prim}}$ with $v \cdot A$ odd. First we claim that $h \cdot A$ is odd: this is equivalent to

$$2h \cdot A = \mathfrak{q}(H + D, H - 2C) = 22 - 2\mathfrak{q}(C, H) \equiv 2 \pmod 4,$$

which follows from the fact that $\mathrm{div}(H) = 2$. Now we can let $h = e_1 + 3f_1$ for (e_1, f_1) a standard basis for a copy of the hyperbolic plane U in $H^2(S, \mathbf{Z})$, and take $v \coloneqq e_1 - 3f_1 \in H^2(S, \mathbf{Z})_{\mathrm{prim}}$. Since $(h + v) \cdot A = 2e_1 \cdot A$ is even, the intersection number $v \cdot A$ is odd. □

We explain in the next section the interpretation of A in terms of a B-field lifting of a Brauer class $\beta \in \mathrm{Br}(S)$.

4.4. **Twisted K3 surface; moduli space of twisted sheaves.** We first recall the notions of Brauer group and B-field lifting. We will only be interested in the case of K3 surfaces. We follow [Hu, Chapter 18] (see also [vG]).

The *Brauer group* of a K3 surface S can be characterized as the cohomology groups

$$\mathrm{Br}(S) \simeq H^2_{\text{ét}}(S, \mathbf{G}_m) \simeq H^2(S, \mathcal{O}_S^*)_{\mathrm{tors}}$$

in the algebraic and analytic context respectively. Since $H^3(S, \mathcal{O}_S) = 0$, the exponential sequence allows us to have another description

$$\mathrm{Br}(S) \simeq \big(H^2(S, \mathbf{Z}) / \mathrm{NS}(S)\big) \otimes (\mathbf{Q}/\mathbf{Z}) \simeq \mathrm{Hom}(\mathrm{NS}(S)^\perp, \mathbf{Q}/\mathbf{Z}).$$

For an element β of $\mathrm{Br}(S)$, a representative $B \in H^2(S, \mathbf{Q})$ is called a *B-field lifting* of β.

Each β of order n in the Brauer group gives a morphism from the transcendental part $\mathrm{NS}(S)^\perp$ (when S is of Picard rank 1, this coincides with $H^2(S, \mathbf{Z})_{\mathrm{prim}}$) to $\mathbf{Q}/\mathbf{Z}$, and the kernel is a sublattice of index n, which in particular does not depend on the choice of the B-field lifting.

In a more geometric setting, each class β gives a *Brauer–Severi variety* $\pi\colon X \to S$, which is a projective fibration that is locally trivial in the étale topology. Equivalently, it is the projectivization $\mathbf{P}(\mathcal{E})$ of some β-twisted vector bundle $\mathcal{E}$ on S. We refer to [HS] for the definitions of β-twisted coherent sheaves as well as the *twisted Chern classes* $c_i^B(E)$ and the *twisted Chern character* $\mathrm{ch}^B(E)$. We only emphasize that the definition of twisted Chern classes depends not just on β but also on the choice of a B-field lifting B.

Back to the situation of Section 4.3, the class $\frac{1}{2}A$ gives a Brauer class β of order 2, and the lattice $\Lambda_{\frac{1}{2}A,\mathrm{prim}}$ is the index-2 sublattice defined by β, as explained above. This is the reason why we adopted the notation $\Lambda_{\frac{1}{2}A}$ instead of Λ_A.

We first find another B-field lifting that is easier to work with.

Lemma 4.15. *There exists another B-field lifting B of the same Brauer class β such that $B \cdot B = B \cdot h = \frac{1}{2}$.*

Proof. Recall that $\frac{1}{2}A \cdot \frac{1}{2}A \equiv \frac{1}{2}A \cdot h \equiv \frac{1}{2} \in \mathbf{Q}/\mathbf{Z}$. We try to find B by adding integral classes to $\frac{1}{2}A$. Denote by (e_1, f_1) and (e_2, f_2) standard bases of two copies of U inside $H^2(S, \mathbf{Z})$. Suppose that $h = e_1 + 3f_1$ and $A = ae_1 + bf_1 + ce_2 + df_2$. By adding e_1 and f_1, we can reduce the coefficients of e_1 and f_1 in $\frac{1}{2}A$ to 0 or $\frac{1}{2}$. The condition $\frac{1}{2}A \cdot h \equiv \frac{1}{2} \in \mathbf{Q}/\mathbf{Z}$ shows that we have either $\frac{1}{2}e_1 + 0f_1$ or $0e_1 + \frac{1}{2}f_1$. We may do the same for e_2 and f_2, and the condition $\frac{1}{2}A \cdot \frac{1}{2}A \equiv \frac{1}{2} \in \mathbf{Q}/\mathbf{Z}$ shows that we always get $\frac{1}{2}e_2 + \frac{1}{2}f_2$. In the two possible situations, we may choose B to be equal to either $\frac{1}{2}e_1 - f_1 + \frac{1}{2}e_2 + \frac{3}{2}f_2$ or $0e_1 + \frac{1}{2}f_1 + \frac{1}{2}e_2 + \frac{1}{2}f_2$. □

So the identifications in Proposition 4.14 become

$$\iota\colon H^2(X_6^\sigma, \mathbf{Z})^{\perp D} \xrightarrow{\sim} \Lambda_B \coloneqq \left\{u \in H^2(S, \mathbf{Z}) \;\middle|\; u \cdot B \in \mathbf{Z}\right\}$$

and

$$\iota\colon H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{trans}} \xrightarrow{\sim} \Lambda_{B,\mathrm{prim}} \coloneqq \left\{u \in H^2(S, \mathbf{Z})_{\mathrm{prim}} \;\middle|\; u \cdot B \in \mathbf{Z}\right\}.$$

We remark that when we write $B = \frac{1}{2}A + u$, with $u \in H^2(S, \mathbf{Z})$, we have

$$2 = 2B \cdot 2B = A \cdot A + 4A \cdot u + 4u \cdot u.$$

Since $A \cdot A \equiv 6 \pmod 8$ while $u \cdot u$ is always even, we see that $A \cdot u$ is odd. In particular $u \neq 0$, so $B \neq \frac{1}{2}A$. Also, the intersection number $A \cdot B$ is even. This gives the following lemma that we will need shortly.

Lemma 4.16. *Since $2B \cdot B = 1 \in \mathbf{Z}$, or equivalently $2B \in \Lambda_B$, we may set $2B = \iota(x_0)$ for some $x_0 \in H^2(X_6^\sigma, \mathbf{Z})^{\perp D}$. The class $D - x_0$ is divisible by 2 in $H^2(X_6^\sigma, \mathbf{Z})$.*

Proof. It is equivalent to show that $D - x_0 + 2C$ is divisible by 2. We have

$$\iota(D - x_0 + 2C) = \iota((D + H) - x_0 - (H - 2C)) = 2(h - B - \tfrac{1}{2}A).$$

Now the class $h - B - \frac{1}{2}A$ is integral and the intersection number $(h - B - \frac{1}{2}A) \cdot B = -\frac{1}{2}A \cdot B$ is also integral since $A \cdot B$ is even. So $h - B - \frac{1}{2}A$ lies in Λ_B and thus comes from a class in $H^2(X_6^\sigma, \mathbf{Z})^{\perp D}$, and $D - x_0 + 2C$ is indeed divisible by 2. □

We now show that, for σ very general in $\mathcal{D}^{1,6,10}$, the projective bundle $\pi\colon D \to S$ is precisely the Brauer–Severi variety for the Brauer class β, which means that the Brauer class that we obtained Hodge-theoretically actually comes from geometry. In particular, for σ very general, the bundle $\pi\colon D \to S$ has non-trivial Brauer class.

Proposition 4.17. *For σ very general in the divisor $\mathcal{D}^{1,6,10}$, the projective bundle $\pi\colon D \to S$ is the Brauer–Severi variety for the Brauer class β.*

Proof. Recall that $K_{D/S} = K_D \otimes \pi^* K_S = i^* D$, since X_6^σ and S both have trivial canonical bundles.

Denote by β' the Brauer class defined by $\pi\colon D \to S$. We may suppose that $D = \mathbf{P}(\mathcal{E})$ with $\mathcal{E}$ a β'-twisted vector bundle on S of rank 2. The relative $\mathcal{O}(1)$ is a $(-\pi^*\beta')$-twisted line bundle on D, and its square $\mathcal{O}(2)$ is a non-twisted line bundle. Moreover, $\mathcal{O}(2) = \omega_{D/S}^\vee \otimes \pi^*\mathcal{L}$ for some line bundle $\mathcal{L}$ on S. We may set $c_1(\mathcal{L}) = kh$ for $k \in \mathbf{Z}$, since for very general σ in the divisor, the K3 surface S has Picard number 1. So the first Chern class $c_1(\mathcal{O}(2))$ is equal to $-i^*D + k(\pi^*h)$.

Consider a B-field lifting B' of β'. We compute the twisted Chern class

$$2c_1^{-\pi^*B'}(\mathcal{O}(1)) = c_1^{-2(\pi^*B')}(\mathcal{O}(2)) = c_1(\mathcal{O}(2)) - 2(\pi^*B') = -i^*D + k(\pi^*h) - 2(\pi^*B').$$

The class $c_1^{-\pi^*B'}(\mathcal{O}(1))$ is necessarily integral, so the last term in the equality is divisible by 2. On the other hand, $i^*(D - x_0) = i^*D - 2\pi^*B$ is also divisible by 2 by Lemma 4.16. Thus the class $\frac{k}{2}h - B' - B$ is integral in $H^2(S, \mathbf{Z})$, which shows that $\beta' = \beta^{-1} = \beta$. □

Finally, we consider the moduli space of twisted sheaves on S, following [MS1, Section 3]. We recall the definition of the twisted Mukai lattice. Consider the map

$$\begin{aligned} \eta_B \colon H^2(S, \mathbf{C}) &\longrightarrow H^*(S, \mathbf{C}) \\ u &\longmapsto (0, u, u \cdot B). \end{aligned}$$

The *twisted Mukai lattice* $\widetilde{H}(S, B, \mathbf{Z})$ is given by the usual Mukai lattice $H^*(S, \mathbf{Z}) := H^0(S, \mathbf{Z}) \oplus H^2(S, \mathbf{Z}) \oplus H^4(S, \mathbf{Z})$, equipped with the Hodge structure given by η_B, that is, its $(2,0)$-part is the image of $H^{2,0}(S)$ under η_B. We recall that the *Mukai pairing* is given by

$$-\chi\big((r_1, c_1, s_1), (r_2, c_2, s_2)\big) := c_1 \cdot c_2 - r_1 s_2 - r_2 s_1.$$

The algebraic part $\mathrm{Pic}(S, B) \subset \widetilde{H}(S, B, \mathbf{Z})$ is generated by the classes $(2, 2B, 0)$, $(0, h, 0)$, and $(0, 0, 1)$. For a β-twisted sheaf E, its *twisted Mukai vector* is defined as

$$v^B(E) := \mathrm{ch}^B(E) \cdot \sqrt{\mathrm{td}(S)},$$

where ch^B is the twisted Chern character. Let $M = M(S, v, B)$ be the moduli space of stable β-twisted sheaves $\mathcal{E}$ on S with Mukai vector $v^B(\mathcal{E}) = v := (2, 2B, 0)$. Here $v^2 = 2$, so by the general theory for moduli of twisted sheaves on K3 surfaces, M is a hyperkähler fourfold, with $H^2(M, \mathbf{Z})$ isometric to $v^\perp \subset \widetilde{H}(S, B, \mathbf{Z})$, the orthogonal of v in the twisted Mukai lattice.

Proposition 4.18. *For σ very general in the divisor $\mathcal{D}^{1,6,10}$, there exists a Hodge isometry between $H^2(X_6^\sigma, \mathbf{Z})$ and $H^2(M, \mathbf{Z})$.*

Proof. The lattice $H^2(M, \mathbf{Z})_{\mathrm{trans}} = \mathrm{Pic}(S, B)^\perp$ consists of elements (a, u, b) satisfying

$$\begin{aligned} -\chi\big((a, u, b), (2, 2B, 0)\big) &= 2u \cdot B - 2b = 0 \\ -\chi\big((a, u, b), (0, h, 0)\big) &= u \cdot h = 0 \\ -\chi\big((a, u, b), (0, 0, 1)\big) &= -a = 0 \end{aligned}$$

which are precisely those in the image of $\Lambda_{B,\mathrm{prim}}$ by the map $\eta_B \colon u \mapsto (0, u, u \cdot B)$. Since we have identified $\Lambda_{B,\mathrm{prim}}$ with $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{trans}}$ via the isometry ι, we can thus define

$$\begin{aligned} \phi \colon H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{trans}} &\longrightarrow H^2(M, \mathbf{Z})_{\mathrm{trans}} \\ x &\longmapsto \eta_B(\iota(x)) = (0, \iota(x), \iota(x) \cdot B) \end{aligned}$$

which is a Hodge isometry onto its image.

For the algebraic part, we may set $\phi(H) = (-2, 2h - 2B, 0)$ and $\phi(D) = (2, 2B, 1)$. It suffices now to extend ϕ to the full lattice. First we notice that the sum $H^2(M, \mathbf{Z})_{\mathrm{trans}} + (\mathbf{Z}\phi(H) + \mathbf{Z}\phi(D))$ is of index 24 in $H^2(M, \mathbf{Z})$. We claim that the quotient is $\mathbf{Z}/24\mathbf{Z}$ by finding some primitive class in the sum which becomes divisible by 24 in the full lattice. Consider the integral class $h - 12B \in H^2(S, \mathbf{Z})_{\mathrm{prim}}$. Its intersection number with B is not integral, so it is not in $\Lambda_{B,\mathrm{prim}}$. Thus $u_1 := 2h - 24B$ is primitive in $\Lambda_{B,\mathrm{prim}}$. We have

$\eta_B(u_1) = (0, 2h - 24B, -11)$ so $\eta_B(u_1) - \phi(H) + 11\phi(D) = (24, 0, 0)$ is indeed divisible by 24 in the full lattice.

Denote by $x_1 \in H^2(X_6^\sigma, \mathbf{Z})_{\text{trans}}$ the preimage $\iota^{-1}(u_1)$ of u_1. Since $\iota(H + D) = 2h$ and $\iota(x_0) = 2B$, we have

$$x_1 = \iota^{-1}(u_1) = \iota^{-1}(2h - 24B) = (H + D) - 12x_0.$$

By Lemma 4.16, the class

$$x_1 - H + 11D = 12(D - x_0)$$

is also divisible by 24. So we may extend the map ϕ to the full lattice by mapping $\frac{1}{2}(D - x_0)$ to $(1, 0, 0)$. □

Now that we have defined a Hodge isometry between the second cohomologies of the hyperkähler fourfolds X_6^σ and M, we may take advantage of the powerful machinery of the Torelli theorem. Notably, the Hodge structure on the second cohomology of a hyperkähler manifold determines its birational model. Furthermore, the birational models are parametrized by the chambers in the chamber decomposition of the movable cone: each chamber corresponds to the ample cone of one birational model (see for example [D, Theorem 3.21]). Moreover, as we are in the type-K3$^{[2]}$ case, the chamber decomposition has an explicit numerical description. Using these facts, we obtain the following result.

Theorem 4.19. *A very general Debarre–Voisin fourfold X_6^σ in the family $\mathcal{C}_{24}$ is isomorphic to the moduli space $M = M(S, v, B)$ of twisted sheaves with Mukai vector $(2, 2B, 0)$ on the twisted K3 surface (S, β).*

Proof. By the Torelli theorem, the existence of a Hodge isometry between second cohomologies shows that X_6^σ and M are birationally isomorphic. Moreover, the number of birational models is given by the number of chambers contained in the movable cone. These chambers are cut out by hyperplanes of the type $\kappa^\perp$, where $\kappa \in \mathrm{Pic}(X_6^\sigma)$ is a primitive class of square -10 and divisibility 2 (see [D, Theorem 3.16]). We show that for a very general X_6^σ with Picard group generated by H and D, there is no such class κ: we may write $\kappa = aH + bD$ and get the equation $22a^2 + 4ab - 2b^2 = -10$. By reduction modulo 5, we verify that this equation has no integral solutions, so no such κ exists. Thus we may conclude that there is only one birational model, and in particular $X_6^\sigma \simeq M$. □

Appendix A. Macaulay2 code

For completeness, we provide the various `Macaulay2` [GS] code used throughout the paper, using mainly the package `Schubert2`.

A.1. **Degree of $\mathrm{SL}(V_{10})$-invariant divisors.** The following code computes the degree of the three $\mathrm{SL}(V_{10})$-invariant divisors in $\mathbf{P}\left(\bigwedge^3 V_{10}^\vee\right)$. The degree 640 of the discriminant is a well-known result.

```
needsPackage "Schubert2";
-- divisor D_{3,3,10}: dual of Grassmannian Gr(3,10)
G = flagBundle{3,7}; (U,Q) = bundles G;
d1 = chern dual(exteriorPower_3 U+exteriorPower_2 U*Q);
-- divisor D^{1,6,10}
G = flagBundle{1,5,4}; (U1,U61,Q) = bundles G;
d2 = chern dual(U1*exteriorPower_2 U61+U1*U61*Q);
-- divisor D^{4,7,7}
```

```
G = flagBundle{4,3,3}; (U4,U74,Q) = bundles G;
d3 = chern dual(exteriorPower_3 U4+exteriorPower_2 U4*U74+U4*exteriorPower_2 U74);
<< (d1,d2,d3) / integral << endl; -- (640, 990, 5500)
```

A.2. **Lemma 3.15.** The following code produces several algebraic classes on X_1^σ realized as relative Schubert classes. Note that the line where the variety X_1^σ is constructed takes quite a while to compute.

```
needsPackage "Schubert2";
(U1,Q) = bundles projectiveBundle 9;
G = flagBundle({3,6},Q); (U41,Q) = bundles G;
time X = sectionZeroLocus dual(U1*exteriorPower_2 U41+U1*U41*Q); -- long time
h = chern_1 dual(U1*OO_X);
s = (X/G)^* schubertCycle_{2,0,0} G;
t = (X/G)^* schubertCycle_{4,0,0} G;
p = 1/3*(6*h^2-s)*h; -- the class pi of a Palatini 3-fold
assert((p*p, p*h^3) / integral == (4,7)); -- verify the intersection numbers
assert(t == 17/3*h^4-7*p*h);
assert(integral((6*h^2-s)*1/3*h^4) == 7);
```

A.3. **Proposition 4.2.** For a general σ in $\mathcal{D}^{1,6,10}$, the following code computes the self-intersection number of a Grassmannian $\mathrm{Gr}(2,7)$ contained in X_3^σ, by considering the top Chern class of the normal bundle using the two normal sequences.

```
needsPackage "Schubert2";
(U,Q) = bundles flagBundle{2,5};
N = dual(U+1)*(Q+2)-det Q-dual U*Q;
<< "c_10(N)=" << integral chern_10 N << endl; -- c_10(N)=2
```

A.4. **Remark 4.6.** The following code verifies that the two tautological bundles on the K3 surface S satisfy $\det(\mathcal{U}_{4/1}^\vee) \simeq \mathcal{O}_S(3)$ and $\det(\mathcal{U}_{8/4}) \simeq \mathcal{O}_S(2)$.

```
needsPackage "Schubert2";
(U,Q) = bundles flagBundle{2,2}; -- first choose V8/V6 in V10/V6
(U3,U4) = bundles flagBundle({3,4},U+5); -- then choose V4/V1 in V8/V1
S = sectionZeroLocus dual(det U+det U3+exteriorPower_2 U3*U4);
h = chern_1(dual U*OO_S);
U41 = U3*OO_S;
U84 = U4*OO_S;
assert(chern_1 dual U41==3*h and chern_1 U84==2*h); -- verify the Chern classes
```

A.5. **Lemma 4.9.** The following code computes the intersection numbers between the classes H and D on X_6^σ.

```
needsPackage "Schubert2";
(U1,Q1) = bundles flagBundle{3,2}; -- first choose U4/V1 in V6/V1
(U2,Q2) = bundles flagBundle({2,4},Q1+4); -- then choose U6/U4 in V10/U4
D = sectionZeroLocus dual((1+U1)*det U2+det U1+exteriorPower_2 U1*U2);
h = chern_1(dual(1+U1+U2)*OO_D);
d = chern_1 cotangentBundle D;
(U,Q) = bundles flagBundle{6,4};
X = sectionZeroLocus dual exteriorPower_3 U;
```

```
h' = chern_1 OO_X(1);
<< (d^3,d^2*h,d*h^2,h^3,h'^4) / integral << endl; -- (12, -12, -36, 132, 1452)
```

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