



# **CALCULUS OF VARIATIONS AND OPTIMAL CONTROL WITH GENERALIZED DERIVATIVE**

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# CALCULUS OF VARIATIONS AND OPTIMAL CONTROL WITH GENERALIZED DERIVATIVE

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## Abstract

In this paper, using the recently defined generalized derivative, we present a generalized formulation of variation of calculus, which includes the classical and conformable formulation as particular cases. In the first part of the article, through the properties of this generalized derivative, we discuss the generalized versions of the Bois-Reymond lemma, a Tonelly's type existence theorem, Euler-Lagrange equation, D'Alembert principle, du Bois-Reymond optimality condition and Noether's theorem. In the second step, we discuss the Picard-Lindelof theorem, Gronwall inequality, Pontryagin maximum principle and Noether principle for optimal control. Finally, to end the article, clarify and make the article more robust and important, we end with an application involving the fractional time Schrödinger equation.

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*Key words and phrases:* Calculus of variations, optimal control, Noether's theorem, time-fractional Schrödinger equation.

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## 1 Introduction and motivation

In 1918, Noether published a paper that strongly influenced the physics of the twentieth century [47]. During the 20th century, Hilbert, Noether, Tonelli, Lebesgue and Hadamard, among others, made notable contributions, Morse applied the calculus of variations to what is now known as Morse theory, and Pontryagin, Rockafellar and Clarke developed new mathematical tools within the theory of optimal control, generalizing the calculus of variations. In particular, Noether’s Theorem is a central result in theoretical physics and in the calculus of variations. It expresses that any differentiable symmetry, coming from a physical system, has its corresponding conservation law, that is, to each (continuous) symmetry there corresponds a conservation law and vice versa. This result constitutes an explanation of why there are conservation laws and physical magnitudes that do not change throughout the temporal evolution of a physical system [29, 41, 44, 61, 68, 70, 71].

It is natural to consider the theory of optimal control as an extension of the theory of the calculus of variations. In this direction, Pontriaguin Maximum (Minimum) principle is used in optimal control theory to find the best possible control to bring a dynamic system from one state to another, especially in the presence of constraints for state controls. or input. It was formulated by the Russian mathematician Lev Pontriaguin and his students in 1956. It has as a special case the Euler-Lagrange equation of the calculus of variations [38].

Speaking more precisely, the calculus of variations deals with the problem of determining the extreme values (maximum or minimum) of certain variable quantities called functionals. By functional, we mean a rule that associates a real number for each element (functions, vectors, tensors, ...) in some class of the so-called spaces of admissible functions, that is, it is a relation that

assigns a real number to the functions of a certain space. Classical calculus shows us that the local extremes of a function can only occur at the points where the first derivative vanishes, hence the Euler-Lagrange equations, the Noether's theorems, Fundamental invariance identities, and the other known classical results are obtained, those that are later expanded, with the optimal control, in the so-called Pontryagin maximum principle. Readers interested in obtaining more details, other historical aspects and more information about the applications, which throughout this time have been varied and extensive, can consult [29, 30, 34, 50, 61].

Since the first ideas of fractional calculus, the importance and relevance in the different areas of science is remarkable. Today, fractional calculus is a well-established theory, both when the approach is theoretical and in the sense of applications. What can be noted is that the vast majority of works involving fractional operators (differentiation and integration), are non-local [42, 45, 46, 62, 65, 66, 67]. One of the special properties of fractional operators is when we take the fractional order of differentiation and integration operators, equal to the whole case, i.e., when the order is no fractional, but integer. So, the natural question that arose in this century, what would happen, if motivated by the definition of the classic derivative by limit, introduce the fractional parameter?.

Then, in 2015 Abdeljawad [1], introduced the so-called conformable derivative, which is nothing more than a disturbance of the classical derivative. However, a problem for such an operator is that a fractional operator could not be considered, as it did not satisfy the criterion, which guarantees whether a given operator is a fractional operator or not [60]. In that sense, countless other derivatives and conformable integrals have been introduced over the past few years, and countless applications have been discussed and reports on advantages compared to the classic case [4, 5, 6, 8, 13, 63, 64].

In 2011 Malinowska and Torres [52], carried out an important work on fractional calculus of variations for a combined Caputo derivative. In this sense, Agrawal et al. [2], dedicated to generalized variational calculus in terms of multi-parameters fractional derivatives. What has been noticed over these years is the growing number of works involving the calculus of variations with fractional calculus. Thus, we can say that it is a well-consolidated and grounded theory. Note that these works are all involving fractional differentiation operators of the type: Caputo, Riesz, Caputo-Katugampola, among others, which are said to be non-local [3, 25, 51, 58, 69].

Now when it comes to calculus of variations via a generalized derivative, the number of works is very limited. In 2016 Lazo and Torres [48], carried out an important approach on variational calculus with conformable fractional derivatives, i.e., they discussed invariant conditions for conformable fractional problems of calculus of variations under the presence of external forces in the dynamics and addressed fractional versions of Noether's symmetry theorem. In this sense, the authors also investigate invariant conditions for fractional optimal control problems, via the Hamiltonian formalism. In 2017 Eroglu et al. [26] discussed an Optimal control problem for a conformable fractional heat conduction equation. In this sense, Eroglu and Yapiskan [28], attacked a local

problem generalization of transversality conditions for optimal control problem. Although this generalized derivative (local) theory has been discussed for more than 10 years, important and relevant work that involves calculus of variations is still very restricted, and this is a problem when looking for results to be used and references. To read some results, see [18, 27, 36, 49, 53, 73, 74] and the references therein.

Due to such restriction and the difficulty of finding relevant results that impact the area, this is one of the main reasons for carrying out this work. Motivated and inspired by the above works and the lack of results in the area, when it involves the calculus of variations and generalized derivatives, we will now highlight what are the main contributions of this paper, which was divided into three essential steps.

In general, the purpose of this paper is to present in a general way and with well-discussed results on the calculus of variations with the generalized derivative.

First, consider the generalized problem of calculus of variations (GPCV), that is, find  $q^* \in \mathcal{Y}$  such that

$$I(q^*) = \min\{I(q) \mid q \in \mathcal{Y}\} \quad (1)$$

where

$$I(q) = \phi_0(q(t_0)) + \phi_1(q(t_1)) + J_{F,t_0}^\alpha \left[ L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \right] (t_1). \quad (2)$$

Here, the Lagrangian  $L_\alpha^1([t_0, t_1]) \ni L : [t_0, t_1] \times \bar{\mathcal{U}} \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\phi_0, \phi_1 : \bar{\mathcal{U}} \rightarrow \mathbb{R}$  are given continuous functions.

In order to make the results of the paper clear and easy to understand, below we explain the main results obtained in the first step, namely:

1. We discussed a generalized du Bois–Reymond lemma.
2. We investigated the existence and uniqueness for GPCV (1)-(2) on necessary and sufficient conditions, presented in Theorem 13 itself, below.
3. Guaranteed that  $q \in \mathcal{Y}$  is a GPCV solution (1)-(2), we investigate the Euler–Lagrange equation and transversality conditions. In that sense, we attack the D’Alembert principle.
4. Finally, to end this step, we present and discuss the proof of the generalized Noether theorem for GPCV (1)-(2) using the Euler-Lagrange equation and the du Bois–Reymond optimality condition, as discussed in the previous items. In this sense, as a consequence of the generalized Noether theorem, we discuss a new and interesting result for autonomous generalized variational problems.

For the second step of the paper, we consider the follows optimization problem, given by

$$\forall u \in \mathcal{A} : J[u] := J_{F,t_0}^\alpha [L(t, q^u(t), u(t))](t_1). \quad (3)$$

$$\text{Find } v \in \mathcal{A} : (\text{CP}) \text{ and } \exists r > 0 \exists l \in \mathbb{N} \forall u \in \mathcal{A} \cap B_r^l(v) : J[v] \leq J[u]. \quad (4)$$

In order to develop the necessary optimality condition for Problem (3)-(4), we first assume that  $u \in \mathcal{A}$ , i.e.

$$\exists r > 0 : B_r(u) = \left\{ \bar{u} \in \mathcal{A} \mid \max_{t_0 \leq t \leq t_1} |\bar{u}(t) - u(t)| < r \right\} \subseteq \mathcal{A}.$$

Thereby, for all control  $\bar{u} \in \mathcal{A}$ , and for some  $\delta = \delta(u, \bar{u}) \in (0, 1)$  sufficiently small

$$\forall h \in (-\delta, \delta) : u + h\bar{u} \in \mathcal{A}, \quad (5)$$

and hence we can evaluate  $J[u + h\bar{u}] \in \mathbb{R}$ .

Continuing the discussion of the results of this paper, we highlight the main results of this second stage, namely:

1. We discussed the extension Picard–Lindelof theorem and Gronwall–Bellman inequality in integral form in the fractional conformable setting.
2. We discussed a proof of the weak Pontryagin Maximum Principle for optimal control problems with generalized derivatives.
3. We investigated the Noether principle for optimal control.
4. Other important results for the discussion of the previous items are discussed during the section.

Finally, since during the paper, the main purpose were to attack and discuss general and theoretical results on calculus of variations via generalized derivative, we prove that, under certain condition, solutions of the fractional linear Schrodinger equation coincides with the extremals of a quantum functional via the time-fractional Hamilton-Jacobi-Bellman equation. In addition we find constants of motion for a fractional quantum variational problem.

## 1.1 Connection with other works and consequences

One of the objectives when discussing a given problem using generalized derivatives, in particular, with  $0 < \alpha \leq 1$ , is to guarantee the special case, that is, when we take  $\alpha = 1$ , the results investigated also are valid. In this present paper, that special property is guaranteed.

A natural consequence and impact in the area of calculus of variations that this paper provides, is the new tools that have become available, in addition to increasing the range of varieties in the area. On the other hand, it is worth mentioning that this work in the structure it is in, with a general approach, is in fact the first. some works mentioned above, which approach the calculus of variations with so-called local derivatives, in most cases are restrictive to a specific theorem or case. Here, we took care and concern in trying to approach in general the results of the calculus of variations and optimal control.

In the rest, in section 2, we present the essential definitions of integral and generalized derivative and some extremely important results that are useful throughout the paper.

## 2 Preliminaries

Fractional and generalized calculus are areas in expansion and continuous development today. An attractive characteristic of this field is that there are numerous fractional operators, and this permits researchers to choose the most appropriate operator for the sake of modeling the problem under investigation, in [10] a fairly complete classification of these fractional operators is presented, with abundant information, on the other hand, in the work [9] some reasons are presented why new operators linked to applications and developments theorists appear every day. These operators had been developed by numerous mathematicians with a barely specific formulation, for instance, the Riemann-Liouville (RL), the Weyl, Erdelyi-Kober, Hadamard integrals, and the Liouville and Katugampola fractional operators and many authors have introduced new fractional operators generated from general classical local derivatives.

In addition, Chapter 1 of [7] presents a history of differential operators, both local and global, from Newton to Caputo and presents a definition of local derivative with new parameter, providing a large number of applications, with a difference qualitative between both types of operators, local and global. Most importantly, Section 1.4 Limitations and Strength of Local and Fractional derivatives concludes, " We can therefore conclude that both the Riemann-Liouville and Caputo operators are not derivatives, and then they are not fractional derivatives, but fractional operators. We agree with the result [72] that, the local fractional operator is not a fractional derivative" (p.24). As we said before, they are new tools that have demonstrated their usefulness and potential in the modeling of different processes and phenomena.

In [75] a generalized derivative was defined in the following way (see also [56]).

**Definition 1** *The generalized derivative of order  $0 < \alpha \leq 1$  of a function  $f : [a, b] \rightarrow \mathbb{R}$  is defined by*

$$\frac{d_F^\alpha f}{dt^\alpha}(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon F(t, \alpha)) - f(t)}{\varepsilon}, \quad (6)$$

where  $F(\alpha, t)$  is a given positive absolutely continuous function. If the limit (6) exist, then we say that  $f$  is  $\alpha$ -differentiable.

Different known local derivatives (conformable or not) can be obtained as particular cases of the kernel  $F(t, \alpha)$ , the interested reader can consult the references [33, 40, 43, 55].

**Remark 2** *If  $f \in C^1[a, b]$ , then we have from (6) that*

$$\frac{d_F^\alpha f}{dt^\alpha}(t) = F(\alpha, t) \frac{df}{dt}(t). \quad (7)$$

It is important to note that for  $F(t, \alpha) \equiv 1$  the generalized derivatives (6) reduces to first order ordinary derivatives (see [1]). Is also important to note that

differently from the majority of definitions of fractional derivative, including the popular Riemann–Liouville and Caputo fractional derivatives, the generalized derivative (6) is local operator and is related to ordinary derivatives if the function is differentiable (see Remark 2). For more on local fractional derivatives, we refer the reader to [1, 5, 12, 16] and references therein.

Usually, we use  $F$  defined as one-parameter Mittag-Leffler function  $\mathbf{E}_\alpha$ , defined by

$$\mathbf{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \alpha k)}, \quad \alpha, z \in \mathbb{C}, \quad \operatorname{Re}(\alpha) > 0,$$

where  $\Gamma$  is the well known Gamma function.

Now, we give the definition of a general fractional integral (see [1, 75]):

**Definition 3** *The generalized integral of order  $0 < \alpha \leq 1$  of a continuous function  $f : [a, t] \rightarrow \mathbb{R}$ , is defined by*

$$J_{F,a}^\alpha f(t) = \int_a^t \frac{f(x)}{F(x, \alpha)} dx. \quad (8)$$

Note that if  $F(t, \alpha) \equiv 1$ , we will have the classic Riemann integral.

**Definition 4** *Let  $E \subset \mathbb{R}$  be a measurable set, and let  $f : E \rightarrow \mathbb{R}$  be a measurable function. We say that  $f$  belongs to  $L_\alpha^1(E, \mathbb{R})$ , if the following property holds*

$$J_{F,E}^\alpha(|f(t)|)(\cdot) = \int_E \frac{|f(x)|}{F(x, \alpha)} dx < +\infty.$$

The following statement is analogous to the one known from the ordinary calculus.

**Theorem 5 (Fundamental theorem of calculus)** *Let  $f \in C^1[a, b]$  and  $0 < \alpha \leq 1$ . Then,*

$$J_{F,a}^\alpha \left( \frac{d_F^\alpha f}{dt^\alpha} \right) (t) = f(t) - f(a)$$

for all  $t \in [a, b]$ .

**Proof.** Since  $f \in C^1[a, b]$ , we have

$$J_{F,a}^\alpha \left[ \frac{d_F^\alpha f}{dt^\alpha} \right] (t) = \int_a^t \frac{1}{F(s, \alpha)} \frac{d_F^\alpha f}{dt^\alpha}(s) ds = \int_a^t f'(s) ds = f(t) - f(a),$$

which is the desired result. ■

**Theorem 6** *Let  $f \in C[a, b]$  and  $0 < \alpha \leq 1$ . Then,*

$$\frac{d_F^\alpha}{dt^\alpha} [J_{F,a}^\alpha f(t)] = f(t),$$

for all  $t \in [a, b]$ .



**Proof.** Let  $f \in C[a, b]$ . Theorem 5 gives for every  $t \in [a, b]$

$$\begin{aligned} (J_{F,a}^\alpha f(t))' &= \left( \int_a^t \frac{f(s)}{F(s, \alpha)} ds \right)' = \frac{f(t)}{F(t, \alpha)}, \\ \frac{d_F^\alpha}{dt^\alpha} [J_{F,a}^\alpha f(t)] &= F(t, \alpha) (J_{F,a}^\alpha f(t))' = F(t, \alpha) \frac{f(t)}{F(t, \alpha)} = f(t). \end{aligned}$$

■

The next result, which we will present, is about integration by parts, being extremely important for the present work (see [75]).

**Theorem 7 (Integration by parts [75])** *Let  $f, g : [a, b] \rightarrow \mathbb{R}$  differentiable functions and  $\alpha \in (0, 1]$ . Then, the following property hold*

$$J_{F,a}^\alpha \left[ f(t) \frac{d_F^\alpha g}{dt^\alpha}(t) \right] (b) = [f(t)g(t)]_a^b - J_{F,a}^\alpha \left[ g(t) \frac{d_F^\alpha f}{dt^\alpha}(t) \right] (b). \quad (9)$$

**Theorem 8** *Let  $\alpha \in (0, 1]$ ,  $g$   $\alpha$ -differentiable at  $t > 0$  and  $f$  differentiable at  $g(t)$  then  $\frac{d_F^\alpha(f \circ g)}{dt^\alpha}(t) = \frac{df(g)}{dt}(t) \frac{d_F^\alpha g}{dt^\alpha}(t)$ .*

**Proof.** We prove the result following a standard limit-approach. First, we consider that the function  $g$  is constant in a neighborhood of  $a > 0$ , then  $\frac{d_F^\alpha(f \circ g)}{dt^\alpha}(t) = 0$ . If  $g$  is not a constant function in a neighborhood of  $a > 0$  we can find an  $\varepsilon_0 > 0$  such that  $g(x_1) \neq g(x_2)$  for any  $x_1, x_2 \in (a - t_0, a + t_0)$ . Now, since  $g$  is continuous at  $a$ , for  $\varepsilon$  sufficiently small, we have

$$\begin{aligned} \frac{d_F^\alpha(f \circ g)}{dt^\alpha}(a) &= \lim_{\varepsilon \rightarrow 0} \frac{f(g((t + \varepsilon F(a, \alpha))) - f(g(a))}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{f(g((t + \varepsilon F(a, \alpha))) - f(g(a))}{g(a + \varepsilon F(a, \alpha)) - g(a)} \frac{g(a + \varepsilon F(a, \alpha)) - g(a)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{f(g((t + \varepsilon F(a, \alpha))) - f(g(a))}{g(a + \varepsilon F(a, \alpha)) - g(a)} \lim_{\varepsilon \rightarrow 0} \frac{g(a + \varepsilon F(a, \alpha)) - g(a)}{\varepsilon}. \end{aligned}$$

Making

$$\varepsilon_1 = g(a + \varepsilon F(a, \alpha)) - g(a)$$

in the first factor we have

$$\lim_{\varepsilon \rightarrow 0} \frac{f(g(a + \varepsilon F(a, \alpha))) - f(g(a))}{g(a + \varepsilon F(a, \alpha)) - g(a)} = \lim_{\varepsilon_1 \rightarrow 0} \frac{f(g(a) + \varepsilon_1) - f(g(a))}{\varepsilon_1}$$

from here

$$\begin{aligned} \frac{d_F^\alpha(f \circ g)}{dt^\alpha}(a) &= \lim_{\varepsilon_1 \rightarrow 0} \frac{f(g(a) + \varepsilon_1) - f(g(a))}{\varepsilon_1} \lim_{\varepsilon \rightarrow 0} \frac{g(a + \varepsilon F(a, \alpha)) - g(a)}{\varepsilon} \\ &= \frac{df(g)}{dt}(a) \frac{d_F^\alpha g}{dt^\alpha}(a). \end{aligned}$$

■

**Theorem 9 (Chain rule for functions of several variables)** *Let  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  ( $N \in \mathbb{N}$ ) be a differentiable function in all its arguments and  $y_1, \dots, y_N : \mathbb{R} \rightarrow \mathbb{R}$  be  $\alpha$ -differentiable functions. Then,*

$$\frac{d_F^\alpha f}{dt^\alpha}(y_1(t), \dots, y_N(t)) = \frac{\partial f}{\partial y_1} \frac{d_F^\alpha y_1}{dt^\alpha}(t) + \frac{\partial f}{\partial y_2} \frac{d_F^\alpha y_2}{dt^\alpha}(t) + \dots + \frac{\partial f}{\partial y_N} \frac{d_F^\alpha y_N}{dt^\alpha}(t).$$

**Proof.** The proof follows the ideas of the classical n-variables calculus and the application of the Theorem 8. ■

### 3 Generalized calculus of variations

Let  $t_0, t_1 \in \mathbb{R}$  with  $0 \leq t_0 < t_1$ . Let a connected open set  $\mathbb{U} \subset \mathbb{R}^n$  and two closed subsets  $\mathbb{U}_0, \mathbb{U}_1 \subset \overline{\mathbb{U}}$ . We denote by  $\text{AC}([t_0, t_1], \mathbb{R}^n)$  the class of all absolutely continuous arcs  $q : [t_0, t_1] \rightarrow \mathbb{R}^n$ .

**Definition 10** *The set of admissible arcs is defined by*

$$\mathcal{Y} = \{q \in \text{AC}([t_0, t_1], \mathbb{R}^n) \mid \forall t \in [t_0, t_1] \, q(t) \in \overline{\mathbb{U}}, \, q(t_0) \in \mathbb{U}_0, \, q(t_1) \in \mathbb{U}_1\}$$

We then consider the following problem of the calculus of variations:

**Problem 11 (Generalized problem of calculus of variations (GPCV))**

*Find  $q^\flat \in \mathcal{Y}$  such that*

$$I(q^\flat) = \min\{I(q) \mid q \in \mathcal{Y}\}$$

where

$$I(q) = \phi_0(q(t_0)) + \phi_1(q(t_1)) + J_{F, t_0}^\alpha \left[ L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \right] (t_1). \quad (10)$$

Here, the Lagrangian  $L_\alpha^1([t_0, t_1]) \ni L : [t_0, t_1] \times \overline{\mathbb{U}} \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\phi_0, \phi_1 : \overline{\mathbb{U}} \rightarrow \mathbb{R}$  are given continuous functions.

In order to obtain a necessary condition for the extremum of (10) we need the following lemma.

**Lemma 12 (Generalized du Bois–Reymond lemma)** *Set  $s, h \in L_\alpha^1([t_0, t_1])$ . If*

$$J_{F, t_0}^\alpha \left[ s(t) \cdot \eta(t) + h(t) \cdot \frac{d_F^\alpha \eta}{dt^\alpha}(t) \right] (t_1) = 0 \quad (11)$$

for any  $\eta \in \text{Lip}([t_0, t_1])$  with  $\eta(t_0) = \eta(t_1) = 0$ , then there exists a continuous representative  $\tilde{h}$  of  $h$  such that  $\tilde{h} \in \text{AC}([t_0, t_1])$  and  $\frac{d_F^\alpha \tilde{h}}{dt^\alpha} = s$  almost everywhere. In particular, for  $h = 0$ , if

$$J_{F, t_0}^\alpha [s(t) \cdot \eta(t)] (t_1) = 0 \text{ for all } \eta \in \text{Lip}([t_0, t_1]),$$

then  $s = 0$ .

**Proof.** We will prove this lemma in two steps:

- Step 1 : Let us assume that  $s \equiv 0$  almost everywhere in  $[t_0, t_1]$  and define the functions

$$h_n(t) = \begin{cases} h(t) & \text{if } |h(t)| \leq n \\ 0 & \text{otherwise,} \end{cases} \quad u_n = \frac{1}{t_1 - t_0} J_{F,t_0}^\alpha [h_n(t)](t_1)$$

and

$$u = \frac{1}{t_1 - t_0} J_{F,t_0}^\alpha [h(t)](t_1).$$

Therefore, we can see that  $h_n \rightarrow h$  and  $u_n \rightarrow u$  as  $n \rightarrow \infty$ .

Moreover, we also set the function

$$\eta_n(t) = J_{F,t_0}^\alpha [h_n(t) - u_n](t).$$

By our assumption,  $\eta_n$  is well defined and its belong to  $Lip([t_0, t_1])$ . We can easily check that  $\eta(t_1) = \eta(t_0) = 0$  and  $\frac{d_F^\alpha \eta_n}{dt^\alpha}(t) = h_n(t) - u_n$  almost everywhere. Hence, by Fatou's Lemma (see e.g. [11]), one has

$$\begin{aligned} J_{F,t_0}^\alpha [(h(t) - u)^2](t_1) &= J_{F,t_0}^\alpha \left[ \lim_{n \rightarrow \infty} (h(t) - u) \cdot (h_n(t) - u_n) \right](t_1) \\ &\leq \lim_{n \rightarrow \infty} J_{F,t_0}^\alpha \left[ (h(t) - u) \cdot \frac{d_F^\alpha \eta_n}{dt^\alpha}(t) \right](t_1) = 0. \end{aligned}$$

So, the only possibility for this to happen is  $h(t) = u$  almost everywhere. Thereby, we can define  $\tilde{h}(t) = u \forall t \in [t_0, t_1]$ .

- Step 2 : Now, we consider the general case  $s \not\equiv 0$  almost everywhere in  $[t_0, t_1]$  and define  $S(t) = J_{F,t_0}^\alpha s(t)(t_1)$ . So,  $\frac{d_F^\alpha S}{dt^\alpha}(t) = s(t)$  almost everywhere. Thus, by our hypotheses one has

$$\begin{aligned} 0 &= J_{F,t_0}^\alpha \left[ (s(t) \cdot \eta(t) + h(t) \cdot \frac{d_F^\alpha \eta}{dt^\alpha}(t)) \right](t_1) \\ &= J_{F,t_0}^\alpha \left[ \frac{d_F^\alpha}{dt^\alpha} (S(t) \cdot \eta(t)) \right](t_1) + J_{F,t_0}^\alpha \left[ (h(t) - S(t)) \cdot \frac{d_F^\alpha \eta}{dt^\alpha}(t) \right](t_1) \\ &= J_{F,t_0}^\alpha \left[ (h(t) - S(t)) \cdot \frac{d_F^\alpha \eta}{dt^\alpha}(t) \right](t_1). \end{aligned}$$

Hence, by the previous step,  $h(t) - S(t)$  is equal to some constant  $u$  almost everywhere and we can define  $\tilde{h}(t) = S(t) + u$ . Hence  $\frac{d_F^\alpha \tilde{h}}{dt^\alpha}(t) = s$  as claimed.

■

### 3.1 Existence and uniqueness of minimizers of (GPCV)

The next theorem is a Tonelli type theorem but the strategies for its proofs are different [15].

**Theorem 13 (Existence and uniqueness theorem)** *Let  $L, \phi_0, \phi_1$  of (GPCV). Let us define the augmented Lagrangian by*

$$\tilde{L}\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t); F(t, \alpha)\right) := \frac{L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right)}{F(t, \alpha)} \quad (12)$$

and assume the following hypotheses:

- **(H<sub>1</sub>)**  $\tilde{L}$  is a convex function with respect to the third variable;
- **(H<sub>2</sub>)** There exists  $k \geq 0$  and a Nagumo function for the functional  $I$   $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , that is,  $\lim_{t \rightarrow +\infty} \frac{\rho(t)}{t} = +\infty$  and

$$\tilde{L}(t, x, v; F(t, \alpha)) \geq \rho(|v|) - k, \quad \phi_0(q), \phi_1(q) \geq -k; \quad (13)$$

- **(H<sub>3</sub>)** For all  $r > 0$  there exists  $K(r) > 0$  such that

$$|\tilde{L}(t, x, v; F(t, \alpha)) - \tilde{L}(t, y, v; F(t, \alpha))| < K(r)\theta(|x - y|)\rho(|v|)$$

for all  $t \in [t_0, t_1]$ ,  $x, y \in \bar{U} \cap B_r$ ,  $v \in \mathbb{R}^n$  and  $F(t, \alpha)$  is defined in Definition 1. Here,  $\theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a nondecreasing function such that  $\lim_{r \rightarrow 0} \theta(r) = 0$ ;

- **(H<sub>4</sub>)** Either  $\mathbb{U}_0$  or  $\mathbb{U}_1$  is a compact set.

Then problem (GPCV) has at least one solution.

In addition, if

- **(H<sub>5</sub>)**  $\tilde{L}$  is strictly convex for any  $t \in [t_0, t_1]$ ,

this solution is unique.

**Proof.** Taking into account **(H<sub>4</sub>)**, we assume that  $\mathbb{U}_0$  is a compact set. In addition, set down any minimizing sequence  $(q_n)_{n \in \mathbb{N}}$  for  $I$ , i.e.,

$$I(q_n) \rightarrow m = \min\{I(q) \mid q \in \mathcal{Y}\} \text{ as } n \rightarrow \infty.$$

Our goal is to show that  $(q_n)_{n \in \mathbb{N}}$  admits a cluster point which is the required minimizer. We recall that  $m$  is a *cluster point* of the sequence  $(q_n)_{n \in \mathbb{N}}$  if, and only if,  $\forall \epsilon > 0$   $\{n \in \mathbb{N} : |q_n - m| < \epsilon\}$  is infinite.

First, we *claim*: the sequence of generalized fractional conformable derivatives  $\left(\frac{d_F^\alpha q_n(t)}{dt^\alpha}\right)_{n \in \mathbb{N}}$  is *equi-absolutely fractional conformable integrable*, that is,  $\forall \varepsilon > 0$  there exists  $\nu > 0$  such that  $A \subset [t_0, t_1]$ , then

$$|A| < \nu \implies J_{F,A}^\alpha \left| \frac{d_F^\alpha q_n}{dt^\alpha}(t) \right|(\cdot) < \varepsilon \text{ for all } n.$$

The hypothesis **(H<sub>2</sub>)** asserts that, for all  $\iota > 0$  there exists  $K_\iota > 0$  such that  $r \leq \frac{\rho(r)}{\iota}$  for all  $r \geq K_\iota$  and for all any measurable set  $A \subset [t_0, t_1]$ , one has

$$\begin{aligned} J_{F,A}^\alpha \left| \frac{d_F^\alpha q_n}{dt^\alpha}(t) \right|(\cdot) &\leq \frac{1}{\iota} J_{F,A \cap \{|\frac{d_F^\alpha q_n}{dt^\alpha}(t)| \geq K_\iota\}}^\alpha \left[ \rho \left( \left| \frac{d_F^\alpha q_n}{dt^\alpha}(t) \right| \right) \right](\cdot) \\ &\quad + J_{F,A \cap \{|\frac{d_F^\alpha q_n}{dt^\alpha}(t)| < K_\iota\}}^\alpha \left| \frac{d_F^\alpha q_n}{dt^\alpha}(t) \right|(\cdot) \\ &\leq \frac{1}{\iota} [I(q_n) + kt_1 + 2k] + |A|K_\iota \\ &\leq \frac{\Lambda}{\iota} + |A|K_\iota \\ &< \varepsilon \end{aligned} \tag{14}$$

for a suitable constant  $\Lambda > 0$  and by arbitrariness of  $\iota$ (large) and  $A$ (small). Thus, our claim is proved.

Notice that, since  $\left(\frac{d_F^\alpha q_n(t)}{dt^\alpha}\right)_{n \in \mathbb{N}} \in L_\alpha^1([t_0, t_1])$  is equi-absolutely fractional conformable integrable, the Dunford-Pettis Theorem (see [23]) remains valid. Its follows that, there exists a subsequence, which we denote again by  $\left(\frac{d_F^\alpha q_n(t)}{dt^\alpha}\right)_{n \in \mathbb{N}}$ , and a function  $\kappa \in L_\alpha^1([t_0, t_1], \mathbb{R}^n)$  such that, in the weak- $L_\alpha^1$  topology,

$$\frac{d_F^\alpha q_n}{dt^\alpha}(t) \rightharpoonup \kappa.$$

Since  $\left(\frac{d_F^\alpha q_n(t)}{dt^\alpha}\right)_{n \in \mathbb{N}}$  is equi-absolutely fractional conformable integrable and  $\mathbb{U}_0$  is a compact set, we have that the sequence  $(q_n)_{n \in \mathbb{N}}$  is equicontinuous and uniformly bounded. Therefore, by the Ascoli–Arzelà theorem (see e.g. [?]), the sequence  $(q_n)_{n \in \mathbb{N}}$  converges uniformly to some absolutely continuous function  $\hat{q}$ . Then, we can conclude that  $\hat{q} \in \mathcal{Y}$  because  $\hat{q}$  is a uniform limit. Having in mind Theorem 7, for all  $\eta \in C_0^1([t_0, t_1], \mathbb{R}^n)$ , one has by integral parts formula

(9)

$$\begin{aligned}
J_{F,t_0}^\alpha[\kappa(t) \cdot \eta(t)](t_1) &= \lim_{n \rightarrow \infty} J_{F,t_0}^\alpha \left[ \frac{d_F^\alpha q_n}{dt^\alpha}(t) \cdot \eta(t) \right] (t_1) \\
&= - \lim_{n \rightarrow \infty} J_{F,t_0}^\alpha \left[ q_n(t) \cdot \frac{d_F^\alpha \eta}{dt^\alpha}(t) \right] (t_1) = -J_{F,t_0}^\alpha \left[ \hat{q}(t) \cdot \frac{d_F^\alpha \eta}{dt^\alpha}(t) \right] (t_1) \\
&\Leftrightarrow J_{F,t_0}^\alpha \left[ \kappa(t) \cdot \eta(t) + \hat{q}(t) \cdot \frac{d_F^\alpha \eta}{dt^\alpha}(t) \right] (t_1) = 0.
\end{aligned}$$

Invoking the generalized du Bois–Reymond lemma (Lemma 12), we conclude that  $\kappa = \frac{d_F^\alpha \hat{q}}{dt^\alpha}$  almost everywhere.

Finally, to conclude the proof, we need to show that  $I(\hat{q}) \leq \liminf_{n \rightarrow \infty} I(q_n)$ . For this to happen, as  $\phi_0$  and  $\phi_1$  are two continuous functions, we just need to prove that

$$J_{F,t_0}^\alpha \left[ L \left( t, \hat{q}(t), \frac{d_F^\alpha \hat{q}}{dt^\alpha}(t) \right) \right] (t_1) \leq \liminf_{n \rightarrow \infty} J_{F,t_0}^\alpha \left[ L \left( t, q_n(t), \frac{d_F^\alpha q_n}{dt^\alpha}(t) \right) \right] (t_1).$$

We have,

$$\begin{aligned}
&J_{F,t_0}^\alpha \left[ L \left( t, \hat{q}(t), \frac{d_F^\alpha \hat{q}}{dt^\alpha}(t) \right) - L \left( t, q_n(t), \frac{d_F^\alpha q_n}{dt^\alpha}(t) \right) \right] (t_1) \\
&= J_{F,t_0}^\alpha \left[ L \left( t, \hat{q}(t), \frac{d_F^\alpha \hat{q}}{dt^\alpha}(t) \right) - L \left( t, \hat{q}(t), \frac{d_F^\alpha q_n}{dt^\alpha}(t) \right) \right] (t_1) \quad (15)
\end{aligned}$$

$$+ J_{F,t_0}^\alpha \left[ L \left( t, \hat{q}(t), \frac{d_F^\alpha q_n}{dt^\alpha}(t) \right) - L \left( t, q_n(t), \frac{d_F^\alpha q_n}{dt^\alpha}(t) \right) \right] (t_1). \quad (16)$$

Now, let's show that both terms in (15) and (16) vanish. Let the functional  $T : L_\alpha^1([t_0, t_1], \mathbb{R}^n) \rightarrow \mathbb{R}$  defined by

$$T(v) = J_{F,t_0}^\alpha [L(t, \hat{q}(t), v)](t_1)$$

and let the sets

$$\Sigma_\varsigma = \{v \in L_\alpha^1([t_0, t_1], \mathbb{R}^n) : T(v) \leq \varsigma \in \mathbb{R}\}.$$

To prove that the term in (15) tends to zero, it suffices to show that the functional  $T$  is weakly lower semicontinuous with respect to the  $L_\alpha^1([t_0, t_1], \mathbb{R}^n)$  topology. Standard arguments used above conduce us to the following: the sets  $\Sigma_\varsigma$  are convex from  $(\mathbf{H}_1)$ , and are closed in the strong  $L_\alpha^1([t_0, t_1], \mathbb{R}^n)$  topology. Then, by a classic property of weak topologies, one obtains that  $\Sigma_\varsigma$  is also closed in the weak  $L_\alpha^1([t_0, t_1], \mathbb{R}^n)$  topology which leads us easily to conclude that  $T$  is weak lower semicontinuous.

To show that the functional in (16) tends to zero, we proceed as follows.

From **(H<sub>2</sub>)** and **(H<sub>3</sub>)**, we have

$$\begin{aligned} & J_{F,t_0}^\alpha \left[ L \left( t, \hat{q}(t), \frac{d_F^\alpha q_n}{dt^\alpha}(t) \right) - L \left( t, q_n(t), \frac{d_F^\alpha q_n}{dt^\alpha}(t) \right) \right] (t_1) \\ & \leq K(r) \theta(\|\hat{q} - q_n\|_\infty) J_{F,t_0}^\alpha \left[ \rho \left( \left| \frac{d_F^\alpha q_n}{dt^\alpha}(t) \right| \right) \right] (t_1) \\ & \leq K(r) \theta(\|\hat{q} - q_n\|_\infty) \Lambda \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

where  $\Lambda$  is given in (14) and  $r$  is an upper bound on  $\|q_n\|_\infty$ .

Now, we will prove the uniqueness. Suppose there exist  $\tilde{q}_1, \tilde{q}_2 \in \mathcal{Y}$  such that

$$\inf\{I(q) : q \in \mathcal{Y}\} = I(\tilde{q}_1) = I(\tilde{q}_2) = m \in \mathbb{R},$$

and show that this implies  $\tilde{q}_1 = \tilde{q}_2$ . Let

$$\tilde{y} = \frac{\tilde{q}_1 + \tilde{q}_2}{2},$$

so  $\tilde{y} \in \mathcal{Y}$ . Since  $\tilde{L}$  is convex, we can say that  $\tilde{y}$  is also a minimum of  $I$  because

$$m \leq I(\tilde{y}) \leq \frac{I(\tilde{q}_1)}{2} + \frac{I(\tilde{q}_2)}{2} = m.$$

We thus obtain

$$\begin{aligned} & \int_{t_0}^{t_1} \left[ \frac{\tilde{L} \left( t, \tilde{q}_1(t), \frac{d_F^\alpha \tilde{q}_1}{dt^\alpha}(t); F(t, \alpha) \right)}{2} + \frac{\tilde{L} \left( t, \tilde{q}_2(t), \frac{d_F^\alpha \tilde{q}_2}{dt^\alpha}(t); F(t, \alpha) \right)}{2} \right. \\ & \quad \left. - \tilde{L} \left( t, \frac{\tilde{q}_1(t) + \tilde{q}_2(t)}{2}, \frac{\frac{d_F^\alpha \tilde{q}_1}{dt^\alpha}(t) + \frac{d_F^\alpha \tilde{q}_2}{dt^\alpha}(t)}{2} \right) \right] dt = 0. \end{aligned}$$

The convexity of  $\tilde{L}$  ensures that the integrand is non-negative. As the integral is null, so the only possibility is

$$\begin{aligned} & \frac{\tilde{L} \left( t, \tilde{q}_1(t), \frac{d_F^\alpha \tilde{q}_1}{dt^\alpha}(t); F(t, \alpha) \right)}{2} + \frac{\tilde{L} \left( t, \tilde{q}_2(t), \frac{d_F^\alpha \tilde{q}_2}{dt^\alpha}(t); F(t, \alpha) \right)}{2} \\ & \quad - \tilde{L} \left( t, \frac{\tilde{q}_1(t) + \tilde{q}_2(t)}{2}, \frac{\frac{d_F^\alpha \tilde{q}_1}{dt^\alpha}(t) + \frac{d_F^\alpha \tilde{q}_2}{dt^\alpha}(t)}{2} \right) = 0 \text{ on } [t_0, t_1]. \end{aligned}$$

We now use **(H<sub>5</sub>)** to get that  $\tilde{q}_1 = \tilde{q}_2$  and  $\frac{d_F^\alpha \tilde{q}_1}{dt^\alpha}(t) = \frac{d_F^\alpha \tilde{q}_2}{dt^\alpha}(t)$  on  $[t_0, t_1]$ , as asserted.

The proof is complete. ■

### 3.2 Euler–Lagrange equation, D’Alembert principle and du Bois–Reymond optimality condition

In this section, we first compute the first-order Gâteaux-differential of  $I$  and thereby we deduce the Euler–Lagrange equations, and the D’Alembert principle.

**Definition 14 (first-order Gâteaux-differential of  $I$ )** *Let  $q, h \in \text{AC}([t_0, t_1], \mathbb{R}^n)$  such that  $q + \epsilon h \in \mathcal{Y}$  and  $I(q + \epsilon h) \in L_\alpha^1([t_0, t_1])$  for  $\epsilon$  in a neighborhood of 0. Then, the first-order Gâteaux-differential of  $I$  at  $q$  in direction  $h$  is defined by*

$$\delta(I(q; h)) := \lim_{\epsilon \rightarrow 0} \frac{I(q + \epsilon h) - I(q)}{\epsilon},$$

*provided that the right-hand side term exists. The function  $q$  is called weak extremal of  $I$  if  $\delta(I(q; h)) = 0$  for all  $h \in \text{AC}([t_0, t_1], \mathbb{R}^n)$ .*

**Lemma 15** *Suppose that the augmented Lagrangian  $\tilde{L}$  given in (12) is of class  $C^1$  and there exists a function  $\rho$  satisfying  $(\mathbf{H}_2)$  of Theorem 13, and the following properties*

$$\left\{ \begin{array}{l} |\partial_2 \tilde{L}(t, x, v; F(t, \alpha))| + |\partial_3 \tilde{L}(t, x, v; F(t, \alpha))| \leq \tilde{K}(r) \rho(|v|) \\ t \in [t_0, t_1], x \in \bar{\mathbb{U}} \cap B_r, v \in \mathbb{R}^n, \text{ and } r, \tilde{K}(r) > 0 \end{array} \right. \quad (17)$$

and

$$\rho(p + m) \leq C_M(1 + \rho(p)), \text{ for all } m \in [0, M], p > 0. \quad (18)$$

*If  $q \in \mathcal{Y}$  is such that  $q(t) \in \mathbb{U}$  for all  $t \in [t_0, t_1]$  and  $I(q) \in L_\alpha^1([t_0, t_1])$ , then for all  $h \in \text{Lip}([t_0, t_1], \mathbb{R}^n)$  such that  $h(t_0) = h(t_1) = 0$ , the first-order Gâteaux-differential of  $I$  at  $q$  in direction  $h$  exists and is given by*

$$\begin{aligned} \delta(I(q; h)) = J_{F, t_0}^\alpha \left[ \partial_2 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \cdot h(t) \right. \\ \left. + \partial_3 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \cdot \frac{d_F^\alpha h}{dt^\alpha}(t) \right] (t_1). \end{aligned} \quad (19)$$

**Proof.** Let  $r > 0$  such that  $q \in \bar{\mathbb{U}} \cap B_r$ . Since  $h \in \text{Lip}([t_0, t_1], \mathbb{R}^n)$  such that  $h(t_0) = h(t_1) = 0$ , the map  $\epsilon \mapsto q + \epsilon h \in \bar{\mathbb{U}}$  is well defined and continuous in its co-domain. Therefore, we can find an  $\epsilon > 0$  sufficiently small such that  $q + \epsilon h \in \bar{\mathbb{U}} \cap B_r$ . We hence have if  $q \in \mathcal{Y}$ , then  $q + \epsilon h \in \mathcal{Y}$ . Now, let us prove that if  $I(q) \in L_\alpha^1([t_0, t_1])$ , then  $I(q + \epsilon h) \in L_\alpha^1([t_0, t_1])$ . Indeed, we have

$$\begin{aligned} & \tilde{L} \left( t, q + \epsilon h, \frac{d_F^\alpha q}{dt^\alpha} + \epsilon \frac{d_F^\alpha h}{dt^\alpha}; F(t, \alpha) \right) \\ &= \tilde{L} \left( t, q, \frac{d_F^\alpha q}{dt^\alpha}; F(t, \alpha) \right) + \int_0^\epsilon \frac{d}{d\nu} \tilde{L} \left( t, q + \nu h, \frac{d_F^\alpha q}{dt^\alpha} + \nu \frac{d_F^\alpha h}{dt^\alpha}; F(t, \alpha) \right) d\nu. \end{aligned}$$



Using the properties (17), (18) and (13), we obtain

$$\begin{aligned}
& \left| \int_0^\epsilon \frac{d}{d\nu} \tilde{L} \left( t, q + \nu h, \frac{d_F^\alpha q}{dt^\alpha} + \nu \frac{d_F^\alpha h}{dt^\alpha}; F(t, \alpha) \right) d\nu \right| \\
&= \left| \int_0^\epsilon \left[ \partial_2 \tilde{L} \left( t, q + \nu h, \frac{d_F^\alpha q}{dt^\alpha} + \nu \frac{d_F^\alpha h}{dt^\alpha}; F(t, \alpha) \right) \cdot h \right. \right. \\
&\quad \left. \left. + \partial_3 \tilde{L} \left( t, q + \nu h, \frac{d_F^\alpha q}{dt^\alpha} + \nu \frac{d_F^\alpha h}{dt^\alpha}; F(t, \alpha) \right) \cdot \frac{d_F^\alpha h}{dt^\alpha} \right] d\nu \right| \\
&\leq \int_0^\epsilon \kappa_1 \left( |h| + \left| \frac{d_F^\alpha h}{dt^\alpha} \right| \right) \rho \left( \left| \frac{d_F^\alpha q}{dt^\alpha} + \nu \frac{d_F^\alpha h}{dt^\alpha} \right| \right) d\nu \\
&\leq \kappa_2 \int_0^\epsilon \rho \left( \left| \frac{d_F^\alpha q}{dt^\alpha} + \nu \frac{d_F^\alpha h}{dt^\alpha} \right| \right) d\nu \\
&\leq \kappa_3 \int_0^\epsilon \left( 1 + \rho \left( \left| \frac{d_F^\alpha q}{dt^\alpha} \right| \right) \right) d\nu \\
&\leq \kappa_3 \left( 1 + \tilde{L} \left( t, q, \frac{d_F^\alpha q}{dt^\alpha}; F(t, \alpha) \right) + k \right) \epsilon,
\end{aligned}$$

for suitable positive constants  $\kappa_i$ ,  $i = 1, 2, 3$ , depending only on  $q$  and on the Lipschitz constant of  $h$ . This shows that if  $I(q) \in L_\alpha^1([t_0, t_1])$ , then  $I(q + \epsilon h) \in L_\alpha^1([t_0, t_1])$ . Thus, the first-order Gâteaux-differential of  $I$  (19) can be derived from a first-order Taylor expansion with integral rest of  $L$  and from the Lebesgue Dominated Convergence theorem. ■

Using the arguments as in the previous proof, we can consider variations where the endpoints are also changed, when it is allowed by the constraints, to obtain for instance the following result.

**Lemma 16** *Under the hypotheses of the previous lemma, suppose in addition that  $q(t_0)$  belongs to the interior of  $\mathbb{U}_0$  and  $q(t_1)$  belongs to the interior of  $\mathbb{U}_1$ . Suppose again that  $\phi_0, \phi_1 \in C^1(\mathbb{U})$ . Then for all  $h \in \text{Lip}([t_0, t_1], \mathbb{R}^n)$ , the first-order Gâteaux-differential of  $I$  at  $q$  in direction  $h$  exists and is given by*

$$\begin{aligned}
\delta(I(q; h)) &= J_{F, t_0}^\alpha \left[ \partial_2 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \cdot h(t) \right. \\
&\quad \left. + \partial_3 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \cdot \frac{d_F^\alpha h}{dt^\alpha}(t) \right] (t_1) \\
&\quad + D\phi_0(q(t_0)) \cdot h(t_0) + D\phi_1(q(t_1)) \cdot h(t_1).
\end{aligned}$$

We are now in a position to state and prove the main result of this section.

**Theorem 17** *Suppose that  $L$  is of class  $C^1$  and that there exists  $\rho$  satisfying **(H<sub>2</sub>)** of Theorem 13 and properties (17), (18). If  $q \in \mathcal{Y}$  is a solution of (GPCV) for all  $t \in [t_0, t_1]$ , then:*

(i) **Euler–Lagrange equation:** the function  $\partial_3 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \in \text{AC}([t_0, t_1])$

with

$$\frac{d_F^\alpha}{dt^\alpha} \left[ \partial_3 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \right] = \partial_2 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right), \quad (20)$$

for almost every  $t \in [t_0, t_1]$ ;

(ii) **Transversality conditions:** the equalities

$$\partial_3 L \left( t_0, q(t_0), \frac{d_F^\alpha q}{dt^\alpha}(t_0) \right) = D\phi_0(q(t_0)) \quad (21)$$

and

$$\partial_3 L \left( t_1, q(t_1), \frac{d_F^\alpha q}{dt^\alpha}(t_1) \right) = -D\phi_1(q(t_1)) \quad (22)$$

hold, if  $q(t_0) \in \mathring{\mathbb{U}}_0$ ,  $q(t_1) \in \mathring{\mathbb{U}}_1$  and  $\phi_0, \phi_1 \in C^1(\mathbb{U})$ .

**Proof.** Since there is no constraint in (GPCV) and since  $q$  is a solution of (GPCV), we can easily see that  $\delta I(q; h) = 0$  for all  $h \in \text{Lip}([t_0, t_1], \mathbb{R}^n)$  such that  $h(t_0) = h(t_1) = 0$ . From (19), it holds that

$$J_{F, t_0}^\alpha \left[ \partial_2 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \cdot h(t) + \partial_3 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \cdot \frac{d_F^\alpha h}{dt^\alpha}(t) \right] (t_1) = 0.$$

Hence statement (i) is consequence of Lemma 12. In order to derive the transversality conditions, we follow the same strategy but with variations in the space  $h \in \text{Lip}([t_0, t_1], \mathbb{R}^n)$ . Then Lemma 16 implies

$$\begin{aligned} J_{F, t_0}^\alpha \left[ \partial_2 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \cdot h(t) + \partial_3 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \cdot \frac{d_F^\alpha h}{dt^\alpha}(t) \right] (t_1) \\ + D\phi_0(q(t_0)) \cdot h(t_0) + D\phi_1(q(t_1)) \cdot h(t_1) = 0. \end{aligned}$$

Integrating by parts (see Theorem 7) and using the Euler–Lagrange equations proved above, we obtain

$$\begin{aligned} 0 &= \left[ \partial_3 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \cdot h(t) \right] \Big|_{t_0}^{t_1} + D\phi_0(q(t_0)) \cdot h(t_0) + D\phi_1(q(t_1)) \cdot h(t_1) \\ &= \left[ \partial_3 L \left( t_1, q(t_1), \frac{d_F^\alpha q}{dt^\alpha}(t_1) \right) + D\phi_1(q(t_1)) \right] \cdot h(t_1) \\ &\quad + \left[ -\partial_3 L \left( t_0, q(t_0), \frac{d_F^\alpha q}{dt^\alpha}(t_0) \right) + D\phi_0(q(t_0)) \right] \cdot h(t_0). \end{aligned}$$

Since  $h(t_0)$  and  $h(t_1)$  are arbitrary, equalities (21) and (22) are proved. ■

**Remark 18** In the literature, the Euler–Lagrange equation and the transversality conditions are known to be first-order necessary optimality conditions. The proof of the first-order necessary optimality conditions in Theorem 17 can easily be adapted to the three following constrained cases:

- (i) fixed initial condition to some  $q_0 \in \mathbb{R}^n$  and free final condition, that is,  $q \in \mathcal{Y}$  such that  $q(t_0) = q_0$ . In that case, the transversality condition (21) cannot be derived in general;
- (ii) free initial condition and fixed final condition to some  $q_1 \in \mathbb{R}^n$ , that is,  $q \in \mathcal{Y}$  such that  $q(t_1) = q_1$ . In that case, the transversality condition (22) cannot be derived in general;
- (iii) fixed initial condition to some  $q_0 \in \mathbb{R}^n$  and fixed final condition to some  $q_1 \in \mathbb{R}^n$ , that is,  $q \in \mathcal{Y}$  such that  $q(t_0) = q_0$  and  $q(t_1) = q_1$ . In that case, none of the transversality conditions is valid in general.

Finally, Lemma 15 yields the D'Alembert principle.

**Corollary 19** *Let  $L_\alpha^1([t_0, t_1]) \ni Q : [t_0, t_1] \times \bar{U} \times \mathbb{R}^n \longrightarrow \mathbb{R}$  and assume that  $I$  satisfies at  $q$  the D'Alembert's principle with generalized forces  $Q$ , i.e.,*

$$\delta(I(q; h)) = J_{F, t_0}^\alpha \left[ Q \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \cdot h(t) \right] (t_1)$$

for all  $h \in \text{Lip}([t_0, t_1], \mathbb{R}^n)$ . Under the hypotheses of the Theorem 17, we get the following generalized Euler–Lagrange equation

$$\frac{d_F^\alpha}{dt^\alpha} \left[ \partial_3 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \right] = \partial_2 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) + Q \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right). \quad (23)$$

**Remark 20** *It is not difficult to prove in this context the common property in the classical calculus of variations [14]: Under suitable assumptions (namely, assume that  $\tilde{L}(t, x, v; F(t, \alpha))$  is strictly convex with respect to  $v$ ), and if  $q \in \mathcal{Y}$  is a weak extremal of  $I$ , then  $q \in C^1([t_0, t_1], \mathbb{R}^n)$ ).*

**Theorem 21** *Let the hypotheses of the Theorem 17 be satisfied. In addition assume that the Lagrangian  $L \in C^k$  and the function  $F(t, \alpha) \in C^{k-1}$ , for some  $k \geq 2$ , and that  $\partial_{33} L(t, x, v)$  is positive definite for all  $t, x, v$ . Then any minimizer  $\tilde{q}$  of  $I$  is of class  $C^k([t_0, t_1], \mathbb{R}^n)$  and is a solution of the Euler–Lagrange equations (20).*

**Proof.** Assuming that  $\tilde{q} \in C^1([t_0, t_1], \mathbb{R}^n)$  (see Remark 20), we just need to prove that  $\tilde{q} \in C^{k-1}([t_0, t_1], \mathbb{R}^n)$ . For this, let us define

$$H(t) = J_{F, t_0}^\alpha \left[ \partial_2 L \left( t, \tilde{q}(t), \frac{d_F^\alpha \tilde{q}}{dt^\alpha}(t) \right) \right]$$

and

$$R \left( t, \frac{d_F^\alpha \tilde{q}}{dt^\alpha}(t) \right) = \partial_3 L \left( t, \tilde{q}(t), \frac{d_F^\alpha \tilde{q}}{dt^\alpha}(t) \right) - H(t), \quad \left( t, \frac{d_F^\alpha \tilde{q}}{dt^\alpha}(t) \right) \in [t_0, t_1] \times \mathbb{R}^n.$$

Notice that, from the previous assumptions, the functions  $H$  and  $R$  are well defined. Hence,  $R \in C^{k-1}([t_0, t_1], \mathbb{R}^n)$  and by the Euler–Lagrange equations (20), we get

$$R\left(t, \frac{d_F^\alpha \tilde{q}}{dt^\alpha}\right) = c, \quad \forall t \in [t_0, t_1],$$

where  $c \in \mathbb{R}$ . Thereby, the classical implicit function theorem and the hypothesis that  $\partial_{33}L(t, x, v)$  is positive definite imply that  $\frac{d_F^\alpha \tilde{q}}{dt^\alpha}$  is a  $C^{k-1}$ -function, that is,  $F(t, \alpha) \frac{dq}{dt} \in C^{k-1}$ . Then,  $q \in C^k$ . ■

We are now in conditions to prove a conformable du Bois–Reymond optimality condition.

**Theorem 22** *If  $q \in \mathcal{Y}$  is a weak extremal of  $I$ , then*

$$\begin{aligned} \frac{d_F^\alpha}{dt^\alpha} \left[ L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right) - \frac{d_F^\alpha q}{dt^\alpha}(t) \cdot \partial_3 L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right) \right] \\ = \partial_1 L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right) F(t, \alpha) \end{aligned} \quad (24)$$

**Proof.** We obtain (24) by direct computations and using Theorem 9, and Euler–Lagrange equation (20). ■

**Remark 23** *If the Lagrangian is autonomous (when  $L$  does not depend explicitly on time  $t$ ), we get by the Du Bois–Reymond optimality condition (24) and Definition 1 that*

$$L\left(q, \frac{d_F^\alpha q}{dt^\alpha}\right) - \frac{d_F^\alpha q}{dt^\alpha} \cdot \partial_3 L\left(q, \frac{d_F^\alpha q}{dt^\alpha}\right) = \text{constant for all } t \in [t_0, t_1]. \quad (25)$$

The quantity (25) corresponds one of the most important conservation law in Mechanics: the autonomous Lagrangian  $L$  correspondent to a mechanical system of conservative points, is invariant under time-translations (time-homogeneity symmetry) i.e., the total energy of a conservative closed system always remain constant in time along all the Euler–Lagrange extremals. This conservation law is known as the 2nd Erdmann necessary condition; in concrete applications, it gains different interpretations: conservation of energy in Mechanics; income-wealth law in Economics; 1st law of Thermodynamics; etc.

### 3.3 Generalized Noether theorem

The theorem of E. Noether [57] identifies a quantity that is preserved along any solution  $q(\cdot)$  of the Euler–Lagrange equations of a variational integral, a so-called first integral of motion, with any differentiable symmetry of the integrand.

**Definition 24 (Variational symmetry group)** *Let  $D \subseteq \mathbb{R}^n$  be an open set. We say that  $\Phi = \{\psi(s, \cdot)\}_{s \in \mathbb{R}} \in (D, D)$  is a one parameter group of diffeomorphisms of  $D$  if it satisfies:*

- (i)  $\psi \in C^2(\mathbb{R} \times D, D)$ ;
- (ii) For each  $s \in \mathbb{R}$ , the map  $\psi(s, \cdot) \in C^2(D, D)$  is invertible, and  $\psi(s, \cdot)^{-1} \in C^2(D, D)$ ;
- (iii)  $\psi(0, \cdot) = \text{Id}_D$ , where  $\text{Id}$  is the identity function;
- (iv)  $\forall s, s' \in \mathbb{R} : s + s' \in \mathbb{R} \Rightarrow \psi(s, \cdot) \circ \psi(s', \cdot) = \psi(s + s', \cdot)$ .

The translation in a spatial direction  $h$  is a typical case of a one-parameter group of diffeomorphisms

$$\psi : q \mapsto q + sh, \quad q, h \in \mathbb{R}^n.$$

Another classical example is the rotations by an angle  $\theta$

$$\psi : q \mapsto qe^{is\theta}, \quad q \in \mathbb{C}, \quad \theta \in \mathbb{R}.$$

In the present paper, we use the concept of a group of diffeomorphisms rather than the concept of infinitesimal transformations as in [35, ?]. These two concepts (see the classical book [59] for more details in particular Chapter 4) can be related by a Taylor expansion of  $\psi(s, q(t))$  in the neighborhood of  $s = 0$ . We have,

$$\psi(s, q(t)) = \psi(0, q(t)) + s \frac{\partial \psi}{\partial s}(0, q(t)).$$

Therefore, for a sufficiently small infinitesimal  $s$ , we can always say that  $\psi(s, t)$  is infinitely close to a transformation of the form  $q(t) \mapsto q(t) + s\eta(q(t))$ , where  $\eta(t) := \frac{\partial \psi}{\partial s}(0, q(t))$ .

With the aim of generalizing the Noether Theorem for our problem (GPCV), firstly, we need to define the concept of symmetry for a Lagrangian of the form (12) under the action of a group of diffeomorphisms.

**Definition 25** Let both  $T = \{\tau(s, \cdot)\}_{s \in \mathbb{R}} \in C^2(T', T')$  and  $S = \{\sigma(s, \cdot)\}_{s \in \mathbb{R}} \in C^2(S', S')$  be one parameter groups of diffeomorphism on the open sets  $T' \subseteq [t_0, t_1]$  and  $S' \subseteq \mathbb{R}^n$  respectively. The variational integral of  $I$  is said to be invariant under the action of  $T$  and  $S$ , if for any weak extremal  $q \in C^2([t_0, t_1], S')$  it satisfies

$$\frac{L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right)}{F(t, \alpha)} = \frac{L\left(\tau(s, t), \sigma(s, q(t)), \frac{d_F^\alpha \sigma(s, q(t))}{d(\tau(s, t))^\alpha}\right) \frac{d_F^\alpha \tau}{dt^\alpha}(s, t)}{F(\tau(s, t), \alpha)} \quad (26)$$

for all  $s \in \mathbb{R}$  and all  $t \in [t_0, t_1]$ , where the expression  $\frac{d_F^\alpha \sigma(s, q(t))}{d(\tau(s, t))^\alpha}$  is defined in the following Remark 26 (ii).

**Remark 26** (i) From Definition 24 (ii) we have  $\tau(s, \tau(s, \cdot)^{-1}(t)) = t$ , thereby the chain rule theorem yields that  $\frac{d_F^\alpha \tau}{dt^\alpha}(s, t)$  is invertible for all  $s \in \mathbb{R}$ ,  $t \in T'$ ;

(ii) Based on the previous remark, in (26) the expression  $\frac{d_F^\alpha \sigma(s, q(t))}{d(\tau(s, t))^\alpha}$  is defined as

$$\frac{d_F^\alpha \sigma(s, q(t))}{d(\tau(s, t))^\alpha} = \frac{F(\tau(s, t), \alpha) \frac{d^\alpha \sigma(s, q(t))}{dt^\alpha}(s, t)}{\frac{d^\alpha \tau}{dt^\alpha}(s, t)}. \quad (27)$$

As usual, because of Definition 24 (iii), we have  $\frac{d_F^\alpha \sigma(0, q(t))}{d(\tau(0, t))^\alpha} = \frac{d_F^\alpha q}{dt^\alpha}(t)$ .

(iii) Notice that, for all  $s \in \mathbb{R}$ , all  $t \in T'$  and for  $h \in$  sufficiently small, we have

$$\begin{aligned} \sigma(s, q(t+h)) &= \sigma(s, q(t)) + h \cdot \frac{\partial \sigma(s, q(t))}{\partial t}[s, t; h] \\ \tau(s, t+h) &= \tau(s, t) + h \cdot \frac{\partial \tau}{\partial t}[s, t; h]. \end{aligned}$$

Therefore, the ratio (of differentials):

$$\frac{\sigma(s, q(t+h)) - \sigma(s, q(t))}{\tau(s, t+h) - \tau(s, t)} = \frac{\frac{\partial \sigma(s, q(t))}{\partial t}[s, t; h]}{\frac{\partial \tau}{\partial t}[s, t; h]} \approx \frac{\frac{\partial \sigma(s, q(t))}{\partial t}(s, t)}{\frac{\partial \tau}{\partial t}(s, t)}$$

for all sufficiently small invertible  $h$ .

The next lemma gives a necessary condition for the variational integral of  $I$  to be invariant under the action of the parameter group of diffeomorphism  $T = \{\tau(s, \cdot)\}_{s \in \mathbb{R}} \in C^2(T', T')$  and  $S = \{\sigma(s, \cdot)\}_{s \in \mathbb{R}} \in C^2(S', S')$ .

**Lemma 27** *If the variational integral of  $I$  is invariant in the sense of Definition 25, then*

$$\begin{aligned} &\partial_1 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \frac{\partial \tau}{\partial t}(0, t) + \partial_2 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \cdot \frac{\partial \sigma}{\partial s}(0, q) \\ &\quad + \partial_3 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \cdot \left[ \frac{d_F^\alpha}{dt^\alpha} \frac{\partial \sigma}{\partial s}(0, q) \right. \\ &\quad \left. + \frac{d_F^\alpha q}{dt^\alpha}(t) \left( \frac{\frac{\partial F}{\partial t} \left( \frac{\partial \tau}{\partial s}(0, t) \right) - \frac{d_F^\alpha}{dt^\alpha} \frac{\partial \tau}{\partial s}(0, t)}{F(t, \alpha)} \right) \right] \\ &\quad + L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \left( \frac{\frac{d_F^\alpha}{dt^\alpha} \frac{\partial \tau}{\partial s}(0, t) - \frac{\partial F}{\partial t} \left( \frac{\partial \tau}{\partial s} \right)}{F(t, \alpha)} \right) = 0 \quad (28) \end{aligned}$$

for all weak extremal  $q \in C^2([t_0, t_1], S')$ .

**Proof.** Differentiating (26) with respect to  $s$  at  $s = 0$ , we obtain

$$\begin{aligned} & \partial_1 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \frac{\partial \tau}{\partial t}(0, t) + \partial_2 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \cdot \frac{\partial \sigma}{\partial s}(0, q) \\ & + \partial_3 L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \cdot \frac{\partial}{\partial s} \left[ \frac{d_F^\alpha \sigma(s, q(t))}{d(\tau(s, t))^\alpha} \right] \Big|_{s=0} \\ & + L \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \left( \frac{\partial}{\partial s} \frac{d_F^\alpha \tau}{dt^\alpha}(s, t) \Big|_{s=0} + \frac{\partial}{\partial s} \frac{1}{F(\tau(s, t), \alpha)} \Big|_{s=0} \right) = 0. \end{aligned} \quad (29)$$

From equality (27) and recalling that  $\tau(s, t), \sigma(s, q) \in C^2$  with respect to  $s$  in the Definition 25, and since  $\frac{d_F^\alpha}{dt^\alpha}$  acts on  $t$  and  $\frac{\partial}{\partial s}$  on variable  $s$ , we deduce that

$$\frac{\partial}{\partial s} \left[ \frac{d_F^\alpha \sigma(s, q(t))}{d(\tau(s, t))^\alpha} \right] \Big|_{s=0} = \frac{d_F^\alpha}{dt^\alpha} \frac{\partial \sigma}{\partial s}(0, q) + \frac{d_F^\alpha q}{dt^\alpha}(t) \left( \frac{\frac{\partial F}{\partial t} \left( \frac{\partial \tau}{\partial s}(0, t) \right) - \frac{d_F^\alpha}{dt^\alpha} \frac{\partial \tau}{\partial s}(0, t)}{F(t, \alpha)} \right), \quad (30)$$

$$\frac{\partial}{\partial s} \left[ \frac{d_F^\alpha \tau}{dt^\alpha}(s, t) \right] \Big|_{s=0} = \frac{\frac{d_F^\alpha}{dt^\alpha} \frac{\partial \tau}{\partial s}(0, t)}{F(t, \alpha)} \quad (31)$$

and

$$\frac{\partial}{\partial s} \frac{1}{F(\tau(s, t), \alpha)} \Big|_{s=0} = - \frac{\frac{\partial F}{\partial t} \left( \frac{\partial \tau}{\partial s}(0, t) \right)}{(F(t, \alpha))^2}. \quad (32)$$

Substituting (30), (31) and (32) into (29), we obtain the desired conclusion (29). ■

**Definition 28** Let  $T' \subseteq [t_0, t_1]$  be an open set. We say that a quantity  $C \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right)$  is a constant of motion on  $T', S'$  if

$$\frac{d_F^\alpha}{dt^\alpha} \left[ C \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \right] = 0 \quad \forall t \in T' \quad (33)$$

along all the extremals  $q \in C^2([t_0, t_1], S')$ .

Note that condition (33) is equivalent to ask that  $C \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right)$  is constant on any interval  $J \subseteq T'$ . In fact, if  $T' \in J$ ,  $G(t) := C \left( t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) - C \left( t', q(t'), \frac{d_F^\alpha q}{dt^\alpha}(t') \right)$  is the unique function such that  $G(t') = 0$  and  $\frac{d_F^\alpha G}{dt^\alpha}(t) = 0$  on any closed interval contained in  $T'$ .

**Theorem 29 (Generalized Noether theorem)** *If the variational integral of  $I$  is invariant in the sense of Definition 25, then the quantity  $C\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right)$  defined for all  $q \in C^2([t_0, t_1], S')$  and  $t \in T'$  by*

$$\begin{aligned} C\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right) &= J_{F, t_0}^\alpha G\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t); \alpha\right)(t_1) \\ &+ \left[ L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right) - \frac{d_F^\alpha q}{dt^\alpha}(t) \cdot \partial_3 L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right) \right] \frac{\partial \tau}{\partial s}(0, t) \\ &+ F(t, \alpha) \partial_3 L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right) \cdot \frac{\partial \sigma}{\partial s}(0, q) \end{aligned} \quad (34)$$

is a constant of motion on  $T'$  and  $S'$ , where  $G$  is a  $L_\alpha^1$ -function defined by

$$\begin{aligned} G\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t); \alpha\right) &:= -\frac{\partial F}{\partial t}\left(\frac{\partial \tau}{\partial s}\right)(0, t) L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right) \\ &+ \partial_3 L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right) \cdot \left[ \frac{\partial F}{\partial t}\left(\frac{\partial \tau}{\partial s}(0, t)\right) \frac{d_F^\alpha q}{dt^\alpha}(t) - \frac{\partial F}{\partial t}(t, \alpha) \cdot \frac{\partial \sigma}{\partial s}(0, q) \right]. \end{aligned} \quad (35)$$

**Proof.** The proof follows directly from the previous results. Using the Euler–Lagrange equation (20) and the du Bois–Reymond optimality condition (24), we deduce that

$$\partial_2 L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right) \cdot \frac{\partial \sigma}{\partial s}(0, q) = \frac{d_F^\alpha}{dt^\alpha} \left[ \partial_3 L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right) \right] \cdot \frac{\partial \sigma}{\partial s}(0, q) \quad (36)$$

and

$$\begin{aligned} &F(t, \alpha) \partial_1 L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right) \frac{\partial \tau}{\partial s}(0, t) \\ &= \frac{d_F^\alpha}{dt^\alpha} \left[ L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right) - \frac{d_F^\alpha q}{dt^\alpha}(t) \cdot \partial_3 L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right) \right] \frac{\partial \tau}{\partial s}(0, t). \end{aligned} \quad (37)$$

Substituting (36) and (37) into (28), and after straightforward calculus, we obtain

$$\begin{aligned} &\frac{d_F^\alpha}{dt^\alpha} \left\{ \left[ L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right) - \frac{d_F^\alpha q}{dt^\alpha}(t) \cdot \partial_3 L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right) \right] \frac{\partial \tau}{\partial s}(0, t) \right. \\ &\quad \left. + F(t, \alpha) \partial_3 L\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t)\right) \cdot \frac{\partial \sigma}{\partial s}(0, q) \right\} + G\left(t, q(t), \frac{d_F^\alpha q}{dt^\alpha}(t); \alpha\right) = 0, \end{aligned}$$

where the function  $G$  is defined in (35). The conclusion follows from the Definition 28. ■



Theorem 29 gives a new and interesting result for autonomous generalized variational problems. Let us consider an autonomous variational problem i.e., the case when Lagrangian  $L$  of  $I$  do not depends explicitly on the independent variable  $t$

$$I(q) = J_{F,t_0}^\alpha \left[ L \left( q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \right] (t_1) \longrightarrow \min . \quad (38)$$

**Corollary 30** *For the autonomous problem (38), one has*

$$L \left( q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) - \frac{d_F^\alpha q}{dt^\alpha}(t) \cdot \partial_3 L \left( q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \quad (39)$$

*is a constant of motion for all  $t \in T'$ .*

**Proof.** As the Lagrangian  $L$  does not depend explicitly on the independent variable  $t$ , we can easily see that the integrate of (38) is invariant under translation of the time variable: the condition of invariance (26) is satisfied with  $\tau(s, t) = t + s$  and  $\sigma(s, q) = q$ . Indeed, given that  $\frac{\partial \tau}{\partial s} = 1$  and  $\frac{\partial \sigma}{\partial s} = 0$ , the invariance condition (26) is verified if  $\frac{d_F^\alpha \sigma(s, q(t))}{d(\tau(s, t))^\alpha} = \frac{d_F^\alpha q}{dt^\alpha}(t)$ . This follows directly from Remark 26, i.e.,

$$\frac{d_F^\alpha \sigma(s, q(t))}{d(\tau(s, t))^\alpha} = \frac{d_F^\alpha \sigma}{dt^\alpha}(s, t + s) = \frac{d_F^\alpha \sigma}{dt^\alpha}(s, t) = \frac{d_F^\alpha q}{dt^\alpha}(t) .$$

Substituting these quantities into (34), we obtain

$$\left[ L \left( q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) - \frac{d_F^\alpha q}{dt^\alpha}(t) \cdot \partial_3 L \left( q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) \right] \underbrace{\left( 1 + \ln \left| \frac{F(t_0, \alpha)}{F(t_1, \alpha)} \right| \right)}_{=\text{constant}} = C .$$

Thus,

$$L \left( q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right) - \frac{d_F^\alpha q}{dt^\alpha}(t) \cdot \partial_3 L \left( q(t), \frac{d_F^\alpha q}{dt^\alpha}(t) \right)$$

is a constant of motion for all  $t \in T'$ . ■

Notice that, we obtain in previous corollary, the same condition than in Remark 23.

**Remark 31** *If  $F(t, \alpha) \equiv 1$ , Problem (GPCV) is reduced to the classical problem of the calculus of variations,*

$$J[q(\cdot)] = \int_{t_0}^{t_1} L \left( t, q(t), \frac{dq}{dt}(t) \right) \longrightarrow \min , \quad (40)$$

*and one obtains from Theorem 29 the standard Noether's theorem [57]:*

$$\begin{aligned} C \left( t, q, \frac{dq}{dt} \right) &= \partial_3 L \left( t, q, \frac{dq}{dt} \right) \cdot \frac{\partial \sigma}{\partial s}(t, q) \\ &+ \left[ L \left( t, q, \frac{dq}{dt} \right) - \partial_3 L \left( t, q, \frac{dq}{dt} \right) \cdot \frac{dq}{dt} \right] \frac{\partial \tau}{\partial s}(t, q) \end{aligned} \quad (41)$$

is a constant of motion, i.e. (41) is constant along all the solutions of the Euler-Lagrange equations

$$\partial_2 L(t, q, \dot{q}) = \frac{d}{dt} \partial_3 L(t, q, \dot{q}) \quad (42)$$

(these classical equations are obtained from (20) putting  $F(t, \alpha) \equiv 1$ ).

## 4 Generalized optimal control

In this section, we adopt the Hamiltonian formalism in order to generalize Noether's principle to the wider context of optimal control, see e.g [22].

Let us consider the optimal problem in Lagrange form, i.e. the minimization of the functional:

$$J_{F, t_0}^\alpha [L(t, q(t), u(t))](t_1) \longrightarrow \min \quad (\text{P})$$

subject to the Cauchy problem

$$\begin{cases} \frac{d_F^\alpha q}{dt^\alpha}(t) = \varphi(t, q(t), u(t)), \\ q(t_0) = q_0, \end{cases} \quad (\text{CP})$$

where  $q_0 \in \mathbb{R}^d$ , the state  $q \in C^1([t_0, t_1], \mathbb{R}^d)$ , the control  $u \in C^0([t_0, t_1], \mathbb{R}^l)$ , the Lagrangian  $L : [t_0, t_1] \times \mathbb{R}^d \times \mathbb{R}^l \rightarrow \mathbb{R}$  and the state equation  $\varphi : [t_0, t_1] \times \mathbb{R}^d \times \mathbb{R}^l \rightarrow \mathbb{R}^d$  are assumed to be  $C^1$ -functions with respect to all their arguments, and  $t_0 < t_1$ .

In particular, we always consider state and control functions in the following spaces

$$\begin{cases} q \in \mathcal{Q} := \{q \in C^1([t_0, t_1], H) : \|q - q_0\| \leq r\} \\ u \in \mathcal{A} := C^0([t_0, t_1], K), \end{cases}$$

where  $H \subseteq \mathbb{R}^d$ ,  $K \subseteq \mathbb{R}^l$  and  $r > 0$  is a fixed radius such that  $B_r(q_0) \subseteq H$ . In this section, we use the notations  $L(\cdot, q, u)$  and  $\varphi(\cdot, q, u)$  to denote  $L(\cdot, q(t), u(t))$  and  $\varphi(\cdot, q(t), u(t))$ , respectively.

In developing this topic, it is therefore essential to already have suitable results about solutions of (CP) such as Picard–Lindelöf theorem and Grönwall–Bellman inequality in integral form in the fractional conformable setting.

**Theorem 32** *Let  $\mathbb{P} \subset \mathbb{R}^d$  be an open and connected set, assume  $(0, a) \subset \mathbb{R}^+$ , and define  $\mathbb{S} = (0, a) \times \mathbb{P}$ . Let  $f(x, y)$  be a real valued vector bounded function on  $\mathbb{S}$ . Consider the following system of ordinary equations of order  $\alpha \in (0, 1]$*

$$\frac{d_F^\alpha y}{dt^\alpha}(x) = f(x, y(x)) \quad (43)$$

*under the conditions,*

- i)  $f(x, y)$  is a continuous function in  $L^1_\alpha(\overline{\mathbb{S}})$ ,  
ii)  $f(x, y)$  satisfies a Lipschitz condition with respect to  $y$  in  $\overline{\mathbb{S}}$ , i.e., there exists a constant  $K_L > 0$ , such that for all  $(x, y), (x, z) \in \overline{\mathbb{S}}$ :

$$\|f(x, y) - f(x, z)\| \leq K_L \|y - z\| \quad (44)$$

where  $\|\cdot\|$  is the supremum norm.

Then, for all  $(x_0, y_0) \in \mathbb{S}$ , there exist a positive number  $\beta$  such that the closed interval  $B = [x_0 - \beta, x_0 + \beta] \subset (0, a)$ , and a unique continuous function  $y : B \rightarrow \mathbb{P}$ , such that

$$\begin{cases} \frac{d_F^\alpha y}{dt^\alpha}(x) = f(x, y(x)), \quad \forall x \in B, \\ y(x_0) = y_0. \end{cases} \quad (45)$$

**Remark 33** Let us consider the function  $g$  defined by

$$g(x, y) = \begin{cases} f(x, y), & \forall (x, y) \in \overline{\mathbb{S}} \text{ with } x \in B, \\ 0 & \text{otherwise,} \end{cases} \quad (46)$$

where the function  $f$  is the same that in the Theorem 32, and the corresponding initial problem

$$\begin{cases} \frac{d_F^\alpha y}{dt^\alpha}(x) = g(x, y(x)), \quad \forall x \in B, \\ y(x_0) = y_0. \end{cases} \quad (47)$$

Then a function  $y : B \rightarrow \mathbb{P}$  verify (46) if, and only if, verify (47).

To prove the previous theorem, we need the following two lemmas.

**Lemma 34** The function  $y(\cdot)$  satisfies condition (47) if, and only if, it satisfies the integral equation

$$y(x) = y_0 + J_{F, x_0 - \beta}^\alpha [g(s, y(s))](x). \quad (48)$$

**Proof.** The conclusion follows directly from Theorems 5 and 6. ■

We define now the set

$$C_B = \{y : y \text{ is a continuous vector function on } B\},$$

which is a complete metric space with the distance  $d(y(x), z(x)) = \|y - z\|_{C_B}$  where  $\|\cdot\|_{C_B} = \max_{x \in B} \|\cdot\|$ . Set

$$C_B^r = \{y \in C_B : \|y(x) - y_0\|_{C_B} \leq r\}$$

for some  $r > 0$ .

**Lemma 35**  $C_B^r$  is a complete metric subspace of  $C_B$ .

**Proof.** As  $C_B^r$  is a closed subspace of  $C_B$ , we obtain the desired conclusion because  $C_B$  is a complete metric space. ■

Next, we present the proof of Theorem 32.

**Proof. (Theorem 32).** As  $f(x, t)$  is bounded on  $\bar{\mathbb{S}}$  and  $F(x, \alpha)$  is a positive function in  $L_\alpha^1(\bar{\mathbb{S}})$ , there exists a positive constant  $M$ , such that  $M = \max_{(x,y) \in \bar{\mathbb{S}}} \left| \frac{f(x, y)}{F(x, \alpha)} \right|$ , is included in  $\bar{\mathbb{S}}$ . For all function  $y \in C_B^K$ , we apply the operator  $T$  defined by

$$T[y(x)] = y_0 + J_{F, x_0 - \beta}^\alpha [g(s, y(s))](x).$$

We claim: the operator  $T$  verifies the following properties:

1.  $T[y] \in C_B^r$  for all  $y \in C_B^r$ ;
2.  $T : C_B^r \rightarrow C_B^r$  is a contractive mapping.

To prove 1. it suffices to check that

$$\|T[y(x)] - y_0\|_{C_B} = \max_{x \in B} \|T[y(x)] - y_0\| = \max_{x \in B} \|J_{F, x_0 - \beta}^\alpha [g(s, y(s))](x)\| \leq Mb \leq r,$$

where  $b = \min\{\beta, \frac{r}{M}\}$ . So  $T[y] \in C_B^r$  for all  $y \in C_B^r$ .

To prove 2., we need to introduce the Bielecki-type norm

$$\|\cdot\|_B =: \text{Sup} \{e^{-2xMK_L} |\cdot| : x \in B\}.$$

Notice that the above norm is equivalent to the standard supremum norm on  $C_B$ , so this vector space is complete under this weighted norm, and by Lemma 35, the set  $C_B^r$  equipped with this norm/metric is a complete metric space. Furthermore, for all  $y, z \in C_B^r$ , we obtain

$$\begin{aligned} \|T[y(x)] - T[z(x)]\|_B &= \|J_{F, x_0 - \beta}^\alpha [g(s, y(s)) - g(s, z(s))](x)\|_B \\ &= e^{-2xMK_L} |J_{F, x_0 - \beta}^\alpha [g(s, y(s)) - g(s, z(s))](x)| \\ &\leq e^{-2xMK_L} |J_{F, x_0 - \beta}^\alpha [g(s, y(s)) - g(s, z(s))](x)| \\ &\leq MK_L e^{-2xMK_L} \int_{x_0 - \beta}^x \|y - z\|_B e^{2sMK_L} ds \\ &\leq \frac{\|y - z\|_B}{2} (1 - e^{-2MK_L(x - x_0 + \beta)}). \end{aligned}$$

Since  $\beta$  was suitably choose 2. is proved.

Finally, Theorem 32 is obtained applying the well known fixed point theorem for contractive mappings in complete metric spaces, and taking into account the Remark 33. ■

Finally, we have the following Grönwall–Bellman inequality in integral form:

**Theorem 36** Let  $r$  be a continuous, nonnegative function and  $c$ ,  $d$  and  $k$  be continuous functions on the interval  $[t_1, t_2]$ ,  $d, k \geq 0$ , and  $\alpha \in (0, 1]$ , such that

$$r(t) \leq c(t) + d(t)J_{F,t_1}^\alpha(kr)(t),$$

for all  $t \in [t_1, t_2]$ . Then we have

$$r(t) \leq c(t) + d(t)J_{F,t_1}^\alpha(cke^{-J_{F,t_1}^\alpha(dk)})(t)e^{J_{F,t_1}^\alpha(dk)(t)},$$

for every  $t \in [t_1, t_2]$ .

**Proof.** Let us define  $R(t) = J_{F,t_1}^\alpha(kr)(t)$ . Thus,  $R(t_1) = 0$ ,

$$r(t) \leq c(t) + d(t)R(t)$$

for all  $t \in [t_1, t_2]$ , then we have

$$\frac{d_F^\alpha}{dt^\alpha}R(t) = k(t)r(t) \leq c(t)k(t) + d(t)k(t)R(t), \quad (49)$$

for every  $t \in [t_1, t_2]$ , since  $k \geq 0$ . Let us define  $E(t) = e^{-J_{F,t_1}^\alpha(dk)(t)}$ . So, we have

$$\begin{aligned} \frac{d_F^\alpha E(t)}{dt^\alpha} &= \frac{d_F^\alpha}{dt^\alpha}(e^{-J_{F,t_1}^\alpha(dk)})(t) \\ &= \frac{d_F^\alpha}{dt^\alpha}(-J_{F,t_1}^\alpha(dk))(t)e^{-J_{F,t_1}^\alpha(dk)(t)} \\ &= -d(t)k(t)E(t). \end{aligned} \quad (50)$$

Since  $R$  and  $E$  are  $N$ -differentiable on  $[t_1, t_2]$ , from above we obtain

$$\begin{aligned} \frac{d_F^\alpha}{dt^\alpha}(ER)(t) &= E(t)\frac{d_F^\alpha}{dt^\alpha}R(t) + R(t)\frac{d_F^\alpha}{dt^\alpha}E(t) = E(t)\frac{d_F^\alpha}{dt^\alpha}R(t) - d(t)k(t)R(t)E(t) \\ &\leq c(t)k(t)E(t) + d(t)k(t)R(t)E(t) - d(t)k(t)R(t)E(t) \\ &= c(t)k(t)E(t), \end{aligned}$$

for every  $t \in [t_1, t_2]$ , since  $E \geq 0$ . Since  $E(t)R(t)$  is differentiable on  $[t_1, t_2]$  we have

$$J_{F,t_1}^\alpha(ckE)(t) \geq J_{F,t_1}^\alpha\left(\frac{d_F^\alpha}{dt^\alpha}(ER)\right)(t) = E(t)R(t) - E(t_1)R(t_1) = E(t)R(t),$$

for any  $t \in [t_1, t_2]$ . Thus, since  $d, E \geq 0$ ,

$$\begin{aligned} r(t) &\leq c(t) + d(t)R(t) \leq c(t) + d(t)\frac{J_{F,t_1}^\alpha(ckE)(t)}{E(t)} \\ &= c(t) + d(t)J_{F,t_1}^\alpha(cke^{-J_{F,t_1}^\alpha(dk)})(t)e^{J_{F,t_1}^\alpha(dk)(t)}, \end{aligned}$$

for every  $t \in [t_1, t_2]$ . ■

In this section, we always assume that the state equation  $\varphi$  satisfies the assumptions of Theorem 32 for each control  $u \in \mathcal{A}$ , i.e setting:

$$\begin{aligned} \forall u \in \mathcal{A} : M_u &:= \max_{\substack{t_1 \leq t \leq t_2 \\ |q - q_0| \leq r}} |\varphi(t, q, u(t))| \\ L_u &:= \max_{\substack{t_0 \leq t \leq t_1 \\ |q - q_0| \leq r}} |\partial_q \varphi(t, q, u(t))| \end{aligned} \quad (51)$$

$$N_u := \max_{t_0 \leq t \leq t_1} |F(t, \alpha)^{-1}|, \quad (52)$$

we assume that

$$\begin{aligned} \forall u \in \mathcal{A} : (t_1 - t_0) N_u M_u &\leq r \\ \lim_{n \rightarrow +\infty} (t_1 - t_0)^n N_u^n L_u^n &= 0. \end{aligned} \quad (53)$$

Therefore, Theorem 32 allows us to state that

$$\forall u \in \mathcal{A} \exists ! q^u \in \mathcal{Q} : \text{ (CP) holds for all } t \in [t_0, t_1]. \quad (54)$$

Finally, observe that the constant  $L_u > 0$  defined in (51) and Taylor expansion yield the Lipschitz condition

$$\forall u \in \mathcal{A} \forall t \in [t_0, t_1] \forall q, \tilde{q} \in \overline{B_r(q_0)} : |\varphi(t, q, u(t)) - \varphi(t, \tilde{q}, u(t))| \leq L_u |q - \tilde{q}|. \quad (55)$$

#### 4.1 Weak Pontryagin Maximum Principle

Invoking (54), we can introduce the notation and the optimization problem

$$\forall u \in \mathcal{A} : J[u] := J_{F, t_0}^\alpha [L(t, q^u(t), u(t))](t_1) \quad (56)$$

$$\text{Find } v \in \mathcal{A} : \text{ (CP) and } \exists r > 0 \exists l \in \mathbb{N} \forall u \in \mathcal{A} \cap B_r^l(v) : J[v] \leq J[u]. \quad (57)$$

In order to develop the necessary optimality condition for Problem (56)-(57), we first assume that  $u \in \mathring{\mathcal{A}}$ , i.e.

$$\exists r > 0 : B_r(u) = \left\{ \bar{u} \in \mathcal{A} \mid \max_{t_0 \leq t \leq t_1} |\bar{u}(t) - u(t)| < r \right\} \subseteq \mathcal{A}.$$

Thereby, for all control  $\bar{u} \in \mathcal{A}$ , and for some  $\delta = \delta(u, \bar{u}) \in (0, 1)$  sufficiently small

$$\forall h \in (-\delta, \delta) : u + h\bar{u} \in \mathcal{A}, \quad (58)$$

and hence we can evaluate  $J[u + h\bar{u}] \in \mathbb{R}$ .

To define the first variation of  $J[-] : \mathcal{A} \rightarrow \mathbb{R}$  at  $u \in \mathcal{A}$  in the direction  $\bar{u} \in \mathcal{A}$ , we can intuitively think at a sort of first order Taylor sum of  $J[u + h\bar{u}]$  at  $u$ :

**Definition 37** [34] Let  $\delta > 0$  be such that (58) holds. We say that  $R : (-\delta, \delta) \rightarrow \mathbb{R}$  is an incremental ratio of  $J[-]$  at  $u \in \mathcal{A}$  in the direction  $\bar{u} \in \mathcal{A}$ , if there exists a function  $\hat{R} : (-\delta, \delta) \rightarrow \mathbb{R}$  (called remainder) such that:

1.  $\forall h \in (-\delta, \delta) : J[u + h\bar{u}] = J[u] + h \cdot R(h) + \hat{R}(h);$
2.  $R$  is continuous at 0;
3.  $\exists A > 0 \forall h \in (-\delta, \delta) : |\hat{R}(h)| \leq A \cdot h^2.$

Based on the Theorem 54 [34], we can state that the incremental ratio  $R(0)$  in the above definition is unique. Now, we can define the first variation of  $J[-]$ .

**Definition 38** We define the first variation of  $J[-]$  at  $u \in \mathring{\mathcal{A}}$  in the direction  $\bar{u} \in \mathcal{A}$  as  $\delta J^u(u; \bar{u}) := R(0)$ , where  $R$  is any incremental ratio of  $J[-]$ . Moreover, we say that  $u \in \mathcal{A}$  is a weak extremal of  $J[-]$  if for any  $\bar{u} \in \mathcal{A}$ ,  $\delta I(u; \bar{u}) = 0$ .

**Theorem 39** [34] If  $v \in \mathring{\mathcal{A}}$  solves the problem (57), then it is a weak extremal of  $J[-]$ .

The next theorem shows the continuity conditions for the map  $u \in \mathcal{A} \mapsto q^u \in \mathcal{Q}$  (see [19] for a similar proof):

**Theorem 40 (Stability of order 1 and 2)** Let  $u \in \mathring{\mathcal{A}}$ ,  $\bar{u} \in \mathcal{A}$  and  $\delta \in (0, 1)$  as in (58). Assume that

$$\forall u, \bar{u} \in \mathcal{A} \exists N \in \mathbb{N} \forall h \in (-\delta, \delta) : L_{u+h\bar{u}}[t_1 - t_0] N_u \leq N. \quad (59)$$

Then, there exist positive constants  $A, \bar{A}$  (depending only on  $u, \bar{u}$  and clearly on  $\phi$ ) such that for all  $|h| < \delta$ , we have

$$\|q^{u+h\bar{u}} - q^u\| \leq |h| A \quad (60)$$

$$\|q^{u+h\bar{u}} - q^u - h\bar{q}\| \leq \bar{A} h^2, \quad (61)$$

where  $\bar{q} \in C^1([t_0, t_1], \mathbb{R}^d)$  is the unique global solution of the following Cauchy's problem

$$\begin{cases} \frac{d_F^\alpha \bar{q}}{dt^\alpha} = \frac{\partial \varphi}{\partial q}(t, q^u, u) \cdot \bar{q} + \frac{\partial \varphi}{\partial u}(t, q^u, u) \cdot \bar{u} \\ \bar{q}(t_0) = 0. \end{cases} \quad (\text{LCP}_1)$$

**Proof.** We will present the proof only for the (60). Similarly, follow the proof of (61).

Let  $h \in (-\delta, \delta)$ . From the evolution ODE (CP), for all  $t \in [t_0, t_1]$ , we have

$$\begin{aligned} |q^{u+h\bar{u}}(t) - q^u(t)| &\leq |J_{F, t_0}^\alpha (\varphi(s, q^{u+h\bar{u}}(s), u(s) + h\bar{u}(s)) - \varphi(s, q^u(s), u(s)))| (t) \\ &\leq J_{F, t_0}^\alpha |\varphi(s, q^{u+h\bar{u}}(s), u(s) + h\bar{u}(s)) - \varphi(s, q^u(s), u(s))| (t) \end{aligned} \quad (62)$$

for all  $s \in [t_0, t]$ , we have

$$\begin{aligned} & \left| \varphi(s, q^{u+h\bar{u}}(s), u(s) + h\bar{u}(s)) - \varphi(s, q^u(s), u(s)) \right| \leq \\ & \leq \left| \varphi(s, q^{u+h\bar{u}}(s), u(s) + h\bar{u}(s)) - \varphi(s, q^u(s), u(s) + h\bar{u}(s)) \right| + \\ & \quad + \left| \varphi(s, q^u(s), u(s) + h\bar{u}(s)) - \varphi(s, q^u(s), u(s)) \right|. \end{aligned} \quad (63)$$

Now, we apply the Lipschitz condition (55) to the first sum, and a first order Taylor expansion with Lagrange remainder to the second one, obtaining

$$\begin{aligned} & \left| \varphi(s, q^{u+h\bar{u}}(s), u(s) + h\bar{u}(s)) - \varphi(s, q^u(s), u(s) + h\bar{u}(s)) \right| \leq \\ & \leq L_{u+h\bar{u}} \cdot \left| q^{u+h\bar{u}}(s) - q^u(s) \right| \end{aligned} \quad (64)$$

$$\left| \varphi(s, q^u(s), u(s) + h\bar{u}(s)) - \varphi(s, q^u(s), u(s)) \right| \leq |h| \left| \frac{\partial \varphi}{\partial u}(s, q^u(s), \xi) \cdot \bar{u}(s) \right| \quad (65)$$

where  $\xi = \xi(u, h, \bar{u}, s) \in [u(s), u(s) + h\bar{u}(s)] \subseteq \mathbb{R}^l$ . The extreme value Theorem applied to  $u, \bar{u}$  on the compact set  $[t_1, t_2]$  yields the existence of positive constants  $B_1, B_2$  such that  $[u(s), u(s) + h\bar{u}(s)] \subseteq [B_1, B_2]^l$ ; since  $|h| < \delta$ , we can always assume that these constants do not depend on  $h$  but only on  $u$  and  $\bar{u}$ .

In the same way, applying the extreme value theorem with  $\frac{\partial \varphi}{\partial u}$  on the compact set  $[t_1, t_2] \times \overline{B_r(q_0)} \times [B_1, B_2]^l$ , we get the existence of a constant  $C_1 \in \mathbb{R}_+^*$  (depending only on  $u, \bar{u}$ ) such that

$$\forall s \in [t_1, t_2] : \left| \frac{\partial \varphi}{\partial u}(s, q^u(s), \xi) \cdot \bar{u}(s) \right| \leq C_1.$$

Thereby, considering (52) and (65), we get

$$J_{F, t_0}^\alpha \left| \varphi(s, q^u(s), u(s) + h\bar{u}(s)) - \varphi(s, q^u(s), u(s)) \right| (s) \leq |h| C_1 N_u(t_1 - t_0). \quad (66)$$

Using inequalities (62), (63), (64), and (66), we have

$$\begin{aligned} \forall t \in [t_0, t_1] : & \left| q^{u+h\bar{u}}(t) - q^u(t) \right| \\ & \leq L_{u+h\bar{u}} \cdot J_{F, t_0}^\alpha \left| q^{u+h\bar{u}}(s) - q^u(s) \right| (s) + |h| C_1 N_u(t_1 - t_0). \end{aligned} \quad (67)$$

Finally, we apply Grönwall-Bellman Theorem 36 to (67) and use assumption (59) to obtain

$$\|q^{u+h\bar{u}} - q^u\|_0 \leq |h| C_1 N_u(t_1 - t_0) e^{L_{u+h\bar{u}}(t_1 - t_0)} \leq |h| C_1 N_u(t_1 - t_0) N. \quad (68)$$

Setting  $A := |h| C_1 N_u(t_1 - t_0) N$ , (68) proves the claim, because also  $N$  does not depend on  $h$  but only on  $u$  and  $\bar{u}$ . ■

**Theorem 41** *In the assumptions of Theorem. 40, the following equality holds*

$$\delta J(u; \bar{u}) = J_{F, t_0}^\alpha \left( \frac{\partial L}{\partial q}(s, q^u(s), u(s)) \cdot \bar{q}(s) + \frac{\partial L}{\partial u}(t, q^u(s), u(s)) \cdot \bar{u}(s) \right) (t_1) \quad (69)$$



where  $\bar{q} = \bar{q}(\bar{u}) \in \mathcal{Q}$  is the unique global solution of problem (LCP<sub>1</sub>). Moreover,  $\bar{u} \in \mathcal{A} \mapsto \delta I(u; \bar{u}) \in \mathbb{R}$  is an  $\mathbb{R}$ -continuous linear map, i.e. it is  $\mathbb{R}$ -linear and satisfies

$$\exists K > 0 \exists r > 0 \forall \bar{u}, \bar{v} \in B_r(0) : |\delta I(u; \bar{u}) - \delta I(u; \bar{v})| \leq K_u \cdot \|\bar{u} - \bar{v}\|. \quad (70)$$

**Proof.** Since  $u \in \mathring{\mathcal{A}}$ , there always exists  $\delta > 0$  such that (58) holds. Define  $v^h := q^{u+h\bar{u}} - q^u - h\bar{q}$  for all  $h \in (-\delta, \delta)$ , so that

$$\|v^h\| \leq \bar{A}h^2 \quad (71)$$

by Theorem 40. For  $t \in [t_0, t_1]$  and  $k = (0, k_q, k_u) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^l$ , let

$$\begin{aligned} R_L(t, q^u(t), u(t); k) &:= L(t, q^u(t) + k_q, u(t) + k_u) - L(t, q^u(t), u(t)) \\ &\quad - \partial_q L(t, q^u(t), u(t)) \cdot k_q - \partial_u L(t, q^u(t), u(t)) \cdot k_u \end{aligned}$$

be the remainder of the first order Taylor formula of  $L$  at the point  $(t, q^u(t), u(t))$  with increment  $k = (0, k_q, k_u)$ . Thereby, for all  $h \in (-\delta, \delta)$ , we get

$$\begin{aligned} J[u + h\bar{u}] - J[u] &= J_{F, t_0}^\alpha \{L(\cdot, q^{u+h\bar{u}}, u + h\bar{u}) - L(\cdot, q^u, u)\}(t_1) \\ &= J_{F, t_0}^\alpha \{L(\cdot, q^u + h\bar{q} + v^h, u + h\bar{u}) - L(\cdot, q^u, u)\}(t_1) \\ &= J_{F, t_0}^\alpha \{\partial_q L(\cdot, q^u, u) \cdot (h\bar{q} + v^h) + \partial_u L(\cdot, q^u, u) \cdot h\bar{u} + \\ &\quad + R_L(\cdot, q^u, u; (\cdot, h\bar{q} + v^h, h\bar{u}))\}(t_1) \\ &= hJ_{F, t_0}^\alpha \{\partial_q L(\cdot, q^u, u) \cdot \bar{q} + \partial_u L(\cdot, q^u, u) \cdot \bar{u}\}(t_0) + \\ &\quad + J_{F, t_0}^\alpha \{\partial_q L(\cdot, q^u, u) \cdot v^h + R_L(\cdot, q^u, u; (\cdot, h\bar{q} + v^h, h\bar{u}))\}(t_1). \end{aligned}$$

Setting

$$\begin{aligned} R &:= J_{F, t_1}^\alpha \{\partial_q L(\cdot, q^u, u) \cdot \bar{q} + \partial_u L(\cdot, q^u, u) \cdot \bar{u}\}(t_2) \text{ and} \\ \hat{R}(h) &:= J_{F, t_1}^\alpha \{\partial_q L(\cdot, q^u, u) \cdot v^h + R_L(\cdot, q^u, u; (\cdot, h\bar{q} + v^h, h\bar{u}))\}(t_2), \end{aligned}$$

Taylor expansion theorem and (71) yield the existence of a constant  $D > 0$  such that  $|\hat{R}(h)| \leq Dh^2$ . This proves that  $R = R(0) = \delta J(u; \bar{u})$  is the incremental ratio of  $J[-]$  with remainder  $\hat{R}$  (Definition 37), which is claim (69). Using the notation  $\bar{q} = \bar{q}(\bar{u})$  and the uniqueness of solution of problem (LCP<sub>1</sub>), it follows that  $\bar{q}(\lambda\bar{u} + \beta\bar{v}) = \lambda\bar{q}(\bar{u}) + \beta\bar{q}(\bar{v})$  for all  $\lambda, \beta \in \mathbb{R}$  and  $\bar{u}, \bar{v} \in \mathcal{A}$ . Thereby, (69) implies that  $\delta J(u; \cdot) : \mathcal{A} \rightarrow \mathbb{R}$  is an  $\mathbb{R}$ -linear map. Finally, we have

$$\begin{aligned} \bar{q}(\bar{u}) - \bar{q}(\bar{v}) &= J_{F, t_1}^\alpha [a \cdot \{\bar{q}(\bar{u}) - \bar{q}(\bar{v})\}](\cdot) + J_{F, t_1}^\alpha [b \cdot (\bar{u} - \bar{v})](\cdot) \\ \|\bar{q}(\bar{u}) - \bar{q}(\bar{v})\| &\leq \|\bar{q}(\bar{u}) - \bar{q}(\bar{v})\| \cdot N_u(t_1 - t_0)L_u + \|b\|N_u(t_1 - t_0)\|\bar{u} - \bar{v}\|, \end{aligned}$$

where

$$\begin{aligned} a(t) &:= \frac{\partial \varphi}{\partial q}(t, q^u(t), u(t)) \\ b(t) &:= \frac{\partial \varphi}{\partial u}(t, q^u(t), u(t)). \end{aligned}$$

Note that  $\|a\| = L_u$  by (51). But assumption (53) implies that  $N_u^N(t_1 - t_0)^N L_u^N < 1$  for some  $N \in \mathbb{N}$  and hence that  $1 - N_u(t_1 - t_0)L_u$  is invertible, yielding

$$\|\bar{q}(\bar{u}) - \bar{q}(\bar{v})\| \leq (1 - N_u(t_1 - t_0)L_u)^{-1} \|b\| N_u(t_1 - t_0) \|\bar{u} - \bar{v}\|.$$

Now, the conclusion (70) follows from this Lipschitz property and (69). ■

In this section, we are interested in giving a proof of the weak Pontryagin Maximum Principle for problems of optimal control with generalized conformable derivatives. For that, we need to introduce the definition of the Hamiltonian using the Lagrange multipliers.

**Definition 42** *Let  $\mathcal{H}$  be the Hamiltonian associated to the Lagrangian  $L$  and the state equation (CP), i.e.*

$$\begin{aligned} \mathcal{H} : [t_0, t_1] \times H \times K \times \mathbb{R}^d &\longrightarrow \mathbb{R} \\ (t, q, u, p) &\longmapsto L(t, q, u) + p \cdot \varphi(t, q, u). \end{aligned} \quad (72)$$

Moreover, for any control  $u \in \mathcal{A}$ , let  $p^u \in \text{PC}^1([t_0, t_1], \mathbb{R}^d)$  denote the adjoint variable (generalized momentum), i.e. the unique solution of the Cauchy problem

$$\begin{cases} \frac{d_F^\alpha p^u}{dt^\alpha} = -\frac{\partial \mathcal{H}}{\partial q}(t, q^u, u, p^u) = -\frac{\partial L}{\partial q}(t, q^u, u) - \left( \frac{\partial \varphi}{\partial q}(t, q^u, u) \right)^T \cdot p^u \\ p^u(t_1) = 0. \end{cases} \quad (\text{LCP}_2)$$

As we showed above, this problem has a unique solution on  $[t_0, t_1]$  because of our assumptions (53).

We want to close this section, giving a proof of the *weak Pontryagin Maximum Principle*, i.e. a theorem where instead of the usual condition

$$\mathcal{H}(t, q^v(t), v(t), p^v(t)) = \min_{k \in K} \mathcal{H}(t, q^v(t), k, p^v(t)) \quad \forall t \in [t_0, t_1]$$

(if  $v$  is an optimal control, i.e. it solves problem (57); see e.g. [41, 61]) we have instead the necessary condition  $\partial_u \mathcal{H}(t, q^v(t), v(t), p^v(t)) = 0$  assuming that  $v \in \mathcal{A}$  is a local minimum of the functional  $J[-]$ .

Directly from the definition of Hamiltonian, we get that, for any control  $u \in \mathcal{A}$ , the pair  $(q^u, p^u)$  is a solution of the following Hamiltonian system:

$$\begin{cases} \frac{d_F^\alpha q}{dt^\alpha} = \frac{\partial \mathcal{H}}{\partial p}(t, q, u, p) \\ \frac{d_F^\alpha p}{dt^\alpha} = -\frac{\partial \mathcal{H}}{\partial q}(t, q, u, p). \end{cases} \quad (\text{HS})$$

Next, we prove the following optimality condition for the functional (56).

**Theorem 43** Let  $u \in \mathring{\mathcal{A}}$  and  $\bar{u} \in \mathcal{A}$ . Then

$$\delta J(u, \bar{u}) = J_{F, t_0}^\alpha \left[ \frac{\partial \mathcal{H}}{\partial u}(\cdot, q^u, u, p^u) \cdot \bar{u} \right] (t_1).$$

Therefore  $u$  is a weak extremal of  $J[-]$  if, and only if,  $(q^u, u, p^u)$  satisfy the equation:

$$\frac{\partial H}{\partial u}(t, q^u, u, p^u) = 0. \quad (\text{CSE})$$

**Proof.** Let  $\bar{u} \in \mathcal{A}$ . Theorem 41 asserts that

$$\delta J(u; \bar{u}) = J_{F, t_0}^\alpha \left[ \frac{\partial L}{\partial q}(\cdot, q^u, u) \cdot \bar{q} + \frac{\partial L}{\partial u}(\cdot, q^u, u) \cdot \bar{u} \right] (t_1). \quad (73)$$

Equation (73) can be written as

$$\begin{aligned} \delta J(u; \bar{u}) = J_{F, t_1}^\alpha & \left[ \left( \frac{\partial L}{\partial q}(\cdot, q^u, u) + \left( \frac{\partial \varphi}{\partial q}(\cdot, q^u, u) \right)^\top \cdot p^u \right) \cdot \bar{q} \right. \\ & \left. - \left( \frac{\partial \varphi}{\partial q}(\cdot, q^u, u) \cdot \bar{q} \right) \cdot p^u + \frac{\partial L}{\partial u}(\cdot, q^u, u) \cdot \bar{u} \right] (t_2). \end{aligned} \quad (74)$$

By (LCP<sub>1</sub>) and (LCP<sub>2</sub>), we have

$$\bar{q} = J_{F, t_0}^\alpha \left[ \frac{\partial \varphi}{\partial q}(\cdot, q^u, u) \cdot \bar{q} + \frac{\partial \varphi}{\partial u}(\cdot, q^u, u) \cdot \bar{u} \right] (t_1) \quad (75)$$

and

$$p^u = -J_{F, t_0}^\alpha \left[ \frac{\partial L}{\partial q}(\cdot, q^u, u) + \left( \frac{\partial \varphi}{\partial q}(\cdot, q^u, u) \right)^\top \cdot p^u \right] (t_1). \quad (76)$$

Now, using in (74) equalities (75), (76) and integrating by parts (Theorem 7) with  $\bar{q}(t_0) = 0$ ,  $p^u(t_1) = 0$ , we get

$$\begin{aligned} \delta J(u; \bar{u}) &= J_{F, t_0}^\alpha \left[ p^u \cdot \left( \frac{\partial \varphi}{\partial q}(\cdot, q^u, u) \cdot \bar{q} + \frac{\partial \varphi}{\partial u}(t, q^u, u) \cdot \bar{u} \right) \right. \\ &\quad \left. - \left( \frac{\partial \varphi}{\partial q}(\cdot, q^u, u) \cdot \bar{q} \right) \cdot p^u + \frac{\partial L}{\partial u}(\cdot, q^u, u) \cdot \bar{u} \right] (t_1) \\ &= J_{F, t_0}^\alpha \left[ \left( \frac{\partial \varphi}{\partial u}(\cdot, q^u, u) \right)^\top \cdot p^u + \frac{\partial L}{\partial u}(\cdot, q^u, u) \right] (t_1) \cdot \bar{u} \\ &= J_{F, t_0}^\alpha \left[ \frac{\partial \mathcal{H}}{\partial u}(\cdot, q^u, u, p^u) \cdot \bar{u} \right] (t_1). \end{aligned}$$

The proof is now completed applying the fundamental lemma (Lemma 12).

■

Summarizing these results yields the weak Pontryagin Maximum Principle.

**Corollary 44 (Weak Pontryagin Maximum Principle)** *If  $v \in \mathring{A}$  is a local extremal of  $J[-]$ , i.e. it solves problem (57), then  $(q^v, v, p^v) \in C^1([t_0, t_1], \mathbb{R}^d) \times C^0([t_0, t_1], \mathbb{R}^l) \times PC^1([t_0, t_1], \mathbb{R}^d)$  are solution of the system*

$$\begin{cases} \frac{d_F^\alpha q}{dt^\alpha} = \frac{\partial \mathcal{H}}{\partial p}(t, q, v, p) \\ \frac{d_F^\alpha p}{dt^\alpha} = -\frac{\partial \mathcal{H}}{\partial q}(t, q, v, p) \\ \frac{\partial \mathcal{H}}{\partial u}(t, q, v, p) = 0 \\ (q(t_0), p(t_1)) = (0, 0). \end{cases} \quad (\text{WPS})$$

Moreover, if  $(q, u, p) \in C^1([t_0, t_1], \mathbb{R}^d) \times C^0([t_0, t_1], \mathbb{R}^l) \times PC^1([t_0, t_1], \mathbb{R}^d)$  solve (WPS), then necessarily  $(q, p) = (q^u, p^u)$ .

**Proof.** By Theorem 39, we get that  $v$  is also a weak extremal of the functional  $J[-]$ . Therefore, the conclusion follows by (HS), Theorem 43 and (LCP<sub>1</sub>), (LCP<sub>2</sub>). ■

**Remark 45** *In the particular case  $\varphi \equiv v$ , i.e., when the generalized optimal control problem (P)-(CP) is reduced to the (GPCV), we obtain from (WPS)*

$$H = L(t, x, v) + p \cdot v, \quad \frac{d_F^\alpha x}{dt^\alpha} = v,$$

and the equations

$$\frac{d_F^\alpha p}{dt^\alpha} = -\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x}, \quad p = -\frac{\partial L}{\partial v}.$$

Comparing the two expressions for  $\frac{d_F^\alpha p}{dt^\alpha}$ , we obtain the Euler–Lagrange equations (20).

**Definition 46** *Any triplet  $(q(\cdot), u(\cdot), p(\cdot))$  satisfying the system (WPS) is called a generalized weak Pontryagin extremal.*

The next proposition generalizes the generalized du Bois–Reymond necessary optimality condition Theorem. 22 (as usual, set in the following theorem:

$\varphi(t, q, u) = u$  so that  $\frac{d_F^\alpha q}{dt^\alpha} = u$ ). Its proof follows directly from (72) and the system (WPS).

**Theorem 47** *The following property holds for any generalized weak Pontryagin extremals  $(q, u, p)$ :*

$$\frac{d_F^\alpha}{dt^\alpha} (\mathcal{H}(t, q(t), u(t), p(t))) = \frac{\partial \mathcal{H}}{\partial t}(t, q(t), u(t), p(t)) F(t, \alpha) \quad \forall t \in [t_0, t_1]. \quad (77)$$

## 4.2 Noether's Principle for optimal control

Using the same arguments of classical optimal control (see e.g. [22, 38]), we can deduce that the optimal problem (57) is equivalent to minimize the augmented functional  $J[q, u, p]$  defined by

$$J[q, u, p] := J_{F, t_0}^\alpha \left[ \mathcal{H}(t, q(t), u(t), p(t)) - p(t) \cdot \frac{d_F^\alpha q}{dt^\alpha}(t) \right] (t_1), \quad (78)$$

with  $\mathcal{H}$  given by (72). The notion of variational invariance for problem (57) is then defined with the help of the augmented functional (78).

**Definition 48** Let  $T = \{\tau(s, \cdot)\}_{s \in \mathbb{R}} \in C^2(T', T')$ ,  $S = \{\sigma(s, \cdot)\}_{s \in \mathbb{R}} \in C^2(S', S')$ ,  $U = \{v(s, \cdot)\}_{s \in \mathbb{R}} \in C^2(U', U')$ , and  $A = \{\pi(s, \cdot)\}_{s \in \mathbb{R}} \in C^2(A', A')$  be one parameter groups of diffeomorphisms of the open sets  $T' \subseteq \mathbb{R}$ ,  $S'$ ,  $A' \subseteq \mathbb{R}^d$  and  $U' \subseteq \mathbb{R}^l$ . The augmented functional (78) is said to be invariant under the action of  $T$ ,  $S$ ,  $U$ ,  $A$ , if for any generalized weak Pontryagin extremal  $(q, u, p)$ , the following equality holds

$$\begin{aligned} & \frac{\left\{ \mathcal{H}(\tau(s, t), \sigma(s, q(t)), v(s, u(t)), \pi(s, p(t))) - \pi(s, p(t)) \cdot \frac{d_F^\alpha \sigma(s, q(t))}{d(\tau(s, t))^\alpha} \right\} \frac{\partial \tau}{\partial t}(s, t)}{F(\tau(s, t), \alpha)} = \\ & = \frac{\mathcal{H}(t, q(t), u(t), p(t)) - p(t) \cdot \frac{d_F^\alpha q}{dt^\alpha}(t)}{F(t, \alpha)}. \end{aligned} \quad (79)$$

for all  $s \in \mathbb{R}$  and all  $t \in T'$ .

**Theorem 49 (Necessary condition of invariance for problem (57))** If the augmented functional (78) is invariant in the sense of Definition 48, then for all generalized weak Pontryagin extremal  $(q, u, p)$ , we have

$$\begin{aligned} & \frac{\partial \mathcal{H}}{\partial t}(\cdot, q, u, p) \frac{\partial \tau}{\partial s}(0, \cdot) + \frac{\partial \mathcal{H}}{\partial q}(\cdot, q, u, p) \cdot \frac{\partial \sigma}{\partial s}(0, q) + \frac{\mathcal{H}(\cdot, q, u, p)}{F(t, \alpha)} \frac{d_F^\alpha}{dt^\alpha} \frac{\partial \tau}{\partial s}(0, \cdot) \\ & - p \cdot \frac{d_F^\alpha}{dt^\alpha} \frac{\partial \sigma}{\partial s}(0, q) + \frac{\partial F}{\partial t} \left( \frac{\partial \tau}{\partial s}(0, q) \right) \left( p \cdot \frac{\partial \sigma}{\partial s}(0, q) - \mathcal{H}(\cdot, q, u, p) \right) \end{aligned} \quad (80)$$

**Proof.** We obtain (80) by differentiating (79) with respect to  $s$  at  $s = 0$  and then considering Rem. 26 and (WPS). ■

Note that for the particular case of calculus of variations ( $\varphi(t, q, u) = u$  and hence  $\mathcal{H} = L + p \cdot \dot{q}$ ) one obtains from (80) the necessary condition of invariance Lemma 27.

**Definition 50 (Constants of Motion/conservation law for problem (57))**

We say that a function  $t \in T' \mapsto C(t, q(t), u(t), p(t)) \in \mathbb{R}$  is a constant of motion on  $T'$ ,  $S'$ ,  $U'$ ,  $A'$  if along any generalized weak Pontryagin extremal  $(q, u, p)$ , we have

$$\frac{d_F^\alpha C}{dt^\alpha}(t, q(t), u(t), p(t)) = 0 \quad \forall t \in T'. \quad (81)$$

The condition 81 implies that

$$t \in J \mapsto C(t, q(t), u(t), p(t)) \in \mathbb{R}$$

is constant along any interval  $J \subseteq T'$ .

**Theorem 51 (Noether's theorem for optimal control)** *If the augmented functional (78) is invariant in the sense of Definition 48, then the quantity defined by*

$$\begin{aligned} C(t, q(t), u(t), p(t)) &:= J_{F, t_1}^\alpha G_1(t, q(t), u(t), p(t); \alpha)(t_2) + \\ &+ \mathcal{H}(t, q(t), u(t), p(t)) \frac{\partial \tau}{\partial s}(0, t) - F(t, \alpha) p(t) \cdot \frac{\partial \sigma}{\partial s}(0, q(t)) \quad \forall t \in T' \end{aligned} \quad (82)$$

is a constant of motion on  $T'$ ,  $S'$ ,  $U'$ ,  $A'$ , where  $G_1$  is a  $L_\alpha^1$ -function defined by

$$\begin{aligned} G_1(t, q(t), u(t), p(t); \alpha) &= \\ &= \frac{\partial F}{\partial t} \left( \frac{\partial \tau}{\partial s}(0, q) \right) \left( p(t) \cdot \frac{\partial \sigma}{\partial s}(0, q) - \mathcal{H}(t, q(t), u(t), p(t)) \right). \end{aligned}$$

**Proof.**

Considering (WPS), we obtain (80) by differentiating (82) with respect to  $t$  in the sense of generalized derivative. ■

**Remark 52** *Similarly to [22, 38], only the functions  $\tau$  and  $\sigma$ , corresponding to the one parameter groups of diffeomorphisms of the time and state variables, appear in Noether's conservation laws (cf. expression (82)).*

## 5 Application-time-fractional Schrödinger equation

In [17, 54], a fractional linear Schrödinger equation, with particular interest in quantum mechanics, is studied. In this section we prove that, under certain condition, solutions of the fractional linear Schrödinger equation coincide with the extremals of a quantum functional via the time-fractional Hamilton–Jacobi–Bellman equation. In addition we use our Corollary 30 to find constants of motion for a fractional quantum variational problem. Precisely, consider the following time-fractional linear Schrödinger equation:

$$i\hbar \frac{\partial_F^\alpha \psi}{\partial t^\alpha}(t, x) + \frac{\hbar^2}{2m} \sum_{j=1}^d \frac{\partial^2 \psi}{\partial x_j^2}(t, x) = U(x) \psi(t, x), \quad (83)$$

where  $\hbar = \frac{h}{2\pi}$ ,  $h$  is the Planck constant,  $m > 0$  the mass of particle, the potential energy  $U : \mathbb{R}^d \rightarrow \mathbb{R}$  and  $\psi : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{C}$  is the wave function associated to the

particle on  $\mathcal{C}^2(\mathbb{R} \times \mathbb{R}^d, \mathbb{C})$ . We know that the action  $S$  obeys the time-fractional Hamilton–Jacobi–Bellman equation [37]

$$\frac{\partial^\alpha S}{\partial t^\alpha}(t, x) + \frac{1}{2m} \left( \frac{\partial \psi}{\partial x_j}(t, x) \right)^2 + U(x) = 0, \quad j = 1, \dots, d. \quad (84)$$

Generally, (84) is the conservation of energy in disguise; the first term equals  $-E$ , and the second term is the kinetic energy. Assume that

$$\frac{\partial S}{\partial x_j} = -\frac{i\hbar}{\psi} \frac{\partial \psi}{\partial x_j}, \quad j = 1, \dots, d \quad (85)$$

and making the allowance for the complex nature of  $\psi$ , we can reformulate (84) as

$$(U - E)\psi\psi^* + \frac{\hbar^2}{2m} \left( \frac{\partial \psi}{\partial x_j} \right) \left( \frac{\partial \psi^*}{\partial x_j} \right) = 0, \quad j = 1, \dots, d, \quad (86)$$

where  $*$  denotes complex conjugation.

The Schrödinger equation is motivated by taking a look at the classical relation between energy and momentum of particle, quantization is done by replacing the physical quantities by operators corresponding to them and state or wave function on which they operate. This corresponding operator for the energy is given by [21, 24, 31]

$$E \rightarrow i\hbar \frac{\partial^\alpha}{\partial t^\alpha}. \quad (87)$$

**Theorem 53** *Let  $L$  be a quantum Lagrangian defined by*

$$L \left( \psi(t, x), \psi^*(t, x), \frac{\partial \psi}{\partial x}, \frac{\partial \psi^*}{\partial x} \right) = (U - E)\psi\psi^* + \frac{\hbar^2}{2m} \sum_{j=1}^d \left( \frac{\partial \psi}{\partial x_j} \frac{\partial \psi^*}{\partial x_j} \right). \quad (88)$$

*Then the solutions of the Schrödinger equation (83) coincide with the extremals of the quantum Lagrangian.*

**Proof.** Considering the problem to minimize a functional with the quantum Lagrangian (88), we obtain from the Euler–Lagrange (20) for the generalized coordinate  $\psi^*$

$$(U - E)\psi - \frac{\hbar^2}{2m} \sum_{j=1}^d \frac{\partial^2 \psi}{\partial x_j^2} = 0. \quad (89)$$

Substituting (87) into (89), we obtain the time-fractional linear Schrödinger equation (83). ■

**Corollary 54** *For an autonomous problem with the quantum Lagrangian (88), we have*

$$(U - E)\psi\psi^* \quad (90)$$

*is a constant of motion along all extremals of the Euler–Lagrange (89).*

**Proof.** As the Lagrangian (88) is autonomous (with respect of  $x$ ), its invariant in the sense of Definition 25 under the symmetries  $(\tau_x, \sigma_{\psi^*}) = (c_j, 0)$ , where  $c_j$  is an arbitrary constant. So, the quantity (90) follows from our Corollary 30. ■

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