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CONTROLLABILITY OF FRACTIONAL IMPULSIVE INTEGRO-DIFFERENTIAL CONTROL SYSTEM

PRISCILA. S. RAMOS, J. VANTERLER DA C. SOUSA AND E. CAPELAS DE OLIVEIRA

ABSTRACT. In 2020 Ramos et al.[5], investigated the existence and uniqueness of mild solutions for a fractional problem of the type Eqs.(1.1)-(1.3), however at the time, it was not considered the control function μ . In this paper, we impose control μ and through necessary and sufficient conditions, we investigate the results of controllability of fractional impulsive integro-differential control system in Banach spaces using the fixed point technique and the (α, θ) -resolvent operator.

1. INTRODUCTION

Over the years, the control theory has been highlighted in the scientific community. Controllability is one of the elementary concepts in mathematical control theory. It is well known that the question of controllability plays a fundamental role in the design of engineering control problems [21, 23, 37, 38]. In fact, the most important property of a control system is only controllability. It is possible to find numerous works of differential and integro-differential equations that discuss the controllability of solutions, as one can see in [22, 37, 39, 40] and references therein.

On the other hand, fractional calculus and fractional differential equations have been investigated extensively due mainly to their demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering such as physics, economics, medicine, control theory, aerodynamics and electromagnetics, [8, 9, 11, 12, 13, 14]. In this sense, what has been noted is that once you have control theory in hand, specifically controllability, it has been noted that investigating solutions of differential and integro-differential equations with non-instantaneous, evolution and abstract impulses, began to be the target of studies in these last decades [15, 16, 17, 18, 19, 20, 21]. For a reading on some works, see [22, 23, 24] and references therein. Controllability problems for different types of differential equations are considered in many papers [25, 26, 27, 28, 29, 30, 31, 32]. In 2011 Debbouche and Baleanu [18], investigated the controllability result of a class of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems in a Banach space, using fixed point technique. Em 2015, Liu and Li [25], investigated the approximate controllability of the following fractional evolution control systems involving Riemann–Liouville fractional derivatives:

$$\begin{aligned} D_t^\alpha x(t) &= Ax(t) + Bu(t) + f(t, x(t)), \quad t \in (0, b], \quad 0 < \alpha < 1, \\ I_t^{1-\alpha} x(t)|_{t=0} &= x_0 \in X \end{aligned}$$

where D_t^α denotes the Riemann-Liouville fractional derivative of order α with the lower limit zero. $A : D(A) \subset X \rightarrow X$ is the infinitesimal generator of a C_0 -semigroup $T(t)_{t \geq 0}$ on a Banach space X . $f : [0, b] \times X \rightarrow X$ is a given function to be specified. The control function u takes value in $V = L^p([0, b], U)$, $p > \frac{1}{\alpha}$, and U is a Banach space. B is a linear operator from V into $L^p([0, b]; X)$.

Mu [10] investigated the existence of mild solutions for the impulsive fractional evolution equations of the form

$$\begin{cases} D_{0+}^\alpha u(t) + Au(t) &= f(t, u(t)), \quad t \in I := [0, T], \quad t \neq t_k \\ u(0) + g(u) &= u_0 \\ \Delta u|_{t=t_k} &= I_k(u(t_k)), \quad k = 1, 2, \dots, m \end{cases}$$

where D_{0+}^α is the Caputo fractional derivative with $0 < \alpha < 1$, $A : D(A) \subset X \rightarrow X$ is a linear closed densely defined operator, $-A$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators $(T(t)_{t \geq 0})$, $0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = T$, $f : I \times X \rightarrow X$ is continuous, $g : PC(I, X) \rightarrow X$ is continuous, the impulsive function $I_k : X \rightarrow X$ is continuous, $\Delta u|_{t=t_k} = u(t^+) - u(t^-)$, where $u(t^+)$, $u(t^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively.

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On necessary and sufficient conditions, in 2021, Kumar et al. [9], investigated the fractional stability damped differential system with non-instantaneous impulsive given by

$$\begin{cases} D^\alpha x(t) &= AD^\beta x(t) + M \left(t, x(t), \int_0^t H(t, r, x(t)) dr \right), t \in \bigcup_{i=0}^m (r_i, t_{i+1}] \\ x(t) &= \mathcal{I}_i(t, x(t_i^-)), t \in (t_i, r_i], i = 1, 2, \dots, m \\ x'(t) &= \mathcal{G}_i(t, x(t_i^-)), t \in (t_i, r_i], i = 1, 2, \dots, m \\ x(0) &= x_0, x'(0) = x_1 \end{cases}$$

and for the controllability, was considered the following system

$$\begin{cases} D^\alpha x(t) &= AD^\beta x(t) + Bu(t) + M \left(t, x(t), \int_0^t H(t, r, x(t)) dr \right), t \in \bigcup_{i=0}^m (r_i, t_{i+1}] \\ x(t) &= \mathcal{I}_i(t, x(t_i^-)), t \in (t_i, r_i], i = 1, 2, \dots, m \\ x'(t) &= \mathcal{G}_i(t, x(t_i^-)), t \in (t_i, r_i], i = 1, 2, \dots, m \\ x(0) &= x_0, x'(0) = x_1 \end{cases}$$

where D^α e D^β denote Caputo fractional derivatives of order $1 < \alpha \leq 2$ and $0 < \beta \leq 1$, respectively. For more details on the parameters A, B, H, \mathcal{I}_i e \mathcal{G}_i , see reference [9]. Other interesting considerations on controllability can be found at [34, 35, ?, 38, 39, 40] and references therein.

However, there are some problems and open questions when discussing the controllability of mild solutions for fractional operators that involve the ψ -Hilfer fractional derivative. Over the years, Sousa and Oliveira [1], introduced the so-called ψ -Hilfer fractional derivative and due to the impact on the scientific community, this derivative served as a motivation to discuss various problems of differential equations. Here we highlight the type Leibniz I and II rule [3]; and the Laplace transform with respect to another function. In this sense, it allowed the discussion of mild solutions of fractional differential equations. However, there are still issues that prevent a closed form for the mild solution of fractional differential equations, which makes the theory still under construction.

Motivated by these questions and by the works presented above, in this present paper, we consider the fractional impulsive integro-differential control system of the form

$$(1.1) \quad {}^C D_{0+}^\alpha \theta(t) + \mathcal{A}(t, \theta(t))\theta(t) = (\mathbf{B}\mu)(t) + \Phi(t, \theta(t)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(t, s, \theta(s)) ds$$

$$(1.2) \quad \theta(0) + \Xi(\theta) = \theta_0$$

$$(1.3) \quad \Delta\theta(t_i) = I_i(\theta(t_i)), i = 1, \dots, n, 0 < t_1 < \dots < t_n < b$$

where ${}^C D_{0+}^\alpha(\cdot)$ is the Caputo fractional derivative of order $0 < \alpha \leq 1, t \in J := [0, b]$, the state $\theta(\cdot)$ takes values in the Banach space Λ , $\theta_0 \in \Lambda, i = 1, 2, \dots, n$ and Λ into Λ such that $D(A)$ is independent of t , it is assumed also that $-\mathcal{A}(t, \cdot)$ generates an addition in the Banach space Λ , the control function μ belongs to the spaces $L^2(J, U)$ a Banach of admissible control functions with U as a Banach space and $\mathbf{B} : U \rightarrow \Lambda$ is a bounded linear operator. In addition, we have $\Phi : [0, b] \times \Lambda \rightarrow \Lambda, g : \Omega \times \Lambda \rightarrow \Lambda, \Xi : \mathcal{PC}([0, b], \Lambda) \times \Lambda \rightarrow \Lambda$ and $\Delta\theta(t_i) = \theta(t_i^+) - \theta(t_i^-)$.

In 2020 Ramos et al. [5], investigated the existence and uniqueness of mild solutions for a fractional problem of the type Eqs.(1.1)-(1.3), however at the time, it was not considered the control function μ . In this sense, we impose μ control to discuss the main objective of this paper. This paper is a natural continuation of the paper [5].

For the discussion of the main result of this paper, we assume the following conditions:

H₁ The bound linear operator $W : L^2(J, U) \rightarrow \Lambda$ defined by

$$W_\mu = \int_0^b \mathcal{R}_{(\alpha, \theta)}(b, s)(\mathbf{B}\mu)(s) ds,$$

has an induced inverse operator \tilde{W}^{-1} which takes values in $L^2(J, U)/\ker W$ and there exist positive constants \mathbf{M}_1 and \mathbf{M}_2 such that $\|\mathbf{B}\| \leq \mathbf{M}_1$ and $\|\tilde{W}^{-1}\| \leq \mathbf{M}_2$.

H₂ $h : \mathcal{PC}(J; \Omega) \rightarrow Y$ is Lipschitz continuous in Λ and bounded in Y , that is, there exist constants $\mathbf{K}_1 > 0$ and $\mathbf{K}_2 > 0$ such that

$$\begin{aligned} \|h(\theta)\|_Y &\leq \mathbf{K}_1, \\ \|h(\theta) - h(v)\|_Y &\leq \mathbf{K}_2 \max_{t \in J} \|\theta - v\|_{\mathcal{PC}}, \quad \theta, v \in \mathcal{PC}(J; \Lambda). \end{aligned}$$

For conditions (\mathbf{H}_3) - (\mathbf{H}_5) let Z be taken as both Λ and Y .

\mathbf{H}_3 $g : \Omega \times Z \rightarrow Z$ is continuous and there exist constants $\mathbf{K}_3 > 0$ and $\mathbf{K}_4 > 0$ such that

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|g(t, s, \theta) - g(t, s, v)\|_Z ds \leq \mathbf{K}_3 \|\theta - v\|_Z, \quad \theta, v \in Z,$$

$$\mathbf{K}_4 = \max \left\{ \int_0^t \|g(t, s, 0)\|_Z ds; (t, s) \in \Omega \right\}.$$

\mathbf{H}_4 $f : J \times Z \rightarrow Z$ is continuous and there exist constants $\mathbf{K}_5 > 0$ and $\mathbf{K}_6 > 0$ such that

$$\|f(t, \theta) - f(t, v)\|_Z \leq \mathbf{K}_5 \|\theta - v\|_Z, \quad \theta, v \in \Omega,$$

$$\mathbf{K}_6 = \max_{t \in J} \|f(t, 0)\|_Z.$$

\mathbf{H}_5 $I_i : \Lambda \rightarrow \Lambda$ are continuous and there exist constants $l_i > 0$, $i = 1, 2, \dots, m$ such that

$$\|I_i(\theta) - I_i(v)\| \leq l_i \|\theta - v\|, \quad \theta, v \in \Lambda.$$

Let us take $\mathbf{M}_0 = \max \|\mathcal{R}_{(\alpha, \theta)(t, s)}\|_{B(Z)}$, $0 \leq s \leq t \leq b$, $\theta \in \Lambda$.

\mathbf{H}_6 There exist positive constants $\delta_1, \delta_2, \delta_3 \in (0, \delta/3]$ such that

$$\delta_1 = \mathbf{M}_0 \|\theta_0\| + \mathbf{M}_0 \mathbf{K}_1,$$

$$\delta_2 = \mathbf{M}_0 \mathbf{M}_1 \mathbf{M}_2 b \left(\|\theta_1\| + \mathbf{M}_0 \|\theta_0\| + \mathbf{M}_0 \mathbf{K}_1 + \mathbf{M}_0 \tilde{\theta} + \mathbf{M}_0 \xi \right) \text{ and}$$

$$\delta_3 = \mathbf{M}_0 \tilde{\theta} + \mathbf{M}_0 \xi,$$

where $\xi = \sum_{i=1}^m (l_i \delta + \|I_i(0)\|)$.

The main contribution of this paper is, through necessary and sufficient conditions, to attack the controllability of mild solutions of fractional impulsive integro-differential control system given by Eqs.(1.1)-(1.3). In other words, we will attack the following result:

Theorem 1.1. *Suppose that the operator $-\mathcal{A}(t, \theta)$ generates an (α, θ) -resolvent family whith $\|\mathcal{R}_{(\alpha, \theta)}(t, s)\| \leq \mathbf{M}e^{-\sigma(t-s)}$ for some constant $\mathbf{M}, \sigma > 0$. If hypotheses (\mathbf{H}_1) - (\mathbf{H}_6) are satisfied, then the fractional control integro-differential system Eq.(1.1) with nonlocal condition Eq.(1.2) and impulsive condition Eq.(1.3) is controllable on J .*

In the rest, the paper is organized as follows. Section 2 presents basic concepts of Riemann-Liouville fractional integral and ψ -Hilfer fractional derivative, in addition to some particular cases. In this sense, we present the concepts of mild solution and family (α, θ) -resolvent. In section 3, we investigate a fundamental Lemma and attack the main objective of the paper, that is, the controllability of mild solutions of fractional impulsive integro-differential control system.

2. PRELIMINARIES

For the preparation of this paper, we will consider X and Y be two Banach spaces such that Y is densely and continuously embedded in X . For any Banach space Z , the norm of Z is denoted by $\|\cdot\|_Z$.

Let $p \in [1, \infty) \subset \mathbb{R}$ and $\mathbf{J} = [a, b] \subset \mathbb{R}$. The space of the real p -integrable functions, in the Lebesgue sense, $L^p(J)$, equipped with its canonical norm, is given by [2, 5]

$$L^p(\mathbf{J}) := \left\{ f : \mathbf{J} \rightarrow \mathbb{R}; \int_a^b |f(t)|^p dt < \infty \right\}$$

e

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{1/p},$$

respectively. The pair $(L^p(\mathbf{J}), \|f\|_p)$ is a Banach space.

Consider the Banach space $(\mathbf{E}, \|\cdot\|)$ and $n \in \mathbb{N}$. The space of continuous functions and the space of continuously differentiable functions n -times:

$$\left(C(\mathbf{J}, \mathbf{E}) := \{f : J \rightarrow \mathbf{E}; f : \text{continuous}\}, \quad \|f\|_C := \sup_{t \in J} |f(t)| \right) \text{ and}$$

$$\left(C^n(\mathbf{J}, \mathbf{E}) := \{f : J \rightarrow \mathbf{E}; f^{(n)} \in C(J, \mathbf{E})\}, \quad \|f\|_{C^n} := \sup_{t \in J} |f^{(n)}(t)| \right)$$

are Banach spaces.

Let $\mathbf{J} = [a, b] \subset \mathbb{R}$ an interval, with $0 < a < b < \infty$, then the space of the functions n -times absolutely continuous is given by

$$AC^n(\mathbf{J}, \mathbb{R}) = AC^n(J) = \left\{ f : \mathbf{J} \rightarrow \mathbb{R}; f^{(n-1)} \in AC(\mathbf{J}) \right\}.$$

Let \mathbf{E} a Banach space and $a = t_0 < t_1 < \dots < t_n = b$ an n -partition of the interval $\mathbf{J} \subset \mathbb{R}$. The space of weighted continuous functions, is given by

$$\mathcal{PC}(\mathbf{J}, \mathbf{E}) := \left\{ \begin{array}{l} f : \mathbf{J} \rightarrow \mathbf{E}; f(t) \text{ be continuous in } t \neq t_k, \text{ left continuous} \\ \text{in } t = t_k \text{ there is the limit on the right, } f(t_k^+), \text{ for } k = 1, 2, \dots, n. \end{array} \right\},$$

equipped with the standard $\|f\|_{\mathcal{PC}} = \{\sup \|f(t)\|; t \in \mathbf{J}\}$ is a Banach space.

Definition 2.1. [1, 4, 6] Let (a, b) ($-\infty \leq a < b \leq \infty$) be a finite or infinite interval of the real line \mathbb{R} and let $\alpha > 0$. In addition, let $\psi(t)$ be an increasing and positive monotone function on (a, b) , having a continuous derivative $\psi'(t)$ on (a, b) . The left-sided fractional integral of function θ with respect to another function ψ on $[a, b]$ is defined by

$$(2.1) \quad I_{a^+}^{\alpha; \psi} \theta(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \theta(s) ds.$$

The right-sided fractional integral is defined in an analogous form [1, 4, 6].

Choosing $\psi(t) = t$ in Eq.(2.1), we have the Riemann-Liouville fractional integral given by [1, 4, 6]

$$(2.2) \quad I_{a^+}^{\alpha} \theta(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \theta(s) ds,$$

where $\Gamma(\cdot)$ is the gamma function and $f \in L^1(\mathbf{J}, \mathbb{R})$.

If $a = 0$, we can write $I^{\alpha} \theta(t) = (g_{\alpha} * \theta)(t)$, where

$$(2.3) \quad g_{\alpha}(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

and as usual $*$ denotes convolution of functions, also we have $\lim_{\alpha \rightarrow 0} g_{\alpha}(t) = \delta(t)$. From choosing $\psi(\cdot)$, we have another fractional integrals.

Here, we restrict to the Riemann-Liouville fractional integral to discuss the results of this paper. However, other formulations of fractional integrals can be obtained by choosing $\psi(\cdot)$ [1, 4, 6].

Also, we begin with the definition of the ψ -Hilfer fractional derivative.

Definition 2.2. [1, 4, 6] Let $n-1 < \alpha < n$, with $n \in \mathbb{N}$, let \mathbf{J} be an interval such that $-\infty \leq a < b \leq \infty$ and let $\theta, \psi \in C^n(\mathbf{J}, \mathbb{R})$ be two functions, such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in \mathbf{J}$. The left-sided ψ -Hilfer fractional derivative ${}^{\mathbf{H}}\mathbb{D}_{a^+}^{\alpha, \beta; \psi}(\cdot)$ of a function θ , of order α and type $0 \leq \beta \leq 1$ is defined by

$$(2.4) \quad {}^{\mathbf{H}}\mathbb{D}_{a^+}^{\alpha, \beta; \psi} \theta(t) := I_{a^+}^{\beta(n-\alpha); \psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{a^+}^{(1-\beta)(n-\alpha); \psi} \theta(t).$$

The right-sided ψ -Hilfer fractional derivative is defined in an analogous form [1, 4, 6].

Choosing $\psi(t) = t$ and taking the limit $\beta \rightarrow 1$, on both sides of the Eq.(2.4), we have the Caputo fractional derivative given by [1, 4, 6]

$$(2.5) \quad {}^{\mathbf{C}}\mathcal{D}_{a^+}^{\alpha} \theta(t) = I_{a^+}^{(n-\alpha); \psi} \left(\frac{d}{dt} \right)^n \theta(t) = I_{a^+}^{(n-\alpha); \psi} \theta^{(n)}(t).$$

To investigate our main result, we use Caputo fractional derivative as in Eq.(2.5).

Definition 2.3. By a mild solution of system Eqs.(1.1)-(1.3) we mean a function $\theta \in \mathcal{PC}([0, b]; \Lambda)$ with values in Ω satisfying the integral equation

$$(2.6) \quad \begin{aligned} \theta_{\mu}(t) &= \mathcal{R}_{(\alpha, \theta)}(t, 0)[\theta_0 - h(\theta)] \\ &+ \int_0^t \mathcal{R}_{(\alpha, \theta)}(t, s) \left[(\mathbf{B}\mu)(s) + f(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\eta)^{\alpha-1} g(s, \eta, \theta(\eta)) d\eta \right] ds \end{aligned}$$

for all $t \in J$, for all $\theta_0 \in \Lambda$ and admissible control $\mu \in L^2(J, U)$.

Lemma 2.4. [5] *If the evolution family $\{U_\theta(t, s)\}_{0 \leq s \leq t \leq b}$ is continuous and $\eta \in \mathcal{L}(J, \mathbb{R}^+)$, then the set $\left\{ \int_0^t U_\theta(t, s)\theta(s)ds \right\}$, $\|\theta(s)\| \leq \eta(s)$ for a.e. $s \in J$ is equicontinuous for $t \in J$.*

From [5] we know that for any fixed $u \in \mathcal{PC}(J, \Lambda)$ there exists a unique continuous function $U_\theta : J \times J \rightarrow \mathbf{B}(\Lambda)$ defined on $J \times J$ such that

$$(2.7) \quad U_\theta(t, s) = I + \int_s^t \mathcal{A}_\theta(w)U_\theta(w, s)dw,$$

where $\mathbf{B}(\Lambda)$ denote the Banach space of bounded linear operator from Λ to Λ with the norm $\|\Theta\| = \sup\{\|\Theta(\theta)\|; \|\theta\| = 1\}$ and I stands for the identity operator on Λ , $\mathcal{A}_\theta = \mathcal{A}(t, \theta(t))$, we have [5]

$$U_\theta(t, t) = I, \quad U_\theta(t, s)U_\theta(s, r) = U_\theta(t, r), \quad (t, s, r) \in J \times J \times J$$

and

$$\frac{\partial U_\theta(t, s)}{\partial t} = \mathcal{A}_\theta(t)U_\theta(t, s), \quad \text{for almost all } t, s \in J.$$

Let \mathbf{E} be the Banach space formed from $D(\mathcal{A})$ with the graph norm. Since, $\mathcal{A}(t)$ is a closed operator, it follows that $\mathcal{A}(t)$ is in the set bounded from \mathbf{E} to Λ .

Definition 2.5. [11, 5] *Let $\mathcal{A}(t, \theta)$ be a closed and linear operator with domain $D(\mathcal{A})$ defined on a Banach space Λ and $\alpha > 0$. Let $\rho(\mathcal{A}(t, \theta))$ be the resolvent set of $\mathcal{A}(t, \theta)$. We call $\mathcal{A}(t, \theta)$ the generator of an (α, θ) -resolvent family if there exists $w \geq 0$ and a strongly continuous function $\mathcal{R}_{(\alpha, \theta)} : \mathbb{R}_+^2 \rightarrow \mathcal{L}(\Lambda)$ such that $\{\lambda^\alpha : \text{Re}(\lambda) > w\} \subset \rho(\mathcal{A})$ and for $0 \leq s \leq t \leq \infty$,*

$$(2.8) \quad \left(\lambda^\alpha I - \mathcal{A}(s, \theta) \right)^{-1} \nu = \int_0^\infty e^{-\lambda(t-s)} \mathcal{R}_{(\alpha, \theta)}(t, s) \nu dt, \quad \text{Re}(\lambda) > w, \quad (\theta, \nu) \in \Lambda^2.$$

In this case, $\mathcal{R}_{(\alpha, \theta)}(t, s)$ is called the (α, θ) -resolvent family generated by $\mathcal{A}(t, \theta)$.

Remark 2.6.

- We can deduce that Eqs.(1.1)-(1.3) is well posed if and only if $\mathcal{A}(t, \theta)$ is the generator of (α, θ) -resolvent family.
- Here $\mathcal{R}_{(\alpha, \theta)}(t, s)$ can be extracted from the evolution operator of the generator $\mathcal{A}(t, \theta)$.
- The (α, θ) -resolvent family is similar to the evolution for non-autonomous differential equation in a Banach space.

3. CONTROLLABILITY RESULT

Definition 3.1. *We shall say that the fractional system Eqs.(1.1)-(1.3) is controllable on the interval $J = [0, b]$ if for all $\theta_0, \theta_1 \in \Lambda$, there exists a constant $\mu \in L^2(J, U)$, such that the mild solution $\theta(\cdot)$ of systems Eqs.(1.1)-(1.3) corresponding to μ , verifies: $\theta(0) + h(\theta) = \theta_0$, $\Delta\theta(t_i) = I_i(\theta(t_i))$, $i = 1, 2, \dots, m$ and $\theta_\mu(b) = \theta_1$.*

Lemma 3.2. *Let $\mathcal{R}_{(\alpha, \theta)}(t, s)$ be the (α, θ) -resolvent family for the fractional problem Eqs.(1.1)-(1.3). There exists a constant $K > 0$ such that*

$$\|\mathcal{R}_{(\alpha, \theta)}(t, s)\omega - \mathcal{R}_{(\alpha, v)}(t, s)\omega\| \leq \mathbf{K}\|\omega\|_Y \int_s^t \|\theta(\tau) - v(\tau)\| d\tau,$$

for every $\theta, v \in \mathcal{PC}(J; \Lambda)$ with values in Ω and every $\omega \in Y$.

Proof. See [33]. □

Lemma 3.3. $\|\varphi(t)\| \leq \tilde{\theta}$, where $\varphi(t) = \int_0^t \left(f(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\eta)^{\alpha-1} g(s, \eta, \theta(\eta)) d\eta \right) ds$

Proof.

$$\begin{aligned} & \|\varphi(t)\| \\ &= \left\| \int_0^t \left(f(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\eta)^{\alpha-1} g(s, \eta, \theta(\eta)) d\eta \right) ds \right\| \\ &= \left\| \int_0^t \left(f(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\eta)^{\alpha-1} (g(s, \eta, \theta(\eta)) - g(s, \eta, 0) + g(s, \eta, 0)) + f(s, 0) - f(s, 0) \right) ds \right\| \end{aligned}$$

$$\begin{aligned} &\leq \int_0^t \left(\|f(s, \eta(s)) - f(s, 0)\| + \|f(s, 0)\| + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\eta)^{\alpha-1} \|g(s, \eta, \theta(\eta)) - g(s, \eta, 0)\| d\eta \right. \\ &+ \left. \frac{1}{\Gamma(\alpha)} \int_0^s (s-\eta)^{\alpha-1} \|g(s, \eta, 0)\| d\eta \right) ds. \end{aligned}$$

Using **(H₃)** and **(H₄)**, we obtain

$$\begin{aligned} \|\phi(t)\| &\leq \int_0^t (\mathbf{K}_5 \|\theta(s)\| + \mathbf{K}_6 + \mathbf{K}_3 \|\theta(s)\| + \mathbf{K}_4) ds \\ &= \mathbf{K}_5 \int_0^t \|\theta(s)\| ds + \mathbf{K}_6 \int_0^t ds + \mathbf{K}_3 \int_0^t \|\theta(s)\| ds + \mathbf{K}_4 \int_0^t ds \\ &\leq K_5 \int_0^t \|\theta(s)\| ds + \mathbf{K}_6 b + \mathbf{K}_4 b + \mathbf{K}_3 \int_0^t \|\theta(s)\| ds. \end{aligned}$$

Therefore, we concluded the proof. \square

Now, let's attack our main result, that is, the proof of Theorem 1.1.

Proof. Using hypothesis **(H₁)**, for an arbitrary function $\theta(\cdot)$, we define the control

$$\begin{aligned} \mu(t) &= \tilde{W}^{-1} \left[\theta_1 - \mathcal{R}_{(\alpha, \theta)}(b, 0) \theta_0 + \mathcal{R}_{(\alpha, \theta)}(b, 0) h(\theta) \right. \\ &- \int_0^b \mathcal{R}_{(\alpha, \theta)}(b, s) \left(f(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\eta)^{\alpha-1} g(s, \eta, \theta(\eta)) d\eta \right) ds \\ &- \left. \sum_{i=1}^m \mathcal{R}_{(\alpha, \theta)}(b, t_i) I_i(\theta(t_i)) \right] (t). \end{aligned}$$

We define an operator $\Xi : \mathbf{S}_\delta \rightarrow \mathbf{S}_\delta$ by

$$\begin{aligned} (\Xi \theta_\mu)(t) &= \mathcal{R}_{(\alpha, \theta)}(t, 0) \theta_0 - \mathcal{R}_{(\alpha, \theta)}(t, 0) h(\theta) \\ &+ \int_0^t \mathcal{R}_{(\alpha, \theta)}(t, \eta) \mathbf{B} \tilde{W}^{-1} \left[\theta_1 - \mathcal{R}_{(\alpha, \theta)}(b, 0) \theta_0 + \mathcal{R}_{(\alpha, \theta)}(b, 0) h(\theta) \right. \\ &- \int_0^b \mathcal{R}_{(\alpha, \theta)}(b, s) \left(f(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s, \tau, \theta(\tau)) d\tau \right) ds \\ &- \left. \sum_{i=1}^m \mathcal{R}_{(\alpha, \theta)}(b, t_i) I_i(\theta(t_i)) \right] (\eta) d\eta \\ &+ \int_0^t \mathcal{R}_{(\alpha, \theta)}(t, s) \left(f(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s, \tau, \theta(\tau)) d\tau \right) ds \\ &+ \sum_{0 < t_i < t} \mathcal{R}_{(\alpha, \theta)}(t, t_i) I_i(\theta(t_i)). \end{aligned}$$

Using this controller the operator Ξ has a fixed point, then a solution of Eq.(2.8). On the other hand, note that, clearly $\Xi \theta_\mu(b) = \theta_1$, which means that the control μ of the system Eqs.(1.1)-(1.3) from the initial state θ_0 to θ_1 in time b , provided we can obtain a fixed of the nonlinear operator Ξ .

Now we show Ξ maps \mathbf{S}_δ into itself.

$$\begin{aligned} \|(\Xi \theta_\mu)(t)\| &\leq \|\mathcal{R}_{(\alpha, \theta)}(t, 0) \theta_0\| + \|\mathcal{R}_{(\alpha, \theta)}(t, 0) h(\theta)\| \\ &+ \int_0^t \|\mathcal{R}_{(\alpha, \theta)}(t, \eta)\| \|\mathbf{B} \tilde{W}^{-1}\| \left[\|\theta_1\| + \|\mathcal{R}_{(\alpha, \theta)}(b, 0) \theta_0\| + \|\mathcal{R}_{(\alpha, \theta)}(b, 0) h(\theta)\| \right. \\ &+ \int_0^b \|\mathcal{R}_{(\alpha, \theta)}(b, s)\| \left(\|f(s, \theta(s))\| + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} (\|g(s, \tau, \theta(\tau))\| d\tau) \right) \\ &+ \left. \sum_{i=1}^m \|\mathcal{R}_{(\alpha, \theta)}(b, t_i)\| \|\{I_i(\theta(t_i)) - I_i(0)\| + \|I_i(0)\|\} \right] d\eta \\ &+ \int_0^t \|\mathcal{R}_{(\alpha, \theta)}(t, s)\| \left(\|f(s, \theta(s))\| + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \|g(s, \tau, \theta(\tau))\| d\tau \right) ds \end{aligned}$$

$$+ \sum_{0 < t_i < t} \|\mathcal{R}_{(\alpha, \theta)}(t, t_i)\| \left\{ \|I_i(\theta(t_i)) - I_i(0)\| + \|I_i(0)\| \right\}.$$

Using **(H₁)**, **(H₂)**, **(H₅)**, **(H₆)** and Lemma 3.3, yields

$$\begin{aligned} \|(\Xi\theta_\mu)(t)\| &\leq \mathbf{M}_0\|\theta_0\| + \mathbf{M}_0\mathbf{K}_1 + \int_0^t \mathbf{M}_0\mathbf{M}_1\mathbf{M}_2 \left[\|\theta_1\| + \mathbf{M}_0\|\theta_0\| + \mathbf{M}_0\mathbf{K}_1 \right. \\ &\quad + \left. \int_0^b \mathbf{M}_0 \left(\|f(s, \theta(s))\| + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \|g(s, \tau, \theta(\tau))\| d\tau \right) ds \right. \\ &\quad + \left. \mathbf{M}_0 \sum_{i=1}^m (I_i\delta + \|I_i(0)\|) \right] d\eta \\ &\quad + \mathbf{M}_0 \int_0^t \left(\|f(s, \theta(s))\| + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \|g(s, \tau, \theta(\tau))\| d\tau \right) ds \\ &\quad + \mathbf{M}_0 \sum_{i=1}^m (I_i\delta + \|I_i(0)\|) \\ &\leq \mathbf{M}_0\|\theta_0\| + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\mathbf{M}_1\mathbf{M}_2 \int_0^t \left[\|\theta_1\| + \mathbf{M}_0\|\theta_0\| + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\tilde{\tau} \right. \\ &\quad + \left. \mathbf{M}_0 \sum_{i=1}^m (I_i\delta + \|I_i(0)\|) \right] d\eta + \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0 \sum_{i=1}^m (I_i\delta + \|I_i(0)\|) \\ &= \mathbf{M}_0\|\theta_0\| + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\mathbf{M}_1\mathbf{M}_2 \int_0^t \left[\|\theta_1\| + \mathbf{M}_0\|\theta_0\| + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0\xi \right] d\eta \\ &\quad + \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0\xi \\ &\leq \mathbf{M}_0\|\theta_0\| + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\mathbf{M}_1\mathbf{M}_2 b (\|\theta_1\| + \mathbf{M}_0\|\theta_0\| + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0\xi) + \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0\xi, \end{aligned}$$

where $\xi = \sum_{i=1}^m (I_i\delta + \|I_i(0)\|)$.

From assumption **(H₆)**, we get $\|(\Xi\theta_\mu)(t)\| \leq \delta$. Thus, Ξ maps \mathbf{S}_δ into itself.

Now for $\theta, v \in \mathbf{S}_\delta$, we have

$$(3.1) \quad \|(\Xi\theta_\mu)(t) - (\Xi v_\mu)(t)\| \leq I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= \|\mathcal{R}_{(\alpha, \theta)}(t, 0)\theta_0 - \mathcal{R}_{(\alpha, v)}(t, 0)\theta_0\| + \|\mathcal{R}_{(\alpha, \theta)}(t, 0)h(\theta) - \mathcal{R}_{(\alpha, v)}(t, 0)h(v)\|, \\ I_2 &= \int_0^t \left\{ \left\| \mathcal{R}_{(\alpha, \theta)}(t, \eta) B\tilde{W}^{-1} \left[\theta_1 - \mathcal{R}_{(\alpha, \theta)}(b, 0)\theta_0 + \mathcal{R}_{(\alpha, \theta)}(t, 0)h(\theta) \right. \right. \right. \\ &\quad - \int_0^b \mathcal{R}_{(\alpha, \theta)}(b, s) \left(f(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s, \tau, \theta(\tau)) d\tau \right) ds \\ &\quad - \left. \left. \sum_{i=1}^m \mathcal{R}_{(\alpha, \theta)}(b, t_i) I_i(\theta(t_i)) \right] - \mathcal{R}_{(\alpha, v)}(t, \eta) B\tilde{W}^{-1} \left[\theta_1 - \mathcal{R}_{(\alpha, v)}(b, 0)\theta_0 + \mathcal{R}_{(\alpha, v)}(b, 0)h(v) \right. \right. \\ &\quad - \int_0^b \mathcal{R}_{(\alpha, v)}(b, s) \left(f(s, v(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s, \tau, v(\tau)) d\tau \right) ds \\ &\quad - \left. \left. \left. \sum_{i=1}^m \mathcal{R}_{(\alpha, v)}(b, t_i) I_i(v(t_i)) \right] \right\} \right\| ds, \\ I_3 &= \int_0^t \left\| \mathcal{R}_{(\alpha, \theta)}(t, s) \left(f(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s, \tau, \theta(\tau)) d\tau \right) \right. \\ &\quad - \left. \mathcal{R}_{(\alpha, v)}(t, s) \left(f(s, v(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s, \tau, v(\tau)) d\tau \right) \right\| ds \end{aligned}$$

and

$$I_4 = \sum_{i=1}^m \|\mathcal{R}_{(\alpha,\theta)}(t, t_i) I_i(\theta(t_i)) - \mathcal{R}_{(\alpha,v)}(t, t_i) I_i(v(t_i))\|.$$

Applying Lemma 3.2 and (\mathbf{H}_2) , we have

$$\begin{aligned} I_1 &\leq \|\mathcal{R}_{(\alpha,\theta)}(t, 0)\theta_0 - \mathcal{R}_{(\alpha,v)}(t, 0)\theta_0\| + \|\mathcal{R}_{(\alpha,\theta)}(t, 0)h(\theta) - \mathcal{R}_{(\alpha,v)}(t, 0)h(v)\| \\ &\quad + \|\mathcal{R}_{(\alpha,v)}(t, 0)h(\theta) - \mathcal{R}_{(\alpha,v)}(t, 0)h(v)\| \\ &\leq \mathbf{K} \|\theta_0\|_Y \int_0^t \|\theta(\tau) - v(\tau)\| d\tau + \mathbf{K} \|h(\theta)\| \int_0^t \|\theta(\tau) - v(\tau)\| d\tau + \|\mathcal{R}_{(\alpha,v)}(t, 0)\| \|h(\theta) - h(v)\| \\ &\leq \mathbf{K} \|\theta_0\| \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| a + \mathbf{K} a \mathbf{K}_1 \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| + \mathbf{M}_0 \mathbf{K}_2 \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| \\ (3.2) \quad &= (\mathbf{K} \|\theta_0\| a + \mathbf{K} a \mathbf{K}_1 + \mathbf{M}_0 \mathbf{K}_2) \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| \end{aligned}$$

Consider $\tilde{\mathbf{A}}(\theta)$ and $\tilde{\mathbf{B}}(v)$ given by

$$\begin{aligned} \tilde{\mathbf{A}}(\theta) &= \theta_1 - \mathcal{R}_{(\alpha,\theta)}(b, 0) \theta_0 + \mathcal{R}_{(\alpha,\theta)}(b, 0) h(\theta) \\ &\quad - \int_0^b \mathcal{R}_{(\alpha,\theta)}(b, s) \left(f(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s, \tau, \theta(\tau)) d\tau \right) ds \\ &\quad - \sum_{i=1}^m \mathcal{R}_{(\alpha,\theta)}(b, t_i) I_i(\theta(t_i)) \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathbf{B}}(v) &= \theta_1 - \mathcal{R}_{(\alpha,v)}(b, 0) \theta_0 + \mathcal{R}_{(\alpha,v)}(b, 0) h(v) \\ &\quad - \int_0^b \mathcal{R}_{(\alpha,v)}(b, s) \left(f(s, v(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s, \tau, v(\tau)) d\tau \right) ds \\ &\quad - \sum_{i=1}^m \mathcal{R}_{(\alpha,v)}(b, t_i) I_i(v(t_i)). \end{aligned}$$

Using Lemma 3.2 and the condition (\mathbf{H}_1) , we have

$$\begin{aligned} I_2 &\leq \int_0^t \|\mathcal{R}_{(\alpha,\theta)}(t, \eta) \mathbf{B} \tilde{W}^{-1} \tilde{\mathbf{A}}(\theta) - \mathcal{R}_{(\alpha,v)}(t, \eta) \mathbf{B} \tilde{W}^{-1} \tilde{\mathbf{A}}(v)\| d\eta \\ &\leq \|\mathbf{B}\| \|\tilde{W}^{-1}\| \mathbf{K}_2 \max \left\{ \|\tilde{\mathbf{A}}(\theta), \tilde{\mathbf{B}}(v)\| \right\} \int_0^t \|\theta(\tau) - v(\tau)\| d\tau \\ (3.3) \quad &\leq \mathbf{M}_1 \mathbf{M}_2 \mathbf{K}_2 \max \left\{ \|\tilde{\mathbf{A}}(\theta), \tilde{\mathbf{B}}(v)\|_Y \right\} a^2 \max_{\tau \in J} \|\theta(\tau) - v(\tau)\|. \end{aligned}$$

Note that,

$$\max \|\tilde{\mathbf{A}}(\theta), \tilde{\mathbf{B}}(v)\|_Y \leq \|\tilde{\mathbf{A}}(\theta)\|_Y + \|\tilde{\mathbf{B}}(v)\|_Y.$$

Using Lemma 3.2, the conditions (\mathbf{H}_2) , (\mathbf{H}_5) and (\mathbf{H}_6) , yields

$$\begin{aligned} \|\tilde{\mathbf{A}}(\theta)\|_Y &= \left\| \theta_1 - \mathcal{R}_{(\alpha,\theta)}(b, 0) \theta_0 + \mathcal{R}_{(\alpha,\theta)}(b, 0) h(\theta) - \int_0^b \mathcal{R}_{(\alpha,\theta)}(b, s) \left(f(s, \theta(s)) \right. \right. \\ &\quad \left. \left. + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s, \tau, \theta(\tau)) d\tau \right) ds - \sum_{i=1}^m \mathcal{R}_{(\alpha,\theta)}(b, t_i) \left[\left(I_i(\theta(t_i)) - I_i(0) \right) + I_i(0) \right] \right\| \\ &\leq \|\theta_1\|_Y + \|\mathcal{R}_{(\alpha,\theta)}(b, 0)\|_Y \|\theta_0\|_Y + \|\mathcal{R}_{(\alpha,\theta)}(b, 0)\|_Y \|h(\theta)\|_Y \\ &\quad + \int_0^b \|\mathcal{R}_{(\alpha,\theta)}(b, s)\| \|f(s, \theta(s))\| ds \\ &\quad + \int_0^b \|\mathcal{R}_{(\alpha,\theta)}(b, s)\| \left(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \|g(s, \tau, \theta(\tau))\| d\tau \right) ds \\ &\quad + \sum_{i=1}^m \mathbf{M}_0 \left(\|I_i(\theta(t_i)) - I_i(0)\| + \|I_i(0)\| \right) \end{aligned}$$

$$\begin{aligned}
& \leq \|\theta_1\|_Y + \mathbf{M}_0\|\theta_0\|_Y + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0 \left\{ \sum_{i=1}^m I_i\delta + \|I_i(0)\| \right\} \\
(3.4) \quad & \leq \|\theta_1\|_Y + \mathbf{M}_0\|\theta_0\|_Y + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0 \xi,
\end{aligned}$$

where $\xi = \sum_{i=1}^m (I_i\delta + \|I_i(0)\|)$ and

$$\begin{aligned}
\|\tilde{\mathbf{B}}(\theta)\|_Y &= \left\| \theta_1 - \mathcal{R}_{(\alpha,v)}(b,0) \theta_0 + \mathcal{R}_{(\alpha,v)}(b,0) h(v) - \int_0^b \mathcal{R}_{(\alpha,v)}(b,s) \left(f(s,v(s)) \right. \right. \\
&\quad \left. \left. + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s,\tau,v(\tau)) d\tau \right) ds - \sum_{i=1}^m \mathcal{R}_{(\alpha,v)}(b,t_i) I_i(v(t_i)) \right\| \\
&\leq \|\theta_1\|_Y + \|\mathcal{R}_{(\alpha,v)}(b,0)\|_Y \|\theta_0\|_Y + \|\mathcal{R}_{(\alpha,v)}(b,0)\|_Y \|h(v)\|_Y \\
&\quad + \int_0^b \|\mathcal{R}_{(\alpha,v)}(b,v)\| \left(\|f(s,v(s))\| + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s,\tau,\theta(\tau)) d\tau \right) ds \\
&\quad + \sum_{i=1}^m \|\mathcal{R}_{(\alpha,v)}(b,t_i)\| \left(\|I_i(v(t_i)) - I_i(0)\| + \|I_i(0)\| \right) \\
&\leq \|\theta_1\|_Y + \mathbf{M}_0\|\theta_0\|_Y + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0 \left\{ \sum_{i=1}^m I_i\delta + \|I_i(0)\| \right\} \\
(3.5) \quad & \leq \|\theta_1\|_Y + \mathbf{M}_0\|\theta_0\|_Y + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0\xi,
\end{aligned}$$

where $\xi = \sum_{i=1}^m (I_i\delta + \|I_i(0)\|)$.

Substituting the inequalities (3.4) and (3.5) in the inequality (3.3), yields

$$\begin{aligned}
I_2 &\leq \mathbf{M}_1\mathbf{M}_2\mathbf{K}2 \max \left\{ \|\tilde{\mathbf{A}}(\theta), \tilde{\mathbf{B}}(v)\|_Y \right\} a^2 \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| \\
(3.6) \quad &\leq \mathbf{M}_1\mathbf{M}_22\mathbf{K}a^2 \{ \|\theta_1\|_Y + \mathbf{M}_0(\|\theta_0\|_Y + \mathbf{K}_1 + \tilde{\tau} + \xi) \} \max_{\tau \in J} \|\theta(\tau) - v(\tau)\|.
\end{aligned}$$

Again, Lemma 3.2 and Lemma 3.3, (H₃), (H₄) and (H₆), yields

$$\begin{aligned}
I_3 &\leq \int_0^t \left\{ \left\| \mathcal{R}_{(\alpha,\theta)}(t,s) \left(f(s,\theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s,\tau,\theta(\tau)) d\tau \right) \right. \right. \\
&\quad \left. \left. - \mathcal{R}_{(\alpha,v)}(t,s) \left(f(s,\theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s,\tau,\theta(\tau)) d\tau \right) \right\| \right. \\
&\quad \left. + \left\| \mathcal{R}_{(\alpha,v)}(t,s) \left(f(s,v(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s,\tau,v(\tau)) d\tau \right) \right. \right. \\
&\quad \left. \left. - \mathcal{R}_{(\alpha,v)}(t,s) \left(f(s,v(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s,\tau,v(\tau)) d\tau \right) \right\| \right\} ds \\
&\leq \mathbf{K} \int_0^t \left\{ \left\| f(s,\theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s,\tau,\theta(\tau)) d\tau \right\|_Y \int_s^t \|\theta(\tau) - v(\tau)\| d\tau \right\} ds \\
&\quad + \mathbf{M}_0 \int_0^t \left(\|f(s,\theta(s)) - f(s,v(s))\| + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \|g(s,\tau,\theta(\tau)) - g(s,\tau,v(\tau))\| d\tau \right) ds \\
&\leq \mathbf{K} \tilde{\tau} a \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| \\
&\quad + \mathbf{M}_0\mathbf{K}_1 \int_0^t \|\theta(s) - v(s)\| ds + \mathbf{M}_0\mathbf{K}_3 \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| \\
&\leq \mathbf{K}\tilde{\tau} a \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| + \mathbf{M}_0\mathbf{K}_1 a \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| + \mathbf{M}_0\mathbf{K}_3 a \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| \\
(3.7) \quad &= \left(\mathbf{K}\tilde{\tau} a + \mathbf{M}_0\mathbf{K}_1 a + \mathbf{M}_0\mathbf{K}_3 a \right) \max_{\tau \in J} \|\theta(\tau) - v(\tau)\|.
\end{aligned}$$

Now, we conclude using Lemma 3.2, (H₅) and (H₆),

$$I_4 \leq \sum_{i=1}^m \left(\|\mathcal{R}_{(\alpha,\theta)}(t,t_i) I_i(\theta(t_i)) - \mathcal{R}_{(\alpha,v)}(t,t_i) I_i(v(t_i))\| + \|\mathcal{R}_{(\alpha,v)}(t,t_i) I_i(\theta(t_i))\| \right)$$

$$\begin{aligned}
& - \mathcal{R}_{(\alpha,v)}(t, t_i) I_i(v(t_i)) \Big\| \\
& \leq \sum_{i=1}^m \left[\mathbf{K} \left(\|I_i(\theta(t_i)) - I_i(0)\| + \|I_i(0)\| \right) \int_{t_i}^t \|\theta(\tau) - v(\tau)\| d\tau \right. \\
& + \mathbf{M}_0 \|I_i(\theta(t_i)) - I_i(v(t_i))\| \Big] \\
& \leq \sum_{i=1}^m \left[\mathbf{K} \left(l_i \delta + \|I_i(0)\| \right) \int_{t_i}^t \|\theta(\tau) - v(\tau)\| d\tau + \mathbf{M}_0 l_i \|\theta(\tau) - v(\tau)\| \right] \\
& \leq \mathbf{K} \sum_{i=1}^m \left(l_i \delta + \|I_i(0)\| \right) a \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| + \mathbf{M}_0 \sum_{i=1}^m l_i \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| \\
& = \mathbf{K} \xi a \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| + \mathbf{M}_0 \sum_{i=1}^m l_i \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| \\
(3.8) \quad & = \left(\mathbf{K} \xi a + \mathbf{M}_0 \sum_{i=1}^m l_i \right) \max_{\tau \in J} \|\theta(\tau) - v(\tau)\|.
\end{aligned}$$

Substituting the inequalities (3.2), (3.6), (3.7) and (3.8) in the inequality (3.1), we obtain

$$\begin{aligned}
\|(\Xi\theta_\mu)(t)\| & \leq I_1 + I_2 + I_3 + I_4 \\
& = \left(\mathbf{K} \|\theta_0\| a + \mathbf{K}\mathbf{K}_1 + \mathbf{M}_0\mathbf{K}_2 \right) \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| \\
& + \mathbf{M}_1\mathbf{M}_2 2a^2 \mathbf{K} \left(\|\theta_1\|_Y + \mathbf{M}_0 (\|\theta_0\|_Y + \mathbf{K}_1 + \tilde{\tau} + \xi) \right) \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| \\
& + \left(\mathbf{K} \xi a + \mathbf{M}_0 \sum_{i=1}^m l_i \right) \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| + \left(\mathbf{K} \xi a + \mathbf{M}_0 \sum_{i=1}^m l_i \right) \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| \\
& = \left\{ \mathbf{K} \|\theta_0\| a + \mathbf{K} a \mathbf{K}_1 + \mathbf{M}_0 \mathbf{K}_2 + \mathbf{M}_1 \mathbf{M}_2 2a \mathbf{K} \left(\|\theta_1\|_Y \right. \right. \\
& + \left. \left. \mathbf{M}_0 (\|\theta_0\|_Y + \mathbf{K}_1 + \tilde{\tau} + \xi) \right) + \mathbf{K} \tilde{\tau} a + \mathbf{M}_0 \mathbf{K}_1 a + \mathbf{M}_0 \mathbf{K}_3 + \mathbf{K} \xi a \right. \\
& + \left. \mathbf{M}_0 \sum_{i=1}^m l_i \right\} \max_{\tau \in J} \|\theta(\tau) - v(\tau)\| \\
& = \lambda \max_{\tau \in J} \|\theta(\tau) - v(\tau)\|,
\end{aligned}$$

where $\lambda = \mathbf{K} \|\theta_0\| a + \mathbf{K} a \mathbf{K}_1 + \mathbf{M}_0 \mathbf{K}_2 + \mathbf{M}_1 \mathbf{M}_2 2a \mathbf{K} \left(\|\theta_1\|_Y + \mathbf{M}_0 (\|\theta_0\|_Y + \mathbf{K}_1 + \tilde{\tau} + \xi) \right) + \mathbf{K} \tilde{\tau} a + \mathbf{M}_0 \mathbf{K}_1 a + \mathbf{M}_0 \mathbf{K}_3 + \mathbf{K} \xi a + \mathbf{M}_0 \sum_{i=1}^m l_i$.

Therefore, Ξ is a contraction mapping and hence there exists a unique fixed point $\theta \in \Lambda$, such that $\Xi\theta(t) = \theta(t)$. Any fixed point of Ξ is a mild solution of Eqs.(1.1)-(1.3) on J which satisfies $\theta(a) = \theta_1$. Thus, system Eqs.(1.1)-(1.3) is controllable on J . \square

Authors' contributions

All authors' contributions to this manuscript are the same. All authors read and approved the final manuscript.

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