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Propagation of singularities for subelliptic wave equations

Cyril Letrouit*[†]

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Abstract

We revisit the paper [Mel86] by R. Melrose, providing a full proof of the main theorem on propagation of singularities for subelliptic wave equations, and linking this result with sub-Riemannian geometry. This result asserts that singularities of subelliptic wave equations only propagate along null-bicharacteristics and abnormal extremal lifts of singular curve.

As a new consequence, for $x \neq y$ and denoting by K_G the wave kernel, we obtain that the singular support of the distribution $t \mapsto K_G(t, x, y)$ is included in the set of lengths of the normal geodesics joining x and y , at least up to the time equal to the minimal length of a singular curve joining x and y .

Contents

1	Introduction	1
2	The cones Γ_m	4
3	A positive commutator	9
4	Proof of Theorem 1	12
5	Proof of Theorem 2	18
6	A consequence for wave equations with sub-Laplacians	22
A	Technical tools	24
B	Further properties of the cones Γ_m	25

1 Introduction

In microlocal analysis, the celebrated propagation of singularities theorem describes the wave-front set $WF(u)$ of a distributional solution u to a partial (or pseudo) differential equation $Pu = f$ in terms of the principal symbol p of P : it says that if p is real and homogeneous, then $WF(u) \setminus WF(f) \subset p^{-1}(0)$, and that if additionally the characteristics are simple ($p = 0 \Rightarrow dp \neq 0$ outside the null section), then $WF(u) \setminus WF(f)$ is invariant under the bicharacteristic flow induced by the Hamiltonian vector field of p .

This result was first proved in [DH72, Theorem 6.1.1] and [Hör71b, Proposition 3.5.1]. However, it leaves open the case where the characteristics of P are not simple. In a very

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short and impressive paper [Mel86], Melrose sketched the proof of an analogous propagation of singularities result for the wave operator $P = D_t^2 - A$ when A is a self-adjoint non-negative real second-order differential operator which is only subelliptic. Such operators P are typical examples for which there exist double characteristic points.

Despite the potential scope of this result, we did not find in the literature any other paper quoting it. The proof provided in [Mel86] is very sketchy, and we thought it would deserve to be written in full details. This is what we do in the present note, before presenting in the last section a new application of this result. Since the publication of [Mel86] in 1986, the development of sub-Riemannian geometry (the geometry associated to subelliptic operators) has brought some tools and concepts which we use here to shed a new light on this result: for example, we explain that singular curves and their abnormal extremal lifts, which are central objects in control theory and played a key role in the discovery of so-called abnormal minimizers (see [Mon94], [Mon02]), appear naturally in [Mel86], although it is not written explicitly.

For the sake of coherence, we borrow nearly all notations to [Mel86]. A is a self-adjoint non-negative real second-order differential operator on a smooth compact manifold X without boundary:

$$\forall u \in C^\infty(X), \quad (Au, u) = (u, Au) \geq 0 \quad (1)$$

with

$$(u, v) = \int_X u(x) \overline{v(x)} d\nu, \quad (2)$$

where ν is some positive C^∞ density. The associated norm is denoted by $\|\cdot\|$.

We also assume that A is subelliptic, in the following sense: there exist a (Riemannian) Laplacian Δ on X and $c, s > 0$ such that

$$\forall u \in C^\infty(X), \quad \|(-\Delta)^{s/2} u\|^2 \leq c((Au, u) + \|u\|^2). \quad (3)$$

Finally, we assume that A has vanishing sub-principal symbol.¹

Example 4. An important class of examples of such operators A is given by sub-Laplacians (or Hörmander's sums of squares, see [RS76] or [LL21]), that is, operators of the form $A = \sum_{i=1}^K Y_i^* Y_i$ for some smooth vector fields Y_i on X (here, Y_i^* denotes the adjoint of Y_i for the scalar product (2)) satisfying Hörmander's condition: the Lie algebra generated by Y_1, \dots, Y_K is equal to the whole tangent bundle TX .

The assumption (1) implies that A has a self-adjoint extension with the domain

$$\mathcal{D}(A) = \{u \in \mathcal{D}'(X); Au \in L^2(X)\}.$$

By the spectral theorem, for any $t \in \mathbb{R}$, the self-adjoint operator

$$G(t) = A^{-1/2} \sin(tA^{1/2})$$

is a well-defined operator bounded on $L^2(X)$, in fact it maps $L^2(X)$ into $\mathcal{D}(A^{1/2})$. Together with the self-adjoint operator $G'(t) = \cos(tA^{1/2})$, this allows to solve the Cauchy problem for the wave operator (here $D_t = \frac{1}{i} \partial_t$)

$$\begin{aligned} (D_t^2 - A)u &= Pu = 0 & \text{in } \mathbb{R} \times X, \\ u &= u_0, \quad \partial_t u = u_1 & \text{at } t = 0 \end{aligned} \quad (5)$$

by

$$u(t, x) = G'(t)u_0 + G(t)u_1.$$

¹Since X is endowed with a smooth density ν , the sub-principal symbol makes sense, see Appendix A.2. Note also that the assumption of vanishing sub-principal symbol is not made in [Mel86], but it simplifies the presentation and it is valid in applications.

For $(u_0, u_1) \in \mathcal{D}(A^{1/2}) \times L^2(X)$, we have $u \in C^0(\mathbb{R}; \mathcal{D}(A^{1/2})) \cap C^1(\mathbb{R}; L^2(X))$.

For $f \in \mathcal{D}'(Y)$ a distribution on a manifold Y (equal to X , $\mathbb{R} \times X$ or $\mathbb{R} \times X \times X$ in the sequel), we denote by $WF(f)$ the usual Hörmander wave-front set (see [Hör71a]); in particular, $WF(f) \subset T^*Y \setminus 0$.

The first main result of [Mel86] is the following (the terminology “null-ray” is explained below):

Theorem 1. *Let $t \mapsto u(t)$ be a solution of (5). For any $t > 0$, if $(x, \xi) \in WF(u(0))$ then there exists $(y, \eta) \in WF(u(-t)) \cup WF(\partial_t u(-t))$ such that (y, η) and (x, ξ) can be joined by a null-ray of length t .*

The second main result of [Mel86], which we state here only in the context of sub-Laplacians², concerns the Schwartz kernel K_G of G , i.e., the distribution $K_G \in \mathcal{D}'(\mathbb{R} \times X \times X)$ defined by

$$\forall u \in C^\infty(X), \quad G(t)u(x) = \int_X K_G(t, x, y)u(y)dy. \quad (6)$$

Theorem 2. [Mel86, Theorem 1.8] *Assume that A is a sub-Laplacian (see Example 4). Then*

$$WF(K_G) \subset \{(t, x, y, \tau, \xi, -\eta) \in T^*(\mathbb{R} \times X \times X) \setminus 0; \\ \text{there is a null-ray from } (0, \tau, y, \eta) \text{ to } (t, \tau, x, \xi)\}. \quad (7)$$

Comments on Theorems 1 and 2. The null-rays which appear in the statements of Theorems 1 and 2 are generalizations of the usual null-bicharacteristics (i.e., integral curves of the Hamiltonian vector field H_p of the principal symbol p of P , contained in the characteristic set $p^{-1}(0)$). Their definition will be given in Section 2: they are paths tangent to a family of convex cones Γ_m introduced in Definition 3. For example, at $m \in T^*(\mathbb{R} \times X)$ which is not in the double characteristic set $p = dp = 0$, Γ_m is simply $\mathbb{R}^+ \cdot H_p(m)$ (or $\mathbb{R}^- \cdot H_p(m)$). In the double characteristic set $\Sigma_{(2)} = \{p = dp = 0\} \subset M$, their definition is more involved, but readers familiar with control theory will recognize that null-rays contained in $\Sigma_{(2)}$ are abnormal extremal lifts of singular curves (as in Pontryagin’s maximum principle). That is, they are integral curves of $\ker(\omega_{\Sigma_{(2)}})$ where $\omega_{\Sigma_{(2)}} = \iota_{\Sigma_{(2)}}^* \omega$ is the pullback of the canonical symplectic form ω on $T^*(\mathbb{R} \times X)$ by the canonical injection $\iota : \Sigma_{(2)} \rightarrow M$.

As a particular case of Theorems 1 and 2, if A is elliptic, then we recover Hörmander’s result [Hör71b, Proposition 3.5.1] already mentioned above (see also [Hör07, Theorem 8.3.1 and Theorem 23.2.9] and [Ler11, Theorem 1.2.23]). In case A has only double characteristics on a symplectic submanifold it was obtained in [Mel84] (in codimension 2) and by B. and R. Lascar [Las82], [LL82] in the general case, using constructions of parametrices (and not positive commutator estimates as in [Mel86]). It is explained in Remark 22 how Theorem 2 implies these results.

Also, in [Mel86], two other results are proved, namely the finite speed of propagation for P and an estimate on the heat kernel $\exp(-tA)$, but it is not our purpose to discuss here these other results, whose proofs are written in details in [Mel86].

Organization of the paper. As said above, the goal of this note is firstly to provide a fully detailed proof of Theorems 1 and 2, and secondly to derive a new consequence on the singular support of the Schwartz kernel K_G of the wave operator.

In Section 2, we define the convex cones Γ_m generalizing bicharacteristics and give an explicit formula (15) for them, then prove their semi-continuity with respect to m , and finally introduce “time functions”, which are by definition non-increasing along these cones. In this section, there is no operator, we work at a purely “classical” level.

The proof of Theorems 1 and 2 is based on a positive commutator argument: the idea, which dates back at least to [Hör71b] (see also [Ivr19, Chapter I.2]), is to derive an *energy*

²This assumption is not made in [Mel86].

inequality from the computation of a quantity of the form $\text{Im}(Pu, Lu)$, where L is some well-chosen (pseudodifferential) operator. In Section 3, we compute this quantity for $L = \text{Op}(\Phi)D_t$ where Φ is a time function, we write it under the form $\frac{1}{2}(Cu, u)$ for an explicit second-order operator C which, up to remainder terms, has non-positive symbol.

In Section 4, we derive from this computation the sought energy inequality, which in turn implies Theorem 1. This proof requires to construct specific time functions and to use the powerful Fefferman-Phong inequality [FP78].

In Section 5, we prove Theorem 2: the main idea is to see K_G itself as the solution of a subelliptic wave equation.

Whether Theorem 2 implies a trace formula in the spirit of [DG75] for subelliptic wave operators is an open question: due to the particular role of the section $\tau = 0$, it is not clear whether the trace $K_G(t, x, x)$ is a well-defined distribution. However, in Section 6, for $x \neq y$, we are able to infer from Theorem 2 that the singular support of the distribution $t \mapsto K_G(t, x, y)$ is included in the set of lengths of the normal geodesics joining x and y , at least up to the time equal to the minimal length of a singular curve joining x and y .

The reader will find in Appendix A the sign conventions for symplectic geometry that we use throughout this note, and a short reminder on pseudodifferential operators. Finally, in Appendices B.1 and B.2, we prove two additional results concerning the inner semi-continuity of the cones Γ_m .

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2 The cones Γ_m

At double characteristic points where $dp = 0$, the Hamiltonian vector field H_p vanishes, and the usual propagation of singularities result [DH72, Theorem 6.1.1] does not provide any information. In [Mel86], Melrose defines convex cones Γ_m which replace the usual propagation cone $\mathbb{R}^+ \cdot H_p$ at these points, and which will indicate the directions in which singularities of the subelliptic wave equation (5) may propagate.

2.1 First definition of the cones Γ_m

In this section, we introduce several notations, and we define the cones Γ_m .

We consider $a \in C^\infty(T^*X)$ satisfying

$$a(x, \xi) \geq 0, \quad a(x, r\xi) = r^2 a(x, \xi), \quad r > 0 \quad (8)$$

in canonical coordinates (x, ξ) . Also we consider

$$p = \tau^2 - a \in C^\infty(M), \quad \text{where } M = T^*(\mathbb{R} \times X) \setminus 0.$$

Of course, a and p will be in the end the principal symbols of the operators A and P introduced in Section 1, but for the moment we work at a purely classical level and forget about operators.

We set

$$M_+ = \{m \in M, p(m) \geq 0, \tau \geq 0\}, \quad M_- = \{m \in M, p(m) \geq 0, \tau \leq 0\};$$

in particular, $M_+ \cup M_- = \{p \geq 0\}$. Let

$$\Sigma = \{m \in M; p(m) = 0, \tau \geq 0\}.$$

Note that $\Sigma \subset \{\tau \geq 0\}$; the next few definitions also hold only at points where $\tau \geq 0$.

For $m \in M_+$, we consider the set

$$\mathcal{H}_m = \mathbb{R}^+ \cdot H_p(m) \subset T_m M,$$

where H_p is the Hamiltonian vector field of p verifying $\omega(H_p, Z) = -dp(Z)$ for any smooth vector field Z (recall that ω is the canonical symplectic form on the cotangent bundle M).

If m verifies $dp(m) = 0$ and $p(m) \geq 0$ (or equivalently $\tau = a = 0$, i.e., m is a double characteristic point), $\mathcal{H}_m = \{0\}$. We therefore extend the notion of ‘‘bicharacteristic direction’’ at m . This will be done first for $m \in M_+$, then also for $m \in M_-$, but never for $m \in \{p < 0\}$: the cones Γ_m are not defined for points $m \in \{p < 0\}$.

Let

$$\Sigma_{(2)} = \{m \in M, \tau = a = 0\} \subset \Sigma.$$

Note that since $a \geq 0$, there holds $\Sigma_{(2)} = M_+ \cap M_-$. At $m \in \Sigma_{(2)}$, we have $\tau = a = da = p = dp = 0$ (this follows from the positivity (8)) and the Hessian of a is well-defined: it is a quadratic form on $T_m M$. We denote by a_m the half of this Hessian, and by $p_m = (d\tau)^2 - a_m$ the half of the Hessian of p . For $m \in \Sigma_{(2)}$, we set

$$\Lambda_m = \{w \in T_m M; d\tau(w) \geq 0, p_m(w) \geq 0\} \quad (9)$$

and, still for $m \in \Sigma_{(2)}$,

$$\Gamma_m := \{v \in T_m M; \omega(v, w) \leq 0 \quad \forall w \in \Lambda_m\}. \quad (10)$$

If $m \in M_+ \setminus \Sigma_{(2)}$, we set

$$\Gamma_m = \mathcal{H}_m. \quad (11)$$

In particular, the cones Γ_m are defined also at points m outside Σ , i.e. for which $p(m) \neq 0$. Note also that the relation (11) says that the cones Γ_m are only *half-tangents*.

In order to extend the definition of the cones Γ_m to M_- , we want this extension to be consistent with the previous definition at points in $M_+ \cap M_- = \Sigma_{(2)}$. We observe that M_- is the image of M_+ under the involution sending τ to $-\tau$. For $(t, \tau, \alpha) \in M_-$, we set

$$\Gamma_m = \Gamma_{m'} \quad \text{where} \quad m' = (t, -\tau, \alpha) \in M_+.$$

It is clear that at points of $M_+ \cap M_- = \Sigma_{(2)}$, the two definitions of Γ_m coincide. With this definition in M_- , note that for $m \in M_- \setminus \Sigma_{(2)}$, there is a sign change:

$$\Gamma_m = -\mathcal{H}_m. \quad (12)$$

In summary, the formulas (10), (11) and (12) define Γ_m at any point $m \in M_+ \cup M_-$, with different definitions for $m \in \Sigma_{(2)}$, $m \in M_+ \setminus \Sigma_{(2)}$ and $m \in M_- \setminus \Sigma_{(2)}$. The cones Γ_m are not defined for $m \notin M_+ \cup M_-$. For any $m \in M_+ \cup M_-$, the cone Γ_m is closed and convex.

Definition 3. A *forward-pointing ray* for p is a Lipschitz curve $\gamma : I \rightarrow M_+$ defined on some interval $I \subset \mathbb{R}$ with (set-valued) derivative $\gamma'(s) \subset \Gamma_{\gamma(s)}$ for all $s \in I$. Such a ray is *forward-null* if $\gamma(s) \in \Sigma$ for any $s \in I$. We define *backward-pointing rays* similarly, with γ valued in M_- , and *backward-null rays*, with γ valued in $\{m \in M; p(m) = 0, \tau \leq 0\}$.

Under the terminology ‘‘ray’’, we mean either a forward-pointing or a backward-pointing ray; under the terminology ‘‘null-ray’’, we mean either a forward-null or a backward-null ray.

In particular null-rays live in $\{p = 0\}$. In Definition 3, the fact that the curve γ is only Lipschitz explains why its derivative can be set-valued.

Remark 4. In the inclusion (7), the null-ray mentioned in the right-hand side is forward if $\tau \geq 0$ and backward if $\tau \leq 0$ (and both forward and backward if $\tau = 0$).

2.2 Formulas for the cones Γ_m

In this section, we derive a formula for the cones Γ_m when $m \in \Sigma_{(2)}$. It is more explicit than (10) and we will give in Section 6 an application of this formula.

It relies on the computation of the polar of a cone defined by a non-negative quadratic form:

Proposition 5. *Let S be a non-negative quadratic form on a real vector space Y , and let $\Theta = (\ker(S))^\perp \subset Y^*$ where \perp is understood in the duality sense. Let $\Lambda = \{\xi = (\xi_0, \eta) \in \mathbb{R} \times Y; \xi_0 \geq S(\eta)^{\frac{1}{2}}\}$ and $\Lambda^0 = \{\xi' \in (\mathbb{R} \times Y)^*; \forall \xi \in \Lambda, \xi'(\xi) \leq 0\}$. Then*

$$\Lambda^0 = \{\xi' = (\xi'_0, \eta') \in (\mathbb{R} \times Y)^*; \eta' \in \Theta \text{ and } -\xi'_0 \geq (S^*(\eta'))^{\frac{1}{2}}\}. \quad (13)$$

where \mathbb{R}^* is identified with \mathbb{R} and

$$S^*(\eta') = \sup_{\eta \notin \ker(S)} \frac{\eta'(\eta)^2}{S(\eta)}. \quad (14)$$

Proof. Let $\xi' = (\xi'_0, \eta') \in (\mathbb{R} \times Y)^*$ such that $\eta' \in \Theta$ and $-\xi'_0 \geq (S^*(\eta'))^{\frac{1}{2}}$, we seek to prove that $\xi' \in \Lambda^0$. Let $\xi = (\xi_0, \eta) \in \Lambda$. In particular, $\xi_0 \geq (S(\eta))^{\frac{1}{2}}$. We have

$$\xi'(\xi) = \xi'_0(\xi_0) + \eta'(\eta) \leq -(S^*(\eta'))^{\frac{1}{2}}(S(\eta))^{\frac{1}{2}} + \eta'(\eta) \leq 0$$

hence $\xi' \in \Lambda^0$, which proves one inclusion.

Conversely, to prove that Λ^0 is included in the expression (13), we first note that if $\eta' \notin \Theta$, then $(\xi'_0, \eta') \notin \Lambda^0$ for any $\xi'_0 \in \mathbb{R}^*$. Indeed, if $\eta' \notin \Theta$, there exists $\eta \in Y$ such that $S(\eta) = 0$ and $\eta'(\eta) > 0$. Thus, considering $\xi = (0, \eta)$, which is in Λ by assumption, we get $\xi'(\xi) = \eta'(\eta) > 0$ for any $\xi'_0 \in \mathbb{R}^*$ and $\xi' = (\xi'_0, \eta')$, proving that $\xi' \notin \Lambda^0$. Now, if $\xi' = (\xi'_0, \eta') \in \Lambda^0$ with $\eta' \in \Theta$, we take $\xi_n = (\xi_{0n}, \eta_n)$ with $\eta_n \notin \ker(S)$ so that $\eta'(\eta_n)^2/S(\eta_n) \rightarrow S^*(\eta')$, and $\eta'(\eta_n) \geq 0$ and $\xi_{0n} = S(\eta_n)^{\frac{1}{2}}$. Then $\xi_n \in \Lambda$. Therefore, $\xi'(\xi_n) \leq 0$, which implies that $-\xi'_0 \geq (S^*(\eta'))^{\frac{1}{2}}$. This proves the result. \square

Applying the previous proposition to $S = a_m$ yields a different definition of the cones Γ_m . First, Λ_m , which has been defined in (9), can be written as

$$\Lambda_m = \{w \in T_m M; d\tau(w) \geq (a_m(w))^{\frac{1}{2}}\},$$

Since the definition of Λ_m does not involve dt , we have $v(\partial_t) = 0$ for any $v \in \Lambda_m^0$. Now, using the notation a_m^* to denote (14) when $S = a_m$, Proposition 5 yields that

$$\Lambda_m^0 = \mathbb{R}^+(-d\tau + B_0),$$

$$B_0 = \{b_0 \in (\ker(a_m))^\perp, a_m^*(b_0) \leq 1\}.$$

The duality \perp is computed with respect to the space $\ker(a_m) \subset T(T^*X)$, i.e., $b_0 \in T^*(T^*X)$.

Comparing the definition of Λ_m^0 as the polar cone of Λ_m and the definition (10) of Γ_m , we see that Γ_m is exactly the image of Λ_m^0 through the canonical isomorphism $\omega(v, \cdot) \mapsto v$ between T_m^*M and $T_m M$. Thus,

$$\Gamma_m = \mathbb{R}^+(\partial_t + B), \quad (15)$$

$$B = \{b \in \ker(a_m)^{\perp_{\omega_X}}, a_m^*(\mathcal{I}(b)) \leq 1\}.$$

Here, \perp_{ω_X} designates the symplectic orthogonal with respect to the canonical symplectic form ω_X on T^*X and $\mathcal{I} : b \mapsto \omega_X(b, \cdot)$ is the canonical isomorphism between $T(T^*X)$ and $T^*(T^*X)$.

In case $A = \sum_{i=1}^K Y_i^* Y_i$ is a sum of squares, the expression $a_m^*(\mathcal{I}(b))$ which appears in (15) can be written in a much simpler form involving the sub-Riemannian metric associated

to the vector fields Y_i , see Lemma 21. For more on formula (15), which plays a key role in the sequel, see also Section 6.3.

Without assuming that A is a sum of squares for the moment, we can already write (15) differently, and for that we introduce the “fundamental matrix” F (see [Hör07, Section 21.5]) defined as follows:

$$\forall Y, Z \in T_m(T^*X), \quad \omega_X(Y, FZ) = a_m(Y, Z). \quad (16)$$

Then, $\omega_X(FY, Z) = -\omega_X(Y, FZ)$. Note that there is a slight abuse of notations here since $T_m(T^*X)$ stands for $T_{\pi_2(m)}(T^*X)$ where $\pi_2 : M \rightarrow T^*X$ is the canonical projection on the second factor (recall that $M = T^*(\mathbb{R} \times X) \setminus \{0\}$).

We now prove the following formula³:

$$\Gamma_m = \mathbb{R}^+(\partial_t + B), \quad B = \text{cxhl} \left\{ \frac{FZ}{a_m(Z)^{\frac{1}{2}}}, Z \notin \ker(a_m) \right\}.$$

Thanks to (15), it is sufficient to prove that if $b \in \ker(a_m)^{\perp_{\omega_X}}$ with $a_m^*(\mathcal{I}(b)) = 1$, then $b = FZ/a_m(Z)^{\frac{1}{2}}$ for some $Z \notin \ker(a_m)$. We set $b_0 = -\mathcal{I}(b) \in \ker(a_m)^\perp$. By Lax-Milgram’s theorem applied to the bilinear form a_m which is continuous and coercive on $T_m(T^*X)/\ker(a_m)$ and b_0 which is a linear form on this space, we get the existence of Z such that $b_0 = a_m(Z, \cdot)$. Using that $a_m^*(b_0) = 1$, we obtain $a_m(Z) = 1$, hence $b_0 = a_m(Z, \cdot)/a_m(Z)^{\frac{1}{2}}$. It follows that $b = -\mathcal{I}^{-1}(b_0) = FZ/a_m(Z)^{\frac{1}{2}}$.

Fixing a norm $|\cdot|$ on TM , the expression (15) implies that near any point $m \in \{p \geq 0\}$, there is a (locally) uniform constant $c > 0$ such that

$$v \in \Gamma_m \Rightarrow v = T\partial_t + v', \quad |v'| \leq cT \quad (17)$$

where v' is tangent to T^*X . Thus, if $\gamma : I \rightarrow M_+$ is a forward-pointing ray (thus a Lipschitz curve) defined for $s \in I$, (17) implies that $dt/ds \geq c'|d\gamma/ds|$, hence $d\gamma/dt = (d\gamma/ds)/(dt/ds)$ is well-defined (possibly set-valued), i.e., γ can be parametrized by t .

Finally, we define the *length* of a ray $\gamma : s \in [s_0, s_1] \rightarrow M_+$ by $\ell(\gamma) := |t(s_1) - t(s_0)|$.

Remark 6. *Thanks to the above parametrization and with a slight abuse in the terminology, we say that there is a null-ray of length $|T|$ from (y, η) to (x, ξ) if there exists a null-ray (in the sense of Definition 3) parametrized by t which joins $(0, \tau, y, \eta)$ to (T, τ, x, ξ) , where τ verifies $\tau^2 = a(y, \eta) = a(x, \xi)$.*

2.3 Inner semi-continuity of the cones Γ_m

Using the formula (15), we can prove a continuity property for the cones Γ_m , inspired by the arguments of [Mel86, Lemma 2.4].

Lemma 7. *Let $a \in C^\infty(T^*X)$ satisfying (8). The assignment $m \mapsto \Gamma_m$ is inner semi-continuous on $M_+ \cup M_- = \{p \geq 0\}$. In other words,*

$$\forall m_j \rightarrow m \quad (m_j \in M_+ \cup M_-), \quad \forall v_j \in \Gamma_{m_j} \text{ such that } v_j \rightarrow v \in T_m M, \text{ there holds } v \in \Gamma_m.$$

Proof of Lemma 7. The assignments $\Sigma_{(2)} \ni m \mapsto \Gamma_m$ and $M_+ \cup M_- \setminus \Sigma_{(2)} \ni m \mapsto \Gamma_m$ are clearly continuous thanks to formula (10) (resp. (11) and (12)). Therefore, we restrict to the case where $m \in \Sigma_{(2)}$ and $m_j \in M_+ \cup M_- \setminus \Sigma_{(2)}$.

The cone Γ_{m_j} at $m_j = (t_j, \tau_j, x_j, \xi_j)$ is given by the positive multiples of the Hamiltonian vector field of p :

$$\Gamma_{m_j} = \mathbb{R}^+[2\tau_j\partial_t - H_a(m_j)] \quad (18)$$

³This is formula (2.6) in [Mel86].

where $H_a(m_j)$ is the Hamiltonian vector field of a at m_j . Dividing by $2\tau_j$, we rewrite it as

$$\Gamma_{m_j} = \mathbb{R}^+ \left(\partial_t - \frac{1}{2} \frac{a(m_j)^{\frac{1}{2}}}{\tau_j} \frac{H_a(m_j)}{a(m_j)^{\frac{1}{2}}} \right) \quad (19)$$

We assume without loss of generality that $\tau_j > 0$, the case $\tau_j < 0$ being similar.

Since $m_j \in \{p \geq 0\}$, we know that $\tau_j \geq (a(m_j))^{\frac{1}{2}}$ (the equality would correspond to null-bicharacteristics) thus the first fraction is bounded. For the second fraction, we consider its image $da(m_j)/a(m_j)^{\frac{1}{2}}$ through the isomorphism \mathcal{I} between the tangent and the cotangent bundle given by the canonical symplectic form on T^*X .

In the sequel, we work in a chart near m . If $m_j - m$ accumulates in a direction where a vanishes at order exactly $k \geq 2$, then a Taylor development yields

$$|H_a(m_j)| = O(\|m_j - m\|_M^k) = o(\|m_j - m\|_M^{(k+1)/2}) = o(a(m_j)^{1/2}) = o(\tau_j)$$

where $\|\cdot\|_M$ is the Euclidean norm on a chart of M near m . Hence, using (18), we obtain that the only limiting direction of the Γ_{m_j} is $\mathbb{R}^+ \cdot \partial_t$, which is contained in Γ_m .

Otherwise, we use the following elementary result.

Lemma 8. *If $\frac{m_j - m}{\|m_j - m\|_M}$ has no accumulation point in $\ker(a_m)$, then for any $v \in T_m M$, there holds $\frac{1}{2} \frac{da(m_j)(v)}{a(m_j)^{1/2}} = \frac{a_m(m_j - m, v)}{a_m(m_j - m)^{1/2}} + o(1)$.*

Proof. Recall that a_m is half the Hessian of a at m . In a chart, we have $da(m_j)(v) = 2a_m(m_j - m, v) + o(\|m_j - m\|_M)$ and $a(m_j) = a_m(m_j - m) + o(\|m_j - m\|_M^2)$, hence the result. \square

In view of (19) and (15), the inner semi-continuity at m is equivalent to proving that

$$a_m^* \left(\frac{1}{2} \frac{a(m_j)^{\frac{1}{2}}}{\tau_j} \frac{da(m_j)}{a(m_j)^{\frac{1}{2}}} \right) \leq 1 + o(1). \quad (20)$$

Using the fact that $a(m_j) \leq \tau_j^2$ and Lemma 8, for any $v \in T_m M \setminus \ker(a_m)$, there holds

$$\frac{1}{a_m(v)} \left(\frac{1}{2} \frac{a(m_j)^{\frac{1}{2}}}{\tau_j} \frac{da(m_j)(v)}{a(m_j)^{\frac{1}{2}}} \right)^2 \leq \frac{a_m(m_j - m, v)^2}{a_m(v)a_m(m_j - m)} + o(1) \leq 1 + o(1)$$

by Cauchy-Schwarz, hence (20) holds, which concludes the proof of Lemma 7.

Remark 9. *We only proved the inner semi-continuity in m , since these arguments do not seem to be sufficiently robust to prove the inner semi-continuity in a . However, we prove in Appendix B.1 that if we make some additional assumptions, the cones Γ_m are also inner semi-continuous with respect to a (and this second proof requires no formula for the cones, just convexity arguments).*

Remark 10. *Let us explain briefly the intuition behind the semi-continuity stated in Lemma 7. Recall that the cones Γ_m generalize bicharacteristic directions at points where $\tau = a = da = p = dp = 0$. To define the cones Γ_m at these points, following formulas (9) and (10), we have first considered directions where p grows (since $p = dp = 0$, we consider the (half) Hessian p_m), yielding Λ_m , and then Γ_m has been defined as the (symplectic) polar cone of Λ_m . This is exactly parallel to a procedure which yields bicharacteristic directions in the non-degenerate case: the directions along which p grows, verifying $dp(v) \geq 0$, form a cone, and it is not difficult to check that its (symplectic) polar consists of a single direction given by the Hamiltonian vector field of p . This unified vision of the cones Γ_m (in the sense that they are obtained in a unified way, no matter whether $m \in \Sigma_{(2)}$ or not) is not used directly in the proof of Lemma 7, but it is at the heart of the proof of Proposition 29.*

Remark 11. *We prove in Appendix B.2 that for any $m \in \Sigma_{(2)}$, the cone Γ_m is exactly given by all limits of the cones Γ_{m_j} for $m_j \notin \Sigma_{(2)}$ tending to m .*

2.4 Time functions

In this section, we introduce time functions which are one of the key ingredients of the proof of Theorems 2 and 1.

Definition 12. A C^∞ function ϕ near $\bar{m} \in \{p \geq 0\} \subset M$ is a time function near \bar{m} if in some neighborhood N of \bar{m} ,

$$\phi \text{ is non-increasing along } \Gamma_m, m \in N \cap \{p \geq 0\}.$$

In particular, ϕ is non-increasing along the Hamiltonian vector field H_p in M_+ but non-decreasing along H_p in M_- (due to (12)).

Note that outside $\{p \geq 0\}$, there is no constraint on the values of ϕ . The following result asserts the existence of (local) time functions.

Proposition 13. Let $\bar{m} \in \{p \geq 0\}$. Then there exists a (non-constant) time function near \bar{m} , and moreover we can choose it independent of τ and homogeneous of any fixed degree.

Proof. We choose $\phi'_t < 0$ and ϕ independent of x, ξ , which is clearly possible even with ϕ 0-homogeneous and independent of τ .

If $\bar{m} \notin \Sigma_{(2)}$, then $m \notin \Sigma_{(2)}$ for m close to \bar{m} . Therefore, we want to check that ϕ is weakly decreasing along H_p when $\tau \geq 0$, and weakly increasing along H_p when $\tau \leq 0$ (because of the sign conventions (11) and (12)). This is the case: if $\tau > 0$ in a small neighborhood of m , then $H_p = 2\tau\phi'_t \leq 0$; and if $\tau < 0$ in a small neighborhood of m , then $H_p = 2\tau\phi'_t \geq 0$.

Let us now consider the case $\bar{m} \in \Sigma_{(2)}$. Firstly, for $m \notin \Sigma_{(2)}$ near \bar{m} , we have $d\phi(H_p) = 2\tau\phi'_t$ is ≤ 0 if $m \in M_+ \setminus \Sigma_{(2)}$ and ≥ 0 if $m \in M_- \setminus \Sigma_{(2)}$. Secondly, for $m \in \Sigma_{(2)}$ near \bar{m} , we have the inequality $d\phi(v) = dt(v)\phi'_t \leq 0$ for any v such that $dt(v) \geq 0$, which is the case for $v \in \Gamma_m$. In any case, ϕ is non-increasing along Γ_m . \square

3 A positive commutator

The proof of Theorems 1 and 2 is based on a “positive commutator” technique, also known as “multiplier” or “energy” method in the literature. The idea is to derive an inequality from the computation of a quantity of the form $\text{Im}(Pu, Lu)$ where L is some well-chosen (pseudodifferential) operator. In the present note, the operator L is related to the time functions introduced in Definition 12.

In the sequel, we use polyhomogeneous symbols, denoted by S_{phg}^m , and the Weyl quantization, denoted by $\text{Op} : S_{\text{phg}}^m \rightarrow \Psi_{\text{phg}}^m$ (see Appendix A.2). For example, we consider the operator $D_t = \frac{1}{i}\partial_t = \text{Op}(\tau)$ (of order 1). The operator $A \in \Psi_{\text{phg}}^2$ has principal symbol $a \in C^\infty(T^*X)$ satisfying (8), and $P = D_t^2 - A$ has principal symbol $p = \tau^2 - a$.

Also, $\Phi(t, x, \xi)$ designates a smooth *real-valued* function on M , homogeneous of degree $\alpha \in \mathbb{R}$ in ξ , compactly supported on the base $\mathbb{R} \times X$, and independent of τ . In Section 4, we will take Φ to be a time function. By the properties of the Weyl quantization, $\text{Op}(\Phi)$ is a compactly supported selfadjoint (with respect to ν) pseudodifferential operator of order α .

As indicated above, our goal in the next section will be to compute C defined by⁴

$$\text{Im}(Pu, \text{Op}(\Phi)D_t u) := \frac{1}{2}(Cu, u), \quad (21)$$

since this will allow us to derive the inequality (51) which is the main ingredient in the proof of Theorems 1 and 2.

⁴In [Mel86], C is explicitly defined as $\text{Im}(\text{Op}(\Phi)D_t u, Pu) := (Cu, u)$; however the formulas (6.1) and (6.2) in [Mel86] are not coherent with this definition, but they are correct if we take the definition (21) for C .

3.1 The operator C

Our goal in this section is to compute C defined by (21). We have

$$\operatorname{Im}(Pu, \operatorname{Op}(\Phi)D_t u) := I_1 - I_2 \quad (22)$$

with

$$I_1 = \operatorname{Im}(D_t^2 u, \operatorname{Op}(\Phi)D_t u) \quad \text{and} \quad I_2 = \operatorname{Im}(Au, \operatorname{Op}(\Phi)D_t u).$$

Noting that

$$[D_t, \operatorname{Op}(\Phi)] = \operatorname{Op}\left(\frac{1}{i}\Phi'_t\right)$$

where $\Phi'_t = \partial_t \Phi$ (see [Zwo12, Theorem 4.6]), we have for I_1 :

$$\begin{aligned} I_1 &= \frac{1}{2i} \left((D_t^2 u, \operatorname{Op}(\Phi)D_t u) - (\operatorname{Op}(\Phi)D_t u, D_t^2 u) \right) \\ &= \frac{1}{2i} \left((D_t \operatorname{Op}(\Phi)D_t^2 u, u) - (D_t^2 \operatorname{Op}(\Phi)D_t u, u) \right) \\ &= -\frac{1}{2i} (D_t [D_t, \operatorname{Op}(\Phi)]D_t u, u) \\ &= -\frac{1}{2i} (D_t \frac{1}{i} \operatorname{Op}(\Phi'_t)D_t u, u) \\ &= \frac{1}{2} (D_t \operatorname{Op}(\Phi'_t)D_t u, u) \end{aligned} \quad (23)$$

Then, we write $\operatorname{Op}(\Phi)D_t = S + iT$ where

$$\begin{aligned} S &= \frac{1}{2} (\operatorname{Op}(\Phi)D_t + D_t \operatorname{Op}(\Phi)) \\ T &= \frac{1}{2i} (\operatorname{Op}(\Phi)D_t - D_t \operatorname{Op}(\Phi)) = \frac{1}{2} \operatorname{Op}(\Phi'_t). \end{aligned} \quad (24)$$

Using that A , S and T are selfadjoint, we compute I_2 :

$$\begin{aligned} I_2 &= \operatorname{Im}(Au, (S + iT)u) = \operatorname{Im}((S - iT)Au, u) = \frac{1}{2i} ([S, A]u, u) - \operatorname{Re}((TAu, u)) \\ &= \frac{1}{2i} ([S, A]u, u) - \frac{1}{2} ((TA + AT)u, u). \end{aligned} \quad (25)$$

First,

$$[S, A] = \frac{1}{2} ([\operatorname{Op}(\Phi), A]D_t + D_t [\operatorname{Op}(\Phi), A]). \quad (26)$$

All in all, combining (22), (23), (24), (25) and (26), we find that C in (21) is given by

$$C = D_t \operatorname{Op}(\Phi'_t)D_t - \frac{i}{2} ([A, \operatorname{Op}(\Phi)]D_t + D_t [A, \operatorname{Op}(\Phi)]) + \frac{1}{2} (A \operatorname{Op}(\Phi'_t) + \operatorname{Op}(\Phi'_t)A). \quad (27)$$

Note that C is of order $2 + \alpha$, although we could have expected order $3 + \alpha$ by looking too quickly at (21).

3.2 The principal and subprincipal symbols of C

In this section, we compute the operator C modulo a remainder term in Ψ_{phg}^α . All symbols and pseudodifferential operators used in the computations are polyhomogeneous (see Appendix A.2); we denote by $\sigma_p(C)$ the principal symbol of C . We use the Weyl quantization in the variables $y = (t, x)$, $\eta = (\tau, \xi)$, hence we have for any $b \in S_{\text{phg}}^m$ and $c \in S_{\text{phg}}^{m'}$:

$$\operatorname{Op}(b)\operatorname{Op}(c) - \operatorname{Op}(bc + \frac{1}{2i}\{b, c\}) \in \Psi_{\text{phg}}^{m+m'-2} \quad (28)$$

and

$$[\text{Op}(b), \text{Op}(c)] - \text{Op}\left(\frac{1}{i}\{b, c\}\right) \in \Psi_{\text{phg}}^{m+m'-3}. \quad (29)$$

Note that in (29), the remainder is in $\Psi_{\text{phg}}^{m+m'-3}$, and not only in $\Psi_{\text{phg}}^{m+m'-2}$ (see [Hör07, Theorem 18.5.4], [Zwo12, Theorem 4.12]). Finally, we recall that $\Phi(t, x, \xi)$ is homogeneous in ξ of degree α .

Now, we compute each of the terms in (27) modulo Ψ_{phg}^α . We prove the following formulas:

$$\frac{1}{2}(A\text{Op}(\Phi'_t) + \text{Op}(\Phi'_t)A) = \text{Op}(a\Phi'_t) \pmod{\Psi_{\text{phg}}^\alpha} \quad (30)$$

$$D_t\text{Op}(\Phi'_t)D_t = \text{Op}(\tau^2\Phi'_t) \pmod{\Psi_{\text{phg}}^\alpha} \quad (31)$$

$$\frac{i}{2}([A, \text{Op}(\Phi)]D_t + D_t[A, \text{Op}(\Phi)]) = \text{Op}(\tau\{a, \Phi\}) \pmod{\Psi_{\text{phg}}^\alpha} \quad (32)$$

Firstly, (30) follows from the fact that $A = \text{Op}(a) \pmod{\Psi_{\text{phg}}^0}$ (since the subprincipal symbol of a vanishes) and from (28) applied once with $b = a$, $c = \Phi'_t$, and another time with $b = \Phi'_t$ and $c = a$.

Secondly, $\text{Op}(\Phi'_t)D_t = \text{Op}(\Phi'_t)\text{Op}(\tau) = \text{Op}(\Phi'_t\tau + \frac{1}{2i}\{\Phi'_t, \tau\}) + \Psi_{\text{phg}}^{\alpha-1}$ thanks to (28). Hence, using again (28), we get

$$\begin{aligned} D_t\text{Op}(\Phi'_t)D_t &= \text{Op}(\tau)\text{Op}(\Phi'_t\tau + \frac{1}{2i}\{\Phi'_t, \tau\}) \pmod{\Psi_{\text{phg}}^\alpha} \\ &= \text{Op}(\tau^2\Phi'_t + \frac{\tau}{2i}\{\Phi'_t, \tau\} + \frac{1}{2i}\{\tau, \Phi'_t\tau\}) \pmod{\Psi_{\text{phg}}^\alpha} \end{aligned}$$

which proves (31).

Thirdly, thanks to $A = \text{Op}(a) \pmod{\Psi_{\text{phg}}^0}$ and (29), we have

$$[A, \text{Op}(\Phi)] = \text{Op}\left(\frac{1}{i}\{a, \Phi\}\right) \pmod{\Psi_{\text{phg}}^{\alpha-1}}$$

(note that the remainder is in Ψ_{phg}^{-1} , not in Ψ_{phg}^0). Using (28), we get

$$[A, \text{Op}(\Phi)]D_t + D_t[A, \text{Op}(\Phi)] = \text{Op}\left(\frac{2\tau}{i}\{a, \Phi\}\right) \pmod{\Psi_{\text{phg}}^\alpha}$$

which proves (32).

In particular, we get the principal symbol

$$\sigma_2(C) = \tau^2\Phi'_t - \tau H_a\Phi + \Phi'_t a.$$

Using $p = \tau^2 - a$, we can write it differently:

$$\begin{aligned} \sigma_p(C) &= \tau^2\Phi'_t - \tau\{\tau^2 - p, \Phi\} + \Phi'_t a \\ &= \tau^2\Phi'_t - \tau\{\tau^2, \Phi\} + \tau H_p\Phi + \Phi'_t a \\ &= \tau^2\Phi'_t - 2\tau^2\Phi'_t + \tau H_p\Phi + \Phi'_t a \\ &= \tau H_p\Phi - \Phi'_t p. \end{aligned} \quad (33)$$

Moreover, the formulas (30), (31) and (32) imply that the subprincipal symbol of C vanishes:

$$\sigma_{\text{sub}}(C) = 0. \quad (34)$$

4 Proof of Theorem 1

The goal of this section is to prove Theorem 1. For $V \subset T^*X$ and $t \in \mathbb{R}$, we set

$$\mathcal{S}^{-t}(V) = \{(-t, y, \eta) \in \mathbb{R} \times T^*X, \text{ there exist } (x, \xi) \in V, \tau \in \mathbb{R} \text{ and a ray} \quad (35)$$

$$\text{from } (-t, \tau, y, \eta) \text{ to } (0, \tau, x, \xi)\}.$$

Also, when we replace the upper index $-t$ in the above definitions by an interval $J \subset \mathbb{R}$, this means that we allow $-t$ to vary in J . Take care that the above notation (35) refers to rays, and not null-rays.

With the above notations, Theorem 1 can be reformulated as follows: for any $t > 0$ and any $(x_0, \xi_0) \in WF(u(0))$, there exists $(y_0, \eta_0) \in WF(u(-t)) \cup WF(\partial_t u(-t))$ such that $(-t, y_0, \eta_0) \in \mathcal{S}^{-t}(\{(x_0, \xi_0)\})$ and one of the rays from (y_0, η_0) to (x_0, ξ_0) is null.

First reduction of the problem. If $a(x_0, \xi_0) \neq 0$, then Theorem 1 follows from the usual propagation of singularities theorem [DH72, Theorem 6.1.1] and the fact that $\Gamma_m = \mathbb{R}^\pm \cdot H_p(m)$ for $m \notin \Sigma_{(2)}$. Therefore, in the sequel we assume that $a(x_0, \xi_0) = 0$.

Also, note that, to prove Theorem 1, it is sufficient to find $T > 0$ independent of (x, ξ) (and possibly small) such that the result holds for any $t \in (0, T)$.

Idea of the proof of Theorem 1. To show Theorem 1, we will prove for $T > 0$ sufficiently small an inequality of the form

$$\|\text{Op}(\Psi_0)u\|_{H^s}^2 \leq c(\|\text{Op}(\Psi_0)u\|_{L^2}^2 + \|\text{Op}(\Psi_1)u\|_{L^2}^2) + \text{Remainder terms} \quad (36)$$

where Ψ_0 and Ψ_1 are functions of t, x, ξ such that

- the function Ψ_0 is supported near $t \in [-T, 0]$ and the function Ψ_1 near $t = -T$;
- on their respective supports in t , the operators $\text{Op}(\Psi_0)$ and $\text{Op}(\Psi_1)$ microlocalize respectively near (x_0, ξ_0) and $\mathcal{S}^{-T}(\{(x_0, \xi_0)\})$.

Then, assuming that u is smooth on the support of Ψ_1 , we deduce by applying (36) for different functions Ψ_0 with different degrees of homogeneity in ξ that u is smooth on the support of Ψ_0 .

The inequality (36), written more precisely as (51) below, will be proved by constructing a time function $\Phi(t, x, \xi)$ such that $\Phi'_t = \Psi_1^2 - \Psi_0^2$, and then by applying the Fefferman-Phong inequality to the operator C given by (27) (for this Φ).

Reduction to $X \subset \mathbb{R}^d$ Let us show that it is sufficient to prove Theorem 1 in the case $X \subset \mathbb{R}^d$. Note first that it is sufficient to prove Theorem 1 “locally”, i.e., for sufficiently short times and in a neighborhood of a fixed point $x \in X$, since null-rays stay close from their departure points for short times (this follows from (11), (12), (15)). Then, working in a coordinate chart $\psi : \Omega \rightarrow \mathbb{R}^d$ where Ω is a neighborhood of x , the differential operator A is pushed forward into a differential operator \tilde{A} on \mathbb{R}^d which is also real, second-order, self-adjoint, non-negative and subelliptic. Moreover, we can lift ψ to a symplectic mapping $\psi_{\text{lift}} : (x, \xi) \mapsto (\psi(x), ((d_x \psi(x))^{-1})^T \xi)$. Through the differential of ψ_{lift} , the cones Γ_m (computed with $a = \sigma_P(A)$, in X) are sent to the same cones, computed this time with $\tilde{a} = \sigma_P(\tilde{A})$ in \mathbb{R}^d . This follows from the “symplectic” definition of the cones in Section 2.1 and the fact that $\sigma_P(\tilde{A})$ is the pushforward of $\sigma_P(A)$. Hence, ψ_{lift} maps also null-rays to null-rays. To sum up, if we prove the Theorem for subsets of \mathbb{R}^d , then pulling back the situation to X proves Theorem 1 in full generality.

In the sequel, we assume $X = \Omega \subset \mathbb{R}^d$.

4.1 Construction of the time function

As explained in the introduction of this section, we construct a time function $\Phi(t, x, \xi)$ which verifies several properties. Some time functions are also constructed in the classical proofs of Hörmander's propagation of singularities theorem [Hör71b, Proposition 3.5.1], but in the present context of subelliptic wave equations, the construction is more involved since the cones Γ_m along which time functions should be non-increasing contain much more than a single direction (compare (11) with (15)). The following lemma summarizes the properties that the time functions we need thereafter should satisfy.

Lemma 14. *Let $(x_0, \xi_0) \in T^*X \setminus 0$ and $V \subset V'$ be sufficiently small open neighborhoods of (x_0, ξ_0) such that $\bar{V} \subset V'$. There exist $T > 0$ and $\delta_1 \ll T$ such that for any $0 \leq \delta_0 \leq \delta_1$ and any $\alpha \in \mathbb{R}$, there exists a smooth function $\Phi(t, x, \xi)$ with the following properties:*

- (1) *it is compactly supported in t, x ;*
- (2) *it is homogeneous of degree α in ξ ;*
- (3) *it is independent of τ ;*
- (4) *there exists $\delta > 0$ such that at any point of M where $p \geq -2\delta a$, there holds $\tau H_p \Phi \leq 0$.*
- (5) *its derivative in t can be written $\Phi'_t = \Psi_1^2 - \Psi_0^2$ with Ψ_0 and Ψ_1 homogeneous of degree $\alpha/2$ in ξ ;*
- (6) *$\Psi_0 = 0$ outside $\mathcal{S}^{(-T, \frac{\delta_0}{2})}(V')$ and $\Psi_1 = 0$ outside $\mathcal{S}^{(-T - \frac{\delta_0}{2}, -T + \frac{\delta_0}{2})}(V')$;*
- (7) *$\Psi_0 > 0$ on $\mathcal{S}^{(-T + \frac{\delta_0}{2}, 0)}(V)$;*
- (8) *Φ is a time function on $\mathcal{S}^{(-T + \frac{\delta_0}{2}, \frac{\delta_0}{2})}(V)$.*

All of the above properties of Φ will be used in Sections 4.2 and 4.4 to prove Theorem 1. The rest of Section 4.1 is devoted to the proof of Lemma 14. The figures may be helpful to follow the explanations.

We fix $(x_0, \xi_0) \in T^*X \setminus 0$. As said in the introduction of Section 4, we assume that $a(x_0, \xi_0) = 0$, and we set $\bar{m} = (0, 0, x_0, \xi_0) \in \Sigma_{(2)}$ where the first two coordinates correspond to the variables t, τ . For m near \bar{m} , the cone $-\Gamma_m$ is the cone with base point m and containing the opposite of the directions of Γ_m .

We are looking for a τ -independent time function; since any ray lives in a slice $\tau = \text{const.}$, we first construct Φ in the slice $\tau = 0$, and then we extend Φ to any τ so that it does not depend on τ . If we start from a time function in $\{\tau = 0\}$, then its extension is also a time function: indeed, the image of a ray contained in $\{\tau \neq 0, a = 0\}$ under the map $\tau \mapsto 0$ is also a ray, this follows from the fact that $\mathbb{R}^+ \partial_t \subset \Gamma_m$ for any $m \in \Sigma_{(2)}$ (see (15)). Thus, the property of being non-increasing along Γ_m is preserved under this extension process.

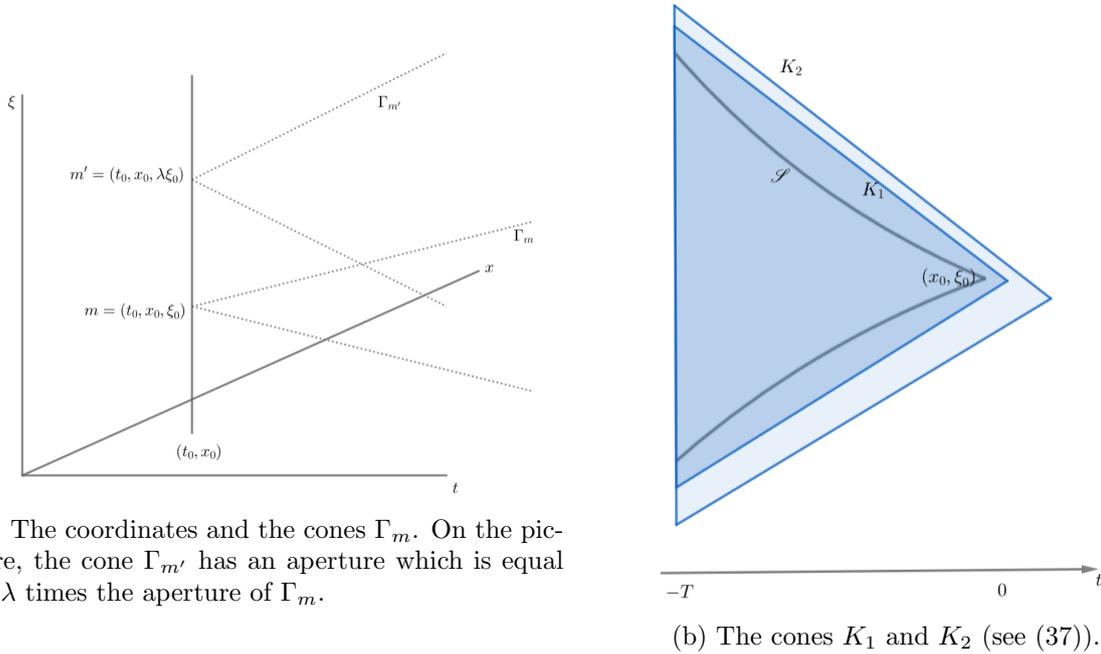
After the τ variable, we turn to the ξ variable. There is a global homogeneity in ξ of the cones Γ_m and consequently of the null-rays:

Homogeneity Property. If $[T_1, T_2] \ni t \mapsto \gamma(t) = (x(t), \xi(t)) \in \{a = 0\}$ is a null-ray parametrized by t , then for any $\lambda > 0$, $[T_1, T_2] \ni t \mapsto \gamma_\lambda(t) = (x(t), \lambda \xi(t))$ is a null-ray parametrized by t and joining the same endpoints as γ (in the same time interval $[T_1, T_2]$).

This property follows from (10). Thanks to this property, we will be able to find Φ satisfying Point (2) in Lemma 14.

Consequently, in our construction, we should have the following picture 1a in mind:

At this point we should say that since we are working in the slice $\{\tau = 0\}$, we will use in the sequel the following convenient abuse of notations: for $m = (t, 0, x, \xi)$, we still denote by m the projection of m on $\mathbb{R} \times T^*\Omega$ obtained by throwing away the coordinate $\tau = 0$. The fact that the whole picture is now embedded in \mathbb{R}^{2d+1} (see Figure 1a) is very convenient: for example, after throwing away the coordinate $\tau = 0$, we see the cones Γ_m as subcones of \mathbb{R}^{2d+1} (and not of its tangent space).



(a) The coordinates and the cones Γ_m . On the picture, the cone $\Gamma_{m'}$ has an aperture which is equal to λ times the aperture of Γ_m .

(b) The cones K_1 and K_2 (see (37)).

Figure 1

Also, in the sequel, we only consider points for which $t \geq -T$ for some (small) $T > 0$.⁵ We set $\delta_1 = T/10$ and take $0 \leq \delta_0 \leq \delta_1$.

The set of all points which belong to a *backward-pointing* ray starting from (x_0, ξ_0) at time 0 and stopped at time $-T$ is denoted by \mathcal{S} :

$$\mathcal{S} = \bigcup_{0 \leq t \leq T} \mathcal{S}^{-t}(\{(x_0, \xi_0)\}).$$

Then, \mathcal{S} is closed according to the first point of the following lemma (the second point will be used later):

Lemma 15. *The following two properties hold:*

1. *For any closed $V \subset T^*X$ and any $T \geq 0$, the set $\mathcal{S}^{-T}(V)$ is closed.*
2. *The mapping $(T, x, \xi) \mapsto \mathcal{S}^{-T}(\{(x, \xi)\})$ is inner semi-continuous, meaning that when $(T_n, x_n, \xi_n) \rightarrow (T, x, \xi)$, any point obtained as a limit, as $n \rightarrow +\infty$, of points of $\mathcal{S}^{-T_n}(\{(x_n, \xi_n)\})$ belongs to $\mathcal{S}^{-T}(\{(x, \xi)\})$.*

Proof. Both properties follow from the locally uniform Lipschitz continuity (17) combined with the extraction of Lipschitz rays as in the Arzelà-Ascoli theorem and the fact that the cones Γ_m are closed. \square

We take two closed convex cones K_1 and K_2 such that

$$\mathcal{S} \subset \text{Int}(K_1) \subset \text{Int}(K_2). \quad (37)$$

(see Figure 1b). It is possible to define Φ going backwards in time from time 0 to time $-T + \frac{\delta_0}{2}$, which is weakly increasing along the directions of K_2 and strictly increasing along the directions of K_1 , and which is compactly supported in (t, x) with support contained in the projection of K_2 on this base.

⁵ T is denoted by ε in [Mel86].

Since $\mathcal{S} \subset \text{Int}(K_1)$, Point 2. of Lemma 15 implies that

$$\begin{aligned} & \text{if } V \text{ is a sufficiently small neighborhood of } (x_0, \xi_0), \\ & \Phi \text{ is strictly increasing from time } 0 \text{ to time } -T + \frac{\delta_0}{2} \end{aligned} \quad (38)$$

along any backward-pointing ray starting from any point $(x, \xi) \in V$.

Also, if V' is a sufficiently small neighborhood of V , then it has the property that $\mathcal{S}^{-t}(V') \subset K_2$ for any $0 \leq t \leq T$, thus Property (6) can be guaranteed.

For $t \geq -T + \frac{\delta_0}{2}$, we have $\Phi'_t \leq 0$ since $\partial_t \in \Gamma_m$, and thus we set $\Psi_0 = \sqrt{-\Phi'_t}$. Then, following the rays backwards in time, we make Ψ_0 fall to 0 between times $-T + \frac{\delta_0}{2}$ and $-T$. Similarly, following the rays backward from time $-T + \frac{\delta_0}{2}$ to time $-T - \frac{\delta_0}{2}$, we extend Φ smoothly and homogeneously (in the fibers in ξ) in a way that Φ is compactly supported in the time-interval $(-T - \frac{\delta_0}{2}, \frac{\delta_0}{2})$ and $\Phi'_t + \Psi_0^2 \geq 0$. Finally, we set $\Psi_1 = \sqrt{\Phi'_t + \Psi_0^2}$. It is clear that points (5), (6), (8) are satisfied. See Figure 2 for the profile of Φ along a ray.

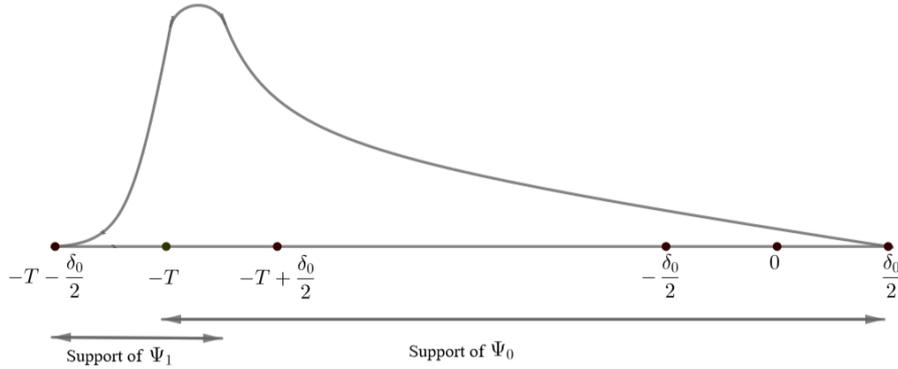


Figure 2: Profile of the function Φ along a ray. The abscissa indicates variable t .

In Lemma 14, Properties (1), (2), (3), (5), (6), (8) follow from the construction. Property (7) follows from (38). Finally, Property (4) follows from the fact that due to (37), we can replace the cones Γ_m by slightly bigger cones in a way that along the rays associated to these new cones, Φ is still non-decreasing.

4.2 A decomposition of C

When Φ satisfies (2), (3), (4) and (5) in Lemma 14, the operator C given by (27) can be expressed as follows:

Proposition 16. *If Φ satisfies (2), (3), (4) and (5) in Lemma 14, then writing $\Phi'_t = \Psi_1^2 - \Psi_0^2$, there holds*

$$C = R + R'P + PR' + C' - \delta(\text{Op}(\Psi_0)A\text{Op}(\Psi_0) + D_t\text{Op}(\Psi_0)^2D_t) \quad (39)$$

where $\delta > 0$ is the same as in (4), $R' = -\frac{\delta}{2}\text{Op}(\Phi'_t) \in \Psi_{\text{phg}}^\alpha$, $R = \delta\text{Op}(\Psi_1)(D_t^2 + A)\text{Op}(\Psi_1) \in \Psi_{\text{phg}}^{2+\alpha}$, and $C' \in \Psi_{\text{phg}}^{2+\alpha}$ has non-positive principal symbol and vanishing subprincipal symbol.

We start the proof of this proposition with the following corrected version of [Mel86, Lemma 5.3]:

Lemma 17. *Let ϕ be a time function near $\bar{m} \in \Sigma_{(2)}$ which does not depend on τ . Then, there holds*

$$\tau H_p \phi \leq \phi'_t p \quad (40)$$

in a neighborhood of \bar{m} .

Proof of Lemma 17. Recalling that $\pm\tau \geq 0$ on M_{\pm} , it follows from the definition of a time function that

$$q = \tau\{p, \phi\} \leq 0 \quad \text{on } \{p \geq 0\}. \quad (41)$$

Now, since ϕ does not depend on τ , we get that q is a quadratic polynomial in τ , vanishing at $\tau = 0$:

$$q = b\tau^2 - c\tau, \quad p = \tau^2 - a, \quad a \geq 0.$$

More explicitly, $b = 2\phi'_t$ and $c = \{a, \phi\}$. From (41), we know that $b \leq 0$. Moreover, (41) also implies that if $b = 0$, then $c = 0$, hence $\phi'_t = H_p\phi = 0$, and (40) is automatically satisfied. Otherwise, $b < 0$. Since $q \leq 0$ on $\tau \notin [-a^{1/2}, a^{1/2}]$ by (41), we get that the other zero of q , $\tau = c/b$, must lie in $[-a^{1/2}, a^{1/2}]$. Thus, $c^2 \leq b^2a$. Then,

$$\tau\{p, \phi\} - \phi'_t p = \frac{1}{2}b(\tau - c/b)^2 + (b^2a - c^2)/2b \leq 0 \quad (42)$$

where we used that $b < 0$. □

Let us come back to the proof of Proposition 16. Following the proof of Lemma 17 and keeping its notations, we replace (41) by the condition that $\tau H_p\Phi \leq 0$ on $\{p \geq -2\delta a\}$ (this is Point (4) in Lemma 14). The proof then gives that in case $b < 0$, there holds $c/b \in [-(1-2\delta)a^{1/2}, (1-2\delta)a^{1/2}]$, hence $c^2 \leq b^2a(1-2\delta)$. Therefore, (42) yields this time

$$\tau\{p, \Phi\} - \Phi'_t p \leq (b^2a - c^2)/2b \leq ba\delta = 2\Phi'_t a\delta.$$

This inequality obviously also holds in case $b = 2\Phi'_t = 0$. Hence, setting $r' = -\frac{\delta}{2}\Phi'_t$, we have

$$\tau\{p, \Phi\} - \Phi'_t p - 2r'p \leq 2\Phi'_t a\delta + \Phi'_t p\delta = \Phi'_t \delta(\tau^2 + a) = \delta(\Psi_1^2 - \Psi_0^2)(\tau^2 + a). \quad (43)$$

We set $R = \delta \text{Op}(\Psi_1)(D_t^2 + A)\text{Op}(\Psi_1)$. It follows from (43), (33), (34) and (28) that the operator

$$C' = C - R - (R'P + PR') + \delta(\text{Op}(\Psi_0)A\text{Op}(\Psi_0) + D_t\text{Op}(\Psi_0)^2D_t) \quad (44)$$

has non-positive principal symbol and vanishing sub-principal symbol. This proves Proposition 16.

4.3 The Fefferman-Phong inequality

The Fefferman-Phong inequality [FP78] (see also [Ler11, Section 2.5.3]) can be stated as follows: for any pseudodifferential operator C'_1 of order $2 + \alpha$ whose (Weyl) symbol is non-positive, there holds for any $u \in C_c^\infty$,

$$(C'_1 u, u)_{L^2} \leq c((\text{Id} - \Delta)^{\alpha/2} u, u)_{L^2} \quad (45)$$

where Δ is a Riemannian Laplacian on X . The following lemma is a simple microlocalization of this inequality.

Lemma 18. *Let $W, W' \subset T^*(\mathbb{R} \times X)$ be conic sets such that W' is a conic neighborhood of W . Let $C' \in \Psi_{\text{phg}}^{2+\alpha}$ with $\text{esssup}(C') \subset W$ such that $\sigma_p(C') \leq 0$ and $\sigma_{\text{sub}}(C') \leq 0$. Then there exists $C_\alpha \in \Psi_{\text{phg}}^{\alpha/2}$ with $\text{esssup}(C_\alpha) \subset W'$ such that*

$$\forall u \in C_c^\infty(\mathbb{R} \times X), \quad (C' u, u)_{L^2} \leq c(\|C_\alpha u\|_{L^2}^2 + \|u\|_{L^2}^2). \quad (46)$$

Proof. Taking a microlocal cut-off χ homogeneous of order 0, essentially supported in W' and equal to 1 on a neighborhood of W , we see that

$$\begin{aligned} (C' u, u) &= (C'(\text{Op}(\chi) + \text{Op}(1-\chi))u, (\text{Op}(\chi) + \text{Op}(1-\chi))u) \\ &= (\text{Op}(\chi)C'\text{Op}(\chi)u, u) + (Q'u, u) \end{aligned} \quad (47)$$

where $Q' \in \Psi^{-\infty}$ is explicit:

$$Q' = \text{Op}(1 - \chi)C' \text{Op}(\chi) + \text{Op}(\chi)C' \text{Op}(1 - \chi) + \text{Op}(1 - \chi)C' \text{Op}(1 - \chi).$$

Since $Q' \in \Psi^{-\infty}$, we have in particular

$$(Q'u, u) \leq c\|u\|_{L^2}^2. \quad (48)$$

Then, we write $C' = C'_1 + C'_2$ where C'_1 has non-positive full Weyl symbol, and $C'_2 \in \Psi_{\text{phg}}^\alpha$. First, we apply (45) with $\text{Op}(\chi)u$ instead of u : we obtain

$$(\text{Op}(\chi)C'_1 \text{Op}(\chi)u, u) \leq c\|C_\alpha u\|_{L^2}^2 \quad (49)$$

with $C_\alpha = (\text{Id} - \Delta)^{\alpha/4} \text{Op}(\chi)$. Secondly, writing $C'_2 = (\text{Id} - \Delta)^{\alpha/4} C''_2 (\text{Id} - \Delta)^{\alpha/4}$ with $C''_2 \in \Psi_{\text{phg}}^0$, we see that

$$(\text{Op}(\chi)C'_2 \text{Op}(\chi)u, u) \leq c\|C_\alpha u\|_{L^2}^2. \quad (50)$$

Combining (47), (48), (49) and (50), we get (46). \square

4.4 End of the proof of Theorem 1

We come back to the proof of Theorem 1. We fix $(x_0, \xi_0) \in T^*X \setminus 0$ and consider u a solution of (5). For the moment, we assume that u is *smooth*. We consider a time function Φ as constructed in Lemma 14.

Using (39), we have

$$\begin{aligned} 0 &= 2\text{Im}(Pu, \text{Op}(\Phi)D_t u) \\ &= (Cu, u) \\ &= ((R + R'P + PR' + C' - \delta(\text{Op}(\Psi_0)A\text{Op}(\Psi_0) + D_t \text{Op}(\Psi_0)^2 D_t))u, u). \end{aligned}$$

Hence, using $Pu = 0$ and applying Lemma 18 to C' , we get:

$$\begin{aligned} (A\text{Op}(\Psi_0)u, \text{Op}(\Psi_0)u) + \|\text{Op}(\Psi_0)D_t u\|_{L^2}^2 &\leq c((R_\alpha + R'P + PR' + C')u, u) \\ &\leq c_\alpha(\|C_\alpha u\|_{L^2}^2 + \|u\|_{L^2}^2 + (R_\alpha u, u)). \end{aligned}$$

with $c_\alpha \geq 1/\delta$ and $R_\alpha = R$, just to keep in mind in the forthcoming inequalities that it depends on α .

But $(A\text{Op}(\Psi_0)u, \text{Op}(\Psi_0)u) \geq \frac{1}{c}((-\Delta)^s \text{Op}(\Psi_0)u, \text{Op}(\Psi_0)u) - \|\text{Op}(\Psi_0)u\|^2$ by subellipticity (3). Hence

$$\|(-\Delta)^{s/2} \text{Op}(\Psi_0)u\|_{L^2}^2 + \|\text{Op}(\Psi_0)D_t u\|_{L^2}^2 \leq c_\alpha(\|C_\alpha u\|_{L^2}^2 + \|u\|_{L^2}^2 + (R_\alpha u, u) + \|\text{Op}(\Psi_0)u\|_{L^2}^2) \quad (51)$$

which we decompose into

$$\|(-\Delta)^{s/2} \text{Op}(\Psi_0)u\|_{L^2}^2 \leq c_\alpha(\|C_\alpha u\|_{L^2}^2 + \|u\|_{L^2}^2 + (R_\alpha u, u) + \|\text{Op}(\Psi_0)u\|_{L^2}^2) \quad (52)$$

and

$$\|\text{Op}(\Psi_0)D_t u\|_{L^2}^2 \leq c_\alpha(\|C_\alpha u\|_{L^2}^2 + \|u\|_{L^2}^2 + (R_\alpha u, u) + \|\text{Op}(\Psi_0)u\|_{L^2}^2). \quad (53)$$

Now, assume that u is a general solution of (5), not necessarily smooth. We have $u \in C^0(\mathbb{R}; \mathcal{D}(A^{1/2})) \cap C^1(\mathbb{R}; L^2(X))$. Recall the following definition.

Definition 19. Let $s_0 \in \mathbb{R}$ and $f \in \mathcal{D}'(\Omega)$. We shall say that f is H^{s_0} at $(x, \xi) \in T^*\Omega \setminus 0$ if there exists a conic neighborhood W of (x, ξ) such that for any 0-th order pseudodifferential operator B with $\text{essupp}(B) \subset W$, we have $Bf \in H_{\text{loc}}^s(\Omega)$.

We shall say that f is smooth at (x, ξ) if it is H^{s_0} at (x, ξ) for any $s_0 \in \mathbb{R}$.

Lemma 20. *Let V, V' be sufficiently small open neighborhoods of (x_0, ξ_0) such that $\bar{V} \subset V'$. Let u be a solution of (5). If u and $\partial_t u$ are smooth in $\mathcal{S}^{(-T-\frac{\delta_0}{2}, -T+\frac{\delta_0}{2})}(V')$, then u is smooth in*

$$U = \mathcal{S}^{(-T+\frac{\delta_0}{2}, 0)}(V).$$

When we say that u is H^{s_0} at (t, y, η) , we mean that $u(t)$ is H^{s_0} at $(y, \eta) \in T^*\Omega$.

Proof of Lemma 20. We set $u_\varepsilon = \rho_\varepsilon * u$ where $\rho_\varepsilon = \varepsilon^{-(d+1)}\rho(\cdot/\varepsilon)$ and $\rho \in C_c^\infty(\mathbb{R}^{d+1})$ is of integral 1 (and depends on the variables t, x). Recall that d is the dimension of X (and of the coordinate patch Ω).

Applying Lemma 14 for any $\alpha \in \mathbb{R}$ yields a function Φ_α which is in particular homogeneous of degree α in ξ ; its derivative in t can be written $\Phi'_\alpha = (\Psi_1^\alpha)^2 - (\Psi_0^\alpha)^2$ (the upper index being not an exponent). Then we apply (52) to u_ε and with $\alpha = 0$: we get

$$\|(-\Delta)^{s/2} \text{Op}(\Psi_0^0)u_\varepsilon\|_{L^2}^2 \leq c_0(\|C_0 u_\varepsilon\|_{L^2}^2 + \|u_\varepsilon\|_{L^2}^2 + (R_0 u_\varepsilon, u_\varepsilon) + \|\text{Op}(\Psi_0^0)u_\varepsilon\|_{L^2}^2) \quad (54)$$

where $R_0 = \delta \text{Op}(\Psi_1^0)(D_t^2 + A)\text{Op}(\Psi_1^0)$ (see Proposition 16) and $c > 0$ does not depend on ε . All quantities

$$\|C_0 u\|_{L^2}, \quad \|u\|_{L^2}, \quad (R_0 u, u), \quad \|\text{Op}(\Psi_0^0)u\|_{L^2}^2$$

are finite. Therefore, taking the limit $\varepsilon \rightarrow 0$ in (54), we obtain $u \in H^{2s}$ in U . Using the family of inequalities (52), we can iterate this argument: first with $\alpha = 2s$, then with $\alpha = 4s, 6s$, etc, and each time we replace Ψ_0^0, R_0, C_0 by $\Psi_0^\alpha, R_\alpha, C_\alpha$. At step k , we deduce thanks to (52) that $u \in H^{2ks}$. In particular, we use the fact that $\|C_\alpha u\|_{L^2}$ and $\|\text{Op}(\Psi_0^\alpha)u\|_{L^2}$ are finite, which comes from the previous step of iteration since C_α is essentially supported close to the essential support of C' (whose essential support is contained in that of Φ thanks to (44)). Thus, $u \in \bigcap_{k \in \mathbb{N}} H^{2ks} = C^\infty$ in U .

Then, using (53) for any $\alpha \in \mathbb{N}$ with Ψ_0^α in place of Ψ_0 , we obtain that $D_t u$ is also H^α in U . Hence, it is C^∞ in U , which concludes the proof of Lemma 20. \square

We conclude the proof of Theorem 1. We assume that

$$u \text{ is smooth in } W = \mathcal{S}^{(-T-\frac{\delta_0}{2}, -T+\frac{\delta_0}{2})}(\{(x_0, \xi_0)\}). \quad (55)$$

Then, u is smooth in a slightly larger set W' , i.e., such that $\bar{W} \subset W'$. By Lemma 15, there exists $V' \subset T^*X \setminus 0$ an open neighborhood of (x_0, ξ_0) such that

$$W \subset \mathcal{S}^{(-T-\frac{\delta_0}{2}, -T+\frac{\delta_0}{2})}(V') \subset W'.$$

Fix also an open set $V \subset T^*X \setminus 0$ such that

$$(x_0, \xi_0) \in V \subset \bar{V} \subset V'.$$

Lemma 20 implies that u is smooth in $\mathcal{S}^{(-T+\frac{\delta_0}{2}, 0)}(V)$. In particular,

$$u \text{ is smooth in } \mathcal{S}^{(-T+\frac{\delta_0}{2}, 0)}(\{(x_0, \xi_0)\}). \quad (56)$$

The fact that (55) implies (56) proves that singularities of (5) propagate only along rays. Using that singularities of P are contained in $\{p = 0\}$, we obtain finally Theorem 1.

5 Proof of Theorem 2

In the last two sections of this note, we assume that A is a sub-Laplacian. As mentioned in the introduction, it means that we assume that A has the form

$$A = \sum_{i=1}^K Y_i^* Y_i \quad (57)$$

where the global smooth vector fields Y_i are assumed to satisfy Hörmander's condition (the Lie algebra generated by Y_1, \dots, Y_K is equal to the whole tangent bundle TX). Here Y_i^* denotes the adjoint of Y_i for the scalar product (2).

5.1 The sub-Riemannian metric

In this preliminary section, we work with a general sub-Laplacian A_\bullet on a smooth compact manifold X_\bullet without boundary. This is because the results of this section will be used in Section 5 also for a sub-Laplacian defined on $X \times X$. We have

$$A_\bullet = \sum_{i=1}^{K_\bullet} Y_{\bullet i}^* Y_{\bullet i}. \quad (58)$$

There is a metric g_\bullet on the distribution $\mathcal{D}_\bullet = \text{Span}(Y_{\bullet 1}, \dots, Y_{\bullet K})$:

$$(g_\bullet)_x(v, v) = \inf \left\{ \sum_{i=1}^{K_\bullet} u_i^2 \mid v = \sum_{i=1}^{K_\bullet} u_i Y_{\bullet i}(x) \right\}. \quad (59)$$

The triple $(X_\bullet, \mathcal{D}_\bullet, g_\bullet)$ is called a sub-Riemannian structure (see [Mon02]).

The principal symbol of A_\bullet , which is also the natural Hamiltonian, is

$$a_\bullet = \sum_{i=1}^{K_\bullet} h_{Y_{\bullet i}}^2.$$

Here, for Y_\bullet a vector field on X_\bullet , we denoted by h_{Y_\bullet} the momentum map given in canonical coordinates (x, ξ) by $h_{Y_\bullet}(x, \xi) = \xi(Y_\bullet(x))$.

Denote by π_\bullet denotes the canonical projection $\pi_\bullet : T^*X_\bullet \rightarrow X_\bullet$ and by $\mathcal{I}_\bullet : b \mapsto \omega_\bullet(b, \cdot)$ the canonical isomorphism between $T(T^*X_\bullet)$ and $T^*(T^*X_\bullet)$. The notation $a_{\bullet m}$ stands for the Hessian of the principal symbol of A_\bullet at m .

Lemma 21. *There holds $a_{\bullet m}^*(\mathcal{I}_\bullet(b)) = g_\bullet(d\pi_\bullet(b))$ for any $b \in (\ker(a_{\bullet m}))^{\perp \omega_\bullet} \subset T(T^*X_\bullet)$.*

Proof. We consider a local g_\bullet -orthonormal frame Z_1, \dots, Z_N . In particular, the Z_j are independent, and the $H_{h_{Z_j}}$ are also independent. We have $a_{\bullet m} = \sum_{j=1}^N (dh_{Z_j})^2$. Hence, $H_{h_{Z_1}}, \dots, H_{h_{Z_N}}$ span $(\ker(a_{\bullet m}))^{\perp \omega_\bullet}$ since

$$\begin{aligned} \ker(a_{\bullet m}) &= \bigcap_{j=1}^N \ker(dh_{Z_j}) = \{\xi \in T(T^*X_\bullet), dh_{Z_j}(\xi) = 0, \forall 1 \leq j \leq N\} \\ &= \{\xi \in T(T^*X_\bullet), \omega_\bullet(\xi, H_{h_{Y_N}}) = 0, \forall 1 \leq j \leq N\} \\ &= \text{span}(H_{h_{Y_1}}, \dots, H_{h_{Y_N}})^{\perp \omega_\bullet}. \end{aligned}$$

We fix $b \in (\ker(a_{\bullet m}))^{\perp \omega_\bullet}$ and we write $b = \sum_{j=1}^N u_j H_{h_{Z_j}}$. Note that $g_\bullet(\sum_{j=1}^N u_j Z_j) = \sum_{j=1}^N u_j^2$. By definition, $\mathcal{I}_\bullet(H_{h_{Z_j}}) = -dh_{Z_j}$ and $d\pi_\bullet(H_{h_{Z_j}}) = Z_j$ for any j , so there holds

$$\begin{aligned} a_{\bullet m}^* \left(\mathcal{I}_\bullet \left(\sum_{j=1}^N u_j H_{h_{Z_j}} \right) \right) &= a_{\bullet m}^* \left(\sum_{j=1}^N u_j dh_{Z_j} \right) = \sup_{\eta \notin \ker(a_{\bullet m})} \frac{\left(\sum_{j=1}^N u_j dh_{Z_j}(\eta) \right)^2}{\sum_{j=1}^N (dh_{Z_j}(\eta))^2} \\ &= \sup_{(\theta_j) \in \mathbb{R}^N} \frac{\left(\sum_{j=1}^N u_j \theta_j \right)^2}{\sum_{j=1}^N \theta_j^2} = \sum_{j=1}^N u_j^2 = g_\bullet \left(\sum_{j=1}^N u_j Z_j \right) \\ &= g_\bullet \left(d\pi_\bullet \left(\sum_{j=1}^N u_j H_{h_{Z_j}} \right) \right) \end{aligned}$$

where, to go from line 1 to line 2, we used that the dh_{Z_j} are independent. \square

5.2 K_G as a solution of a wave equation

The rest of Section 5 is devoted to the proof of Theorem 2, i.e., we deduce the wave-front set of the Schwartz kernel K_G from the “geometric” propagation of singularities given by Theorem 1. The idea is to consider K_G itself as the solution of a wave equation to which we can apply Theorem 1.

We consider the product manifold $X \times X$, with coordinate x on its first copy, and coordinate y on its second copy. We set

$$A^\otimes = \frac{1}{2}(A_x \otimes \text{Id}_y + \text{Id}_x \otimes A_y)$$

and we consider the operator

$$P = \partial_{tt}^2 - A^\otimes$$

acting on functions of $\mathbb{R} \times X_x \times X_y$. Using (6), we can check that the Schwartz kernel K_G is a solution of

$$K_G|_{t=0} = 0, \quad \partial_t K_G|_{t=0} = \delta_{x-y}, \quad PK_G = 0.$$

The operator A^\otimes is a self-adjoint non-negative real second-order differential operator on $X \times X$. Moreover it is subelliptic: it is immediate that the vector fields $Y_1 \otimes \text{Id}_y, \dots, Y_K \otimes \text{Id}_y, \text{Id}_x \otimes Y_1, \dots, \text{Id}_x \otimes Y_K$ verify Hörmander’s Lie bracket condition, since it is satisfied by Y_1, \dots, Y_K . Hence, Theorem 1 applies to P , with the null-rays being computed with A^\otimes in $T^*(X \times X)$ (see (62) for the associated cones). We denote by \sim_t the relation of existence of a null-ray of length $|t|$ joining two given points of $T^*(X \times X) \setminus 0$ (see Remark 6 for the omission of the variables t and τ in the null-rays).

Since $WF(K_G(0)) = \emptyset$ and

$$WF(\partial_t K_G(0)) = \{(z, z, \zeta, -\zeta) \in T^*(X \times X) \setminus 0\},$$

we have

$$WF(K_G(t)) \subset \{(x, y, \xi, -\eta) \in T^*(X \times X) \setminus 0, \exists(z, \zeta) \in T^*X \setminus 0, (z, z, \zeta, -\zeta) \sim_t (x, y, \xi, -\eta)\}. \quad (60)$$

Let us denote by g^1 the sub-Riemannian metric on X_x and by g^2 the sub-Riemannian metric on X_y . The sub-Riemannian metric on $X_x \times X_y$ is $g^\otimes = \frac{1}{2}(g^1 \oplus g^2)$. In other words, if $q = (q_1, q_2) \in X \times X$ and $v = (v_1, v_2) \in T_q(X \times X) \approx T_{q_1}X \times T_{q_2}X$, we have

$$g_q^\otimes(v) = \frac{1}{2}(g_{q_1}^1(v_1) + g_{q_2}^2(v_2)). \quad (61)$$

Now, the cones Γ_m^\otimes associated to A^\otimes are given by

$$\begin{aligned} \Gamma_m^\otimes &= \mathbb{R}^+(\partial_t + B), \\ B &= \{b \in \ker(a_m^\otimes)^\perp_{\omega^\otimes}, g^\otimes(d\pi^\otimes(b)) \leq 1\}. \end{aligned} \quad (62)$$

Here, \perp_{ω^\otimes} designates the symplectic orthogonal with respect to the canonical symplectic form ω^\otimes on $T^*(X \times X)$, and $\pi^\otimes : T^*(X \times X) \rightarrow X \times X$ is the canonical projection.

To evaluate the right-hand side of (60), we denote by \approx_t the relation of existence of a null-ray of length $|t|$ joining two given points of $T^*X \setminus 0$ (the cones Γ_m are subsets of $T(T^*(\mathbb{R} \times X))$ as defined in Section 2). Let us prove that

$$\begin{aligned} \{(x, y, \xi, -\eta) \in T^*(X \times X) \setminus 0, \exists(z, \zeta) \in T^*X \setminus 0, (z, z, \zeta, -\zeta) \sim_t (x, y, \xi, -\eta)\} \\ \subset \{(x, y, \xi, -\eta) \in T^*(X \times X) \setminus 0, (x, \xi) \approx_t (y, \eta)\}. \end{aligned} \quad (63)$$

Combining with (60), it will immediately follow that

$$WF(K_G(t)) \subset \{(x, y, \xi, -\eta) \in T^*(X \times X) \setminus 0, (x, \xi) \approx_t (y, \eta)\}. \quad (64)$$

5.3 Proof of (63).

We denote by $\gamma : [0, t] \rightarrow T^*(X \times X) \setminus 0$ a null-ray from $(z, z, \zeta, -\zeta)$ to $(x, y, \xi, -\eta)$, parametrized by time. Our goal is to construct a null-ray of length $|t|$ in $T^*X \setminus 0$, from (y, η) to (x, ξ) . It is obtained by concatenating a null-ray from (y, η) to (z, ζ) with another one, from (z, ζ) to (x, ξ) . However, there are some subtleties hidden in the parametrization of this concatenated null-ray.

We write $\gamma(s) = (\alpha_1(s), \alpha_2(s), \beta_1(s), \beta_2(s))$, and for $i = 1, 2$ and $0 \leq s \leq t$, we set $\gamma_i(s) = (\alpha_i(s), \beta_i(s)) \in T^*X$. We also set $\delta_i(s) = g^i(d\pi_i(\dot{\gamma}_i(s)))$, where $\pi_i : T^*X \rightarrow X$ (here X is the i -th copy of X). The upper dot denotes here and in the sequel the derivative with respect to the time variable. Since $g^\otimes(d\pi^\otimes(\dot{\gamma}(s))) \leq 1$ for any $s \in [0, t]$, we deduce from (61) that

$$\frac{1}{2}(\delta_1(s) + \delta_2(s)) \leq 1.$$

We are going to construct a null-ray $\varepsilon : [0, t] \rightarrow T^*X$ of the form

$$\begin{aligned} \varepsilon(s) &= (\alpha_2(\theta(s)), -\beta_2(\theta(s))), & 0 \leq s \leq s_0 \\ \varepsilon(s) &= (\alpha_1(\theta(s)), \beta_1(\theta(s))), & s_0 \leq s \leq t. \end{aligned} \quad (65)$$

The parameter s_0 and the parametrization θ will be chosen so that the first part of ε joins (y, η) to (z, ζ) and the second part joins (z, ζ) to (x, ξ) . We choose $\theta(0) = t$, hence $\varepsilon(0) = (y, \eta)$. Then, for $0 \leq s \leq s_0$, we choose $\theta(s) \leq t$ in order to guarantee that $g^1(d\pi_1(\dot{\varepsilon}(s))) = 1$. This defines s_0 in a unique way as the minimal time for which $\varepsilon(s_0) = (z, \zeta)$. In particular, $\theta(s_0) = 0$. A priori, we do not know that $s_0 \leq t$, but we will prove it below. Then, for $s_0 \leq s_1$, we choose $\theta(s) \geq 0$ in order to guarantee that $g^2(d\pi_2(\dot{\varepsilon}(s))) = 1$. This defines s_1 in a unique way as the minimal time for which $\varepsilon(s_1) = (x, \xi)$. Finally, if $s_1 \leq t$, we extend ε by $\varepsilon(s) \equiv (x, \xi)$ for $s_1 \leq s \leq t$.

We check that ε is a null-ray in T^*X . We come back to the definition of null-rays as tangent to the cones Γ_m . It is clear that

$$\ker(a_m^\otimes)^{\perp \omega^\otimes} = \ker(a_m)^{\perp \omega_1} \times \ker(a_m)^{\perp \omega_2}$$

where ω_i is the canonical symplectic form on T^*X_i . Therefore, $\dot{\varepsilon}(s) \in \ker(a_m)^{\perp \omega_i}$ for $i = 1$ when $0 \leq s \leq s_0$ and for $i = 2$ when $s_0 \leq s \leq t$. Thanks to Lemma 21, the inequality in (15) (but for the cones in X_1 and X_2) is verified by $\dot{\varepsilon}(s)$ for any $0 \leq s \leq t$ by definition. There is a ‘‘time-reversion’’ (or ‘‘path reversion’’) in the first line of (65); the property of being a null-ray is preserved under time reversion together with momentum reversion. Hence ε is a null-ray in T^*X .

The fact that $s_0, s_1 \leq t$ follows from the following computation:

$$\begin{aligned} t &\geq \int_0^t g^\otimes(d\pi^\otimes(\dot{\gamma}(s))) ds = \frac{1}{2} \int_0^t g^1(d\pi_1(\dot{\gamma}_1(s))) ds + \frac{1}{2} \int_0^t g^2(d\pi_2(\dot{\gamma}_2(s))) ds \\ &= \frac{1}{2} \int_0^{s_0} g^1(d\pi_1(\dot{\varepsilon}(s))) ds + \frac{1}{2} \int_{s_0}^{s_1} g^2(d\pi_2(\dot{\varepsilon}(s))) ds \\ &= s_0 + (s_1 - s_0) = s_1. \end{aligned}$$

where the second equality follows from the fact that ε is a reparametrization of γ_1 (resp. γ_2) for $s \in [0, s_0]$ (resp. $[s_0, s_1]$). This concludes the proof of (63).

5.4 Conclusion of the proof of Theorem 2

Let us finish the proof of Theorem 2. We fix (x_0, ξ_0) , (y_0, η_0) and t_0 such that there is no null-ray from $(y_0, \eta_0) \in T^*X$ to $(x_0, \xi_0) \in T^*X$ in time t_0 .

Claim. There exist a conic neighborhood V of $(x_0, y_0, \xi_0, -\eta_0)$ and a neighborhood V_0 of t_0 such that for any $N \in \mathbb{N}$ and any $t \in V_0$, $\partial_t^{2N} K_G(t)$ is smooth in V .

Proof. We choose V so that for $(x, y, \xi, -\eta) \in V$ and $t \in V_0$, there is no null-ray from (y, η) to (x, ξ) in time t . Such a V exists, since otherwise by extraction of null-rays (which are Lipschitz with a locally uniform constant, see (17)), there would exist a null-ray from (y_0, η_0) to (x_0, ξ_0) in time t_0 . Then, we can check that for any $N \in \mathbb{N}$, $K_G^{(2N)} = \partial_t^{2N} K_G$ is a solution of

$$K_G^{(2N)}|_{t=0} = 0, \quad \partial_t K_G^{(2N)}|_{t=0} = (A^\otimes)^N \delta_{x-y}, \quad PK_G^{(2N)} = 0.$$

Repeating the above argument leading to (64) with $K_G^{(2N)}$ instead of K_G , we obtain

$$WF(K_G^{(2N)}(t)) \subset \{(x, y, \xi, -\eta) \in T^*(X \times X) \setminus 0, (x, \xi) \approx_t (y, \eta)\},$$

which proves the claim. \square

We deduce from the claim that if there is no null-ray from $(y_0, \eta_0) \in T^*X$ to $(x_0, \xi_0) \in T^*X$ in time t_0 , then $(t_0, \tau_0, x_0, y_0, \xi_0, -\eta_0) \notin WF(K_G)$ for any $\tau_0 \in \mathbb{R}$.

Finally, if there is a null-ray from (y_0, η_0) to (x_0, ξ_0) in time t_0 , then $a(x_0, \xi_0) = a(y_0, \eta_0)$, and due to the fact that $WF(K_G)$ is included in the characteristic set of $\partial_{tt}^2 - A^\otimes$, the only τ_0 's for which $(t_0, \tau_0, x_0, y_0, \xi_0, -\eta_0) \in WF(K_G)$ is possible are the ones satisfying $\tau_0^2 = a(x_0, \xi_0) = a(y_0, \eta_0)$. This concludes the proof of Theorem 2.

Remark 22. *Theorem 2 allows to recover some results already known in the literature.*

In the situations studied in [Las82], [LL82] and [Mel86], $\Sigma_{(2)}$ is a symplectic manifold. In this case, thanks to (15), we see that the only null-rays starting from points in $\Sigma_{(2)}$ are lines in t . Therefore Theorem 2 implies:

- *the “wave-front part” of the main results of [Las82] and [LL82] (but not the effective construction of parametrices handled in these papers).*
- *Theorem 1.8 in [Mel84], which can be reformulated as follows: if Σ_2 (in the notations of [Mel84]) is of codimension 2, then*

*singularities outside Σ_2 propagate along bicharacteristics,
and singularities inside Σ_2 propagate along lines in t .*

This is exactly the content of Theorem 2 in this case. To see that Theorem 1.8 of [Mel84] can be reformulated as above, we must notice that on Σ_2 , $\overset{\circ}{\chi}_t \pm$ extends as the identity for any $t \in \mathbb{R}$, which follows from the following property (denoting by $U_{x_0}^ X$ the set of covectors of norm 1 with base point x_0):*

$$\forall t > 0, \forall x_0 \in X, \exp_{x_0}^t : U_{x_0}^* X \rightarrow X \text{ is proper} \quad (66)$$

(when restricted to minimizers), which implies that for any open neighborhood V of x_0 , $(\exp_{x_0}^t)^{-1}(X \setminus V)$ is compact, at positive distance from Σ_2 . The property (66) is always true in the absence of singular curves (defined in Section 6.1).

6 A consequence for wave equations with sub-Laplacians

We now turn to the consequences of Theorem 2. For that purpose, we briefly introduce notations and concepts from sub-Riemannian geometry. Our presentation is inspired by [Mon02, Chapter 5 and Appendix D]. In this last section, we continue to assume that A is a sub-Laplacian on X (see Example 4). The associated sub-Riemannian metric (see (59)) is denoted by g .

6.1 Sub-Riemannian geometry and horizontal curves

Fix an interval $I = [b, c]$ and a point $x_0 \in X$. We denote by $\Omega(I, x_0; \mathcal{D})$ the space of all absolutely continuous curves $\gamma : I \rightarrow X$ that start at $\gamma(b) = x_0$ and whose derivative is square integrable with respect to g , implying that the length

$$\int_I \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} dt$$

of γ is finite. Such a curve γ is called *horizontal*. The *endpoint map* is the map

$$\text{End} : \Omega(I, x_0; \mathcal{D}) \rightarrow X, \quad \gamma \mapsto \gamma(c).$$

The metric (59) induces a distance d on X , and $d(x, y) < +\infty$ for any $x, y \in X$ thanks to Hörmander's condition (this is the Chow-Rashevskii theorem).

Two types of curves in $\Omega(I, x_0; \mathcal{D})$ will be of particular interest: the critical points of the endpoint map, and the curves which are projections of the Hamiltonian vector field H_a associated to a .

Projections of integral curves of H_a are geodesics:

Theorem 23. [Mon02, Theorem 1.14] *Let $\gamma(s)$ be the projection on X of an integral curve (in T^*X) of the Hamiltonian vector field H_a . Then γ is a horizontal curve and every sufficiently short arc of γ is a minimizing sub-Riemannian geodesic (i.e., a minimizing path between its endpoints in the metric space (X, d)).*

Such horizontal curves γ are called *normal* geodesics, and they are smooth.

The differentiable structure on $\Omega(I, x_0; \mathcal{D})$ described in [Mon02, Chapter 5 and Appendix D] allows to give a sense to the following notion:

Definition 24. *A singular curve is a critical point for the endpoint map.*

Note that in Riemannian geometry (i.e., for a elliptic), there exist no singular curves.

In the next definition, we use the notation \mathcal{D}^\perp for the annihilator of \mathcal{D} (thus a subset of the cotangent bundle T^*X), and $\bar{\omega}_X$ denotes the restriction to \mathcal{D}^\perp of the canonical symplectic form ω_X on T^*X .

Definition 25. *A characteristic for \mathcal{D}^\perp is an absolutely continuous curve $\lambda(t) \in \mathcal{D}^\perp$ that never intersects the zero section of \mathcal{D}^\perp and that satisfies $\dot{\lambda}(t) \in \ker(\bar{\omega}_X(\lambda(t)))$ at every point t for which the derivative $\dot{\lambda}(t)$ exists.*

Theorem 26. [Mon02, Theorem 5.3] *A curve $\gamma \in \Omega$ is singular if and only if it is the projection of a characteristic λ for \mathcal{D}^\perp with square-integrable derivative. λ is then called an abnormal extremal lift of the singular curve γ .*

Normal geodesics and singular curves are particularly important in sub-Riemannian geometry because of the following fact (Pontryagin's maximum principle):

any minimizing geodesic in (X, d) is either a singular curve or a normal geodesic.

The existence of geodesics which are singular curves but not normal geodesics was proved in [Mon94].

6.2 The singular support of $K_G(\cdot, x, y)$

When A is a sub-Laplacian (57), the cones Γ_m defined in Section 2.1 have an additional geometric interpretation, which we now explain.

We consider a null-ray, as introduced in Definition 3. It is necessarily of one of the following types (depending on the value of τ , which is a constant):

- either a null-bicharacteristic with (constant) $\tau \neq 0$, since $\Gamma_m = \mathbb{R}^\pm \cdot H_p(m)$ in this case;
- or contained in $\Sigma_{(2)}$ and tangent to the cones Γ_m given by (15), with $\tau \equiv 0$ since $d\tau(v) = 0$ for any $v \in \Gamma_m$ according to (15).

In the second case, setting $n = \pi_2(m)$ and writing $v = c(\partial_t + b)$ as in (15), we have $b \in T_n \mathcal{D}^\perp$ since $a \equiv 0$ along the path. There holds $\ker(a_m) = T_n \mathcal{D}^\perp$ and, plugging into the above formula, we also get $b \in (T_n \mathcal{D}^\perp)^\perp_{\omega_X}$. It follows that $b \in T_n \mathcal{D}^\perp \cap (T_n \mathcal{D}^\perp)^\perp_{\omega_X} = \ker \bar{\omega}_X$, i.e., the trajectory of the null-ray (forgetting the time variable) is a characteristic curve.

In summary, when A is a sub-Laplacian (57), Theorem 2 asserts that singularities of the wave equation (5) propagate only along integral curves of H_a and characteristics for \mathcal{D}^\perp . From that, we can infer the following proposition, in the spirit of Duistermaat-Guillemin's trace formula [DG75]:

Proposition 27. *We fix $x, y \in X$ with $x \neq y$. We denote by \mathcal{L} the set of lengths of normal geodesics from x to y and by T_s the minimal length of a singular curve joining x to y . Then $\mathcal{G} : t \mapsto K_G(t, x, y)$ is well-defined as a distribution on $(-T_s, T_s)$, and*

$$\text{sing supp}(\mathcal{G}) \subset \mathcal{L} \cup -\mathcal{L}.$$

Note that this proposition does not say anything about times $|t| \geq T_s$: it could happen a priori that $t \mapsto K_G(t, x, y)$ is not a distribution after T_s .

Proof. As said above, null-rays living in $\{\tau = 0\}$ are characteristic curves (in the sense of Definition 25) tangent to Γ_m . Now, it follows from (15), Theorem 26 and Lemma 21 that the least $t > 0$ for which there exists a null-ray of length t with $\tau \equiv 0$ joining x and y is equal to the length of the shortest singular curve joining x and y .

We consider $\varphi : \mathbb{R} \rightarrow \mathbb{R} \times X \times X$, $t \mapsto (t, x, y)$ which has conormal set $N_\varphi = \{(t, x, y, 0, \xi, \eta)\}$ (in other words N_φ corresponds to $\tau = 0$). Thus, using Theorem 2, we see that $WF(\mathcal{G})$ does not intersect the conormal set of $\varphi|_{(-T_s, T_s)}$. Then, [Hör71a, Theorem 2.5.11] ensures that \mathcal{G} , which is the pull-back of K_G by $\varphi|_{(-T_s, T_s)}$, is well-defined as a distribution over $(-T_s, T_s)$. Of course, $\text{sing supp}(\mathcal{G})$ is the projection of $WF(\mathcal{G})$ (for $|t| < T_s$).

For $|t| < T_s$, null-rays between x and y are contained in $\{\tau \neq 0\}$, thus they are tangent to the cones $\Gamma_m = \mathbb{R}^\pm \cdot H_p(m)$. Hence, the singularities of \mathcal{G} occur at times belonging to the set \mathcal{L} of lengths of normal geodesics (for $\tau > 0$, we obtain normal geodesics from y to x , and for $\tau < 0$, normal geodesics from x to y). \square

Remark 28. *If $x = y$, the same reasoning as in the proof of Proposition 27 says nothing more than $\text{sing supp}(K_G(\cdot, x, x)) \subset \mathbb{R}$ since for any point $(x, \xi) \in \mathcal{D}^\perp$ and any $t \in \mathbb{R}$, the constant path joining (x, ξ) to (x, ξ) in time t is a null-ray (with $\tau = 0$).*

6.3 Comments on the inequality in (15)

In the formula (15) for the cones Γ_m , the inequality $a_m^*(\mathcal{I}(b)) \leq 1$ may seem surprising at first sight. When A is a sub-Laplacian, according to Lemma 21, it is equivalent to $g(d\pi(b)) \leq 1$. In rough terms, Theorem 1 does not exclude that singularities contained in \mathcal{D}^\perp propagate at speeds < 1 , which would be in strong contrast with the usual propagation ‘‘at speed 1’’ of singularities of wave equations with elliptic Laplacian. In a joint work with Yves Colin de Verdière [CdVL21], we give explicit examples of initial data of a subelliptic wave equation whose singularities effectively propagate at any speed between 0 and 1 along singular curves.

A Technical tools

A.1 Sign conventions in symplectic geometry

In the present note, we take the following conventions (the same as [Hör07], see Chapter 21.1): on a symplectic manifold with canonical coordinates (x, ξ) , the symplectic form is $\omega = d\xi \wedge dx$, and the Hamiltonian vector field H_f of a smooth function f is defined by the relation $\omega(H_f, \cdot) = -df(\cdot)$. In coordinates, it reads

$$H_f = \sum_j (\partial_{\xi_j} f) \partial_{x_j} - (\partial_{x_j} f) \partial_{\xi_j}.$$

In these coordinates, the Poisson bracket is

$$\{f, g\} = \omega(H_f, H_g) = \sum_j (\partial_{\xi_j} f) (\partial_{x_j} g) - (\partial_{x_j} f) (\partial_{\xi_j} g),$$

which is also equal to $H_f g$ and $-H_g f$.

A.2 Pseudodifferential operators

This appendix is a short reminder on basic properties of pseudodifferential operators. Most proofs can be found in [Hör07]. In this note, we work with the class of polyhomogeneous symbols, which is slightly smaller than the usual class of symbols but has the advantage that the subprincipal symbol can be read easily when using the Weyl quantization (see [Hör07], the paragraph before Section 18.6).

We consider Ω an open set of a d -dimensional manifold, and μ a smooth volume on Ω . The variable in Ω is denoted by q . Let $\pi : T^*\Omega \rightarrow \Omega$ be the canonical projection.

$S_{\text{hom}}^n(T^*\Omega)$ stands for the set of homogeneous symbols of degree n with compact support in Ω . We also denote by $S_{\text{phg}}^n(T^*\Omega)$ the set of polyhomogeneous symbols of degree n with compact support in Ω . Hence, $a \in S_{\text{phg}}^n(T^*\Omega)$ if $a \in C^\infty(T^*\Omega)$, the projection $\pi(\text{supp}(a))$ is a compact of Ω , and there exist $a_j \in S_{\text{hom}}^{n-j}(T^*\Omega)$ such that for any $N \in \mathbb{N}$, $a - \sum_{j=0}^N a_j \in S_{\text{phg}}^{n-N-1}(T^*\Omega)$. We denote by $\Psi_{\text{phg}}^n(\Omega)$ the space of polyhomogeneous pseudodifferential operators of order n on Ω , with a compactly supported kernel in $\Omega \times \Omega$.

We use the Weyl quantization denoted by $\text{Op} : S_{\text{phg}}^n(T^*\Omega) \rightarrow \Psi_{\text{phg}}^n(\Omega)$. It is obtained by using partitions of unity and the formula in local coordinates

$$\text{Op}(a)f(q) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}_{q'}^d \times \mathbb{R}_p^d} e^{i\langle q-q', p \rangle} a\left(\frac{q+q'}{2}, p\right) f(q') dq' dp.$$

If a is real-valued, then $\text{Op}(a)^* = \text{Op}(a)$. Moreover, with this quantization, the principal and subprincipal symbols of $A = \text{Op}(a)$ with $a \sim \sum_{j \leq n} a_j$ are simply $\sigma_p(A) = a_n$ and $\sigma_{\text{sub}}(A) = a_{n-1}$ (usually, the subprincipal symbol is defined for operators acting on half-densities, but we make here the identification $f \leftrightarrow f d\nu^{1/2}$).

We also have the following properties:

1. If $A \in \Psi_{\text{phg}}^l(\Omega)$ and $B \in \Psi_{\text{phg}}^n(\Omega)$, then $[A, B] \in \Psi_{\text{phg}}^{l+n-1}(\Omega)$. Moreover, $\sigma_p([A, B]) = \frac{1}{i} \{\sigma_p(a), \sigma_p(b)\}$ where the Poisson bracket is taken with respect to the canonical symplectic structure of $T^*\Omega$.
2. If X is a vector field on Ω and X^* is its formal adjoint in $L^2(\Omega, \mu)$, then X^*X is a second order pseudodifferential operator, with $\sigma_p(X^*X) = h_X^2$ and $\sigma_{\text{sub}}(X^*X) = 0$.
3. If $A \in \Psi_{\text{phg}}^n(\Omega)$, then A maps continuously the space $H^s(\Omega)$ to the space $H^{s-n}(\Omega)$.

Finally, we define the essential support of A , denoted by $\text{essupp}(A)$, as the complement in $T^*\Omega$ of the points (q, p) which have a conic-neighborhood W so that A is of order $-\infty$ in W .

B Further properties of the cones Γ_m

B.1 Inner semi-continuity of the cones Γ_m in a

In this appendix, we prove that if we make some additional assumptions, the convex cones Γ_m are inner semi-continuous with respect to a (in addition to their inner semi-continuity with respect to m proved in Lemma 7). For that, we introduce the following class of functions on T^*X (for $k \in \mathbb{N}$):

$$\mathcal{A}_k = \left\{ \sum h_{Y_j}^2, (Y_j) \in \mathcal{D}_k \right\},$$

where \mathcal{D}_k is the set of families of smooth vector fields generating a regular (i.e., constant rank) distribution of rank k . Note that any $a \in \mathcal{A}_k$ automatically satisfies (8).

Proposition 29. *The mapping $(M_+ \cup M_-) \times \mathcal{A}_k \ni (m, a) \mapsto \Gamma_m^{(a)}$ is inner semi-continuous (for the C^∞ topology in $a \in \mathcal{A}_k$). In other words,*

$$\forall m_j \rightarrow m_* \quad (m_j, m_* \in M_+ \cup M_-), \quad \forall a_j \in \mathcal{A}_k, \quad a_j \xrightarrow{C^\infty} a_*, \quad \forall v_j \in \Gamma_{m_j}^{(a_j)}, \quad v_j \rightarrow v \in T_{m_*} M, \\ \text{there holds } v \in \Gamma_{m_*}^{(a_*)}$$

where we temporarily denoted by $\Gamma_m^{(a)}$ the cone computed with the Hamiltonian a at point m .

Proposition 29 follows quite directly from the computations done in the proof of Lemma 7. However, we give here a different proof which has the advantage of requiring no formula, and which illustrates Remark 10.

Definition 30. Let F be a manifold and $E \subset F$ be a closed set. For $x \in E$, the tangent cone $C(x)$ is the \mathbb{R}^+ -subcone of the tangent space $T_x F$ consisting of all the vectors $\gamma'(0)$ where $\gamma: [0, a[\rightarrow F$ is a C^1 curve so that $\gamma(0) = x$ and $\gamma(t) \in E$ for $t \geq 0$ small enough. The dual tangent cone $C^o(x)$ is the subcone of $T_x^* F$ of all covectors ξ so that $\xi(v) \leq 0$ for all $v \in C(x)$.

Let us remark that if ∂E is smooth at x , then $C^o(x)$ is generated by the normal outgoing covectors at x .

Proof of Proposition 29. We set $Y = T^*X$. The statement clearly holds if $m_* = (t_*, \tau_*, y_*)$ does not verify $\tau_* = 0$ and $a_*(y_*) = 0$. Hence we assume in the sequel that $\tau_* = 0$ and $y_* \in a_*^{-1}(0)$. Writing $m_j = (t_j, \tau_j, y_j)$, we can also assume that for any j , $\tau_j \neq 0$ since otherwise $0 = \tau_j^2 \geq a_j(y_j) \geq 0$, meaning that all cones Γ_m are computed according to the formula (10), and in this case we even have continuity of the cones $\Gamma_{m_j}^{(a_j)}$ towards $\Gamma_{m_*}^{(a_*)}$. In other words, with transparent notations, we assume in the sequel that $m_j \notin \Sigma_{(2)}^{(a_j)}$ and $m_* \in \Sigma_{(2)}^{(a_*)}$.

For $a \in \mathcal{A}_k$, $p_0 \geq 0$, and $b = (a, p_0)$, we consider

$$E_b = \{(t, \tau, y) \in T^*\mathbb{R} \times Y; \tau \geq 0 \text{ and } \tau^2 - a(y) \geq p_0\}.$$

There are two steps: 1) prove that the mapping $(m, a) \mapsto C_b^o(m)$ is inner semi-continuous where $b = (a, p(m))$ and $C_b^o(m)$ is the dual tangent cone of E_b at m (which is in ∂E_b); 2) conclude the proof of Proposition 29.

1) Since $a_* \in \mathcal{A}_k$, the characteristic manifold $Z = a_*^{-1}(0) \subset Y$ is smooth (see [ABB19], below Definition 4.33), and it is non-degenerate. Thus, the Morse-Bott Lemma (see [BH04]) guarantees the existence of local coordinates $y = (\tilde{y}, z) \in N$ such that $y_* = (0, 0)$ and $a_*(y) = \|\tilde{y}\|^2$. In these coordinates (valid for $(t, \tau, y) \in N$), the set $E_{b_*} \cap N$ is convex.

The boundary ∂E_{b_j} is smooth at m_j and, for any j , the tangent cone of E_{b_j} at $m_j \in \partial E_{b_j}$ is a set $H_j \subset T_{m_j} M$ which is nearly a half tangent space⁶. Indeed, the convergence $a_j \rightarrow a_*$ (in the C^∞ topology) implies that any set H_∞ which is the limit of a convergent subsequence of (H_j) is a half-space, and $E_{b_*} \subset H_\infty$. Hence, by convexity of $E_{b_*} \cap N$, the tangent cone at m_* is contained in H_∞ . By taking duals, we get the opposite inclusion: any limit of the dual tangent cones $C_{b_j}^o(m_j)$ belongs to $C_{b_*}^o(m_*)$. This proves the result: the mapping $(m, a) \mapsto C_b^o(m)$ is inner semi-continuous at (m_*, a_*) .

2) Let us compute $C_b^o(m)$ depending on m and a .

If $m \notin \Sigma_{(2)}$, then its tangent cone is $C_b(m) = \{w \in T_m M, dp(w) \geq 0\}$. Hence $C_b^o(m) = \{\lambda \in T_m^* M, \lambda(w) \leq 0 \forall w \text{ such that } dp(w) \geq 0\} = \{-dp\}$ where this last differential is taken at m .

If $m \in \Sigma_{(2)}$, then $C_b(m) = \{w \in T_m M; d\tau(w) \geq 0, p_m(w) \geq 0\} = \Lambda_m$ and $C_b^o(m) = \{\lambda \in T_m^* M; \lambda(w) \leq 0 \forall w \in \Lambda_m\}$.

Then, identifying $T_m^* M$ and $T_m M$ through the isomorphism $\omega(v, \cdot) \mapsto v$, we see that in both cases $C_b^o(m)$ identifies with Γ_m (see the sign conventions for symplectic geometry in Appendix A.1). Since this identification between $T_m^* M$ and $T_m M$ is continuous in m , we get the result. \square

⁶The formula for the tangent cone in point 2) at $m \notin \Sigma_{(2)}$ is perturbed since we take coordinates, but this perturbation is smooth since $a_j \rightarrow a_*$ in the C^∞ topology.

B.2 What is there exactly in the cone Γ_m when $m \in \Sigma_{(2)}$?

Lemma 7 and Proposition 29 state that the cones Γ_m are inner semi-continuous. It is natural to wonder whether a cone Γ_m can be much bigger than the set of limits of the cones Γ_{m_j} for m_j tending to m . The answer is given by the following:

Proposition 31. *For any $m \in \Sigma_{(2)}$, the cone Γ_m (resp. its boundary) is exactly given by all limits of the cones Γ_{m_j} for $m_j \notin \Sigma_{(2)}$ (resp. $m_j \in \Sigma \setminus \Sigma_{(2)}$) converging to m .*

Proof. As in Section 2.3, we work in a chart near m . Let $v \in \Gamma_m$, which, up to multiplication by a constant, we can take equal to $\partial_t + b$ according to (15). According to (19), we have to prove that b is the limit of $\frac{1}{2} \frac{a(m_j)^{\frac{1}{2}}}{\tau_j} \frac{H_a(m_j)}{a(m_j)^{\frac{1}{2}}}$ for some well-chosen $m_j \rightarrow m$. Playing with the multiplication factor $a(m_j)^{\frac{1}{2}}/\tau_j$, it is sufficient to show that if $a_m^*(\mathcal{I}(b)) = 1$, then b is the limit of $\frac{1}{2} \frac{H_a(m_j)}{a(m_j)^{1/2}}$ for some well-chosen $m_j \rightarrow m$.

Since our computations do not depend on t, τ , we replace m_j, m by $\pi_2(m_j), \pi_2(m)$ (omitted in the notations).

Following the computations of Lemma 8 and using the notation F for the “fundamental matrix” introduced in (16), we get

$$\begin{aligned} \frac{1}{2} \omega_X(H_a(m_j), w) &= -\frac{1}{2} da(m_j)(w) = -a_m(m_j - m, w) + o(m_j - m) \\ &= \omega_X(F(m_j - m), w) + o(m_j - m), \end{aligned}$$

and finally $\frac{1}{2} \frac{H_a(m_j)}{a(m_j)^{1/2}} = \frac{F(m_j - m)}{a_m(m_j - m)^{1/2}} + o(1)$. But it follows from (16) that

$$F : T_m(T^*X)/\ker(a_m) \rightarrow (\ker(a_m))^{\perp \omega_X}$$

is an isomorphism⁷. Thus, choosing the sequence (m_j) adequately, we can take $F(m_j - m)$ colinear to $b \in (\ker(a_m))^{\perp \omega_X}$, and then we compute

$$\begin{aligned} a_m^*(\mathcal{I}(F(m_j - m))) &= \sup_{w \notin \ker(a_m)} \frac{\omega(F(m_j - m), w)^2}{a_m(w)} = \sup_{w \notin \ker(a_m)} \frac{a_m(m_j - m, w)^2}{a_m(w)} \\ &= a_m(m_j - m). \end{aligned}$$

Hence, with this choice of m_j , any limit v' of $F(m_j - m)/a_m(m_j - m)^{1/2}$ is colinear to b and the above computation implies that

$$a_m^*(\mathcal{I}(v')) = a_m^*(\mathcal{I}(b)),$$

which implies that $F(m_j - m)/a_m(m_j - m)^{1/2}$ tends to b . \square

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⁷It follows for example from Lax-Milgram’s lemma applied in the space $T_m(T^*X)/\ker(a_m)$, see Section 2.2.

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